EL PROBLEMA DE CAUCHY ASOCIADO A LA ECUACIÓN DE GARDNER-BURGERS PERIÓDICO.

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Introducción

En este trabajo trataremos el buen planteamiento tanto local como global en el caso periódico de la ecuación

$$u_t + uu_x + u^2u_x + u_{xxx} - u_{xx} = 0, \tag{1}$$

conocida como la ecuación de *Gardner-Burgers (G-B)*, o también como la *mKdv-Kdv-Burgers*. La ecuación *G-B* modela fenómenos que se presentan en teoría de fluidos [4] y en elasticidad [1]. Específicamente estudiaremos el problema de valor inicial

$$\begin{cases} u_t + uu_x + u^2u_x + u_{xxx} = u_{xx}, & x \in \mathbb{T}, t > 0 \\ u(x,0) = \varphi(x) \in H^s(\mathbb{T}), \end{cases} \tag{2}$$

donde $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$
Recientemente, en [5] estudiaron el problema de valor inicial

$$\begin{cases} u_t + u^2u_x + u_{xxx} = u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x,0) = \varphi(x) \in H^s(\mathbb{R}), \end{cases} \tag{3}$$

empleando técnicas del tipo Bourgain-Kenig-Ponce-Vega en espacios de Bourgain. En dicho artículo prueban la existencia de soluciones en $H^s(\mathbb{R})$, para $s > -1/4$. Por tal razón, basandonos en [5] emprendimos la labor de estudiar el buen planteamiento de (2).
Este trabajo se ha dividido en cuatro capítulos de la siguiente manera: En el primer capítulo recordamos algunos resultados los cuales serán de utilidad en el desarrollo posterior del trabajo. En el segundo capítulo se estudia la buena colocación tanto local como global del problema (2) en los espacios de Sobolev H^s, para $s > \dfrac{1}{2}$ en el caso local y para $s \geq 1$ en el caso global pero con dato inicial suficientemente pequeño. Dato curioso, pues en el caso de la mKdV la ley de conservación ayuda para que sus soluciones persistan en todo tiempo sin importar el tamaño del dato inicial. El tercer capítulo, aborda la existencia de soluciones del problema (2) en espacios de Bourgain utilizando las técnicas de Kenig-Ponce-Vega [3], siguiendo los argumentos expuestos por Molinet y Ribaud en [8] y Junfeng Li en [5] para el caso no periódico,

obteniendo buena colocación local de (3) en $H^s(\mathbb{R})$ para $s > -\frac{1}{4}$. En nuestro caso obtenemos un resultado semejante con la misma cota para el s. Por último, el texto presenta un Apéndice en el que se compilan los principales resultados utilizados en el capítulo tres.

Notación

Antes de abordar el problema de nuestro interés presentamos la notación que usaremos, organizada dependiendo del escenario en el que estemos trabajando.

1. Notaremos con $\mathbb{R}$ al cuerpo de los números reales.

2. Notaremos con $\mathbb{C}$ al cuerpo de los números complejos.

3. Para $x, y \in \mathbb{R}$, escribiremos $x \sim y$ para significar que existen constantes $C_1, C_2 > 0$ tales que
$$C_1|x| \leq |y| \leq C_2|x|.$$

4. Para un espacio de Banach X y un intervalo I cerrado y acotado, notamos con $C(I, X)$ al espacio de las funciones continuas definidas en I con valores en X, dotado de la norma
$$\|f\|_{\infty,X} = \sup_{t\in I} \|f(t)\|_X.$$

5. Notaremos con $\mathbb{T}$ al toro unidimensional.

6. $\mathcal{P} = C^\infty(\mathbb{T}) = \{f : \mathbb{T} \longrightarrow \mathbb{C};\ f \text{ es infinitamente diferenciable en } \mathbb{T}\}$ y $\mathcal{P}'$ el espacio de las distribuciones periódicas.

7. Para $f \in \mathcal{P}'$, $\mathcal{F}_x(f) = \widehat{f}$ denota la *transformada de Fourier de f con respecto a la variable espacial x*, es decir,
$$\widehat{f}(k) = \left(f; e^{ikx}\right),\ k \in \mathbb{Z},$$
donde $(\ ;\ )$ es el paréntesis de dualidad. En particular, si $f \in L^1(\mathbb{T})$,
$$\mathcal{F}_x(f)(k) = \widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)\ dx.$$

8. Escribiremos
$$\sum_{k\in\mathbb{Z}} a_k = \sum_{k=-\infty}^{\infty} a_k = \lim_{N\to\infty} \sum_{|k|\leq N} a_k.$$

9. Para $s \in \mathbb{R}$:

- $H^s(\mathbb{T}) = \left\{ f \in \mathcal{P}';\ (1+k^2)^{\frac{s}{2}}\ \widehat{f} \in \ell^2(\mathbb{Z}) \right\}$ es el *Espacio de Sobolev periódico de orden* s, dotado con la norma

$$\|f\|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} (1+k^2)^s \left|\widehat{f}(k)\right|^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left|\widehat{f}(k)\right|^2$$

  donde $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$ y $\ell^2(\mathbb{Z}) = \left\{ (C_m)_{m\in\mathbb{Z}};\ \sum_{m\in\mathbb{Z}} |C_m|^2 < +\infty \right\}$.

- $\dot{H}^s(\mathbb{T})$ es el *Espacio de Sobolev homogéneo periódico de orden* s, provisto con la norma

$$\|f\|^2_{\dot{H}^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} k^{2s} \left|\widehat{f}(k)\right|^2.$$

10. Si $f \in \mathcal{P}'$ entonces definimos el operador Λ^s por

$$\Lambda^s f = \mathcal{F}_x^{-1}\left(\left(1+k^2\right)^{\frac{s}{2}}\ \widehat{f}(k) \right), \tag{4}$$

por lo tanto $\|f\|_s^2 \sim \|\Lambda^s f\|_0^2$.

11. El espacio de Schwartz será notado como es usual por

$$\mathcal{S}(\mathbb{R}) = \{g : \mathbb{R} \longrightarrow \mathbb{C};\ g \in C^\infty(\mathbb{R})\} \text{ y}$$
$$\|g\|_{\alpha,\beta} = \sup_{t\in\mathbb{R}} \left\{ \left|t^\alpha g^{(\beta)}(t)\right| < +\infty, \text{ para todo } \alpha,\beta \in \mathbb{N} \right\}.$$

y $\mathcal{S}'(\mathbb{R})$ será el espacio de las distribuciones temperadas $\mathcal{S}'(\mathbb{R})$.

12. Para $g \in \mathcal{S}'(\mathbb{R})$, $\mathcal{F}_t(g) = \widehat{g}$ denota la *transformada de Fourier de* f *con respecto a la variable temporal* t. En particular, si $g \in L^1(\mathbb{R})$,

$$\mathcal{F}_t(g)(\tau) = \widehat{g}(\tau) = \int_{\mathbb{R}} e^{-it\tau} g(t)\ dt.$$

13. Análogamente, para $s \in \mathbb{R}$,

- $H^s(\mathbb{R}) = \left\{ g \in \mathcal{S}'(\mathbb{R});\ (1+|\tau|^2)^{\frac{s}{2}}\ \widehat{g} \in L^2(\mathbb{R}) \right\}$, es el *Espacio de Sobolev de orden* s, dotado con la norma

$$\begin{aligned}\|g\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} \left(1+|\tau|^2\right)^s |\widehat{g}(\tau)|^2\ d\tau \\ &= \int_{\mathbb{R}} \langle\tau\rangle^{2s} |\widehat{g}(\tau)|^2\ d\tau = \|\langle\tau\rangle^s\ \widehat{g}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

- $\dot{H}^s(\mathbb{R})$ es el *Espacio de Sobolev homogéneo de orden* s, provisto de la norma

$$\|g\|^2_{\dot{H}^s(\mathbb{R})} = \int_{\mathbb{R}} |\tau|^{2s} |\widehat{g}(\tau)|^2\ d\tau = \||\tau|^s\ \widehat{g}\|^2_{L^2(\mathbb{R})}.$$

Observe que en estos espacios tenemos para $\lambda > 0$ que:

$$\begin{aligned} \|g(\lambda t)\|_{H^s(\mathbb{R})} &\leq \left(\lambda^{-\frac{1}{2}} + \lambda^{s-\frac{1}{2}}\right) \|g(t)\|_{H^s(\mathbb{R})} \\ \|g(\lambda t)\|_{\dot{H}^s(\mathbb{R})} &\sim \lambda^{s-\frac{1}{2}} \|g(t)\|_{H^s(\mathbb{R})} . \end{aligned} \tag{5}$$

14. Denotamos por $\mathcal{F}(u)(k,\tau) = \widehat{u}(k,\tau)$ la *Transformada de Fourier con respecto a ambas variables* (temporal t y espacial x).

15. Notaremos con $\mathcal{U}$ el grupo unitario en $H^s(\mathbb{T})$, $s \in \mathbb{R}$, generado por el operador $-\partial_x^3$, es decir
$$\mathcal{U}(t) = e^{-t\partial_x^3}.$$

16. Sea $\mathcal{X}$ el espacio de todas las u tales que

    i) $u : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{C}$.

    ii) $u(x,\cdot) \in \mathcal{S}(\mathbb{R})$ para cada $x \in \mathbb{T}$.

    iii) La aplicación $x \longmapsto u(x,t)$ es de clase C^∞ para cada $t \in \mathbb{R}$.

    iv) $\widehat{u}(0,\tau) = 0$ para todo $\tau \in \mathbb{R}$.

17. Para $s, b \in \mathbb{R}$, el completado de $\mathcal{X}$ con respecto a la norma

$$\|u\|^2_{H^{b,s}} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \int_{\mathbb{R}} \langle\tau\rangle^{2b} \left|\widehat{u}(\tau,k)\right|^2 \, d\tau. \tag{6}$$

lo notaremos con $H^{b,s}(\mathbb{T}\times\mathbb{R})$ y corresponde al *Espacio de Sobolev espacio-temporal.*

18. Para $s, b \in \mathbb{R}$, $X^{b,s}$ denota el completado de $\mathcal{X}$ con respecto a la norma

$$\|u\|^2_{X^{b,s}} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \int_{\mathbb{R}} \left\langle \left|\tau - k^3\right| + k^2 \right\rangle^{2b} \left|\widehat{u}(k,\tau)\right|^2 \, d\tau$$

o equivalentemente,

$$\|u\|^2_{X^{b,s}} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \int_{\mathbb{R}} \left\langle i\tau + k^2 \right\rangle^{2b} \left|\mathcal{F}\left(\mathcal{U}(-t)u\right)(k,\tau)\right|^2 \, d\tau. \tag{7}$$

19. Para $T > 0$ consideramos el *espacio localizado* $X_T^{b,s}$

$$X_T^{b,s} := \left\{u;\ \text{existe } w \in X^{b,s} \text{ tal que } u(t) = w(t) \text{ para todo } t \in [0,T]\right\},$$

dotado de la norma

$$\|u\|_{X_T^{b,s}} = \inf_{w\in X^{b,s}} \left\{\|w\|_{X^{b,s}}\ ;\ w(t) = u(t) \text{ para } t \in [0,T]\right\}. \tag{8}$$

20. Denotamos por $\mathcal{W}$ el semigrupo asociado con la evolución libre de (1), esto es, para todo $t \geq 0$

$$\mathcal{F}_x\left(\mathcal{W}(t)\varphi\right)(k) = e^{(-k^2+ik^3)t}\widehat{\varphi}(k), \quad \varphi \in \mathcal{P}',$$

o visto de otra manera,

$$\mathcal{W}(t)\varphi = e^{t(-\partial_x^3+\partial_x^2)}\varphi = \mathcal{F}_x^{-1}\left(e^{ik^3t}e^{-k^2t}\widehat{\varphi}\right) = \mathcal{U}(t)\mathcal{V}(t)\varphi. \tag{9}$$

donde

$$\mathcal{U}(t)\varphi = e^{-t\partial_x^3}\varphi = \mathcal{F}_x^{-1}\left(e^{ik^3t}\widehat{\varphi}\right)$$
$$\mathcal{V}(t)\varphi = e^{t\partial_x^2}\varphi = \mathcal{F}_x^{-1}\left(e^{-k^2t}\widehat{\varphi}\right).$$

Además, extendemos $\mathcal{W}$ a toda la recta de la siguiente forma:

$$\mathcal{F}_x(\mathcal{W}(t)\varphi)(k) = e^{(-k^2+ik^3)|t|}\widehat{\varphi}(k), \quad \varphi \in \mathcal{P}',\ t \in \mathbb{R}$$

y a esta extensión la seguiremos notando con $\mathcal{W}$.

21. A lo largo de este trabajo el operador A, esta dado por:

$$A = \partial_x^2 - \partial_x^3. \tag{10}$$

Capítulo 1

Preliminares

Proposición 1.0.1.

$$\lim_{h\to 0}\left\|\frac{e^{(t+h)A}\varphi - e^{tA}\varphi}{h} - Ae^{tA}\varphi\right\|_{s-3} = 0$$

uniformemente con respecto a $t \geq 0$

Demostración. Ver [9] teorema 4.15 página 224-425. □

Proposición 1.0.2. $E : [0,\infty) \longrightarrow \mathbb{B}(H^s(\mathbb{T}))$, *donde* $E(t)\varphi = e^{tA}\varphi = \mathcal{F}_x^{-1}\left(e^{(ik^3-k^2)t}\widehat{\varphi}\right)$, *es un semigrupo de contracción.*

Proposición 1.0.3 (Desigualdad de Gronwall). *Sean* $k \in L^1([a,b],\mathbb{R})$, $k \geq 0$ *y* $f, g \in C([a,b],\mathbb{R})$ *tales que*

$$f(t) \leq g(t) + \int_a^t k(\tau)f(\tau)\ d\tau, \qquad a \leq t \leq b$$

Entonces

$$f(t) \leq g(t) + \int_a^t k(\tau)\ e^{\int_\tau^t k(r)dr} g(\tau)\ d\tau, \qquad a \leq t \leq b$$

En particular, si $g(t) \equiv g$ *es constante en* $[a,b]$, *se tiene que*

$$f(t) \leq g\ e^{\int_a^t k(\tau)d\tau}, \qquad a \leq t \leq b$$

Proposición 1.0.4 (Desigualdad de Young). *Sean* $a, b > 0$, $\epsilon > 0$ *y* $1 < p < \infty$, *entonces*

$$ab \leq \epsilon\ a^p + \epsilon^{-\frac{1}{p-1}}\ b^{p'}$$

donde $\frac{1}{p} + \frac{1}{p'} = 1$.

Teorema 1.0.1 (Desigualdad de Hausdorff-Young para la convolución). *Sean* $p, q, r \in [1,\infty]$ *tales que* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. *Si* $f \in L^p(\mathbb{R}^n)$ *y* $g \in L^q(\mathbb{R}^n)$ *entonces* $f * g \in L^r(\mathbb{R}^n)$ *y*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\ \|g\|_{L^q}$$

En particular, si $f \in L^p(\mathbb{R}^n)$*, con* $1 \leq p \leq \infty$*, y* $g \in L^1(\mathbb{R}^n)$*, entonces* $f * g \in L^p(\mathbb{R}^n)$ *y*

$$\|f * g\|_{L^p} \leq \|f\|_{L^p} \ \|g\|_{L^1}$$

Teorema 1.0.2. *Sean* $f, u \in \mathcal{P}$ *de valor real,* $r > \frac{3}{2}$ *y* $s \geq 1$*. Entonces existe una constante* $C = C(s)$ *tal que*

$$|\langle u, f\partial_x u\rangle_s| \leq C\left[\|\partial_x f\|_{r-1} \|u\|_s^2 + \|\partial_x f\|_{s-1} \|u\|_r \|u\|_s\right]$$

Lema 1. *Supongamos que* $\beta > 0$*,* $\gamma > 0$*,* $\beta + \gamma > 1$*, que* $a \geq 0$*,* $b \geq 0$ *y* g *son no negativos, que* $t^{\gamma-1}g(t)$ *es integrable localmente sobre* $0 \leq t < T$ *y que*

$$g(t) \leq a + b\int_0^t (t-\tau)^{\beta-1}\tau^{\gamma-1}g(\tau)\ d\tau$$

en $(0, T)$*. Entonces*

$$g(t) \leq a\ E_{\beta,\gamma}\left((b\Gamma(\beta))^{\frac{1}{\nu}}\, t\right)$$

donde $\nu = \beta + \gamma - 1 > 0$*,* $E_{\beta,\gamma}(s) = \sum_{m=0}^{\infty} c_m s^{m\nu}$ *con* $c_0 = 1$ *y* $\frac{c_{m+1}}{c_m} = \frac{\Gamma(m\nu+\gamma)}{\Gamma(m\nu+\gamma+\beta)}$ *para* $m \geq 0$*.*

Capítulo 2

Estudio clásico del Problema

En este capítulo trataremos el buen planteamiento tanto local como global del problema (2) en los espacios de Sobolev $H^s(\mathbb{T})$, para $s > \dfrac{1}{2}$.

2.1. Estimativa para la parte lineal

La ecuación lineal

$$\begin{cases} u_t + u_{xxx} = u_{xx}, & x \in \mathbb{T}, t > 0 \\ u(x,0) = \varphi(x) \in H^s(\mathbb{T}) \end{cases}$$

Tiene como solución a

$$u(t,x) = e^{tA}\varphi = e^{(\partial_x^2 - \partial_x^3)t}\varphi := \sum_{k\in\mathbb{Z}} e^{(ik^3-k^2)t}\widehat{\varphi}(k)e^{ikx} \tag{2.1}$$

Lema 2. *(**Regularización**). Sean $\varphi \in H^s(\mathbb{T})$, $\lambda \geq 0$ y $t > 0$. Entonces, existe una constante k_λ que depende de λ tal que*

$$\|e^{tA}\varphi\|_{s+\lambda} \leq k_\lambda \left(1 + \frac{1}{t^{\frac{\lambda}{2}}}\right) \|\varphi\|_s,$$

Demostración. Ver en [9] Teorema 4.17, página 225. □

2.2. Ecuación no lineal

Por comidad escriberemos el problema (2) en la forma

$$\begin{cases} u_t = Au + F(t), \\ u(x,0) = \varphi(x) \in H^s(\mathbb{T}), \end{cases} \tag{2.2}$$

donde $F(t) = -u^2u_x - uu_x$ y A es como en (10) Observe que dicha ecuación es equivalente a la ecuación integral

$$u(t) = e^{tA}\varphi + \int_0^t e^{(t-\tau)A}\partial_x \left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)d\tau. \tag{2.3}$$

Más precisamnete, tenemos:

Teorema 2.2.1. *Si $s > \frac{1}{2}$, el problema (2) es equivalente a la ecuación integral (2.3) en el siguiente sentido: Si $u \in C([0,T], H^s)$ es la solución de (2) entonces u es solución de (2.3) y recíprocamente, si $u \in C([0,T], H^s)$ es solución de la ecuación integral (2.3) entonces u es solución de (2) donde la derivada en el tiempo es tomada en la topología de H^{s-3}, es decir,*

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h} + uu_x + u^2u_x + u_{xxx} - u_{xx}\right\|_{s-3} = 0$$

Demostración. Ver [9] los Teoremas 4.19 y 4.20, páginas 227 a 230. □

Antes de abordar la demostración del buen planteamiento local de (2) definiremos cierto espacio y cierta aplicación que serán de utilidad en el transcurso de la prueba. Con esto en mente, definimos para $T > 0, s > \frac{1}{2}$ y $M > 0$ el espacio:

$$\mathfrak{X}_s(T,M) = \{u \in C[0,T], H^s) \mid \|u - e^{tA}\varphi\|_s \leq M\} \tag{2.4}$$
$$= \overline{B}(e^{tA}\varphi, M), \tag{2.5}$$

dotado de la métrica

$$d(f,g) = \|f-g\|_{\infty,s} = \sup_{[0,T]}\|f(t)-g(t)\|_s$$

y la aplicación:

$$\Psi(u)(t) = e^{tA}\varphi + \int_0^t e^{(t-\tau)A}\partial_x \left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)d\tau, \tag{2.6}$$

para $0 \leq t \leq T$.
La prueba del buen planteamiento local la haremos en varias etapas. Por esta razón comenzamos con:

Proposición 2.2.1. *Si $u \in C([0,T], H^s)$, $s > \frac{1}{2}$, entonces, $\Psi(u) \in C([0,T], H^s)$*

Demostración. La prueba de este resultado es estándar y es consecuencia de la definición de Ψ en (2.6), del Lema de regularización 2, del lema de Sobolev, del Teorema de la convergencia dominada de Lebesgue y de la desigualdad triangular. □

Proposición 2.2.2. *Existe $\widetilde{T} > 0$ tal que $\Psi : \mathfrak{X}_s(\widetilde{T}, M) \longrightarrow \mathfrak{X}_s(\widetilde{T}, M)$ es una contracción.*

Demostración. Primero veamos que existe $T_1 > 0$ tal que $\Psi : \mathfrak{X}_s(T_1) \to \mathfrak{X}_s(T_1)$. Con esto en mente, supongamos que $u \in \mathfrak{X}_s(T)$, entonces

$$\begin{aligned}\left\|\Psi(u)(t) - e^{tA}\varphi\right\|_s &= \left\|-\int_0^t e^{(t-\tau)A}\partial_x\left(\frac{u^2}{2}+\frac{u^3}{3}\right)(\tau)d\tau\right\|_s \\ &\leq \int_0^t \left\|e^{(t-\tau)A}\partial_x(u^2+u^3)(\tau)\right\|_s d\tau. \end{aligned} \tag{2.7}$$

El integrando en la desigualdad (2.7) es estimado a partir del lema de regularización (2) con $\lambda = 1$ y del Lema de Sobolev, pues $s > \frac{1}{2}$. En efecto,

$$\begin{aligned}\left\|e^{(t-\tau)A}\partial_x\left(\frac{u^2}{2}+\frac{u^3}{3}\right)(\tau)\right\|_s &\leq C\left[1+\frac{1}{\sqrt{2(t-\tau)}}\right]\left\|\partial_x\left(\frac{u^2}{2}+\frac{u^3}{3}\right)(\tau)\right\|_{s-1} \\ &\leq C_s\left[1+\frac{1}{\sqrt{2(t-\tau)}}\right]\left[\|u(\tau)\|_s^2+\|u(\tau)\|_s^3\right]. \end{aligned} \tag{2.8}$$

Las desigualdades (2.8) y $\left\|u(\tau) - e^{\tau A}\varphi\right\|_s \leq M$ transforman a (2.7) en:

$$\begin{aligned}&\left\|\Psi(u)(t) - e^{tA}\varphi\right\|_s \\ &\qquad \leq C_s\left[(M+\|\varphi\|_s)^2+(M+\|\varphi\|_s)^3\right]\int_0^t\left[1+\frac{1}{\sqrt{2(t-\tau)}}\right]d\tau \\ &\qquad \leq C_s\left[(M+\|\varphi\|_s)^2+(M+\|\varphi\|_s)^3\right](T+\sqrt{2T}). \end{aligned} \tag{2.9}$$

Ahora, eligiendo $T_1 > 0$ tal que

$$C_s\left[(M+\|\varphi\|_s)^2+(M+\|\varphi\|_s)^3\right](T+\sqrt{2T}) \leq M$$

tenemos que $\Psi(\mathfrak{X}_s(T)) \subseteq \mathfrak{X}_s(T)$, para cada $T \leq T_1$.
Veamos ahora que existe T_2 tal que Ψ es una contracción. Sean $u, v \in \mathfrak{X}_s(T)$, debemos mostrar que

$$\|\Psi(u) - \Psi(v)\|_{s,\infty} \leq \alpha\|u-v\|_{s,\infty} \qquad 0 < \alpha < 1$$

Con esto en mente, procedemos a estimar $\|\Psi(u) - \Psi(v)\|_s$. Para ello procedemos de forma similar que en la primera parte de esta prueba y observando que apartir de la desigualdad triangular, del lema de regularización (2) con $\lambda = 1$, del Lema de Sobolev (pues $s > \frac{1}{2}$) y de las identidades algebraicas $u^2 - v^2 = (u-v)(u+v)$, $u^3 - v^3 = (u-v)(u^2+uv+v^2)$ tenemos para $u, v \in \mathfrak{X}_s(T, M)$ que

$$\begin{aligned}&\|\Psi(u) - \Psi(v)\|_s \\ &\qquad \leq C_s(M+\|\varphi\|_s)\left[1+(M+\|\varphi\|_s)\right]\|u-v\|_{\infty,s}\int_0^t\left[1+\frac{1}{\sqrt{2(t-\tau)}}\right]d\tau \\ &\qquad \leq C_s(M+\|\varphi\|_s)\left[1+(M+\|\varphi\|_s)\right](T+\sqrt{2T})\|u-v\|_{\infty,s} \end{aligned}$$

Eligiendo, $T_2 > 0$, tal que

$$\left[C_s(M + \|\varphi\|_s) + C_s(M^2 + \|\varphi\|_s^2)\right](T + \sqrt{2T}) = \frac{1}{2} < 1 \tag{2.10}$$

se tiene que Ψ es una contracción. Por lo tanto, si $\widetilde{T} = min\{T_1, T_2\}$ se tiene que: $\Psi : \mathfrak{X}_s(T) \longrightarrow \mathfrak{X}_s(T)$ es una contracción para cada $0 < T \leq \widetilde{T}$. □

Nota 1. *Observe que la proposición anterior implica via el teorema del punto fijo de Banach, que existen $T > 0$ y $u \in C([0,T], H^s(\mathbb{T}))$ tal que*

$$u(t) = \Psi(u)(t) = e^{tA}\varphi - \int_0^t e^{(t-\tau)A}\partial_x \left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)d\tau.$$

Por lo tanto, el teorema 2.2.1 y la proposición 2.2.2 implican que (2) tiene solución en $H^s(\mathbb{T})$, para $s > 1/2$

Teorema 2.2.2. *El problema (2)es localmente bien planteado en $H^s(\mathbb{T})$ para $s > \frac{1}{2}$.*

Demostración. De acuerdo con la proposición 2.2.2 vemos que solo falta demostrar la unicidad y la dependencia continua del dato inicial. Comenzaremos abordando el asunto de la unicidad. Con este fin, sean $u, v \in C([0,T], H^s)$ soluciones de (2) con datos iniciales $u(0) = \varphi$ y $v(0) = \psi$ respectivamente. Aplicando, la norma de $H^s(\mathbb{T})$ a $u(t) - v(t)$ y posteriormente usando la ecuación integral(2.3), la desigualdad triangular y el lema de regularización con $\lambda = 1$ y $s > \dfrac{1}{2}$ se tiene que

$$\begin{aligned}
\|u(t) - v(t)\|_s &\leq \left\|e^{tA}(\varphi - \psi)\right\|_s + \int_0^t \left\|e^{(t-\tau)A}\partial_x \left(\frac{u^3 - v^3}{3} - \frac{u^2 - v^2}{2}\right)(\tau)\right\|_s d\tau \\
&\leq \|\varphi - \psi\|_s + \int_0^t \left\|e^{(t-\tau)A}\partial_x \left(\frac{u^3 - v^3}{3} - \frac{u^2 - v^2}{2}\right)(\tau)\right\|_s d\tau \\
&\leq \|\varphi - \psi\|_s + C_s \int_0^t \left[1 + \frac{1}{\sqrt{2(t-\tau)}}\right] \|u(\tau) - v(\tau)\|_s \, d\tau,
\end{aligned} \tag{2.11}$$

donde $C_s = C_s\left(\|u\|_{\infty,s} + \|v\|_{\infty,s} + \|u\|_{\infty,s}^2 + \|v\|_{\infty,s}^2\right)$.
Eligiendo, $T > 0$ si es necesario de tal manera que $1 < \frac{1}{\sqrt{2(t-\tau)}}$, (2.11) se transforma en

$$\|u(t) - v(t)\|_s \leq \|\varphi - \psi\|_s + C_s \int_0^t \frac{1}{\sqrt{t-\tau}} \|u(\tau) - v(\tau)\|_s \, d\tau \tag{2.12}$$

El lema 1 con $g(t) = \|u(t) - v(t)\|_s$, $a = \|\varphi - \psi\|_s$, $b = C_s$, $\beta = \frac{1}{2}$ y $\gamma = 1$ junto con (2.12) implican que

$$\|u(t) - v(t)\|_s \leq \|\varphi - \psi\|_s \; E\left(C_s^2 \pi t\right)$$

donde $E(x) = \sum_{m=0}^{\infty} c_m \, x^{\frac{m}{2}}$ con $c_0 = 1$ y $\frac{c_{m+1}}{c_m} = \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+1+\frac{1}{2}\right)}$. Por lo tanto,

$$\|u(t) - v(t)\|_s \leq \|\varphi - \psi\|_s \; E\left(\left(C_s\sqrt{\pi}\right)^2 t\right) \leq \|\varphi - \psi\|_s \sum_{m=0}^{\infty} c_m \left(\pi C_s^2 T\right)^{\frac{m}{2}} \tag{2.13}$$

Observe que (2.13) implica la unicidad. La dependencia continua es consecuencia de observar que $T > 0$ el tiempo de existencia de u garantizado por la proposición (2.2.2), es una función continua de $||\phi||_s$, de desigualdad(2.12) y de observar que $u \in \mathfrak{X}_s(T, M)$. □

Proposición 2.2.3. *Sea $u \in C([0,T], H^s)$, $s > \dfrac{1}{2}$ que satisface (2) entonces $u \in C((0,T], \mathcal{P})$.*

Demostración. Para $\lambda \in (0,1)$, la ecuación integral (2.3) y el lema de regularización (2) implican que

$$\begin{aligned}
&\|u(t)\|_{s+\lambda} \leq ||e^{tA}\varphi||_{s+\lambda} + \int_0^t \|e^{(t-\tau)A}\partial_x \left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)\|_{s+\lambda} d\tau \\
&\leq C_\lambda \left(1 + \frac{1}{t^{\frac{1+\lambda}{2}}}\right) \|\varphi\|_{s-1} + C_\lambda \int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\lambda}{2}}}\right) ||\partial_x(u^2 + u^3)(\tau)||_{s-1} d\tau \\
&\leq C_\lambda \left(1 + \frac{1}{t^{\frac{1+\lambda}{2}}}\right) \|\varphi\|_{s} + C_\lambda \int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\lambda}{2}}}\right) ||(u^2 + u^3)(\tau)||_{s} d\tau \\
&\leq C_\lambda \left(1 + \frac{1}{t^{\frac{1+\lambda}{2}}}\right) \|\varphi\|_{s} + C_\lambda \int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\lambda}{2}}}\right) (||u||_s^2 + ||u||_s^3) d\tau \\
&\leq C_\lambda \left(1 + \frac{1}{t^{\frac{1+\lambda}{2}}}\right) \|\varphi\|_{s} + C_\lambda (||u||_{\infty,s}^2 + ||u||_{\infty,s}^3) \int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\lambda}{2}}}\right) d\tau \\
&\leq C_\lambda \left(1 + \frac{1}{\epsilon^{\frac{1+\lambda}{2}}}\right) \|\varphi\|_{s} + K_\lambda (T + T^{\frac{1-\lambda}{2}}),
\end{aligned}$$

donde $0 < \epsilon < t \leq T$ y $K_\lambda = C_\lambda(||u||_{\infty,s}^2 + ||u||_{\infty,s}^3)$. Iteración de este argumento implica el resultado. □

Teorema 2.2.3. *El problema de valor inicial (2)es globalmente bien planteado en $H^1(\mathbb{T})$, si $\|\phi\|_0$ es suficientemente pequeña.*

Demostración. La prueba de este resultado la haremos en varias etapas, a saber:

- Estimativa para $\|u\|_0$.

  Haciendo el producto interno en $H^0 = L^2$ de la ecuación en (2) con u que es real, la proposición 2.2.3, integración por partes y el hecho de ser ∂_x^3 un operador antisimétrico se tiene que

$$\partial_t \int_0^{2\pi} u^2 dx = -2 \int_0^{2\pi} (u'(x))^2 dx,$$

es decir;

$$\|u(t)\|_0^2 \leq \|\phi\|_0^2. \tag{2.14}$$

- Estimativa para $\|u_x\|_0$

  Derivamos la ecuación en (2), la multiplicamos por u_x e integramos para obtener

$$\int_0^{2\pi} (u_x u_{xt} + u u_x u_{xx} + u_x^3 + 2u u_x^3 + u^2 u_x u_{xx}) dx = \int_0^{2\pi} (u_x)_{xx} u_x dx,$$

  que es posible hacerlo gracias al lema 2.2.3. Esta identidad se transforma en

$$\frac{1}{2}\partial_t \|u_x\|_0^2 + \int_0^{2\pi} (u u_x u_{xx} + u_x^3) dx + \int_0^{2\pi} 2u u_x^3 dx + \underbrace{\int_0^{2\pi} u^2 u_x u_{xx}\, dx}_{(1)} = -\|u_{xx}\|_0^2. \tag{2.15}$$

  Observe que (1) en (2.15) se transforma via integración por partes en

$$\begin{aligned} \int_0^{2\pi} u^2 u_x u_{xx} &= \frac{1}{2}\int_0^{2\pi} u^2 \partial_x (u_x^2) dx \\ &= -\frac{1}{2}\int_0^{2\pi} (u^2)_x u_x^2 dx \\ &= -\frac{1}{2}\int_0^{2\pi} 2u u_x^3 dx \\ &= -\int_0^{2\pi} u u_x^3 dx. \end{aligned}$$

  Por lo tanto (2.15) se transforma en

$$\frac{1}{2}\partial_t \|u_x\|_0^2 + \int_0^{2\pi} (u u_x u_{xx} + u_x^3) dx = -\|u_{xx}\|_0^2 - \int_0^{2\pi} u u_x^3 dx. \tag{2.16}$$

  Integración por partes en (2.16) implica que,

$$\frac{1}{2}\partial_t \|u_x\|_0^2 = -\|u_{xx}\|_0^2 + \underbrace{\int_0^{2\pi} u u_x u_{xx} dx}_{(2)} - \underbrace{\int_0^{2\pi} u u_x^3 dx}_{(3)}. \tag{2.17}$$

  Acotar (2) en (2.17), es consecuencia de las desigualdades de Cauchy-Schwartz,

de Gagliardo-Niremberg, de Young y de (2.16). En efecto,

$$\begin{aligned}
\int_0^{2\pi} uu_xu_{xx}dx &\leq \int_0^{2\pi} |u||u_x||u_{xx}|dx \\
&\leq \|u_x\|_\infty \left(\int_0^{2\pi} |u_{xx}|^2dx\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |u|^2dx\right)^{\frac{1}{2}} \\
&\leq \|u_x\|_\infty \|u_{xx}\|_0\|u\|_0 \\
&\leq \|u_x\|_0^{\frac{1}{2}}\|u_{xx}\|_0^{\frac{1}{2}}\|u_{xx}\|_0\|u\|_0 \\
&\leq \|u_x\|_0^{\frac{1}{2}}\|u_{xx}\|_0^{\frac{3}{2}}\|\varphi\|_0 \\
&\leq \frac{1}{4\epsilon^4}\|u_x\|_0^2\|\varphi\|_0^4 + \frac{3}{4}\epsilon^{\frac{4}{3}}\|u_{xx}\|_0^2, \qquad (2.18)
\end{aligned}$$

La acotación de (3) en (2.17) se obtiene a partir de las desigualdades de Cauchy-Schwartz, de Gagliardo-Niremberg, y de (2.16). En efecto,

$$\begin{aligned}
\int_0^{2\pi} uu_x^3dx &\leq \left(\int_0^{2\pi} |u|^2\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |u_x|^6\right)^{\frac{1}{2}} \\
&\leq \|\varphi\|_0 \left(\int_0^{2\pi} |u_x|^6\right)^{\frac{3}{6}} = \|\varphi\|_0\|u_x\|_{L^6}^3 \\
&\leq \|\varphi\|_0\|\varphi\|_0\|u_{xx}\|_0^2 = \|\varphi\|_0^2\|u_{xx}\|_0^2. \qquad (2.19)
\end{aligned}$$

Las desigualdades (2.18) y (2.19) transforman (2.17) en

$$\begin{aligned}
\partial_t\|u_x\|_0^2 &\leq -2\|u_{xx}\|_0^2 + 2\left(\frac{1}{4\epsilon^4}\|u_x\|_0^2\|\varphi\|_0^4 + \frac{3}{4}\epsilon^{\frac{4}{3}}\|u_{xx}\|_0^2\right) + 2\|\varphi\|_0^2\|u_{xx}\|_0^2 \\
&\leq (\frac{3}{2}\epsilon^{\frac{4}{3}}\|\varphi\|_0 + 2\|\varphi\|_0^2 - 2)\|u_{xx}\|_0^2 + \frac{\epsilon^{-4}}{4}\|\varphi\|_0^4\|u_x\|_0^2.
\end{aligned}$$

Si elegimos φ tal que $\|\varphi\|_0$ sea suficientemente pequeña y ϵ suficientemente pequeño tenemos que $\frac{3}{2}\epsilon^{\frac{4}{3}}\|\varphi\|_0 + 2\|\varphi\|_0^2 - 2 < 0$ y por lo tanto,

$$\partial_t\|u_x\|_0^2 \leq C_\epsilon\|\varphi\|_0^4\|u_x\|_0^2.$$

Integrando esta desigualdad entre 0 y t y aplicando a este resultado la desigualdad de Gronwall obtenemos

$$\|u_x\|_0^2 \leq \|\varphi'\|^2 e^{C_\epsilon\|\varphi\|_0^4 t}. \qquad (2.20)$$

Por lo tanto (2.16) y (2.20) implican el resultado.

□

Corolario 2.2.1. *El problema de valor inicial (2) es globalmente bien planteado en H^s, $s \geq 1$, siempre que $\|\varphi\|_0$ sea suficientemente pequeño.*

Demostración. Este resultado es consecuencia del Teorema 2.2.3, de la proposición 2.2.3 y de la ecuación integral (2.3). En efecto, si $\theta \in (0,1)$ entonces,

$$\begin{aligned}
\|u\|_{1+\theta} &\leq \|e^{tA}\varphi\|_{1+\theta} + \int_0^t \left\| e^{(t-\tau)A}\partial_x \left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)\right\|_{1+\theta} d\tau \\
&\leq \|\varphi\|_{1+\theta} + C\int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\theta}{2}}}\right) \left\|\partial_x \left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)\right\|_0 d\tau \\
&\leq \|\varphi\|_{1+\theta} + C\int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\theta}{2}}}\right) \left\|\left(\frac{u^2}{2} + \frac{u^3}{3}\right)(\tau)\right\|_{H^1} d\tau \\
&\leq \|\varphi\|_{1+\theta} + C\int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{1+\theta}{2}}}\right) \left(\|u\|_{H^1}^2 + \|u\|_{H^1}^3\right) d\tau \\
&\leq \|\varphi\|_{1+\theta} + K\|\varphi\|_0 \left(T + T^{\frac{1-\theta}{2}}\right).
\end{aligned}$$

Iterando este argumento se obtiene el resultado. □

Capítulo 3

G-B vía Espacios de Bourgain

3.1. Resultados Principales

Trabajaremos principalmente con la ecuación integral (2.3) asociada a la ecuación diferencial (2). Con la ayuda del semigrupo $\mathcal{W}$ definido en (9) podemos reescribir la ecuación integral como

$$u(t) = \mathcal{W}(t)\varphi - \int_0^t \mathcal{W}(t-t')\partial_x \left(\left(\frac{u^3}{3} + \frac{u^2}{2} \right)(t') \right) dt', \quad t \geq 0 \tag{3.1}$$

con el fin de aplicar la técnica de Bourgain-Kenig-Ponce-Vega, no trabajaremos directamente con (3.1) sino con la siguiente ecuación integral equivalente:

$$u(t) = \psi \left[\mathcal{W}(t)\varphi - \chi_{\mathbb{R}_+}(t) \int_0^t \mathcal{W}(t-t')\partial_x \left(\frac{\psi_T^3(t')u^3(t')}{3} + \frac{\psi_T^2(t')u^2(t')}{2} \right) dt' \right], \tag{3.2}$$

donde $t \in \mathbb{R}$ y, en lo que sigue, ψ es una función en el tiempo que satisface

$$\psi \in C_0^\infty(\mathbb{R}), \quad \sup \psi \subset [-1,1], \quad \psi \equiv 1 \text{ en } \left[-\frac{1}{2}, \frac{1}{2} \right]$$

y además $\psi_T(\cdot) = \psi\left(\frac{\cdot}{T}\right)$. En primer lugar notemos que si u es solución de (3.2) entonces u es solución de (3.1) sobre $[0,T]$ para $T < 1$.
El objeto de este capítulo es probar el siguiente resultado

Teorema 3.1.1. *Sean $\varphi \in H^s(\mathbb{T})$ y $s > -\frac{1}{4}$. Existen $T > 0$ y u solución de (3.1) en*

$$Z_T = C\left([0,T], H^s(\mathbb{T})\right) \cap X_T^{s,\frac{1}{2}}$$

Para abordar la prueba de este teorema necesitamos unos lemas y teoremas previos los cuales se establecerán a continuación

3.2. Estimativas Lineales

En esta parte estudiaremos tanto el operador $\psi\mathcal{W}$ como el operador $\mathfrak{L}$ definido por

$$\mathfrak{L} : f \longmapsto \chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t \mathcal{W}(t-t')f(t')\ dt'.$$

3.2.1. Estimativa lineal para el término libre

Proposición 3.2.1. *Sea $s \in \mathbb{R}$. Existe $C > 0$ tal que*

$$\|\psi(t)\mathcal{W}(t)\varphi\|_{X^{\frac{1}{2},s}} \leq C\|\varphi\|_s\ , \tag{3.3}$$

para toda $\varphi \in H^s(\mathbb{T})$.

Demostración. De la definición de $\|\cdot\|_{X^{\frac{1}{2},s}}$ tenemos

$$\|\psi(t)\mathcal{W}(t)\varphi\|^2_{X^{\frac{1}{2},s}} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\left\langle i\left(\tau-k^3\right)+k^2\right\rangle\left|\mathcal{F}\left(\psi(t)\mathcal{W}(t)\varphi\right)(k,\tau)\right|^2\ d\tau$$

pero,

$$\begin{aligned}\mathcal{F}(\psi(t)\mathcal{W}(t)\varphi)(k,\tau) &= \mathcal{F}_t\left(\psi(t)\mathcal{F}_x\left(\mathcal{W}(t)\varphi\right)(k)\right)(\tau)\\ &= \mathcal{F}_t\left(\psi(t)e^{-k^2|t|}e^{ik^3t}\widehat{\varphi}(k)\right)(\tau)\\ &= \widehat{\varphi}(k)\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}e^{ik^3t}\right)(\tau)\end{aligned}$$

por tanto

$$\begin{aligned}&\|\psi(t)\mathcal{W}(t)\varphi\|^2_{X^{\frac{1}{2},s}}\\ &\quad= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left|\widehat{\varphi}(k)\right|^2\int_{\mathbb{R}}\left\langle i\left(\tau-k^3\right)+k^2\right\rangle\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}e^{ik^3t}\right)(\tau)\right|^2\ d\tau\\ &\quad= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left|\widehat{\varphi}(k)\right|^2\int_{\mathbb{R}}\left\langle i\tau+k^2\right\rangle\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}e^{ik^3t}\right)\left(\tau-k^3\right)\right|^2\ d\tau\end{aligned}$$

Como

$$\begin{aligned}\mathcal{F}_t\left(\psi(t)e^{-k^2t}e^{ik^3t}\right)\left(\tau+k^3\right) &= \int_{\mathbb{R}} e^{-it(\tau+k^3)}\psi(t)e^{-k^2|t|}e^{ik^3t}\ dt\\ &= \mathcal{F}_t\left(\psi(t)e^{-k^2t}\right)(\tau)\end{aligned}$$

y $\langle i\tau + k^2\rangle \leq C\left(\langle\tau\rangle + \langle k\rangle^2\right)$, entonces

$$\begin{aligned}
\|\psi(t)\mathcal{W}(t)\varphi\|^2_{X^{\frac{1}{2},s}} &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\varphi}(k)|^2\int_{\mathbb{R}}\left(\langle\tau\rangle + \langle k\rangle^2\right)\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)(\tau)\right|^2\,d\tau \\
&\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\varphi}(k)|^2\int_{\mathbb{R}}\langle\tau\rangle\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)(\tau)\right|^2\,d\tau \\
&\quad + C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\varphi}(k)|^2\int_{\mathbb{R}}\langle k\rangle^2\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)(\tau)\right|^2\,d\tau \\
&= C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\varphi}(k)|^2\left\|\langle\tau\rangle^{\frac{1}{2}}\,\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|^2_{L^2_\tau(\mathbb{R})} \\
&\quad + C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s+2}|\widehat{\varphi}(k)|^2\left\|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|^2_{L^2_\tau(\mathbb{R})} \qquad (3.4)
\end{aligned}$$

y dado que $\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\varphi}(k)|^2 = \|\varphi\|^2_s$, tenemos que controlar la norma $\|\cdot\|_{L^2_\tau(\mathbb{R})}$.
Con esto en mente, consideremos la función $g_k(t) = \psi(t)e^{-k^2|t|}$. Vamos a estimar $\|g_k\|_{H^b(\mathbb{R})}$ para $b = 0$ y $b = \frac{1}{2}$.
Sea $0 \leq b \leq 1$. Supongamos en primer lugar que $|k| \geq 1$. Tenemos que

$$\|g_k\|_{H^b(\mathbb{R})} = \left\|\langle\tau\rangle^b\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})} = \left\|\langle\tau\rangle^b\left[\widehat{\psi} * \mathcal{F}_t\left(e^{-k^2|t|}\right)\right]\right\|_{L^2(\mathbb{R})}$$

y como $\langle\tau\rangle^b \leq C_b\left(\langle\tau - z\rangle^b + |z|^b\right)$ entonces

$$\begin{aligned}
\langle\tau\rangle^b\left[\left|\widehat{\psi}\right| * \left|\mathcal{F}_t\left(e^{-k^2|t|}\right)\right|\right](\tau) &= \langle\tau\rangle^b\int_{\mathbb{R}}\left|\widehat{\psi}(\tau - z)\right|\left|\mathcal{F}_t\left(e^{-k^2|t|}\right)(z)\right|\,dz \\
&\leq C_b\left[\int_{\mathbb{R}}\langle\tau - z\rangle^b\left|\widehat{\psi}(\tau - z)\right|\left|\mathcal{F}_t\left(e^{-k^2|t|}\right)(z)\right|\,dz\right. \\
&\qquad \left. + \int_{\mathbb{R}}|z|^b\left|\widehat{\psi}(\tau - z)\right|\left|\mathcal{F}_t\left(e^{-k^2|t|}\right)(z)\right|\,dz\right] \\
&\leq C_b\left[\langle\tau\rangle\left|\widehat{\psi}\right| * \left|\mathcal{F}_t\left(e^{-k^2|t|}\right)(\tau)\right| + \left|\widehat{\psi}\right| * |\tau|^b\left|\mathcal{F}_t\left(e^{-k^2|t|}\right)(\tau)\right|\right]
\end{aligned}$$

esta desigualdad y la de Young, implican que

$$\begin{aligned}
\|g_k\|_{H^b(\mathbb{R})} &\leq C_b\left[\left\|\langle\tau\rangle\widehat{\psi} * \mathcal{F}_t\left(e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})} + \left\|\widehat{\psi} * |\tau|^b\mathcal{F}_t\left(e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})}\right] \\
&\leq C_b\left[\left\|\langle\tau\rangle\widehat{\psi}\right\|_{L^1(\mathbb{R})}\left\|\mathcal{F}_t\left(e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})} + \left\|\widehat{\psi}\right\|_{L^1(\mathbb{R})}\left\||\tau|^b\mathcal{F}_t\left(e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})}\right],
\end{aligned}$$

pero

$$\left\|\mathcal{F}_t\left(e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})} = \left\|e^{-k^2|t|}\right\|_{L^2(\mathbb{R})} = \left\|e^{-k^2|t|}\right\|_{H^0(\mathbb{R})}$$

$$\left\||\tau|^b\mathcal{F}_t\left(e^{-k^2|t|}\right)\right\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}}|\tau|^{2b}\left|\mathcal{F}_t\left(e^{-k^2|t|}\right)(\tau)\right|^2\,d\tau\right)^{\frac{1}{2}} = \left\|e^{-k^2|t|}\right\|_{\dot{H}^b(\mathbb{R})}$$

y $\left\|\langle\tau\rangle\widehat{\psi}\right\|_{L^1(\mathbb{R})}, \left\|\widehat{\psi}\right\|_{L^1(\mathbb{R})} < +\infty$ ya que $\psi \in \mathcal{S}(\mathbb{R})$ (pues $\psi \in C_0^\infty(\mathbb{R})$). Además, tomando $\lambda = k^2 > 0$ en (3.37) tenemos que

$$\left\|e^{-k^2|t|}\right\|_{H^0(\mathbb{R})} \leq \left[\left(k^2\right)^{-\frac{1}{2}} + \left(k^2\right)^{-\frac{1}{2}}\right] \left\|e^{-|t|}\right\|_{H^0(\mathbb{R})} \leq 2|k|^{-1} \left\|e^{-|t|}\right\|_{L^2(\mathbb{R})}$$
$$\left\|e^{-k^2|t|}\right\|_{\dot{H}^b(\mathbb{R})} \sim \left(k^2\right)^{b-\frac{1}{2}} \left\|e^{-|t|}\right\|_{\dot{H}^b(\mathbb{R})} = |k|^{2b-1} \left\|e^{-|t|}\right\|_{\dot{H}^b(\mathbb{R})}$$

y dado que $|k|^{-1} < |k|^{2b-1}$ (pues $-1 \leq 2b-1 \leq 1$ y $|k| \geq 1$), entonces

$$\|g_k\|_{H^b(\mathbb{R})} \leq C_b \left[|k|^{-1} + |k|^{2b-1}\right] \leq C \left[|k|^{2b-1} + |k|^{2b-1}\right] \leq C|k|^{2b-1}$$

esto es

$$\|g_k\|_{H^b(\mathbb{R})} \leq C|k|^{2b-1}, \quad \text{para } |k| \geq 1. \tag{3.5}$$

Si $k = 0$ entonces $g_0(t) = \psi(t)$ y así

$$\|g_0\|_{H^b(\mathbb{R})} = \|\psi\|_{H^b(\mathbb{R})} < +\infty \tag{3.6}$$

ya que $\psi \in \mathcal{S}(\mathbb{R})$.
Afirmamos que $|k|^{2b-1} < C\langle k\rangle^{2b-1}$ para $|k| \geq 1$. Para demostrarlo consideremos dos casos para b:
Caso $0 \leq b < \frac{1}{2}$.
Como $2b-1 < 0$ y $\langle k\rangle \leq C|k|$ si $|k| \geq 1$ entonces $|k|^{2b-1} \leq C\langle k\rangle^{2b-1}$ con lo cual $C|k|^{2b-1} \leq C\langle k\rangle^{2b-1}$.
Caso $\frac{1}{2} \leq b \leq 1$.
Tenemos que $2b-1 \geq 0$ y como $|k|^{2b-1} \leq 1 + |k|^{2b-1}$, entonces

$$\begin{aligned} C|k|^{2b-1} &\leq C\left(1 + |k|^{2b-1}\right) \\ &\leq C\left(1 + |k|^2\right)^{\frac{2b-1}{2}} \leq C\langle k\rangle^{2b-1}. \end{aligned}$$

Entonces de (3.5) se sigue que

$$\|g_k\|_{H^b(\mathbb{R})} \leq C\langle k\rangle^{2b-1}, \quad \text{para todo } 0 \leq b \leq 1 \tag{3.7}$$

Tomando $b = 0$ y $b = \frac{1}{2}$, tenemos

$$\left\|\mathcal{F}_t\left(\psi(t)e^{-k^2t}\right)\right\|_{L^2(\mathbb{R})} = \|g_k\|_{H^0(\mathbb{R})} \leq C\langle k\rangle^{-1}$$
$$\left\|\langle\tau\rangle^{\frac{1}{2}}\ \mathcal{F}_t\left(\psi(t)e^{-k^2t}\right)\right\|_{L^2(\mathbb{R})} = \|g_k\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq C$$

luego de (3.4) se sigue que

$$\begin{aligned} \|\psi(t)\mathcal{W}(t)\varphi\|^2_{X^{s,\frac{1}{2}}} &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left|\widehat{\varphi}(k)\right|^2 C^2 + C\sum_{k\in\ \mathbb{Z}}\langle k\rangle^{2s+2}C^2\langle k\rangle^{-2} \\ &\leq C\|\varphi\|^2_{H^s(\mathbb{T})} + C\|\varphi\|^2_{H^s(\mathbb{T})} \leq C\|\varphi\|^2_{H^s(\ \mathbb{T})} \end{aligned}$$

y de aquí

$$\|\psi(t)\mathcal{W}(t)\varphi\|_{X^{s,\frac{1}{2}}} \leq C\|\varphi\|_{H^s(\mathbb{T})}, \quad \text{para toda } \varphi \in H^s(\mathbb{T}) \qquad \square$$

3.2.2. Estimativa para el caso no homogéneo

Proposición 3.2.2. *Sean $w \in \mathfrak{X}$ y h_k definida sobre $\mathbb{R}$ por:*

$$h_k(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-k^2|t|}}{i\tau + k^2} \widehat{w}(\tau) d\tau,$$

donde $\widehat{w}(\tau)$ denota $\widehat{w}(k,\tau)$. Entonces,

$$\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(h_k) \right\|^2_{L^2(\mathbb{R})} \leq C \left[\left(\int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|}{\langle i\tau + k^2 \rangle} \, d\tau \right)^2 + \left(\int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau \right) \right], \quad (3.8)$$

para todo $k \in \mathbb{Z}$.

Demostración. Reescribimos a h_k de la siguiente manera:

$$\begin{aligned}
h_k(t) &= \psi(t) \int_{\mathbb{R}} \frac{e^{it\tau} - e^{-k^2|t|}}{i\tau + k^2} \widehat{w}(\tau) \, d\tau \\
&= \psi(t) \int_{|\tau| \leq 1} \frac{e^{it\tau} - e^{-k^2|t|}}{i\tau + k^2} \widehat{w}(\tau) \, d\tau + \psi(t) \int_{|\tau| \geq 1} \frac{e^{it\tau} - e^{-k^2|t|}}{i\tau + k^2} \widehat{w}(\tau) \, d\tau \\
&= \underbrace{\psi(t) \int_{|\tau| \leq 1} \frac{e^{it\tau} - 1}{i\tau + k^2} \widehat{w}(\tau) \, d\tau}_{I} + \underbrace{\psi(t) \int_{|\tau| \leq 1} \frac{1 - e^{-k^2|t|}}{i\tau + k^2} \widehat{w}(\tau) \, d\tau}_{II} \\
&\quad + \underbrace{\psi(t) \int_{|\tau| \geq 1} \frac{e^{it\tau}}{i\tau + k^2} \widehat{w}(\tau) \, d\tau}_{III} - \underbrace{\psi(t) \int_{|\tau| \geq 1} \frac{e^{-k^2|t|}}{i\tau + k^2} \widehat{w}(\tau) \, d\tau}_{IV} \\
&= I + II + III - IV
\end{aligned}$$

A continuación estableceremos la contribución de cada uno de estos cuatro términos a (3.8).

1. Contribución de IV.

   Tenemos que,

$$\begin{aligned}
\left\| \langle i\tau' + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(IV) \right\|^2_{L^2_{\tau'}(\mathbb{R})} &= \left\| \langle i\tau' + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t \left(\psi(t) e^{-k^2|t|} \right) \int_{|\tau| \geq 1} \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right\|_{L^2_{\tau'}(\mathbb{R})} \\
&= \int_{\mathbb{R}} \langle i\tau' + k^2 \rangle \left| \mathcal{F}_t \left(\psi(t) e^{-k^2|t|} \right) (\tau') \right|^2 \left| \int_{|\tau| \geq 1} \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right|^2 d\tau' \\
&= \left| \int_{|\tau| \geq 1} \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right|^2 \int_{\mathbb{R}} \langle i\tau' + k^2 \rangle \left| \mathcal{F}_t \left(\psi(t) e^{-k^2|t|} \right) (\tau') \right|^2 d\tau'.
\end{aligned}$$

   Dado que $\langle i\tau + k^2 \rangle \leq C|i\tau + k^2|$ (pues $|\tau| \geq 1$, entonces,

$$\begin{aligned}
\left| \int_{|\tau| \geq 1} \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right| &\leq \int_{|\tau| \geq 1} \frac{|\widehat{w}(\tau)|}{|i\tau + k^2|} \, d\tau \leq C \int_{|\tau| \geq 1} \frac{|\widehat{w}(\tau)|}{\langle i\tau + k^2 \rangle} \, d\tau \\
&\leq C \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|}{\langle i\tau + k^2 \rangle} \, d\tau.
\end{aligned}$$

Luego,

$$\begin{aligned}
&\left\|\langle i\tau+k^2\rangle^{\frac{1}{2}}\ \mathcal{F}_t(IV)\right\|^2_{L^2_\tau(\mathbb{R})}\\
&\qquad\leq\left(C\int_{\mathbb{R}}\frac{|\widehat{w}(\tau)|}{\langle i\tau+k^2\rangle}\ d\tau\right)^2\int_{\mathbb{R}}\langle i\tau'+k^2\rangle\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)(\tau')\right|^2\ d\tau'
\end{aligned}$$

Ahora bien, haciendo un análisis similar al hecho en la demostración de la Proposición 3.2.1 tenemos que

$$\int_{\mathbb{R}}\langle i\tau+k^2\rangle\left|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)(\tau)\right|^2\leq C$$

y por lo tanto

$$\left\|\langle i\tau+k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(IV)\right\|^2_{L^2_\tau(\mathbb{R})}\leq C\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\tau)|}{\langle i\tau+k^2\rangle}\ d\tau\right)^2 \tag{3.9}$$

2. Contribución de III.

   Para $\psi\in\mathcal{S}(\mathbb{R})$ y $w\in\mathfrak{X}$ tenemos que,

$$\begin{aligned}
\mathcal{F}_t(III)(\tau')&=\int_{\mathbb{R}}e^{-it\tau'}\left(\psi(t)\int_{|\tau|\geq1}\frac{e^{it\tau}}{i\tau+k^2}\widehat{w}(\tau)\ d\tau\right)\ dt\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-it\tau'}\psi(t)\frac{e^{it\tau}}{i\tau+k^2}\widehat{w}(\tau)\ \chi_{|\tau|\geq1}\ d\tau\ dt\\
&=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}e^{-it(\tau'-\tau)}\psi(t)\ dt\right)\frac{\widehat{w}(\tau)}{i\tau+k^2}\ \chi_{|\tau|\geq1}\ d\tau\\
&=\int_{\mathbb{R}}\widehat{\psi}(\tau'-\tau)\frac{\widehat{w}(\tau)}{i\tau+k^2}\ \chi_{|\tau|\geq1}\ d\tau\\
&=\left(\widehat{\psi}*\frac{\widehat{w}}{i(\cdot)+k^2}\ \chi_{|\cdot|\geq1}\right)(\tau'),
\end{aligned}$$

   así que,

$$\mathcal{F}_t(III)(\tau)=\left(\widehat{\psi}*\frac{\widehat{w}}{i(\cdot)+k^2}\ \chi_{|\cdot|\geq1}\right)(\tau).$$

   Por lo tanto,

$$\begin{aligned}
&\left\|\langle i\tau+k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(III)\right\|^2_{L^2_\tau(\mathbb{R})}\\
&\qquad=\int_{\mathbb{R}}\langle i\tau+k^2\rangle\left|\left(\widehat{\psi}(\tau')*_{\tau'}\frac{\widehat{w}(\tau')}{i\tau'+k^2}\ \chi_{|\tau'|\geq1}\right)(\tau)\right|^2\ d\tau
\end{aligned}$$

Ahora bien, como $\langle i\tau + k^2\rangle \leq C\left[\langle\tau'\rangle + |i(\tau-\tau') + k^2|\right]$ se sigue que:

$$\begin{aligned}
&\langle i\tau + k^2\rangle^{\frac{1}{2}} \left|\left(\widehat{\psi}(\tau') *_{\tau'} \frac{\widehat{w}(\tau')}{i\tau' + k^2}\ \chi_{|\tau'|\geq 1}\right)(\tau)\right| \\
&\qquad = \left|\int_{\mathbb{R}} \langle i\tau + k^2\rangle^{\frac{1}{2}} \widehat{\psi}(\tau') \frac{\widehat{w}(\tau-\tau')}{i(\tau-\tau') + k^2}\ \chi_{|\tau-\tau'|\geq 1}\ d\tau'\right| \\
&\qquad \leq \int_{\mathbb{R}} \langle i\tau + k^2\rangle^{\frac{1}{2}} \left|\widehat{\psi}(\tau')\right| \frac{|\widehat{w}(\tau-\tau')|}{|i(\tau-\tau') + k^2|}\ \chi_{|\tau-\tau'|\geq 1}\ d\tau' \\
&\qquad \leq C\left[\int_{\mathbb{R}} \langle\tau'\rangle^{\frac{1}{2}} \left|\widehat{\psi}(\tau')\right| \frac{|\widehat{w}(\tau-\tau')|}{|i(\tau-\tau') + k^2|}\ \chi_{|\tau-\tau'|\geq 1}\ d\tau'\right. \\
&\qquad\qquad \left. + \int_{\mathbb{R}} \left|\widehat{\psi}(\tau')\right| \left|i(\tau-\tau') + k^2\right|^{\frac{1}{2}} \frac{|\widehat{w}(\tau-\tau')|}{|i(\tau-\tau') + k^2|}\ \chi_{|\tau-\tau'|\geq 1}\ d\tau'\right] \\
&\qquad \leq C\left[\left(\langle\cdot\rangle^{\frac{1}{2}} \left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|}\ \chi_{|\cdot|\geq 1}\right)(\tau)\right. \\
&\qquad\qquad \left. + \left(\left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|^{\frac{1}{2}}}\ \chi_{|\cdot|\geq 1}\right)(\tau)\right].
\end{aligned}$$

Luego,

$$\begin{aligned}
\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}} \mathcal{F}_t(III)\right\|^2_{L^2_\tau(\mathbb{R})} \leq C\Bigg[&\int_{\mathbb{R}} \left|\left(\langle\cdot\rangle^{\frac{1}{2}} \left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|}\ \chi_{|\cdot|\geq 1}\right)(\tau)\right|^2 d\tau \\
&+ \int_{\mathbb{R}} \left|\left(\left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|^{\frac{1}{2}}}\ \chi_{|\cdot|\geq 1}\right)(\tau)\right|^2 d\tau\Bigg]
\end{aligned}$$

es decir,

$$\begin{aligned}
\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}} \mathcal{F}_t(III)\right\|^2_{L^2_\tau(\mathbb{R})} \leq C\Bigg[&\left\|\langle\cdot\rangle^{\frac{1}{2}} \left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|}\ \chi_{|\cdot|\geq 1}\right\|^2_{L^2_\tau(\mathbb{R})} \\
&+ \left\|\left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|^{\frac{1}{2}}}\ \chi_{|\cdot|\geq 1}\right\|^2_{L^2_\tau(\mathbb{R})}\Bigg].
\end{aligned}$$

La desigualdad de Young implica que:

$$\begin{aligned}
\left\|\langle\cdot\rangle^{\frac{1}{2}} \left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|}\ \chi_{|\cdot|\geq 1}\right\|_{L^2_\tau} &\leq \left\|\langle\tau\rangle^{\frac{1}{2}} \widehat{\psi}\right\|_{L^1_\tau} \left\|\frac{\widehat{w}(\tau)}{|i\tau + k^2|}\ \chi_{|\tau|\geq 1}\right\|_{L^2_\tau} \\
\left\|\left|\widehat{\psi}\right| * \frac{|\widehat{w}|}{|i(\cdot) + k^2|^{\frac{1}{2}}}\ \chi_{|\cdot|\geq 1}\right\|_{L^2_\tau} &\leq \left\|\widehat{\psi}\right\|_{L^1_\tau} \left\|\frac{\widehat{w}(\tau)}{|i\tau + k^2|^{\frac{1}{2}}}\ \chi_{|\tau|\geq 1}\right\|_{L^2_\tau}.
\end{aligned}$$

Además, dado que $|\tau| \geq 1$ entonces

$|i\tau + k^2|^{\frac{1}{2}} \leq |i\tau + k^2|$ y $\langle i\tau + k^2\rangle \leq C\,|i\tau + k^2|$, luego

$$\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(III)\right\|^2_{L^2_\tau(\mathbb{R})} \leq C\left\|\frac{\widehat{w}(\tau)}{\langle i\tau + k^2\rangle^{\frac{1}{2}}}\right\|^2_{L^2_\tau(\mathbb{R})} \tag{3.10}$$

3. Contribución de II.

En primer lugar tenemos que

$$\begin{aligned}\mathcal{F}_t(II) &= \mathcal{F}_t\left(\psi(t)\left(1 - e^{-k^2|t|}\right)\int_{|\tau|\leq 1}\frac{\widehat{w}(\tau)}{i\tau + k^2}\,d\tau\right)\\ &= \left(\int_{|\tau|\leq 1}\frac{\widehat{w}(\tau)}{i\tau + k^2}\,d\tau\right)\mathcal{F}_t\left[\psi(t)\left(1 - e^{-k^2|t|}\right)\right]\end{aligned}$$

entonces

$$\begin{aligned}&\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\;\mathcal{F}_t(II)\right\|^2_{L^2_\tau(\mathbb{R})}\\ &\qquad = \left|\int_{|\tau|\leq 1}\frac{\widehat{w}(\tau)}{i\tau + k^2}\,d\tau\right|^2\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left[\psi(t)\left(1 - e^{-k^2|t|}\right)\right]\right\|^2_{L^2_\tau}\end{aligned}$$

pero usando la desigualdad de Cauchy-Schwarz tenemos

$$\begin{aligned}\left|\int_{|\tau|\leq 1}\frac{\widehat{w}(\tau)}{i\tau + k^2}\,d\tau\right|^2 &\leq \left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|}{\langle i\tau + k^2\rangle^{\frac{1}{2}}}\frac{\langle i\tau + k^2\rangle^{\frac{1}{2}}}{|i\tau + k^2|}\,d\tau\right)^2\\ &\leq \left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2\rangle}\,d\tau\right)\left(\int_{|\tau|\leq 1}\frac{\langle i\tau + k^2\rangle}{|i\tau + k^2|^2}\,d\tau\right)\\ &\leq \left(\int_{\mathbb{R}}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2\rangle}\,d\tau\right)\left(\int_{|\tau|\leq 1}\frac{\langle i\tau + k^2\rangle}{|i\tau + k^2|^2}\,d\tau\right)\end{aligned}$$

luego

$$\begin{aligned}\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(II)\right\|^2_{L^2_\tau(\mathbb{R})} &\leq \left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left[\psi(t)\left(1 - e^{-k^2|t|}\right)\right]\right\|^2_{L^2_\tau}\\ &\quad\times\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2\rangle}\,d\tau\right)\left(\int_{|\tau|\leq 1}\frac{\langle i\tau + k^2\rangle}{|i\tau + k^2|^2}\,d\tau\right)\end{aligned} \tag{3.11}$$

Ahora bien, dado que estamos trabajando con $k \in \mathbb{Z}$, consideremos dos casos: $k = 0$ y $|k| \geq 1$.

Si $k = 0$

$$\psi(t)\int_{|\tau|\leq 1}\frac{1 - e^{-0^2|t|}}{i\tau + 0^2}\widehat{w}(\tau)\,d\tau = 0$$

con lo cual $\mathcal{F}_t(II) = 0$ y de aquí $\left\|\langle i\tau + 0^2\rangle^{\frac{1}{2}}\mathcal{F}_t(II)\right\|^2_{L^2_\tau} = 0$.

Supongamos ahora que $|k| \geq 1$. De una parte tenemos que

$$\begin{aligned}\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left[\psi(t)\left(1 - e^{-k^2|t|}\right)\right]\right\|^2_{L^2_\tau} \\ &= \left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\psi(t)\right) - \langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|^2_{L^2_\tau} \\ &\leq C\left[\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\widehat{\psi}\right\|^2_{L^2_\tau} + \left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|^2_{L^2_\tau}\right]\end{aligned}$$

y como $\langle i\tau + k^2\rangle \leq C\left[|\tau| + \langle k^2\rangle\right]$ y $\langle i\tau + k^2\rangle \leq C\left[\langle\tau\rangle + \langle k^2\rangle\right]$ entonces

$$\begin{aligned}\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\widehat{\psi}\right\|_{L^2_\tau} &\leq C\left\|\left(|\tau|^{\frac{1}{2}} + \langle k^2\rangle^{\frac{1}{2}}\right)\widehat{\psi}\right\|_{L^2_\tau} \leq C\left[\left\||\tau|^{\frac{1}{2}}\widehat{\psi}\right\|_{L^2_\tau} + \langle k^2\rangle^{\frac{1}{2}}\left\|\widehat{\psi}\right\|_{L^2_\tau}\right] \\ &\leq C\langle k^2\rangle^{\frac{1}{2}}\end{aligned}$$

y

$$\begin{aligned}\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\widehat{\psi}(t)e^{-k^2|t|}\right)\right\|_{L^2_\tau} &\leq C\left\|\left(\langle\tau\rangle^{\frac{1}{2}} + \langle k^2\rangle^{\frac{1}{2}}\right)\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|_{L^2_\tau} \\ &\leq C\left[\left\|\langle\tau\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|_{L^2_\tau} + \langle k^2\rangle^{\frac{1}{2}}\left\|\mathcal{F}_t\left(\psi(t)e^{-k^2|t|}\right)\right\|_{L^2_\tau}\right].\end{aligned}$$

Usando la estimativa (3.7) obtenida en la demostración de la Proposición (3.2.1) (con $b = \frac{1}{2}$ y $b = 0$ respectivamente) tenemos

$$\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left(\widehat{\psi}(t)e^{-k^2|t|}\right)\right\|_{L^2_\tau} \leq C\langle k^2\rangle^{\frac{1}{2}}$$

con lo cual

$$\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t\left[\psi(t)\left(1 - e^{-k^2|t|}\right)\right]\right\|^2_{L^2_\tau} \leq C\langle k^2\rangle \leq C\langle k\rangle^2$$

Adicionalmente, para $|k| \geq 1$ y $|\tau| \leq 1$ tenemos que $\dfrac{\langle i\tau + k^2\rangle}{|i\tau + k^2|^2} \leq C\dfrac{1}{|k|^2}$ entonces

$$\int_{|\tau|\leq 1}\frac{\langle i\tau + k^2\rangle}{|i\tau + k^2|^2}\,d\tau \leq C\frac{1}{|k|^2}\int_{|\tau|\leq 1}d\tau \leq C\frac{1}{|k|^2} \tag{3.12}$$

Luego, (3.11) se transforma en

$$\left\|\langle i\tau + k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(II)\right\|^2_{L^2_\tau(\mathbb{R})} \leq C\frac{\langle k\rangle^2}{|k|^2}\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2\rangle}\,d\tau\right) \leq C\int_{\mathbb{R}}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2\rangle}\,d\tau,$$

donde la última desigualdad es consecuencia de ser $|k| \geq 1$ lo que implica que $\langle k\rangle \leq C|k|$.

Por lo tanto, para todo $k \in \mathbb{Z}$ tenemos

$$\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(II) \right\|^2_{L^2_\tau(\mathbb{R})} \leq C \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau. \tag{3.13}$$

4. Contribución de I.

En primer lugar, como

$$\begin{aligned} I &= \psi(t) \int_{|\tau| \leq 1} \left(e^{it\tau} - 1 \right) \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau = \psi(t) \int_{|\tau| \leq 1} \sum_{n \geq 1} \frac{(it\tau)^n}{n!} \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau \\ &= \sum_{n \geq 1} \psi(t) \int_{|\tau| \leq 1} \frac{t^n (i\tau)^n}{n!} \frac{\widehat{w}(\tau)}{i\tau + k^2} \, d\tau \leq \sum_{n \geq 1} \frac{\psi(t) t^n}{n!} \int_{|\tau| \leq 1} \frac{(i\tau)^n \widehat{w}(\tau)}{i\tau + k^2} \, d\tau \end{aligned}$$

entonces,

$$\begin{aligned} \mathcal{F}_t(I) &= \sum_{n \geq 1} \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \int_{|\tau| \leq 1} \frac{(i\tau)^n \widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right) \\ &= \sum_{n \geq 1} \left(\int_{|\tau| \leq 1} \frac{(i\tau)^n \widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right) \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \end{aligned}$$

y de esta manera

$$\begin{aligned} &\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(I) \right\|_{L^2_\tau(\mathbb{R})} \\ &\qquad = \left\| \sum_{n \geq 1} \left(\int_{|\tau| \leq 1} \frac{(i\tau)^n \widehat{w}(\tau)}{i\tau + k^2} \, d\tau \right) \langle i\tau + k^2 \rangle^{\frac{1}{2}} \, \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \right\|_{L^2_\tau} \\ &\qquad \leq \sum_{n \geq 1} \left(\int_{|\tau| \leq 1} \frac{|i\tau|^n \, |\widehat{w}(\tau)|}{|i\tau + k^2|} \, d\tau \right) \left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \right\|_{L^2_\tau} \\ &\qquad \leq \left(\int_{|\tau| \leq 1} \frac{|\widehat{w}(\tau)|}{|i\tau + k^2|} \, d\tau \right) \sum_{n \geq 1} \left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \right\|_{L^2_\tau} \end{aligned}$$

Dado que $\langle i\tau + k^2 \rangle \leq C \left[\langle \tau \rangle + |k^2| \right]$ entonces,

$$\begin{aligned} &\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \right\|_{L^2_\tau} \\ &\qquad \leq C \left[\left\| \langle \tau \rangle^{\frac{1}{2}} \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \right\|_{L^2_\tau} + |k| \left\| \mathcal{F}_t \left(\frac{\psi(t) t^n}{n!} \right) \right\|_{L^2_\tau} \right] \\ &\qquad \leq C \left[\left\| \frac{\psi(t) t^n}{n!} \right\|_{H^{\frac{1}{2}}_t} + |k| \left\| \frac{\psi(t) t^n}{n!} \right\|_{L^2_t} \right] \end{aligned}$$

con lo cual,

$$\left\|\langle i\tau+k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(I)\right\|_{L^2_\tau(\mathbb{R})} \leq C\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|}{|i\tau+k^2|}\,d\tau\right)\sum_{n\geq 1}\left[\left\|\frac{\psi(t)t^n}{n!}\right\|_{H_t^{\frac{1}{2}}}+|k|\left\|\frac{\psi(t)t^n}{n!}\right\|_{L^2_t}\right]. \tag{3.14}$$

Ahora bien, tenemos que

$$\left\|\frac{\psi(t)t^n}{n!}\right\|_{L^2_t}\leq\left\|\frac{\psi(t)t^n}{n!}\right\|_{H_t^{\frac{1}{2}}}\leq\left\|\frac{\psi(t)t^n}{n!}\right\|_{H^1_t}\leq\frac{1}{n!}\left\|\psi(t)t^n\right\|_{H^1_t}$$

y además, $\|\psi(t)t^n\|_{H^1_t}\leq Cn$. En efecto,

$$\|t^n\psi(t)\|_{H^1}\leq C\left[\|t^n\psi(t)\|_{H^0}+\|\partial_t\left(t^n\psi(t)\right)\|_{H^0}\right]$$

pero,

$$\begin{aligned}\|t^n\psi(t)\|^2_{H^0}&=\int_{\mathbb{R}}|t^n|^2|\psi(t)|^2\,dt=\int_{-1}^{1}t^{2n}|\psi(t)|^2\,dt\leq\int_{\mathbb{R}}|\psi(t)|^2=\|\psi\|^2_{H^0}\\ \|\partial_t\left(t^n\psi(t)\right)\|_{H^0}&=\left\|nt^{n-1}\psi(t)+t^n\psi'(t)\right\|_{H^0}\leq n\left\|t^{n-1}\psi(t)\right\|_{H^0}+\|t^n\psi'(t)\|_{H^0}\\ &\leq n\|\psi\|_{H^0}+\|\psi'\|_{H^0}\end{aligned}$$

luego,

$$\begin{aligned}\|t^n\psi(t)\|_{H^1}&\leq C\left[(n+1)\|\psi\|_{H^0}+\|\psi'\|_{H^0}\right]\leq C(n+1)\left[\|\psi\|_{H^0}+\|\psi'\|_{H^0}\right]\\ &\leq 2Cn\left[\|\psi\|_{H^0}+\|\psi'\|_{H^0}\right]\leq Cn,\end{aligned}$$

pues $\psi\in\mathcal{S}(\mathbb{R})$. Retomando la estimación en (3.14) y usando la desigualdad de Cauchy-Schwarz junto con (3.12), para $|k|\geq 1$ tenemos que:

$$\begin{aligned}\left\|\langle i\tau+k^2\rangle^{\frac{1}{2}}\mathcal{F}_t(I)\right\|_{L^2_\tau(\mathbb{R})}&\leq C\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|}{|i\tau+k^2|}\,d\tau\right)\sum_{n\geq 1}\left[\frac{1}{n!}Cn+|k|\frac{1}{n!}Cn\right]\\ &\leq C\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|}{|i\tau+k^2|}\,d\tau\right)(1+|k|)\sum_{n\geq 1}\frac{n}{n!}\\ &\leq C(1+|k|)\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|}{|i\tau+k^2|}\,d\tau\right)\\ &\leq C\langle k\rangle\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|}{\langle i\tau+k^2\rangle^{\frac{1}{2}}}\frac{\langle i\tau+k^2\rangle^{\frac{1}{2}}}{|i\tau+k^2|}\,d\tau\right)\\ &\leq C\langle k\rangle\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau+k^2\rangle}\,d\tau\right)^{\frac{1}{2}}\left(\int_{|\tau|\leq 1}\frac{\langle i\tau+k^2\rangle}{|i\tau+k^2|^2}\,d\tau\right)^{\frac{1}{2}}\\ &\leq C\langle k\rangle\left(\int_{|\tau|\leq 1}\frac{|\widehat{w}(\tau)|^2}{\langle i\tau+k^2\rangle}\,d\tau\right)^{\frac{1}{2}}C\frac{1}{|k|}\end{aligned}$$

$$\leq \left(\int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau \right)^{\frac{1}{2}}.$$

Así que,

$$\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(I) \right\|^2_{L^2_\tau(\mathbb{R})} \leq C \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau. \tag{3.15}$$

Por lo tanto, las estimativas (3.15), (3.13), (3.10) y (3.9) implican que:

$$\begin{aligned}
\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(h_k) \right\|^2_{L^2(\ \mathbb{R})} &= \left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(I + II + III + IV) \right\|^2_{L^2(\mathbb{R})} \\
&\leq C \left[\left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(I) \right\|^2_{L^2_\tau} + \left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(II) \right\|^2_{L^2_\tau} \right. \\
&\qquad \left. + \left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(III) \right\|^2_{L^2_\tau} + \left\| \langle i\tau + k^2 \rangle^{\frac{1}{2}} \mathcal{F}_t(IV) \right\|^2_{L^2_\tau} \right] \\
&\leq C \left[C \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau + C \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau \right. \\
&\qquad \left. + C \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau + C \left(\int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|}{\langle i\tau + k^2 \rangle} \, d\tau \right)^2 \right] \\
&\leq C \left[\left(\int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|}{\langle i\tau + k^2 \rangle} \, d\tau \right)^2 + \int_{\mathbb{R}} \frac{|\widehat{w}(\tau)|^2}{\langle i\tau + k^2 \rangle} \, d\tau \right]
\end{aligned}$$

□

Proposición 3.2.3. *Sea $s \in \mathbb{R}$,*

(a) *Existe $C > 0$ tal que para toda $v \in (\mathbb{R} \times \mathbb{T})$*

$$\begin{aligned}
&\left\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t \mathcal{W}(t - t')v(t') \, dt' \right\|_{X^{\frac{1}{2},s}} \\
&\qquad \leq C \left[\|v\|_{X^{-\frac{1}{2},s}} + \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \left(\int_{\mathbb{R}} \frac{|\widehat{v}(\tau)|}{\langle i\tau + k^2 \rangle} \, d\tau \right)^2 \right)^{\frac{1}{2}} \right]
\end{aligned} \tag{3.16}$$

(b) *Para cada $0 < \delta < \frac{1}{2}$ existe $C_\delta > 0$ tal que, para toda $v \in X^{-\frac{1}{2}+\delta,s}$*

$$\left\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t \mathcal{W}(t - t')v(t')dt' \right\|_{X^{\frac{1}{2},s}} \leq C_\delta \|v\|_{X^{-\frac{1}{2}+\delta,s}} \tag{3.17}$$

Demostración. Ver [2] páginas 45-49. □

Proposición 3.2.4. *Sean $s \in \mathbb{R}$ y $\delta > 0$. Entonces para $f \in X^{-\frac{1}{2}+\delta,s}$, la aplicación*

$$t \in \mathbb{R} \longmapsto \int_0^t \mathcal{W}(t - t')f(t') \, dt' \in H^{s+2\delta}(\mathbb{T}) \tag{3.18}$$

es continua. Más aún, si (f_n) *es una sucesión tal que* $f_n \xrightarrow[n\to\infty]{} 0$ *en* $X^{-\frac{1}{2}+\delta,s}$ *entonces*

$$\left\| \int_0^t \mathcal{W}(t-t') f_n(t')\ dt' \right\|_{L^\infty(\mathbb{R}_+, H^{s+2\delta})} \xrightarrow[n\to\infty]{} 0 \tag{3.19}$$

Demostración. Ver [2] páginas 50-53. □

3.3. Estimativa Bilineal

Proposición 3.3.1. *Dado* $s \in (-1,0]$*, existen* $C, \mu, \delta > 0$ *tales que para todo par de funciones* $u, v \in X^{\frac{1}{2},s}$ *con soporte compacto en* $[-T,T]$

$$\|\partial_x(uv)\|_{X^{-\frac{1}{2}+\delta,s}} \leq C\ T^\mu\ \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s}} \tag{3.20}$$

Vamos a usar la siguiente versión, que es consecuencia de la Proposición (3.3.1) junto con la desigualdad

$$\forall s \geq s_c, \qquad \langle k\rangle^s \leq \langle k\rangle^{s_c^+} \langle k_1\rangle^{s-s_c^+} + \langle k\rangle^{s_c^+} \langle k-k_1\rangle^{s-s_c^+} \tag{3.21}$$

Proposición 3.3.2. *Dado* $s_c^+ \in (-1,0]$*, existen* $C, \mu, \delta > 0$ *tales que*

$$\|\partial_x(uv)\|_{X^{-\frac{1}{2}+\delta,s}} \leq C\ T^\mu\ \left(\|u\|_{X^{\frac{1}{2},s_c^+}} \|v\|_{X^{\frac{1}{2},s}} + \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s_c^+}} \right), \tag{3.22}$$

para todo $s \geq s_c^+$ *y todo par de funciones* $u, v \in X^{\frac{1}{2},s}$ *con soporte compacto en* $[-T,T]$.

Tendremos presente la siguiente relación algebraica

$$\text{máx}\,(|\sigma|, |\sigma_1|, |\sigma_2|) \geq |k|\ |k_1|\ |k-k_1| \tag{3.23}$$

donde

$$\sigma = \tau - k^3, \qquad \sigma_1 = \tau_1 - k_1^3, \qquad \sigma_2 = (\tau - \tau_1) - (k-k_1)^3 \tag{3.24}$$

Demostración. (Proposición 3.3.1) Esta prueba la realizaremos por dualidad. como el dual del espacio $X^{b,s}$ es $X^{-b,-s}$, entonces

$$\|\partial_x(uv)\|_{X^{-\frac{1}{2}+\delta,s}} = \sup_{\|w\|_{X^{\frac{1}{2}-\delta,-s}}=1} |(\partial_x(uv), w)|$$

$$\begin{aligned}
\|u\|_{b,s}^2 &= \|\langle|\tau - k^3| + k^2\rangle^b \langle k\rangle^s \widehat{u}\|_{L^2(\mathbb{R}\times\mathbb{Z})}^2 \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle|\tau - k^3| + k^2\rangle^{2b} \langle k\rangle^{2s} |\widehat{u}(k,\tau)|^2 d\tau \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} (1 + (|\tau - k^3| + k^2)^2)^b (1+k^2)^s |\widehat{u}(k,\tau)|^2 d\tau
\end{aligned}$$

entonces tenemos que

$$\|u\|_{\frac{1}{2},s} = \|\langle i(\tau - k^3) + k^2\rangle^{\frac{1}{2}}\langle k\rangle^s \widehat{u}\|_{L^2(\mathbb{R}\times\mathbb{Z})}$$

$$\begin{aligned}|(\partial(uv), w)| &= \left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \widehat{\partial(uv)}(k,\tau)\widehat{w}(k,\tau)d\tau\right| \\ &\leq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |\widehat{\partial(uv)}(k,\tau)||\widehat{w}(k,\tau)d\tau|\end{aligned}$$

Como $|\widehat{\partial(uv)}(k,\tau)| = |(ik)\widehat{uv}(k,\tau)| = |(ik)\widehat{u} * \widehat{v}(k,\tau)|$ entonces

$$|(\partial(uv), w)| \leq |k|\sum_{k_1}\int_{\mathbb{R}} |\widehat{u}(k_1,\tau_1)||\widehat{v}(k-k_1,\tau-\tau_1)|d\tau_1$$

Haciendo

$$\begin{aligned}\widehat{f}(k,\tau) &= \langle i(\tau - k^3) + k^2\rangle^{\frac{1}{2}}\langle k\rangle^s \widehat{u}(k,\tau) \\ \widehat{g}(\tau,k) &= \langle i(\tau - k^3) + k^2\rangle^{\frac{1}{2}}\langle k\rangle^s \widehat{v}(k,\tau) \\ \widehat{h}(\tau,k) &= \langle i(\tau - k^3) + k^2\rangle^{\frac{1}{2}-\delta}\langle k\rangle^s \widehat{w}(k,\tau)\end{aligned}$$

y teniendo en cuenta las definiciones de σ, σ_1 y σ_2 de (3.24) se tiene :

$$\begin{aligned}|(\partial(uv), w)| &\leq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_{k_1}\int_{\mathbb{R}} |\widehat{u}(k_1,\tau_1)||\widehat{v}(k-k_1,\tau-\tau_1)|d\tau_1\right)|\widehat{w}(k,\tau)d\tau| \\ &= \sum_{k,k_1}\int_{\mathbb{R}\times\mathbb{R}} |k||\widehat{u}(k_1,\tau_1)||\widehat{v}(k-k_1,\tau-\tau_1)||\widehat{w}(k,\tau)d\tau d\tau_1 \\ &= \sum_{k,k_1}\int_{\mathbb{R}\times\mathbb{R}} |k|\frac{|\widehat{f}(k_1,\tau_1)|}{\langle i(\tau_1 - k_1^3) + k_1^2\rangle^{\frac{1}{2}}\langle k_1\rangle^s} \\ &\quad \frac{|\widehat{g}(k-k_1,\tau-\tau_1)|}{\langle i(\tau-\tau_1-(k-k_1)^3 + (k-k_1)^2\rangle^{\frac{1}{2}}\langle k-k_1\rangle^s}\frac{|\widehat{h}(k,\tau)|}{\langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}-\delta}\langle k\rangle^{-s}}d\tau d\tau_1 \qquad (3.25)\end{aligned}$$

$$\begin{aligned}&= \sum_{k,k_1}\int_{\mathbb{R}\times\mathbb{R}} |k|\frac{|\widehat{f}(k_1,\tau_1)|}{\langle i\sigma_1 + k_1^2\rangle^{\frac{1}{2}}\langle k_1\rangle^s}\frac{|\widehat{g}(k-k_1,\tau-\tau_1)|}{\langle i\sigma_2 + (k-k_1)^2\rangle^{\frac{1}{2}}\langle k-k_1\rangle^s}\frac{|\widehat{h}(k,\tau)|}{\langle i\sigma + k^2\rangle^{\frac{1}{2}-\delta}\langle k\rangle^{-s}}d\tau d\tau_1 \\ &= \sum_{k,k_1}\int_{\mathbb{R}\times\mathbb{R}} \frac{|k|\langle k\rangle^s|\widehat{h}(k,\tau)|\langle k\rangle^{-s}|\widehat{f}(k_1,\tau_1)|\langle k-k_1\rangle^{-s}|\widehat{g}(k-k_1,\tau-\tau_1)|}{\langle i\sigma + k^2\rangle^{\frac{1}{2}-\delta}\langle i\sigma_1 + k_1^2\rangle^{\frac{1}{2}}\langle i\sigma_2 + (k-k_1)^2\rangle^{\frac{1}{2}}} = I \qquad (3.26)\end{aligned}$$

Veamos que

$$I \leq \|f\|_{L^2(\mathbb{T}\times\mathbb{R})}\|g\|_{L^2(\mathbb{T}\times\mathbb{R})}\|h\|_{L^2(\mathbb{T}\times\mathbb{R})}$$

Dividamos $\mathbb{Z}^2 \times \mathbb{R}^2$ en las siguientes regiones

$$\begin{aligned}
\mathcal{A} &= \left\{(\tau,\tau_1,k,k_1) \in \mathbb{R}^2\times\mathbb{Z}^2;\ |k_1| \geq 1,\ |k-k_1| \geq 1,\ |k| \leq 2|k_1|\right\}\\
\mathcal{B} &= \left\{(\tau,\tau_1,k,k_1) \in \mathbb{R}^2\times\mathbb{Z}^2;\ |k_1| \geq 1,\ |k-k_1| \geq 1,\ |k| \geq 2|k_1|\right\}\\
\mathcal{D} &= \left\{(\tau,\tau_1,k,k_1) \in \mathbb{R}^2\times\mathbb{Z}^2;\ k_1 = 0 \ \text{ o } \ |k-k_1| \leq 1\right\}
\end{aligned}$$

y estimemos I en cada una de las anteriores regiones. Notemos que por simetría podemos suponer que $|\sigma_1| \geq |\sigma_2|$ en $A \cup B$

1. Estimación de I en A

   Como $(\tau,\tau_1,k,k_1) \in \mathcal{A}$, el teorema de Fubini implica que:

$$\begin{aligned}
I_{\mathcal{A}} &= \sum_{k_1}\int \frac{\langle k_1\rangle^{-s}\left|\widehat{f}(\tau_1,k_1)\right|}{\langle i\sigma_1+k_1^2\rangle^{\frac{1}{2}}}\\
&\quad\times\left[\sum_{k}\int \frac{|k|\langle k-k_1\rangle^{-s}\left|\widehat{g}(\tau-\tau_1,k-k_1)\right|\langle k\rangle^{s}\left|\widehat{h}(\tau,k)\right|}{\langle i\sigma_2+(k-k_1)^2\rangle^{\frac{1}{2}}\langle i\sigma+k^2\rangle^{\frac{1}{2}-\delta}}\ d\tau\right]\ d\tau_1.\\
&\leq \|f\|_{L^2_tL^2_x}\left\|\left(\sum_{k}\int \frac{\langle k_1\rangle^{-2s}|k|^2\langle k-k_1\rangle^{-2s}\langle k\rangle^{2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+(k-k_1)^2\rangle\langle i\sigma+k^2\rangle^{1-4\delta}}\ d\tau\right)^{\frac{1}{2}}\right.\\
&\qquad\left.\times\left(\sum_{k}\int \frac{\left|\widehat{g}(\tau-\tau_1,k-k_1)\right|^2\left|\widehat{h}(\tau,k)\right|^2}{\langle i\sigma+k^2\rangle^{2\delta}}\ d\tau\right)^{\frac{1}{2}}\right\|_{L^2_{\tau_1}\ell^2_{k_1}}\\
&\leq \|f\|_{L^2_tL^2_x}\left\|\sum_{k}\int \frac{\langle k_1\rangle^{-2s}|k|^2\langle k-k_1\rangle^{-2s}\langle k\rangle^{2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+(k-k_1)^2\rangle\langle i\sigma+k^2\rangle^{1-4\delta}}\ d\tau\right\|^{\frac{1}{2}}_{L^\infty_{\tau_1}\ell^\infty_{k_1}}\\
&\qquad\times\left\|\sum_{k}\int \frac{\left|\widehat{g}(\tau-\tau_1,k-k_1)\right|^2\left|\widehat{h}(\tau,k)\right|^2}{\langle i\sigma+k^2\rangle^{2\delta}}\ d\tau\right\|^{\frac{1}{2}}_{L^1_{\tau_1}\ell^1_{k_1}}
\end{aligned}$$

$$\begin{aligned}
I_{\mathcal{A}} &\leq \|f\|_{L^2_tL^2_x}\left(\sup_{\tau_1,k_1}\frac{\langle k_1\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle}\sum_{k}\int \frac{|k|^2\langle k-k_1\rangle^{-2s}\langle k\rangle^{2s}}{\langle i\sigma_2+(k-k_1)^2\rangle\langle i\sigma+k^2\rangle^{1-4\delta}}\ d\tau\right)^{\frac{1}{2}}\\
&\qquad\times\left[\sum_{k}\int \frac{\left|\widehat{h}(\tau,k)\right|^2}{\langle i\sigma+k^2\rangle^{2\delta}}\left(\sum_{k_1}\int \left|\widehat{g}(\tau-\tau_1,k-k_1)\right|^2\ d_{\tau_1}\right)\ d\tau\right]^{\frac{1}{2}}.
\end{aligned}$$

Las desigualdades:

$$\sum_{k_1}\int |\widehat{g}(\tau-\tau_1,k-k_1)|^2\ d\tau_1 \leq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |\widehat{g}(\tau,k)|^2\ d\tau = \|\widehat{g}\|^2_{L^2_\tau \ell^2_k} = \|g\|^2_{L^2_t L^2_x}$$

$$\sum_{k}\int \frac{\left|\widehat{h}(\tau,k)\right|^2}{\langle i\sigma+k^2\rangle^{2\delta}}\ d\tau \leq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \frac{\left|\widehat{h}(\tau,k)\right|^2}{\langle\sigma\rangle^{2\delta}}\ d\tau = \left\|\frac{\widehat{h}}{\langle\sigma\rangle^{\delta}}\right\|^2_{L^2_\tau \ell^2_k},$$

y el Lema 4, con $\theta=\delta$ implican que:

$$I_{\mathcal{A}} \leq C\ T^{\mu}\,\|f\|_{L^2_tL^2_x}\,\|g\|_{L^2_tL^2_x}\,\|h\|_{L^2_tL^2_x}$$
$$\times\left(\sup_{\tau_1,k_1}\frac{\langle k_1\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle}\sum\int_{\mathcal{A}(\tau_1,k_1)}\frac{|k|^2\langle k\rangle^{2s}\langle k-k_1\rangle^{-2s}}{\langle i\sigma+k^2\rangle^{1-4\delta}\langle i\sigma_2+(k-k_1)^2\rangle}\ d\tau\right)^{\frac{1}{2}},$$

donde $\mathcal{A}(\tau_1,k_1) = \{(\tau,k)\in\mathbb{R}\times\mathbb{Z};\ (\tau,\tau_1,k,k_1)\in\mathcal{A}\}$. El Lema 3, con $s=-1+\epsilon$ y $\delta=\frac{\epsilon}{32}$ conduce a:

$$I_{\mathcal{A}} \leq C\ T^{\mu}\,\|f\|_{L^2_tL^2_x}\ \|g\|_{L^2_tL^2_x}\ \|h\|_{L^2_tL^2_x} \tag{3.27}$$

2. Estimación de I en B

De (3.26) se sigue que:

$$I_{\mathcal{B}} = \sum_{k}\int \frac{\left|\widehat{h}(\tau,k)\right|}{\langle i\sigma+k^2\rangle^{\delta}}$$
$$\times\left[\sum_{k_1}\int \frac{|k|\langle k\rangle^{s}\langle k_1\rangle^{-s}\langle k-k_1\rangle^{-s}\left|\widehat{f}(\tau_1,k_1)\right|\,|\widehat{g}(\tau-\tau_1,k-k_1)|}{\langle i\sigma_1+k_1^2\rangle^{\frac{1}{2}}\langle i\sigma_2+(k-k_1)^2\rangle^{\frac{1}{2}}\langle i\sigma+k^2\rangle^{\frac{1}{2}-2\delta}}\ d\tau_1\right]\ d\tau.$$

Nuevamente, las desigualdad de Cauchy-Schwarz, la desigualdad de Hölder, el teorema de Fubini y la desigualdad:

$$\sum_{k}\int \frac{\left|\widehat{h}(\tau,k)\right|^2}{\langle i\sigma+k^2\rangle^{2\delta}}\ d\tau \leq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \frac{\left|\widehat{h}(\tau,k)\right|^2}{\langle\sigma\rangle^{2\delta}}\ d\tau = \left\|\frac{\widehat{h}}{\langle\sigma\rangle^{\delta}}\right\|^2_{L^2_\tau \ell^2_k}$$

implican que:

$$I_{\mathcal{B}} = \left\|\frac{\widehat{h}}{\langle\sigma\rangle^{\delta}}\right\|_{L^2_\tau \ell^2_k}$$
$$\times\left\|\sum_{k_1}\int \frac{|k|\langle k\rangle^{s}\langle k_1\rangle^{-s}\langle k-k_1\rangle^{-s}\left|\widehat{f}(\tau_1,k_1)\right|\,|\widehat{g}(\tau-\tau_1,k-k_1)|}{\langle i\sigma_1+k_1^2\rangle^{\frac{1}{2}}\langle i\sigma_2+(k-k_1)^2\rangle^{\frac{1}{2}}\langle i\sigma+k^2\rangle^{\frac{1}{2}-2\delta}}\ d\tau_1\right\|_{L^2_\tau \ell^2_k}$$

$$= \left\| \frac{\widehat{h}}{\langle\sigma\rangle^{\delta}} \right\|_{L^2_\tau \ell^2_k} \left\| \left(\sum_{k_1} \int \frac{|k|^2 \langle k\rangle^{2s} \langle k_1\rangle^{-2s} \langle k-k_1\rangle^{-2s}}{\langle i\sigma_1 + k_1^2\rangle \langle i\sigma_2 + (k-k_1)^2\rangle \langle i\sigma + k^2\rangle^{1-4\delta}} \, d\tau_1 \right)^{\frac{1}{2}} \right.$$

$$\left. \times \left(\sum_{k_1} \int \left| \widehat{f}(\tau_1, k_1) \right|^2 \left| \widehat{g}(\tau - \tau_1, k-k_1) \right|^2 \, d\tau_1 \right)^{\frac{1}{2}} \right\|_{L^2_\tau \ell^2_k}$$

$$= \left\| \frac{\widehat{h}}{\langle\sigma\rangle^{\delta}} \right\|_{L^2_\tau \ell^2_k} \left\| \sum_{k_1} \int \frac{|k|^2 \langle k\rangle^{2s} \langle k_1\rangle^{-2s} \langle k-k_1\rangle^{-2s}}{\langle i\sigma_1 + k_1^2\rangle \langle i\sigma_2 + (k-k_1)^2\rangle \langle i\sigma + k^2\rangle^{1-4\delta}} \, d\tau_1 \right\|_{L^\infty_\tau \ell^\infty_k}$$

$$\left\| \sum_{k_1} \int \left| \widehat{f}(\tau_1, k_1) \right|^2 \left| \widehat{g}(\tau - \tau_1, k-k_1) \right|^2 \, d\tau_1 \right\|_{L^1_\tau \ell^1_k},$$

y razonando de manera análoga a como se hizo en la estimación para I en A obtenemos que:

$$I_{\mathcal{B}} \leq C \; T^{\mu} \, \|f\|_{L^2_t L^2_x} \, \|g\|_{L^2_t L^2_x} \, \|h\|_{L^2_t L^2_x}$$

$$\times \left(\sup_{\tau,k} \frac{|k|^2 \langle k\rangle^{2s}}{\langle i\sigma + k^2\rangle^{1-4\delta}} \sum_{\mathcal{B}(\tau,k)} \int \frac{\langle k_1\rangle^{-2s} \langle k-k_1\rangle^{-2s}}{\langle i\sigma_1 + k_1^2\rangle \langle i\sigma_2 + (k-k_1)^2\rangle} \, d\tau_1 \right)^{\frac{1}{2}} \tag{3.28}$$

donde $\mathcal{B}(\tau,k) = \{(\tau_1,k_1) \in \mathbb{R}\times\mathbb{Z}; (\tau,\tau_1,k,k_1) \in \mathcal{B}\}$. Notemos que $\mathcal{B}(\tau,k)$ es el mismo conjunto definido en el Lema (3), luego con base en este lema y tomando $s = -1+\epsilon$ y $\delta = \frac{\epsilon}{32}$, tenemos que

$$I_{\mathcal{B}} \leq C \; T^{\mu} \, \|f\|_{L^2_t L^2_x} \; \|g\|_{L^2_t L^2_x} \; \|h\|_{L^2_t L^2_x} \tag{3.29}$$

3. Estimación de I en $\mathcal{D}$. Por simetría, supongamos que $|k_1| \leq |k-k_1|$. Retomando la demostración en (3.28), para este caso tenemos que

$$I_{\mathcal{D}} \leq C \; T^{\mu} \, \|f\|_{L^2_t L^2_x} \, \|g\|_{L^2_t L^2_x} \, \|h\|_{L^2_t L^2_x}$$

$$\times \left(\sup_{\tau,k} \frac{|k|^2 \langle k\rangle^{2s}}{\langle i\sigma + k^2\rangle^{1-4\delta}} \sum_{\mathcal{D}(\tau,k)} \int \frac{\langle k_1\rangle^{-2s} \langle k-k_1\rangle^{-2s}}{\langle i\sigma_1 + k_1^2\rangle \langle i\sigma_2 + (k-k_1)^2\rangle} \, d\tau_1 \right)^{\frac{1}{2}}$$

donde $\mathcal{D}(\tau,k) = \{(\tau_1,k_1) \in \mathbb{R}\times\mathbb{Z};\; (\tau,\tau_1,k,k_1) \in \mathcal{D}\}$. De manera similar a lo que ocurrió en las estimaciones anteriores, $\mathcal{D}(\tau,k)$ corresponde al mismo conjunto introducido en el Lema 3, entonces por este lema, con $s = -1+\epsilon$ y $\delta = \frac{\epsilon}{32}$, concluimos que

$$I_{\mathcal{D}} \leq C \; T^{\mu} \, \|f\|_{L^2_t L^2_x} \; \|g\|_{L^2_t L^2_x} \; \|h\|_{L^2_t L^2_x} \tag{3.30}$$

Por lo tanto, de (3.25), (3.27), (3.29) y (3.30) se sigue que

$$|\partial_x(uv), w| \leq C \; T^{\mu} \, \|f\|_{L^2_t L^2_x} \; \|g\|_{L^2_t L^2_x} \; \|h\|_{L^2_t L^2_x}$$

pero

$$\|f\|^2_{L^2_tL^2_x} = \left\|\widehat{f}\right\|^2_{L^2_\tau \ell^2_k} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle i(\tau - k^3) + k^2\rangle \langle k\rangle^{2s} \left|\widehat{u}(\tau,k)\right|^2 \ d\tau = \|u\|^2_{X^{\frac{1}{2},s}}$$

$$\|g\|^2_{L^2_tL^2_x} = \|\widehat{g}\|^2_{L^2_\tau \ell^2_k} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle i(\tau - k^3) + k^2\rangle \langle k\rangle^{2s} \left|\widehat{v}(\tau,k)\right|^2 \ d\tau = \|u\|^2_{X^{\frac{1}{2},s}}$$

$$\|h\|^2_{L^2_tL^2_x} = \left\|\widehat{h}\right\|^2_{L^2_\tau \ell^2_k} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle i(\tau - k^3) + k^2\rangle^{1-2\delta} \langle k\rangle^{-2s} \left|\widehat{w}(\tau,k)\right|^2 \ d\tau$$

$$= \|w\|^2_{X^{\frac{1}{2}-\delta,-s}}$$

entonces

$$|\partial_x(uv), w| \leq C\ T^\mu \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s}} \|w\|_{X^{\frac{1}{2}-\delta,-s}}$$

y de esta manera

$$\|\partial_x(uv)\|_{X^{-\frac{1}{2}+\delta,s}} = \sup_{\|w\|_{X^{\frac{1}{2}-\delta,-s}}=1} |\partial_x(uv), w| \leq C\ T^\mu \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s}} \qquad \square$$

Demostración. Proposición (3.3.2)
En primer lugar veamos que (3.21) es válida.
Como $|k| \leq |k-k_1| + |k_1|$ y $s - s_c^+ > 0$ entonces

$$\langle k\rangle^{s-s_c^+} \leq C\left[\langle k-k_1\rangle^{s-s_c^+} + \langle k_1\rangle^{s-s_c^+}\right]$$

y así

$$\langle k\rangle^{s} = \langle k\rangle^{s_c^+}\langle k\rangle^{s-s_c^+} \leq C\left[\langle k\rangle^{s_c^+}\langle k_1\rangle^{s-s_c^+} + \langle k\rangle^{s_c^+}\langle k-k_1\rangle^{s-s_c^+}\right]$$

Ahora bien, para demostrar (3.3.2) vamos a retomar la demostración de la Proposición (3.3.1) en (3.25) y usamos la desigualdad (3.21) para I, entonces

$$|(\partial_x(uv), w)| \leq I \leq C(I_1 + I_2)$$

donde, I es como en (3.26),

$$I_1 = \sum_{k,k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{|k|\langle k_1\rangle^{-s_c^+}\langle k-k_1\rangle^{-s}\langle k\rangle^{s_c^+} \left|\widehat{f}(\tau_1,k_1)\right| \left|\widehat{g}(\tau-\tau_1,k-k_1)\right| \left|\widehat{h}(\tau,k)\right|}{\langle i\sigma_1 + k_1^2\rangle^{\frac{1}{2}}\langle i\sigma_2 + (k-k_1)^2\rangle^{\frac{1}{2}}\langle i\sigma + k^2\rangle^{\frac{1}{2}-\delta}}\ d\tau_1\ d\tau$$

$$I_2 = \sum_{k,k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{|k|\langle k_1\rangle^{-s}\langle k-k_1\rangle^{-s_c^+}\langle k\rangle^{s_c^+} \left|\widehat{f}(\tau_1,k_1)\right| \left|\widehat{g}(\tau-\tau_1,k-k_1)\right| \left|\widehat{h}(\tau,k)\right|}{\langle i\sigma_1 + k_1^2\rangle^{\frac{1}{2}}\langle i\sigma_2 + (k-k_1)^2\rangle^{\frac{1}{2}}\langle i\sigma + k^2\rangle^{\frac{1}{2}-\delta}}\ d\tau_1\ d\tau$$

Teniendo en cuenta que $s_c^+ < s < 0$ entonces

$$\langle k_1\rangle^{-2s} \leq \langle k_1\rangle^{-2s_c^+}, \quad \langle k-k_1\rangle^{-2s} \leq \langle k-k_1\rangle^{-2s_c^+}$$

y así podemos estimar I_1 e I_2 en los conjuntos $\mathcal{A}$, $\mathcal{B}$ y $\mathcal{D}$ de manera análoga a como se hizo en la demostración de la Proposición 3.3.1:

$$\begin{aligned}
I_{1\mathcal{A}} &\leq C\ T^{\mu}\, \|f\|_{L_t^2L_x^2}\, \|g\|_{L_t^2L_x^2}\, \|h\|_{L_t^2L_x^2} \\
&\quad \times \left(\sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s_c^+}}{\langle i\sigma_1+k_1^2\rangle} \sum \int_{\mathcal{A}(\tau_1,k_1)} \frac{|k|^2\langle k\rangle^{2s_c^+}\langle k-k_1\rangle^{-2s}}{\langle i\sigma+k^2\rangle^{1-4\delta}\langle i\sigma_2+(k-k_1)^2\rangle}\, d\tau \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\, \|f\|_{L_t^2L_x^2}\, \|g\|_{L_t^2L_x^2}\, \|h\|_{L_t^2L_x^2} \\
&\quad \times \left(\sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s_c^+}}{\langle i\sigma_1+k_1^2\rangle} \sum \int_{\mathcal{A}(\tau_1,k_1)} \frac{|k|^2\langle k\rangle^{2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma+k^2\rangle^{1-4\delta}\langle i\sigma_2+(k-k_1)^2\rangle}\, d\tau \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s_c^+}}\|v\|_{X^{\frac{1}{2},s}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}$$

$$\begin{aligned}
I_{1\mathcal{B}} &\leq C\ T^{\mu}\, \|f\|_{L_t^2L_x^2}\, \|g\|_{L_t^2L_x^2}\, \|h\|_{L_t^2L_x^2} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma+k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{B}(\tau,k)} \frac{\langle k_1\rangle^{-2s_c^+}\langle k-k_1\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+(k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\, \|f\|_{L_t^2L_x^2}\, \|g\|_{L_t^2L_x^2}\, \|h\|_{L_t^2L_x^2} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma+k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{B}(\tau,k)} \frac{\langle k_1\rangle^{-2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+(k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s_c^+}}\|v\|_{X^{\frac{1}{2},s}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}$$

$$\begin{aligned}
I_{1\mathcal{D}} &\leq C\ T^{\mu}\, \|f\|_{L_t^2L_x^2}\, \|g\|_{L_t^2L_x^2}\, \|h\|_{L_t^2L_x^2} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma+k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{D}(\tau,k)} \frac{\langle k_1\rangle^{-2s_c^+}\langle k-k_1\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+(k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\, \|f\|_{L_t^2L_x^2}\, \|g\|_{L_t^2L_x^2}\, \|h\|_{L_t^2L_x^2} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma+k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{D}(\tau,k)} \frac{\langle k_1\rangle^{-2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+(k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s_c^+}}\|v\|_{X^{\frac{1}{2},s}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}$$

$$
\begin{aligned}
I_{2\mathcal{A}} &\leq C\ T^{\mu}\, \|f\|_{L^2_tL^2_x}\, \|g\|_{L^2_tL^2_x}\, \|h\|_{L^2_tL^2_x} \\
&\quad \times \left(\sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s}}{\langle i\sigma_1 + k_1^2\rangle} \sum \int_{\mathcal{A}(\tau_1,k_1)} \frac{|k|^2\langle k\rangle^{2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma + k^2\rangle^{1-4\delta}\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\, \|f\|_{L^2_tL^2_x}\, \|g\|_{L^2_tL^2_x}\, \|h\|_{L^2_tL^2_x} \\
&\quad \times \left(\sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s_c^+}}{\langle i\sigma_1 + k_1^2\rangle} \sum \int_{\mathcal{A}(\tau_1,k_1)} \frac{|k|^2\langle k\rangle^{2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma + k^2\rangle^{1-4\delta}\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}
$$

$$
\begin{aligned}
I_{2\mathcal{B}} &\leq C\ T^{\mu}\, \|f\|_{L^2_tL^2_x}\, \|g\|_{L^2_tL^2_x}\, \|h\|_{L^2_tL^2_x} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma + k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{B}(\tau,k)} \frac{\langle k_1\rangle^{-2s}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma_1 + k_1^2\rangle\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\, \|f\|_{L^2_tL^2_x}\, \|g\|_{L^2_tL^2_x}\, \|h\|_{L^2_tL^2_x} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma + k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{B}(\tau,k)} \frac{\langle k_1\rangle^{-2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma_1 + k_1^2\rangle\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}
$$

$$
\begin{aligned}
I_{2\mathcal{D}} &\leq C\ T^{\mu}\, \|f\|_{L^2_tL^2_x}\, \|g\|_{L^2_tL^2_x}\, \|h\|_{L^2_tL^2_x} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma + k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{D}(\tau,k)} \frac{\langle k_1\rangle^{-2s}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma_1 + k_1^2\rangle\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\, \|f\|_{L^2_tL^2_x}\, \|g\|_{L^2_tL^2_x}\, \|h\|_{L^2_tL^2_x} \\
&\quad \times \left(\sup_{\tau,k} \frac{|k|^2\langle k\rangle^{2s_c^+}}{\langle i\sigma + k^2\rangle^{1-4\delta}} \sum \int_{\mathcal{D}(\tau,k)} \frac{\langle k_1\rangle^{-2s_c^+}\langle k-k_1\rangle^{-2s_c^+}}{\langle i\sigma_1 + k_1^2\rangle\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau_1 \right)^{\frac{1}{2}} \\
&\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}
$$

Por lo tanto,

$$
\begin{aligned}
I_1 &\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s_c^+}}\|v\|_{X^{\frac{1}{2},s}}\|w\|_{X^{\frac{1}{2}-\delta,-s}} \\
I_2 &\leq C\ T^{\mu}\|u\|_{X^{\frac{1}{2},s}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2}-\delta,-s}}
\end{aligned}
$$

y de esta manera

$$|(\partial_x(uv), w)| \leq C\ T^{\mu} \left(\|u\|_{X^{\frac{1}{2},s_c^+}} \|v\|_{X^{\frac{1}{2},s}} + \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s_c^+}} \right) \|w\|_{X^{\frac{1}{2}-\delta,-s}}.$$

Luego,

$$\begin{aligned} \|\partial_x(uv)\|_{X^{-\frac{1}{2}+\delta,s}} &= \sup_{\|w\|_{X^{\frac{1}{2}-\delta,-s}}=1} |(\partial_x(uv), w)| \\ &\leq C\ T^{\mu} \left(\|u\|_{X^{\frac{1}{2},s_c^+}} \|v\|_{X^{\frac{1}{2},s}} + \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s_c^+}} \right) \end{aligned}$$ □

3.4. Estimativa Trilineal

Proposición 3.4.1. *Dado $s \in (-\frac{1}{4}, 0]$, existen $C, \mu, \delta > 0$ tales que para toda tripla de funciones $u, v, w \in X^{s,\frac{1}{2}}$ con soporte compacto en $[-T, T]$*

$$\|\partial_x(uvw)\|_{X^{-\frac{1}{2}+\delta,s}} \leq C\ T^{\mu}\ \|u\|_{X^{\frac{1}{2},s}} \|v\|_{X^{\frac{1}{2},s}} \|w\|_{X^{\frac{1}{2},s}} \tag{3.31}$$

Demostración. La prueba de este lema la realizaremos por dualidad y nos basaremos en la siguiente relación algebraica

$$max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|) \gtrsim |k - k_1||k - k_2||k - k_3| \tag{3.32}$$

$$= |k_1 + k_2||k_1 + k_3||k_2 + k_2|, \tag{3.33}$$

donde

$$k = k_1 + k_2 + k_3,\ \tau = \tau_1 + \tau_2 + \tau_3 \quad k_i \in \mathbb{Z},\ \tau_i \in \mathbb{R},\ i = 1, 2, 3$$

y

$$\sigma = \tau - k^3, \quad \sigma_i = \tau_i - k_i^3,\ i = 1, 2, 3.$$

Como el dual del espacio $X^{b,s}$ es $X^{-b,-s}$, entonces

$$\|\partial_x(uvw)\|_{X^{-\frac{1}{2}+\delta,s}} = \sup_{\|h\|_{X^{\frac{1}{2}-\delta,s}}=1} |(\partial_x(uvw), h)|$$

$$\begin{aligned} (\partial(uvw), h) &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{\partial(uvw)}(k,\tau)\widehat{h}(k,\tau)d\tau \\ &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} k\widehat{uvw}(k,\tau)\widehat{h}(k,\tau)d\tau \\ &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} k\widehat{u} * \widehat{v} * \widehat{w}(k,\tau)\widehat{h}(k,\tau)d\tau \end{aligned}$$

Haciendo

$$\begin{aligned}
\widehat{f}_1(k,\tau) &= \langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}}\langle k\rangle^s\widehat{u}(k,\tau)\\
\widehat{f}_2(\tau,k) &= \langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}}\langle k\rangle^s\widehat{v}(k,\tau)\\
\widehat{f}_3(\tau,k) &= \langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}}\langle k\rangle^s\widehat{w}(k,\tau)\\
\widehat{f}(k,\tau) &= \langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}-\delta}\langle k\rangle^{-s}\widehat{h}(k,\tau)
\end{aligned}$$

tenemos

$$(\partial(uvw),h) = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \frac{k\widehat{f}(k,\tau)\langle k\rangle^s}{\langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}-\delta}}\ \widehat{u}*\widehat{v}*\widehat{w}(k,\tau)d\tau \tag{3.34}$$

Como:

$$\widehat{u}*\widehat{v}*\widehat{w}(k,\tau) = \sum_{k_1,k_2}\int \widehat{u}(k_1,\tau_1)\widehat{v}(k_2,\tau_2)\widehat{w}(k-k_1-k_2),\tau-\tau_1-\tau_2)d\tau_1 d\tau_2 \tag{3.35}$$

sustituyendo (3.35) en (3.34), obtenemos

$$\begin{aligned}
I = (\partial(uvw),h) &= \sum_{k,k_1,k_2}\int_{\tau,\tau_1,\tau_2} \underbrace{\frac{k\widehat{f}(k,\tau)\langle k\rangle^s}{\langle i(\tau-k^3)+k^2\rangle^{\frac{1}{2}-\delta}}\ \frac{\prod_j\langle k_j\rangle^{-s}\widehat{f}_j(k_j,\tau_j)}{\prod_j\langle i((\sigma_j,k_j)^2)\rangle^{\frac{1}{2}}}}_{G(k,k_1,k_2,\tau,\tau_1,\tau_2)}\ d\tau\ d\tau_1\ d\tau_2\\
&= \sum_A\int G(k,k_1,k_2,\tau,\tau_1,\tau_2)d\tau\ d\tau_1\ d\tau_2 + \sum_B\int G(k,k_1,k_2,\tau,\tau_1,\tau_2)d\tau d\tau_1 d\tau_2\\
&\quad + \sum_D\int G(k,k_1,k_2,\tau,\tau_1,\tau_2)d\tau d\tau_1 d\tau_2 + \sum_E\int G(k,k_1,k_2,\tau,\tau_1,\tau_2)d\tau d\tau_1 d\tau_2
\end{aligned}$$

donde:

$$\begin{aligned}
A &= \{(k,k_1,k_2,\tau,\tau_1,\tau_2)\in\mathbb{Z}^3\times\mathbb{R}^3,\ |k_1| = max(|k_1|,|k_2|,|k_3|)\geq 1\}\\
B &= \{(k,k_1,k_2,\tau,\tau_1,\tau_2)\in\mathbb{Z}^3\times\mathbb{R}^3,\ |k_2| = max(|k_1|,|k_2|,|k_3|)\geq 1\}\\
D &= \{(k,k_1,k_2,\tau,\tau_1,\tau_2)\in\mathbb{Z}^3\times\mathbb{R}^3,\ |k_3| = max(|k_1|,|k_2|,|k_3|)\geq 1\}\\
E &= \{(k,k_1,k_2,\tau,\tau_1,\tau_2)\in\mathbb{Z}^3\times\mathbb{R}^3,\ max(|k_1|,|k_2|,|k_3|)\leq 1\}
\end{aligned}$$

Las regiones A, B y D son simétricas por ello nosotros sólo estimaremos a I en la región A y E

1. Estimativa de I sobre A

$$\sum_{k_1\in A}\int_{\tau_1\in A}|\widehat{f}_1(k_1,\tau_1)|g(k_1,\tau_1)d\tau_1 \leq$$

$$\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\langle k_1\rangle^{-s}}{\langle i\sigma_1+k_1^2\rangle^{1/2}}\sum\int\frac{|k|\langle k\rangle^s|\widehat{f}(k,\tau)|}{\langle i\sigma+k^2\rangle^{1/2-\delta}}m(k_j,\tau_j)d\tau d\tau_2\right\|_{l^2_{k_1}(A)L^2_{\tau_1}(A)}$$

donde

$$g(k_1,\tau_1)=\frac{\langle k_1\rangle^{-s}}{\langle i\sigma_1+k_1^2\rangle^{1/2}}\sum_{k,k_2}\int\frac{|k|\langle k\rangle^s|\widehat{f}(k)|}{\langle i\sigma+k^2\rangle^{1/2-\delta}}\frac{\prod_{j=2,3}\langle k_j\rangle^{-s}|\widehat{f}(k_j,\tau_j)|}{\prod_{j=2,3}\langle i\sigma_j+k_j^2\rangle^{1/2}}d\tau d\tau_2$$

y

$$m(k_j,\tau_j)=\frac{\prod_{j=2,3}\langle k_j\rangle^{-s}|\widehat{f}(k_j,\tau_j)|}{\prod_{j=2,3}\langle i\sigma_j+k_j^2\rangle^{1/2}}$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\sum_{k,k_2}\int\left\|\frac{\langle k_1\rangle^{-s}|k|\langle k\rangle^s|\widehat{f}(k,\tau)|}{\langle i\sigma+k^2\rangle^{1/2-\delta}}m(k_j,\tau_j)\right\|_{l^2_{k_1}(A)L^2_{\tau_1}(A)}d\tau d\tau_2$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}$$

$$\sum_{k_1,k_2}\int\frac{|\widehat{f}(k,\tau)|}{\langle i\sigma+k^2\rangle^a}\left\|\frac{|k|\langle k\rangle^s}{\langle i\sigma+k^2\rangle^{1/2-\delta-a}}\frac{\langle k_1\rangle^{-s}}{\langle i\sigma_1+k_1^2\rangle^{1/2}}m(k_j,\tau_j)\right\|_{l^2_{k_1}(A)L^2_{\tau_1}(A)}d\tau d\tau_2$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\sum_{k_2}\int\left(\sum_k\int\frac{|\widehat{f}(k,\tau)|}{\langle i\sigma+k^2\rangle^a}F(k,k_2,\tau,\tau_2)d\tau\right)^{1/2}d\tau_2$$

donde

$$F(k,k_2,\tau,\tau_2)=\frac{|k|\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}}\sum_{k_1\in A}\int\frac{\langle k_1\rangle^{-2s}}{\langle i\sigma+k^2\rangle}\frac{\prod_{j=2,3}\langle k_j\rangle^{-2s}|\widehat{f}(k_j,\tau_j)|^2}{\prod_{j=2,3}\langle i\sigma_j+k_j^2\rangle}$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left(\sum_k\int\frac{|\widehat{f}(k,\tau)|^2}{\langle i\sigma+k^2\rangle^{2a}}\right)^{1/2}\sum_{k_2}\int\left(\sum_k\int F(k,k_2,\tau,\tau_2)^2d\tau\right)^{1/2}d\tau_2$$

$$=\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f}(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_k(A)}\times$$

$$\sum_{k_2}\int\left[\sum_k\int\left(\frac{|k|\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}}\sum_k\int\frac{\langle k_1\rangle^{-2s}}{\langle i\sigma_1+k_1\rangle^2}m(k_j,\tau_j)\right)d\tau\right]^{1/2}d\tau_2$$

$$=\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f}(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_k(A)}\times\sum_{k_2}\int(\frac{\langle k_2\rangle^{-2s}|\widehat{f}_2(k_2,\tau_2)|^2}{\langle i\sigma_2+k_2^2\rangle}$$

$$\sum_{k,k_1}\int\frac{|k|^2\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\sigma-2a}}\frac{\langle k_1\rangle^{-2s}}{i\sigma_1+k_1^2}\frac{\langle k_3\rangle^{-2s}|\widehat{f}(k_3,\tau_3)|^2}{\langle i\sigma_3+k_3\rangle}d\tau_1d\tau)^{1/2}d\tau_2$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f}_2(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_k(A)}\left(\sum_{k_2}\int|\widehat{f}_2(k_2,\tau_2)|\left[G(k_2,\tau_2)\right]^2\right)^{1/2}d\tau_2$$

donde

$$G(k_2,\tau_2)=$$

$$\left[\frac{\langle k_2\rangle{-2s}}{\langle i\sigma_2+k_2\rangle}\sum_{k,k_1}\int\frac{|k|^2\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}}\frac{\langle k_1\rangle^{-2s}|\langle k_3\rangle^{-2s}}{\langle i\sigma_1+k_1\rangle\langle i\sigma_3+k_3\rangle}|\widehat{f_3}(k_3,\tau_3)|^2d\tau_1d\tau\right]^{1/2}$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f_2}(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_\tau(A)}\|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\left(\sum_{k_2}\int\left[G(k_2,\tau_2)\right]^2d\tau_2\right)^{1/2}$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f_2}(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_\tau(A)}\|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\times(\sum_{k_2}\int\frac{\langle k_2\rangle^{-2s}}{\langle i\sigma_2+k_2\rangle}$$

$$\sum_{k,k_1}\int\frac{|k|\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}}\frac{\langle k_1\rangle^{-2s}\langle k_3\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_3+k_3^2\rangle}|\widehat{f_3}(k_3,\tau_3)|^2d\tau_1d\tau d\tau_2)^{1/2}$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f_2}(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_\tau(A)}\|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\times$$

$$\left(\sum_{k,k_1,k_2}\int\frac{|k|^2\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}}\frac{\prod_{j=1}^3\langle k_j\rangle^{-2s}}{\prod_{j=1}^3\langle i\sigma_j+k_j^2\rangle}|\widehat{f_3}(k_3,\tau_3)|^2d\tau d\tau_1d\tau_2\right)^{1/2}$$

$$\leq\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\frac{\widehat{f_2}(k,\tau)}{\langle i\sigma+k^2\rangle^a}\right\|_{l^2_k(A)L^2_\tau(A)}\|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\times$$

$$\left[\sum_{k,k_2}\int\frac{|k|^2\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}}\frac{\langle k_2\rangle^{-2s}}{\langle i\sigma_2+k_2^2\rangle}N(k,k_1,k_2,\tau,\tau_1,\tau_2)d\tau d\tau_2\right]^{1/2}$$

donde

$$N(k,k_1,k_2,\tau,\tau_1,\tau_2)=\sum_{k_1}\int\frac{\langle k_1\rangle^{-2s}\langle k_3\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_3+k_3^2\rangle}|\widehat{f_3}(k_3,\tau_3)|^2d\tau_1$$

$$=\sum_{k_1}\int\frac{\langle k_1\rangle^{-2s}\langle k_3\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_3+k_3^2\rangle}|\widehat{f_3}(k-k_1-k_2,\tau-\tau_1-\tau_2)|^2d\tau_1,$$

como

$$N(k,k_1,k_2,\tau,\tau_1,\tau_2)$$

$$\leq\sup_{k_1,\tau_1\in A}\frac{\langle k_1^{-2s}\rangle\langle k_3^{-2s}\rangle}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_3+k_3^2\rangle}\sum_{k_1}\int|\widehat{f_3}(k-k_1-k_2,\tau-\tau_1-\tau_2)|^2d\tau_1$$

y además

$$\sum_{k_1}\int |\widehat{f_3}(k-k_1-k_2,\tau-\tau_1-\tau_2)|^2 d\tau_1 \leq$$
$$\sum_{k_1}\int_{-\infty}^{\infty} |\widehat{f_3}(k-k_1-k_2,\tau-\tau_1-\tau_2)|^2 d\tau_1$$

haciendo $-k_1=\tilde{k}_1$ y $-\tau_1=\tilde{\tau}_1$ tenemos:

$$\sum_{k_1}\int |\widehat{f_3}(k-k_1-k_2,\tau-\tau_1-\tau_2)|^2 d\tau_1 \leq$$
$$\sum_{k_1}\int_{-\infty}^{\infty} |\widehat{f_3}(\tilde{k}_1+k-k_2,\tilde{\tau}_1+\tau-\tau_2)|^2 d\tilde{\tau}_1$$

Ahora haciendo el cambio de variable $\overset{*}{k_1}=\tilde{k}_1+k-k_2$ y $\overset{*}{\tau_1}=\tilde{\tau}_1+\tau-\tau_2$ tenemos que:

$$\sum_{k_1}\int |\widehat{f_3}(k-k_1-k_2,\tau-\tau_1-\tau_2)|^2 d\tau_1 \leq$$
$$\sum_{\overset{*}{k_1}}\int_{-\infty}^{\infty} |\widehat{f_3}(\overset{*}{k_1},\overset{*}{\tau_1})|^2 d\,\overset{*}{\tau_1}=\|\widehat{f_3}\|_{l^2_{k_1}L^2_\tau}=\|f_3\|_{L^2(\mathbb{T}\times\mathbb{R})}$$

por lo tanto tenemos que:

$$N(k,k_1,k_2,\tau,\tau_1,\tau_2)\leq \sup_{k_1,\tau_1\in A}\frac{\langle k_1^{-2s}\rangle\langle k_3^{-2s}\rangle}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_3+k_3^2\rangle}\|f_3\|_{L^2(\mathbb{T}\times\mathbb{R})}$$

es decir que

$$I \leq \|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})} \left\| \frac{\widehat{f_2}(k,\tau)}{\langle i\sigma + k^2\rangle^a} \right\|_{l_k^2(A)L_\tau^2(A)} \|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\|f_3\|_{L^2(\mathbb{T}\times\mathbb{R})} \times$$

$$\left[\sum_{k,k_2}\int \frac{|k|^2\langle k\rangle^{2s}\langle k_2\rangle^{-2s}}{\langle i\sigma+k^2\rangle^{1-2\delta-2a}\langle i\sigma_2+k_2^2\rangle}\right]^{1/2} \times$$

$$\left[\sup_{k_1,\tau_1\in A} \frac{\langle k_1\rangle^{-2s}}{\langle i\sigma_2+k_2^2\rangle}\frac{\langle k_3\rangle^{-2s}\langle k_3\rangle^{-2s}}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma_3+k_3^2\rangle} d\tau d\tau_2\right]^{1/2}$$

$$\leq \sup_{k_1,\tau_1\in A}\left[\frac{\langle k_1\rangle^{-s}}{\langle i\sigma_1+k_1^2\rangle^{1/2}}\left(\sum_{k,k_2}\int \frac{\langle k_2\rangle^{-2s}\langle k_3\rangle^{-2s}|k|^2\langle k\rangle^{2s}}{\langle i\sigma+k^2\rangle^{1-2\sigma-2a}\prod_{j=2,3}\langle i\sigma_j+k_j^2\rangle} d\tau d\tau_2\right)^{1/2}\right] \times$$

$$\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\|f_3\|_{L^2(\mathbb{T}\times\mathbb{R})} \left\| \frac{\widehat{f_2}(k,\tau)}{\langle i\sigma + k^2\rangle^a} \right\|_{l_k^2(A)L_\tau^2(A)}$$

por el lema (5) del apéndice se tiene:

$$I \leq T^\mu \|f\|_{L^2(\mathbb{T}\times\mathbb{R})}\|f_1\|_{L^2(\mathbb{T}\times\mathbb{R})}\|f_2\|_{L^2(\mathbb{T}\times\mathbb{R})}\|f_3\|_{L^2(\mathbb{T}\times\mathbb{R})}$$

2. Estimativa de I sobre E

   Como $max\{|k_1|,|k_2|,|k_3|\} < 1$, entonces $|k_1| = |k_2| = |k_3| = 0$.

□

La siguiente proposición es consecuencia directa de la proposición (3.4.1) junto con la desigualdad triangular, que implica

$$\forall s \geq s_c^+, \langle k\rangle^s \leq \langle k\rangle^{s_c^+}\langle k_1\rangle^{s-s_c^+} + \langle k\rangle^{s_c^+}\langle k_2\rangle^{s-s_c^+} + \langle k\rangle^{s_c^+}\langle k-k_1-k_2\rangle^{s-s_c^+} \tag{3.36}$$

Proposición 3.4.2. *Dado $s_c^+ \in (-\frac{1}{4}, 0]$, existen $C, \mu, \delta > 0$ tales que para cada $s \geq s_c^+$ y cada tripla de funciones $u, v, w \in X^{s,\frac{1}{2}}$ con soporte compacto en $[-T, T]$*

$$\|\partial_x(uvw)\|_{X^{-\frac{1}{2}+\delta,s}} \leq$$
$$C\ [T^\mu\ \|u\|_{X^{s,\frac{1}{2}}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2},s_c^+}} + \|u\|_{X^{\frac{1}{2},s}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2},s_c^+}}$$
$$+ \|u\|_{X^{\frac{1}{2},s_c^+}}\|v\|_{X^{\frac{1}{2},s_c^+}}\|w\|_{X^{\frac{1}{2},s}}]$$

3.5. Demostración del Teorema 3.1.1

Sea $\varphi \in H^s(\mathbb{T})$, con $s > -\frac{1}{4}$. Probaremos la existencia de una solución u para la ecuación integral (3.1) de la G-B en algún intervalo $[0, T]$ para $T \leq 1$ suficientemente pequeño.

En primer lugar, es claro que si u es solución de la ecuación integral $u = F(u)$, donde

$$F(u) = \psi(t)\left[\mathcal{W}(t)u_0 - \chi_{\mathbb{R}_+}(t)\int_0^t \mathcal{W}(t-t')\partial_x\left(\psi_T^3(t')u^3(t') + \psi_T^2(t')u^2(t')\right)\ dt'\right] \tag{3.37}$$

entonces u es solución de la ecuación integral (3.1) en $[0,T]$.
Así pues, vamos a resolver (3.37) en el espacio

$$Z = \left\{u \in X^{s,\frac{1}{2}};\ \|u\|_Z = \|u\|_{X^{\frac{1}{2},s_c^+}} + \nu\|u\|_{X^{\frac{1}{2},s}} < +\infty\right\} \tag{3.38}$$

donde $s_c^+ \in (-\frac{1}{4}, \text{mín}(0,s))$ es fijo, y donde la constante ν está definida para toda φ no trivial por

$$\nu = \frac{\|\varphi\|_{H^{s_c^+}}}{\|\varphi\|_{H^s}}$$

El objetivo es mostrar que F está bien definida y que es una contracción en $\overline{B}(0,R) \subset Z$,donde $\overline{B}(0,R)$ es la bola cerrada de radio R, el cual elegiremos adecuadamente mas adelante.

$$\begin{aligned}\|F(u)\|_{X^{1/2,s_c^+}} &\leq \|\psi(t)\mathcal{W}\varphi\|_{X^{1/2},s_c^+} \\ &+ \left\|\chi_{\mathbb{R}_+(t)}\int_0^t \mathcal{W}(t-t')\partial_x\left(\psi_T^3(t')u^3(t') + \psi_T^2(t')u^2(t')\right)\ dt'\right\|_{X^{\frac{1}{2},s_c^+}}\end{aligned}$$

por la proposición (3.2.1) tenemos que $\|\psi(t)\mathcal{W}(t)\varphi\|_{X^{1/2,s_c^+}} \leq C\|\varphi\|_{H^{s_c^+}}$ y por la proposición (3.2.3) parte b, con $v = \partial_x\left(\psi_T^3(t')u^3(t') + \psi_T^2(t')u^2(t')\right)$ se tiene que

$$\begin{aligned}&\left\|\chi_{\mathbb{R}_+(t)}\int_0^t \mathcal{W}(t-t')\partial_x\left(\psi_T^3(t')u^3(t') + \psi_T^2(t')u^2(t')\right)\ dt'\right\|_{X^{\frac{1}{2},s_c^+}} \\ &\qquad\qquad \leq C_\delta\|\partial_x(\psi_T^3u^3 + \psi_T^2u^2)\|_{X^{-1/2+\delta,s_c^+}}\end{aligned}$$

De las proposiciones (3.3.1)y (3.4.1), existen $C, \mu > 0$ tales que:

$$\begin{aligned}&\|\partial_x(\psi_T^3u^3 + \psi_T^2u^2)\|_{X^{-1/2+\delta,s_c^+}} \\ &\quad\leq C\ T^\mu\ (2\|\psi_Tu\|_{X^{\frac{1}{2},s_c^+}}\|\psi_Tu\|_{X^{\frac{1}{2},s_c^+}} + \|\psi_Tu\|_{X^{\frac{1}{2},s_c^+}}\|\psi_Tu\|_{X^{\frac{1}{2},s_c^+}}\|\psi_Tu\|_{X^{\frac{1}{2},s_c^+}}) \\ &\qquad\qquad \leq CT^\mu[\|\psi_Tu\|^2_{X^{\frac{1}{2},s_c^+}} + \|\psi_Tu\|^3_{X^{\frac{1}{2},s_c^+}}] \\ &\qquad\qquad \leq CT^\mu[\|u\|^2_{X^{\frac{1}{2},s_c^+}} + \|u\|^3_{X^{\frac{1}{2},s_c^+}}]\end{aligned}$$

Entonces

$$\|F(u)\|_{X^{1/2,s_c^+}} \leq C\|\varphi\|_{H^{s_c^+}} + CT^\mu[\|u\|^2_{X^{\frac{1}{2},s_c^+}} + \|u\|^3_{X^{\frac{1}{2},s_c^+}}] \tag{3.39}$$

De igual manera por las proposiciones (3.2.1) y (3.2.3) parte b tenemos que

$$\|F(u)\|_{X^{1/2,s}} \leq C\|\varphi\|_{H^s} + C_\delta\|\partial_x(\psi_T^3 u^3 + \psi_T^2 u^2)\|_{X^{-1/2+\delta,s}}$$

y por las proposiciones (3.3.1) y (3.4.1) que, existen $C, \mu > 0$ tales que,

$$\begin{aligned}\|\partial_x(\psi_T^3 u^3 + \psi_T^2 u^2)\|_{X^{-1/2+\delta,s}} &\leq C\ T^\mu\ (2\|\psi_T u\|_{X^{\frac{1}{2},s_c^+}}\|\psi_T u\|_{X^{\frac{1}{2},s}} \\ &\quad + 3(\|\psi_T u\|^2_{X^{\frac{1}{2},s_c^+}}\|\psi_T u\|_{X^{\frac{1}{2},s}}) \\ &\leq C\ T^\mu\ (\|u\|_{X^{\frac{1}{2},s_c^+}}\|u\|_{X^{\frac{1}{2},s}} + \|u\|^2_{X^{\frac{1}{2},s_c^+}}\|u\|_{X^{\frac{1}{2},s}}).\end{aligned}$$

Por lo tanto,

$$\|F(u)\|_{X^{1/2,s}} \leq C\|\varphi\|_{H^s} + CT^\mu[(\|u\|_{X^{\frac{1}{2},s_c^+}}\|u\|_{X^{\frac{1}{2},s}} + \|u\|^2_{X^{\frac{1}{2},s_c^+}}\|u\|_{X^{\frac{1}{2},s}})]. \tag{3.40}$$

Así que, (3.39) y (3.40) implican que,

$$\begin{aligned}\|F(u)\|_Z &= \|F(u)\|_{X^{1/2,s}} + \nu\|F(u)\|_{X^{1/2,s}} \\ &\leq C(\|\varphi\|_{H^{s_c^+}} + \nu\|\varphi\|_{H^s}) + CT^\mu(\|u\|_Z^3 + \|u\|_Z^2)\end{aligned} \tag{3.41}$$

Por otra parte

$$\begin{aligned}&F(u) - F(v) = \\ &\quad - \chi_{\mathbb{R}_+}(t)\int_0^t \mathcal{W}(t-t')\partial_x\left(\psi_T^3(t')(u^3-v^3)(t') + \psi_T^2(t')(u^2-v^2)(t')\right)\ dt',\end{aligned}$$

por la proposición (3.2.3) parte b, existe C_δ tal que

$$\begin{aligned}&\|F(u) - F(v)\|_{X^{1/2,s_c^+}} = \\ &\quad \|\chi_{\mathbb{R}_+}(t)\int_0^t \mathcal{W}(t-t')\partial_x\left(\psi_T^3(t')(u^3-v^3)(t') + \psi_T^2(t')(u^2-v^2)(t')\right)\ dt'\|_{X^{1/2,s}} \\ &\qquad \leq C_\delta\|\partial_x\left(\psi_T^3(u^3-v^3) + \psi_T^2(u^2-v^2)\right)\|_{X^{-1/2+\delta,s_c^+}},\end{aligned}$$

por las propiedades (3.3.1)y (3.4.1), existen $C, \mu > 0$ tales que:

$$\begin{aligned}&\|\partial_x\left(\psi_T^3(u^3-v^3) + \psi_T^2(u^2-v^2)\right)\|_{X^{-1/2+\delta,s_c^+}} \leq \\ &\|\partial_x\left(\psi_T(u-v)\psi_T^2(u^2+uv+v^2)\right)\|_{X^{-1/2+\delta,s_c^+}} + \|\partial_x\left(\psi_T(u-v)\psi_T(u+v)\right)\|_{X^{-1/2+\delta,s_c^+}} \\ &\quad \leq CT^\mu[\|u-v\|_{X^{1/2,s_c^+}}(\|u\|^2_{X^{1/2,s_c^+}} + \|u\|_{X^{1/2,s_c^+}}\|v\|_{X^{1/2,s_c^+}} + \|v\|^2_{X^{1/2,s_c^+}}) \\ &\qquad + \|u-v\|_{X^{1/2,s_c^+}}\|u+v\|_{X^{1/2,s_c^+}}],\end{aligned}$$

entonces

$$\begin{aligned}&\|F(u) - F(v)\|_{X^{1/2,s_c^+}} \\ &\leq CT^\mu[\|u-v\|_{X^{1/2,s_c^+}}(\|u\|^2_{X^{1/2,s_c^+}} + \|u\|_{X^{1/2,s_c^+}}\|v\|_{X^{1/2,s_c^+}} + \|v\|^2_{X^{1/2,s_c^+}}) + \|u+v\|_{X^{1/2,s_c^+}}].\end{aligned} \tag{3.42}$$

De manera análoga

$$\begin{aligned}
&\|F(u)-F(v)\|_{X^{1/2,s}}\\
&\leq CT^{\mu}\|u-v\|_{X^{1/2,s_c^+}}[\|u\|_{X^{1/2,s_c^+}}\|u\|_{X^{1/2,s}}+\|u\|_{X^{1/2,s_c^+}}\|v\|_{X^{1/2,s}}+\|u\|_{X^{1/2,s}}\|v\|_{X^{1/2,s_c^+}}\\
&+\|v\|_{X^{1/2,s_c^+}}\|v\|_{X^{1/2,s}}]+CT^{\mu}\|u-v\|_{X^{1/2,s}}[\|u\|^2_{X^{1/2,s_c^+}}+\|u\|_{X^{1/2,s_c^+}}\|v\|_{X^{1/2,s_c^+}}+\|v\|^2_{X^{1/2,s_c^+}}]\\
&\qquad+CT^{\mu}(\|u-v\|_{X^{1/2,s_c^+}}\|u+v\|_{X^{1/2,s}}+\|u-v\|_{X^{1/2,s}}\|u+v\|_{X^{1/2,s_c^+}}). \qquad (3.43)
\end{aligned}$$

luego de (3.42) y (3.43), tenemos:

$$\begin{aligned}
\|F(u)-F(v)\|_Z &= \|F(u)-F(v)\|_{X^{1/2,S_c^+}}+\nu\|F(u)-F(v)\|_{X^{1/2,s}}\\
&\leq CT^{\mu}\|u-v\|_Z\|u+v\|_Z+\|u-v\|_Z(\|u\|_Z+\|v\|_Z)^2\\
&= CT^{\mu}\|u-v\|_Z(\|u+v\|_Z+(\|u\|_Z+\|v\|_Z)^2). \qquad (3.44)
\end{aligned}$$

Sean $u,v\in\overline{B}(0,R)=\{u\in Z;\ \|u\|_Z<R\}$, de la definición de ν y (3.41) se tiene

$$\|F(u)\|_Z<2C\|\varphi\|_{H^{s_c^+}}+CT^{\mu}(R^2+R^3). \qquad (3.45)$$

Haciendo,

$$R=4C\|\varphi\|_{H^{s_c^+}} \quad \text{y} \quad T^{\mu}=2\theta(16C^2\|\varphi\|_{H^{s_c^+}}+64C^3\|\varphi\|^2_{H^{s_c^+}})^{-1}, \qquad (3.46)$$

donde $\theta=\dfrac{1+4c\|\varphi\|_{H^{s_c^+}}}{2(1+8c\|\varphi\|_{H^{s_c^+}})}$ tenemos

$$\begin{aligned}
\|F(u)\|_Z &< 2C\|\varphi\|_{H^{s_c^+}}\\
&\qquad+\frac{C}{16C^2\|\varphi\|_{H^{s_c^+}}+64C^3\|\varphi\|^2_{H^{s_c^+}}}(16C^2\|\varphi\|^2_{H^{s_c^+}}+64C^3\|\varphi\|^2_{H^{s_c^+}})\\
&= 3C\|\varphi\|_{H^{s_c^+}}<R \qquad (3.47)
\end{aligned}$$

(3.44) implica que

$$\begin{aligned}
\|F(u)-F(v)\|_Z &\leq CT^{\mu}\|u-v\|_Z(\|u+v\|_Z+(\|u\|_Z+\|v\|_Z)^2)\\
&\leq CT^{\mu}\|u-v\|_Z(\|u\|_Z+\|v\|_Z+(\|u\|_Z+\|v\|_Z)^2)\\
&\leq CT^{\mu}\|u-v\|_Z(2R+4R^2)\\
&< \frac{1}{2}\|u-v\|_Z \qquad (3.48)
\end{aligned}$$

Luego la aplicación F es estrictamente contractiva en $\overline{B}(0,4C\|\varphi\|_{H^{s_c^+}})$, y por el teorema del punto fijo existe $u\in X^{\frac{1}{2},s}$ tal que $F(u)=u$, es decir existe una solución de la ecuación $G-B$ en el intervalo $[0,T]$, con $T=T(\|\varphi\|_{H^{s_c^+}})>0$

Apéndice

Lema 3. *Para toda pareja fija* (τ_1, k_1) *con* $|k_1| \geq 1$*, definimos el siguiente subconjunto de* $\mathbb{R} \times \mathbb{Z}$

$$\mathcal{A}(\tau_1, k_1) = \{(\tau, k) \in \mathbb{R} \times \mathbb{Z};\ |k| \leq 2|k_1|,\ |k - k_1| \geq 1,\ |\sigma_2| \leq |\sigma_1|\}\,.$$

Entonces, para todo $\epsilon > 0$*, suficientemente pequeño, existe* $C > 0$*, que depende únicamente de* ϵ*, tal que*

$$I = \frac{\langle k_1\rangle^{2-2\epsilon}}{\langle i\sigma_1 + k_1^2\rangle} \sum \int_{(\tau,k)\in\mathcal{A}} \frac{\langle k - k_1\rangle^{2-2\epsilon}|k|^2\langle k\rangle^{-2+2\epsilon}}{\langle i\sigma + k^2\rangle^{1-\frac{\epsilon}{8}}\langle i\sigma_2 + (k-k_1)^2\rangle}\, d\tau \leq C \tag{3.49}$$

Demostración. Ver [2], páginas 60-62. □

Lema 4. *Sea* f *una función con soporte compacto (en el tiempo) en* $[-T, T]$*. Para cada* $0 < \theta < \frac{1}{2}$ *existe* $\mu = \mu(\theta) > 0$ *tal que*

$$\left\| \mathfrak{F}^{-1}_{t,x}\left(\frac{\widehat{f}(\tau,k)}{\langle \tau - k^3\rangle^{\theta}}\right)\right\|_{L^2_t(\mathbb{R})L^2_x(\mathbb{T})} \leq C\ T^{\mu}\|f\|_{L^2_t(\mathbb{R})L^2_x(\mathbb{T})} \tag{3.50}$$

Demostración. ver [2], páginas 63-64. □

Lema 5. *Para todo* $\epsilon > 0$*, existe* $C > 0$ *tal que* $\forall |k_1| \geq 1$

$$I = \frac{\langle k_1\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1 + k_1^2\rangle} \sum_{k,k_2} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}|k|^2\langle k\rangle^{-\frac{1}{2}+2\epsilon}}{\langle i\sigma + k^2\rangle^{1-\delta}\langle i\sigma + k_2^2\rangle\langle i\sigma_3 + k_3^2\rangle} d\tau\ d\tau_2 \leq C,$$

donde

$$\begin{aligned} A = A(k_1, \tau_1) = \{&(k, k_2, \tau, \tau_2) \in \mathbb{Z}^2 \times \mathbb{R}^2; |k_1| \geq max(|k_2|, |k_3|), \\ &k = k_1 + k_2 + k_3\}, y\ \ 0 < \delta \ll \epsilon \end{aligned}$$

Demostración. La demostración de este lema la realizaremos en varias etapas para ello vamos a estimar a I en cuatro casos: $|k_1| \gg |k_2| \gg |k_3|$, $|k_1| \gg |k_2| \sim |k_3|$, $|k_1| \sim |k_2| \gg |k_3|$ y $|k_1| \sim |k_1| \sim |k_3|$.
Como k_2 y k_3 juegan el mismo papel, podemos asumir que $|k_2| \geq |k_3|$.

1. Si $|k_1| \gg |k_2| \gg |k_3|$, tenemos $k \sim |k_1| \gg |k_2| \gg |k_3|$

   - Si $|\sigma| = max(|\sigma|, |\sigma|_1, |\sigma|_2, |\sigma|_3)$, entonces por (3.33) tenemos $|\sigma| \gtrsim \langle k_1\rangle^2 |k_2|$, y además $\dfrac{|k_1|}{(1+\sigma_1^2+k_1^2)^{1/2}} \leq 1$.

$$I = \frac{1}{(i\sigma_1+\sigma_1^2+k_1^4)^{1/2}} \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle} d\tau\ d\tau_2$$
$$\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle} d\tau\ d\tau_2. \tag{3.51}$$

   Si $|k_2| < 1$, $k_2 = 0$.
   Si $|k_2| \geq 1$

$$\begin{aligned}\langle i\sigma+k^2\rangle^{1-\delta} &\geq \langle i\sigma+k^2\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim (1+k_2^2)^{3(\frac{1-3\delta}{2})}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &= \langle k_2\rangle^{3(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &= \langle k_2\rangle^{\frac{3}{2}(1-3\delta)}\langle k_2\rangle^{\frac{3}{2}(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_2\rangle^{\frac{3}{2}(1-3\delta)}\langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\end{aligned} \tag{3.52}$$

   (3.51), (3.52) y el hecho que $k_2 = 0$ implican que

$$\begin{aligned}I &\lesssim \sum_{A\cap|k_2|\geq 1} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_2\rangle^{\frac{3}{2}(1-3\delta)}\langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\langle i\sigma+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle} d\tau\ d\tau_2\\ &\lesssim \sum_{A\cap|k_2|\geq 1} \int_A \frac{d\tau\ d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\\ &\lesssim \sum_{k_2} \frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \sum_{k} \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C,\end{aligned} \tag{3.53}$$

   puesto que cada sumatoria como cada integral convergen, en efecto:

   ◇

$$\sum_{k_2} \frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \leq \sum_{k_2} \frac{1}{(1+|k_2|)^{1+2\epsilon-\frac{9}{2}\delta}} < C \tag{3.54}$$

   ◇

$$\begin{aligned}\sum_{k} \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}} &\leq \sum_{k} \frac{1}{(1+|k_3|)^{1+2\epsilon-\frac{9}{2}\delta}}\\ &\leq \sum_{k} \frac{1}{(1+|k-k_1-k_2|)^{1+2\epsilon-\frac{9}{2}\delta}}\\ &\leq \sum_{j} \frac{1}{(1+|j|)^{1+2\epsilon-\frac{9}{2}\delta}} < C\end{aligned}$$

◇

$$\int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \leq \int_{-\infty}^{\infty} \frac{d\tau_2}{(1+|\tau_2-k_2|)^{1+\delta}}$$

Haciendo el cambio de variable $\eta = \tau_2 - k_2$, $d\eta = -d\tau_2$

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\tau_2}{(1+|\tau_2-k_2|)^{1+\delta}} &\leq \int_{\infty}^{-\infty} -\frac{d\eta}{(1+|\eta|)^{1+\delta}} \\ &\leq \int_{-\infty}^{\infty} \frac{d\eta}{(1+|\eta|)^{1+\delta}} < C \end{aligned} \tag{3.55}$$

◇

$$\begin{aligned}\int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} &\leq \int_A \frac{d\tau}{(1+|\tau_3-k_3|)^{1+\delta}} \\ &\leq \int_A \frac{d\tau}{(1+|\tau-\tau_1-\tau_2-k-k_1-k_2|)^{1+\delta}} \\ &\leq \int_{-\infty}^{\infty} \frac{d\tau}{(1+|\tau-\tau_1-\tau_2-k-k_1-k_2|)^{1+\delta}},\end{aligned}$$

Haciendo el cambio de variable $\eta = \tau-\tau_1-\tau_2-k-k_1-k_2$, $d\eta = -d\tau_2$

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\tau}{(1+|\tau-\tau_1-\tau_2-k-k_1-k_2|)^{1+\delta}} &\leq \int_{\infty}^{-\infty} -\frac{d\eta}{(1+|\eta|)^{1+\delta}} \\ &\leq \int_{-\infty}^{\infty} \frac{d\eta}{(1+|\eta|)^{1+\delta}} < C\end{aligned} \tag{3.56}$$

De manera análoga se obtiene el acotamiento si $|\sigma_2|$ o $|\sigma_3|$ es el máximo de $(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|)$.

- Si $|\sigma_1| = max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|)$,

$$I \lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}|k|^2}{\langle\sigma_1\rangle\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle} d\tau\ d\tau_2 \tag{3.57}$$

$$\begin{aligned}\langle i\sigma+k^2\rangle^{1-\delta} &= (1+(i\sigma+k^2)^2)^{\frac{1-\delta}{2}} = (1+\sigma^2+k^4)^{\frac{1-\delta}{2}} \\ &\geq (1+k^4)^{\frac{1-\delta}{2}} \geq (1+k^4)^{\frac{1-\delta}{2}} = \langle k\rangle^{2(1-\delta)}\end{aligned} \tag{3.58}$$

(3.58) implica

$$\frac{1}{\langle k\rangle^{2(1-\delta)}} \geq \frac{1}{\langle i\sigma+k^2\rangle^{1-\delta}} \tag{3.59}$$

(3.59), junto con la desigualdad $|k|^2 \leq \langle k\rangle^2$ transforman a (3.57) en

$$I \lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}\langle k\rangle^2}{\langle\sigma_1\rangle\langle k\rangle^{2(1-\delta)}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}d\tau\ d\tau_2$$
$$\lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}\langle k\rangle^{2\delta}}{\langle\sigma_1\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}d\tau\ d\tau_2 \tag{3.60}$$

$$\begin{aligned}\langle\sigma_1\rangle &= \langle\sigma_1\rangle^{1-2\delta}\langle\sigma_1\rangle^{\delta}\langle\sigma_1\rangle^{\delta}\\ &\geq \langle\sigma_1\rangle^{1-2\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} = (1+\sigma_1^2)^{\frac{1-2\delta}{2}}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &= (1+(\tau_1-k_1^3)^2)^{\frac{1-2\delta}{2}}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\geq (1+k_1^6)^{\frac{1-2\delta}{2}}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\geq ((1+k_1^2)^3)^{\frac{1-2\delta}{2}}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} \geq \langle k_1\rangle^{3(1-2\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_2\rangle^{3(1-2\delta)-2\delta}\langle k_1\rangle^{2\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_1\rangle^{2\delta}\langle k_2\rangle^{\frac{3}{2}(1-2\delta)-\delta}\langle k_3\rangle^{\frac{3}{2}(1-2\delta)-\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}.\end{aligned} \tag{3.61}$$

Luego (3.61) y $|k_2| < 1$ transforma a (3.60) en

$$\begin{aligned}I &\lesssim \sum_{A\cap|k_2|\geq 1}\int \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}\langle k\rangle^{2\delta}}{\langle k_1\rangle^{2\delta}\langle k_2\rangle^{\frac{3}{2}(1-2\delta)-\delta}\langle k_3\rangle^{\frac{3}{2}(1-2\delta)-\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\langle\sigma_2\rangle\langle\sigma_3\rangle}d\tau\ d\tau_2\\ &\lesssim \sum_{A\cap|k_2|\geq 1}\int \frac{\langle k_2\rangle^{-1+4\delta-2\epsilon}\langle k_3\rangle^{-1+4\delta-2\epsilon}}{\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}d\tau\ d\tau_2\\ &\lesssim \sum_{k_2}\frac{1}{\langle k_2\rangle^{1+2\epsilon-4\delta}}\sum_{k}\frac{1}{\langle k_3\rangle^{1+2\epsilon-4\delta}}\int\frac{d\tau}{\langle\sigma_2\rangle^{1+\delta}}\int\frac{1}{\langle\sigma_3\rangle^{1+\delta}} \lesssim C\end{aligned} \tag{3.62}$$

2. $|k_1| \gg |k_2| \sim |k_3|$

   En este caso tenemos que

$$|k| \sim |k_1| \gg |k_2| \sim |k_3| \tag{3.63}$$

- Consideremos $|\sigma| = max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|)$,

$$|\sigma| \gtrsim |k_1+k_2||k_1+k_3||k-k_1| \sim \langle k_1\rangle^2|k-k_1| \tag{3.64}$$

$$I \lesssim \frac{|k_1|^2}{\langle i\sigma_1+k_1^2\rangle}\sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}d\tau\ d\tau_2, \tag{3.65}$$

$$\langle i\sigma_1 + k_1^2\rangle = (1+\sigma_1^2+k_1^4)^{1/2} \geq (1+|k_1|^4)^{1/2} \sim 1+|k_1|^2 > |k_1|^2 \quad (3.66)$$

(3.66) transforma a (3.65) en

$$\begin{aligned}
I &\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\
I &\lesssim \sum_{\substack{k,k_2\in A\\ |k-k_1|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\
&\qquad + \sum_{\substack{k,k_2\in A\\ |k-k_1|\geq 1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2
\end{aligned} \quad (3.67)$$

Si $|k-k_1|<1$, $k=k_1$

$$\begin{aligned}
&\sum_{\substack{k,k_2\in A\\ |k-k_1|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim \\
&\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1-4\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim \\
&\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1-4\epsilon}}{\langle i\sigma+k^2\rangle^{1-3\delta}\langle i\sigma+k^2\rangle^{\delta}\langle i\sigma+k^2\rangle^{\delta}\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim \\
&\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1-4\epsilon}}{\langle k_1\rangle^{2(1-3\delta)}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim \\
&\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1-4\epsilon}}{\langle k_2\rangle^{2(1-3\delta)}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim \\
&\sum_{k_2} \frac{1}{\langle k_2\rangle^{1+4\epsilon-6\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C
\end{aligned} \quad (3.68)$$

Si $|k-k_1|\geq 1$,

$$\begin{aligned}
\langle i\sigma+k^2\rangle^{1-\delta} &= \langle i\sigma+k^2\rangle^{1-3\delta}\langle i\sigma+k^2\rangle^{\delta}\langle i\sigma+k^2\rangle^{\delta} \\
&\gtrsim \langle k\rangle^{2(1-3\delta)}|k-k_1|^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} \\
&= \langle k\rangle^{2(1-5\delta)}\langle k_1\rangle^{4\delta}|k-k_1|^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} \\
&\gtrsim \langle k_2\rangle^{\frac{3}{2}(1-5\delta)}\langle k_3\rangle^{\frac{1}{2}(1-5\delta)}\langle k_1\rangle^{4\delta}|k-k_1|^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} \\
&\gtrsim \langle k_2\rangle^{\frac{3}{2}(1-5\delta)}\langle k_3\rangle^{\frac{1}{2}(1-5\delta)}\langle k_1\rangle^{1+\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}
\end{aligned} \quad (3.69)$$

$$\sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_2\rangle^{\frac{3}{2}(1-5\delta)}\langle k_3\rangle^{\frac{1}{2}(1-5\delta)}\langle k_1\rangle^{1+\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{k,k_2\in A}\int_A \frac{d\tau\ d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{15}{2}\delta}\langle k_3\rangle^{2\epsilon-\frac{15}{2}\delta}\langle k_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$
$$\sum_{k,k_2\in A}\int_A \frac{d\tau\ d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{15}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{13}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$
$$\sum_{k_2}\frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{15}{2}\delta}}\sum_{k}\frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{13}{2}\delta}}\int_A\frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}}\int_A\frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C \quad (3.70)$$

por (3.84), (3.68) y (3.70) se tiene que

$$I \lesssim C \quad (3.71)$$

- $|\sigma_1| = max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|)$

$$|\sigma_1| \gtrsim |k_1+k_2||k_1+k_3||k-k_1| \sim \langle k_1\rangle|k-k_1| \quad (3.72)$$

Se tiene

$$I \lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2$$
$$\lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2$$
$$\lesssim \sum_{\substack{k,k_2\in A\\ |k-k_1|<1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2$$
$$+ \sum_{\substack{k,k_2\in A\\ |k-k_1|\geq 1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2. \quad (3.73)$$

Si $|k-k_1|<1$, $k=k_1$ y además

$$\langle i\sigma_1+k_1^2\rangle = \langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}$$
$$\gtrsim \langle i\sigma_1+k_1^2\rangle^{1-3\delta}|k_1|^{2\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} \quad (3.74)$$

$$\langle i\sigma+k^2\rangle^{(1-\delta)} = (1+\sigma^2+k^4)^{\frac{1}{2}(1-\delta)} \sim (1+|k|^2)^{(1-\delta)} \geq |k|^{2(1-\delta)} \quad (3.75)$$

(3.75) y (3.74) tenemos

$$\sum_{\substack{k,k_2\in A\\ |k-k_1|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$

$$\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle \sigma_2\rangle^{1+\delta}\langle \sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \tag{3.76}$$

$$\begin{aligned}\langle i\sigma_1+k_1^2\rangle^{1-3\delta} &= (1+\sigma_1^2+k_1^4)^{\frac{1-3\delta}{2}}\\ &\gtrsim (1+k_1^4)^{\frac{1}{2}(1-3\delta)} \sim (1+k_1^2)^{1-3\delta} = \langle k_1\rangle^{2(1-3\delta)}\end{aligned} \tag{3.77}$$

(3.77) y (3.76) implica que

$$\begin{aligned}&\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle \sigma_2\rangle^{1+\delta}\langle \sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2\\ &\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1-4\epsilon}}{\langle k_1\rangle^{2(1-3\delta)}\langle \sigma_2\rangle^{1+\delta}\langle \sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim\\ &\sum_{k_2} \int_A \frac{\langle k_2\rangle^{1-4\epsilon}}{\langle k_2\rangle^{2(1-3\delta)}\langle \sigma_2\rangle^{1+\delta}\langle \sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim\\ &\sum_{k_2} \frac{1}{\langle k_2\rangle^{1+4\epsilon-6\delta}} \int_A \frac{d\tau_2}{\langle \sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle \sigma_3\rangle^{1+\delta}} < C\end{aligned} \tag{3.78}$$

Si $|k-k_1|\geq 1$

$$\begin{aligned}\langle i\sigma_1+k_1^2\rangle &= \langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\\ &\gtrsim \langle k_1\rangle^{2(1-3\delta)}|k-k_1|^{1-3\delta}\langle k_1\rangle^{2\delta}\langle \sigma_2\rangle^{\delta}\langle \sigma_3\rangle^{\delta}\\ &= \langle k_1\rangle^{2(1-5\delta)}\langle k_1\rangle^{4\delta}|k-k_1|^{1-3\delta}\langle k_1\rangle^{2\delta}\langle \sigma_2\rangle^{\delta}\langle \sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_1\rangle^{2(1-5\delta)}|k-k_1|^{4\delta}|k-k_1|^{1-3\delta}\langle k_1\rangle^{2\delta}\langle \sigma_2\rangle^{\delta}\langle \sigma_3\rangle^{\delta}\\ &= \langle k_1\rangle^{\frac{3}{2}(1-5\delta)}\langle k_1\rangle^{\frac{1}{2}(1-5\delta)}\langle k_1\rangle^{2\delta}|k-k_1|^{1+\delta}\langle \sigma_2\rangle^{\delta}\langle \sigma_3\rangle^{\delta}\end{aligned} \tag{3.79}$$

$$\langle i\sigma+k^2\rangle \gtrsim |k|^{2(1-\delta)} \tag{3.80}$$

(3.79) y (3.80) implican que

$$\sum_{\substack{k,k_2\in A\\|k-k_1|\geq 1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$

$$\sum_{\substack{k,k_2\in A\\|k-k_1|\geq 1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_2\rangle^{\frac{3}{2}(1-5\delta)}\langle k_3\rangle^{\frac{1}{2}(1-5\delta)}|k-k_1|^{1+\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$

$$\sum_{\substack{k,k_2\in A\\|k-k_1|\geq 1}}\int_A \frac{d\tau d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{15}{2}\delta}\langle k_3\rangle^{2\epsilon-\frac{5}{2}\delta}|k-k_1|^{1+\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$

$$\sum_{\substack{k,k_2\in A\\|k-k_1|\geq 1}}\int_A \frac{d\tau d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{15}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{3}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$

$$\sum_{\substack{k_2\\|k-k_1|\geq 1}}\frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{15}{2}\delta}}+\sum_{\substack{k\\|k-k_1|\geq 1}}\frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{3}{2}\delta}}\int_A\frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}}\int_A\frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}}<C \tag{3.81}$$

luego de (3.73), (3.78) y (3.81) tenemos que

$$I \lesssim C \tag{3.82}$$

3. $|k_1|\sim|k_2|\gg|k_3|$

- Si $|\sigma|=max(|\sigma|,|\sigma_1|,|\sigma_2|,|\sigma_3|)$
  de (3.33) tenemos

$$|\sigma|\gtrsim\langle k_1\rangle\langle k_2\rangle|k_1+k_2| \tag{3.83}$$

$$I\lesssim\sum_{k,k_2\in A}\int_A\frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2$$

$$\begin{aligned}I\lesssim&\sum_{\substack{k,k_2\in A\\|k_1+k_2|<1}}\int_A\frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2\\&+\sum_{\substack{k,k_2\in A\\|k_1+k_2|\geq 1}}\int_A\frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2\end{aligned} \tag{3.84}$$

Si $|k_1+k_2|<1$, $k_2=-k_1$

$$\begin{aligned}\langle i\sigma+k^2\rangle^{1-\delta}&\gtrsim\langle i\sigma+k^2\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_2\rangle^{\delta}\\&\gtrsim\langle k_1\rangle^{2(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_2\rangle^{\delta}\\&\gtrsim\langle k_1\rangle^{\frac{1}{2}(1-3\delta)}\langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma_2\rangle^{\delta},\end{aligned} \tag{3.85}$$

entonces

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_k \int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_1\rangle^{\frac{1}{2}(1-3\delta)}\langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 =$$
$$\frac{1}{\langle k_1\rangle^{2\epsilon-\frac{3}{2}\delta}}\sum_k \int_A \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \leq$$
$$\frac{1}{(1+k_1^2)^{\epsilon-\frac{3}{4}\delta}}\sum_k \int_A \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \leq$$
$$M\sum_k \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}}\int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}}\int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C \tag{3.86}$$

Si $|k_1+k_2|\geq 1$

$$\begin{aligned}\langle i\sigma+k^2\rangle^{1-\delta} &= \langle i\sigma+k^2\rangle^{1-3\delta}\langle i\sigma+k^2\rangle^{\delta}\langle i\sigma+k^2\rangle^{\delta}\\ &\gtrsim \langle k_1\rangle^{1-3\delta}\langle k_2\rangle^{1-3\delta}|k_1+k_2|^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_3\rangle^{1-3\delta}\langle k_2\rangle^{1-3\delta-4\delta}\langle k_2\rangle^{4\delta}|k_1+k_2|^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle k_2\rangle^{-\frac{1}{2}(1-3\delta)}\langle k_2\rangle^{1-7\delta}\langle k_2\rangle^{4\delta}|k_1+k_2|^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle k_2\rangle^{-\frac{1}{2}-\frac{11}{2}\delta}|k_1+k_2|^{1+\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\end{aligned} \tag{3.87}$$

de (3.87) se tiene

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_3\rangle^{\frac{3}{2}(1-3\delta)}\langle k_2\rangle^{-\frac{1}{2}-\frac{11}{2}\delta}|k_1+k_2|^{1+\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}} \int_A \frac{d\tau\ d\tau_2}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_2\rangle^{2\epsilon-\frac{11}{2}\delta}|k_1+k_2|^{1+\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$
$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}} \int_A \frac{d\tau\ d\tau_2}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$
$$\sum_{\substack{k_2\\ |k_1+k_2|\geq 1}} \frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}}\sum_{\substack{k\\ |k_1+k_2|\geq 1}} \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}}\int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}}\int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C \tag{3.88}$$

(3.84), (3.86) y (3.88), implican que

$$I \lesssim C \tag{3.89}$$

- Si $|\sigma_1| = max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|)$

$$\begin{aligned}
I &\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma_1+k_1^2\rangle\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\, d\tau\ d\tau_2 \\
&\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\, d\tau\ d\tau_2 \\
&\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}|k|^2}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon}\langle i\sigma+k^2\rangle^{\frac{1}{4}(3+4\epsilon)}\langle i\sigma+k^2\rangle^{-\delta}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle\sigma_3\rangle}\, d\tau\ d\tau_2 \\
&\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}\langle k\rangle^2}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle k\rangle^{2(\frac{1}{4}(3+4\epsilon))}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle\sigma_3\rangle}\, d\tau\ d\tau_2
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}\langle k\rangle^2}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle k\rangle^{\frac{3}{2}+2\epsilon}\langle i\sigma_1+k_1^2\rangle\langle i\sigma_2+k_2^2\rangle\langle\sigma_3\rangle}\, d\tau\ d\tau_2 \\
&\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}\langle k_1\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\langle i\sigma_2+k_2^2\rangle\langle\sigma_3\rangle}\, d\tau\ d\tau_2
\end{aligned} \tag{3.90}$$

$$\begin{aligned}
\langle i\sigma_2+k_2^2\rangle &= \langle i\sigma_2+k_2^2\rangle^{3/4+\epsilon}\langle i\sigma_2+k_2^2\rangle^{1/4-\epsilon} \\
&\geq \langle i\sigma_2+k_2^2\rangle^{3/4+\epsilon}\langle k_2\rangle^{1/2-2\epsilon}
\end{aligned} \tag{3.91}$$

(3.91) y (3.90) implica

$$\begin{aligned}
I &\lesssim \sum_{k,k_2\in A} \int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\, d\tau\ d\tau_2 \\
&\lesssim \sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1}} \int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\, d\tau\ d\tau_2 \\
&\quad + \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}} \int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\, d\tau\ d\tau_2
\end{aligned} \tag{3.92}$$

Si $|k_1+k_2|<1$, $k_1=-k$, supongamos que $|\sigma_2|\geq|\sigma_1|$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{k}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_1\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{k}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle\sigma\rangle^{1-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \tag{3.93}$$

$$\begin{aligned}\langle i\sigma_1+k_1^2\rangle &= \langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle i\sigma_1+k_1^2\rangle^{2\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\\ &\geq \langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle\sigma\rangle^{2\delta}\langle\sigma_3\rangle^{\delta}\\ &\geq \langle k_1\rangle^{2(1-3\delta)}\langle\sigma\rangle^{2\delta}\langle\sigma_3\rangle^{\delta}\\ &\gtrsim \langle k_1\rangle^{1/2-3/2\delta}\langle k_3\rangle^{3/2(1-3\delta)}\langle\sigma\rangle^{2\delta}\langle\sigma_3\rangle^{\delta}\end{aligned} \tag{3.94}$$

(3.94) y (3.93) implican

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{k}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle\sigma\rangle^{1+\delta}\langle k_1\rangle^{1/2-3/2\delta}\langle k_3\rangle^{3/2-9/2\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2$$
$$=\frac{1}{\langle k_1\rangle^{\epsilon-\frac{3}{2}\delta}}\sum_{k}\int_A \frac{d\tau\ d\tau_2}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}}\langle\sigma\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\leq C \tag{3.95}$$

Si $|k_1+k_2|\geq 1$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle\sigma\rangle^{\frac{1}{4}-\epsilon-\delta}\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_1\rangle\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle\sigma\rangle^{1-\delta}\langle\sigma_1\rangle\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim$$
$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle\sigma\rangle^{1+\delta}\langle\sigma_1\rangle^{1-3\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \tag{3.96}$$

$$\begin{aligned}\langle\sigma_1\rangle^{1-3\delta} &\gtrsim \langle k_1\rangle^{2(1-3\delta)}|k_1+k_2|^{3/4-3\delta}\\ &\gtrsim \langle k_1\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{3}{2}-6\delta}|k_1+k_2|^{1+2\epsilon-3\delta}\end{aligned} \tag{3.97}$$

(3.96) y (3.97) implican

$$\begin{aligned}&\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{\frac{1}{4}-\epsilon-\delta}\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle^{3/4+\epsilon}\langle\sigma_3\rangle}\ d\tau\ d\tau_2 \lesssim\\ &\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{\langle k_1\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle\sigma\rangle^{1+\delta}\langle k_1\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{3}{2}-6\delta}|k_1+k_2|^{1+2\epsilon-3\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim\\ &\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1}}\int_A \frac{d\tau\ d\tau_2}{\langle k_3\rangle^{1+2\epsilon-6\delta}|k_1+k_2|^{1+2\epsilon-3\delta}\langle\sigma\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim\\ &\sum_{\substack{k_2\\ |k_1+k_2|\geq 1}}\frac{1}{|k_1+k_2|^{1+2\epsilon-3\delta}}\sum_{\substack{k\\ |k_1+k_2|\geq 1}}\frac{1}{\langle k_3\rangle^{1+2\epsilon-6\delta}}\int_A\frac{d\tau\ d\tau_2}{\langle\sigma\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\end{aligned} \tag{3.98}$$

4. $|k_1|\sim|k_1|\sim|k_3|$

- Si $|\sigma| = max(|\sigma|,|\sigma_1|,|\sigma_2|,|\sigma_3|)$

$$|\sigma| \gtrsim |k-k_1||k-k_2||k-k_3| \tag{3.99}$$

a) Si $|k_1|\gg|k|$

$$\begin{aligned}\langle i\sigma+k^2\rangle^{1-\delta} &\geq \langle\sigma\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\geq \langle k_1\rangle^{1-3\delta}\langle k_2\rangle^{1-3\delta}\langle k_3\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\geq \langle k_2\rangle^{\frac{1}{2}-\frac{3}{2}\delta}\langle k_3\rangle^{\frac{1}{2}-\frac{3}{2}\delta}\langle k_2\rangle^{1-3\delta}\langle k_3\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\ &\geq \langle k_2\rangle^{\frac{3}{2}-\frac{9}{2}\delta}\langle k_3\rangle^{\frac{3}{2}-\frac{9}{2}\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\end{aligned} \tag{3.100}$$

$$\begin{aligned}I &\lesssim \sum_{k,k_2\in A}\int_A\frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2\\ &\lesssim \sum_{k,k_2\in A}\int_A\frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_2\rangle^{3/2-\frac{9}{2}\delta}\langle k_3\rangle^{3/2-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2\\ &= \sum_{k,k_2\in A}\int_A\frac{d\tau\ d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\\ &= \sum_{k_2}\frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}}\sum_{k}\frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}}\int_A\frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}}\int_A\frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} \leq C\end{aligned} \tag{3.101}$$

b) Si $|k| \backsim |k_1|$

$$|\sigma| \gtrsim |k_1+k_3||k_1+k_2||k_2+k_3| \tag{3.102}$$

dado que $k = k_1 + k_2 + k_3$, si los k_i tienen el mismo signo se tiene

$$|\sigma| \gtrsim \langle k_1\rangle\langle k_2\rangle\langle k_3\rangle \tag{3.103}$$

(3.103) y (3.100) implican que

$$\begin{aligned} I &\lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\ &\lesssim \sum_{k,k_2\in A}\int_A \frac{d\tau\ d\tau_2}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} < C \end{aligned} \tag{3.104}$$

Supongamos que k_1 tiene signo diferente con k_2 y k_3, entonces tenemos

$$|\sigma| \gtrsim |k_1+k_2||k_1+k_3|\langle k_1\rangle \tag{3.105}$$

$$\begin{aligned} I &\lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\ &= \sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\ &+ \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|<1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\ &+ \sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \\ &+ \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}}\int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \end{aligned} \tag{3.106}$$

Analicemos por separado cada una de estas sumas,

$$\begin{aligned} \langle i\sigma+k^2\rangle^{1-\delta} \geq \langle i\sigma+k^2\rangle^{1-3\delta}\langle i\sigma+k^2\rangle^{\delta}\langle i\sigma+k^2\rangle^{1-\delta} \\ \langle k\rangle^{2(1-3\delta)}\langle\sigma_2\rangle^{\delta}\langle\sigma\rangle^{\delta} \end{aligned} \tag{3.107}$$

por tanto

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k\rangle^{2(1-3\delta)}\langle \sigma_2\rangle^{1+\delta}\langle \sigma\rangle^{1+\delta}}\ d\tau\ d\tau_2 =$$

$$\sum_{\substack{k\\ |k_1+k_2|<1\\ |k_1+k_3|<1}} \frac{\langle k\rangle^{1-4\delta}}{\langle k\rangle^{2(1-\delta)}} \int_A \frac{d\tau_2}{\langle \sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle \sigma_3\rangle^{1+\delta}} \leq C \tag{3.108}$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|<1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 =$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|<1}} \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_1\rangle^{\frac{1}{2}-2\epsilon}}{\langle k\rangle^{2(1-3\delta)}} \int_A \frac{d\tau_2}{\langle \sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle \sigma_3\rangle^{1+\delta}} \lesssim$$

$$\lesssim \sum_{\substack{k_2\\ |k_1+k_2|\geq 1\\ |k_1+k_3|<1}} \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_1\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_2\rangle^{\frac{3}{2}(1-3\delta)}\langle k_1\rangle^{\frac{1}{2}(1-3\delta)}} \int_A \frac{d\tau_2}{\langle \sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle \sigma_3\rangle^{1+\delta}}$$

$$\lesssim \frac{1}{\langle k_1\rangle^{2\epsilon-3\delta}} \sum_{\substack{k_2\\ |k_1+k_2|\lesssim 1\\ |k_1+k_3|<1}} \frac{1}{\langle k_2\rangle^{1+3\epsilon-\frac{9}{2}\delta}} \int_A \frac{d\tau_2}{\langle \sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle \sigma_3\rangle^{1+\delta}} \leq C \tag{3.109}$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \leq$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}} \frac{\langle k_1\rangle^{\frac{1}{2}-2\epsilon}\langle k\rangle^{\frac{1}{2}-2\epsilon}}{\langle k\rangle^{2(1-3\delta)}} \int_A \frac{d\tau_2}{\langle \sigma_2\rangle^{1+\delta}} \frac{d\tau}{\langle \sigma_3\rangle^{1+\delta}} \lesssim$$

$$\lesssim \frac{\langle k_1\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_1\rangle^{\frac{1}{2}-\frac{3}{2}\delta}} \sum_{\substack{k\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}} \frac{\langle k\rangle^{\frac{1}{2}-2\epsilon}}{\langle k\rangle^{\frac{3}{2}(1-3\delta)}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}}$$

$$\lesssim \frac{1}{\langle k_1\rangle^{2\epsilon-\frac{3}{2}\delta}} \sum_{\substack{k\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}} \frac{1}{\langle k\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} \leq C \quad (3.110)$$

$$\begin{aligned}
|\sigma|^{1-\delta} &\gtrsim \langle\sigma\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\
&\gtrsim |k_1+k_2|^{1-3\delta}|k_1+k_3|^{1-3\delta}\langle k_1\rangle^{1-3\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\
&= |k_1+k_2|^{1-3\delta}|k_1+k_3|^{1-3\delta}\langle k_1\rangle^{1-3\delta}\langle k_1\rangle^{1-11\delta}\langle k_1\rangle^{4\delta}\langle k_1\rangle^{4\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\
&\gtrsim \langle k_2\rangle^{1-3\delta}\langle k_3\rangle^{1-3\delta}\langle k_1\rangle^{1-11\delta}\langle k_2\rangle^{4\delta}\langle k_3\rangle^{4\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta}\\
&\gtrsim \langle k_2\rangle^{1+\delta}\langle k_3\rangle^{1+\delta}\langle k_1\rangle^{1-11\delta}\langle\sigma_2\rangle^{\delta}\langle\sigma_3\rangle^{\delta} \qquad (3.111)
\end{aligned}$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle i\sigma+k^2\rangle^{1-\delta}\langle i\sigma_2+k_2^2\rangle\langle i\sigma_3+k_3^2\rangle}\ d\tau\ d\tau_2 \lesssim$$

$$\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{1/2-2\epsilon}\langle k_3\rangle^{1/2-2\epsilon}}{\langle k_2\rangle^{1+\delta}\langle k_3\rangle^{1+\delta}\langle k_1\rangle^{1-11\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau\ d\tau_2 \lesssim$$

$$\begin{aligned}
&\lesssim \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \frac{1}{\langle k_2\rangle^{\frac{1}{2}+2\epsilon+\delta}\langle k_3\rangle^{\frac{1}{2}+2\epsilon+\delta}\langle k_2\rangle^{\frac{1}{2}-\frac{11}{2}\delta}\langle k_3\rangle^{\frac{1}{2}-\frac{11}{2}\delta}} \int_A \frac{d\tau\ d\tau_2}{\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\\
&\lesssim \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} \leq C
\end{aligned} \qquad (3.112)$$

- Si $|\sigma_1| = max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|)$

$$|\sigma_1| \gtrsim |k_1+k_2||k_1+k_3|\langle k_1\rangle \qquad (3.113)$$

$$
\begin{aligned}
I &\lesssim \sum_{k,k_2\in A}\int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ \ d\tau_2 \lesssim \\
&= \sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}}\int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ \ d\tau_2 \\
&\quad + \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|<1}}\int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ \ d\tau_2 \\
&\quad + \sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}}\int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ \ d\tau_2 \\
&\quad + \sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}}\int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ \ d\tau_2
\end{aligned}
\tag{3.114}
$$

Analicemos cada una de estas sumas por separado
Si $|k_1+k_2|<1$y $|k_1+k_3|<1$, entones $k_1=-k_2$ y $k_1=-k_3$, además

$$
\begin{aligned}
\langle i\sigma_1+k_1^2\rangle &= \langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta} \\
&= \langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle k_1\rangle^{2\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}\langle i\sigma_1+k_1^2\rangle^{\delta}
\end{aligned}
\tag{3.115}
$$

entonces tenemos

$$
\begin{aligned}
&\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}}\int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau\ \ d\tau_2 \lesssim \\
&\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}}\int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}d\tau\ d\tau_2 \lesssim \\
&\sum_{\substack{k,k_2\in A\\ |k_1+k_2|<1\\ |k_1+k_3|<1}}\int_A \frac{\langle k_1\rangle^{\frac{1}{2}-2\epsilon}\langle k_1\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_1\rangle^{2(1-3\delta)}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}d\tau\ d\tau_2 \lesssim
\end{aligned}
$$

$$\lesssim \sum_{\substack{k \\ |k_1+k_2|<1 \\ |k_1+k_3|<1}} \int_A \frac{d\tau \ d\tau_2}{\langle k\rangle^{1+4\epsilon-6\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}$$

$$\lesssim \sum_{\substack{k \\ |k_1+k_2|<1 \\ |k_1+k_3|<1}} \frac{1}{\langle k\rangle^{1+4\epsilon-6\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C \tag{3.116}$$

Si $|k_1 + k_3 < 1$, tenemos que $k_1 = -k_3$, por tanto $k = k_2$, de esta manera tenemos

$$\sum_{\substack{k,k_2\in A \\ |k_1+k_2|\geq 1 \\ |k_1+k_3|<1}} \int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1 + k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle} \ d\tau \ \ d\tau_2 \lesssim$$

$$\sum_{\substack{k \\ |k_1+k_2|\geq 1 \\ |k_1+k_3|<1}} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1 + k_1^2\rangle^{1-3\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \ d\tau \ \ d\tau_2 \lesssim$$

$$\sum_{\substack{k \\ |k_1+k_2|\geq 1 \\ |k_1+k_3|<1}} \int_A \frac{d\tau \ \ d\tau_2}{\langle k\rangle^{1+4\epsilon-6\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$

$$\sum_{\substack{k \\ |k_1+k_2|\geq 1 \\ |k_1+k_3|<1}} \frac{1}{\langle k\rangle^{1+4\epsilon-6\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C \tag{3.117}$$

Si $k_1 + k_2 < 1$, $K_1 = -k_2$, $k_3 = k$, entonces tenemos que

$$\sum_{\substack{k,k_2\in A \\ |k_1+k_2|<1 \\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1 + k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle} \ d\tau \ \ d\tau_2 \lesssim$$

$$\sum_{\substack{k,k_2\in A \\ |k_1+k_2|<1 \\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_1\rangle^{2(1-3\delta)}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \ d\tau \ \ d\tau_2 \lesssim$$

$$\sum_{\substack{k \\ |k_1+k_2|<1 \\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_1\rangle^{\frac{1}{2}-2\epsilon}\langle k\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_1\rangle^{\frac{3}{2}(1-3\delta)}\langle k_1\rangle^{\frac{1}{2}(1-3\delta)}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} d\tau \ \ d\tau_2 \lesssim$$

$$\tag{3.118}$$

$$\lesssim \frac{1}{\langle k_1\rangle^{2\epsilon-\frac{3}{2}\delta}} \sum_{\substack{k\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}} \int_A \frac{d\tau \quad d\tau_2}{\langle k\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}} \lesssim$$

$$\lesssim \frac{1}{\langle k_1\rangle^{2\epsilon-\frac{3}{2}\delta}} \sum_{\substack{k\\ |k_1+k_2|<1\\ |k_1+k_3|\geq 1}} \frac{1}{\langle k\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C \tag{3.119}$$

Si $|k_1+k_2|\geq 1$ y $|k_1+k_3|\geq 1$,

$$\begin{aligned}\langle i\sigma_1+k_1^2\rangle^{1-3\delta} &\gtrsim |k_1+k_2|^{1-3\delta}|k_1+k_2|^{1-3\delta}\langle k_1\rangle^{1-3\delta}\\ &\gtrsim |k_1+k_2|^{1-3\delta}|k_1+k_2|^{1-3\delta}\langle k_1\rangle^{1-11\delta}\langle k_1\rangle^{1-4\delta}\langle k_1\rangle^{1-4\delta}\\ &\quad \langle k_1\rangle^{1-11\delta}\langle k_2\rangle^{1+\delta}\langle k_1\rangle^{1+\delta}\end{aligned} \tag{3.120}$$

$$\begin{aligned}&\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k\rangle^{2\delta}\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle}\ d\tau \quad d\tau_2 \lesssim\\ &\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle i\sigma_1+k_1^2\rangle^{1-3\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau \quad d\tau_2 \lesssim\\ &\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_3\rangle^{1+\delta}\langle k_2\rangle^{1+\delta}\langle k_1\rangle^{1-11\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau \quad d\tau_2 \lesssim\\ &\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{\langle k_2\rangle^{\frac{1}{2}-2\epsilon}\langle k_3\rangle^{\frac{1}{2}-2\epsilon}}{\langle k_2\rangle^{1+\delta}\langle k_3\rangle^{1+\delta}\langle k_2\rangle^{\frac{1}{2}(1-11)\delta}\langle k_3\rangle^{\frac{1}{2}(1-11)\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau \quad d\tau_2 \lesssim\\ &\sum_{\substack{k,k_2\in A\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \int_A \frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}\langle\sigma_2\rangle^{1+\delta}\langle\sigma_3\rangle^{1+\delta}}\ d\tau \quad d\tau_2 \lesssim\\ &\sum_{\substack{k_2\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \frac{1}{\langle k_2\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \sum_{\substack{k\\ |k_1+k_2|\geq 1\\ |k_1+k_3|\geq 1}} \frac{1}{\langle k_3\rangle^{1+2\epsilon-\frac{9}{2}\delta}} \int_A \frac{d\tau_2}{\langle\sigma_2\rangle^{1+\delta}} \int_A \frac{d\tau}{\langle\sigma_3\rangle^{1+\delta}} < C\end{aligned} \tag{3.121}$$

Luego por lo obtenido en los items 1,2,3 y 4 tenemos el resultado.

□

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