UNIVERSIDAD NACIONAL DE COLOMBIA

Minimization of Energy Functionals in Elasticity

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Año 2013

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Trabajo final presentado como requisito parcial para optar al título de:
Magister en Matemáticas

Director:
Leonid P. Lebedev

Línea de Investigación:
Mecánica de cuerpos elásticos

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Año 2013

Acknowledgements

Above all, I would like to thank God, who is the driving force in my life. I appreciate the spiritual support of my family throughout my life, without whose encouragement I would not pursue a masters degree.

I must acknowledge my advisor Leonid Petrovich Lebedev, whose patience and extensive knowledge were essential during my thesis. Our mathematical debates strongly shaped the way I see and feel mathematics. Appreciation goes out to Felix Soriano and Leonardo Rendón, for their many insights in functional analysis and theory of Sobolev spaces.

I would also like to thank my classmates and friends Frank Trujillo, Óscar Riaño, Daniel Montenegro and Daniel Garzón, for our exchanges of mathematical knowledge; their proposals of problems enriched my understanding and helped me to develop some tricks. I appreciate scientific discussion with Diego Tapias, who is an excellent chemist and whose advice was always accurate.

This thesis was financially supported by a grant from the MAZDA Foundation for Art and Science.

Title. Minimization of Energy Functionals in Elasticity

Abstract

Existence of equilibrium states of elastic bodies under a load is a main concern in the mathematical theory of elasticity, however the large number of models makes it necessary to develop a general theory that includes as many elastic models as possible. This thesis is devoted to linear models, and the methods exposed here cover most of the linear problems so far described in the literature. Energy spaces are a suitable setup for finding minimum points of energy functionals, so providing existence theorems. The central theorem is a generalization of Korn's inequality, yielding sufficient and necessary conditions for a linear functional to have an equilibrium state in a Sobolev space. Some problems in elasticity are studied with the aid of the theory developed, including a nonlinear energy functional.

Keywords: elasticity, energy spaces, Korn type inequalities, existence of solution

Título. Minimización de funcionales de energía en elasticidad

Resumen

Uno de los principales problemas en la teoría matemática de cuerpos elásticos es probar la existencia de estados de equilibrio de un cuerpo elástico sometido a fuerzas externas, sin embargo debido a la gran cantidad de modelos en elasticidad, es necesario desarrollar una teoría que incluya el mayor número de modelos. El objetivo de esta tesis es exponer métodos que permitan estudiar una amplia gama de problemas lineales descritos en la literatura. Los espacios de energía resultan ser adecuados para encontrar puntos mínimos, obteniendo así teoremas de existencia. El teorema central es una generalización de la desigualdad de Korn, con el cual se encuentran condiciones necesarias y suficientes para que un funcional tenga un estado de equilibrio en un espacio de Sobolev. Usando la teoría desarrollada se estudian algunos problemas en elasticidad, incluyendo funcionales de energía no lineales.

Palabras clave: elasticidad, espacios de energía, desigualdades tipo Korn, existencia de solución.

Contents

List of Symbols

a.e. almost everywhere, page 21

$\Subset$ Compact inclusion, page 17

$B(u,R) = \{v \mid \|u-v\| < R\}$, page 7

$S(u,R) = \{v \mid \|u-v\| = R\}$, page 7

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$, page 18

$\boldsymbol{a}_i$ Convariant basis vector, page 2

$\boldsymbol{a}^i$ Contravariant basis vector, page 2

a_{ij} First fundamental form, page 3

b_{ij} Second fundamental form, page 4

Γ^k_{ij} Christoffel symbols, page 3

s.w.l.s-c Sequentially weakly lower semi-continuous, page 6

s.w.c sequentially weakly continuous, page 40

$K(r,h)$ Standard cone with aperture r and vector direction h, page 13

$\mathbb{M}_+$ Set of matrices with positive determinant, page 5

$\mathcal{D}(T)$ Domain of a linear operator T, page 9

$\mathfrak{G}(T) = \{(x,Tx) \mid x \in \mathcal{D}(T)\}$ Graph of a linear operator T, page 9

$C^\infty(\Omega)$ Space of infinitely differentiable functions in Ω, page 12

$C^l_b(\Omega)$ Space of bounded functions with bounded continuous derivatives of order l, page 13

$C_c(\Omega)$ Space of continuous functions with compact support in Ω, page 12

$C^\infty_c(\Omega)$ Space of test functions, page 12

$W^{l,p}(\Omega)$ Sobolev space of derivatives of order l and exponent p, page 13

$H^l(\Omega) = W^{l,2}(\Omega)$. , page 22

$W^{-l,p'}(\Omega)$ Dual space of $W^{l,p}(\Omega)$, page 13

E_h Operator of translation by $-h$, page 17

$\triangle_h^l = (E_h - 1)^l$ Finite difference operator, page 17

tr Trace operator, page 18

1 Introduction

The theory of elastic bodies goes back to Galileo Galilei, who early in the sixteenth century conducted experiments on beams. His work paved the way toward subsequent investigations, but the first landmark discovery was Hooke's law, first published as an anagram in 1660. The theory was then enriched with contributions of mathematicians like Euler, Jacob Bernoulli, Daniel Bernoulli, Cauchy, Navier and Green, among others. The reader interested in the history of elasticity can consult Love [27].

Elasticity is not only important in structural designs, but it is also a huge source of problems of mathematical interest. The theory still lacks existence theorems for important models and even less is known about regularity in the nonlinear case. The purpose of this thesis is to review progress in the linear theory of elastic bodies.

1.1 Preliminaries in Elasticity

Elasticity is mainly devoted to materials whose stress state depends on its present moment, regardless of the process the material undergoes to reach it. Although this is experimentally false, predictions based on this assumption are acceptable in practice. Additionally, we restrict ourselves to materials satisfying Hooke's law. This is not a great disadvantage, since metals are well described under this assumption and they are mainly used within its linear range. Nonlinear materials such as rubbers will be only briefly discussed.

We are interested only in materials whose state can be described by an energy density function. These materials are called hyperelastic. The simplest example in the theory of elasticity is the spring. Before deforming the spring, it is in its reference configuration, and after stretching it a length x, the energy stored by the spring is $\frac{1}{2}kx^2$. The problem of the minimization of energy here is fairly simple, since the total energy of the spring under a load f is given by $E(x) = \frac{1}{2}kx^2 - fx$. In the one-dimensional case the deformation is thus described by the change of length.

The general case of three-dimensional bodies is preferably described in curvilinear coordinates. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and $\boldsymbol{\varphi} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism onto its image $\boldsymbol{\varphi}(\Omega) = \hat{\Omega}$. The function $\boldsymbol{\varphi}$ is the parameterization of the manifold $\hat{\Omega}$, which represents the elastic body in its reference configuration. The derivative with respect to x_i is denoted by $\partial\boldsymbol{\varphi}/\partial x_i = \partial_i\boldsymbol{\varphi}$.

The linearly independent vectors $\boldsymbol{a}_i = \partial_i\boldsymbol{\varphi}$ are called covariant basis vectors of the tangent space. The contravariant basis vectors $\boldsymbol{a}^i$ are completely determined by the relation

$\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j$. The first fundamental form is given by $a_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$ and the volume element by $\sqrt{a}\,dx$, where $a = \det(a_{ij})$. The covariant derivative of a function $\boldsymbol{v} = (v_i \boldsymbol{a}^i) : \Omega \to \mathbb{R}^3$ is denoted by

$$v_{i|j} = \partial_i v_j - v_k \Gamma^k_{ij},$$

where Γ^k_{ij} are the Christoffel symbols. We will use hereinafter the Einstein convention about summations.

Once a force is applied on the body, it undergoes a deformation carrying the point $\boldsymbol{\varphi}(x)$ into the point x^d, so defining a deformation function $\boldsymbol{\varphi}^d(x) = x^d$. The deformed configuration is $\boldsymbol{\varphi}^d(\Omega) = \hat{\Omega}^d$ and the function $\boldsymbol{u} = \boldsymbol{\varphi}^d - \boldsymbol{\varphi}$ is the displacement. The first fundamental form of the deformed configuration is denoted by $a_{ij}(\boldsymbol{u}) = (\boldsymbol{a}_i + \partial_i \boldsymbol{u}) \cdot (\boldsymbol{a}_j + \partial_j \boldsymbol{u})$.

Since the distance between two points depends on the first fundamental form, the change of it measures the stretching or deformation of an elastic body

$$a_{ij}(\boldsymbol{u}) - a_{ij} = u_{i|j} + u_{j|i} + \partial_i \boldsymbol{u} \cdot \partial_j \boldsymbol{u}.$$

The last term $\partial_i \boldsymbol{u} \cdot \partial_j \boldsymbol{u}$ is nonlinear and hard to deal with, so linear theory of elasticity assumes small deformations and rules out this nonlinearity. This approximation has been used for years in the design of buildings, yielding good results. Thus, instead of the equation above, we use the linearized strain tensor

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i|j} + u_{j|i}),$$

which is the symmetric part of the jacobian matrix $\nabla \boldsymbol{u}$. The energy stored by the elastic body after a displacement $\boldsymbol{u}$ is

$$\frac{1}{2} \int_\Omega E^{ijkl} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{kl}(\boldsymbol{u}) \sqrt{a}\,dx,$$

where E^{ijkl} is the tensor of elastic moduli. Additionally, we have the relation $E^{ijkl} = E^{jikl} = E^{klij}$, which can be deduced from physical considerations. If body forces $\boldsymbol{f}$, like gravity, and an external load $\boldsymbol{g}$ are applied respectively in the body and on a part of the surface $\Gamma \subset \partial\Omega$, then the total energy is

$$E(\boldsymbol{u}) = \frac{1}{2} \int_\Omega E^{ijkl} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{kl}(\boldsymbol{u}) \sqrt{a}\,dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \sqrt{a}\,dx - \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{u}\,d\sigma, \tag{1-1}$$

where $d\sigma$ is the surface area element. For more detailed discussion on general elasticity, see Lebedev & Cloud [24], Green & Zerna [18], Nečas & Hlaváček [32] and Ciarlet [10].

In the design of ships or rockets, the metallic cover is modelled as a thin shell, so reducing a three-dimensional problem to a two-dimensional one. Let $\boldsymbol{\varphi} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a diffeomorphism with image $\hat{\Omega}$, the middle surface of the shell. The normal vector to the surface is

$$\boldsymbol{a}_3 = \boldsymbol{a}^3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}. \tag{1-2}$$

A shell of thickness $2h$ occupies the volume

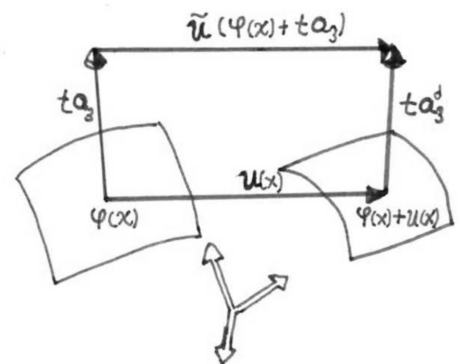
$$\mathcal{S} = \{\boldsymbol{\varphi}(x) + \theta \boldsymbol{a}_3 \mid x \in \Omega \text{ and } -h \leq \theta \leq h\}.$$

The second fundamental form of the middle surface is $b_{ij} = \partial_{ij}\boldsymbol{\varphi} \cdot \boldsymbol{a}_3$, and the curvature at a point x in the direction $\boldsymbol{l} = l^1\boldsymbol{a}_1 + l^2\boldsymbol{a}_2$ is

$$k(x, \boldsymbol{l}) = \frac{b_{ij}l^i l^j}{a_{ij}l^i l^j}.$$

The radius of curvature is $R(x, \boldsymbol{l}) = 1/k(x, \boldsymbol{l})$. The maximum of the function $R(x, \boldsymbol{l})$ is called the radius of curvature of the surface $R_{\max}$. The equations of elasticity are greatly simplified if $h/R_{\max}$ is negligible, however this assumption does not hold if the shell has corners where $R \to 0$. The reader can consult Truesdell [38].

Since we want the middle surface to be sufficient for describing the shell, we suppose that the normal section of the shell is preserved after deformations and that the thickness of the shell is constant. These hypotheses are called the Kirchhoff–Love approximation. To see why it is sufficient to know the displacement of the middle surface, let $\tilde{\boldsymbol{u}} : \mathcal{S} \to \mathbb{R}^3$ define the displacement of the shell and denote its restriction to the middle surface by $\boldsymbol{u}$. The normal vector after deformation is



$$\boldsymbol{a}_3^d = \frac{(\boldsymbol{a}_1 + \partial_1\boldsymbol{u}) \times (\boldsymbol{a}_2 + \partial_2\boldsymbol{u})}{|(\boldsymbol{a}_1 + \partial_1\boldsymbol{u}) \times (\boldsymbol{a}_2 + \partial_2\boldsymbol{u})|}.$$

Since $\tilde{\boldsymbol{u}}(\boldsymbol{\varphi}(x) + t\boldsymbol{a}_3)$ must lie in the normal at $\boldsymbol{\varphi}^d(x) = \boldsymbol{\varphi}(x) + \boldsymbol{u}(x)$, it is clear from the figure that

$$\boldsymbol{u}(x) + t\boldsymbol{a}_3^d = t\boldsymbol{a}_3 + \tilde{\boldsymbol{u}}(\boldsymbol{\varphi}(x) + t\boldsymbol{a}_3),$$

we can solve for $\tilde{\boldsymbol{u}}$ and so the middle surface determines the displacements in the whole shell.

The measure of stretching in the middle surface is, as before, the change of the first fundamental form; after linearization we have

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i|j} + u_{j|i}) - b_{ij}u_3.$$

On the other hand, the bending of the shell is related to the change of the second fundamental form, therefore we introduce the linearized tensor of change of curvature $\rho_{ij}(\boldsymbol{u})$ as we did for the first fundamental form

$$\rho_{ij}(\boldsymbol{u}) = u_{3|ij} - b_i^k b_{kj} u_3 + b_i^k u_{k|j} + b_j^k u_{k|i} + b_{j|i}^k u_k.$$

The total energy of a shell subjected to an external load $\boldsymbol{f}$, and to a force $\boldsymbol{g}$ and a moment M along a segment of the edge Γ is

$$E(\boldsymbol{u}) = \frac{1}{2}\int_\Omega E^{ijkl}\varepsilon_{ij}(\boldsymbol{u})\varepsilon_{kl}(\boldsymbol{u}) + D^{ijkl}\rho_{ij}(\boldsymbol{u})\rho_{kl}(\boldsymbol{u})\,dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\,dx - \int_\Gamma (\boldsymbol{g} \cdot \boldsymbol{u} + M\partial_{\boldsymbol{n}} u_3)\,d\sigma$$

where E^{ijkl} and D^{ijkl} are symmetric tensors, and $\partial_{\boldsymbol{n}} u_3$ is the derivative in the outward normal direction. Some relevant books on the theory of shells are Novozhilov *et al.* [33], Ciarlet [11] and Timoshenko & Woinowsky [36].

Further simplifications in the theory of shells lead to different models, such as membrane shells, purely bending shells, revolution shells, shallow shells and plates, among others. The last model, namely plates, was the first "shell" theory and is still used in structural designs, for example floors in buildings are plates.

All models so far have the same general form, that is, the internal energy of the body is a bilinear form B whose argument is the displacement in the elastic body, and the external forces $\boldsymbol{f}$ are linear functionals, therefore the problem of seeking the minimum energy of an elastic body turns into the abstract problem of minimizing the *energy functional*

$$E(\boldsymbol{u}) = \frac{1}{2} B(\boldsymbol{u}, \boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{u}). \tag{1-3}$$

This is the central problem discussed in this thesis.

A Short Note on Nonlinearity

Nonlinear problems are difficult in general, but since some materials do not obey linear models, even for very small deformations, it is necessary to study the energy functional of these materials asunder. Let us assume that the problem is three dimensional and that the parameterization $\boldsymbol{\varphi}$ is the identity function. The internal energy of a body is usually written as

$$\int_\Omega W(x, \nabla \boldsymbol{\varphi}^d)\, dx,$$

where $W : \Omega \times \mathbb{M}_+ \to \mathbb{R}$ is called the stored energy functional. The symbol $\mathbb{M}_+$ stands for the set of matrices with positive determinant. The function W satisfies the condition of frame-indifference or objectivity if $W(x, \boldsymbol{F}) = W(x, \boldsymbol{QF})$ for every $\boldsymbol{F} \in \mathbb{M}_+$ and every proper orthogonal transformation $\boldsymbol{Q}$. The objectivity says that the behavior of the material does not depend on the frame of reference, whereby every realistic model should satisfy this relation.

When the function $W(x, \boldsymbol{F})$ is convex with respect to the variable $\boldsymbol{F}$, theorems of existence are well known, but convexity conflicts with important physical phenomena [10, ch. 4]. The breakthrough was made by John Ball, who introduced the concept of polyconvexity, which allowed him to prove existence theorems for many realistic non-convex models [4].

1.2 Remarks on Existence of Solution

Minimization of the functional (1-3) was discussed by Mikhlin [28], who applied methods in functional analysis to prove existence of a unique minimum point for an elastic body under a load. The following theorem provides the guidelines to prove existence of a minimum point;

in fact, in the next chapter we will try to find sufficient conditions, so that a functional satisfies the hypotheses in this theorem.

Theorem 1.1 ([8])**.** *Let B be a continuous bilinear form in a Hilbert space H. Assume that B is a coercive bilinear form, that is, there exists a constant $C > 0$ such that*

$$B(u,u) \geq C\|u\|^2 \tag{1-4}$$

for every $u \in H$. Let $K \subset H$ be a nonempty closed and convex subset. Then, given any $w \in H$, there exists a unique element $u_0 \in K$ such that

$$B(u_0, v - u_0) \geq (w, v - u_0) \qquad \text{for every } v \in K. \tag{1-5}$$

In particular, if $K = H$ then

$$B(u_0, v) = (w, v) \qquad \text{for every } v \in H. \tag{1-6}$$

Moreover, if B is symmetric, then u_0 is the minimum point of the functional

$$I(v) = \frac{1}{2}B(v,v) - (w,v). \tag{1-7}$$

Although there is not such a general theorem for nonlinear functionals, the theory of calculus of variations has been so highly developed that a wide class of nonlinear functionals is covered; the reader can consult Giaquinta [16]. The lack of compactness of the unit ball in the norm topology of infinite-dimensional spaces makes it difficult to minimize functionals, however by weakening the topology, the collection of compact sets increases, hence it is convenient to focus on continuous functionals with respect to the weak topology.

Definition 1.2. A functional $I : X \to \mathbb{R}$ in a Banach space X is sequentially weakly lower semi-continuous (s.w.l.s-c) if for every sequence $u_n \rightharpoonup \tilde{u}$ we have $\liminf_n I(u_n) \geq I(\tilde{u})$.

As for finite-dimensional spaces, we can assert existence of a minimum point using the weak topology.

Lemma 1.3. *Let K be a weakly compact set in a Banach space X. If $I : X \to \mathbb{R}$ is s.w.l.s-c. then there exists a minimum point in K.*

Proof. Suppose I does not have a lower bound in K, then there exists a sequence $\{u_n\} \in K$ such that $I(u_n) \to -\infty$. By the compactness of K we select a subsequence, say $\{u_n\}$, such that $u_n \rightharpoonup \tilde{u}$; then $\liminf_n I(u_n) \geq I(\tilde{u}) > -\infty$, contradicting our initial assumption.

Define $d = \inf_{u \in K} I(u)$ and let $\{u_n\}$ be a minimizing sequence, i.e. $I(u_n) \to d$. Choose a subsequence such that $u_n \rightharpoonup \tilde{u}$, then $\liminf_n I(u_n) = d \geq I(\tilde{u})$ and so $I(\tilde{u}) = d$. □

The following theorem shows that convex functionals are well behaved with respect to the weak topology.

Theorem 1.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and define a continuous function $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ satisfying the conditions:*

i) W is bounded from below;

ii) $W(x, \boldsymbol{p}, \boldsymbol{q})$ is convex with respect to $\boldsymbol{q}$.

Then the functional

$$I(\boldsymbol{u}) = \int_\Omega W(x, \boldsymbol{u}, \nabla \boldsymbol{u}) \tag{1-8}$$

is s.w.l.s-c with respect to the weak topology in $[W^{1,q}(\Omega)]^m$ for $1 \leq q < \infty$.

In finite-dimensional spaces, the fact that a function is bounded from below and continuous does not ensure existence of a minimum point; we need to be sure the function does not approach the lower bound uniquely at infinity. Let us introduce the concept of growing functionals.

Definition 1.5. A functional $I : X \to \mathbb{R}$ in a Banach space X is growing whenever $\liminf_{\|u\|=R} I(u) \to \infty$ as $R \to \infty$.

Suppose a functional $I : X \to \mathbb{R}$ is s.w.l.s-c and growing in a reflexive Banach space X, then $I(u) \geq A$ for some A and $\|u\| \geq R$ if R is large enough; since $B(0, R) = \{u \mid \|u\| \leq R\}$ is weakly compact, from Lemma 1.3 the functional I is bounded from below in $B(0, R)$ and thus also in X. Now define $d = \inf_{u \in X} I(u)$ and take R such that if $\|u\| \geq R$ then $I(u) > d$, so that I attains its global minimum in $B(0, R)$.

The reader might notice that in Theorem 1.1 we required the functional I to be growing through condition (1-4). The bottleneck in proving existence theorems is often to verify that the functional at hand is growing, and in the next chapter we will devote a big effort to this.

Homology theory can be used to prove existence of critical points rather than global minimums; nevertheless, we can still be interested in finding at least local minimums, like in the problem of buckling.

Suppose that the functional I in a Hilbert space H has continuous derivative I' in Fréchet sense, so by Riesz representation theorem the derivative at each point $u \in H$ is a vector $\nabla I(u)$ such that $I'(u; v) = (\nabla I(u), v)$. Therefore we have a vector field ∇I and we must seek the zeros of ∇I. In finite-dimensional spaces we try to calculate the degree of the image of ∇I for some large enough sphere $S(0, R) = \{u \mid \|u\| = R\}$; if the degree is different from zero, the set $\nabla I(S(0, R))$ encloses the zero point and so there exists a critical point of I. Essentially, this method works well because $H - \{0\}$ is not contractible whenever H is finite dimensional, which is intuitively obvious, however infinite-dimensional spaces are not so simple, as the following well-known theorem shows.

Theorem 1.6. *The space $l^2(\mathbb{N}) - \{0\}$ is contractible.*

Proof. Define the right shift operator

$$E : l^2(\mathbb{N}) - \{0\} \to l^2(\mathbb{N}) - \{0\} \tag{1-9}$$

as $E(u_1, u_2, \ldots) = (0, u_1, u_2, \ldots)$ and define the homotopy $F(u, t) = (1 - t)u + tEu$. Since $F(u, t) = 0$ is impossible, the homotopy is well defined. Now define $G(u, t) = (1-t)Eu + te$, where $e = (1, 0, \ldots)$. By transitivity the identity map is null-homotopic. □

To prove existence theorems using homology it is necessary to impose conditions of compactness to avoid this weird behavior of infinite-dimensional spaces. We refer the reader to Krasnosel'skii [23] for detailed discussion.

2 Energy Spaces and Generalized Solutions

Energy spaces are maybe the most suitable spaces to find a minimum point of an energy functional; these spaces take advantage of the bilinearity involved in the energy functional to define an inner product and then we apply Riesz representation theorem or Lax-Milgram theorem to prove existence of a minimum point. The purpose in this chapter is not only to construct energy spaces, but also investigate their relationship with Sobolev spaces, which are standard and thus well characterized.

The hardest problem we will face is to prove coerciveness of bilinear forms, and for that we will use theory of singular integrals, obtaining a general theorem in theory of Sobolev spaces. As an additional outcome, we will show that existence of non-mixed derivatives bounded in L^p implies existence of all other derivatives also bounded in L^p.

2.1 Unbounded Operators

This is only a brief review on unbounded operators, see Yosida [41] for detailed discussion. Let X and Y be Banach spaces and define a linear operator $T : \mathcal{D}(T) \subset X \to Y$ whose domain $\mathcal{D}(T) \subset X$ is a dense subspace of X. A linear operator is bounded if $\sup\{\|Tx\| \mid \|x\| \leq 1\} < \infty$. Since every bounded operator can be uniquely and continuously extended to the whole space X, then we assume that every bounded operator is defined in the whole X.

If an operator is not bounded, then it is referred to as unbounded; in such a case, we would like the operator to be at least closed, i.e. the graph $\mathfrak{G}(T)$ of the operator is a closed subspace in $X \times Y$. An operator T is closable if there exists a closed operator S such that $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $S = T$ in $\mathcal{D}(T)$; the operator S is called a closed extension, and the minimal closed extension $\overline{T}$ is called the closure of T. The graph space is the closure of the vector space $\mathfrak{G}(T)$ with the norm $\|(x, Tx)\|_T = \|x\|_X + \|Tx\|_Y$. If X and Y are L^p spaces, we prefer the equivalent norm $\|(x, Tx)\|_T^p = \|x\|^p + \|Tx\|^p$. The projection of $X \times Y$ onto X restricted to $\overline{\mathfrak{G}(T)}$ is denoted by π; whenever the operator is not closable, there is an element $y \neq 0 \in Y$ such that $(0, y) \in \overline{\mathfrak{G}(T)}$ and thus π is not injective. On the other hand if T is closable, then π is injective and we can make the identification $x \leftrightarrow (x, Tx)$, therefore in this case we will not distinguish between x and (x, Tx).

The dual space of X is denoted by X^*. The adjoint operator $T^* : \mathcal{D}(T^*) \subset Y^* \to X^*$ has as domain the subspace

$$\mathcal{D}(T^*) = \{y^* \in Y^* \mid y^* \circ T \in X^*\};$$

the adjoint operator is defined by $T^*y^* = y^* \circ T$. When X and Y are Hilbert spaces, there is other definition of adjoint, called geometrical adjoint $T' : \mathcal{D}(T') \subset Y \to X$. Let us first introduce the function $\psi_X : X \to X^*$ that maps every x into the linear functional $(\cdot, x)$. By the Riesz representation theorem, the function ψ is well defined, semi-linear and isometric onto Y. The geometrical adjoint is defined by $T' = \psi_X^{-1} \circ T^* \circ \psi_Y$, for which we have the relation

$$(x, T'y) = \psi_X(T'y)(x) = ((T^* \circ \psi_Y)y)(x) = (\psi_Y y \circ T)(x) = (Tx, y)$$

whenever $y \in \mathcal{D}(T')$. So we conclude that T' is just the representation of T^* in Y, then we write T^* for both adjoints.

Lemma 2.1. *Let $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator.*

a) If T is closable, then $\mathfrak{G}(\overline{T}) = \overline{\mathfrak{G}(T)}$.

b) T is closable if and only if the domain of the adjoint T^* is total, i.e. $y^*(y) = 0$ for every $y^* \in \mathcal{D}(T^*)$ implies that $y = 0$.

c) Suppose that T is closed. If π is compact then $\ker T$ is finite-dimensional.

d) If T is closed, then $\mathfrak{G}(T)^*$ is isomorphic to the space $(X^* \times Y^*)/\mathfrak{G}(T)^\perp$ with norm

$$\|(x^*, y^*) + \mathfrak{G}(T)^\perp\| = \inf\{\|(x^*, y^*) + (u^*, v^*)\|_{X^* \times Y^*} \mid (u^*, v^*) \in \mathfrak{G}(T)^\perp\}. \tag{2-1}$$

Proof. b) The orthogonal space of a set $M \subset X$ is

$$M^\perp = \{x^* \in X^* \mid x^*(x) = 0 \text{ for every } x \in M\}.$$

Notice that $(X \times Y)^* = X^* \times Y^*$ and define the function $V : X^* \times Y^* \to Y^* \times X^*$ mapping $(x^*, y^*) \mapsto (-y^*, x^*)$, then we claim that $V\mathfrak{G}(T)^\perp = \mathfrak{G}(T^*)$. In fact, $(y^*, x^*) \in V\mathfrak{G}(T)^\perp$ if and only if $x^*(x) = y^*Tx$ for every $x \in \mathcal{D}(T)$, which holds if and only if $(y^*, x^*) \in \mathfrak{G}(T^*)$.

Now, T is closable if and only if $(0, y) \in \overline{\mathfrak{G}(T)}$ implies that $y = 0$, which in turns holds by the preceding paragraph if and only if $y^*(y) = 0$ for every $y^* \in \mathcal{D}(T^*)$ implies that $y = 0$. We recall that $\mathfrak{G}(T)^\perp = \overline{\mathfrak{G}(T)}^\perp$.

c) Define the set $M = \{(x, 0) \in \mathfrak{G}(T)\}$. The restriction of π to M is a compact isometry between M and $\ker T$, therefore by the open mapping theorem we must only prove that M is a closed space, because the ball in $\ker T$ would be compact. If $(x_n, 0) \to (x, y)$ then $(x, y) \in \mathfrak{G}(T)$, because $\mathfrak{G}(T)$ is closed; hence $y = 0$ and $(x, 0) \in M$.

□

Once an operator T is proved to be closable, we will replace it by its closure and denote it simply as T. The following somewhat obscure lemma will be useful when studying energy spaces. This is a generalization of a result due to Vorovich [39, p. 80].

Lemma 2.2. *Let $T : \mathcal{D}(T) \subset X \to Y_1$ be a closed operator and $M \subset \mathfrak{G}(T)$ a closed subspace. Suppose that*

a) π is compact.

b) $Q : \mathcal{D}(Q) = \mathcal{D}(T) \subset X \to Y_2$ is such that

$$C_1 \|x\|_T \leq \|x\|_Q \leq C_2 \|x\|_T. \tag{2-2}$$

c) $R : \mathcal{D}(R) \subset X \to Y_2$ is such that $\mathcal{D}(R) \supset \mathcal{D}(Q)$ and $R\pi$ is compact.

d) The operator $\tilde{Q}\pi = (Q + R)\pi$ is injective in M.

Then the norm $\|x\|_{\tilde{Q}} = \|\tilde{Q}\pi x\|_{Y_2}$ and the graph norm are equivalent in M.

Remark. Note that we write $\|x\|_T$ instead of $\|(x, Tx)\|_T$, because we can make the identification $x \leftrightarrow (x, Tx)$ through π. The inclusion $\mathcal{D}(R) \supset \mathcal{D}(Q)$ is also consequence of the compactness of $R\pi$.

Proof. The inequality $\|\tilde{Q}\pi x\|_{Y_2} \leq C\|x\|_T$ follows immediately from the continuity of $R\pi$. For the reverse inequality, assume it is false; hence there exists a sequence $\{x_n\} \in M$ such that $\|x_n\|_T = 1$ and $\|\tilde{Q}\pi x_n\|_{Y_2} \to 0$. Since π is compact, there exist $x^* \in X$ and a subsequence, say $\{x_n\}$, such that $x_n \to x^*$ in X. Furthermore, the compactness of $R\pi$ allows us to assume that $Rx_n \to y^*$ in Y_2 for some $y^* \in Y_2$, therefore $Qx_n \to -y^*$ and since Q is closed we have $x^* \in \mathcal{D}(Q) \subset \mathcal{D}(R)$. Obviously $Qx^* = -y^*$ and to see that $Rx^* = y^*$, note that $\|R\pi(x_n - x^*)\| \leq C\|x_n - x^*\|_T \to 0$. From the injectivity of $\tilde{Q}\pi$ in M we conclude that $x^* = 0$.

For large enough n we get

$$\|\tilde{Q}\pi x_n\|_{Y_2} \geq \|x_n\|_Q - \|Rx_n\|_{Y_2} - \|x_n\|_X \geq C_1\|x_n\|_T - \varepsilon;$$

letting $n \to 0$ we get $\|x_n\|_T \to 0$, contradicting our initial assumption $\|x_n\|_T = 1$. □

Corollary 2.3. *Let $T : \mathcal{D}(T) \subset X \to Y$ be a closed operator with compact π. Suppose that $M \subset \mathfrak{G}(T)$ is a closed subspace such that $M \cap \ker T = 0$, then*

$$\|Tx\|_Y \geq C\|x\|_X \tag{2-3}$$

for every $x \in M$.

The reader might recognize in (2-3) a generalization of Friedrich's inequality. Indeed, Lemma 2.2 could be seen as a generalization of Poincaré inequality. The drawback in the preceding proof is that the constant C remains unknown.

2.2 Sobolev Spaces

As early as the beginning of the twenty century, it was clear that the idea of a differentiable function, with classical derivative at every point, was not at all suitable to solve problems involving derivatives; functions with corners would be ruled out, even if they had well defined derivative almost everywhere. Long time before Sobolev defined what today we call weak or distributional derivative, Banach constructed the Sobolev space for functions in the real line. Jean Leray proved the existence of weak solution for Navier-Stokes equations using his own definition of "weak" derivative, which he called quasi-derivative, before Sobolev published his outstanding work on Sobolev spaces.

The space of infinitely differentiable functions on an open set $\Omega \subset \mathbb{R}^n$ is denoted by $C^\infty(\Omega)$. For a function $u \in C^\infty(\Omega)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we use the notation

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$ is the order of the derivative. The operator derivative of order l in $L^p(\Omega)$, where $1 \leq p \leq \infty$, has as domain

$$\mathcal{D}(D^l) = \{u \in C^\infty(\Omega) \cap L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for every } |\alpha| = l\};$$

this operator is defined by

$$\begin{aligned} D^l : \mathcal{D}(D^l) \subset L^p(\Omega) &\to [L^p(\Omega)]^m \\ u \quad &\mapsto (D^\alpha u), \end{aligned}$$

where m is the number of multi-indices of order l. The operator D^l is unbounded but closable. In fact, define the space of test functions $[C_c^\infty(\Omega)]^m$ as the set of smooth functions $\boldsymbol{\varphi} = (\varphi_\alpha)$ with compact support in Ω; if $\boldsymbol{\varphi} \in [C_c^\infty(\Omega)]^m$ then we have after integration by parts

$$\int_\Omega \sum_{|\alpha|=l} D^\alpha u \varphi_\alpha \, dx = (-1)^l \int_\Omega u \big(\sum_{|\alpha|=l} D^\alpha \varphi_\alpha \big) \, dx,$$

the integral in the right-hand side is a continuous functional in $L^p(\Omega)$, hence the set of test functions belongs to the domain of the adjoint operator of D^l, which is to say $[C_c^\infty(\Omega)]^m \subset \mathcal{D}(D^{l*})$. Since the set of test functions is complete in $[L^p(\Omega)]^m$ for $1 \leq p \leq \infty$, the operator D^l is closable by Lemma 2.1.b. To see that $[C_c^\infty(\Omega)]^m$ is complete in $[L^p(\Omega)]^m$, it suffices to check it for $m = 1$; suppose $\int_\Omega f\phi \, dx = 0$ for every $\phi \in C_c^\infty(\Omega)$, then this equality also holds for every $\phi \in C_c(\Omega)$. Take any compact set $K \subset \Omega$ with non-empty interior, thus the restriction of f to K belongs to $L^1(K)$ and by the density of $C_c(\mathring{K})$ we can choose a sequence $\{g_n\}$ such that $g_n \to f$ pointwise, hence by dominated convergence theorem

$$\int_\Omega \frac{g_n}{\frac{1}{n} + |g_n|} f \, dx = 0 \to \int_K |f| \, dx,$$

therefore $f = 0$ in K and since K is arbitrary, $f = 0$ in the whole Ω.[1]

As we said before, hereafter we will denote by D^l the closure of the derivative operator for $1 \le p < \infty$. The graph space of D^l is referred to as Sobolev space $W^{l,p}(\Omega)$ for $1 \le p < \infty$. If $p = \infty$ then the graph space is the space $C_b^l(\Omega)$ of all bounded functions with bounded continuous derivatives of order l, so we have recovered the classical derivative in this case.

Let us introduce a new definition of weak derivative D_w^α. Let $u \in L^1_{loc}(\Omega)$, then its weak derivative $D_w^\alpha u = v$, if it exists, is a function $v \in L^1_{loc}(\Omega)$ such that for every test function φ we have

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v \varphi \, dx.$$

The weak derivative D_w^l in $L^p(\Omega)$ has as domain all functions $u \in L^p(\Omega)$ with all weak derivatives of order l also in $L^p(\Omega)$. It is not hard to prove that D_w^l is closed and extends the classical derivative $D^l_{\text{classical}}$, therefore[2] $D^l \subset D_w^l$ for $1 \le p \le \infty$; if $p = \infty$ then $D^l \subsetneq D_w^l$. It is not trivial however to prove that $D^l = D_w^l$ for $1 \le p < \infty$, whence we will not make difference between both derivatives for $1 \le p < \infty$, but when referring to the Sobolev space $W^{l,\infty}(\Omega)$, we will use the definition of D_w^l and if there is no confusion, we will drop the subscript w.

Using Lemma 2.1.d, the dual space of $W^{l,p}(\Omega)$ consists of the set of linear functionals $\boldsymbol{f} = (f_0, \dots, f_\alpha) \in [L^{p'}(\Omega)]^{m+1}$, where p' is conjugated to p, defined by

$$\boldsymbol{f}(u) = \int_\Omega f_0 u \, dx + \sum_{|\alpha|=l} \int_\Omega f_\alpha D^\alpha u \, dx,$$

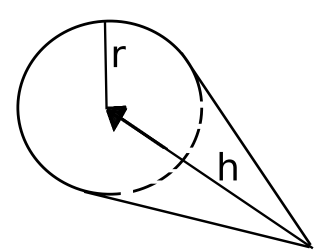
with norm

$$\|\boldsymbol{f}\|_{W^{-l,p'}(\Omega)} = \inf\{ \{ \|f_0 + (-1)^{|\alpha|+1} \sum_{|\alpha|=l} D^\alpha \varphi_\alpha\|^{p'}_{L^{p'}(\Omega)} + \|f_\alpha + \varphi_\alpha\|^{p'}_{L^{p'}(\Omega)} \}^{1/p'} \mid \\ \mid \boldsymbol{\varphi} = (\varphi_\alpha) \in [C_c^\infty(\Omega)]^m \}.$$

The dual space of $W^{l,p}(\Omega)$ is denoted by $W^{-l,p'}(\Omega)$. The dual space can be seen as consisting of distributions.

A standard cone with aperture r and vector direction h is defined as

$$K(r,h) = \{x \in \mathbb{R}^n \mid x = ty, \text{ where } y \in B(h,r) \text{ and } 0 \le t \le 1\}.$$



An open set Ω satisfies the cone condition with parameters r and $|h|$ if for every $x \in \Omega$ there exists a cone such that $x + K(r,h) \subset \Omega$; note that the direction of h can change, but not its magnitude. A set that satisfies the cone condition allows us to represent the function at each point in terms

[1]Felix Soriano showed me this nice proof.
[2]Remember that D^l is the closure of $D^l_{\text{classical}}$.

of the derivatives of the function through the Sobolev's integral representation, as the next theorem shows. We can think of the cone as a pencil that draws the function.

Theorem 2.4. *Let the open set $\Omega \subset \mathbb{R}^n$ satisfy the cone condition with parameters r and $|h|$, and let ω satisfy*

$$\omega \in C_c^\infty(B(0,r)) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \omega\, dx = 1. \tag{2-4}$$

Then for every $u \in C^\infty(\Omega)$ and every $x \in \Omega$ such that $x + K(r,h) \subset \Omega$ we have

$$u(x) = \int_{\mathbb{R}^n} \Big(\sum_{|\alpha|<l} \frac{(-1)^{|\alpha|}}{\alpha!} D_y^\alpha [(x-y)^\alpha \omega(y-x-h)] \Big) u(y)\, dy + \\ + \sum_{|\alpha|=l} \int_{\mathbb{R}^n} (D^\alpha u)(x-y) \frac{\zeta_\alpha(y)}{|y|^{n-l}}\, dy \tag{2-5}$$

where

$$\zeta_\alpha(y) = \frac{|\alpha|}{\alpha!} \tilde{y}^\alpha \zeta(y) \tag{2-6}$$

and

$$\zeta(y) = \int_{|y|}^\infty \omega\Big(-h - \rho\tilde{y} \Big) \rho^{n-1}\, d\rho. \tag{2-7}$$

Remark. Perhaps the reader does not perceive clearly where the cone is important, besides we said it is. The support of ζ is just $K(r,h)$ and the integral over $\mathbb{R}^n$ in the second integral of (2-5) makes sense because the support of the integrand, namely $x + K(r,h)$, lies in Ω.

Proof. Suppose $(1-t)x + ty \in \Omega$ for $0 \le t \le 1$, then multidimensional Taylor's formula allows us to write

$$u(x) = \sum_{|\alpha|<l} \frac{D^\alpha f(y)}{\alpha!}(x-y)^\alpha + l \sum_{|\alpha|=l} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{l-1} D^\alpha f(y + t(x-y))\, dt.$$

Multiply now both sides by $\omega(y-x-h)$ and integrate with respect to y in $\mathbb{R}^n$, so we have

$$u(x) = \int_{\mathbb{R}^n} \sum_{|\alpha|<l} \frac{D^\alpha u(y)}{\alpha!}(x-y)^\alpha \omega(y-x-h)\, dy + \\ + l \sum_{|\alpha|=l} \int_{\mathbb{R}^n} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{l-1} D^\alpha u(y+t(x-y))\, dt\, \omega(y-x-h)\, dy;$$

as regard the first integral, after integration by parts it becomes

$$\int_{\mathbb{R}^n} \Big(\sum_{|\alpha|<l} \frac{(-1)^{|\alpha|}}{\alpha!} D_y^\alpha [(x-y)^\alpha \omega(y-x-h)] \Big) u(y)\, dy.$$

As for the second integral, apply Fubinni's theorem, set the change of variable $z = y+t(x-y)$ and use again Fubinni's theorem, obtaining

$$\int_{\mathbb{R}^n} D^\alpha u(z)(x-z)^\alpha\Big(\int_0^1 \omega\Big(\frac{z-x}{1-t}-h\Big)\frac{dt}{(1-t)^{n+1}}\Big)\,dz.$$

Replace $|x-z|/(1-t)$ by ρ and define $\tilde{z} = z/|z|$, so obtaining

$$\int_{\mathbb{R}^n} D^\alpha u(z)\frac{(x-z)^\alpha}{|x-z|^n}\Big(\int_{|x-z|}^\infty \omega\Big(\rho\frac{z-x}{|z-x|}-h\Big)\rho^{n-1}\Big)\,dz =$$
$$= \int_{\mathbb{R}^n}\frac{D^\alpha u(x-z)}{|z|^{n-l}}\tilde{z}^\alpha\Big(\int_{|z|}^\infty \omega\Big(-h-\rho\tilde{z}\Big)\rho^{n-1}\Big)\,dz,$$

concluding so the theorem. □

If Ω satisfies the cone condition, then the spaces $\cap_{k=1}^l W^{k,p}(\Omega)$ and $W^{l,p}(\Omega)$ coincide, where the norm of the former space is

$$\|u\|_{\cap_{k=1}^l W^{k,p}(\Omega)} = \Big\{\sum_{|\alpha|\le l}\|D^\alpha u\|^p_{L^p(\Omega)}\Big\}^{1/p}.$$

Many authors use the space $\cap_{k=1}^l W^{k,p}(\Omega)$ as definition of Sobolev space, but we prefer the definition we provided, because it resembles more exactly the original norm used by Sobolev[3] and both of them are equivalent for every connected Ω. Indeed, in the norm that Sobolev used, a polynomial projection replaced the term $\int_\Omega u\,dx$ with the idea of using only the necessary information about the function.

We state now a fragment of the Sobolev embedding theorem, to which Sobolev contributed the main part, however this theorem is actually a collection of many results due to many mathematicians. The embedding theorem is one of the cornerstones in the theory of Sobolev spaces.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition, then the following embeddings are continuous for $1 \le p < \infty$*

a) If either $lp > n$ or $l = n$ and $p = 1$, then

$$W^{j+l,p}(\Omega) \to C_b^j(\Omega) \tag{2-8}$$

b) If $lp = n$, then

$$W^{j+l,p}(\Omega) \to W^{j,q}(\Omega), \qquad \textit{for } 1 \le q < \infty \tag{2-9}$$

c) If $lp < n$, then

$$W^{j+l,p}(\Omega) \to W^{j,q}(\Omega), \qquad \textit{for } 1 \le q \le p^* = \frac{np}{n-lp} \tag{2-10}$$

[3]Sobolev originally defined the norm only for connected spaces.

Sketch of the proof. Actually, the proof of this theorem is very long, so we only show what is the idea of the proof and how to use the Sobolev's integral representation. It is not hard to prove that for a set satisfying the cone condition there is a finite collection of open sets $\{\Omega_i\}_{i=1,\dots,N}$ such that $\Omega = \cup_{i=1}^N \Omega_i$ and that for every $x \in \Omega_i$ it holds $x + K(r, h_i) \subset \Omega$, where r and h_i are held fixed; this is stronger than saying that r and $|h_i|$ are held fixed. Sobolev's integral representation can be written for each $x \in \Omega_i$ as

$$u(x) = (Q_l * u)(x) + \sum_{|\alpha|=l} (S_\alpha * D^\alpha u)(x), \tag{2-11}$$

where

$$Q = \sum_{|\alpha|<l} \frac{(-1)^l}{\alpha!} D^\alpha[(-x)^\alpha \omega(x-h)] \in L^q(\mathbb{R}^n), \quad 1 \le q \le \infty,$$

$$S_\alpha = \sum_{|\alpha|=l} \frac{\zeta_\alpha(y)}{|y|^{n-l}} \in L^q(\mathbb{R}^n), \quad \begin{cases} 1 \le q < n/(n-l) & \text{if } l < n, \\ 1 \le q \le \infty & \text{if } l \ge n, \end{cases}$$

we have used the notation in Theorem 2.4.

If either $lp > n$ or $l = n$ and $p = 1$ then using Hölder inequality in (2-11) we have for every smooth function u

$$|u(x)| \le \|Q\|_{L^q(\mathbb{R}^n)} \|u\|_{L^p(\Omega)} + \sum_{|\alpha|=l} \|S_\alpha\|_{L^q(\mathbb{R}^n)} \|D^\alpha u\|_{L^p(\Omega)} \le C\|u\|_{W^{l,p}(\Omega)},$$

since smooth functions are dense in Sobolev spaces, then extending to the whole space we conclude that $W^{l,p}(\Omega) \to C_b(\Omega)$; the general case is straightforwards.

On the other hand, if $lp < n$ and $p > 1$ use Hardy–Littlewood–Sobolev inequality, which says that for $1 < p < q < \infty$ and $\lambda = n(1/p + 1/q)$, if $u \in L^p(\mathbb{R}^n)$ then

$$\||x|^{-\lambda} * u\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}.$$

The proof of the embeddings can be found in the literature provided at the end of this section. □

Taking the adjoint of the above embeddings, we get the corresponding continuous embeddings of $L^q(\Omega)$ spaces into $W^{-l,p}(\Omega)$. Another fundamental theorem concerning the compactness of the embedding is the Rellich-Kondrachov theorem.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition, then the following embeddings are compact for $1 \le p < \infty$*

a) If $lp > n$, then

$$W^{j+l,p}(\Omega) \to C_b^j(\Omega) \tag{2-12}$$

b) If $lp \leq n$, then

$$W^{j+l,p}(\Omega) \to W^{j,q}(\Omega), \qquad \text{for } 1 \leq q < p^* = \frac{np}{n-lp} \tag{2-13}$$

Weak derivatives can also be studied by using finite differences. The translation operator is defined as $(E_h u)(x) = u(x+h)$, and as we want to avoid any problem with the domain of the function u, we define E_h for functions in the whole $\mathbb{R}^n$; as regards a function with domain Ω, it is extended by defining the function as zero outside Ω.

Definition 2.7. The finite difference operator of order l is defined as

$$\triangle_h^l u = (E_h - 1)^l u, \tag{2-14}$$

or equivalently

$$(\triangle_h^l u)(x) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} u(x+kh) \tag{2-15}$$

A subset $V \subset \Omega$ is said to be compactly contained and is denoted by $V \Subset \Omega$ if $\overline{V}$ is compact and $\overline{V} \subset \Omega$.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < \infty$. If $u \in W^{l,p}(\Omega)$ then for every $V \Subset \Omega$ and $|h| < \frac{1}{l} \inf\{|x-y| \mid x \in V$ and $y \in \Omega^c\}$ we have*

$$\left\| \frac{\triangle_h^l u}{|h|^l} \right\|_{L^p(V)} \leq C \|u\|_{W^{l,p}(\Omega)} \tag{2-16}$$

Remark. Later on we will see that a partial converse holds if V satisfies the cone condition and if $1 < p < \infty$.

Proof. We prove it for $l = 1$ and the rest of the proof is by induction. Using the definition of finite difference operator then for every smooth u we obtain

$$\begin{aligned}
\|\triangle_h u\|_{L^p(V)} = \|u(x+h) - u(x)\|_{L^p(V)} &\leq \| \int_0^1 \nabla u(x+th) \cdot h \, dt \|_{L^p(V)} \\
&\leq \int_0^1 \|\nabla u(x+th) \cdot h\|_{L^p(V)} \, dt \leq |h| \int_0^1 \|\nabla u(x+th)\|_{L^p(V)} \\
&\leq |h| \|\nabla u\|_{L^p(\Omega)} \leq |h| \|u\|_{W^{1,p}(\Omega)}
\end{aligned}$$

In the preceding sequence of inequalities we have used Minkowski's integral inequality. The inequality for every $u \in W^{1,p}(\Omega)$ follows using a sequence a smooth functions. □

When studying boundary value problems, it is not clear what is the value of a function in the boundary, because we must extend the function from Ω to $\overline{\Omega}$, and for functions in $L^p(\overline{\Omega})$ we can change the values of the function in the boundary without changing the function itself, because the measure of the boundary is zero.[4]

[4]We suppose Ω has a *well behaved* boundary.

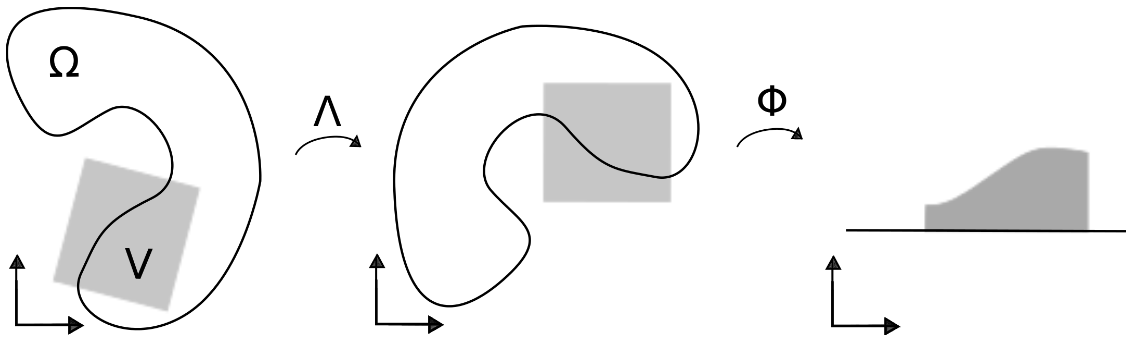


Figure **2-1**

Definition 2.9. An open set $\Omega \subset \mathbb{R}^n$ is called a bounded elementary domain with C^l-boundary, if for some bounded open set $W \subset \mathbb{R}^{n-1}$ there exists a function $\psi \in C^l(W)$ and a constant c such that

$$\Omega = \{x \mid \overline{x} \in W \text{ and } \psi(\overline{x}) < x_n < c\}, \tag{2-17}$$

where $x = (\overline{x}, x_n)$.

The flattening function $\mathbf{\Phi}(\overline{x}, x_n) = (\overline{x}, x_n - \psi(\overline{x}))$ of a C^l-boundary is a C^l-diffeomorphism.

Definition 2.10. A bounded open set $\Omega \subset \mathbb{R}^n$ has a C^l-boundary if for every $x \in \partial\Omega$ there exist an open set V and an orthogonal transformation Λ such that $\Lambda(V \cap \Omega)$ is a bounded elementary domain with C^l-boundary.

It can be proved that sets with C^1-boundary also satisfy the cone condition. Before defining the meaning of the boundary values of a function, we must pass through an intermediate definition.

Definition 2.11. Let $u \in L^1_{loc}(\mathbb{R}^n_+)$ and $v \in L^1_{loc}(\mathbb{R}^{n-1})$. The function v is said to be the trace function of u if there exists a function $\tilde{u}$ equivalent to u, which is such that

$$\tilde{u}(\overline{x}, x_n) \to v(\overline{x}) \quad \text{ in } L^1_{loc}(\mathbb{R}^{n-1}) \text{ as } x_n \to 0^+. \tag{2-18}$$

The trace of a function is denoted by $\operatorname{tr} u$.

To define the trace of a function we use the typical method of partition of unity. Given a bounded open set with C^1-boundary, take a finite covering of $\partial\Omega$ consisting of sets V_i as in Definition 2.10. Suppose now that $\{\vartheta_j\}$ is a C^∞-partition of unity of $\partial\Omega$ subordinated to $\{V_i\}$.

Definition 2.12. Let $u \in L^1_{loc}(\Omega)$ and $v \in L^1_{loc}(\partial\Omega)$. The function v is said to be the trace function of u if for every j the function $(\vartheta_j v) \circ \Lambda^{-1} \circ \mathbf{\Phi}^{-1}$ is the trace of $(\vartheta_j u) \circ \Lambda^{-1} \circ \mathbf{\Phi}^{-1}$ as in Definition 2.11; see Figure **2-1**.

It is possible to verify that the Definition 2.12 does not depend on the partition. Now consider a Banach space of functions $Z(\Omega)$ such that for every compact set $K \Subset \Omega$ the inequality $\|u\|_{L^1(K)} \leq C_K \|u\|_{Z(\Omega)}$ holds, and that $C^\infty(\Omega) \cap Z(\Omega)$ is dense in $Z(\Omega)$; in particular, every Sobolev space with $1 \leq p < \infty$ satisfies both conditions. The following lemma is a useful tool for proving embedding theorems.

Lemma 2.13. *Let $Z(\mathbb{R}^n_+)$ be a normed space of functions such that*

> *for every $u \in Z(\mathbb{R}^n_+)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < h < \delta$ then $\|u - E_{h\boldsymbol{e}_n} u\|_{Z(\mathbb{R}^n_+)} < \varepsilon$, where $\boldsymbol{e}_n = (0, \cdots, 0, 1)$.*

Suppose that there exists a constant C such that for every $u \in C^\infty(\mathbb{R}^n_+) \cap Z(\mathbb{R}^n_+)$ and every $x_n > 0$ we have

$$\|u(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq C\|u\|_{Z(\mathbb{R}^n_+)}. \tag{2-19}$$

Then there exists a trace for every $u \in Z(\mathbb{R}^n_+)$ and

$$\|tr u\|_{L^p(\mathbb{R}^{n-1})} \leq C\|u\|_{Z(\mathbb{R}^n_+)}. \tag{2-20}$$

Sketch of the proof. Fix $u \in Z(\mathbb{R}^n_+)$ and take a sequence of smooth functions such that $u_k \to u$ in $Z(\mathbb{R}^n_+)$. Using (2-19) construct a function $\tilde{u}$ equivalent to u such that $u_k(\cdot, x_n) \to \tilde{u}(\cdot, x_n)$ in $L^p(\mathbb{R}^{n-1})$ for every $x_n > 0$; hence for any sequence $\{x_n^{(s)}\}$ converging to zero we get

$$\begin{aligned}
\|\tilde{u}(\cdot, x_n^{(s)}) - \tilde{u}(\cdot, x_n^{(r)})\|_{L^p(\mathbb{R}^{n-1})} &\leq \|\tilde{u}(\cdot, x_n^{(s)}) - u_k(\cdot, x_n^{(s)})\|_{L^p(\mathbb{R}^{n-1})} + \\
&+ \|u_k(\cdot, x_n^{(s)}) - u_k(\cdot, x_n^{(r)})\|_{L^p(\mathbb{R}^{n-1})} + \|u_k(\cdot, x_n^{(r)}) - \tilde{u}(\cdot, x_n^{(r)})\|_{L^p(\mathbb{R}^{n-1})} \\
&\leq \|\tilde{u}(\cdot, x_n^{(s)}) - u_k(\cdot, x_n^{(s)})\|_{L^p(\mathbb{R}^{n-1})} + \\
&+ C\|u_k - E_{|x_n^{(s)} - x_n^{(r)}|\boldsymbol{e}_n} u_k\|_{Z(\mathbb{R}^n_+)} + \|u_k(\cdot, x_n^{(r)}) - \tilde{u}(\cdot, x_n^{(r)})\|_{L^p(\mathbb{R}^{n-1})},
\end{aligned}$$

letting $k \to \infty$ we have

$$\|\tilde{u}(\cdot, x_n^{(s)}) - \tilde{u}(\cdot, x_n^{(r)})\|_{L^p(\mathbb{R}^{n-1})} \leq C\|\tilde{u} - E_{|x_n^{(s)} - x_n^{(r)}|\boldsymbol{e}_n} \tilde{u}\|_{Z(\mathbb{R}^n_+)},$$

then $\{\tilde{u}(\cdot, x_n^{(s)})\}$ is a Cauchy sequence, concluding so the lemma. □

Now we state, without proof, the Sobolev embedding theorem for the boundary of the set. From the preceding lemma, the proof of this theorem relies on the embedding of Sobolev spaces into hyperplanes.

Theorem 2.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^l-boundary, then the following embeddings are continuous for $1 < p < \infty$*

a) If $lp > n$, then

$$W^{l,p}(\Omega) \to L^\infty(\partial\Omega) \tag{2-21}$$

b) If $lp = n$, then

$$W^{l,p}(\Omega) \to L^q(\partial\Omega), \qquad \text{for } 1 \le q < \infty \tag{2-22}$$

c) If $lp < n$, then

$$W^{l,p}(\Omega) \to L^q(\partial\Omega), \qquad \text{for } 1 \le q \le p^* = \frac{(n-1)p}{n-lp} \tag{2-23}$$

Finally, let us prove a classical result in analysis, using instead Sobolev spaces. We will need this theorem in Chapter 3.

Theorem 2.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-boundary. If $u \in W^{1,p}$ for $1 < p < \infty$ then*

$$\int_\Omega \partial_i u \, dx = \int_{\partial\Omega} u n_i \, d\sigma, \tag{2-24}$$

where $\boldsymbol{n}$ is the outward normal vector and $d\sigma$ is the measure in the boundary.

Remark. This theorem also holds for some irregular sets, like cubes.

Sketch of the proof. It is possible to prove that there exist smooth functions such that $u_k \to u$ in $W^{1,p}(\Omega)$ and that $\operatorname{tr} u_k = \operatorname{tr} u$. Since the trace is integrable, apply classical Gauss-Green formula to u_k and approximate to u. □

The classical book in the theory of Sobolev space was written by Sobolev himself [35], however with the development of the theory of these spaces, this book is now old fashioned and more recent treatises have been published, see Burenkov [9] and Adams & Fournier [2].

2.3 Energy Spaces

Let Ω represent an elastic body subjected to external forces f. We know from Section 1.1 that if the displacement is $\boldsymbol{u}$, then the total energy of the body is

$$E(\boldsymbol{u}) = \frac{1}{2} B(\boldsymbol{u}, \boldsymbol{u}) - f(\boldsymbol{u}),$$

where B is a symmetric bilinear form. In a general setting, the field of displacements is a vector field $\boldsymbol{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, where n and m are any natural numbers. The current task is to make it clear what $\boldsymbol{u}$ and f are; is the field of displacements smooth? what exactly does external forces mean?

We must first establish what are the bilinear forms we will study, but the extensive variety of bilinear forms in elasticity makes it impossible to give the most general form that these functionals can take, however the methods used here might suffice to deal with most of the possible bilinear forms.

In what follows assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set satisfying the cone condition. Let us introduce the energy operator $\boldsymbol{A} : \mathcal{D}(\mathbf{A}) \subset [L^2(\Omega)]^m \to [L^2(\Omega)]^M$ whose component A_i for $i = 1, \dots, M$ is given by

$$A_i \boldsymbol{u} = \sum_{\substack{|\alpha| \leq l \\ 1 \leq k \leq m}} a^k_{i\alpha} D^\alpha u_k,$$

where l is the highest order derivative in the family of operators $\boldsymbol{A} = (A_i)$ and $a^k_{i\alpha}$ are measurable functions. The domain of $\boldsymbol{A}$ is defined as

$$\mathcal{D}(\mathbf{A}) = \{\boldsymbol{u} \in [C^\infty(\Omega) \cap L^2(\Omega)]^m \mid A_i \boldsymbol{u} \in L^2(\Omega) \text{ for every } i = 1, \dots, M\}$$

Instead of the codomain $[L^2(\Omega)]^M$ for the energy operator we could consider $\prod_{i=1}^M L^2(Z_i)$, where $Z_i \subset \overline{\Omega}$ are measurable subsets, like the boundary, however in including this kind of operators we would complicate the notation without adding something interesting, so we do not try to extend our definition of $\boldsymbol{A}$.

The symmetric bilinear forms we are interested in are

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sum_{i,j} E^{ij} A_i \boldsymbol{u} A_j \boldsymbol{v} \, dx.$$

where the measurable functions E^{ij} satisfy the relations $E^{ij} = E^{ji}$. We usually remove the summation symbol and write simply $E^{ij} A_i \boldsymbol{u} A_j \boldsymbol{v}$. Suppose furthermore that the functions E^{ij} are bounded and uniformly positive definite, i.e. there exists a constant $C > 0$ such that

$$E^{ij}(x) \xi_i \xi_j \geq C|\xi|^2 \text{ a.e.}$$

for every $\xi = (\xi_1, \dots, \xi_M)$ and every $x \in \Omega$. Physically, this assumption says that the material can not get arbitrarily fragile.

The statement of the problem of minimization of E is not complete yet, as we are not given boundary conditions. These conditions can be expressed in terms of a linear operator $\partial B : \mathcal{D}(\mathbf{A}) \subset [L^2(\Omega)]^m \to Y$ being Y a Banach space. The following example illustrates this fact.

Example. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary. Define a plate of thickness $2h$ as the set $\Omega \times [-h, h]$. The normal displacement of the middle surface of the plate is $u : \Omega \to \mathbb{R}$ and the total energy of the plate subjected to a normal force f is

$$E(u) = \int_\Omega E^{ijkl} \rho_{ij}(u) \rho_{kl}(u) \, dx - \int_\Omega f u \, dx,$$

where E^{ijkl} is the tensor of elastic moduli and

$$\rho_{ij}(u) = \partial_{ij} u$$

is the tensor of change of curvature.

If the edge of the plate is held fixed, then the boundary conditions are $u|_{\partial\Omega} = 0$. Additionally, we would require the edge to be built-in and then the additional boundary conditions are $\partial_{\boldsymbol{n}} u|_{\partial\Omega} = 0$, where $\partial_{\boldsymbol{n}} u$ is the derivative in the direction of outward normal $\boldsymbol{n}$. The boundary operator is thus $\partial B u = (u|_{\partial\Omega}, \partial_{\boldsymbol{n}} u|_{\partial\Omega})$, which is clearly linear.

Boundary operators are not constrained to have their image in the boundary of Ω; notice that we did not require the set Ω to have in general smooth boundary. For example, other boundary operator is the average of the function u, i.e.

$$\partial B u = \frac{1}{m(\Omega)} \int_{\Omega} u \, dx,$$

where $m(\Omega)$ is the volume or measure of Ω. Boundary conditions are said to be homogeneous if $\partial B \boldsymbol{u} = 0$ and non-homogeneous otherwise. Since non-homogeneous conditions can be transformed into homogeneous, we will limit ourselves to this latter case. In fact, let w belong to the range of ∂B and fix the condition $\partial B \boldsymbol{u} = w$; take $\boldsymbol{u}_0$ such that $\partial B \boldsymbol{u}_0 = w$ and consider now the problem of finding the minimum of the functional $E(\boldsymbol{u} + \boldsymbol{u}_0)$ subjected to the condition $\partial B \boldsymbol{u} = 0$. If $\boldsymbol{v}$ solves this homogeneous problem, then the function $\boldsymbol{v} + \boldsymbol{u}_0$ solves the non-homogeneous original problem.

The boundary conditions ∂B are compatible if $\boldsymbol{A}\boldsymbol{u} = 0$ and $\partial B \boldsymbol{u} = 0$ imply that $\boldsymbol{u} = \boldsymbol{0}$, whence the bilinear form B is an inner product in the set $\mathcal{D}_\partial(\boldsymbol{A}) = \mathcal{D}(\boldsymbol{A}) \cap \ker \partial B$. The energy space $\boldsymbol{E}(\Omega)$ is the completion of $\mathcal{D}_\partial(\boldsymbol{A})$ with the inner product B, therefore $\boldsymbol{E}(\Omega)$ is a Hilbert space.

Definition 2.16. A *generalized solution* of the problem of minimization of the energy functional

$$E(\boldsymbol{u}) = \frac{1}{2} B(\boldsymbol{u}, \boldsymbol{u}) - f(\boldsymbol{u}) \tag{2-25}$$

with boundary conditions ∂B is an element in the energy space $\boldsymbol{E}(\Omega)$ minimizing the energy functional.

From Theorem 1.1 we get immediately the following existence result.

Theorem 2.17. *Let B be the bilinear form described above and $\boldsymbol{E}(\Omega)$ its corresponding energy space with boundary conditions ∂B. If $f \in \boldsymbol{E}^*(\Omega)$ represents the external forces, then there exists a unique generalized solution of the energy functional* (2-25) *in $\boldsymbol{E}(\Omega)$.*

Despite this seemingly strong existence and uniqueness result, so far we have not progressed in understanding what $\boldsymbol{u}$ and f are. We do not know how are neither the elements in $\boldsymbol{E}(\Omega)$ nor the continuous functionals in $\boldsymbol{E}^*(\Omega)$. Since the energy operator $\boldsymbol{A}$ involves derivatives of order at most l, it is reasonable to wonder if there is some resemblance between $\boldsymbol{E}(\Omega)$ and $[W^{l,2}(\Omega)]^m = [H^l(\Omega)]^m$. While the structure of energy spaces in general is unknown, the Sobolev spaces have been well studied, so the aim in the remainder of this section is to prove that certain energy spaces are isomorphic to a subspace of its corresponding Sobolev space.

The boundedness of the functions E^{ij} and its uniform positive definiteness allow us to get the inequalities

$$C_1 \int_\Omega |A_i \boldsymbol{u}|^2 \, dx \leq \int_\Omega E^{ij} A_i \boldsymbol{u} A_j \boldsymbol{u} \, dx \leq C_2 \int_\Omega |A_i \boldsymbol{u}|^2 \, dx,$$

hence it is enough to consider the bilinear form

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega A_i \boldsymbol{u} A_i \boldsymbol{v} \, dx.$$

For the sake of generality, we will replace $L^2(\Omega)$ by $L^p(\Omega)$ for $1 \leq p < \infty$, so that the energy operator is now $\boldsymbol{A} : \mathcal{D}(\mathbf{A}) \subset [L^p(\Omega)]^m \to [L^p(\Omega)]^M$ and its domain is changed in the obvious way. The space of interest is $\mathcal{D}_\partial(\boldsymbol{A}) = \mathcal{D}(\boldsymbol{A}) \cap \ker \partial B$, and on it we define the energy norm

$$\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)} = \|\boldsymbol{A}\boldsymbol{u}\|_{[L^p(\Omega)]^M} = \Big\{ \int_\Omega \sum_{i=1}^{M} |A_i \boldsymbol{u}|^p \, dx \Big\}^{1/p}.$$

Our interest is to apply Lemma 2.2 to the previous norm, and to this end we must impose additional conditions on $\boldsymbol{A}$ and ∂B.

Let l be the highest order derivative in the operator $\mathbf{A}$. Let us split the operator $\mathbf{A}$ into its higher order part $\mathbf{A}^H$ and its lower order part $\mathbf{A}^L$. The components of $\mathbf{A}^H$ are thus

$$A_i^H \boldsymbol{u} = \sum_{\substack{|\alpha|=l \\ 1\leq k\leq m}} a_{i\alpha}^k D^\alpha u_k,$$

whereas the components of $\mathbf{A}^L$ are

$$A_i^L \boldsymbol{u} = \sum_{\substack{|\alpha|<l \\ 1\leq k\leq m}} a_{i\alpha}^k D^\alpha u_k.$$

In the notation of Lemma 2.2, we pretend that $T = D^l$, $M = \ker \partial B$, $Q = \mathbf{A}^H$ and $R = \mathbf{A}^L$.

Suppose that $a_{i\alpha}^k \in L^\infty(\Omega)$ for $|\alpha| = l$, hence

$$\int_\Omega |a_{i\alpha}^k D^\alpha u_k|^p \, dx \leq C \int_\Omega |D^\alpha u_k|^p \, dx \leq C \|\boldsymbol{u}\|_{[W^{l,p}(\Omega)]^m}. \tag{2-26}$$

Summing up all terms in $\mathbf{A}^H$ we conclude that $\|\boldsymbol{u}\|_{\boldsymbol{A}^H} \leq C\|\boldsymbol{u}\|_{[W^{l,p}(\Omega)]^m}$. Concerning the operator $\mathbf{A}^L$, we recall the compact embeddings

a) For $(l - j)p > n$

$$W^{l,p}(\Omega) \to W^{j,\infty}(\Omega).$$

b) For $(l-j)p \leq n$

$$W^{l,p}(\Omega) \to W^{j,q}(\Omega) \qquad \text{if } 1 \leq q < \frac{np}{n-(l-j)p}.$$

Since we want the function $\boldsymbol{u} \to a^k_{i\alpha} D^\alpha u_k$ for $|\alpha| = j < l$ to be compact from $W^{l,p}(\Omega) \to L^p(\Omega)$, we must show that the multiplication operator

$$\begin{aligned} T_{a^k_{i\alpha}} : L^q(\Omega) &\to L^p(\Omega) \\ f &\mapsto a^k_{i\alpha} f \end{aligned}$$

is continuous, because the composition of a compact and a continuous operator is compact. The value of q in the operator $T_{a^k_{i\alpha}}$ depends on the compact embedding of Sobolev spaces.

Given any two real numbers q_1 and q_2 such that $\frac{1}{q_1} + \frac{1}{q_2} = 1$, we have by Hölder inequality

$$\int_\Omega |a^k_{i\alpha} f|^p \, dx \leq \Big\{ \int_\Omega |a^k_{i\alpha}|^{pq_1} \, dx \Big\}^{1/q_1} \Big\{ \int_\Omega |f|^{pq_2} \, dx \Big\}^{1/q_2}.$$

Since we need the operator $T_{a^k_{i\alpha}}$ to be well defined, necessarily $pq_2 = q$. Solving for q_1, we conclude that $pq_1 = qp/(q-p)$ and thus it suffices to prove that $a^k_{i\alpha} \in L^{qp/(q-p)}(\Omega)$. By the compact embedding theorem, we have $1 \leq q < r$, where r depends on n, p, l and j; since the function $qp/(q-p)$ is decreasing on q, we require then that $a^k_{i\alpha} \in L^s(\Omega)$, where $s > rp/(r-p)$. After a bit of algebra we get

a) If $(l-j)p > n$ and $|\alpha| = j$ then $a^k_{i\alpha} \in L^1(\Omega)$.

b) If $(l-j)p \leq n$ and $|\alpha| = j$ then $a^k_{i\alpha} \in L^s(\Omega)$, where $s > n/(l-j)$

With these assumptions $\boldsymbol{A}^L$ is compact.

The operator $\boldsymbol{A}$ is well defined for every function in $W^{l,p}(\Omega)$, therefore the operator ∂B is densely defined in $W^{l,p}(\Omega)$. Since we want the space $M = \ker \partial B$ to be closed in $W^{l,p}(\Omega)$, we further assume that ∂B is bounded, so it can be extended to a bounded operator in $W^{l,p}(\Omega)$.

It remains to ensure the injectivity of $\mathbf{A}$ in $\ker \partial B$ and the inequality $\|\boldsymbol{u}\|_{[W^{l,p}(\Omega)]^m} \leq C\|\boldsymbol{u}\|_{\boldsymbol{A}^H}$. The latter inequality is usually referred to as $[W^{l,p}(\Omega)]^m$-coerciveness and is in general very hard to prove; we turn in next section completely to this question. For the injectivity we do not have a general theorem, so we postpone it until Chapter 3, where only particular operators are investigated.

Theorem 2.18. *Let* $\boldsymbol{A}$ *be an energy operator and define the norm*

$$\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)} = \Big\{ \int_\Omega \sum_{i=1}^{M} |A_i \boldsymbol{u}|^p \, dx \Big\}^{1/p}, \tag{2-27}$$

where $\Omega \subset \mathbb{R}^n$ *is a bounded open set with smooth enough boundary, depending on* ∂B. *Suppose furthermore that* *a) If* $|\alpha| = l$ *then* $a^k_{i\alpha} \in L^\infty(\Omega)$.

b) If $|\alpha| = j$ *and* $0 < (l-j)p \le n$ *then* $a^k_{i\alpha} \in L^s(\Omega)$, *where* $s > n/(l-j)$

c) If $|\alpha| = j$ *and* $(l-j)p > n$ *then* $a^k_{i\alpha} \in L^1(\Omega)$.

d) ∂B *is bounded in* $W^{l,p}(\Omega)$ *and compatible, i.e.* $\ker \partial B \cap \ker \boldsymbol{A} = \boldsymbol{0}$.

e) $\|\boldsymbol{u}\|_{[W^{l,p}(\Omega)]^m} \le C\|\boldsymbol{u}\|_{\boldsymbol{A}^H}$.

Then the spaces $[W^{l,p}(\Omega)]^m$ *and* $\boldsymbol{E}^p(\Omega)$ *are isomorphic for* $1 \le p < \infty$.

Proof. Apply Lemma 2.2 with $M = \ker \partial B$, $T = D^l$, $Q = \boldsymbol{A}^H$ and $R = \boldsymbol{A}^L$. The projection π is compact by Rellich-Kondrachov theorem. □

Now we improve Theorem 2.17.

Corollary 2.19. *Let* $\boldsymbol{A}$ *be an energy operator and define the bilinear form*

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega E^{ij} A_i \boldsymbol{u} A_j \boldsymbol{v}\, dx. \tag{2-28}$$

Besides the conditions in the preceding theorem for $p = 2$, *suppose that the coefficients* $E^{ij} \in L^\infty(\Omega)$ *are uniformly positive definite. If* $\boldsymbol{f} \in [H^{-l}(\Omega)]^m$ *represents the external forces, then there exists a unique generalized solution of the energy functional* (2-25) *in* $[H^l(\Omega)]^m$ *satisfying the boundary conditions* ∂B.

Proof. Use Theorems 2.17. □

Despite the generality of Lemma 2.2, it could happen that the graph space of an operator is not equivalent to the corresponding Sobolev space, but it is equivalent in a subspace.

Example. Define the operator $Au = \Delta u$ with boundary conditions $\partial B u = u|_{\partial\Omega} = 0$. We claim that the norm

$$\|u\|^2_{E^2(\Omega)} = \int_\Omega (\Delta u)^2\, dx$$

is equivalent to the norm of $W^{2,2}_0(\Omega)$, even though the norm

$$\|u\|^2_\Delta = \int_\Omega u^2\, dx + \int_\Omega (\Delta u)^2\, dx$$

is not equivalent to the norm of $W^{2,2}(\Omega)$. In fact, suppose u is smooth and integrate by parts, obtaining

$$\begin{aligned}
\int_\Omega (\Delta u)^2\, dx &= \int_\Omega \sum_{i=1}^n (\partial_i^2 u)^2 + 2\sum_{i\neq j} \partial_i^2 u \partial_j^2 u\, dx \\
&= \int_\Omega \sum_{i=1}^n (\partial_i^2 u)^2 + 2\sum_{i\neq j} (\partial_{ij} u)^2\, dx \\
&\ge \int_\Omega \sum_{i,j} (\partial_{ij} u)^2\, dx.
\end{aligned}$$

Since the last norm is equivalent to the norm in $W_0^{2,2}(\Omega)$, we get the desired equivalence.

On the other hand, if the graph space of Δ and $W^{2,2}(\Omega)$ are isomorphic, then by Lemma 2.1.c the kernel of Δ is finite-dimensional, but it is false, because the kernel of Δ is the set of all harmonic functions.

Example. It happens a similar anomaly with the operator $A_1\boldsymbol{u} = \partial_1 u_1 - \partial_2 u_2$ and $A_2\boldsymbol{u} = \partial_2 u_1 + \partial_1 u_2$ with boundary conditions $\partial Bu = u|_{\partial\Omega} = 0$.

2.3.1 Coerciveness of Operators

As mentioned before, the inequality $\|\boldsymbol{u}\|_{\boldsymbol{A}^H} \geq C\|\boldsymbol{u}\|_{[W^{l,p}(\Omega)]^m}$ is anything but trivial. Even particular cases of this inequality have been subject of numerous investigations. By far, the most important case relates to the operator

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$$

where $\boldsymbol{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. The operator $\boldsymbol{\varepsilon}$ is just the strain tensor. The inequality

$$\int_\Omega \sum_{i,j} \varepsilon_{ij}(\boldsymbol{u})^2\,dx + \int_\Omega \sum_i u_i^2\,dx \geq C\int_\Omega \sum_{i,j} \partial_i u_j^2\,dx$$

is known as first Korn's inequality. It was partially proved by Korn [21, 22], however the first successful proof was given by Friedrichs [15]. The number of papers devoted to this inequality makes it impossible to provide an extensive bibliography on the subject, and we only refer the reader to some works; Acosta *et al.* [1], Conti *et al.* [12], Horgan [19], Nečas & Hlaváček [32] and Kondrat'ev & Oleynik [20].

Assuming $\boldsymbol{u}$ is smooth enough, the strain tensor satisfies the equality

$$\partial_{ik}u_j = \partial_k\varepsilon_{ij}(\boldsymbol{u}) + \partial_i\varepsilon_{jk}(\boldsymbol{u}) - \partial_j\varepsilon_{ki}(\boldsymbol{u}).$$

The following theorem shows that this is basically the crucial property. In the sequel, the coefficients of the operator A_i^H are constant and we will drop the letter H.

Theorem 2.20. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. The spaces $[W^{l,p}(\Omega)]^m$ and $\boldsymbol{E}(\Omega)$ are isomorphic for $1 < p < \infty$ if and only if for some $\hat{l} \in \mathbb{N}$ there exist constant coefficients $b_{\lambda k}^{\alpha i}$ such that for every $|\alpha| = l + \hat{l}$*

$$D^\alpha u_k = \sum_{|\lambda|=\hat{l}} \sum_{i=1}^m b_{\lambda k}^{\alpha i} D^\lambda A_i\boldsymbol{u}. \tag{2-29}$$

In principle, the summation in equation (2-29) makes no sense, since derivatives of order higher than l are not defined in $[W^{l,p}(\Omega)]^m$, so equation (2-29) should be understood in distributional sense. To avoid an explosion of summations, we write (2-29) simply as $D^\alpha u_k = b_{\lambda k}^{\alpha i} D^\lambda A_i$.

Theorem 2.20 was first proved by Aranszajn [3] for real-valued functions and extended by Schechter [34]. The first proof I know for vector-valued functions was published by Nečas [30], but I did not have access to this article. Later, he published a proof for the case $p = 2$ [31]. The statement of the theorem (2.20) in the preceding works is in algebraic terms, but readily equivalent to that given here.

The proof I provide is inspired by the proofs of Korn's inequality given by Gobert [17], Mosolov & Myasnikov [29] and Ting [37]. The proof relies strongly on the theory of singular integrals developed by Zygmund and Calderón.

To appreciate the strength of Theorem 2.20, we will use it to get ride from mixed derivatives in Sobolev spaces.

Theorem 2.21. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set satisfying the cone condition. Then the norm*

$$\left\{ \int_\Omega |u|^p \, dx + \sum_{i=1}^n \int_\Omega |\partial_i^l u|^p \, dx \right\}^{1/p} \tag{2-30}$$

is equivalent to the norm in $W^{l,p}(\Omega)$ for $1 < p < \infty$. Moreover, removing a derivative $\partial_i^l u$ or replacing it by a mixed derivative in the norm, will yield a non-equivalent norm.

Remark. The theorem can be re-read as: if non-mixed derivatives exist and belong to $L^p(\Omega)$, then mixed derivatives also exist and belong to $L^p(\Omega)$.

Proof. We will show that we can write down all derivatives of certain high enough order. Suppose $\hat{l} \geq (n-1)l$ and take any derivative of order $|\alpha| = \hat{l} + l$. Since each time we can choose only among n indices to derive, then at least some index, say k, is taken $\lfloor (\hat{l}+l)/n \rfloor \geq l$ times, therefore we arrange all these indices in such a way that $D^\alpha u = D^\beta(\partial_k^l u)$ for some index β, concluding the first claim of theorem.

If we remove or replace a derivative $\partial_i^l u$ by some mixed derivative, then the derivative $\partial_i^{l+\hat{l}} u$ can not be written for any $\hat{l}$. □

It is worth noting that Theorem 2.20 seems to be not well known. Nowadays the standard method to prove Korn type inequalities is through Lions' lemma, nevertheless this lemma is only for $p = 2$ and sets with Lipschitz boundary; see also [13]. Even more, Lions' lemma cannot furnish Theorem 2.21, but only the isomorphism for the case $W^{2,2}(\Omega)$.

The long proof of Theorem 2.20 is now detailed discussed.

Necessity

The idea of the proof is to exhibit an infinite set of linearly independent functions in the kernel of the energy operator $\boldsymbol{A}$, whence by Lemma 2.1.c the graph spaces of the operator $\boldsymbol{A}$ and D^l cannot be isomorphic. Both operators act in vector functions $\boldsymbol{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$.

A l-form is a polynomial in the variables $\xi = (\xi_1, \ldots, \xi_n)$ defined as

$$\sum_{|\alpha|=l} a_\alpha \xi^\alpha,$$

where α is a multi-index and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Let us introduce the space $S(n,m,l)$ as the set of 1-forms in the variables $\lambda = (\lambda_1, \ldots, \lambda_m)$ whose coefficients are l-forms in the variables $\xi = (\xi_1, \ldots, \xi_n)$, i.e. the space of polynomials

$$P^1(\xi)\lambda_1 + \cdots + P^m(\xi)\lambda_m$$

where

$$P^i(\xi) = \sum_{|\alpha|=l} a^i_\alpha \xi^\alpha.$$

There is a bijective correspondence between operators and the space $S(n,m,l)$ by means of the map

$$A\boldsymbol{u} = \sum_{|\alpha|=l} a^k_\alpha D^\alpha u_k \longleftrightarrow A(\xi,\lambda) = \sum_{|\alpha|=l} a^k_{i,\alpha} \xi^\alpha \lambda_k.$$

The image of the operator is called the characteristic of A. A function $\boldsymbol{u}$ annihilate a polynomial in $S(n,m,l)$ if $\boldsymbol{u}$ belongs to the kernel of the corresponding operator.

Lemma 2.22. *If $Z \subset S(n,m,l)$ is a proper subspace, then there exists a polynomial vector function annihilating every element in Z.*

Proof. Assume $\{R_s\}$ is a basis of Z and fix a vector polynomial $\boldsymbol{P}(x) = (\frac{1}{\beta!} c^k_\beta x^\beta)$ where $k = 1, \ldots, m$ and $|\beta| = l$. Evaluating $\boldsymbol{P}$ in the operator corresponding to

$$R^s(\xi,\lambda) = \sum_{|\alpha|=l} q^k_{s,\alpha} \xi_\alpha \lambda_k \tag{2-31}$$

we have

$$R_s(\boldsymbol{P}(x)) = q^k_{s,\alpha} D^\alpha \big(\frac{1}{\beta!} c^k_\beta x^\beta\big) = q^k_{s,\alpha} c^k_\alpha = 0,$$

since this is an underdetermined system of equations in the unknowns c^k_α, there is at least one non-zero solution. Moreover, each component of the polynomial $\boldsymbol{P}$ is a l-form. □

All is ready to prove half of the main theorem.

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. If $[W^{l,p}(\Omega)]^m$ and $\boldsymbol{E}^p(\Omega)$ are isomorphic for $1 \le p < \infty$, then for some $\hat{l} \in \mathbb{N}$ there exist constant coefficients $b^{\alpha i}_{\lambda k}$ such that for every $|\alpha| = l + \hat{l}$*

$$D^\alpha u_k = b^{\alpha i}_{\lambda k} D^\lambda A_i(\boldsymbol{u}). \tag{2-32}$$

Proof. Suppose that for every $\hat{l}$ it is impossible to know all derivatives of $\boldsymbol{u}$ of order $\hat{l}+l$. Define the space $Z_{\hat{l}} \subset S(n,m,l+\hat{l})$ as the set of all linear combinations

$$Q^1(\xi)A_1(\xi,\lambda) + \cdots + Q^r(\xi)A_r(\xi,\lambda),$$

where $Q^j(\xi)$ is a $\hat{l}$-form. By the hypothesis of theorem, the space $Z_{\hat{l}}$ does not contain every $\xi^\alpha \lambda_t$, hence $Z_{\hat{l}}$ is properly contained for every $\hat{l} = 0, 1, \ldots$ Using Lemma 2.22, there exists a sequence of linearly independent vector polynomials annihilating each space $Z_{\hat{l}}$.

Fix a polynomial $\boldsymbol{P}$ whose components are $(l+\hat{l})$-forms annihilating some $Z_{\hat{l}}$, then $D^\lambda A_i(\boldsymbol{P}) = 0$ for every $|\lambda| = \hat{l}$ and necessarily $A_i(\boldsymbol{P})$ is a polynomial of degree less than $\hat{l}$, but $A_i(\boldsymbol{P})$ can only be zero or a polynomial of degree $\hat{l}$, so $A_i(\boldsymbol{P}) = 0$ and $\boldsymbol{P}$ lies in the kernel of $\boldsymbol{A}$.

If $[W^{l,p}(\Omega)]^m$ and $\boldsymbol{E}^p(\Omega)$ are isomorphic, the projection operator of $\boldsymbol{E}^p(\Omega)$ is compact, which contradicts the fact that ker $\boldsymbol{A}$ has infinite dimension, see Lemma 2.1.c. □

Sufficiency

For this part of the proof, we will make use of the theory of singular integrals. In the remainder we will use the notations $\tilde{x} = x/|x|$ and S^n for the set of unit vectors in $\mathbb{R}^{n+1}$.

Theorem 2.24 ([14, p. 79])**.** *Let φ be a real-valued function on S^{n-1} with zero average such that its odd part is in $L^1(S^{n-1})$ and its even part is in $L^q(S^{n-1})$ for some $q > 1$. Then the singular integral*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\varphi(\tilde{y})}{|y|^n} f(x-y)\, dy, \tag{2-33}$$

is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Next, we will prove a lemma that makes it easier to compute derivatives and gives us one of the conditions in the preceding theorem.

Lemma 2.25 ([28, p. 91] or [37])**.** *Let φ be a real-valued function on S^{n-1} such that $\varphi(\tilde{x}) \in C^l(\mathbb{R}^n)$, then for every $|\alpha| = l$ there exists a continuous function ψ on S^{n-1} such that*

$$D^\alpha\Big(\frac{\varphi(\tilde{x})}{|x|^k}\Big) = \frac{\psi(\tilde{x})}{|x|^{k+l}}. \tag{2-34}$$

Moreover, if $k = n - l$ then

$$\int_{S^{n-1}} \psi(\tilde{x})\, d\sigma = 0. \tag{2-35}$$

Proof. Define $h(x) = \varphi(\tilde{x})$ and suppose x and y are unitary vector, then using similarity of triangles

$$\frac{h(x+ty) - h(x)}{t} = r\frac{h(rx+rty) - h(rx)}{rt},$$

letting t tends to zero we obtain $(\partial_y h)(x) = r(\partial_y h)(rx)$ and so also equation (2-34).

For any multi-index $|\beta| = l-1$, equation (2-34) gives $D^\beta(\varphi(\tilde{x})/|x|^{n-l}) = \psi_0(\tilde{x})/|x|^{n-1}$. After integration by parts and defining $\boldsymbol{n}$ as the outer normal to the sphere

$$\int_{r_1\le|x|\le r_2} \partial_i\Big(\frac{\psi_0(\tilde{x})}{|x|^{n-1}}\Big)\,dx = \frac{1}{r_1^{n-1}}\int_{r_1 S^{n-1}} \psi_0(\tilde{x}) n_i\,d\sigma - \frac{1}{r_2^{n-1}}\int_{r_2 S^{n-1}} \psi_0(\tilde{x}) n_i\,d\sigma = 0,$$

because the average of $\psi_0 n_i$ on rS^{n-1} is the same for every $r>0$. On the other hand, since $\partial_i(\psi_0(\tilde{x})/|x|^{n-1}) = \psi(\tilde{x})/|x|^n$ we get

$$\int_{r_1\le|x|\le r_2} \frac{\psi(\tilde{x})}{|x|^n}\,dx = \int_{r_1}^{r_2}\frac{dr}{r}\int_{S^{n-1}} \psi(\tilde{x})\,d\sigma = 0,$$

which shows that ψ has zero average on S^{n-1}. □

We do not need boundedness of the set Ω for the sufficiency part of the theorem, thus it holds also for sets like $\mathbb{R}^n_+$.

Theorem 2.26. *Let $\Omega\subset\mathbb{R}^n$ be an open set satisfying the cone condition. The spaces $[W^{l,p}(\Omega)]^m$ and $\boldsymbol{E}^p(\Omega)$ are isomorphic for $1<p<\infty$ if for some $\hat{l}\in\mathbb{N}$ there exist constant coefficients $b^{\alpha i}_{\lambda k}$ such that for every $|\alpha| = l+\hat{l}$*

$$D^\alpha u_k = b^{\alpha i}_{\lambda k} D^\lambda A_i \boldsymbol{u}. \tag{2-36}$$

Proof. It suffices to prove the inequalities

$$C_1\|\boldsymbol{u}\|_{W^{l,p}(\Omega)} \le \|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)} \le C_2\|\boldsymbol{u}\|_{W^{l,p}(\Omega)}$$

for every infinitely differentiable function. The inequality $\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)} \le C_2\|\boldsymbol{u}\|_{W^{l,p}(\Omega)}$ is readily seen. The reverse inequality $\|\boldsymbol{u}\|_{W^{l,p}(\Omega)} \le C_1\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)}$ will be proved at first locally. Additionally, the reader should be aware that sometimes we replace $\boldsymbol{u}$ defined on Ω by its extension, defined as zero outside Ω, to evaluate integrals in $\mathbb{R}^n$.

Let $\{\Omega_k\}$ be as in Theorem 2.5. Fix one set Ω_s with its cone $K(h,r)$ and let $\omega:\mathbb{R}^n\to\mathbb{R}$ be an infinitely differentiable function with support in the ball $B(0,r)$ and such that $\int_{\mathbb{R}^n}\omega\,dx = 1$. Assume, without loss of generality, that the length of the direction vector of the cone is $2r$. For every $x\in\Omega_s$ the Sobolev's integral representation yields

$$u_k(x) = \int_{\mathbb{R}^n}\Big(\sum_{|\alpha|<l+\hat{l}} \frac{(-1)^{|\alpha|}}{\alpha!} D^\alpha_y[(x-y)^\alpha\omega(y-x-h)]\Big) u_k(y)\,dy + \\ + \sum_{|\alpha|=l+\hat{l}} \int_{\mathbb{R}^n} (D^\alpha u_k)(x-y)\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\,dy$$

where

$$\zeta_\alpha(y) = \frac{|\alpha|}{\alpha!}\tilde{y}^\alpha\zeta(y)$$

and

$$\zeta(y) = \int_{|y|}^{\infty} \omega\Big(-h - \rho\tilde{y}\Big)\rho^{n-1}\,d\rho.$$

Replacing equation (2-36) in Sobolev's integral representation we have

$$u_k(x) = (Gu_k)(x) + \int_{\mathbb{R}^n} b^{\alpha t}_{\lambda k}(D^{\lambda}A_t\boldsymbol{u})(x-y)\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\,dy, \tag{2-37}$$

where we have dropped the symbol $\sum_{|\alpha|=l+\hat{l}}$.

For any $|\gamma| = l$ the derivative of u_k is

$$D^{\gamma}u_k(x) = D^{\gamma}(Gu_k)(x) + \int_{\mathbb{R}^n} (D^{\gamma+\lambda}A_t\boldsymbol{u})(x-y)b^{\alpha t}_{\lambda k}\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\,dy. \tag{2-38}$$

The operator $D^{\gamma}(Gu_k)$ is

$$D^{\gamma}(Gu_k)(x) = (Q * u_k)(x) = \int_{\mathbb{R}^n} Q(x-y)u_k(y)\,dy$$

where $Q \in L^1(\mathbb{R}^n)$, then $\|Q * u_k\|_{L^p(\Omega_s)} \le \|Q * u_k\|_{L^p(\mathbb{R}^n)} \le \|Q\|_{L^1(\mathbb{R}^n)}\|u_k\|_{L^p(\Omega)}$. It remains to evaluate the continuity of the terms in the second summand in equation (2-38), which we call $T^t_\lambda A_t$.

Note that ζ is a compactly supported infinitely differentiable function. Moreover, since $\operatorname{supp}(\omega) \subset B(0,r)$ and $|h| = 2r$ we have that $-h - t\tilde{y} \notin B(0,r)$ for every $0 < t \le r$ and therefore

$$\begin{aligned}
\zeta(t\tilde{y}) &= \int_{t}^{\infty} \omega\Big(-h-\rho\tilde{y}\Big)\rho^{n-1}\,d\rho \\
&= \int_{t}^{r} \omega\Big(-h-\rho\tilde{y}\Big)\rho^{n-1}\,d\rho + \int_{r}^{\infty} \omega\Big(-h-\rho\tilde{y}\Big)\rho^{n-1}\,d\rho \\
&= \int_{r}^{\infty} \omega\Big(-h-\rho\tilde{y}\Big)\rho^{n-1}\,d\rho = \zeta(r\tilde{y}),
\end{aligned}$$

in this way we can define a function $\varphi : S^{n-1} \to \mathbb{R}$ as $\varphi_\alpha(\tilde{y}) = \zeta_\alpha(r\tilde{y})$.

We write $T^t_\lambda A_t$ as

$$\begin{aligned}
T^t_\lambda A_t\boldsymbol{u} = \int_{|x-y|\le\epsilon} (D^{\gamma+\lambda}A_t\boldsymbol{u})(y)b^{\alpha t}_{\lambda k}\frac{\zeta_\alpha(x-y)}{|x-y|^{n-l-\hat{l}}}\,dy + \\
+ \int_{|x-y|>\epsilon} (D^{\gamma+\lambda}A_t\boldsymbol{u})(y)b^{\alpha t}_{\lambda k}\frac{\zeta_\alpha(x-y)}{|x-y|^{n-l-\hat{l}}}\,dy,
\end{aligned}$$

first term tends to zero as $\epsilon \to 0$, so we integrate by parts the second term with respect to

∂_i, which is a derivative in $D^{\gamma+\lambda}$, and we have

$$
\begin{aligned}
T^t_\lambda A_t\boldsymbol{u} &= \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}(\partial_i D^{\gamma+\lambda'}A_t\boldsymbol{u})(y)b^{\alpha t}_{\lambda k}\frac{\zeta_\alpha(x-y)}{|x-y|^{n-l-\hat{l}}}\,dy\\
&= -\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}(D^{\gamma+\lambda'}A_t\boldsymbol{u})(y)b^{\alpha t}_{\lambda k}\partial_{y,i}\Big(\frac{\zeta_\alpha(x-y)}{|x-y|^{n-l-\hat{l}}}\Big)\,dy+\\
&\qquad\qquad + \lim_{\epsilon\to 0}\frac{1}{\epsilon^{n-l-\hat{l}}}\int_{\epsilon S^{n-1}}(D^{\gamma+\lambda'}A_t\boldsymbol{u})(x-y)b^{\alpha t}_{\lambda k}\zeta_\alpha(y)n_i\,d\sigma,
\end{aligned}
$$

second integral tends to zero and if we set $g(z)=\zeta_\alpha(z)/|z|^{n-l-\hat{l}}$ we get for the first integral

$$
\begin{aligned}
T^t_\lambda A_t &= -\int_{\mathbb{R}^n}(D^{\gamma+\lambda'}A_t\boldsymbol{u})(y)b^{\alpha t}_{\beta\lambda k}\partial_{y,i}(g(x-y))\,dy\\
&= \int_{\mathbb{R}^n}(D^{\gamma+\lambda'}A_t\boldsymbol{u})(y)b^{\alpha t}_{\lambda k}(\partial_i g)(x-y)\,dy\\
&= \int_{\mathbb{R}^n}(D^{\gamma+\lambda'}A_t\boldsymbol{u})(x-y)b^{\alpha t}_{\lambda k}\partial_i\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\,dy
\end{aligned}
$$

Integrating by parts $l+\hat{l}-1$ times, which is legal because each time the singularity is less than $n-1$, we have

$$
T^t_\lambda A_t\boldsymbol{u} = \int_{\mathbb{R}^n}(\partial_i A_t\boldsymbol{u})(x-y)b^{\alpha t}_{\lambda k}D^{\gamma+\lambda'}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\,dy,
$$

where ∂_i is a derivative in $D^{\gamma+\lambda}$. If we derive one more time as before we get

$$
\begin{aligned}
T^t_\lambda A_t &= \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}(\partial_i A_t\boldsymbol{u})(y)b^{\alpha t}_{\lambda k}(D^{\gamma+\lambda'}g)(x-y)\,dy\\
&= -\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}A_t\boldsymbol{u}(y)b^{\alpha t}_{\lambda k}\partial_{y,i}[(D^{\gamma+\lambda'}g)(x-y)]\,dy+\\
&\qquad\qquad + \lim_{\epsilon\to 0}\int_{\epsilon S^{n-1}}A_t\boldsymbol{u}(x-y)b^{\alpha t}_{\lambda k}(D^{\gamma+\lambda'}g)(y)n_i\,d\sigma,
\end{aligned}
$$

in the second integral we can replace $g(y)$ by $\varphi_\alpha(\tilde{y})/|y|^{n-l-\hat{l}}$, obtaining

$$\begin{aligned}
T^t_\lambda A_t \boldsymbol{u} &= \lim_{\epsilon\to 0}\int_{|y|>\epsilon} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\, dy + \\
&\qquad + \lim_{\epsilon\to 0}\int_{\epsilon S^{n-1}} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda'}\Big(\frac{\varphi_\alpha(\tilde{y})}{|y|^{n-l-\hat{l}}}\Big) n_i\, d\sigma \\
&= \lim_{\epsilon\to 0}\int_{|y|>\epsilon} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\beta\lambda k} D^{\gamma+\lambda}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\, dy + \\
&\qquad + \lim_{\epsilon\to 0}\frac{1}{\epsilon^{n-1}}\int_{\epsilon S^{n-1}} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k}\psi_{0,\alpha}(\tilde{y}) n_i\, d\sigma \\
&= \lim_{\epsilon\to 0}\int_{|y|>\epsilon} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\, dy + \\
&\qquad + A_t\boldsymbol{u}(x)\lim_{\epsilon\to 0}\frac{1}{\epsilon^{n-1}}\int_{\epsilon S^{n-1}} b^{\alpha t}_{\lambda k}\psi_{0,\alpha}(\tilde{y}) n_i\, d\sigma
\end{aligned}$$

where $\psi_{0,\alpha}$ is defined as in Lemma 2.25. Notice that second summand is no more than $A_t(x)$ multiplied by a constant, so we turn to prove L^p continuity of the first integral, which is a singular integral.

Let RA_t be the operator

$$(RA_t\boldsymbol{u})(x) = \lim_{\epsilon\to 0}\int_{|y|>\epsilon} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\, dy$$

We break up the integral above

$$\begin{aligned}
(RA_t\boldsymbol{u})(x) &= \int_{|y|>r} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\, dy + \\
&\qquad + \lim_{\epsilon\to 0}\int_{\epsilon<|y|\le r} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda}\Big(\frac{\varphi_\alpha(\tilde{y})}{|y|^{n-l-\hat{l}}}\Big)\, dy \\
&= \int_{|y|>r} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k} D^{\gamma+\lambda}\Big(\frac{\zeta_\alpha(y)}{|y|^{n-l-\hat{l}}}\Big)\, dy + \\
&\qquad + \lim_{\epsilon\to 0}\int_{\epsilon<|y|\le r} A_t\boldsymbol{u}(x-y) b^{\alpha t}_{\lambda k}\frac{\psi_\alpha(\tilde{y})}{|y|^n}\, dy
\end{aligned}$$

where ψ_α has zero average on the sphere by Lemma 2.25 and thus the singular integral is well defined. Since ψ_α is bounded on S^{n-1}, we apply Theorem 2.24 to conclude that $\|RA_t\boldsymbol{u}\|_{L^p(\Omega_s)} \le \|RA_t\boldsymbol{u}\|_{L^p(\mathbb{R}^n)} \le C\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)}$.

We have proved the local result $\|D^l\boldsymbol{u}\|_{L^p(\Omega_s)} \le C\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)}$, so we paste these fragments

$$\|D^l\boldsymbol{u}\|_{L^p(\Omega)} \le \sum_s \|D^l\boldsymbol{u}\|_{L^p(\Omega_s)} \le C\|\boldsymbol{u}\|_{\boldsymbol{E}^p(\Omega)}$$

from which the theorem follows. □

As I promised earlier, we will prove a partial converse of Theorem 2.8 as an application of Theorem 2.21. For simplicity, we define the operator $D_h^l = (1/|h|^l)\triangle_h^l$; the reader might verify that for smooth functions $D_h^l u \to \partial_h^l u$ uniformly over compact sets as $|h| \to 0$. If $u, v \in L^2(\Omega)$ and v has compact support in Ω then for small enough $|h|$ we get

$$\int_\Omega v D_h^l u\, dx = (-1)^l \int_\Omega u D_{-h}^l v\, dx$$

Theorem 2.27. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and that $1 < p < \infty$. If $V \Subset \Omega$ satisfies the cone condition, $u \in L^p(V)$ and there exists a constant C such that for every $|h| < \frac{1}{l}\inf\{|x-y| \mid x \in V \text{ and } y \in \Omega^c\}$ we have*

$$\|D_h^l u\|_{L^p(V)} \leq C, \tag{2-39}$$

then $u \in W^{l,p}(V)$.

Proof. Assume the vector $h = |h|e_i$ for some canonical basis vector e_i. Since $D_h^l u$ is bounded and $L^p(V)$ is reflexive, there exists a sequence $h_k \to 0$ such that

$$D_{h_k}^l u \rightharpoonup v_i \qquad \text{in } L^p(V).$$

If $\varphi \in C_c^\infty(V)$, then

$$\int_V u\partial_i^l \varphi\, dx = \lim_{h_k \to 0} \int_\Omega u D_{h_k}^l \varphi\, dx = (-1)^l \lim_{h_k \to 0} \int_V D_{h_k}^l u \varphi\, dx = (-1)^l \int_V v_i \varphi\, dx,$$

hence $v_i = \partial_i^l u$ and $\|\partial_i^l u\|_{L^p(V)} \leq C$, therefore by Theorem 2.21 we prove what we wanted. □

3 Applications to Elasticity

The machinery developed in Chapter 2 can be used to prove existence of equilibrium states for many linear models in elasticity, as the reader might notice from Chapter 1. We prove first existence of solution for a general linear problem in elasticity. This example serves as a guideline for proving existence theorems in linear theories; the reader is also referred to Bîrsan [7] to see a recent example in theory of linear Cosserat shells. Nevertheless, some linear systems resist being studied exactly with these methods, like conical shells, in which case weighted Sobolev spaces must be used.

Although we are dealing with linear problems, we can study nonlinear problems constructing suitable energy spaces for them. We will prove existence of at least an equilibrium point for a nonlinear shallow shell constrained by a frictionless obstacle.

3.1 Existence of Solution for Linear Elasticity

Let us consider the problem of existence of a unique minimum point, where the energy functional describes an elastic body under a load. The displacement of a body is given by a function $\boldsymbol{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and the energy operator, which we will call in this case strain operator, is

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) + c_{ij}^k u_k;$$

it can be written as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) + \boldsymbol{C} \cdot \boldsymbol{u},$$

where $\boldsymbol{C} \cdot \boldsymbol{u} = (c_{ij}^k u_k)$ is a matrix of size $n \times n$. The coefficients c_{ij}^k are often replaced by Christoffel symbols. The body forces are $\boldsymbol{f}$ and the load applied on $\Gamma_2 \subset \partial\Omega$ is $\boldsymbol{g}$, hence the total energy of the body is

$$E(\boldsymbol{u}) = \int_\Omega E^{ijkl} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{kl}(\boldsymbol{u})\, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, dx - \int_{\Gamma_2} \boldsymbol{g} \cdot \boldsymbol{u}\, d\sigma. \tag{3-1}$$

We assume all the conditions in Section 2.3 and additionally $c_{ij}^k \in L^\infty(\Omega)$; this requirement will be clear below in Lemma 3.2. We impose the Dirichlet boundary conditions

$$\boldsymbol{u}|_{\Gamma_1} = 0, \tag{3-2}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$.

Theorem 3.1. *Suppose that* Ω *is a bounded open set with* C^1*-boundary and that* $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ *and* $\boldsymbol{g} : \Gamma_2 \to \mathbb{R}^n$ *are measurable functions such that*

$$\begin{aligned} &\boldsymbol{f} \in [L^p(\Omega)]^n \text{ and } \boldsymbol{g} \in [L^q(\Gamma_2)]^n \text{ for } 1 < p, q \leq \infty, && \text{if } n = 2, \\ &\boldsymbol{f} \in [L^{\frac{2n}{n+2}}(\Omega)]^n \text{ and } \boldsymbol{g} \in [L^{\frac{2(n-1)}{n}}(\Gamma_2)]^n, && \text{if } n > 2. \end{aligned} \tag{3-3}$$

Then there exists a unique minimum point of the functional (3-1) *with boundary conditions* (3-2).

The above theorem follows from Theorem 2.18, Korn's inequality and the injectivite of the strain operator in next lemma. The fact that $\boldsymbol{f}$ and $\boldsymbol{g}$ represent bounded functionals is consequence of the Sobolev embedding theorem, as we remarked in Section 2.2.

Lemma 3.2. *The strain operator is injective in the space*

$$[W^{1,2}_{\partial}(\Omega)]^n = \{\boldsymbol{u} \in [W^{1,2}(\Omega)]^n \mid \boldsymbol{u}|_{\Gamma_1} = \boldsymbol{0}\}. \tag{3-4}$$

Proof. Since the operator $\boldsymbol{\varepsilon}$ describes the strain state of a body, we expect its norm is invariant under orthogonal transformations, and certainly this is the case. Let us write the norm as

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L^2(\Omega)} &= \int_\Omega \varepsilon_{ij}(\boldsymbol{u})\varepsilon^{ij}(\boldsymbol{u})\, dx \\ &= \int_\Omega \operatorname{Tr} \boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{\varepsilon}(\boldsymbol{u})^T\, dx \end{aligned}$$

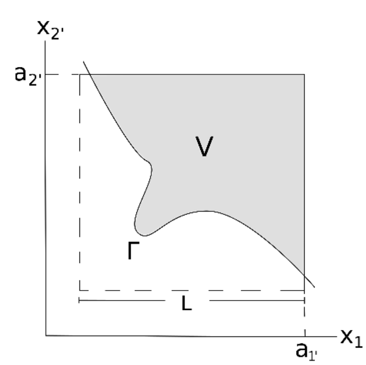
If $\{e_i\}$ and $\{e_{i'}\}$ are two orthonormal bases for $\mathbb{R}^n$, whose rules of transformation are $e_{i'} = q^i_{i'} e_i$ and $e_i = q^{i'}_i e_{i'}$, then by the invariance of the trace under orthogonal transformation we have

$$\int_\Omega (\partial_i u_j + \partial_j u_i + c^k_{ij} u_k)^2\, dx = \int_{\Omega'} (\partial_{i'} u_{j'} + \partial_{j'} u_{i'} + c^{k'}_{i'j'} u_{k'})^2\, dx',$$

where $c^{k'}_{i'j'} = c^k_{ij} q^{k'}_k q^i_{i'} q^j_{j'}$. Furthermore, notice that $|\boldsymbol{C}|^2 = c^k_{ij} c^{ij}_k$ is also invariant under orthogonal transformations.

If $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{0}$ then its norm is also zero, and by the invariance above we have for any orthonormal coordinate system

$$2\partial_{i'} u_{i'} = -c^{k'}_{i'i'} u_{k'}. \tag{3-5}$$



Draw a cube K, rotating if it is necessary, with sides of length L such that Γ_1 passes through adjacent sides of the cube and then define the set $V = K \cap \Omega'$. The sides of the cube intersecting Ω' are $x_{i'} = a_{i'}$.

The function $(x_{i'} - a_{i'}) u^2_{i'} n_{i'}$ is zero on ∂V, thus using Gauss-Green formula we have

$$\int_V \partial_{i'}((x_{i'} - a_{i'}) u^2_{i'})\, dx' = \int_V u^2_{i'} + 2(x_{i'} - a_{i'}) u_{i'} \partial_{i'} u_{i'}\, dx' = 0.$$

It follows

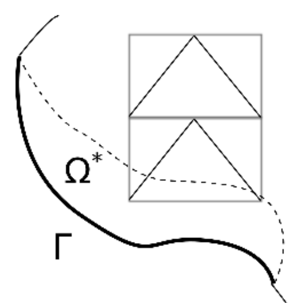
$$\begin{aligned}\int_V u_{i'}^2\,dx' &\le 2\int_V |(x_{i'}-a_{i'})u_{i'}\partial_{i'}u_{i'}|\,dx' = \int_V \left|(x_{i'}-a_{i'})u_{i'}c_{i'i'}^{k'}u_{k'}\right|dx'\\ &\le \frac{L}{2}\int_V \sum_{k'=1}^{n} |c_{i'i'}^{k'}|(u_{i'}^2+u_{k'}^2)\,dx'\\ &= \frac{L}{2}\int_V (|c_{i'i'}^{i'}|+\sum_{k'=1}^{n}|c_{i'i'}^{k'}|)u_{i'}^2 + \sum_{k'\neq i'}|c_{i'i'}^{k'}|u_{k'}^2\,dx'\\ &\le L\sum_{k'=1}^{n}\|c_{i'i'}^{k'}\|_\infty \int_V |\boldsymbol{u}|^2\,dx' \le nL\|\boldsymbol{C}\|_{L^\infty(\Omega)}\int_V |\boldsymbol{u}|^2\,dx'.\end{aligned}$$

Take the sum over all the components of $\boldsymbol{u}$, obtaining

$$\int_V |\boldsymbol{u}|^2\,dx' \le n^2L\|\boldsymbol{C}\|_{L^\infty(\Omega)}\int_V |\boldsymbol{u}|^2\,dx'.$$

Choose L small enough so that $n^2L\|\boldsymbol{C}\|_{L^\infty(\Omega)} < 1$, then necessarily $\boldsymbol{u} = \boldsymbol{0}$ in V. Notice that the length L does not depend on the location of the cube in Ω.

Since for every point in Γ_1 there exists a neighborhood where $\boldsymbol{u} = \boldsymbol{0}$, then in a neighborhood Ω^* of Γ_1 we have $\boldsymbol{u} = \boldsymbol{0}$. Using Ω^* we can extend the equality $\boldsymbol{u} = \boldsymbol{0}$ by means of rectangles as the figure shows. The set Ω can then be covered by a net of rectangles stemming from Ω^*, covering a subset Ω_1. Next we use smaller rectangles covering a greater set $\Omega_2 \subset \Omega$ and so on, using at most countable many rectangles, obtaining in this way $\boldsymbol{u} = \boldsymbol{0}$ in Ω. □



The energy functional (3-1) is a simple case in elasticity, and for structural designs more energy terms can be added, involving the boundary of the body, without altering seriously the proof.

3.2 Nonlinear Shallow Shells

The mathematical theory of shallow shells was thoroughly studied by Vorovich [39] and his students [40, 25]. As we saw in Chapter 1, this theory assumes Kirchhoff-Love hypothesis and thinness of the shell, as well as other assumptions well explained in the book of Vorovich. In some applications we must include additional constraints for displacements; this is the case for an obstacle whose contact area with the shell is unknown in advance. Owing to the practical importance of this problem, it has begun to receive intensive study [25, 6, 26, 5].

For the time being, let greek indices take values in $\{1, 2\}$ and latin indices in $\{1, 2, 3\}$. We remind the reader that the midsurface S^* of a shell is given by a smooth function $\boldsymbol{\varphi} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and that the displacement in this surface determines the displacement in the remainder of the shell. The covariant basis vectors are denoted by $\boldsymbol{a}_\alpha$ and the normal vector by $\boldsymbol{a}_3$. The displacement of the body is given by a function $\boldsymbol{u} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$, where the components u_α represent tangential displacement and u_3 normal displacement. In theory of shallow shells the displacement $\boldsymbol{u}^\theta$ of a point $\boldsymbol{\varphi}(x) + \theta\boldsymbol{a}_3$ can be approximated by

$$\boldsymbol{u}^\theta = \boldsymbol{u} + \theta\partial_\alpha u_3 \boldsymbol{a}^\alpha. \tag{3-6}$$

The strain operator is

$$\varepsilon_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(u_{\beta|\alpha} + u_{\alpha|\beta}) - b_{\alpha\beta}u_3 + \frac{1}{2}\partial_\alpha u_3 \partial_\beta u_3,$$
$$\rho_{\alpha\beta}(\boldsymbol{u}) = u_{3|\alpha\beta}.$$

Because of the nonlinearity of this operator, we cannot use the operator itself to define an energy space, but we will set up an energy space using the linear part of it.

The external load on the shell is $\boldsymbol{f} : \Omega \to \mathbb{R}^3$, the force along the edge is $\boldsymbol{g} : \Gamma_2 \subset \partial\Omega \to \mathbb{R}^3$ and the moment is $M : \Gamma_2 \subset \partial\Omega \to \mathbb{R}$. Consequently, the total energy of the shallow shell is

$$E(\boldsymbol{u}) = \int_\Omega E^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}(\boldsymbol{u})\varepsilon_{\gamma\delta}(\boldsymbol{u}) + D^{\alpha\beta\gamma\delta}\rho_{\alpha\beta}(\boldsymbol{u})\rho_{\delta\gamma}(\boldsymbol{u})\,dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx - \int_{\Gamma_2}(\boldsymbol{g}\cdot\boldsymbol{u} + M\partial_{\boldsymbol{n}}u_3)\,d\sigma,$$

where $E^{\alpha\beta\gamma\delta} \in L^\infty(\Omega)$ and $D^{\alpha\beta\gamma\delta} \in L^\infty(\Omega)$ are uniformly positive definite. We assume that the shell is clamped along the edge $\Gamma_1 \subset \partial\Omega$, so the boundary conditions are

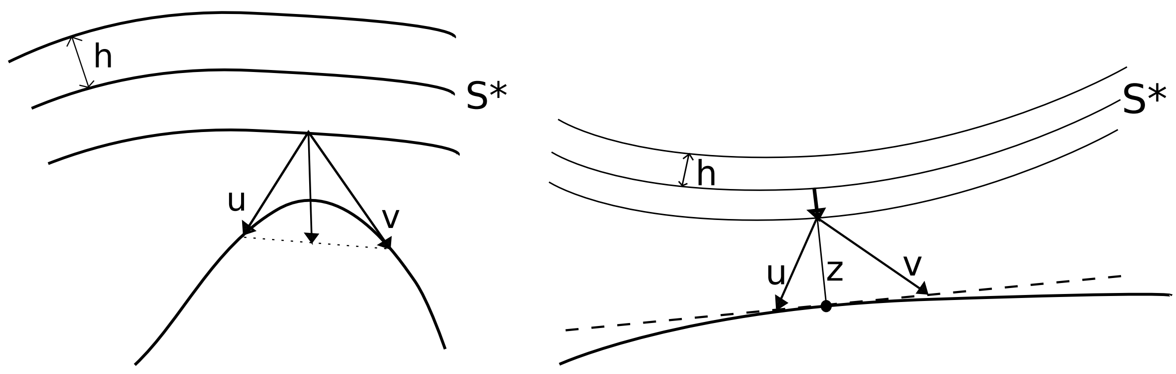
$$\boldsymbol{u}|_{\Gamma_1} = \boldsymbol{0} \qquad\qquad \partial_{\boldsymbol{n}}u_3|_{\Gamma_1} = 0.$$

By the Sobolev embedding theorem, the boundary conditions are a bounded operator, satisfying the requirement in Chapter 2.

The contact area between the shell and the obstacle is unknown in advance. The obstacle is assumed rigid and unmovable, so it maintains its form. Let us assume a frictionless contact between the obstacle and shell. The face of the shell exposed to the obstacle is $\boldsymbol{\varphi} - h\boldsymbol{a}^3$, where $2h$ is the thickness of the shell. The surface of the obstacle is described by the function $\boldsymbol{\psi} = \boldsymbol{\varphi} + z\boldsymbol{a}^3$, where z is a smooth function. Notice that $z(x) \le -h$ at every $x \in \Omega$. The displacement of a point $\boldsymbol{\varphi} - h\boldsymbol{a}^3$ in the face of the shell can be written in terms of the displacement in the midsurface using (3-6), so that

$$\boldsymbol{u}^{-h} = \boldsymbol{u} - h\boldsymbol{a}^\alpha\partial_\alpha u_3 = (u_\alpha - h\partial_\alpha u_3)\boldsymbol{a}^\alpha + u_3\boldsymbol{a}^3. \tag{3-7}$$

In general the condition of a non-penetrating obstacle is not convex, as Fig. **3-1**a shows. However, if we limit ourselves to small tangential displacements, as assumed in the theory of shallow shells, the obstacle restriction can be approximated by a convex restriction. Since the tangential displacements are small, we can replace the obstacle at each point by the corresponding tangent plane. See Fig. **3-1**b.



(a) A vector lying between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, penetrating the obstacle.

(b) Linearization of obstacle condition, yielding a convex restriction.

Figure **3-1**

The tangent plane is spanned by the vectors

$$\partial_\alpha \boldsymbol{\psi} = (a_{\alpha\beta} - z b_{\alpha\beta})\boldsymbol{a}^\beta + \partial_\alpha z \boldsymbol{a}^3. \tag{3-8}$$

A perpendicular vector in the tangent plane pointing toward the shell is given by

$$\begin{aligned}\boldsymbol{r} = \partial_1 \boldsymbol{\psi} \times \partial_2 \boldsymbol{\psi} = [&\partial_1 z(a_{2\alpha} - z b_{2\alpha}) - \partial_2 z(a_{1\alpha} - z b_{1\alpha})]\epsilon^{\alpha\beta}\boldsymbol{a}_\beta \\ &+ (a_{1\alpha} - z b_{1\alpha})(a_{2\beta} - z b_{2\beta})\epsilon^{\alpha\beta}\boldsymbol{a}_3,\end{aligned}$$

where $\epsilon^{\alpha\beta}$ is the alternating tensor

$$\epsilon^{\alpha\beta} = \begin{cases} 1/\sqrt{a} & \text{if } (\alpha, \beta) = (1, 2), \\ -1/\sqrt{a} & \text{if } (\alpha, \beta) = (2, 1), \\ 0 & \text{if } \alpha = \beta. \end{cases} \tag{3-9}$$

Since the shell does not penetrate the obstacle and the tangential displacements are small, the vector $\boldsymbol{u}^{-h}(x)$ must lie over the tangent plane of the point $\boldsymbol{\psi}(x)$. Hence we require the condition

$$\boldsymbol{u}^{-h} \cdot \boldsymbol{r} \geq (z + h)\boldsymbol{a}_3 \cdot \boldsymbol{r} \tag{3-10}$$

or

$$N(\boldsymbol{u}) = P^\alpha(\boldsymbol{\varphi}, z)(u_\alpha - h\partial_\alpha u_3) + P^3(\boldsymbol{\varphi}, z)u_3 \geq P^3(\boldsymbol{\varphi}, z)(z + h) \tag{3-11}$$

where

$$\begin{aligned} P^1(\boldsymbol{\varphi}, z) &= \partial_2 z(a_{12} - z b_{12}) - \partial_1 z(a_{22} - z b_{22}), \\ P^2(\boldsymbol{\varphi}, z) &= \partial_1 z(a_{21} - z b_{21}) - \partial_2 z(a_{11} - z b_{11}), \\ P^3(\boldsymbol{\varphi}, z) &= (a_{11} - z b_{11})(a_{22} - z b_{22}) - (a_{12} - z b_{12})(a_{21} - z b_{21}). \end{aligned}$$

The set $\mathbf{K}$ of functions satisfying (3-11) is convex and closed. In fact, for every $t \in (0,1)$ we have

$$N((1-t)\boldsymbol{u} + t\boldsymbol{v}) = (1-t)N(\boldsymbol{u}) + tN(\boldsymbol{v}) \geq P^3(\boldsymbol{\varphi}, z)(z+h); \tag{3-12}$$

to see that $\mathbf{K}$ is closed, take a sequence of functions in $\mathbf{K}$ converging to $\boldsymbol{u}$, so we can take a subsequence converging almost everywhere to $\boldsymbol{u}$ and then $\boldsymbol{u}$ satisfies inequality (3-11).

Lemma 3.3. *The space*

$$[W^{1,2}_{\partial}(\Omega)]^2 \times W^{2,2}_{\partial}(\Omega) = \{\boldsymbol{u} \in [W^{1,2}(\Omega)]^2 \times W^{2,2}(\Omega) \mid \boldsymbol{u}|_{\Gamma_1} = \mathbf{0} \text{ and } \partial_{\boldsymbol{n}} u_3 = 0\} \tag{3-13}$$

is isomorphic to the energy space $\boldsymbol{E}(\Omega)$ generated by the bilinear form

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega E^{\alpha\beta\gamma\delta} \hat{\varepsilon}_{\alpha\beta}(\boldsymbol{u}) \hat{\varepsilon}_{\gamma\delta}(\boldsymbol{v}) + D^{\alpha\beta\gamma\delta} \rho_{\alpha\beta}(\boldsymbol{u}) \rho_{\gamma\delta}(\boldsymbol{v})\, d\boldsymbol{x}, \tag{3-14}$$

where $\hat{\boldsymbol{\varepsilon}}$ is the linear part of $\boldsymbol{\varepsilon}$, i.e.

$$\hat{\varepsilon}_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(u_{\beta|\alpha} + u_{\alpha|\beta}) - b_{\alpha\beta} u_3 \tag{3-15}$$

Proof. It suffices to prove the equivalence using the norm

$$\|\boldsymbol{u}\|^2_{\boldsymbol{E}(\Omega)} = \int_\Omega \hat{\varepsilon}_{\alpha\beta}(\boldsymbol{u}) \hat{\varepsilon}^{\alpha\beta}(\boldsymbol{u}) + \rho_{\alpha\beta}(\boldsymbol{u}) \rho^{\alpha\beta}(\boldsymbol{u})\, d\boldsymbol{x}.$$

Using Lemma 2.2 as in the preceding chapter and Korn's inequality, it remains to prove the injectivity of the operator.

As in Lemma 3.2, the norm of $\boldsymbol{\rho}(\boldsymbol{u})$ is invariant under orthogonal transformations, and if $\boldsymbol{\rho}(\boldsymbol{u}) = \mathbf{0}$ then $\partial_{\alpha\alpha} u_3 = \Gamma^{\delta}_{\alpha\alpha} \partial_\delta u_3$, which is just equation (3-5) with $\partial_\alpha u_3$ instead of $u_{i'}$, therefore we conclude that u_3 is constant in Ω, and the constant is necessarily zero.

Now we readily have $u_\alpha = 0$ by Lemma 3.2 and then the operator is injective. □

Expanding the energy functional and using the Sobolev embedding theorem, we can write the energy functional as $E(\boldsymbol{u}) = \|\boldsymbol{u}\|^2_{\boldsymbol{E}(\Omega)} + \Psi(\boldsymbol{u})$, where the functional Ψ is sequentially weakly continuous (s.w.c), so the energy functional is s.w.l.s-c in $\boldsymbol{E}(\Omega)$. The existence of a minimum point in the closed and convex set $\mathbf{K}$ follows from Lemma 1.3, the reflexivity of Hilbert spaces and the fact that the functional E is growing, whose long proof can be found in [39]. Although we have already proved existence of a minimum point, we can actually prove a result somewhat stronger.

Theorem 3.4. *If $\boldsymbol{u}_k$ is a minimizing sequence in $\mathbf{K}$, then we can select a subsequence that converges strongly to a minimum point of E in $\boldsymbol{K}$.*

This theorem is consequence of the next lemma.

Lemma 3.5. *Let* $\Psi : H \to \mathbb{R}$ *be a weakly continuous functional in a Hilbert space* H *restricted to some weakly compact set* K. *Then for every minimizing sequence* $\{u_n\}$ *of the functional* $I(u) = \|u\|^2 + \Psi(u)$, *there exists a subsequence* $\{u_{n_k}\} \subset \{u_n\}$ *converging strongly to a minimum point* u_0.

Proof. Define $d = \inf_{u \in K} I(u)$ and let $\{u_n\}$ be a minimizing sequence. Without loss of generality, we can suppose that $u_n \rightharpoonup u_0$. Indeed, as $\{u_n\}$ is bounded in the Hilbert space H, we can select from $\{u_n\}$ a subsequence that converges weakly to some u_0. Re-denoting this subsequence as $\{u_n\}$, we get $\{u_n\}$ with the needed property.

Once again, from $\{u_n\}$ take another subsequence such that $\|u_{n_k}\| \to a$, where $a \in \mathbb{R}$.

Notice that $\|u_0\| \leq a$. In fact,

$$\|u_0\|^2 = \lim_{n \to \infty} |(u_{n_k}, u_0)| \leq \lim_{n \to \infty} \|u_{n_k}\| \|u_0\| = a\|u_0\|.$$

On the other hand, since $\{u_{n_k}\}$ is a minimizing sequence and Ψ is weakly continuous, we have

$$d = \lim_{k \to \infty} I(u_{n_k}) = a^2 + \Psi(u_0).$$

However, $I(u_0) \geq d = a^2 + \Psi(u_0)$ so that $\|u_0\| \geq a$, and from the inequality above we have $a = \|u_0\|$, hence u_0 is a minimum point. Since $\|u_{n_k}\| \to \|u_0\|$ as $n_k \to \infty$, the sequence $\{u_{n_k}\}$ converges strongly to u_0 [8, p. 78]. □

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