Non-commutative differential calculus of some algebras of polynomial type having PBW bases

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Cálculo differencial no conmmutativo de algunas álgebras de tipo polinomial con bases PBW

Abstract: In this work, we study the notion of differential calculus associated to an associative algebra, from its origin in manifolds geometry, to some generalizations in non commutative differential geometry. In particular, we inquire the notion of differentially smoothness of an algebra, which treats about the existence of differential calculus structures that satisfies conditions relative to the Gelfand-Kirillov dimension of the base algebra, a condition of connectedness over the differential, and the existence of a volume form that allow to construct isomorphisms between the homogeneous sets of forms and the dual of these sets, such as in manifolds theory. We also study the Brzezinski's differential calculus, which is a differential calculus constructed from a finite set of skew derivations, and the Brzezinski's integral calculus, that is a pair of a cokernel and a hom-connection that induces a complex of integral forms over the Brzezinski's differential calculus. Finally, we study automorphisms and skew derivations of some 3-dimensional diffusion algebras, generalized Weyl algebras and skew polynomial algebras, which are objects having PBW bases.

Resumen: En este trabajo, estudiamos la noción de cálculo diferencial asociado a un álgebra asociativa, desde su origen en la geometría de variedades, hasta algunas generalizaciones en la geometría diferencial no conmutativa. En particular, investigamos la noción de álgebra diferencialmente suave, que consiste en la existencia de estructuras de cálculo diferencial que satisfacen condiciones relativas a la dimensión de Gelfand-Kirillov del álgebra base, una condición de conexidad sobre la diferencial, y la existencia de una forma de volumen que permite construir isomorfismos entre los conjuntos homogéneos de formas y el dual de estos conjuntos, tal cual como en la teoría de variedades. También estudiamos el cálculo diferencial de Brezezinski, el cual es un cálculo diferencial construido a partir de un conjunto finito de derivaciones torcidas, y el cálculo integral de Brzezinski, que consta de una pareja de un conúcleo y una conexión-hom que permite inducir un complejo de formas integrales sobre el cálculo differencial de Brzezinski. Finalmente, estudiamos automorfismos y derivaciones torcidas de algunas álgebras de difusión, álgebras de Weyl generalizadas y álgebras polinomiales torcidas que son 3-dimensionales, las cuales son objetos que poseen bases PBW.

Keywords: Noncommutative geometry, differentially smooth, integral calculus, Brzezinski's calculus, diffusion algebra, skew derivation.

Palabras clave: Geometría no conmutativa, diferenciablemente suave, cálculo integral, cálculo de Brzezinski, álgebra de difusión, derivación torcida.

Dedicatory

To my parents, Rosalba Santiago Ladino and Enrique Sarmiento Pineda

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Contents

Introduction

It is possible to give an interpretation of a phenomenon with the following two objects: space and change notions; the first one, to fix the place where the objects live, and the second one, to understand the interaction between the objects in the space. In the manifolds geometry, these roles are played by the Euclidean locality of charts, that fix the local coordinates systems, and the differentiability that allows the existence of vector fields, and therefore, the notion of movement. Frequently, in the study of this theory, we observe that these treatments are governed by purely algebraic objects, which let an issue in the air: how many can we "algebraize" this theory? In the literature, we found many different options to solve this question, being this one of the bigger problems of the noncommutative geometry, the *non-unanimity*. For instance, we found different treatments in Connes [Con85], Dubois et al. [DV88], [DVKM90] and Schelter [Sch86], in the 80's, or in Brzeziński [Brz14], [Brz16b], [Brz16a], [BL18], [AB18] or Dubois et al., in [DVKMM01], in the latest decades, among many others. In these algebraic descriptions, we have graded algebras such that the homogeneous components modules form a complex of modules which boundary operator satisfies a graded Leibniz's rule, that generalizes the set of differential forms of a smooth manifold. This structure plays the key role of the change notion. In this work, we are interested in algebras having PBW bases, a set of generators that play the role of a coordinate system in the noncommutative context, covering the notion of space. Then, with the aim of studying differential calculus structures related to algebras with PBW bases, first, we review some constructions of differential calculus over general noncommutative algebras, and later, we study objects involved in differential calculus such as automorphisms or skew derivations of noncommutative algebras having PBW bases. We do this by studying some ideas that we found in literature, and developing original results that we found in the realization of this work. It is important to say that all these results have been submitted for publication.

We constructed an extended bibliography in the process of clarify most of the concepts treated in this document, and we consider that the main texts that we study in the realization of this work are [DVKM90], [Brz14], [Brz16b], [BL18], [AB18], [IPR01] and [Art15].

Next, we present the structure of the thesis:

In the first chapter, we study algebraic generalizations of differential geometry objects that we can obtain for any $\mathbb{K}$-algebra, like differential operator modules, the Lie algebra of derivations, the jet modules, the universal and the Chevalley-Eilenberg differential calculus, among others. All of these objects appear in the pursuit of differential calculus structures over an algebra that work as well known objects in differential geometry (see Section 1.1). In order, to describe a complete generalization of the manifolds calculus situation, we present the Brzeziński's calculus in Section 1.2, which posses a differential and an integral part that are constructed from the skew derivations of the algebra. This Brzeziński's calculus is a structure that allows us to define the smoothness of an affine algebra in Section 1.3. As it was mentioned in the Connes's non commutative differential geometry [Con85], we close the chapter showing a result that presents an explicit connection between the differential structure of an algebra and trace operators in Section 1.4.

In the second chapter, we study the principal tool of Brzeziński's calculus, the skew derivations, over three types of noncommutative algebras. In Section 2.1, generalized Weyl algebras, where we present an example of Brzeziński's calculus construction and a characterization of σ-skew derivations over some extended automorphisms of generalized Weyl algebras; in Section 2.2, diffusion algebras, where we track their origins from an specific experiment of stochastic flow of particles. We separate these algebras in two types and we obtain, necessary conditions of derivations and graded automorphisms of these algebras. Skew polynomial algebras in Section 2.3, where we present a classification theorem that splits their in either Ore extensions or generalized Weyl algebras; we characterize some extended automorphisms of the Ore extensions cases. Finally, in Section 2.4, we study the skew derivations, objects that we have considered in the previous sections, but in a more general algebras that involve the diffusion algebras, generalized Weyl algebras and skew polynomial algebras. We present a counter argument and we discuss about one proposition established in [Art15].

Our original results are the following: in Section 2.2.2, we describe the inner derivations of the generators of diffusion algebras in both types, in terms of its PBW bases in Remark 2.2.13 and, to do that, we obtained the Lemma 2.2.9, Proposition 2.2.10 and Proposition 2.2.12; in Section 2.2.5, in Propositions 2.2.23, 2.2.24, 2.2.25, 2.2.26 and 2.2.27, and Corollary 2.2.28, we developed conditions of derivations and skew derivations of diffusion algebras of type 2 of two generators, with aim to obtain conditions to discard skew derivations, in order to guarantee orthogonal systems of skew derivations when the automorphisms are linear; in all the results of Section 2.3.1, we describe extended endomorphisms and automorphisms of 3-dimensional skew polynomial algebras that are Ore extensions, and summarized in Remark 2.3.25; also, we point out the differentially smoothness of the most of 3-dimensional skew algebras in Section 2.3.2, as we do in Examples 1.3.35, 1.3.40, 1.3.43 and 1.3.46, and we close the Section 2.3.3 with a generalization of these ideas in our Theorem 2.3.39. In Section 2.4, in Proposition 2.4.7, we correct a mistake that we found in [Art15].

Throughout this document, we use the following notation:

- $\mathbb{K}$ denotes a field of characteristic zero.
- $\mathbb{R}$ denotes de the set of real numbers.
- $\mathcal{A}$ denotes a $\mathbb{K}$-algebra.
- $\mathcal{D}$ denotes a diffusion algebra.

- M represents a compact smooth manifold.
- σ denotes an algebra automorphism.
- P is used to denote an auxiliary number in a brief argument.
- $M_n(\mathbb{C})$ denotes the algebra of matrix of size $n \times n$ with entries in the set of complex numbers $\mathbb{C}$.
- $\operatorname{tr}(A)$ denotes the trace of a square matrix A.
- $sl(n, \mathbb{C})$ denotes the set of square matrix of size $n \times n$ with complex entries and trace zero.
- $\operatorname{Hom}_A(B, C)$ denotes the set of A-linear maps with domain B and codomain C.
- $\operatorname{End}_{\mathbb{K}}(\mathcal{A})$ denotes the $\mathbb{K}$-linear endomorphism of $\mathcal{A}$.
- δ_a denotes the inner skew derivation of an element a.
- ∇ denotes a connection of an algebra.
- $\Omega(\mathcal{A})$ denotes a differential calculus over an algebra $\mathcal{A}$.
- $\Omega^j(\mathcal{A})$ denotes the homogeneous components of graded j of a differential calculus over $\mathcal{A}$.
- d denotes the differential of a differential calculus.
- $R[w; \sigma, \delta]$ denotes an Ore extension of R.

CHAPTER 1

Algebraic calculus

1.1 Differential calculus

In this section, we consider different algebraic treatments of differential calculus over commutative and non-commutative $\mathbb{K}$-algebras. First in the commutative case, we give the geometrical inspiration of these objects from the study of differential operators. In the non-commutative case, we present different options of differential calculus such as the universal differential calculus, the Chevalley-Eilenberg differential calculus (also known as the cochain complex of a Lie algebra) and the Brzezinski's differential calculus, among others. The most of this section was revised from [GMS05] and the geometrical interpretation for commutative structures was consulted in [Nes03]. We start presenting a key notion into all this document.

Definition 1.1.1. ([Brz16b], p. 7) Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, Ω an $\mathcal{A}$-bimodule and $d : \mathcal{A} \to \Omega$ a $\mathbb{K}$-linear map. We say that (Ω, d) is a *first-order differential calculus* on $\mathcal{A}$, if:

1. For all $a, b \in \mathcal{A}$, $d(ab) = ad(b) + d(a)b$.
2. Ω satisfies the density condition denoted by $\Omega = \mathcal{A}d(\mathcal{A})$: For all $\omega \in \Omega$, there exist finite $a_i, b_i \in \mathcal{A}$ such that $\omega = \sum_i b_i d(a_i)$.

1.1.1 Commutative case

In this subsection, we show how the concept of derivation appears from the study of differential operators over a commutative $\mathbb{K}$-algebra, and how the $\mathcal{A}$-module of derivations can decompose the first order differential operators. We finish with the definition of the universal differential calculus over a commutative algebra.

Let $\mathcal{A}$ be a commutative $\mathbb{K}-$algebra (in [GMS05] the treatment is realized assuming $\mathbb{K}$ a ring) and let P and Q be two $\mathcal{A}$-modules. The $\mathbb{K}-$module of $\mathbb{K}$-module homomorphisms $\mathrm{Hom}_{\mathbb{K}}(P, Q)$ can be endowed with two $\mathcal{A}-$modules structures in the following way:

$$(a\phi)(p) := a\phi(p), \qquad (\phi \cdot a)(p) := \phi(ap), \qquad \phi \in \mathrm{Hom}_{\mathbb{K}}(P, Q), \qquad a \in \mathcal{A}, \quad p \in P \tag{1.1.1}$$

With this, we define for each $a \in \mathcal{A}$,

$$\delta_a \phi := a\phi - \phi \cdot a : P \to Q.$$

We have that $a\phi - \phi \cdot a \in \mathrm{Hom}_{\mathbb{K}}(P, Q)$ because $a\phi$ is a $\mathbb{K}$-map, and for all $p_1, p_2 \in P$ and all $\alpha \in \mathbb{K}$ we have that,

$$\begin{aligned}(\phi \cdot a)(p_1 + \alpha p_2) &= \phi(a(p_1 + \alpha p_2)) = \phi(a(p_1)) + \phi(\alpha p_2)) \\ &= \phi(a(p_1)) + \phi(a\alpha p_2)) = (\phi \cdot a)(p_1) + \alpha(\phi \cdot a)(p_2),\end{aligned}$$

which means that $\phi \cdot a \in \mathrm{Hom}_{\mathbb{K}}(P, Q)$, and as $\mathrm{Hom}_{\mathbb{K}}(P, Q)$ is a $\mathbb{K}$-module, we get that $a\phi - \phi \cdot a \in \mathrm{Hom}_{\mathbb{K}}(P, Q)$.

Definition 1.1.2 ([GMS05], p. 24)**.** An element $\Delta \in \mathrm{Hom}_{\mathbb{K}}(P, Q)$ is called a *$Q-$valued differential operator* of order s on P, if $\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$, for any tuple of $s + 1$ elements $a_0, ..., a_s$ of $\mathcal{A}$. The set of $Q-$valued differential operators of order s on P is noted $\mathrm{Diff}_s(P, Q)$.

Example 1.1.3. If $\mathbb{K} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}[x]$, $P = \mathbb{R}[x]$ and $Q = \mathbb{R}[x, y]$, a $\mathbb{R}[x, y]-$valued differential operator of order 0 on $\mathbb{R}[x]$ is a $\Delta \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[x], \mathbb{R}[x, y])$ such that for all $f \in \mathbb{R}[x] = A$, and for all $g \in \mathbb{R}[x] = P$, it satisfies $\delta_f(\Delta)(g) = f\Delta(g) - \Delta(fg) = 0$, i.e., $\Delta(fg) = f\Delta(g)$, i.e., Δ is an $\mathbb{R}[x]$-linear map. Now, if Δ is a differential operator of order 2, then, for all $f, g, p \in \mathbb{R}[x]$, we obtain that,

$$\begin{aligned}0 &= \delta_f \circ \delta_g \circ \delta_p(\Delta) \\ &= \delta_f \circ \delta_g(p\Delta - \Delta \cdot p) \\ &= \delta_f(g(p\Delta - \Delta \cdot p) - (p\Delta - \Delta \cdot p) \cdot g) \\ &= \delta_f(gp\Delta - g(\Delta \cdot p) - (p\Delta) \cdot g + \Delta \cdot (pg)) \\ &= f(gp\Delta - g(\Delta \cdot p) - (p\Delta) \cdot g + \Delta \cdot (pg)) \\ &- (gp\Delta - g(\Delta \cdot p) - (p\Delta) \cdot g + \Delta \cdot (pg)) \cdot f \\ &= fgp\Delta - fg(\Delta \cdot p) - f(p\Delta) \cdot g + f\Delta \cdot (pg) \\ &- (gp\Delta) \cdot f + (g(\Delta \cdot p)) \cdot f + ((p\Delta) \cdot g) \cdot f - \Delta \cdot (pgf).\end{aligned}$$

Then, for all $q \in \mathbb{R}[x]$, we get,

$$\begin{aligned}0 &= [fgp\Delta - fg(\Delta \cdot p) - f(p\Delta) \cdot g + f\Delta \cdot (pg) \\ &- (gp\Delta) \cdot f + (g(\Delta \cdot p)) \cdot f + ((p\Delta) \cdot g) \cdot f - \Delta \cdot (pgf)](q) \\ &= fgp\Delta(q) - fg\Delta(pq) - fp\Delta(gq) + f\Delta(pgq) \\ &- gp\Delta(fq) + g\Delta(pfq) + p\Delta(gfq) - \Delta(pgfq).\end{aligned} \tag{1.1.2}$$

Reorganizing expression (1.1.2), we note that

$$\begin{aligned}0 = fgp\Delta(q) &- [fg\Delta(pq) + fp\Delta(gq) + gp\Delta(fq)] \\ &+ [f\Delta(pgq) + g\Delta(pfq) + p\Delta(gfq)] - \Delta(pgfq)\end{aligned}$$

Remark 1.1.4. In general, the zero order differential operators satisfies,

$$\delta_a(\Delta)(p) = a\Delta(p) - \Delta(ap) = 0, \quad a \in \mathcal{A}.$$

This means that Δ is an $\mathcal{A}-$module homomorphism, i.e., $\mathrm{Diff}_0(\mathcal{A},Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},Q)$. Then, if $P=\mathcal{A}$, we can describe completely each $\Delta \in \mathrm{Diff}_0(\mathcal{A},Q)$ with $\Delta(1)$ and obtain an $\mathcal{A}-$module isomorphism between Q and $\mathrm{Diff}_0(\mathcal{A},Q)$ such that we identify $q \in Q$ with $\Delta \in \mathrm{Diff}_0(\mathcal{A},Q)$ that $\Delta(1)=q$. Then, we can say that $Q=$Diff$_0(\mathcal{A},Q)$.

As for any $a,b \in \mathcal{A}$, $p \in P$ and $\Delta \in \mathrm{Hom}_{\mathbb{K}}(P,Q)$,

$$\begin{aligned}\delta_b \circ \delta_a(\Delta)(p) &= \delta_b(a\Delta - \Delta\cdot a)(p)\\ &= [b(a\Delta - \Delta\cdot a) - (a\Delta - \Delta \cdot a)\cdot b](p)\\ &= ba\Delta(p) - b(\Delta\cdot a)(p) - [(a\Delta)\cdot b](p) + [(\Delta\cdot a)\cdot b](p)\\ &= ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp),\end{aligned} \tag{1.1.3}$$

we have that a first order differential operator $\Delta \in \mathrm{Diff}_1(P,Q)$ satisfies the following condition for all $a,b \in \mathcal{A}$ and for all $p \in P$:

$$ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0,$$

and, if $P=\mathcal{A}$ with $p=1$, we obtain:

$$\delta_b \circ \delta_a(\Delta)(1) = ba\Delta(1) - b\Delta(a) - a\Delta(b) + \Delta(ab) = 0, \qquad a,b \in \mathcal{A}.$$

In this way, we have

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(1).$$

Remark 1.1.5. We note in this work that, if in expression (1.1.2), we get that for all $\Delta \in \mathrm{Diff}_2(\mathcal{A},Q)$, with Q be an $\mathcal{A}$-module, such that $\Delta(1)=0$, taking $q=1$, then, for all $f,g,p \in \mathcal{A}$,

$$\begin{aligned}\Delta(pgf) &= f(\Delta(pg) - g\Delta(p) - p\Delta(g))\\ &\quad - g(p\Delta(f) - \Delta(pf)) + p\Delta(gf)\\ &= f\delta_p \circ \delta_g(\Delta)(1) - g\delta_p(\Delta)(f) + p\Delta(gf).\end{aligned}$$

Definition 1.1.6 ([GMS05], p. 25)**.** A $Q-$valued differential operator Δ of order 1 on $\mathcal{A}$ is called a *$Q-$valued derivation* of $\mathcal{A}$, if $\Delta(1)=0$, i.e., if $\Delta(ab) = b\Delta(a) + a\Delta(b)$, for all $a,b \in \mathcal{A}$. The set of $Q-$valued derivations of $\mathcal{A}$ is denoted by $\mathfrak{d}(\mathcal{A},Q)$, and if $Q=\mathcal{A}$, then $\mathfrak{d}(\mathcal{A})$.

Example 1.1.7 ([Fre06], p. 77)**.** If $\mathcal{A} = \mathbb{K}[x_1,...,x_n]$, then $\mathfrak{d}(\mathcal{A}) = span_{\mathcal{A}}\{\partial_1,...,\partial_n\}$ where $\partial_i = \frac{\partial}{\partial x_i}$ is the usual partial polynomial derivation by x_i.

Remark 1.1.8. If $\mathcal{A}$ is commutative, in the Definition 1.1.1 d is an Ω-valued derivation of $\mathcal{A}$.

The Proposition 1.1.10 is mentioned without proof in [GMS05]. In this work, we present its proof and the Proposition 1.1.9 for completeness of the document.

Proposition 1.1.9. *If $\Delta \in \mathrm{Diff}_1(\mathcal{A},Q)$, then the assignment $\beta(p) := \Delta(p) - p\Delta(1) \in Q$ defines a map $\beta : \mathcal{A} \to Q$ such that $\beta \in \mathfrak{d}(\mathcal{A},Q)$.*

Proof. We first check that $\beta \in \mathrm{Diff}_1(\mathcal{A}, Q)$. β is $\mathbb{K}$-linear because Δ and α are $\mathbb{K}$-linear maps. Also, for $a, b \in \mathcal{A}$ we have that

$$\delta_a \circ \delta_b(\beta) = \delta_a((b\beta) - (\beta \cdot b)) = a((b\beta) - (\beta \cdot b)) - [((b\beta) - (\beta \cdot b)) \cdot a].$$

Then, for any $p \in \mathcal{A}$, we obtain

$$\begin{aligned}
\delta_a \circ \delta_b(\beta)(p) &= a((b\beta) - (\beta \cdot b))(p) - [((b\beta) - (\beta \cdot b)) \cdot a](p) \\
&= a((b\beta)(p) - (\beta \cdot b)(p)) - (b\beta)(ap) + (\beta \cdot b)(ap) \\
&= ab\beta(p) - a\beta(bp) - b\beta(ap) + \beta(bap) \\
&= ab[\Delta(p) - p\Delta(1)] - a[\Delta(bp) - bp\Delta(1)] \\
&- b[\Delta(ap) - ap\Delta(1)] + [\Delta(bap) - bap\Delta(1)] \\
&= ab\Delta(p) - abp\Delta(1) - a\Delta(bp) + abp\Delta(1) \\
&- b\Delta(ap) + bap\Delta(1) + \Delta(bap) - bap\Delta(1) \\
&= \delta_a \circ \delta_b(\Delta)(p) \\
&= 0,
\end{aligned}$$

where the eighth equation is given by the commutativity of $\mathcal{A}$, and the last equation is due to Δ is a Q-valued first order differential operator. Then $\beta \in \mathrm{Diff}_1(\mathcal{A}, Q)$, and as $\beta(1) = \Delta(1) - 1\Delta(1) = 0$, we have that $\beta \in \mathfrak{d}(\mathcal{A}, Q)$. □

Proposition 1.1.10. ([GMS05], p. 25) *For any $\mathcal{A}$-module Q, it satisfies that* $\mathrm{Diff}_1(\mathcal{A}, Q) = \mathrm{Diff}_0(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q)$.

Proof. If $\Delta \in \mathrm{Diff}_1(\mathcal{A}, Q)$, for all $a \in \mathcal{A}$, we have that $\Delta(a) = a\Delta(1) + [\Delta(a) - a\Delta(1)]$, and then we can say, by the Proposition 1.1.9, that $\Delta = \alpha + \beta$, where $\alpha \in \mathrm{Diff}_0(\mathcal{A}, Q)$ and $\beta \in \mathfrak{d}(\mathcal{A}, Q)$ such that for all $a \in \mathcal{A}$, $\alpha(a) = a\Delta(1)$ and $\beta(a) = \Delta(a) - a\Delta(1)$, then $\mathrm{Diff}_1(\mathcal{A}, Q) \subset \mathrm{Diff}_0(\mathcal{A}, Q) + \mathfrak{d}(\mathcal{A}, Q)$. Also, if $\alpha \in \mathrm{Diff}_0(\mathcal{A}, Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, Q)$ and $\beta \in \mathfrak{d}(\mathcal{A}, Q)$ such that $\alpha + \beta = 0$, since $\beta(1) = 0$, we have that $\alpha(1) = 0$ and then $\alpha = 0$. Therefore $\beta = 0$. This proves that $\mathrm{Diff}_0(\mathcal{A}, Q) + \mathfrak{d}(\mathcal{A}, Q)$ is a direct sum. By the last, if $\alpha \in \mathrm{Diff}_0(\mathcal{A}, Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, Q)$ and $\beta \in \mathfrak{d}(\mathcal{A}, Q)$, i.e., for all $a, b \in \mathcal{A}$, $\delta_b(\alpha) = 0$ and $\delta_a \circ \delta_b(\beta) = 0$. Then, we obtain that

$$\delta_a \circ \delta_b(\alpha + \beta) = \delta_a \circ \delta_b(\alpha) + \delta_a \circ \delta_b(\beta) = 0.$$

Hence $\alpha + \beta \in \mathrm{Diff}_1(\mathcal{A}, Q)$, which proves that $\mathrm{Diff}_1(\mathcal{A}, Q) \supset \mathrm{Diff}_0(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q)$. □

Remark 1.1.11. If $Q = \mathcal{A}$, we obtain that $\mathfrak{d}(\mathcal{A})$ has a Lie algebra structure with the commutator (Lie bracket) $[\Delta, \Delta'] = \Delta \circ \Delta' - \Delta' \circ \Delta$, for all $\Delta, \Delta' \in \mathfrak{d}(\mathcal{A})$. Also, by the Proposition 1.1.10, $\mathrm{Diff}_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q)$, we have that $\mathrm{Diff}_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}(\mathcal{A})$.

Now, we discuss a way to represent any differential operator on a module P of order s with a differential operator of order 0 on the module of s-order jets of P (see Definition 1.1.12 below).

If P is an $\mathcal{A}$-module, we can consider $\mathcal{A}$ and P as $\mathbb{K}$-modules, and therefore to construct $\mathcal{A} \otimes_{\mathbb{K}} P$. We define for each $b \in \mathcal{A}$ the $\mathcal{A}-$module homomorphisms $\delta^b \in \mathrm{End}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathbb{K}} P)$

on the simple tensors as $\delta^b(a\otimes p) = (ba)\otimes p - a\otimes bp$, and for a fix $k\in\mathbb{N}$, we consider the $\mathcal{A}-$submodule of $\mathcal{A}\otimes_{\mathbb{K}}P$ generated by the elements $\delta^{b_0}\circ\cdots\circ\delta^{b_k}(a\otimes p)$, for all $b_0,...,b_k\in\mathcal{A}$, all $p\in P$ and for all $a\in A$; this submodule is denoted by μ^{k+1}.

Definition 1.1.12 ([GMS05], p. 26)**.** The $\mathcal{A}-$module $\mathcal{J}^k(P) = (\mathcal{A}\otimes_{\mathbb{K}} P)/\mu^{k+1}$ is called the *k-jet module* of P. The elements of $\mathcal{J}^k(P)$ are denoted $a\otimes_k p$.

For instance, by the computation

$$\begin{aligned}\delta^a\circ\delta^b(1\otimes p) &= \delta^a(b\otimes p - 1\otimes bp)\\ &= (ab)\otimes p - b\otimes ap - a\otimes bp + 1\otimes(abp)\end{aligned}\tag{1.1.4}$$

in $\mathcal{J}^1(P)$, then

$$(ab)\otimes_1 p + 1\otimes_1(abp) = b\otimes_1(ap) + a\otimes_1(bp).$$

If we consider the $\mathcal{A}-$module structure $\bullet$ on $\mathcal{J}^k(P)$ such that for $b\in\mathcal{A}$, $b\bullet(a\otimes_k p) = a\otimes_k bp$, we obtain the $\mathcal{A}-$module morphism:

$$J^k : P\to\mathcal{J}^k(P) : p\mapsto 1\otimes_k p,$$

and by this, when we consider the natural $\mathcal{A}-$module structure of $\mathcal{J}^k(P)$, $b(a\otimes p) = (ba)\otimes p$, we obtain that J^kp, with $p\in P$, generate $\mathcal{J}^k(P)$ (if we consider the normal $\mathcal{A}$-module structure on $\mathcal{J}^1(P)$, then J^k is not an $\mathcal{A}$-module map).

Now, we introduce the concept of connection on modules and rings. If $P=\mathcal{A}$, we denote by $\mathcal{J}^s$ the $s-$order jet module $\mathcal{J}^s(\mathcal{A})$. If $s=1$, we can consider the morphisms (which in real are monomorphisms, c.f. [GMS05], p. 27):

$$i:\mathcal{A}\to\mathcal{J}^1 : a\mapsto a\otimes_1 1,\qquad J^1:\mathcal{A}\to\mathcal{J}^1 : a\mapsto 1\otimes_1 a.$$

In this work, we present a proof of Proposition 1.1.13, which is only mentioned in [GMS05].

Proposition 1.1.13 ([GMS05], p. 27)**.** *$\mathcal{J}^1 = i(\mathcal{A})\oplus\mathcal{O}^1$ where $\mathcal{O}^1\subset\mathcal{J}^1$ is the $\mathcal{A}-$module generated by the elements of form $1\otimes_1 b - b\otimes_1 1$, i.e., $\mathcal{O}^1 = \operatorname{span}_{\mathcal{A}}((J^1-i)(\mathcal{A}))$.*

Proof. To prove the direct sum, first, we see that $a\otimes_1 b = a\otimes_1 b - ab\otimes_1 1 + ab\otimes_1 1 = a[1\otimes_1 b - b\otimes 1] + ab\otimes 1 = J^1(ab) + i(a)$, for all simple tensor of $a\otimes_1 b\in\mathcal{J}^1$, proving that $\mathcal{J}^1 = i(\mathcal{A}) + \mathcal{O}^1$. Now, we have the following exact sequence,

$$0\to\mathcal{O}^1\hookrightarrow\mathcal{J}^1\xrightarrow{f} i(\mathcal{A})\to 0,$$

where the value of f on a simple tensor is $f(a\otimes_1 b) := ab\otimes_1 1$. This map is well defined because

$$\begin{aligned}ab\otimes_1 1 &= ab\otimes_1 1 - ba\otimes_1 a + ab\otimes_1 1\\ f(a\otimes_1 b) &= f(ab\otimes_1 1) - f(b\otimes_1 a) + f(1\otimes_1 ab)\\ f(a\otimes_1 b) &= f(ab\otimes_1 1 - b\otimes_1 a + 1\otimes_1 ab).\end{aligned}$$

Also, f is an $\mathcal{A}$-linear map because the tensor product is linear in the first argument, $cf(a\otimes_1 b) = c(ab\otimes_1 1) = cab\otimes_1 1 = f(ac\otimes_1 b)$, for all $c\in\mathcal{A}$, and we extend from the simple tensor the definition of f to all $\mathcal{J}^1$. We have that this exact sequence splits because if $i' : i(\mathcal{A})\hookrightarrow\mathcal{A}^1$ is the inclusion map, then $f\circ i' = id_{i(\mathcal{A})}$, whence we obtain that $\mathcal{J}^1\cong i(\mathcal{A})\oplus\mathcal{O}^1$, and since $\mathcal{J}^1 = i(\mathcal{A}) + \mathcal{O}^1$, it follows $\mathcal{J}^1 = i(\mathcal{A})\oplus\mathcal{O}^1$. ☐

The following argument and the Proposition 1.1.14 are presented in this work to fulling the details of the proof of Corollary 1.1.15 presented in [GMS05].

If $b \in \mathcal{A}$, the assignment $\delta^b(a \otimes p)$, for all $a \otimes p \in \mathcal{A} \otimes P$, defines an $\mathcal{A}$-module homomorphism $\delta^b \in \mathrm{End}_{\mathcal{A}}(\mathcal{A} \otimes P)$. Also, with the $\mathcal{A}$-module map $J : P \to \mathcal{A} \otimes P : p \mapsto 1 \otimes p$, if $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes P, Q)$, it satisfies

$$\begin{aligned}\delta_b(f \circ J)(p) &= b(f \circ J)(p) - (f \circ J)(bp)\\ &= bf(1 \otimes p) - f(1 \otimes bp)\\ &= f(b \otimes p - 1 \otimes bp)\\ &= f(\delta^b(1 \otimes p))\\ &= f \circ \delta^b \circ J(p),\end{aligned}$$

and since $f \circ \delta^b$ is again an $\mathcal{A}$-module map, applying iteratively this argument we obtain, for each $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes P, Q)$, for all $a_0, ..., a_k \in \mathcal{A}$ and for every $p \in P$, that

$$\delta_{a_0} \circ \cdots \circ \delta_{a_k}(f \circ J)(p) = f \circ \delta^{a_k} \circ \cdots \circ \delta^{a_0} \circ J(p). \tag{1.1.5}$$

Proposition 1.1.14. *$J^k : P \to \mathcal{J}^k(P) : p \mapsto 1 \otimes_1 p$ is a $\mathcal{J}^k(P)$ valued differential operator on P of order k. Moreover, If $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^1, Q)$, then $f \circ J^k \in \mathrm{Diff}_k(P, Q)$.*

Proof. As $J^k = \pi \circ J$, where $\pi : \mathcal{A} \otimes P \to \mathcal{J}^k(P)$ is the canonical $\mathcal{A}$-module map, by (1.1.5) we have that

$$\begin{aligned}\delta_{a_0} \circ \cdots \circ \delta_{a_k}(f \circ J^k)(p) &= \delta_{a_0} \circ \cdots \circ \delta_{a_k}(f \circ \pi \circ J)(p)\\ &= f \circ pi \circ \delta^{a_k} \circ \cdots \circ \delta^{a_0} \circ J(p) = 0,\end{aligned}$$

since $\delta^{a_k} \circ \cdots \circ \delta^{a_0} \circ J(p) = \delta^{a_k} \circ \cdots \circ \delta^{a_0}(1 \otimes p) \in \mu^{k+1} = \mathrm{Ker}(\pi)$. ☐

Corollary 1.1.15 ([GMS05], p. 27)**.** *If $\Delta \in \mathrm{Diff}_k(P, Q)$ then there exists a unique $\mathcal{A}$-module map $f_\Delta : \mathcal{J}^k(P) \to Q$ such that $f_\Delta \circ J^k = \Delta$.*

Proof. We define $f_\Delta : \mathcal{J}^1(P) \to Q$ where $a \otimes_1 p \mapsto a\Delta(p)$, and we extend linearly to all element of $\mathcal{J}^1$. This is an $\mathcal{A}$-module map, and by Proposition 1.1.14, we obtain that $\Delta = f_\Delta \circ J^k \in \mathrm{Diff}_k(P, Q)$. If there is another $g \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^k, q)$ such that $\Delta = g \circ J^k$, we obtain that for all $p \in P$, $f(1 \otimes_1 p) = g(1 \otimes_1 p)$, and since $J^k(P)$ is an $\mathcal{A}$ generator of $\mathcal{J}^k(P)$, we obtain that $f = g$. ☐

Remark 1.1.16. We have that the injective assignment $\Delta \mapsto f_\Delta$ is an identification that allows us to think that $\mathrm{Diff}_k(P, Q) \hookrightarrow \mathrm{Hom}(\mathcal{J}^k(P), Q)$. Actually, we have the following proposition that we present without proof.

Proposition 1.1.17. ([KVL86], p. 13) *We have the $\mathcal{A}$-module isomorphism $\mathrm{Diff}_k(P, Q) \cong \mathrm{Hom}(\mathcal{J}^k, Q)$.*

Now, we can construct a differential on $\mathcal{O}^1$ as follows: the map $h^1 : \mathcal{J} \to \mathcal{O}^1 : 1 \otimes_1 b \mapsto 1 \otimes_1 b - b \otimes_1 1$ is an $\mathcal{A}-$module morphism, and considering the $\mathcal{A}-$module morphism

$d^1 = h^1 \circ J^1 : \mathcal{A} \to \mathcal{O}^1 : b \mapsto 1 \otimes_1 b - b \otimes_1 1$, we obtain that for all $a, b \in \mathcal{A}$, we have the following:

$$\begin{aligned} d^1(ab) &= 1 \otimes_1 ab - ab \otimes_1 1 \\ &= 1 \otimes_1 ab - ab \otimes_1 1 + a \otimes_1 b - a \otimes_1 b \\ &= (a \otimes_1 b - ab \otimes_1 1) + (1 \otimes_1 ab - a \otimes_1 b) \\ &= a(1 \otimes_1 b - b \otimes_1 1) + (1 \otimes_1 a - a \otimes_1 1)b \\ &= ad^1(b) + d^1(a)b. \end{aligned} \tag{1.1.6}$$

With this Leibniz's rule and with the fact that the elements of the form $d^1 a$, where $a \in A$, are generators of the $\mathcal{A}-$module $\mathcal{O}^1$, these play the role of the one-forms of the ring $\mathcal{A}$. The $k-$forms set is the exterior product of the $\mathcal{A}-$module $\mathcal{O}^1$ k times, $\mathcal{O}^k = \bigwedge_{i=1}^{k} \mathcal{O}^1$ and there exists for each $k \in \mathbb{N}$, an $\mathcal{A}-$module morphism $h^k : \mathcal{J}^1(\mathcal{O}^{k-1}) \to \mathcal{O}^k$ such that when we define $d^k = h^k \circ J^1 : \mathcal{O}^{k-1} \to \mathcal{O}^k$, it satisfies that $d^k \circ d^{k-1} = 0$, for all $k \in \mathbb{N}$. The pair $(\bigwedge \mathcal{O}, d)$ is a kind of a De Rham calculus[1] over the ring $\mathcal{A}$, where $\bigwedge \mathcal{O} = \bigoplus_{k \in \mathbb{N}} \mathcal{O}^k$ and $d = \bigoplus_{k \in \mathbb{N}} d^k$ (c.f. [GMS05] p. 28).

The construction of $\mathcal{O}$ is very close to the construction of a relevant structure associated to any algebra, called the *universal differential calculus*.

1.1.2 Differential geometric-algebraic dictionary

We summary the algebraic objects mentioned before with their respective geometrical object. All of them were consulted in [Nes03], Chapter 9, from the theory of smooth manifolds (see Table 1.1), and the undefined objects are clarified below of the table.

Algebraic object	**Geometric object**
Commutative ring $\mathbb{K}$	$\mathbb{R}$
Commutative $\mathbb{K}$-algebra $\mathcal{A}$	$C^\infty(M)$
$\lvert\mathcal{A}\rvert = \{h : \mathcal{A} \to \mathbb{K} : h(1) = 1\}$	M
$h_z \in \lvert\mathcal{A}\rvert$	$z \in M$
$T_h\mathcal{A}$	T_zM
$\mu_h = \mathrm{Ker}(h)$	$\mu_z = \{f \in C^\infty(M) : f(z) = 0\}$
$T_h^*\mathcal{A} = \mu_h/\mu_h^2$	$T_z^*M \cong \mu_z/\mu_z^2$
$v_h : T_h^{**}\mathcal{A} \to T_h\mathcal{A}$ surjective	$v_z : T_z^{**}M \to T_zM$ isomorphism
If $F : \mathcal{B} \to \mathcal{A}$, then $\lvert F\rvert : \lvert\mathcal{A}\rvert \to \lvert\mathcal{B}\rvert : h \mapsto h \circ F$	$f : M \to N$ smooth map
$d_h\lvert F\rvert : T_h\mathcal{A} \to T_{h\circ F}\mathcal{B} : \varepsilon \mapsto \varepsilon \circ F$	$d_zf : T_zM \to T_{f(z)}N$
$\mathcal{J}_h^1(\mathcal{A}) = \mathbb{K} \oplus T_h^*\mathcal{A}$	$\mathcal{J}_z^1(M) = \mathbb{R} \oplus \mu_z/\mu_z^2 \cong \mathbb{R} \oplus T_h^*M$
$\mathfrak{d}(\mathcal{A})$	$\mathfrak{X}(M)$
$\nabla : \mathfrak{d}(\mathcal{A}) \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{d}(\mathcal{A}), \mathfrak{d}(\mathcal{A}))$	$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

Table 1.1: Dictionary 1.

In the **Algebraic objects**, we clarify that:

---

[1] A de Rham Calculus is a cochain complex of modules such that the coboundary map satisfies a Leibniz's rule.

1. $|\mathcal{A}| \subset \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$ called the $\mathbb{K}$-*spectrum* (see [Nes03], p. 110). The reason to define $|\mathcal{A}|$ such as the space of measures is because a state of a classic physical system is determined by measurements ([Nes03], p. 22).
2. $T_h\mathcal{A} := \{\varepsilon : \mathcal{A} \to \mathbb{K} : \forall f, g \in \mathcal{A}, \varepsilon(fg) = h(f)\varepsilon(g) + \varepsilon(f)h(g)\}$, for $h \in |\mathcal{A}|$ (see [Nes03], p. 110). The reason to define
3. $F : \mathcal{B} \to \mathcal{A}$ is a $\mathbb{K}$-homomorphism (see [Nes03], p. 112).

In [Nes03], p. 110, it was established that the module of first order jets of a manifold M is $\mathcal{J}^1 M = \bigcup_{z\in} \mathcal{J}_z M$. We deduce that in the algebraic approach we obtain the first order jet module.

$$\mathcal{J}^1\mathcal{A} = \bigcup_{h\in|\mathcal{A}|} \mathcal{J}^1_h\mathcal{A} = \bigcup_{h\in|\mathcal{A}|} \mathbb{K} \oplus T^*_h\mathcal{A}$$

On the other hand, in Section 1.1.1, we found the notion of jet module and $\mathcal{J}^1 = i(\mathcal{A}) \oplus \mathcal{O}^1$ (see Proposition 1.1.13). Since i is a monomorphism, we say that $\mathcal{J}^1 = \mathcal{A} \oplus \mathcal{O}^1$.

Remark 1.1.18. We warn that in this remark, we are going to let free the creativity, and give not rigorous innocent ideas. If we identify the two interpretations of the first order jet modules, and let a distribution of the union with the sum, we get

$$\mathcal{A} \oplus T^*\mathcal{A} = \left(\bigcup_{h\in|\mathcal{A}|} \mathbb{K}\right) \oplus \left(\bigcup_{h\in|\mathcal{A}|} T^*_h\mathcal{A}\right) = \bigcup_{h\in|\mathcal{A}|} \mathbb{K} \oplus T^*_h\mathcal{A} = \mathcal{A} \oplus \mathcal{O}^1$$

One of the many problems of this idea is that in the case of $\mathcal{A} = C^\infty(M)$, in the identification $C^\infty(M) = \bigcup_{z\in M} \mathbb{R}$, it has been forgotten the continuity of the elements of $C^\infty(M)$, but we want to recall the beauty of this idea, because again with a little bit of innocence, we could identify $T^*\mathcal{A}$ with the module $\mathcal{O}^1$. This is (maybe) an example of the incompatibility between different algebraic generalizations of the well known manifolds geometry.

Remark 1.1.19. We point out that in the previous idea of Remark 1.1.18 the continuity or smoothness of the elements of $\mathcal{A} = C^\infty(M)$ (really, only in the treatment of [GMS05], see [Nes03], Chapter 4), and this could be a main problem in the splice of the proposals. The smoothness of M in yields the continuity or differentiability of the elements $\mathcal{A}$ and it is well known that the smoothness of M can be recovered from the algebra $C^\infty(M)$, such as is treated in [Nes03], Chapther 7. This justify our interest in the different notions of smoothness of more generic algebras. In Section 1.3, we list some different definitions of smoothness of algebras and in particular, in Section 1.3.2, we study the Brzezinski's notion of *differentially smoothness* which is based on the ideas of Connes [Con85] and Dubois et al [DVKM90].

1.1.3 Universal differential calculus

Consider $\mathrm{Ker}(\mu)$, where $\mu : \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A}$ is the multiplication $\mathbb{K}$-map, $\mu(a \otimes b) = ab$, and let $d : \mathcal{A} \to \mathrm{Ker}(\mu)$, the $\mathbb{K}$-map defined by $d(b) = 1 \otimes b - b \otimes 1$, for all $b \in \mathcal{A}$.

Remark 1.1.20. If we consider the right and left actions of $\mathcal{A}$ over $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$ given by

$$x(a \otimes b)y = (xa) \otimes (by), \qquad \text{for all } a, b, x, y \in \mathcal{A},$$

we obtain that,

1. μ is an $\mathcal{A}$-bimodule homomorphism.
2. For all $a, b \in \mathcal{A}$, $d(ab) = ad(b) + d(a)b$.
3. I is is left-generated by $d(\mathcal{A})$, i.e., $\mathrm{Ker}(\mu) = \mathcal{A}d(\mathcal{A})$.

With this, we can define the following.

Definition 1.1.21 ([Bou89], p. 567)**.** The first differential calculus on $\mathcal{A}$, $(\Omega^1(\mathcal{A}), d)$, where $\Omega^1(\mathcal{A}) = \mathrm{Ker}(\mu)$ and $d(b) = 1 \otimes b - b \otimes 1$, for all $b \in \mathcal{A}$, it is called the *universal differential calculus* of $\mathcal{A}$.

For a non-commutative $\mathbb{K}$-algebra $\mathcal{A}$, the universal differential algebra $(\Omega^1(\mathcal{A}), d)$ is defined such as in the commutative case of Definition 1.1.24. Since $d(ab) = ad(b) + (da)b$, for all $a, b \in \mathcal{A}$, with $a = b = 1$, we obtain $d(1) = 0$.

Proposition 1.1.22. ([Lan97], p. 106; [DV88], p. 405) *Let P be an $\mathcal{A}$-bimodule. Any P-valued derivation Δ of $\mathcal{A}$ can be expressed as $\Delta = f^{\Delta} \circ d$ where $f^{\Delta} : \Omega^1(\mathcal{A}) \to P$ is an $\mathcal{A}$-bimodule homomorphism.*

Remark 1.1.23. This structure is very useful to characterize the derivations of a non commutative algebra $\mathcal{A}$, but this cannot be consider as the generalization of the universal forms of the commutative case because for $a, b \in \mathcal{A}$, we have $dadb = 1 \otimes ab - a \otimes b - b \otimes a + ab \otimes 1$ and $dbda = 1 \otimes ba - b \otimes a - a \otimes b + ba \otimes 1$. Therefore $dadb \neq -dbda$, which it is true in the commutative case.

Definition 1.1.24 ([GMS05], p. 56)**.** A graded algebra $\Omega^* = \bigoplus_k \Omega^k$, where $\Omega^0 \cong \mathcal{A}$, it is said to be a *differential calculus* over $\mathcal{A}$, if it is a cochain complex of $\mathbb{K}-$modules

$$0 \to \mathbb{K} \to \mathcal{A} \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots \Omega^k \xrightarrow{\delta} \cdots$$

for a co-boundary operator δ such that for all $\alpha, \beta \in \Omega$, it satisfies that $\delta(\alpha\beta) = \delta(\alpha)\beta + (-1)^{\deg(\alpha)}\alpha\delta(\beta)$ (Leibniz's rule). It is also known as the de Rham complex of the differential graded algebra (Ω^*, δ).

Following [Kar87], Sections 1.3 and 1.24, we construct the extension $\Omega^*(\mathcal{A})$ of $\Omega^1(\mathcal{A})$ considering the direct sum of $\mathcal{A}$-tensor products of $\Omega^1(\mathcal{A})$, i.e., $\Omega^*(\mathcal{A}) = \bigoplus_{k \in \mathbb{N}} \Omega^k(\mathcal{A})$, where,

$$\Omega^k(\mathcal{A}) = \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) = \bigotimes_{i=1}^{k} \Omega^1(\mathcal{A}),$$

for $k > 0$ and $\Omega^0(\mathcal{A}) = \mathcal{A}$. For an element of $\Omega^*(\mathcal{A})$, we omit the symbol $\otimes$ and we say that $\Omega^*(\mathcal{A})$ consists of the elements of the form

$$a_0 da_1 \cdots da_k, \qquad a_i \in \mathcal{A}, \quad k \in \mathbb{N}.$$

As we have for all $a_1, b_0 \in \mathcal{A}$ that $(da_1)b_0 = d(a_1b_0) - a_1db_0$, then we obtain that for all $a_0, a_1, b_0, b_1 \in \mathcal{A}$, it satisfies that $(a_0da_1)(b_0db_1) = a_0d(a_1b_0)db_1 - a_0a_1db_0db_1$. Also, we can extend d for all $\Omega^*(\mathcal{A})$ defining

$$d(a_0da_1\cdots da_k) = da_0da_1\cdots da_k.$$

The pair $(\Omega^*(\mathcal{A}), d)$ (which can be seen also as a complex $\Omega^k(\mathcal{A}) \xrightarrow{d} \Omega^{k+1}(\mathcal{A})$ because with the extended structure and $d(1) = 0$, we obtain that $d^2 = 0$) is called the *universal differential graded calculus* of $\mathcal{A}$, and this structure has also a universal property as follows.

Proposition 1.1.25 ([DV88], p. 405). *Let $\varphi_0 : \mathcal{A} \to \Omega'^0$ be an algebra homomorphism, where $\Omega' = \bigoplus \Omega'^n$ is a differential graduate algebra. Then, there exists a unique homomorphism of differential graduate algebras $\varphi : \Omega(\mathcal{A}) \to \Omega'$ such that $\varphi|_{\mathcal{A}} = \varphi_0$.*

Remark 1.1.26 ([DV88], p. 405). We can construct a notion of Lie derivative over the universal calculus as follows. If $\delta \in \mathfrak{D}\mathcal{A}$, the unique homomorphism of bimodules $i_\delta : \Omega^1(\mathcal{A}) \to \mathcal{A} = \Omega^0(\mathcal{A})$, such that $\delta = i_\delta \circ d$ (see Proposition 1.1.22), it is extended to an antiderivative of $\Omega(\mathcal{A})$[2], $i_\delta : \Omega^*(\mathcal{A}) \to \Omega^{*-1}(\mathcal{A})$, i.e., $i_\delta : \Omega^{k+1}(\mathcal{A}) \to \Omega^k(\mathcal{A})$. The Lie derivative for each $\delta \in \mathfrak{D}\mathcal{A}$ is defined as $\mathrm{L}_\delta = i_\delta \circ d + d \circ i_\delta$. We obtain that, for any $\delta_1, \delta_2 \in \mathfrak{D}\mathcal{A}$, the following relations:

$$i_{\delta_1} \circ i_{\delta_2} + i_{\delta_2} \circ i_{\delta_1} = 0, \quad \mathrm{L}_{\delta_1} \circ i_{\delta_2} - i_{\delta_2} \circ \mathrm{L}_{\delta_1} = i_{[\delta_1,\delta_2]}, \quad \mathrm{L}_{\delta_1} \circ \mathrm{L}_{\delta_2} - \mathrm{L}_{\delta_1} \circ \mathrm{L}_{\delta_2} = \mathrm{L}_{[\delta_1,\delta_2]}.$$

These relations coincide with the behavior of interior product between a form and a vector space ([Mor98], p.72), and the Cartan formula for the Lie derivative in manifolds theory ([Mor98], p. 74 and 75).

Universal property of Proposition 1.1.25 is also useful to construct $\Omega_D(\mathcal{A})$ in Section 1.1.5, which in the case of $\mathcal{A} = M_n(\mathbb{C})$, it is an example of a differential calculus that is integrable in Section 1.3.2.

1.1.4 Chevalley-Eilenberg differential calculus

In this subsection, we present the Chevalley-Eilenberg differential calculus and show how it comes from the manifolds geometry. We finish with the definition of the action of the modules of derivations over the Chevalley-Eilenberg differential calculus.

There is a canonical way to construct a differential calculus over an algebra, which comes to manifolds geometry as follows (c.f. [GMS05] p. 55). Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ such that a $\mathbb{K}-$module P is a $\mathfrak{g}-$module, i.e., $\mathfrak{g}$ acts on P on the left by endomorphism $\mathfrak{g}\times P \to P : (\varepsilon, p) \mapsto \varepsilon p$, where for $\varepsilon, \varepsilon' \in \mathfrak{g}$, we denote $[\varepsilon, \varepsilon']p = (\varepsilon\circ\varepsilon' - \varepsilon'\circ\varepsilon)p$. With this, we can form the $\mathfrak{g}-$modules of $P-$valued k-cochains on the Lie algebra $\mathfrak{g}$ denoted by $C^k[\mathfrak{g}; P]$, which is the set of $\mathbb{K}$-k-multilinear skew-symmetric maps $c^k : (\mathfrak{g} \times \cdots \times \mathfrak{g}) \to P$. If we define $C^0[\mathfrak{g}; P] = P$, we obtain the cochain complex Chevalley-Eilenberg with coefficients in P:

$$0 \to P \xrightarrow{\delta^0} C^1[\mathfrak{g}; P] \xrightarrow{\delta^1} \cdots C^k[\mathfrak{g}; P] \xrightarrow{\delta^k} \cdots$$

---

[2] A $\mathbb{K}$-linear map such that $d(ab) = d(a)b + (-1)^{\deg(a)}ad(b)$, for all $a, b \in B$ is called an antiderivative of a graded algebra B (see [Bou89], p. 552).

We define the Chevalley-Eilenberg co-boundary operators for each $k \in \mathbb{N}$, $\delta^k : C^k[\mathfrak{g}; P] \to C^{k+1}[\mathfrak{g}; P]$ as follows:

$$\delta^k c^k(\varepsilon_0, ..., \varepsilon_k) = \sum_{i=0}^{k} (-1)^i \varepsilon_i c^k(\varepsilon_0, ..., \widehat{\varepsilon_i}, ..., \varepsilon_k) + \sum_{i<j\leq k} (-1)^{i+j} c^k([\varepsilon_i, \varepsilon_j], \varepsilon_0, ..., \widehat{\varepsilon_i}, ..., \widehat{\varepsilon_j}..., \varepsilon_k).$$

With this definition, we can show that a $\mathbb{K}-$algebra $\mathcal{A}$ has associated a differential calculus. As we mentioned before, the $\mathcal{A}-$module $\mathfrak{d}\mathcal{A}$ is a Lie $\mathbb{K}$-algebra, and therefore we obtain the Chevalley-Eilenberg complex over $\mathcal{A}$:

$$0 \to \mathbb{K} \hookrightarrow \mathcal{A} \xrightarrow{\delta} C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}],$$

where the injection of $\mathbb{K}$ in $\mathcal{A}$ is given by the assignment $k \mapsto k \cdot 1$. The Chevalley-Eilenberg co-boundary $d^k : C^k[\mathfrak{d}\mathcal{A}, \mathcal{A}] \to C^{k+1}[\mathfrak{d}\mathcal{A}, \mathcal{A}]$, in this case, works as follows:

- If $a \in \mathcal{A}$ and $u \in \mathfrak{d}\mathcal{A}$, then $da(u) = \sum_{i=0}^{0} (-1)^i u_i(a) = u(a)$.
- If $\phi \in C^1[\mathfrak{d}\mathcal{A}, \mathcal{A}]$ and $u_0, u_1 \in \mathfrak{d}\mathcal{A}$, we obtain that

$$\begin{aligned} d\phi(u_0, u_1) &= (-1)^0 u_0(\phi(u_1) + (-1)^1 u_1(\phi(u_0) + (-1)^{0+1}\phi([u_0, u_1]) \\ &= u_0(\phi(u_1)) - u_1(\phi(u_0)) - \phi([u_0, u_1]), \end{aligned}$$

and so $C^0[\mathfrak{d}\mathcal{A}, \mathcal{A}] = \mathcal{A}$ and $C^1[\mathfrak{d}\mathcal{A}, \mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A}, \mathcal{A})$. As we can show that $d(1) = 0$ (because if $u \in \mathfrak{d}\mathcal{A}$ we have that $u(1) = u(1 \cdot 1) = u(1) + u(1)$, therefore $u(1) = 0$ and $d(1)(u) = u(1) = 0$), then d is a $C^1[\mathfrak{d}\mathcal{A}, \mathcal{A}]-$derivation of $\mathcal{A}$.

We can consider $C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}]$ with a structure of a graded $\mathcal{A}-$algebra with the following product: If $\phi \in C^r[\mathfrak{d}(\mathcal{A}), \mathcal{A}]$ and $\phi' \in C^s[\mathfrak{d}(\mathcal{A}), \mathcal{A}]$, we define

$$\phi \wedge \phi'(u_1, ..., u_{r+s}) = \sum_{i_1<\cdots<i_r; j_1<\cdots<j_s} sgn^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s} \phi(u_{i_1}, ..., u_{i_r})\phi'(u_{j_1}, ..., u_{j_s}),$$

where sgn is the sign of the permutation. With this product we obtain that,

- $d(\phi \wedge \phi') = d(\phi) \wedge \phi' + (-1)^{\deg(\phi)}\phi \wedge d(\phi')$. This relation makes the structure $(C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}], d)$ a differential graded $\mathbb{K}$-algebra.
- $\phi \wedge \phi' = (-1)^{\deg(\phi)\deg(\phi')}\phi' \wedge \phi$. This relation makes $(C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}], d)$ a graded commutative algebra.

Definition 1.1.27 ([GMS05], p. 58)**.** The structure $(C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}], d)$ is called the *Chevalley-Eilenberg differential calculus* over a $\mathbb{K}$-algebra $\mathcal{A}$.

The minimal Chevalley-Eilenberg differential calculus over a $\mathcal{A}$ is the subcomplex $\mathcal{O}^*\mathcal{A}$ of $\mathcal{A}-$submodules generated by the monomials $a_0 da_1 \wedge \cdots \wedge da_k$ with $a_i \in \mathcal{A}$.

$$0 \to \mathbb{K} \to \mathcal{A} \xrightarrow{d} \mathcal{O}^*\mathcal{A}.$$

Example 1.1.28 ([Mor98], p. 71)**.** Let M be a C^∞ manifold and let $\mathcal{A} = C^\infty(M)$ be the algebra of continuous functions $f : M \to \mathbb{K}$. Then, the Chevalley-Eilenberg complex of $\mathcal{A}$ and $\mathfrak{d}(\mathcal{A}) = \mathfrak{X}(M)$ is the de Rham complex of M, where $\delta = (k+1)d$ is the exterior differentiation defined, as follows:

$$\begin{aligned} d\omega(X_0, ..., X_k) = & \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\ & + \sum_{i<j\leq k} (-1)^{i+j} \phi^k([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j} ..., X_k), \end{aligned}$$

where $\omega \in C^k[\mathfrak{d}\mathcal{A}, \mathcal{A}] = \Omega^k(M)$ is a k-form on M and $X_0, ..., X_k \in \mathfrak{d}\mathcal{A}$.

If $\mathcal{A}$ is a graded commutative $\mathbb{K}$-algebra ([GMS05], p. 352), we obtain that $\mathfrak{d}\mathcal{A}$ is a Lie superalgebra with a superbracket operation; the jet modules are defined in a similar way as we show before, but using the graded of the elements to define the δ'_as operators, and the inclusion $\mathcal{O}^*[\mathfrak{d}\mathcal{A}] \subset C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}]$ is a bigraded differential algebra, where the coboundary operator is defined as follows:

$$d\phi(u, u') = -u'(u(\phi)) + (-1)^{[u][u']} u(u'(\phi)) + [u', u](\phi).$$

In Definition 1.1.35, if we choose $Q = \mathcal{A}$, we can, as in the commutative case, build the Chevelley-Eilenberg complex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}, \mathcal{A}]$, where the co-boundary operator d and the wedge product $\wedge$ are defined in the same way. Also, we obtain that $\mathcal{O}^1[\mathfrak{d}\mathcal{A}, \mathcal{A}] = \mathrm{Hom}_{Z(\mathcal{A})}(\mathfrak{d}\mathcal{A}, \mathcal{A})$.

Remark 1.1.29. In this case, we obtain that this algebra is not commutative graded: if $u \in \mathfrak{d}\mathcal{A}$ and $a, b \in \mathcal{A}$, we have that $(a \wedge db)(u) = adb(u) = au(b)$ and $[(db) \wedge a](u) = (db)(u)a = u(b)a$; as $\mathcal{A}$ is non-commutative we do not have that $au(b) = u(b)a$. Therefore, we cannot ensure that $a \wedge db = (-1)^{0 \cdot 1}[(db) \wedge a]$ in general (but if $a \in Z(\mathcal{A})$, we obtain that $adb = (db)a$).

Remark 1.1.30. For the Chevalley-Eilenberg differential calculus, we can define an analogous of a Lie derivative as follows. For $\delta, u_1, ..., u_{k-1} \in \mathfrak{d}\mathcal{A}$ and $\phi \in C^k[\mathfrak{d}\mathcal{A}, \mathcal{A}]$,

$$\bar{i}_\delta(\phi)(u_1, ..., u_{k-1}) = \phi(u, u_1, ..., u_{k-1}).$$

The function $\bar{i}_\delta : C^k[\mathfrak{d}\mathcal{A}, \mathcal{A}] \to C^{k-1}[\mathfrak{d}\mathcal{A}, \mathcal{A}]$ is denoted in the literature by $(\delta \rfloor)$, and for $\phi \in C^k[\mathfrak{d}\mathcal{A}]$, $\bar{i}_\delta(\phi)$ is called the *interior product* of ϕ by δ ([Mor98], p. 73). Then the Lie derivative is expressed as,

$$\mathbb{L}_u(\phi) = d(u \rfloor \phi) + u \rfloor d(\phi).$$

Remark 1.1.31. We note that, apparently, this construction of L works only if $\mathcal{A}$ is a graded algebra (c.f. [GMS05], p. 357).

With this notion of Chevalley-Eilenberg differential calculus, we can obtain some interesting differential calculus structures, as we will see in the next section.

1.1.5 Differential calculi associated to $\Omega(\mathcal{A})$

We present some differential calculus structures that arise when we compare the universal calculus and the Chevalley-Eilenberg calculus over an algebra $\mathcal{A}$. This is found in [DV88], p. 406.

Considering the universal calculus over $\mathcal{A}$, $\Omega(\mathcal{A})$, we define the following objects that are graded subalgebras of $\Omega(\mathcal{A})$ (see Remark 1.1.26 to the definition of i_δ and L).

1. $\Omega_H(\mathcal{A}) = \{x \in \Omega(\mathcal{A}) : i_\delta(x) = 0, \text{ for all } \delta \in \mathfrak{d}\mathcal{A}\}$ is the set of *horizontal* elements of $\Omega(\mathcal{A})$. This is a submodule of $\Omega(\mathcal{A})$, because $\Omega^0_H(\mathcal{A}) = \mathcal{A}$.
2. $\Omega_I(\mathcal{A}) = \{x \in \Omega(\mathcal{A}) : \mathrm{L}_\delta(x) = 0, \text{ for all } \delta \in \mathfrak{d}\mathcal{A}\}$ is the set of *invariant* elements of $\Omega(\mathcal{A})$. This is a graded differential subalgebra of $\Omega(\mathcal{A})$.
3. $\Omega_B(\mathcal{A}) = \Omega_H(\mathcal{A}) \cap \Omega_I(\mathcal{A})$ is the set of *basic* elements of $\Omega(\mathcal{A})$. This is a graded differential subalgebra of $\Omega_I(\mathcal{A})$, and therefore of $\Omega(\mathcal{A})$.

By the universal property of $\Omega(A)$ (see Proposition 1.1.25), since both complex satisfy $\Omega^0(\mathcal{A}) = \mathcal{A} = C^0[\mathfrak{d}\mathcal{A}, \mathcal{A}]$, the identity map is extended to a unique homomorphism of graded differential algebras $\varphi : \Omega(\mathcal{A}) \to C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}]$. In general, this map is not either injective or surjective. $\mathrm{Im}\varphi := \Omega_D$ is the smallest graded differential subalgebra of $C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}]$ which contains $\mathcal{A}$; $\Omega^n_D(\mathcal{A})$ is $\mathcal{A}$-generated by the elements $a_0 d_D a_1 \cdots d_D a_n$, where d_D denotes the boundary operator of $C^*[\mathfrak{d}\mathcal{A}, \mathcal{A}]$.

Remark 1.1.32. In terms of the previous section, $\Omega_D(\mathcal{A})$ is the same minimal Chevalley-Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$, and if $\mathcal{A} = C^\infty(M)$ is the algebra of continuous functions of a smooth manifold M, this complex is the classical de Rham differential calculus of M (see Example 1.1.28).

It is satisfied that for any $\delta \in \mathfrak{d}\mathcal{A}$, we have that $\bar{i}_\delta \circ \varphi = \varphi \circ i_\delta$, where $\bar{i}_\delta$ is defined in Remark 1.1.30 and i_δ is as in Remark 1.1.26. In this way, we obtain that $\Omega_D(\mathcal{A})$ is stable under all $\bar{i}_\delta$. Also, we get that if $i_\delta(\alpha) = 0$, for all $\delta \in \mathfrak{d}\mathcal{A}$, then $\alpha = 0$.

Example 1.1.33 ([DV88], p. 406)**.** If $\mathcal{A} = M_n(\mathbb{C})$, since $\mathfrak{d}\mathcal{A} = sl(n, \mathbb{C})$[3] (in this work, we give a explanation of this fact in Example 1.1.37 and Remark 1.1.38), we obtain that $\varphi^1 : \Omega(\mathcal{A}) \to C^1[sl(n, \mathbb{C}), M_n(\mathbb{C})]$ is injective, and since

$$\dim_{\mathbb{C}}(\Omega^1(M_n(\mathbb{C}))) = n^2(n^2 - 1) = \dim_{\mathbb{C}}(C^1[sl(n, \mathbb{C}), M_n(\mathbb{C})]), \tag{1.1.7}$$

then $\Omega(\mathcal{A}) = C^1[sl(n, \mathbb{C}), M_n(\mathbb{C})]$. Therefore, the minimal Chevalley-Eilenberg differential calculus $\Omega_D(\mathcal{A})$ and the Chevalley-Eilenberg differential calculus $C^*[sl(n, \mathbb{C}), M_n(\mathbb{C})]$ are the same.

Remark 1.1.34. In [DVKM90], it was presented a deep development of the noncommutative differential geometry of the algebra $M_n(\mathbb{C})$ using as a principal tool the differential calculus $\Omega_D(M_n(\mathbb{C}))$. It was established a set of generators of $\Omega_D(M_n(\mathbb{C}))$ that are elements of $\Omega^1_D(M_n(\mathbb{C}))$. In particular, these elements are identified with the basis $\{\theta_k\}$ of

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[3] $sl(n, \mathbb{C}) = \{C \in M_n(\mathbb{C}) : \mathrm{tr}(C) = 0\}$.

$\mathfrak{d}\mathcal{A}^* = sl(n,\mathbb{C})^*$ (which is identified with $1 \otimes sl(n,\mathbb{C})^*$ in $\Omega^1_D(\mathcal{A})$) of dual elements of the basis $\{\mathrm{ad}(iE_k)\}$ of $\mathfrak{d}\mathcal{A} = sl(n,\mathbb{C})$ (see Example 1.1.37 below), where we consider the hermitian traceless basis (this means that it is a basis of $sl(n,\mathbb{C})$ over $\mathbb{C}$, and over $\mathbb{R}$ generate all the hermitian traceless matrices, i.e., the elements $A \in M_n(\mathbb{C})$ $\overline{A^t} = A$ and $\mathrm{tr}(A) = 0$)) of $sl(n,\mathbb{C})$ $\{E_k : k = 1, ..., n^2-1\}$, where E_k is an element of

$$\{E_{rj} + E_{jr} : r \neq j\} \cup \{iE_{rj} - iE_{jr} : r \neq j\} \cup \{E_{rr} - E_{nn} : r = 1, ..., n-1\},$$

where the set $\{E_{ij}\}$ is te canonical basis of $M_n(\mathbb{C})$. With this basis, we have that for all $1 \leq k, t, m \leq n^2 - 1$, there exists $s^m_{kt}, C^m_{kt} \in \mathbb{C}$ such that $s^m_{kt} = s^m_{tk}$ and $C^m_{kt} = -C^m_{tk}$, and,

$$E_k E_t = g_{kl} id + (s^m_{kt} - C^m_{kt}) E_m, \quad \text{where, } g_{kt} = g_{ik} = \frac{1}{n}\mathrm{tr}(E_k E_t). \tag{1.1.8}$$

Using the inverse of the matrix $\hat{g} = (g_{kl})$, it was constructed a star isomorphism $* : \Omega_D(M_n(\mathbb{C})) \to \Omega_D(M_n(\mathbb{C}))$, that allows us to define a structure of graded finite dimensional complex Hilbert space, a dual operator of d denoted δ, and therefore a Laplacian operator as in the manifolds theory, that is, $\Delta = d\delta + \delta d$ (but in this case $*(*\alpha) = (-1)^{n^2 p}\alpha$, if $\alpha \in \Omega^p_D(M_n(\mathbb{C}))$[4]). With this, it was constructed a Hodge theory of $M_n(\mathbb{C})$ (see [DVKM90], p. 319).

Considering the space of *harmonic elements* $\mathrm{Ker}\Delta$ (see [Mor98], p. 155), we have that the map

$$\eta : \mathrm{Ker}\Delta \to H_D(M_n(\mathbb{C})) : \alpha \mapsto [\alpha],$$

is an isomorphism of graded vector spaces, where $H_D(M_n(\mathbb{C}))$ is the cohomology[5] complex of $\Omega_D(\mathcal{A})$. $\mathrm{Ker}\Delta$ is also the set of elements invariant of the subalgebra $1 \otimes \wedge sl(n,\mathbb{C})^*$, which as we say before, it is the wedge generation of the dual basis of $\mathfrak{d}\mathcal{A}$ (see [DVKM90], p. 319).

Also, it is characterized some *connections* (see [Con85], p. 326), i.e., $\nabla : \mathcal{M} \to \mathcal{M} \otimes_{M_n(\mathbb{C})} \Omega^1_D(M_n(\mathbb{C}))$, is a linear map, where $\mathcal{M}$ is a right $M_n(\mathbb{C})$-module, and $\nabla(\phi A) = (\nabla\phi)A + \phi \otimes dA$, for all $\phi \in \mathcal{M}$ and all $A \in M_n(\mathbb{C})$. This characterization is do it when there exists a *Hermitian form* $h : \mathcal{M} \times \mathcal{M} \to M_n(\mathbb{C})$ such that $(\mathcal{M}, h)$ is a *Hermitian structure* with $\mathcal{M}$ being a free Hermitian module of rank one generated by a gauge element, and with this, it is described the behavior of any gauge invariant connection ∇, from the shape of a particular Hermitian connection ∇^0. As in [Con85], p. 327 it is extended connections to all the structure $\mathcal{M} \otimes_{M_n(\mathbb{C})} \Omega_D(M_n(\mathbb{C}))$, and with this the notion of curvature is discused, showing that a Hermitian connection ∇ over the simplest Hermitian module $\mathcal{M}$ with vanishing curvature $\nabla^2 = 0$, must be a gauge connection or must be $\nabla = \nabla^0$.

1.1.6 Skew derivations

In this section, we present the general notion of skew derivation and a universal property of the universal differential calculus over these skew derivations.

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[4] In Hodge theory over a manifold M, if $\omega \in \Omega^p(M)$, then $*(*\omega) = (-1)^{(n-p)p}\omega$ (see [Mor98], p. 151).

[5] $H^k_D(M_n(\mathbb{C})) = \mathrm{Ker}d^{k+1}/\mathrm{Im}d^k$.

Definition 1.1.35 ([GMS05], p. 493)**.** Given Q an $\mathcal{A}$-bimodule, the set $\mathfrak{d}(\mathcal{A},Q)$ of *Q-valued derivations* of $\mathcal{A}$, is the set of $\mathbb{K}$-module homomorphisms $\Delta \in \mathrm{Hom}_{\mathcal{K}}(\mathbb{A},Q)$ such that

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad a,b \in A.$$

If $\Delta \in \mathfrak{d}(\mathcal{A},Q)$ and $c \in Z(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba \text{ for all } b \in \mathcal{A}\}$, then we have that $c\Delta \in \mathfrak{d}(\mathcal{A},Q)$, and therefore, $\mathfrak{d}(\mathcal{A},Q)$ has a $Z(\mathcal{A})$-bimodule structure.

Definition 1.1.36. If $\delta : \mathcal{A} \to \mathcal{A}$ is an $\mathcal{A}$-valued derivation of $\mathcal{A}$ or simply, a *derivation* of $\mathcal{A}$, we say that δ is a *inner derivation* of $\mathcal{A}$, if there exists $a \in \mathcal{A}$ such that for all $b \in \mathcal{A}$, $\delta(b) = \delta_a(b) := ab - ba$.

The following example was mentioned in [DV88], p. 406, without proof and in this work we present a detailed treatment of it.

Example 1.1.37. If we consider $\mathbb{C}$ and $\mathcal{A} = Q = M_n(\mathbb{C})$, we get that any derivation of $\mathcal{A}$ is an inner derivation.

If δ is a derivation of $\mathcal{A}$, since $M_n(\mathbb{C}) = \mathbb{C}I \oplus sl(n,\mathbb{C})$, where $sl(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \mathrm{tr}(A) = 0\}$, and $\delta(I) = \delta(II) = \delta(I)I + i\delta(I) = 2\delta(I)$, we get that $\delta(I) = 0$. Therefore, δ is completely determined by the restriction $\delta|_{sl(n,\mathbb{C})} := \delta'$.

We also have that δ' is an endomorphism of $sl(n,\mathbb{C})$: for any pair $A, B \in M_n(\mathbb{C})$,

$$\mathrm{tr}(\delta'([A,B])) = \mathrm{tr}(\delta'(AB - BA)) = \mathrm{tr}(\delta'(A)B + A\delta'(B) - \delta'(B)A - B\delta'(A)) = 0,$$

where the last equation is due to $\mathrm{tr}(CD) = \mathrm{tr}(DC)$, for any $C, D \in M_n(\mathbb{C})$. Now, considering the canonical basis $\{E_{ij} \in M_n(\mathbb{C}) : 1 \leq i,j \leq n\}$, we have that the set $\mathcal{B} = \{E_{ij}\}_{1 \leq i \neq j \leq n} \cup \{E_{11} - E_{ii}\}_{2 \leq i \leq n}$ is a basis of $sl(n,\mathbb{C})$. Since for any E_{pm} and E_{rs}, $E_{pm}E_{rs} = 0$ if $m \neq r$, and $E_{pm}E_{rs} = E_{ps}$ if $m = r$, we get that

$$[E_{pm}, E_{rs}] = \begin{cases} E_{pp} - E_{rr} & \text{if } m = r \text{ and } s = p \\ -E_{rm} & \text{if } m \neq r \text{ and } s = p \\ E_{ps} & \text{if } m = r \text{ and } s \neq p \\ 0 & \text{if } m \neq r \text{ and } s \neq p. \end{cases}$$

Therefore, any element of $\mathcal{B}$ is a bracket, and then $\mathrm{tr}(\delta'(\mathcal{B})) = \{0\}$, which implies that $\delta'(sl(n,\mathbb{C})) \subseteq sl(n,\mathbb{C})$.

From the Lie algebras theory, we have that all the derivations of a semisimple[6] Lie algebra are inner (see [Jam72], p 23), and since $sl(n,\mathbb{C})$ is a semisimple Lie algebra, we get that δ' is inner. In consequence, δ is inner too, as we desired.

Remark 1.1.38. In Lie algebras theory, the inner derivation δ_a for a an element of a Lie algebra $\mathcal{A}$ is called the *adjoint* of a, and it is denoted by $\mathrm{ad}_a : \mathcal{A} \to \mathcal{A} : b \mapsto [a,b]$, where $[\cdot,\cdot]$ denotes the bracket of $\mathcal{A}$. In fact, if $\dim_{\mathbb{K}}(\mathcal{A}) = n$, the map $\mathrm{ad} : \mathcal{A} \to \mathrm{Der}(\mathcal{A}) \subset M_n(\mathbb{K})$ is called the *adjoint representation* which is a main object of the study of Lie algebras. In this case, the set of inner derivations is $\mathrm{Im}(\mathrm{ad})$, and if $\mathcal{A}$ is a semisimple Lie algebra, we get that not only ad is onto $\mathrm{Der}(\mathcal{A})$ but it is an isomorphism of lie algebras (c.f. [Jam72]).

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[6] A Lie algebra is called *semisimple*, if the radical of the algebra, i.e. the maximal soluble ideal, it is the zero ideal (see [Jam72], p.11).

In the case of $P = \mathcal{A}$, we can define a more general object than the Q-valued derivations of $\mathcal{A}$ as follows.

Definition 1.1.39 ([Brz16b], p. 4)**.** Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a ring automorphism, P an $\mathcal{A}$-bimodule and $\Delta \in \mathrm{Hom}_{\mathcal{K}}(\mathcal{A}, P)$. We say that Δ is a *right σ-skew derivation*, if for all $a, b \in \mathcal{A}$, $\Delta(ab) = \Delta(a)\sigma(b) + a\Delta(b)$.

Example 1.1.40 ([Art15], p.2)**.** Let σ be a ring automorphism of $\mathcal{A}$ and $a \in \mathcal{A}$. The assignment $\mathrm{ad}_\sigma(a)(r) := ax - \sigma(x)a$ defines a right σ-skew derivation $\mathrm{ad}_\sigma(a) : \mathcal{A} \to \mathcal{A}$. Let $r, s \in \mathcal{A}$, then

$$\begin{aligned}
[(\mathrm{ad}_\sigma a)(r)]s + \sigma(r)(\mathrm{ad}_\sigma a)(s) &= [ar - \sigma(r)a]s + \sigma(r)(as - \sigma(s)a) \\
&= ars - \sigma(r)as + \sigma(r)as - \sigma(r)\sigma(s)a) \\
&= ars - \sigma(r)\sigma(s)a \\
&= (\mathrm{ad}_\sigma a)(rs).
\end{aligned}$$

As $\mathrm{ad}_\sigma a$ is linear and satisfies the σ-Leibniz's rule, $\mathrm{ad}_\sigma a$ is a left σ-skew derivation of $\mathcal{A}$.

Remark 1.1.41. The right σ-skew derivations $\mathrm{ad}_\sigma(a)\mathcal{A} \to \mathcal{A}$ are called *inner skew derivations* (see [AB18], p. 4).

The following two examples are mentioned in [AB18]. Nevertheless, in this work we present their proof for completeness of the document.

Example 1.1.42. Let $\mathcal{A} = \mathbb{K}[h]$ and let $q \in \mathbb{K}$ such that q is not a n-root of unity, for a fixed $n \in \mathbb{N}$. We define the *Jackson's derivative* or the *q-derivative* J_{q^n} of a polynomial $f(h) \in \mathcal{A}$ as

$$J_{q^n}(f(h)) = f'_{q^n}(h) = \frac{f(q^n h) - f(h)}{(q^n - 1)h},$$

if $n \geq 1$, and as the usual polynomial derivation, if $n = 0$. If we consider the automorphism $\varphi^n : \mathbb{K}[h] \to \mathbb{K}[h] : g(h) \mapsto g(q^n h)$, we obtain that the Jackson's derivative is a right φ-skew derivation. The proof of the $\mathbb{K}$-linearity is short. If $f(h), g(h) \in \mathbb{K}[h]$, we obtain that

$$\begin{aligned}
f'_{q^n}(h)g(q^n h) + f(h)g'_{q^n}(h) &= \frac{f(q^n h) - f(h)}{(q^n - 1)h} g(q^n h) + f(h)\frac{g(q^n h) - g(h)}{(q^n - 1)h} \\
&= \frac{f(q^n h)g(q^n h) - f(h)g(h)}{(q^n - 1)h} \\
&= (f(h)g(h))'_{q^n}(h).
\end{aligned}$$

Example 1.1.43. If $\partial : \mathbb{K}[h] \to \mathbb{K}[h]$ is a right φ^n-skew derivation of $\mathbb{K}[h]$ (φ^n defined as in Example 1.1.42), then we obtain that $\partial = a(h)J_{q^n}$: We prove this fact for the $\mathbb{K}[h]$-basis $\{h^k : k \in \mathbb{N}\}$, and then the assumption will be hold for all $f(h) \in \mathbb{K}[h]$ by the $\mathbb{K}$-linearity of J_{q^n}. We claim that for all positive $k \in \mathbb{N}$,

$$\partial(h^k) = \left[\sum_{i=0}^{k-1} (q^n)^i\right] h^{k-1}\partial(h).$$

By induction, since the case $k = 1$ is immediate, we suppose the assumption holds for k. Then

$$\begin{aligned}\partial(h^{k+1}) &= \partial(h^k)\varphi^n(h) + h^k\partial(h)\\ &= \left[\sum_{i=0}^{k-1}(q^n)^i\right]h^{k-1}\partial(h)q^n h + h^k\partial(h)\\ &= \left[\sum_{i=0}^{k-1}(q^n)^{i+1}\right]h^k\partial(h) + h^k\partial(h)\\ &= \left[\sum_{i=0}^{k}(q^n)^i\right]h^k\partial(h),\end{aligned}$$

which proves the assumption. Since

$$\left[\sum_{i=0}^{k}(q^n)^i\right]h^k = \frac{(q^n)^{k+1}-1}{q^n-1}h^k = \frac{(q^n h)^{k+1}-h^{k+1}}{(q^n-1)h} = J_{q^n}(h^k),$$

it follows that $\partial(h^k) = J_{q^n}(h^k)a(h)$, with $a(h) = \partial(h)$.

In the realization of this work, we tried to generalize Proposition 1.1.22 (following faithfully the proof of this in [Lan97]) to the right σ-skew derivations. With this in mind, we consider the following: as P is an $\mathcal{A}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{A}$ is a ring homomorphism (isomorphism), we can consider P with the left-$\mathcal{A}$ structure $*$ defined as $p * f = p\sigma(f)$, for all $p \in P$ and $f \in \mathcal{A}$. We denote P_*^σ the $\mathcal{A}$-bimodule constructed from take P with its original right-$\mathcal{A}$ operation and the left-$\mathcal{A}$ $*$. We obtain the following result.

Proposition 1.1.44. *Let P be an $\mathcal{A}$-bimodule. Every right σ-skew derivation $\Delta : \mathcal{A} \to P$ can be decomposed as $\Delta = \rho^\Delta \circ d$, where $\rho^\Delta : \Omega^1(\mathcal{A}) \to P$ is an A-module homomorphism.*

Proof. We have that the map $id_P : P \to P_*^\sigma : p \mapsto p$ is a right $\mathcal{A}$-module isomorphism. Then, if $\Delta : \mathcal{A} \to P$ is a right σ-skew derivation we have that $\overline{\Delta} = id_P \circ \Delta : \mathcal{A} \to P_*^\sigma$ is a P_*^σ-valued derivation of $\mathcal{A}$:

$$\begin{aligned}\overline{\Delta}(ab) = id_P(\Delta(ab)) &= id_P(\Delta(a)\sigma(b) + a\Delta(b))\\ &= \overline{\Delta}(a) * b + a\overline{\Delta}(b).\end{aligned}$$

Therefore, by Proposition 1.1.22, there exists an $\mathcal{A}$-bimodule homomorphism $\rho^{\overline{\Delta}} : \Omega(\mathcal{A}) \to P_*^\sigma$ such that $\overline{\Delta} = \rho^{\overline{\Delta}} \circ d$, and then $\Delta = (id_{P*}^\sigma)^{-1} \circ \rho^{\overline{\Delta}} \circ d$, where $id_Q^\alpha : Q \to Q_*^\alpha : q \mapsto q$ is a right $\mathcal{A}$ isomorphism, and $\rho^\Delta = (id_{P*}^\sigma)^{-1} \circ \rho^{\overline{\Delta}} : \Omega(\mathcal{A}) \to P$ is an $\mathcal{A}$-bimodule homomorphism. □

1.2 Integral calculus

In this section, we study the Brzezinski's construction of a differential and integral calculus over a $\mathbb{K}$- algebra $\mathcal{A}$ from a hom-connection, which is an algebraic connection constructed with twisted multi-derivations of a $\mathcal{A}$.

1.2.1 Hom-connections

In [Brz08] the *hom-connection* is motivated as a second part, through the functor Hom (which is right-adjoint of the tensor product), of a connection (see [Con85], p. 326), which is a linear map such that $\nabla : M \to M \otimes \Omega^1(\mathcal{A})$, where $\Omega(\mathcal{A})$ is a differential calculus, M is an left $\mathcal{A}$-module and ∇ satisfy the condition $\nabla(am) = da \otimes_{\mathcal{A}} m + a\nabla(m)$, for all $a \in \mathcal{A}$ and all $m \in M$. This non-commutative algebraic definition of connection is a replace of definitions of of commutative connections that appears in differential geometry in terms of global sections of vector bundles over a smooth manifold. In [Brz08] the pairing between connections and hom-connections was established in some cases when M is a $(\mathcal{A}, \mathcal{A}')$-bimodule or a comodule.

Our interest in hom-connections is due to a definition of first order integral calculus introduced in [Brz16b], where the cokernel of the hom-connection is the space of integrals of $\mathcal{A}$. These are algebraic objects defined in [Brz16b] as a dual algebraic notion of right connections on supermanifolds.

Definition 1.2.1 ([BKL10], p. 284)**.** Let M be a right $\mathcal{A}$-module and $(\Omega(\mathcal{A}), d)$ a differential $\mathbb{N}-$graded algebra with $\Omega^0(\mathcal{A}) = \mathcal{A}$.

1. We say that (M, ∇) is a *right hom-connection*, if $\nabla : \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M) \to M$ is a $\mathcal{K}$-linear map such that, for all $f \in \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M)$ and $a \in \mathcal{A}$,
$$\nabla(fa) = \nabla(f)a + f(da).$$
2. We define for $n > 0$, $\nabla_n : \mathrm{Hom}_{\mathcal{A}}(\Omega^{n+1}(\mathcal{A}), M) \to \mathrm{Hom}_{\mathcal{A}}(\Omega^n(\mathcal{A}), M) : f \mapsto \nabla_n(f)$ such that, for all $\omega \in \Omega^n(\mathcal{A})$, $\nabla_n(f)(\omega) = \nabla(f\omega) + (-1)^{n+1} f(d\omega)$. Considering $n = 1$, we define the curvature of (M, ∇) as the map $\nabla \circ \nabla_1$ and say that (M, ∇) is a *flat hom-connection*, if $\nabla \circ \nabla_1 = 0$.
3. If (M, ∇) is a flat connection, we define the complex $(\bigoplus_{n\geq 0} \mathrm{Hom}_{\mathcal{A}}(\Omega^n(\mathcal{A}), M), \nabla)$. The homology of this complex when $M = \mathcal{A}$ is called a *complex of integral forms* on $\mathcal{A}$, denoted $H_*(\mathcal{A}, \mathcal{A}, \nabla)$, and the canonical map $\Lambda : \mathcal{A} \to \mathit{coker}(\nabla) = H_0(\mathcal{A}, \mathcal{A}, \nabla)$ is called a ∇-*integral* on $\mathcal{A}$. Calling $\mathcal{I}_k\mathcal{A} := \mathrm{Hom}_{\mathcal{A}}(\Omega^k\mathcal{A}, \mathcal{A})$, we denote this complex of integral forms as $(\mathcal{I}\mathcal{A}, \nabla)$.

We say some words about the previous definition.

- The right $\mathcal{A}$-module structure of $\mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M)$ is given by $(fa)(\omega) := f(a\omega)$, for each map $f \in \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M)$ and $a \in \mathcal{A}$.
- The right $\Omega(A)$-module structure of $\bigoplus_{n\geq 0} \mathrm{Hom}_{\mathcal{A}}(\Omega^n(\mathcal{A}), M)$, with the action
$$f\omega(\omega') := f(\omega\omega'),$$
for $\omega \in \Omega^n(\mathcal{A})$, $f \in \mathrm{Hom}_{\mathcal{A}}(\Omega^{n+m}(\mathcal{A}), M)$ and $\omega' \in \Omega^m(\mathcal{A})$.
- The following detail was completed in this work. If we want to see that the chain of homomorphisms is a complex, we have to verify that $\nabla_n \circ \nabla_{n+1} = 0$. Let us see. $\nabla_n : \mathrm{Hom}_{\mathcal{A}}(\Omega^{n+1}(\mathcal{A}), M) \to \mathrm{Hom}_{\mathcal{A}}(\Omega^n(\mathcal{A}), M)$ and $\nabla_{n+1} : \mathrm{Hom}_{\mathcal{A}}(\Omega^{n+2}(\mathcal{A}), M) \to$

$\mathrm{Hom}_{\mathcal{A}}(\Omega^{n+1}(\mathcal{A}), M)$.

If $g \in \mathrm{Hom}_{\mathcal{A}}(\Omega^{n+2}(\mathcal{A}), M)$, then $\nabla_{n+1}(g) \in \mathrm{Hom}_{\mathcal{A}}(\Omega^{n+1}(\mathcal{A}), M)$ and therefore

$$\nabla_n(\nabla_{n+1}(g)) \in \mathrm{Hom}_{\mathcal{A}}(\Omega^n(\mathcal{A}), M).$$

Let $\omega \in \Omega^n(\mathcal{A})$. Then $\nabla_{n+1}(g)\omega \in \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M)$ and $d\omega \in \Omega^{n+2}(\mathcal{A})$, whence,

$$\begin{aligned}\nabla_n(\nabla_{n+1}(g))(\omega) &= \nabla(\nabla_{n+1}(g)\omega) + (-1)^{n+1}\nabla_{n+1}(g)(d\omega)\\ &= \nabla(\nabla_{n+1}(g)\omega) + (-1)^{n+1}(\nabla(gd\omega) + (-1)^n g(dd\omega))\\ &= \nabla(\nabla_{n+1}(g)\omega) + (-1)^{n+1}\nabla(gd\omega)\\ &= \nabla(\nabla_{n+1}(g)\omega + (-1)^{n+1}g(d\omega)).\end{aligned}$$

Hence, $\nabla_{n+1}(g)\omega + (-1)^{n+1}g(d\omega) \in \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M)$. If $\alpha \in \Omega^1(\mathcal{A})$, we obtain that,

$$\begin{aligned}&[\nabla_{n+1}(g)\omega + (-1)^{n+1}g(d\omega)](\alpha)\\ &\quad= [\nabla_{n+1}(g)\omega](\alpha) + [(-1)^{n+1}g(d\omega)](\alpha)\\ &\quad= \nabla_{n+1}(g)(\omega\alpha) + (-1)^{n+1}g((d\omega)\alpha)\\ &\quad= \nabla(g\omega\alpha) + (-1)^n g(d(\omega\alpha)) + (-1)^{n+1}g((d\omega)\alpha)\\ &\quad= \nabla(g\omega\alpha) + (-1)^n g(d(\omega)\alpha + (-1)^n\omega d\alpha) + (-1)^{n+1}g((d\omega)\alpha)\\ &\quad= \nabla(g\omega\alpha) + (-1)^n g(d(\omega)\alpha) + g(\omega d\alpha) + (-1)^{n+1}g((d\omega)\alpha)\\ &\quad= \nabla(g\omega\alpha) + g(\omega d\alpha)\\ &\quad= \nabla_1(g\omega)(\alpha),\end{aligned}$$

and so $[\nabla_{n+1}(g)\omega + (-1)^{n+1}g(d\omega)] = \nabla_1(g\omega)$. In this way,

$$\nabla_n(\nabla_{n+1}(g))(\omega) = \nabla(\nabla_{n+1}(g)\omega + (-1)^{n+1}g(d\omega)) = \nabla(\nabla_1(g\omega)) = (\nabla \circ \nabla_1)(g\omega).$$

If (M, ∇) is a flat connection, then $\nabla_n \circ \nabla_{n+1} = 0$, for all $n \geq 0$, (where $\nabla_0 = \nabla$) and we obtain the complex :

$$\begin{tikzcd}\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(\Omega^3(\mathcal{A}), M) \arrow[r, "\nabla_2"'] & \mathrm{Hom}_{\mathcal{A}}(\Omega^2(\mathcal{A}), M) \arrow[r, "\nabla_1"'] & \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), M) \arrow[r, "\nabla"'] & M\end{tikzcd}$$

Remark 1.2.2. In the literature, we found the following important facts.

- In [BKL10], Theorem 2.2, a right $\mathcal{A}$-module M has a right hom-connection with respect to the universal differential graded algebra if and only if M is $\mathcal{A}$-injective.
- In [BKL10], Corollary 2.3, it was shown that for an injective and finitely cogenerated $\mathcal{A}$-module M, there exists a right hom-connection (this for any differential graded algebra).
- In [BKL10], Proposition 2.4, it was proved that a direct sum of $\mathcal{A}$-modules, such that each adding up has a right hom-connection, both respect to the same arbitrary differential graded algebra, has a right hom-connection.

Remark 1.2.3. We formulate the following questions about hom-connections.

1. In the proof of Theorem 2.2 of [BKL10], it was used the following fact that is not clear for us: *if M is a right $\mathcal{A}$-injective module then* $\mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M)$ *is injective.*
2. Is there some relation between the different hom-connections that we can associate to an $\mathcal{A}$-module M?

Now, with this notion of hom-connection, we can define a notion of integral calculus as follows.

Definition 1.2.4 ([Brz16b], p. 9)**.** Let (Ω, d) be a first order differential calculus. The pair (∇, Λ) is called a *first-order integral calculus* on $\mathcal{A}$, where:

1. $\nabla : \mathrm{Hom}_{\mathcal{A}}(\Omega, \mathcal{A}) \to \mathcal{A}$ is a hom-connection.
2. $\Lambda : \mathcal{A} \to \mathrm{coker}(\nabla) : a \mapsto \overline{a}$, where $\mathrm{coker}(\nabla) = \mathcal{A}/\mathrm{Img}(\nabla)$.

In the next section, we present a way of constructing a hom-connection over an affine algebra using a matrix notion of a skew derivation.

1.2.2 Brzezinski's calculus

In [BKL10], we found a way to obtain a first order differential and integral calculus of an algebra $\mathcal{A}$ as follows.

Definition 1.2.5 ([BKL10], p. 287)**.** If $\sigma : \mathcal{A} \to M_n(\mathcal{A})$ is an algebra homomorphism and $\partial : \mathcal{A} \to \mathcal{A}^n$ is a $\mathcal{K}$-linear map, we say that the pair (∂, σ) is a *right twisted multi-derivation*, if for all $a, b \in \mathcal{A}$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b). \tag{1.2.1}$$

If we denote $\partial(a) = (\partial_1(a), ..., \partial_n(a))$ and $\sigma(b) = (\sigma_{ij}(b))_{i,j \leq n}$, where $\sigma_{ij} \in \mathrm{End}_{\mathbb{K}}(\mathcal{A})$ we write

$$(\partial_1(ab), ..., \partial_n(ab)) = (\partial_1(a), ..., \partial_n(a)) \begin{pmatrix} \sigma_{11}(b) & \cdots & \sigma_{1n}(b) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(b) & \cdots & \sigma_{nn}(b) \end{pmatrix} + a(\partial_1(b), ..., \partial_n(b))$$

Then, we have that the expression (1.2.1) is equivalent to say that, for all $1 \leq i \leq n$, we have:

$$\partial_i(ab) = \left[\sum_{r=1}^{n} \partial_r(a)\sigma_{ri}(b) \right] + a\partial_i(b). \tag{1.2.2}$$

If we begin with (∂, σ) a right-multi derivation on $\mathcal{A}$, and show that there exists $\overline{\sigma} \in M_n(\mathrm{End}(\mathcal{A}))$ such that the matrix product between σ and $\overline{\sigma}$ satisfies $\sigma\overline{\sigma} = \overline{\sigma}\sigma = \mathbb{I}$ (where σ is also seen as an element of $M_n(\mathrm{End}(\mathcal{A}))$, having that the composition is the product in $\mathrm{End}(\mathcal{A})$ and $\mathbb{I} = (\delta_{ij})$ is the identity in $M_n(\mathrm{End}(\mathcal{A}))$), from Lemma 3.2 it follows that from a freely left $\mathcal{A}$-module $\mathcal{A}\omega_1 \oplus \cdots \oplus \mathcal{A}\omega_l$, we have that for the canonical σ right structure, the same left base $\{\omega_1, ..., \omega_l\}$ is a right base with the right action defined by

$$\omega_i a = \sum_{j=1}^{n} \sigma_{ij}(a)\omega_j. \tag{1.2.3}$$

Remark 1.2.6. By [BKL10], Proposition 3.3, the existence of $\overline{\sigma}$ can be ensured when $\sigma \in M_n(\mathcal{A})$ is an upper-triangular matrix with non zero diagonal components, if we show that the diagonal entries σ_{ii} are invertible, for all $1 \leq i \leq n$.

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ 0 & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_{nn} \end{pmatrix}.$$

If $\omega_1 = (1, 0, 0, ..., 0)$, $\omega_2 = (0, 1, 0, ..., 0)$, and so on, we can define ω_i, for $i \leq n$. The $\mathcal{A}$-bimodule $\mathcal{A}^n = \mathcal{A}\omega_1 \oplus \cdots \oplus \mathcal{A}\omega_l$ will be our first order differential calculus $\Omega^1(\mathcal{A})$, defining

$$d : \mathcal{A} \to \Omega^1(\mathcal{A}) : a \mapsto d(a) = \sum_{i=1}^{n} \partial_i(a)\omega_i. \tag{1.2.4}$$

Since all of these ideas were developed in Brzezinski's works, in this work we called with his surname the differential calculus constructed before.

Definition 1.2.7. When the structure $(\mathcal{A}^n, d)$ constructed above is a first order differential calculus $(\mathcal{A}^n, d)$, i.e., when it satisfies the density condition of Definition 1.1.1, it is called the *Brzezinski's differential calculus* of (∂, σ).

The following proposition is very important to our work, because this provides a way to construct, from a Brzezinski's differential calculus, an integral calculus on a noncommutative algebra $\mathcal{A}$.

Proposition 1.2.8. ([BKL10], p. 290) *Let (∂, σ) be a right multiderivation on $\mathcal{A}$ such that there exist $\overline{\sigma}, \widehat{\sigma} : \mathcal{A} \to M_n(\mathcal{A})$ which satisfies $\overline{\sigma}\sigma^T = \mathbb{I}$, $\sigma^T\overline{\sigma} = \mathbb{I}$, $\widehat{\sigma}\overline{\sigma}^T = \mathbb{I}$ and $\overline{\sigma}^T\widehat{\sigma} = \mathbb{I}$. If $(\Omega(\mathcal{A}), d)$ is the Brzezinski's differential calculus of (∂, σ) and $\varepsilon_i : \Omega(\mathcal{A}) \to \mathcal{A}$ are the right $\mathcal{A}$-module maps defined by $\varepsilon_i(\omega_j) = \delta_{ij}$, for $i, j = 1, 2, ..., n$, then there exists a unique hom-connection $\nabla : \mathrm{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), \mathcal{A}) \to \mathcal{A}$ such that $\nabla(\varepsilon_i) = 0$, for all $i = 1, ..., n$.*

Before presenting the proof of the Proposition 1.2.8, in this work we found relevant fulling some details that we present in the following two lemmas under the same assumptions.

Lemma 1.2.9. ([BKL10], p. 288) *For each $a \in \mathcal{A}$ and all $i = 1, 2, ..., n$, then $a\omega_i = \sum_l \omega_l\overline{\sigma}_{li}(a)$.*

Proof. By (1.2.3), we know that $\omega_l\overline{\sigma}_{li}(a) = \sum_k \sigma_{lk}(\overline{\sigma}_{li}(a))\omega_k$. As by assumption $\sigma^T\overline{\sigma} = \mathbb{I}$, we have that $\sum_l \sigma_{lk} \circ \overline{\sigma}_{li} = \delta_{ki}$, for all $k, i = 1, ..., n$. Therefore,

$$\sum_l \omega_l\overline{\sigma}_{li}(a) = \sum_{l,k} \sigma_{lk}(\overline{\sigma}_{li}(a))\omega_k = \sum_k \left[\sum_l \sigma_{lk}(\overline{\sigma}_{li}(a))\right]\omega_k = \sum_k \delta_{ki}(a)\omega_k = a\omega_i. \tag{1.2.5}$$

□

Lemma 1.2.10. ([BKL10], p. 291) *For all* $f \in Hom_{\mathcal{A}}(\Omega^1(\mathcal{A}), \mathcal{A})$,

$$f = \sum_{i,k} \varepsilon_i \widehat{\sigma}_{ik}(f(\omega_k)). \tag{1.2.6}$$

Proof. We show that the equality holds when we evaluate on the basic elements ω_j. We know from Lemma 1.2.9 that $\widehat{\sigma}_{ik}(f(\omega_k))\omega_j = \sum_l \omega_l \overline{\sigma}_{lj}(\widehat{\sigma}_{ik}(f(\omega_k)))$, for all $i, k = 1, ..., n$, whence

$$\varepsilon_i(\widehat{\sigma}_{ik}(f(\omega_k))\omega_j) = \varepsilon_i(\sum_l \omega_l \overline{\sigma}_{lj}(\widehat{\sigma}_{ik}(f(\omega_k)))) = \sum_l \varepsilon_i(\omega_l)\overline{\sigma}_{lj}(\widehat{\sigma}_{ik}(f(\omega_k))) = \overline{\sigma}_{ij}(\widehat{\sigma}_{ik}(f(\omega_k))),$$

where the second equation is due the fact that ε_i is a right $\mathcal{A}$-modules map, for all $i = 1, ..., n$, and the last equation follows from the definition of ε_i. Also, as $\overline{\sigma}^T\widehat{\sigma} = \mathbb{I}$, we have that $\sum_i \overline{\sigma}_{ij} \circ \widehat{\sigma}_{ik} = \delta_{jk}$. Hence,

$$\begin{aligned}
\left[\sum_{i,k} \varepsilon_i \widehat{\sigma}_{ik}(f(\omega_k))\right](\omega_j) &= \sum_{i,k} \varepsilon_i(\widehat{\sigma}_{ik}(f(\omega_k))\omega_j) \\
&= \sum_{i,k} \overline{\sigma}_{ij}(\widehat{\sigma}_{ik}(f(\omega_k))) \\
&= \sum_{k} \delta_{jk}(f(\omega_k)) \\
&= f(\omega_j).
\end{aligned}$$

By the right $\mathcal{A}$-linearity of the two maps in mention, this equation holds for all $\omega \in \Omega^1(\mathcal{A})$. Therefore, we conclude the proof. ☐

Now, we proceed to prove Proposition 1.2.8.

Proof. For each $i = 1, ..., n$, we write $\partial_i^\sigma = \sum_{j,k} \overline{\sigma}_{kj} \circ \partial_j \circ \widehat{\sigma}_{ki}$, and define $\nabla(f) =$

$\sum_i \partial_i^\sigma(f(\omega_i))$. For $a \in \mathcal{A}$ and $f \in \operatorname{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), \mathcal{A})$ we have:

$$\begin{aligned}
\nabla(fa) &= \sum_i \partial_i^\sigma(fa(\omega_i)) \\
&= \sum_i \partial_i^\sigma(f(a\omega_i)) \\
&= \sum_{i,j,k} \overline{\sigma}_{kj} \circ \partial_j \circ \widehat{\sigma}_{ki}(f(a\omega_i)) \\
&= \sum_{i,j,k,l} \overline{\sigma}_{kj} \circ \partial_j \circ \widehat{\sigma}_{ki}(f(\omega_l)\overline{\sigma}_{li}(a)) \\
&= \sum_{j,k,l} \overline{\sigma}_{kj} \circ \partial_j(\widehat{\sigma}_{kl}(f(\omega_l))a) \\
&= \sum_{j,k,l,r} \overline{\sigma}_{kj}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\sigma_{rj}(a)) + \sum_{j,k,l} \overline{\sigma}_{kj}(\widehat{\sigma}_{kl}(f(\omega_l))\partial_j(a)) \\
&= \sum_{k,l,r} \overline{\sigma}_{kr}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l))))a + \sum_{j,l} f(\omega_l)\overline{\sigma}_{lj}(\partial_j(a)) \\
&= \sum_l \partial_l^\sigma(f(\omega_l))a + \sum_{j,l} f(\omega_l\overline{\sigma}_{lj}(\partial_j(a))) \\
&= \nabla(f)a + \sum_j f(\partial_j(a)\omega_l) \\
&= \nabla(f)a + f(da),
\end{aligned} \tag{1.2.7}$$

where the equalities hold by the following reasons:

1. Definition of ∇.
2. The right $\mathcal{A}$-module structure of $\operatorname{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), \mathcal{A})$.
3. Definition of ∂_i^σ.
4. First, apply Lemma 1.2.9 and then the equality is obtained by the $\mathcal{A}$-linearity of f and the linearity of both $\overline{\sigma}_{kj}, \partial_j$ and $\widehat{\sigma}_{ki}$.
5. $\sum_i \widehat{\sigma}_{ki}(f(\omega_l)\overline{\sigma}_{li}(a)) = \widehat{\sigma}_{kl}(f(\omega_l))a$: As $\widehat{\sigma}$ is multiplicative, we have that

   $$\widehat{\sigma}(f(\omega_l)\overline{\sigma}_{li}(a)) = \widehat{\sigma}(f(\omega_l))\widehat{\sigma}(\overline{\sigma}_{li}(a)).$$

   If we denote $\widehat{\sigma}(f(\omega_l)\overline{\sigma}_{li}(a)) = (\widehat{\sigma}_{pq}(f(\omega_l)\overline{\sigma}_{li}(a)))$ and $\widehat{\sigma}(f(\omega_l))\widehat{\sigma}(\overline{\sigma}_{li}(a)) = (C_{pq})$, where $C_{pq} = \sum_s \widehat{\sigma}_{ps}(f(\omega_l))\widehat{\sigma}_{sq}(\overline{\sigma}_{li}(a))$, for all $p, q = 1, ..., n$, then

   $$\overline{\sigma}_{ki}(f(\omega_l)\overline{\sigma}_{li}(a)) = \sum_s \widehat{\sigma}_{ks}(f(\omega_l))\widehat{\sigma}_{si}(\overline{\sigma}_{li}(a)),$$

   for all $i = 1, ..., n$. As $\widehat{\sigma}\overline{\sigma}^T = \mathbb{I}$, we have that $\sum_i \widehat{\sigma}_{si} \circ \overline{\sigma}_{li} = \delta_{sl}$, for all $s, l = 1, ..., n$, and therefore, summing on i,

$$\begin{aligned}\sum_i \overline{\sigma_{ki}}(f(\omega_l)\overline{\sigma}_{li}(a)) &= \sum_{i,s} \widehat{\sigma}_{ks}(f(\omega_l))\widehat{\sigma}_{si}(\overline{\sigma}_{li}(a)) \\ &= \sum_s \widehat{\sigma}_{ks}(f(\omega_l))[\sum_i \widehat{\sigma}_{si}(\overline{\sigma}_{li}(a))] \\ &= \sum_s \widehat{\sigma}_{ks}(f(\omega_l))\delta_{sl}(a) \\ &= \widehat{\sigma}_{kl}(f(\omega_l))a.\end{aligned}$$

6. This equation is obtained by the rule (1.2.2) and the linearity of $\overline{\sigma}_{kj}$ and ∂_j.

7. First, we have that $\sum_j \overline{\sigma}_{kj}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\sigma_{rj}(a)) = \overline{\sigma}_{kr}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l))))a$: As by assumption $\overline{\sigma}\sigma^T = \mathbb{I}$, it follows $\sum_j \overline{\sigma}_{sj} \circ \sigma_{rt} = \delta_{sr}$. On the other hand, as $\overline{\sigma}$ is multiplicative, we have that

$$\overline{\sigma}_{kj}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\sigma_{rj}(a)) = \sum_s \overline{\sigma}_{ks}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\overline{\sigma}_{sj}(\sigma_{rj}(a))$$

and therefore, when we sum over j,

$$\begin{aligned}\sum_j \overline{\sigma}_{kj}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\sigma_{rj}(a)) &= \sum_{s,j} \overline{\sigma}_{ks}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\overline{\sigma}_{sj}(\sigma_{rj}(a)) \\ &= \sum_{s,j} \overline{\sigma}_{ks}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\overline{\sigma}_{sj}(\sigma_{rj}(a)) \\ &= \sum_s \overline{\sigma}_{ks}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))[\sum_j \overline{\sigma}_{sj}(\sigma_{rj}(a))] \\ &= \sum_s \overline{\sigma}_{ks}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))\delta_{sr}(a) \\ &= \overline{\sigma}_{kr}(\partial_r(\widehat{\sigma}_{kl}(f(\omega_l)))a.\end{aligned}$$

For the second term, by assumption $\overline{\sigma}^T\widehat{\sigma} = \mathbb{I}$, so $\sum_k \overline{\sigma}_{ks}\widehat{\sigma}_{kl} = \delta_{sl}$, for all $s, l = 1, ..., n$. Also, as $\overline{\sigma}$ is multiplicative, we have that

$$\overline{\sigma}_{kj}(\widehat{\sigma}_{kl}(f(\omega_l))\partial_j(a)) = \sum_s \overline{\sigma}_{ks}(\widehat{\sigma}_{kl}(f(\omega_l)))\overline{\sigma}_{sj}(\partial_j(a)).$$

Therefore, summing on k,

$$\begin{aligned}\sum_k \overline{\sigma}_{kj}(\widehat{\sigma}_{kl}(f(\omega_l))\partial_j(a)) &= \sum_k \sum_s \overline{\sigma}_{ks}(\widehat{\sigma}_{kl}(f(\omega_l)))\overline{\sigma}_{sj}(\partial_j(a)) \\ &= \sum_s [\sum_k \overline{\sigma}_{ks}(\widehat{\sigma}_{kl}(f(\omega_l)))]\overline{\sigma}_{sj}(\partial_j(a)) \\ &= \sum_s \delta_{sl}(f(\omega_l))\overline{\sigma}_{sj}(\partial_j(a)) \\ &= f(\omega_l)\overline{\sigma}_{lj}(\partial_j(a)).\end{aligned}$$

8. This is by definition of ∂_l^σ and the $\mathcal{A}$-linearity of f.

9. The first term is due to the definition of $\nabla(f)$ and the second term is obtained by (1.2.9).
10. This equality follows from the linearity of f and the definition of d in the Brzezinski calculus.

This shows that ∇ is a hom-connection. On the other hand, as $\partial_j(1) = 0$, for each $j = 1, \ldots, n$, we have

$$\begin{aligned}\nabla(\varepsilon_j) &= \sum_i \partial_i^{\sigma}(\varepsilon_j(\omega_i))\\ &= \partial_j^{\sigma}(1)\\ &= \sum_{p,k} \overline{\sigma}_{kp} \circ \partial_p \circ \widehat{\sigma}_{kj}(1) = 0.\end{aligned}$$

Then $\nabla(\varepsilon_j) = 0$, for all $j = 1, \ldots, n$. Now, if $\overline{\nabla}$ is a hom connection such that $\overline{\nabla}(\varepsilon_i) = 0$, for all $i = 1, \ldots, n$, then

$$\begin{aligned}\overline{\nabla}(f) &= \sum_{i,k} \overline{\nabla}(\varepsilon_i \widehat{\sigma}_{ik}(f(\omega_k)))\\ &= \sum_{i,k} \overline{\nabla}(\varepsilon_i)\widehat{\sigma}_{ik}(f(\omega_k)) + \sum_{i,k} \varepsilon_i(d\widehat{\sigma}_{ik}(f(\omega_k)))\\ &= 0 + \sum_{i,j,k} \varepsilon_i(\partial_j(\widehat{\sigma}_{ik}(f(\omega_k)))\omega_j)\\ &= \sum_{i,j,k,l} \varepsilon_i(\omega_l \overline{\sigma}_{lj}(\partial_j(\widehat{\sigma}_{ik}(f(\omega_k)))))\\ &= \sum_{i,j,k,l} \varepsilon_i(\omega_l)\overline{\sigma}_{lj}(\partial_j(\widehat{\sigma}_{ik}(f(\omega_k))))\\ &= \sum_{i,j,k} \overline{\sigma}_{ij}(\partial_j(\widehat{\sigma}_{ik}(f(\omega_k))))\\ &= \nabla(f),\end{aligned}$$

where each equality follows by the following reasons:

1. Lemma 1.2.10 and linearity of $\overline{\nabla}$.
2. Definition of hom-connection.
3. By assumption over $\overline{\nabla}$ for the first term and due to definition of d in the second term.
4. Lemma 1.2.9 and linearity of ε_i.
5. By the right $\mathcal{A}$-linearity of ε_i.
6. Due to $\varepsilon_i(\omega_l) = \delta_{il}$.
7. Definition of ∇.

Then $\overline{\nabla} = \nabla$, which concludes the proof of the uniqueness of ∇. □

Definition 1.2.11. Let $\mathcal{A}$ be a $\mathcal{K}$-algebra and (∂, σ) a free multiderivation on $\mathcal{A}$. The first order integral calculi (see Definition 1.2.4) in Proposition 1.2.8 is called the *Brzezinski's integral calculus* of (∂, σ) on $\mathcal{A}$.

Remark 1.2.12. If we found a set $\{\sigma_1, ..., \sigma_n\}$ of automorphism of algebras of $\mathcal{A}$, and a set $\{\partial_1, ..., \partial_n\}$ of $\mathbb{K}$-linear maps such that ∂_i is a σ_i-right skew derivation of $\mathcal{A}$, for each $i = 1, ..., n$, we construct a diagonal matrix σ such that the diagonal elements are automorphisms of $\mathcal{A}$. In this way,

$$\sigma = \widehat{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_{nn} \end{pmatrix}, \quad \overline{\sigma} = \begin{pmatrix} \sigma_{11}^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_{22}^{-1} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_{nn}^{-1} \end{pmatrix},$$

and hence, we obtain a multiderivation (∂, σ) on $\mathcal{A}$. Just like in [BKL10], the hom-connection of Proposition 1.2.8 in this case is defined by

$$\nabla(f) = \sum_{i=1}^{n} \sigma_i^{-1} \circ \partial_i \circ \sigma_i(f(\omega_i)). \tag{1.2.8}$$

Remark 1.2.13. In the strategy of Remark 1.2.12, if we want to show that the Brzezinski differential calculus of (∂, σ) effectively satisfies the density condition, then we have to prove that there exist two finite subsets $\{a_{it}\}, \{b_{it}\} \subset \mathcal{A}$ such that

$$\sum_{t=1}^{n} a_{it}\partial_k(b_{it}) = \delta_{ik}, \quad \text{for all } i, k = 1, ..., n \tag{1.2.9}$$

In [AB18], p. 6, we found that this is equivalent to guarantee that there exist sets $\{a_{it}\}, \{b_{it}\}, \{c_{it}\} \subset \mathcal{A}$ such that

$$\sum_{i} a_{it}\partial_k(b_{it})\sigma_k(\sigma_i^{-1}(c_{it})) = \delta_{ik}, \qquad \text{for all } i, k = 1, ..., n. \tag{1.2.10}$$

As we can see, it is clear that (1.2.9) imply (1.2.10), taking $c_{it} = 1$, for all pair i and t. On the other hand, if (1.2.10) holds, then by the twisted Leibniz's rules,

$$\sum_{t} a_{it}\partial_k(b_{it}\sigma_i^{-1}(c_{it})) - \sum_{t} a_{it}b_{it}\partial_k(\sigma_i^{-1}(c_{it})) = \sum_{i} a_{it}\partial_k(b_{it})\sigma_k(\sigma_i^{-1}(c_{it})) = \delta_{ik},$$

and we obtain that $\{a_{it} - a_{it}b_{it}\}, \{b_{it}\sigma_i^{-1}(c_{it}), \sigma_i^{-1}(c_{it})\}$ are the required two sets. A twisted multiderivation (∂_i, σ_i) as in Remark 1.2.12 is called an *orthogonal system of skew derivations* (see [AB18], p. 5).

The following theorem gives us sufficiency conditions to guarantee the existence of an ortogonal system of skew derivations.

Proposition 1.2.14 ([AB18], p. 18)**.** *Let* $(\partial_i, \sigma_i)_{i=1}^n$ *be a family of skew derivations on a ring* $\mathcal{A}$*. If there exists* $\{b_1, ..., b_n\} \subset \mathcal{A}$ *such that,*

$$\mathcal{A}\{\partial_i(b_i) : i = 1, ..., n\}\mathcal{A} = \mathcal{A}, \qquad \partial_k(b_i) = 0, \qquad \textit{for all } i \neq k,$$

then $(\partial_i, \sigma_i)_{i=1}^n$ *is an orthogonal system of skew derivations.*

Proof. The fact that the bilateral ideal generated by $\{\partial_i(b_i) : i = 1, ..., n\}$ is equal to $\mathcal{A}$ is equivalent to the existence of finite subsets $\{a_{it}\}, \{c_{it}\} \subset \mathcal{A}$ such that

$$\begin{aligned}
1 &= \sum_t a_{it}\partial_i(b_i)c_{it} \\
&= \sum_t a_{it}\partial_i(b_i)c_{it} + \left[\sum_t a_{it}b_i\partial_i(\sigma_i^{-1}(c_{it})) - \sum_t a_{it}b_i\partial_i(\sigma_i^{-1}(c_{it}))\right] \\
&= \left[\sum_t a_{it}\partial_i(b_i)\sigma_i(\sigma_i^{-1}(c_{it})) + \sum_t a_{it}b_i\partial_i(\sigma_i^{-1}(c_{it}))\right] - \sum_t a_{it}b_i\partial_i(\sigma_i^{-1}(c_{it})) \\
&= \sum_t a_{it}\partial_i(b_i\sigma_i^{-1}(c_{it})) - \sum_t a_{it}b_i\partial_i(\sigma_i^{-1}(c_{it})).
\end{aligned}$$

This gives us the condition (1.2.9) with $i = k$, considering the subsets $\{a_{it}\} \cup \{a_{it}b_i\}$ and $\{b_i\sigma_i^{-1}(c_{it})\} \cup \{\sigma_i^{-1}(c_{it})\}$. With these sets, we check the case when $i \neq k$:

$$\begin{aligned}
&\sum_t a_{it}\partial_k(b_i\sigma_i^{-1}(c_{it})) - \sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it})) \\
&= \left[\sum_t a_{it}\partial_k(b_i)\sigma_k(\sigma_i^{-1}(c_{it})) + \sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it}))\right] - \sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it})) \\
&= \sum_t a_{it}\ 0\ \sigma_k(\sigma_i^{-1}(c_{it})) + \left[\sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it})) - \sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it}))\right] \\
&= \sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it})) - \sum_t a_{it}b_i\partial_k(\sigma_i^{-1}(c_{it})) = 0,
\end{aligned}$$

which completes the proof. □

The idea of Remark 1.2.12 will be our principal tool in the next chapter to verify the smoothness of some PBW algebras, as it was observed in [Brz16b] (see Example 2.1.16).

1.3 Smoothness

There exist multiples notions of smoothness of an algebra $\mathcal{A}$; for instance, in Remark 1.3.1, we list some of them. In this section, we study the notion of *differentially smoothness*, an algebraic structure that, without topological tools, guarantees the existence of an analogous of the star Hodge isomorphism (see [Brz14], p. 2).

Remark 1.3.1. The following are some notions of smoothness of an algebra that we can found in literature.

- A $\mathbb{K}$-algebra $\mathcal{A}$ is said to be *smooth* (see [Sch86], p. 678), if for all nilpotent ideal $\mathcal{N}$ of $\mathcal{A}$, and for all $\mathcal{C}$ of the same nature of $\mathcal{A}$[7], such that for all $\phi : \mathcal{C} \to \mathcal{A}$ that allows construct exact sequences (see [Bou89], 197)

$$0 \to \mathcal{N} \to \mathcal{C} \xrightarrow{\phi} \mathcal{A} \to 0,$$

  then, we obtain that these exact sequences splits, i.e., exist $\phi' : \mathcal{A} \to \mathcal{C}$, such that $\phi' \circ \phi = id_{\mathcal{A}}$ (see [Bou89], p. 211). We want to remark that in [Sch86], p. 678, it was established that the smoothness of $\mathcal{A}$ is equivalent to that $\Omega(\mathcal{A})$ (see Definition 1.1.21) it is a projective module as an $\mathcal{A}^e := \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}^{op}$-module.

- If $\mathbb{K}$ is algebraically closed, a $\mathbb{K}$-algebra $\mathcal{A}$ is said *homologically smooth*, if as a $\mathcal{A}$-bimodule, there exists a finite length projective resolution of finitely generated $\mathcal{A}$-modules (c.f. [B$^+$14]).

- In [DVKMM01], p. 3, there is an approach to the notion of *smooth $*$-algebras*: a C^*-algebra $\mathcal{A}$ is called *smooth-$*$*, if there exists a set of functionals $\mathcal{S}$ with some norm conditions that is Hausdorff under a special topology, also a special condition over a subset of derivations $\mathcal{P}$ of $\mathcal{A}$ and a completeness of a topology on $\mathcal{A}$ constructed from the pair $(\mathcal{S}, \mathcal{P})$ with the $\mathcal{A}$.

In this section, we present a notion of smoothness of an algebra, called the *differentially smooth*, which treats about the existence of a particular differential graded structure associated to an affine algebra $\mathcal{A}$ that satisfies integral and connectivity conditions. This differential graded structure, as in manifolds theory[8], must satisfy that there exists some minimal non negative integer such that the elements of higher degree vanish. In this algebraic approach, that minimal integer is obtained using the *Gelfand-Kirillov dimension* of the algebra.

1.3.1 Gelfand-Kirillov dimension

Let V be generating subspace of a finite generated or affine algebra A, i.e., $V \subset A$, is a $\mathbb{K}$-subspace such that

$$\bigcup_{i=0}^{\infty} A_n = A, \qquad \text{where } A_n = \mathbb{K} + \sum_{i=1}^{n} V^n,$$

where V^n denotes the $\mathbb{K}$-subspace of A spanned by all words in generators of A of length at most n.

Definition 1.3.2 ([KL00], p. 5)**.** For any pair of functions $f, g : \mathbb{N} \to \mathbb{R}$ such that there exist $n', m' \in \mathbb{N}$ for which $f|_{\mathbb{N}_{\geq n'}}$ and $g|_{\mathbb{N}_{\geq m'}}$ are monotone and positive valued functions, where $\mathbb{N}_{\geq n'} = \{n \in \mathbb{N} : n \geq n'\}$, we say that f and g have the same *growth*, if there exist $c, c, m, m' \in \mathbb{R}^+$ such that $f(n) \leq cg(mn)$ and $g(n) \leq c' f(m'n)$, for almost all $n \in \mathbb{N}$. If we have that $f(n) \leq cg(mn)$ for almost $n \in \mathbb{N}$ where $c, n \in \mathbb{R}^+$ fixed, we say that f has minor growth than g and we denote it by $f \leq^* g$.

---

[7] In [Sch86], $\mathcal{A}$ and $\mathcal{C}$ must satisfy the same operations properties of $M_n(\mathbb{K})$.

[8] For a n-dimensional manifold, we have that for all $n' > n$, any n'-differential form is vanish (see [Mor98], p. 64)

Example 1.3.3 ([KL00], p. 6)**.** If $f, g : \mathbb{N} \to \mathbb{R}$ are polynomial functions such as in the Definition 1.3.2, f and g have the same growth if and only if $\deg(f) = \deg(g)$.

For a $\mathbb{K}$-algebra A and V a generating subspace, we define the map $\dim_V : \mathbb{N} \to \mathbb{R}$ with $\dim_v(n) := \dim_{\mathbb{K}}(A_n)$. The algebra A is said to have *polynomial growth*, if $\dim_V$ and n^ν have the same growth for a fixed $\nu \in \mathbb{N}$, i.e., if there exist $c \in \mathbb{R}$ and $\nu \in \mathbb{N}$ such that $\dim_{\mathbb{K}}(A_n) \leq cn^\nu$ for all sufficiently large n.

Definition 1.3.4 ([KL00], p. 14)**.** The *Gelfand-Kirillov dimension* of an algebra A is the real number defined as

$$\mathrm{GKdim}(A) = \overline{\lim} \log_n(\dim_v(n)),$$

where $\overline{\lim}$ the upper limit of a sequence[9], or equivalently

$$\begin{aligned} \mathrm{GKdim}(A) :&= \inf\{\nu \in \mathbb{R} : \dim_{\mathbb{K}}(A_n) \leq n^\nu, n >> 0\} \\ &= \inf\{\nu \in \mathbb{R} : \dim_V(n) \leq^* n^\nu\}. \end{aligned}$$

In the case of commutative affine algebras with polynomial growth, the Gelfand-Kirillov dimension coincides with the dimension of the underlying affine space (the Krull dimension of its coordinate algebra).

Example 1.3.5. For $\mathbb{K}[x_1, ..., x_n]$, $\mathrm{GKdim}(\mathbb{K}[x_1, ..., x_n]) = n$.

Example 1.3.6. In the case of $M_n(\mathbb{K})$, we have $\mathrm{GKdim}(M_n(\mathbb{K})) = 0$, because $V = M_n(\mathbb{C})$, a generating finite dimensional vector space, satisfies $V^k = V$, for all $k \in \mathbb{N}$, due to $1 \in M_n(\mathbb{K})$. Therefore, $\overline{\lim} \log_k(\dim_v(k)) = \overline{\lim} \log_k(n) = 0$.

The following result is very useful to compute the Gelfand-Kirillov dimension.

Proposition 1.3.7 ([KL00], p. 13)**.** *If f, g are as in Definition 1.3.2, then f and g have the same growth if and only if $\overline{\lim} \log_n(f(n)) = \overline{\lim} \log_n(g(n)) = \overline{\lim} \log_n(g(n))$.*

1.3.2 Differentially smoothness

We proceed to give preliminaries to the differentially smoothness. First, given a left A-module X with action $a \cdot x$, for all $a \in A$, $x \in X$, and an algebra automorphism ν of A, the notation ${}^\nu X$ stands for X with the A-module structure twisted by ν, i.e., with the action $a \otimes x \mapsto \nu(a) \cdot x$.

Definition 1.3.8 ([BS17], p. 416)**.** We say that a differential calculus $(\Omega\mathcal{A}, d)$, where $\Omega\mathcal{A} = \bigoplus_{i\in\mathbb{N}} \Omega^i\mathcal{A}$, with $\Omega^i\mathcal{A} = \bigwedge_{j=1}^{i} \Omega^1\mathcal{A}$ and $\Omega^0\mathcal{A} = \mathcal{A}$, it is of dimension n, if $\Omega^n\mathcal{A} \neq 0$, and $\Omega^m = 0$, for all $n < m$.

Definition 1.3.9 ([BL18], p 2)**.** We say that $\Omega^n\mathcal{A}$ *admits a volume form*, if $\Omega^n\mathcal{A}$ is isomorphic to $\mathcal{A}$ as a right and left module (not necessary as a bimodule). If this is the case, and ω is a right generator of $\Omega^n\mathcal{A}$, we say that $\omega \in \Omega^n\mathcal{A}$ is a *volume form*.

We call a first order differential calculus (Ω, d) *connected*, if $\mathrm{Ker}(d|_{\Omega^0}) = \mathbb{C}$.

---

[9] For a sequence $(x_n)_{n\in\mathbb{N}}$, the upper limit is defined as $\overline{\lim}\, x_n = \lim_{n\to\infty} \sup_{k\geq n} x_n$.

Remark 1.3.10. For a differential calculus that admits a volume form, we define for each right generator $\omega \in \Omega^n\mathcal{A}$, the algebra automorphism $\nu_\omega : \mathcal{A} \to \mathcal{A}$ such that for each $a \in \mathcal{A}$, $a\omega = \omega\nu_\omega(a)$.

Definition 1.3.11 ([BS17], p 417)**.** An n-dimensional differential calculus ΩA is said to be *integrable*, if ΩA admits a complex of integral forms $(\mathcal{I}A, \nabla)$ (see Definition 1.2.1) for which there exist an algebra automorphism ν of A and A-bimodule isomorphism $\Theta_k : \Omega^k A \to^\nu \mathcal{I}_{n-k}A$, $k = 0, ..., n$, rendering commutative the following diagram, for each $k = 0, ..., n-1$:

$$\begin{tikzcd}
\Omega^k A \arrow[r, "d"] \arrow[d, "\Theta_k"] & \Omega^{k+1} A \arrow[d, "\Theta_{k+1}"] \\
\mathcal{I}_{n-k}A \arrow[r, "\nabla_k"] & \mathcal{I}_{n-(k+1)}A
\end{tikzcd}$$

The n-form $\omega := \Theta_n^{-1}(1) \in \Omega^n A$ is called an *integrating volume form.*

Remark 1.3.12. In [BS17] there are some examples of integral differential calculus considering $M_n(\mathbb{C})$ (see Remark 1.1.34).

We present the following theorem of equivalence with the proof for just one direction, only with the aim of understanding a construction of it.

Theorem 1.3.13 ([BS17], p 417)**.** *ΩA is an n-dimensional integrable differentiable calculus over A if and only if ΩA has a volume form ω such that all left multiplication maps $l^k_{\pi_\omega} : \Omega^k A \to \mathcal{I}_{n-k}A : \omega' \mapsto \pi_\omega \cdot \omega'$, for $k = 1, ..., n-1$, are bijective, where π_ω is the right A-module isomorphism $\Omega^n A \to A$ corresponding to the volume form ω with $\pi_\omega(\omega a) = a$.*

Proof. $(\Leftarrow)$: This proof has four steps:

1. For each $k = 0, ..., n$, we define Θ_k from $l^k_{\pi_\omega}$ and show that are bijective:

   We define for each $k = 0, ..., n$ the map $\Theta_k = (-1)^{(n-1)k} l^k_{\pi_\omega} : \Omega^k\mathcal{A} \to \mathcal{I}_{n-k}\mathcal{A}$. Since $l^k_{\pi_\omega}$ is bijective for $k = 1, ..., n-1$, then Θ_k is a bijective map for $k = 1, ..., n-1$. Also, since $n(n-1) \in 2\mathbb{Z}$, for all $n \in \mathbb{N}$, we have that $\Theta_n(\omega') = l^n_{\pi_\omega}(\omega') = \pi_\omega \cdot \omega' = \pi_\omega(\omega')$, i.e., $\Theta_n = \pi_\omega$, then Θ_n is bijective too. Rest to verify the bijectivity of Θ_0. We Define $\Theta_0^{-1} : \mathcal{I}_n\mathcal{A} \to \mathcal{A} : \phi \mapsto \nu_\omega^{-1}(\phi(\omega))$, where ν_ω is the automorphism of Remark 1.3.10. By definition, for any $a \in \mathcal{A}$ we get

   $$\Theta_0(a)(\omega) = (-1)^{(n-1)0} l^0_{\pi_\omega}(a)(\omega) = (\pi_\omega \cdot a)(\omega) = \pi_\omega(a\omega).$$

   Then, for any $a \in \mathcal{A}$, we have

   $$\Theta_0^{-1}(\Theta_0(a)) = \nu_\omega^{-1}(\Theta_0(a)(\omega)) = \nu_\omega^{-1}(\pi_\omega(a\omega)) = \nu_\omega^{-1}(\pi_\omega(\omega\nu(a))) = \nu_\omega^{-1}(\nu_\omega(a)) = a,$$

   which implies that $\Theta_0^{-1} \circ \Theta_0 = id$. On the other hand, for all $\phi \in \mathcal{I}_n\mathcal{A}$ and $a \in \mathcal{A}$,

   $$\begin{aligned} \Theta_0(\Theta_0^{-1}(\phi))(\omega a) &= l^0_{\phi_\omega}(\Theta_0^{-1}(\phi))(\omega a) = [\pi_\omega \cdot (\Theta_0^{-1}(\phi))](\omega a) \\ &= [\pi_\omega \cdot (\nu_\omega^{-1}(\phi(\omega)))](\omega a) = \pi_\omega(\nu_\omega^{-1}(\phi(\omega))(\omega a)) \\ &= \pi_\omega(\omega\phi(\omega)a) = \phi(\omega)a = \phi(\omega a). \end{aligned}$$

   Since ω is a right generator of $\Omega^n\mathcal{A}$, then $\Theta_0(\Theta_0^{-1}(\phi)) = \phi$, for all $\phi \in \mathcal{I}_n\mathcal{A}$ and therefore, $\Theta_0 \circ \Theta_0^{-1} = id$. Thus, we get that Θ_0 is a bijection.

2. To show that $\Theta_{k+m}(\omega' \wedge \omega'') = (-1)^{(n-1)m}\ \Theta_k(\omega') \cdot \omega''$, for all $\omega' \in \Omega^k\mathcal{A}$, $\omega'' \in \Omega^m\mathcal{A}$:

$$\begin{aligned}\Theta_{k+m}(\omega' \wedge \omega'') &= (-1)^{(n-1)(k+m)}\ l^{k+m}_{\pi_\omega}(\omega' \wedge \omega'') = (-1)^{(n-1)(k+m)}\ \pi_\omega \cdot (\omega' \wedge \omega'')\\ &= (-1)^{(n-1)(k+m)}\ (\pi_\omega \cdot \omega') \cdot \omega'' = (-1)^{(n-1)(k+m)}\ l^k_{\pi_\omega}(\omega')) \cdot \omega''\\ &= (-1)^{(n-1)m}\ \Theta_k(\omega') \cdot \omega''.\end{aligned}$$

3. To show that Θ_k are right $\mathcal{A}$-maps and left $\mathcal{A}$-maps by an automorphism $\nu : \mathcal{A} \to \mathcal{A}$:

   By the previous step, if $m = 0$, i.e., for all $\omega'' = a \in \mathcal{A}$, and all $\omega' \in \Omega^k\mathcal{A}$, we get $\Theta_k(\omega' a) = \Theta_k(\omega') \cdot a$. Therefore, Θ_k is an $\mathcal{A}$-module map, for all $k = 0, ..., n$. In other words, since π_ω is a right $\mathcal{A}$-map and for all $\omega'' = \omega b \in \Omega^n\mathcal{A}$, we get that

$$\pi_\omega(a\omega'') = \pi_\omega(a\omega b) = \pi_\omega(\omega\nu_\omega(a)b) = \nu_\omega(a)b = \nu_\omega(a)\pi_\omega(\omega'').$$

   With this, we obtain that for all $a \in \mathcal{A}$, $\omega' \in \Omega^k\mathcal{A}$ and $\omega'' \in \Omega^{n-k}\mathcal{A}$,

$$\begin{aligned}\Theta_k(a\omega')(\omega'') &= (-1)^{(n-1)k}\ l^k_{\pi_\omega}(a\omega')(\omega'') = (-1)^{(n-1)k}\ (\pi_\omega \cdot a\omega')(\omega'')\\ &= (-1)^{(n-1)k}\ \pi_\omega(a\omega' \wedge \omega'') = (-1)^{(n-1)k}\nu_\omega(a)\pi_\omega(\omega' \wedge \omega'')\\ &= \nu_\omega(a)(-1)^{(n-1)k} l^k_{\pi_\omega}(\omega)(\omega'') = \nu_\omega(a)\Theta_k(\omega)(\omega''),\end{aligned}$$

   i.e., $\Theta_k(a\omega') = \nu_\omega(a)\Theta_k(\omega')$, we have that $\Theta_k : \Omega^k\mathcal{A} \to^\nu \mathcal{I}_{n-k}\mathcal{A}$ are isomorphisms of bimodules, with $\nu = \nu_\omega$.

4. Define ∇_k, for each $k = 0, ..., n$, and show that ∇_k is a extension of ∇_0 which is a hom-connection:

   We define for each $k = 0, ..., n$, the map $\nabla_k = \Theta_{n-k} \circ d \circ \Theta^{-1}_{n-k-1} : \mathcal{I}_{k+1}\mathcal{A} \to \mathcal{I}_k\mathcal{A}$. In this way, by definition, we have that $\nabla_k \circ \Theta_{n-k-1} = \Theta_{n-k} \circ d$, for all $k = 0, ..., n$. Since $d \circ d = 0$, we get that for all $k = 1, ..., n$,

$$\begin{aligned}\nabla_{k-1} \circ \nabla_k &= (\Theta_{n-k+1} \circ d \circ \Theta^{-1}_{n-k}) \circ (\Theta_{n-k} \circ d \circ \Theta^{-1}_{n-k-1})\\ &= \Theta_{n-k+1} \circ (d \circ d) \circ \Theta^{-1}_{n-k-1} = 0.\end{aligned}$$

   First, we prove that $\nabla := \nabla_0 : \mathcal{I}_1\mathcal{A} \to \mathcal{A}$ is a divergence (or a hom-connection). For all $a \in \mathcal{A}$ and $\phi \in \mathcal{I}_1\mathcal{A}$, we have that

$$\begin{aligned}\nabla(\phi \cdot a) &= \Theta_n \circ d \circ \Theta^{-1}_{n-1}(\phi \cdot a)\\ &= \Theta_n(d(\Theta^{-1}_{n-1}(\phi \cdot a)))\\ &= \Theta_n(d(\Theta^{-1}_{n-1}(\phi) \cdot a))\\ &= \Theta_n(d(\Theta^{-1}_{n-1}(\phi)) \wedge a + (-1)^{n-1}\Theta^{-1}_{n-1}(\phi) \wedge da)\\ &= \Theta_n(d(\Theta^{-1}_{n-1}(\phi)) \wedge a) + (-1)^{n-1}\Theta_n(\Theta^{-1}_{n-1}(\phi) \wedge da)\\ &= \Theta_n(d(\Theta^{-1}_{n-1}(\phi))a) + (-1)^{n-1}\Theta_n(\Theta^{-1}_{n-1}(\phi) \wedge da)\\ &= \Theta_n(d(\Theta^{-1}_{n-1}(\phi)))a + (-1)^{2(n-1)}\phi \cdot da\\ &= \nabla(\phi)a + \phi \cdot da,\end{aligned}$$

where the sixth equality is by the second step, because we have that

$$\begin{aligned}\Theta_n(\Theta_{n-1}^{-1}(\phi) \wedge da) &= \Theta_{(n-1)+1}(\Theta_{n-1}^{-1}(\phi) \wedge da)\\ &= (-1)^{(n-1)1}\Theta_{(n-1)}(\Theta_{n-1}^{-1}(\phi) \cdot da\\ &= (-1)^{(n-1)}\phi \cdot da.\end{aligned}$$

This proves that $\nabla(\phi \cdot a) = \nabla(\phi)a + \phi \cdot da$, i.e., that ∇ is a hom-connection. Now, we are going to prove that ∇_m is a extension of ∇ in the way that for all $\phi \in \mathcal{I}_{m+1}\mathcal{A}$ and $\omega'' \in \Omega^m\mathcal{A}$, $\nabla_m(\phi)(\omega'') = \nabla(\phi \cdot \omega'') + (-1)^{m+1}\phi(d\omega)$. To do it, we first note that since $\Theta_{k+m}(\omega' \wedge \omega'') = (-1)^{(n-1)m}\Theta_k(\omega') \cdot \omega''$, fixing $\omega' = \Theta_k^{-1}(\phi)$, and using that Θ_{k+m} is an isomorphism, we get

$$\Theta_{k+m}^{-1}(\phi \cdot \omega'') = (-1)^{(n-1)m}\Theta_k^{-1}(\phi) \wedge \omega''.$$

Then, we can show that for all $\phi \in \mathcal{I}_{m+1}\mathcal{A}$ and $\omega'' \in \Omega^m\mathcal{A}$,

$$\begin{aligned}\nabla(\phi \cdot \omega'') &= \Theta_n \circ d \circ \Theta_{n-1}^{-1}(\phi \cdot \omega'')\\ &= (\Theta_n \circ d)(\Theta_{(n-1-m)+m}^{-1}(\phi \cdot \omega''))\\ &= (\Theta_n \circ d)((-1)^{(n-1)m}\Theta_{(n-1-m)}^{-1}(\phi) \wedge \omega'')\\ &= (-1)^{(n-1)m}\Theta_n(d(\Theta_{(n-1-m)}^{-1}(\phi) \wedge \omega''))\\ &= (-1)^{(n-1)m}\Theta_n(d(\Theta_{(n-1-m)}^{-1}(\phi)) \wedge \omega'' + (-1)^{n-1-m}\Theta_{(n-1-m)}^{-1}(\phi) \wedge d\omega'')\\ &= (-1)^{(n-1)m}\Theta_n(d(\Theta_{(n-1-m)}^{-1}(\phi)) \wedge \omega'')\\ &+ (-1)^{(n-1)m+n-1-m}\Theta_n(\Theta_{(n-1-m)}^{-1}(\phi) \wedge d\omega'')\\ &= (-1)^{2(n-1)m}\Theta_{n-m}(d(\Theta_{(n-1-m)}^{-1}(\phi))) \cdot \omega''\\ &+ (-1)^{(m+1)n-1}(-1)^{(n-1)(m+1)}\Theta_{n-m-1}(\Theta_{(n-1-m)}^{-1}(\phi)) \cdot d\omega''\\ &= \nabla_m(\phi) \cdot \omega'' + (-1)^m\phi \cdot d\omega''\\ &= \nabla_m(\phi)(\omega'') + (-1)^m\phi \cdot d\omega'',\end{aligned}$$

as we desired.

□

With this in mind, we can present the notion of smoothness of an algebra as follows.

Definition 1.3.14 ([BS17], p. 421)**.** An affine algebra R of integer Gelfand-Kirillov dimension m, it is said to be *differentially smooth*, if there exists a connected, m-dimensional, integrable differential calculus on R.

The following example is obtained in [BL18]. In this work, we make explicit its differentially smooth structure, because this is useful in the next chapter.

Example 1.3.15 ([BL18], p. 6)**.** The algebra $\mathbb{C}[x]$ is differentially smooth. Since we have that $\text{GKdim}(\mathbb{C}[x]) = 1$ (see Example 1.3.5), considering the Remark 1.2.12, we need one skew derivation ∂ such that there exists $b(x) \in \mathbb{C}[x]$ which satisfies $\mathbb{C}[x]b(x) = \mathbb{C}[x]$.

If we choose $\sigma : \mathbb{C}[x] \to \mathbb{C}[x] : f(x) \mapsto f(q^{-1})$, with $\mathbb{C} \setminus \{0\}$, by the Example 1.1.43 we have that $\partial = a(x)J_{q^{-1}}$ where $J_{q^{-1}}$ is the Jackson's derivative (see Example 1.1.42). To guarantee that the Brzezinski's differential calculus satisfies the density condition, we need that $a(x) = a \in \mathbb{C}$, and in that case, if $b(x)$ is a linear polynomial, we get that the left ideal generated by $\partial(b(x))$ is all $\mathbb{C}[x]$. Then, we define the Brzezinski's differential calculus $(\Omega^1(\mathbb{C}[x]), d)$ $\Omega^1(\mathbb{C}[x]) = \mathbb{C}[x]\omega$, where the right action is defined by $\omega f(x) = \sigma(f(x))\omega = f(q^{-1}x)\omega$, as in equation (1.2.3), and with

$$d : \mathbb{C}[x] \to \Omega^1(\mathbb{C}[x]) : f(x) \mapsto \partial(f(x))\omega,$$

like in (1.2.4). We verify the Leibniz's rule:

$$\begin{aligned} d(f(x)g(x)) = \partial(f(x)g(x))\omega &= [\partial(f(x))\sigma(g(x)) + f(x)\partial(g(x))]\omega \\ &= \partial(f(x))[\sigma(g(x))\omega] + f(x)\partial(g(x))\omega \\ &= [\partial(f(x))\omega]g(x) + f(x)[\partial(g(x))\omega] \\ &= d(f(x))g(x) + f(x)d(g(x)). \end{aligned}$$

Now, in order to prove the differentially smoothness, we check that the differential calculus $\Omega(\mathbb{C}[x]) = \mathbb{C}[x] \oplus \Omega^1(\mathbb{C}[x])$ satisfies the following conditions:

1. 1-dimensional: Since $\Omega^1(\mathbb{C}[x])$ is a bimodule generated by ω, it is finitely generated, and so the exterior product $\Omega^k(\mathbb{C}[x]) = \bigwedge^k \Omega^1(\mathbb{C}[x]) = 0$, for all $k \geq 1$ (see [Bou89], p. 511).

2. Integrablility: Since σ is an automorphism, $\Omega^1(\mathbb{C}[x])$ is a $\mathbb{C}[x]$-free bimodule of dimension 1. Then, we have that ω is a volume form of $\Omega^1(\mathbb{C}[x])$. In language of Theorem 1.3.13, we have that $\nu_\omega = \sigma^{-1} : \mathbb{C}[x] \to \mathbb{C}[x] : f(x) \mapsto f(qx)$. Note that the map $l^k_{\pi_\Omega}$ are bijective, for $k = 0, 1$, because $l^1_{\pi_\omega}(f(x))(\omega g(x)) = f(qx)g(x) = l^0_{\pi_\omega}(f(x)\omega)(g(x))$, but this is not necessary to apply Theorem 1.3.13 because $n = 1$, and the Theorem requires the revision of bijectivity of $l^k_{\pi_\omega}$, for $k = 1..., n-1$; since $\Omega^1(\mathbb{C}[x])$ admits a volume form, we get that this is an integrable differential calculus.

3. Connecteness: If $f(x) \in \mathbb{C}[x]$ such that $d(f(x)) = \partial(f(x))\omega = 0$, we get that $aJ_{q^{-1}}(f(x)) = 0$. Thus $f(x) = f \in \mathbb{C}$, which means that $\Omega^1(\mathbb{C}[x])$ is connect.

Example 1.3.16 ([BS17], p. 16)**.** The noncommutative pillow algebra $\mathcal{O}(\mathbb{P}_\theta)$ is differentially smooth, which is a subalgebra of the coordinate algebra of the noncommutative torus $\mathcal{O}(\mathbb{T}^2_\theta)$. The algebra $\mathcal{O}(\mathbb{T}^2_\theta)$ is a complex C^*-algebra[10] generated by unitaries elements U and V (i.e., $|U| = |V| = 1$) such that $UV = \lambda VU$, with $\lambda = e^{2\pi i\theta}$, where θ is an irrational number. The operator $*$ is an isomorphism and $\mathcal{O}(\mathbb{P}_\theta)$ is the set of fixing points of $*$, that are generated by $U + U^*$ and $V + V^*$.

Example 1.3.17 ([BS17], p. 22)**.** For a positive $N \in \mathbb{N}$, the *coordinate algebra of the quantum cone* $\mathcal{O}(C^N_{q,k})$ is a complex $*$-algebra generated by a, b and b^*, such that $a^* = a$, which satisfies the relations,

$$ab = q^N ba + k[N]_q b, \quad bb^* = \prod_{l=0}^{N-1} (q^{-l}a + k[-l]_q), \quad b^*b = \prod_{l=1}^{N} (q^l a + k[l]_q),$$

---

[10] A complex algebra $\mathcal{A}$ is called C^*, if there exists a norm $||\cdot||$ that makes $\mathcal{A}$ a Banach algebra and a map $* : \mathcal{A} \to \mathcal{A}$ such that $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda}x^*$, $(xy)^* = y^*x^*$, $(x^*)^* = x$ and $||x^*x|| = ||x||^2$, for all $x, y \in \mathcal{A}$.

where, $q > 0$, $k \in \mathbb{R}$, are parameters and, for all $n \in \mathbb{Z}$, $[n]_q = (1 - q^n)/(1 - q)$. If $k \neq 0$, the quantum cone algebra $\mathcal{O}(C^N_{q,k})$ is differentially smooth.

Remark 1.3.18. The differentially smoothness of quantum cones was established in [AB18], p. 21, proving the integrable condition of a differential calculus which was used in a previous document to prove that these algebras are also homologically smooth (see Remark 1.3.1). This with the aim to saying that the notions of smoothness of algebras are not necessarily exclusive.

Example 1.3.19 ([BS17], p. 8)**.** The algebra $\mathcal{A} = \mathbb{C}[x, y]/\langle xy\rangle$ is not a differentially smooth algebra. Since in this case $V = \mathbb{C}x + \mathbb{C}y$ is a generator subspace, we get that $\dim_{\mathbb{C}}(V^n) = \dim_{\mathbb{C}}(\mathbb{C}x^n + \mathbb{C}y^n) = 2$, and therefore, $\dim_V(n) = \dim_{\mathbb{C}}(\mathcal{A}_n) = 2n - 1$. Since the functions $f(n) = 2n - 1$ and $g(n) = n$ have the same growth, we obtain that $\operatorname{GKdim}(\mathcal{A}) = \overline{\lim} \log_n(n) = 1$. If we suppose that $\mathcal{A}$ is differentially smooth, then there exists a one dimensional integrable, connected differential calculus on $\mathcal{A}$. By the integral condition we can assert features over the maps Θ_1 and ν, but these implies also the non connected of the differential calculus.

Remark 1.3.20. We want to remark that although in [BS17], the algebra $M_n(\mathbb{C})$ is used as an example for the integrability definition, for the differentially smooth is not a good example because $\operatorname{GKdim}(M_n(\mathbb{C})) = 0$ (see Example 1.3.6).

We found in [BL18] a way to obtain information about the differentially smoothness of a tensor product $R \otimes S$, if R and S are differentially smooth algebras and satisfy some additional conditions as follows.

Remark 1.3.21. In [BL18], p. 4, we found that if R and S have n and m-dimensional differential calculi $(\Omega R, d_R)$ and $(\Omega S, d_S)$ with $\Omega R = \sum_{k=0}^{n} \Omega^k R$ and $\Omega S = \sum_{k=0}^{m} \Omega^k S$, respectively, then we can define

$$\Omega(R \otimes S) := \Omega R \otimes \Omega S = \bigoplus_{k=0}^{n+m} \Omega^k(R \otimes S), \qquad \Omega^k(R \otimes S) := \sum_{i=0}^{k} \Omega^i R \otimes \Omega^{k-i} S.$$

By Definition 1.3.8, if $i > n$ or $k - i > m$, we have that $\Omega^i R \otimes \Omega^{k-i} S = 0$. $\Omega(R \otimes S)$ is a graded differential algebra with graded multiplication defined over homogeneous elements $\omega, \omega' \in \Omega R$ and $\nu, \nu' \in \Omega S$,

$$(\omega \otimes \nu)(\omega' \otimes \nu') = (-1)^{|\nu||\omega'|} \omega\omega' \otimes \nu\nu'. \tag{1.3.1}$$

We can extended the derivations to a derivation d of $\Omega(R \otimes S)$ defining, for $\omega \in \Omega^i R$ and $\nu \in \Omega^j S$,

$$d(\omega \otimes \nu) = \omega \otimes d_S(\nu) + (-1)^i d_R(\omega) \otimes \nu. \tag{1.3.2}$$

The density conditions of $(\Omega R, d_R)$ and $(\Omega S, d_S)$ are guarantee if,

$$\omega = \sum_t r_0^t d_R(r_1^t) \cdots d_R(r_i^t), \qquad \nu = \sum_u s_0^u d_S(s_1^u) \cdots d_S(s_j^u). \tag{1.3.3}$$

Then, by (1.3.1) and (1.3.2) we get

$$\omega \otimes \nu = \sum_{t,u} (r_0^t \otimes s_0^u) d(r_1^t \otimes 1) \cdots d(r_i^t \otimes 1) d(1 \otimes s_1^u) \cdots d(1 \otimes s_j^u).$$

Thus, we obtain that $(\Omega R \otimes \Omega S, d)$ is a differential calculus for $R \otimes S$.

With this structure in our hands, we obtain the following useful result that we present without proof.

Theorem 1.3.22 ([BL18], 4)**.** *Let R and S be algebras with integrable differential calculi $(\Omega R, d_R)$ and $(\Omega S, d_S)$. Suppose that ΩR is a finitely generated projective right R-module and that ΩS is a finitely generated projective right S-module. Then $(\Omega R \otimes \Omega S, d)$ is an integrable differential calculus for $R \otimes S$.*

Corollary 1.3.23 ([BL18], p. 5)**.** *If R and S are differentially smooth algebras with respect to calculi which are finitely generated projective as right modules and*

$$\mathrm{GKdim}(R \otimes S) = \mathrm{GKdim}(R) + \mathrm{GKdim}(S),$$

then the tensor product algebra $R \otimes S$ is differentially smooth.

The differentially smoothness of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is mentioned in [BS17], p. 8, and concluded in [BL18], p. 6. In this work, we want to get the explicit differential calculus following Remark 1.3.21, Theorem 1.3.22 and Corollary 1.3.23.

Example 1.3.24. By Example 1.3.15, we have that $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are differentially smooth, with 1-dimensional integrable connect differential calculus $\Omega(\mathbb{C}[z]) = \mathbb{C}[z] \oplus \Omega^1(\mathbb{C}[z])$ free $\mathbb{C}[z]$-right modules, for $z = x, y$. Since these differential calculus are right-free, then these are right-projective too. We have that the canonical map

$$\varphi : \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y] \to \mathbb{C}[x, y] : x^k \otimes y^j \mapsto x^k y^j,$$

is an isomorphism of $\mathbb{C}$-algebras (see [Bou89], p. 469). Therefore

$$\mathrm{GKdim}(\mathbb{C}[x] \otimes \mathbb{C}[y]) = \mathrm{GKdim}(\mathbb{C}[x, y]) = 2 = 1 + 1 = \mathrm{GKdim}(\mathbb{C}[x]) + \mathrm{GKdim}(\mathbb{C}[y]).$$

By Corollary 1.3.23, we get that $\mathbb{C}[x, y]$ is differentially smooth, and the differential calculus is, following Remark 1.3.21, given by

$$\begin{aligned}
\Omega(\mathbb{C}[x, y]) &= \Omega^0(\mathbb{C}[x, y]) \oplus \Omega^1(\mathbb{C}[x, y]) \oplus \Omega^2(\mathbb{C}[x, y]) \\
&= \Big(\Omega^0(\mathbb{C}[x]) \otimes \Omega^0(\mathbb{C}[y])\Big) \oplus \Big([\Omega^0(\mathbb{C}[x]) \otimes \Omega^1(\mathbb{C}[y])] + [\Omega^1(\mathbb{C}[x]) \otimes \Omega^0(\mathbb{C}[y])]\Big) \\
&\oplus \Big([\Omega^0(\mathbb{C}[x]) \otimes \Omega^2(\mathbb{C}[y])] + [\Omega^1(\mathbb{C}[x]) \otimes \Omega^1(\mathbb{C}[y])] + [\Omega^2(\mathbb{C}[x]) \otimes \Omega^0(\mathbb{C}[y])]\Big) \\
&= \Big(\mathbb{C}[x] \otimes \mathbb{C}[y]\Big) \oplus \Big([\mathbb{C}[x] \otimes \Omega^1(\mathbb{C}[y])] + [\Omega^1(\mathbb{C}[x]) \otimes \mathbb{C}[y]]\Big) \\
&\oplus \Big([\mathbb{C}[x] \otimes 0] + [\Omega^1(\mathbb{C}[x]) \otimes \Omega^1(\mathbb{C}[y])] + [0 \otimes \mathbb{C}[y]]\Big).
\end{aligned}$$

In a more explicit way, we have

$$\Omega(\mathbb{C}[x, y]) = \Big(\mathbb{C}[x] \otimes \mathbb{C}[y]\Big) \oplus \Big(\mathbb{C}[x] \otimes \omega_y \mathbb{C}[y] + \omega_x \mathbb{C}[x] \otimes \mathbb{C}[y]\Big) \oplus \Big(\omega_x \mathbb{C}[x] \otimes \omega_y \mathbb{C}[y]\Big),$$

where all tensors are over $\mathbb{C}$, and the derivative $d : \mathbb{C}[x, y] \to \Omega^1(\mathbb{C}[x, y]) = \mathbb{C}[x] \otimes \omega_y \mathbb{C}[y] + \omega_x \mathbb{C}[x] \otimes \mathbb{C}[y]$, it is described in simple tensors as

$$\begin{aligned}
d(x^k y^j) &= d(x^k \otimes y^j) = x^k \otimes \partial_y(y^j)\omega_y + \partial_y(x^k)\omega_x \otimes y^j, \\
d(f(x) \otimes g(y)\omega_y) &= f(x) \otimes d(g(y)\omega_y) + \partial_x(f(x))\omega_x \otimes g(y)\omega_y = \partial_x(f(x))\omega_x \otimes g(y)\omega_y, \\
d(f(x)\omega_x \otimes g(y)) &= f(x)\omega_x \otimes \partial_y(g(y))\omega_y - d(f(x)\omega_x) \otimes g(y) = f(x)\omega_x \otimes \partial_y(g(y))\omega_y.
\end{aligned}$$

Now, we present the definition of an Ore extension (also known as skew polynomial rings), which was introduced in [Ore33], and it is used in the next chapter.

Definition 1.3.25 ([GWJ04], p. 34)**.** If R is a ring, $\sigma : R \to R$ a ring endomorphism and $\delta : R \to R$ a left σ-skew derivation, we define $R[x, \sigma, \delta]$, as the quotient algebra of the free algebra $R\{x\}$ subject to the product $xr = \sigma(r)x + \delta(r)$, such that $\{x^j : l \in \mathbb{N}\}$ forms a left R-basis of $R[x, \sigma, \delta]$. The R-algebra $R[x; \sigma, \delta]$ is called an *Ore extension* of R or a *skew polynomial ring.*

Remark 1.3.26. In [GWJ04], we found that for any ring endomorphism σ of R and δ a left σ-skew derivation, there exists an Ore extension $R[x; \sigma, \delta]$ in Proposition 2.3, and by Proposition 2.4 is established that if we consider A the quotient algebra of the free algebra $R\langle x\rangle$ such that $xr - \sigma(r)x = \delta(r)$, then there exists an unique ring isomorphism $\phi : R[x; \sigma, \delta] \to A$ such that $\phi|_R = id$. Therefore A is an Ore extension (cf. [GWJ04] , Corollary 2.5).

Now, we present without proof some results that guarantee the differential smoothness of particular Ore extensions.

Theorem 1.3.27 ([BL18], p. 6)**.** *Let R be an algebra with an integrable differential calculus $(\Omega R, d)$ such that ΩR is a finitely generated right R-module. For any automorphism σ of R that extends to a degree-preserving automorphism of ΩR, which commutes with d, there exists an integrable differential calculus $(\Omega A, d)$ on the skew-polynomial ring $A = R[z; \sigma]$. If R is differentially smooth with respect to $(\Omega R, d)$ and $\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + 1$, then A is also differentially smooth.*

For an illustration of this theorem, see Example 1.3.38.

Theorem 1.3.28 ([BL18], p. 13)**.** *Let R be an algebra with an integral differential calculus $(\Omega R, d)$ such that ΩR is a finitely generated right R-module. Let σ be an automorphism of R that extends to a degree-preserving automorphism of ΩR, which commutes with d. If R is differentially smooth with respect to $(\Omega R, d)$ and $\mathrm{GKdim}(R[z; \sigma]) = \mathrm{GKdim}(R) + 1$, then $R[z; \sigma][t; \sigma_q]$ is also differentially smooth, where*

$$\sigma_q : R[z; \sigma] \to R[z; \sigma], \qquad wf(z) \mapsto \sigma(w)f(qz), \qquad \textit{for all } w \in R, \quad f(z) \in \mathbb{K}[z].$$

Example 1.3.29. The $\mathbb{C}$-algebra generated by $x_1, ..., x_n$ such that, for all pair $1 \leq i < j \leq n$, there exist $q_{ij} \in \mathbb{C} \setminus \{0\}$, it is differentially smooth (see [BL18], p. 13).

Remark 1.3.30. Let us show the differential calculus used in the proofs of Theorems 1.3.27 and 1.3.28. If $(\Omega R, d)$ is the GKdim(R)-dimensional, integral, connect differential calculus of R, for $R[z; \sigma]$ we build the structure $\Omega(R[z; \sigma]) = (\Omega R[z; \hat{\sigma}]) \oplus (\Omega R[z; \hat{\sigma}])^{\overline{\sigma}}$ in the following way:

1. $\hat{\sigma} : \Omega R \to \Omega R$ must be a degree-preserving automorphism that commutes with d and such that $\hat{\sigma}|_R = \sigma$.
2. Construct the Ore extension $(\Omega R)[z; \hat{\sigma}]$, and define the automorphism

$$\overline{\sigma} : (\Omega R)[z; \hat{\sigma}] \to (\Omega R)[z; \hat{\sigma}] : \omega z^n \mapsto \overline{\sigma}(\omega z^n) = (-1)^{|\omega|}\hat{\sigma}(\omega)z^n, \quad \text{for all } \omega \in \Omega^j R.$$

3. Consider $((\Omega R)[z;\hat{\sigma}])^{\overline{\sigma}}$ the $(\Omega R)[z;\hat{\sigma}]$-bimodule $(\Omega R)[z;\hat{\sigma}]$ with the actions $\omega\omega'\omega'' := \omega\omega'\overline{\sigma}(\omega'')$, for all $\omega, \omega', \omega'' \in (\Omega R)[z;\hat{\sigma}]$.

4. Construct the *trivial extension* (see [BL18], p. 6) $((\Omega R)[z;\hat{\sigma}]) \oplus ((\Omega R)[z;\hat{\sigma}])^{\overline{\sigma}}$, which is that direct sum of bimodules with the product,

$$(\omega z^n, \nu z^m)(\omega' z^{n'}, \nu' z^{m'}) := (\omega z^n \omega' z^{n'}, \omega z^n \nu' z^{m'} + \nu z^m \overline{\sigma}(\omega' z^{n'})).$$

   we can check $((\Omega R)[z;\hat{\sigma}]) \oplus ((\Omega R)[z;\hat{\sigma}])^{\overline{\sigma}}$ is a grading algebra, with the grading is given by

$$|(\omega z^n, 0)| = |\omega|, \qquad |(0, \nu z^m)| = |\nu| + 1,$$

   for all homogeneous $\omega, \nu \in \Omega R$.

5. The differential $d : ((\Omega R)[z;\hat{\sigma}]) \oplus ((\Omega R)[z;\hat{\sigma}])^{\overline{\sigma}} \to ((\Omega R)[z;\hat{\sigma}]) \oplus ((\Omega R)[z;\hat{\sigma}])^{\overline{\sigma}}$, is defined, for all homogeneous $\omega, \nu \in \Omega R$, by

$$d(\omega z^n, \nu z^m) = (d(\omega) z^n, (-1)^{|\omega|}\omega\partial_z(z^n) + d(\nu) z^m)$$

Now, we present a treatment of differentially smoothness of Ore extensions $R[x;\sigma,\delta]$ where $R = \mathbb{C}[x]$ and $\delta \neq 0$. First, we present the Lemma 1.3.31 without proof, and later we consider Theorem 1.3.32 following the proof given in [Brz14], filling the details that are omitted in that document.

Lemma 1.3.31 ([AB18], p. 9). *Let $\Omega(\mathcal{A})$ be an n-dimensional calculus over $\mathcal{A}$ admitting a volume form ω. Assume that, for all $k = 1, \ldots, n-1$, there exists a finite number of forms $\omega_i^k, \overline{\omega}_i^k \in \Omega^k(\mathcal{A})$ such that, for all $\omega' \in \Omega^k(\mathcal{A})$,*

$$\omega' = \sum_i \omega_i^k \pi_\omega(\overline{\omega}_i^{n-k} \wedge \omega') = \sum_i \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_i^{n-k}))\overline{\omega}_i^k, \tag{1.3.4}$$

then, ω is an integrating form and all the $\Omega^k(\mathcal{A})$ are finitely generated and projective as left and right $\mathcal{A}$-modules.

Theorem 1.3.32 ([Brz14], p. 4). *If $\mathcal{A} = \mathbb{C}[x][y;\sigma,\delta]$ is an Ore extension satisfying one of the next conditions*

1. *$\sigma = id$ and $p(x) \in \mathbb{C}[x]$.*
2. *$\sigma(x) = x + r$, and $\delta(x) = p(x) = c$, with $r, c \in \mathbb{C}$ and $r \neq 0$.*
3. *$\sigma(x) = qx + r$, and $\delta(x) = p(x) = c(x + (r/(q-1)))$, with $q, r, c \in \mathbb{C}$ and $q \neq 1$,*

then $\mathcal{A}$ is differentially smooth.

Proof. We work with the general description $\sigma(x) = qx + r$ for $a, r \in \mathbb{C}$. Since in this case $\mathrm{GKdim}(\mathcal{A}) = 2$, we have to construct a 2-dimensional differential calculus. We do that in the following steps:

1. The following assignments define two automorphisms of $\mathcal{A}$,

$$\nu_x(x) = x, \quad \nu_x(x) = qy + p'(x) \text{ and } \nu_y(x) = \sigma^{-1}(x), \quad \nu_y(y) = y,$$

where $p'(x)$ is the usual derivative of $p(x)$, and $[\nu_x, \nu_y] = 0$, if $q = 1$ (see [Brz14], p. 3).

2. Construct the homogeneous component of degree 1 as the free right $\mathcal{A}$-module with generators $d(x)$ and $d(y)$, i.e., $\Omega^1(\mathcal{A}) = d(x)\mathcal{A} + d(y)\mathcal{A}$. Following the action defined in expression (1.2.3), define the left $\mathcal{A}$-module structure by

$$ad(x) = d(x)\nu_x(a), \quad ad(y) = d(y)\nu_y(a), \quad \text{for all } a \in \mathcal{A}, \tag{1.3.5}$$

which, by definition of automorphisms ν_x and ν_y, it turns to

$$\begin{aligned} xd(x) &= d(x)\nu_x(x) = d(x)x, \\ xd(y) &= d(y)\nu_y(x) = d(y)\sigma^{-1}(x) = d(y)q^{-1}x - d(y)q^{-1}r, \\ yd(x) &= d(x)\nu_x(y) = d(x)qy + d(x)p'(x), \\ yd(y) &= d(y)\nu_y(y) = d(y)y. \end{aligned} \tag{1.3.6}$$

3. Define the differential map $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ such that $x \mapsto d(x)$ and $y \mapsto d(y)$. By relations (1.3.6) and the Leibniz's rule of d, we have that d respects $yx = (qx+r)y + p(x)$, as follows

$$\begin{aligned} d(yx) &= d((qx + r)y + p(x)) \\ d(y)x + yd(x) &= qd(x)y + qxd(y) + rd(y) + d(p(x)) \\ d(y)x + d(x)qy + d(x)p'(x) &= qd(x)y + d(y)x + (-d(y)r + rd(y)) + d(x)p'(x) \\ d(y)x + d(x)qy + d(x)p'(x) &= qd(x)y + d(y)x + d(x)p'(x) \\ 0 &= 0 \end{aligned}$$

where, since $xd(x) = d(x)x$, by an application of Leibniz's rule, we have that for any $h(x) \in \mathbb{C}[x]$, $d(h(x)) = h'(x)d(x) = d(x)h'(x)$, in particular $d(p(x)) = d(x)p'(x)$. Then $(\Omega^1(\mathcal{A}), d)$ is a first order differential calculus of $\mathcal{A}$.

4. Define linear maps $\partial_x, \partial_y : \mathcal{A} \to \mathcal{A}$, such that

$$d(a) = d(x)\partial_x(a) + d(y)\partial_y(a), \quad \text{for all } a \in \mathcal{A}.$$

Since $\mathcal{A}$ is right free with generators $d(x)$ and $d(y)$, the maps ∂_x and ∂_y are well defined.

5. $(\Omega^1(\mathcal{A}), d)$ satisfies $\mathrm{Ker}(d) = \mathbb{C}$: since $\Omega^1(\mathcal{A})$ is right free with generators $d(x)$ and $d(y)$, then, $d(a) = 0$ if and only if $\partial_x(a) = \partial_y(a) = 0$. Using relations (1.3.6) and the definition of ν_x and ν_y, since $d(x^k y^l) = d(x)\partial_x(x^k y^l) + d(y)\partial_y(x^k y^l)$, and

$$\begin{aligned} d(x^k y^l) &= d(x^k)y^l + x^k d(y^l) \\ &= d(x)kx^{k-1}y^l + x^k d(y)ly^{l-1} \\ &= d(x)kx^{k-1}y^l + d(y)\sigma^{-1}(x^k)ly^{l-1} \\ &= d(x)kx^{k-1}y^l + d(y)\sigma^{-1}(x^k)ly^{l-1} \\ &= d(x)kx^{k-1}y^l + d(y)(q^{-1}x - q^{-1}r)^k ly^{l-1}, \end{aligned}$$

we obtain, $\partial_x(x^k y^l) = kx^{k-1}y^l$ and $\partial_y(x^k y^l) = l(q^{-1}x - q^{-1}r)^k y^{l-1}$. In particular, $\partial_x(\sum_{k,l} c_{k,l}x^k y^l) = \sum_{k,l} c_{k,l}kx^{k-1}y^l = 0$ if and only if $c_{k,l} = 0$, whenever $k \neq 0$, and for $y^l \in \mathcal{A}$, we have that $\partial_y(y^l) = \sigma^{-1}(1)ly^{l-1} = 0$ if and only if $l \neq 0$. Therefore, $d(a) = 0$ if and only if $a \in \mathbb{C}$.

6. Considering the universal extension of d to higher forms compatible with relations (1.3.6), we obtain the following rules for $\Omega^2(\mathcal{A}) = \Omega^1(\mathcal{A}) \wedge \Omega^1(\mathcal{A})$, where $d(a \wedge b) = d(a) \wedge b + (-1)^{\deg(a)} a \wedge d(b)$,

$$\begin{aligned} d(wd(w)) &= d(d(w)w) \\ d(w) \wedge d(w) + w \wedge d^2(w) &= d^2(w) \wedge w - d(w) \wedge d(w) \\ d(w) \wedge d(w) &= -d(w) \wedge d(w). \end{aligned}$$

Therefore, $d(w) \wedge d(w) = 0$, for $w \in \{x, y\}$, since $d^2 = 0$. Also, due to that $xd(y) = d(y)q^{-1}x - d(y)q^{-1}r$, we have that

$$\begin{aligned} d(xd(y)) &= d(d(y)q^{-1}x - d(y)q^{-1}r) \\ d(x) \wedge d(y) + x \wedge d^2(y) &= q^{-1}d^2(y) \wedge x - q^{-1}d(y) \wedge d(x) - q^{-1}rd^2(y) \\ d(x) \wedge d(y) &= -q^{-1}d(y) \wedge d(x) \\ d(y) \wedge d(x) &= -qd(x) \wedge d(y). \end{aligned} \tag{1.3.7}$$

7. Note that the last of the necessary equations (1.3.7), does not induce any additional constraints, since $[\nu_x, \nu_y] = 0$, for all $a \in \mathcal{A}$,

$$\begin{aligned} a(d(y) \wedge d(x) + qd(x) \wedge d(y)) &= ad(y) \wedge d(x) + qad(x) \wedge d(y) \\ &= d(y)\nu_y(a) \wedge d(x) + qd(x)\nu_x(a) \wedge d(y) \\ &= d(y) \wedge \nu_y(a)d(x) + qd(x) \wedge \nu_x(a)d(y) \\ &= d(y) \wedge d(x)[\nu_x \circ \nu_y(a) - \nu_y \circ \nu_x(a)] \\ &= d(y) \wedge d(x)[\nu_x, \nu_y](a) = 0. \end{aligned}$$

8. We have that $\omega := d(x) \wedge d(y)$ freely generates $\Omega^2(\mathcal{A})$ as a right $\mathcal{A}$-module (see [Bou89], p. 511), and also,

$$a\omega = \omega(\nu_x \circ \nu_y)(a), \qquad \text{for all } a \in \mathcal{A},$$

and since both ν_x and ν_y are automorphisms, ω is a volume form with $\nu_\omega = \nu_y \circ \nu_x$.

9. As a matter of fact, since $\Omega^2(\mathcal{A}) = \omega\mathcal{A}$, and $d(d(x) \wedge d(y)) = d^2(x) \wedge d(y) - d(x) \wedge d^2(y) = 0$, we have been obtained that $(\Omega(\mathcal{A}), d)$ is a 2-dimensional connect differential calculus with volume form $\omega = d(x) \wedge d(y)$.

10. In order to apply the Lemma 1.3.31 with the aim of obtaining the integrable condition, we define,

$$\omega_1 = d(x), \quad \overline{\omega}_1 = -q^{-1}d(y), \text{ and } \omega_2 = d(y), \quad \overline{\omega}_2 = d(x).$$

Then, for all $\omega' = d(x)a + d(y)b$,

$$\begin{aligned}
&\omega_1\pi_\omega(\overline{\omega}_1 \wedge \omega') + \omega_2\pi_\omega(\overline{\omega}_2 \wedge \omega')\\
&= d(x)\pi_\omega(-q^{-1}d(y) \wedge (d(x)a + d(y)b)) + d(y)\pi_\omega(d(x) \wedge (d(x)a + d(y)b))\\
&= d(x)\pi_\omega(-q^{-1}d(y) \wedge d(x)a) + d(y)\pi_\omega(d(x) \wedge d(y)b)\\
&= d(x)\pi_\omega(d(x) \wedge d(x)a) + d(y)\pi_\omega(d(x) \wedge d(y)b)\\
&= d(x)a + d(y)b = \omega'.
\end{aligned}$$

Furthermore, using relations (1.3.5), we compute

$$\begin{aligned}
&\nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_1))\overline{\omega}_1 + \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_2))\overline{\omega}_2\\
&= \nu_\omega^{-1}(\pi_\omega((d(x)a + d(y)b) \wedge d(x)))(-q^{-1}d(y)) + \nu_\omega^{-1}(\pi_\omega((d(x)a + d(y)b) \wedge d(y)))d(x)\\
&= \nu_\omega^{-1}(\pi_\omega(d(y)b \wedge d(x)))(-q^{-1}d(y)) + \nu_\omega^{-1}(\pi_\omega(d(x)a \wedge d(y)))d(x)\\
&= \nu_\omega^{-1}(\pi_\omega(d(x) \wedge d(y)\nu_x(b)))d(y) + \nu_\omega^{-1}(\pi_\omega(d(x) \wedge d(y)\nu_y(a)))d(x)\\
&= \nu_\omega^{-1}(\nu_x(b))d(y) + \nu_\omega^{-1}(\nu_y(a))d(x)\\
&= \nu_y^{-1}(b)d(y) + \nu_x^{-1}(a)d(x)\\
&= d(y)b + d(x)a = \omega'.
\end{aligned}$$

By Lemma 1.3.31, we obtain that ω is an integrating form, i.e., that $(\Omega(\mathcal{A}), d)$ is integral.

Then, since $(\Omega(\mathcal{A}), d)$ is 2-dimensional, integral, connect differential calculus for $\mathcal{A}$, then $\mathcal{A}$ is differentially smooth, which concludes the proof. □

Remark 1.3.33. About the proof of Theorem 1.3.32, we want to say some things.

1. The reason that these algebras have Gelfand-Kirillov dimension 2 is because these algebras are isomorphic to Hopf algebras that have this Gelfand-Kirillov dimension (see [Brz14], p. 1).

2. By the Lemma 1.3.31, we obtain that each homogeneous $\mathcal{A}$-module $\Omega^k(\mathcal{A})$ is right and left projective. Therefore, since $\Omega(\mathcal{A})$ is a direct sum of projective modules, we obtain $\Omega(\mathcal{A})$ is a right and left $\mathcal{A}$-projective finitely generated module.

3. We note that the maps ∂_x and ∂_y are left ν_x and ν_y-skew derivations, respectively, this because, for all pair $a, b \in \mathcal{A}$,

   $$\begin{aligned}
   d(x)\partial_x(ab) + d(y)\partial_y(ab) &= d(ab)\\
   &= d(a)b + ad(b)\\
   &= (d(x)\partial_x(a) + d(y)\partial_y(a))b + a(d(x)\partial_x(b) + d(y)\partial_y(b))\\
   &= d(x)\partial_x(a)b + d(y)\partial_y(a)b + ad(x)\partial_x(b) + ad(y)\partial_y(b)\\
   &= d(x)\partial_x(a)b + d(y)\partial_y(a)b + d(x)\nu_x(a)\partial_x(b) + d(y)\nu_y(a)\partial_y(b)\\
   &= d(x)[\partial_x(a)b + \nu_x(a)\partial_x(b)] + d(y)[\partial_y(a)b + \nu_y(a)\partial_y(b)].
   \end{aligned}$$

   Since $d(x)$ and $d(y)$ form a right $\mathcal{A}$-base for $\Omega^1(\mathcal{A})$, we obtain the assertion.

4. Since for all left φ-skew derivation ∂, if φ is an algebra automorphism, we get that $\varphi^{-1} \circ \partial$ is a right φ^{-1}-skew derivation, this because for all pair $a, b \in \mathrm{Dom}(\partial)$,

   $$\begin{aligned}
   \varphi^{-1} \circ \partial(ab) &= \varphi^{-1}(\partial(a)b + \varphi(a)\partial(b))\\
   &= \varphi^{-1}(\partial(a))\varphi^{-1}(b) + a\varphi^{-1}(\partial(b)),
   \end{aligned}$$

we conclude that $\nu_x^{-1} \circ \partial_x$ and $\nu_y^{-1} \circ \partial_y$ are right ν_x^{-1} and ν_y^{-1}-skew derivations, respectively. If we consider the set of right skew derivations $\{\nu_x^{-1} \circ \partial_x, \nu_y^{-1} \circ \partial_y\}$ in Remark 1.2.12, we note that the Brzezinski's differential calculus constructed from $\{\nu_x^{-1} \circ \partial_x, \nu_y^{-1} \circ \partial_y\}$ is precisely the calculus constructed in the proof of Theorem 1.3.32, as it was mentioned in [Brz14], p. 6.

We want to pointing out the following about the hom-connection intrinsic used in the proof of Theorem 1.3.32, and its Brzezinski's integral calculus.

Remark 1.3.34. In Lemma 1.3.31, it was established the hom-connection (see [BS17], p. 10) as

$$\nabla : \mathcal{I}_1\mathcal{A} \to \mathcal{A} : f \mapsto (-1)^{n-1} \sum_i \pi_\omega(d(\nu_\omega^{-1}(f(\omega_i^1)))\overline{\omega}_i^{n-1}),$$

which, in this case $\nabla(\xi_x) = \nabla(\xi_y) = 0$, where, ξ_x and ξ_y are the dual basis of $d(x)$ and $d(y)$, i.e., for all $d(x)a + d(x)b \in \Omega^1(\mathcal{A})$,

$$\xi_x(d(x)a + d(x)b) = a, \qquad \xi_y(d(x)a + d(x)b) = b,$$

then, $\{\xi_x, \xi_y\}$ is an $\mathcal{A}$-right basis of $\mathcal{I}_1\mathcal{A} = \operatorname{Hom}_{\mathcal{A}}(\Omega^1(\mathcal{A}), \mathcal{A})$. Therefore, by linearity and the definition of a hom-connection (see Definiton 1.2.1), for all $\xi_x \cdot a + \xi_y \cdot b \in \mathcal{I}_1\mathcal{A}$,

$$\begin{aligned}
\nabla(\xi_x \cdot a + \xi_y \cdot b) &= \nabla(\xi_x \cdot a) + \nabla(\xi_y \cdot b) \\
&= \nabla(\xi_x)a + \xi_x(da) + \nabla(\xi_y)b + \xi_y(db) \\
&= \xi_x(da) + \xi_y(db) \\
&= \partial_x(a) + \partial_y(a),
\end{aligned}$$

where ∂_x and ∂_y are the skew derivations defined in point 6 of proof of Theorem 1.3.32, and treated in point 3 of Remark 1.3.33. We have that ∇ is surjective: because ∂_x it is surjective (see point 5 of Theorem 1.3.32), then, if $a, c \in \mathcal{A}$ such that $\partial_x(a) = c$, $\nabla(\xi_x \cdot a) = \partial_x(a) = c \in \operatorname{Im}(\nabla)$. This implies that $\operatorname{coker}(\nabla) = 0$. Hence, the Brzezinski's integral calculus (see Definition 1.2.11) in this case is zero, as it was observed in [Brz14], p. 6.

The following examples correspond to algebras that will be used in the next chapter.

Example 1.3.35. Consider the Ore extension $\mathbb{C}[x][z;\sigma_1,\delta_1]$ such that $\sigma_1(x) = \beta x$ and $\delta_1(x) = 0$, with $\beta \in \mathbb{C} \setminus \{0\}$. Then, by Theorem 1.3.32, we get that $\mathbb{C}[x][z;\sigma_1,\delta_1]$ is differentially smooth.

Example 1.3.36. Considering the Ore extension $\mathbb{C}[x][y;\sigma_1,\delta_1]$ such that $\sigma_1(x) = \alpha^{-1}x$ and $\delta_1(x) = -\alpha^{-1}(a_3x + b_3)$, where $\alpha \in \mathbb{C} \setminus \{0\}$ and $a_3, b_3 \in \mathbb{C}$, then, by Theorem 1.3.32, we get that $\mathbb{C}[x][y;\sigma_1,\delta_1]$ is differentially smooth.

Example 1.3.37. The Ore extension $\mathbb{C}[z][y;\sigma_1,\delta_1]$ such that $\sigma_1(z) = z$ and $\delta_1(z) = z$. By Theorem 1.3.32, $\mathbb{C}[z][y;\sigma_1,\delta_1]$ is differentially smooth. As we see in the proof of Theorem 1.3.32,

$$\Omega(\mathbb{C}[z][y;\sigma_1,\delta_1]) = \mathcal{A} \bigoplus \Big(d(z)\mathcal{A} \oplus d(y)\mathcal{A} \Big) \bigoplus (d(z) \wedge d(y))\mathcal{A},$$

where the two automorphisms are defined by

$$\nu_z(z) = z, \quad \nu_z(y) = y + 1, \quad \nu_y(z) = z, \quad \nu_y(y) = y,$$

and the left module structure is defined by $ad(w) := d(w)\nu_w(a)$, for all $a \in \mathcal{A}$, for $w \in \{z, y\}$, where,

$$zd(z) = d(z)z, \quad yd(z) = d(z)(y + 1), \quad zd(y) = d(y)z, \quad yd(y) = d(y)y.$$

For $a \in \mathcal{A}$, $d(a) = d(z)\partial_z(a) + d(y)\partial_y(a)$, where in this case, ∂_w represents the usual partial derivative respect to w, for $w \in \{z, y\}$. In degree 2, we have that $d(z) \wedge d(y) = -d(y) \wedge d(z)$. Therefore, for $a, b \in \mathcal{A}$,

$$\begin{aligned} d(d(z)a + d(y)b) &= d(d(z)a) + d(d(y)b) \\ &= d(z) \wedge d(a) + d(y) \wedge d(b) \\ &= d(z) \wedge (d(z)\partial_z(a) + d(y)\partial_y(a)) + d(y) \wedge (d(z)\partial_z(b) + d(y)\partial_y(b)) \\ &= d(z) \wedge d(y)\partial_y(a) + d(y) \wedge d(z)\partial_z(b) \\ &= d(z) \wedge d(y)[\partial_y(a) - \partial_z(b)]. \end{aligned} \tag{1.3.8}$$

Also, since $\omega = d(z) \wedge d(y)$ is the integrating volume form, $d(\Omega^2(\mathcal{A})) = 0$.

The following example is due to a direct application of Theorem 1.3.27 to the Example 1.3.36, and it is obtained in this work. The Gelfand-Kirillov dimension condition of Theorem 1.3.27 is from [Rey13], Theorem 14.

Example 1.3.38. The iterated Ore extension $\mathbb{C}[z][y; \sigma_1, \delta_1][x; \sigma_2, \delta_2]$ such that $\sigma_1(z) = z$, $\delta_1(z) = z$, $\sigma_2(y) = y$, $\delta_2(y) = 0$, $\sigma_2(z) = \beta^{-1}z$ and $\delta_2(z) = 0$, for $\beta \in \mathbb{C}$. Since, by Example 1.3.37, the first iteration $\mathbb{C}[z][y; \sigma_1, \delta_1]$ is differentially smooth, and since by Remark 1.3.33, $\Omega(\mathbb{C}[z][y; \sigma_1, \delta_1])$ it is finitely generated as right $\mathbb{C}[z][y; \sigma_1, \delta_1]$-module, we can apply the Theorem 1.3.27. We only have to check that there exists an automorphism σ of $\mathbb{C}[z][y; \sigma_1, \delta_1]$ such that it can be extended to an automorphism of $\Omega(\mathbb{C}[z][y; \sigma_1, \delta_1])$, $\hat{\sigma}$, that $[\hat{\sigma}, d] = 0$. We show that for the identity map of $\mathbb{C}[z][y; \sigma_1, \delta_1]$ there exists only one extension of this type. We call $\mathcal{B} = \mathbb{C}[z][y; \sigma_1, \delta_1]$. For $\sigma = id_{\mathcal{B}}$,

$$\hat{\sigma} \circ d(a) = d(id_{\mathcal{B}}(a)) = d(a), \quad \text{for all } a \in \mathcal{B},$$

hence, $\hat{\sigma}(d(z)a + d(y)b) = \hat{\sigma}(d(z))\sigma(a) + \hat{\sigma}(d(y))\sigma(b) = d(z)a + d(y)b$, which means, $\hat{\sigma}|_{\Omega^1(\mathcal{B})} = id_{\Omega^1(\mathcal{B})}$. Now, for $d(z)a + d(y)b \in \Omega^1(\mathcal{B})$,

$$\hat{\sigma}(d(d(z)a + d(y)b)) = d(\hat{\sigma}(d(z)a + d(y)b)) = d(d(z)a + d(y)b).$$

Since $d(d(z)a + d(y)b) = d(z) \wedge d(y)[\partial_y(a) - \partial_z(b)]$ (see expression (1.3.8)), with $a = z$ and $b = 0$, we obtain that $\hat{\sigma}(d(z) \wedge d(y)) = d(z) \wedge d(y)$, which implies that $\hat{\sigma}|_{\Omega^2(\mathcal{B})} = id_{\Omega^2(\mathcal{B})}$. Therefore, the unique extension of the identity map of $\mathbb{C}[z][y; \sigma_1, \delta_1]$ to a graded automorphism of $\Omega(\mathbb{C}[z][y; \sigma_1, \delta_1])$, such that it commutes with d, is the identity map of $\Omega(\mathbb{C}[z][y; \sigma_1, \delta_1])$. Therefore, as such automorphism exists, by Theorem 1.3.27, the iterated Ore extension $\mathbb{C}[z][y; \sigma_1, \delta_1][x; \sigma_2, \delta_2]$ is differentially smooth.

Remark 1.3.39. We want to clarify that not always it is possible to extend an automorphism such that commutes with the differential, even when we try to extend the automorphism that defines the Ore extension (see Remark 2.3.29).

The next example is an original product of this work, following the ideas of the proof of Theorem 1.3.32.

Example 1.3.40. Consider the Ore extension, $\mathcal{A} = \mathbb{C}[y,z][x;\sigma,\delta]$, such that $\sigma(y) = y$, $\delta(y) = 0$ $\sigma(z) = \beta^{-1}z$ and with $\beta, \delta(z) \in \mathbb{C}$. From [Rey13], p. 101, we have that has $\mathrm{GKdim}(\mathcal{A}) = 3$. Hence, following the ideas developed in the proof of Theorem 1.3.32,

1. We set the right free $\mathcal{A}$-module $\Omega^1(\mathcal{A}) = d(y)\mathcal{A} \bigoplus d(z)\mathcal{A} \bigoplus d(x)\mathcal{A}$.
2. We establish the $\mathbb{C}$-algebra automorphisms $\nu_x, \nu_y, \nu_z : \mathcal{A} \to \mathcal{A}$, defined by,
$$\begin{aligned} &\nu_x(x) = \beta^{-1}x, && \nu_x(y) = y, && \nu_x(z) = \beta z,\\ &\nu_y(x) = x, && \nu_y(y) = y, && \nu_y(z) = z,\\ &\nu_z(x) = \beta^{-1}x, && \nu_z(y) = y, && \nu_z(z) = \beta z. \end{aligned}$$
We want to note that $\nu_x = \nu_z$, $\nu_y = id_{\mathcal{A}}$ and $[\nu_x,\nu_y] = [\nu_x,\nu_z] = [\nu_y,\nu_z] = 0$.
3. We define the left $\mathcal{A}$-action over $\Omega^1(\mathcal{A})$, $ad(w) = d(w)\nu_w(a)$, for all $a \in \mathcal{A}$ and $w \in \{x,y,z\}$. Since the ν_w's are automorphisms, then $\Omega^1(\mathcal{A})$ is also left free, with the explicit relations
$$\begin{aligned} &xd(x) = d(x)\beta^{-1}x, && yd(x) = d(x)y, && zd(x) = d(x)\beta z,\\ &xd(y) = d(y)x, && yd(y) = d(y)y, && zd(y) = d(y)z,\\ &xd(z) = d(z)\beta^{-1}x, && yd(z) = d(z)y, && zd(z) = d(z)\beta z. \end{aligned} \tag{1.3.9}$$
4. With these objects, we can extend to a derivation $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ that satisfies $yz = zy$, $xy = yx$ and $xz = \beta^{-1}zx + \delta(z)$,
$$\begin{aligned} d(yz - zy) &= d(y)z + yd(z) - d(z)y - zd(y) = d(y)z + d(z)\nu_z(y) - d(z)y - d(y)\nu_y(z)\\ &= d(y)z + d(z)\nu_z(y) - d(z)y - d(y)\nu_y(z)\\ &= d(y)[z - \nu_y(z)] + d(z)[\nu_z(y) - y] = 0, \end{aligned}$$
and in the same form, $d(xy - yx) = 0$. Also,
$$\begin{aligned} d(xz) &= d(\beta^{-1}zx + \delta(z))\\ d(x)z + xd(z) &= \beta^{-1}d(z)x + \beta^{-1}zd(x)\\ d(x)z + d(z)\nu_z(x) &= \beta^{-1}d(z)x + \beta^{-1}d(x)\nu_x(z)\\ 0 &= 0. \end{aligned}$$
With this, we have that $(\Omega^1(\mathcal{A}), d)$ is a first order differential calculus.
5. For $w \in \{x,y,z\}$, we have that $\nu_w(w) = \theta_w w$, with $\theta_w \in \mathbb{C}$. In this way, for all $i \in \mathbb{N} \setminus \{0\}$,
$$d(w^i) = d(w)\frac{(\theta_w^i - 1)}{(\theta_w - 1)}w^{i-1},$$

and with this, we define maps $\partial_x, \partial_x, \partial_x : \mathcal{A} \to \mathcal{A}$, such that $d(a) = d(x)\partial_x(a) + d(y)\partial_y(a) + d(z)\partial_z(a)$, for all $a \in \mathcal{A}$ where, for all $y^i z^j x^k \in \mathcal{A}$,

$$\partial_x(y^i z^j x^k) = \beta^j \frac{(\beta^{-k}-1)}{(\beta^{-1}-1)} y^i z^j x^{k-1}, \quad \partial_z(y^i z^j x^k) = \frac{(\beta^j - 1)}{(\beta - 1)} y^i z^{j-1} x^k,$$
$$\partial_y(y^i z^j x^k) = i y^{i-1} z^j x^k.$$

These maps satisfy that $d(a) = 0$ if and only if $\partial_x(a) = \partial_y(a) = \partial_z(a) = 0$, and this is equivalent in this case to $\mathrm{Ker} d = \mathbb{C}$.

6. With this, we extend the structure of $(\Omega^1(\mathcal{A}), d)$ to the 2-degree homogeneous calculus applying the operator d to elements of $\Omega^1(\mathbb{A})$, following the rules $d^2 = 0$ and $d(ab) = d(a) \wedge b + (-1)^{\deg(a)} a \wedge d(b)$, for all $a, b \in \Omega^1(\mathcal{A})$. Then, applying d to the relations 1.3.9, for $\beta \neq -1$, $d(w) \wedge d(w) = 0$ for $w \in \{x, y, z\}$, and

$$d(x) \wedge d(y) = -d(y) \wedge d(x), \quad d(x) \wedge d(z) = -\beta^{-1} d(z) \wedge d(x), \quad d(z) \wedge d(y) = -d(y) \wedge d(z),$$

as in the proof of Theorem 1.3.32, these relations do not impose conditions over the algebra due to the fact that $[\nu_w, \nu_{w'}] = 0$, for all pair $w, w' \in \{x, y, z\}$. Then it was established that $\Omega^2(\mathcal{A}) = d(y) \wedge d(z)\mathcal{A} \bigoplus d(y) \wedge d(x)\mathcal{A} \bigoplus d(z) \wedge d(x)\mathcal{A}$.

7. Now, we consider $\Omega^3(\mathcal{A}) = d(y) \wedge d(z) \wedge d(x)\mathcal{A}$, where by the relations in the previous item,

$$\begin{aligned} d(x) \wedge d(y) \wedge d(z) &= \beta^{-1} d(y) \wedge d(z) \wedge d(x) \\ d(y) \wedge d(x) \wedge d(z) &= -\beta^{-1} d(y) \wedge d(z) \wedge d(x) \\ d(z) \wedge d(y) \wedge d(x) &= -d(y) \wedge d(z) \wedge d(x) \\ d(z) \wedge d(x) \wedge d(y) &= d(y) \wedge d(z) \wedge d(x) \\ d(x) \wedge d(z) \wedge d(y) &= -\beta^{-1} d(y) \wedge d(z) \wedge d(x), \end{aligned}$$

we have that $\omega := d(y) \wedge d(z) \wedge d(x)$ is a volume form of $\Omega(\mathcal{A}) = \bigoplus_{i=0}^{3} \Omega^i(\mathcal{A})$, with $\nu_\omega = \nu_x \circ \nu_z \circ \nu_y$.

8. With the aim of applying Lemma 1.3.31, we set the elements,

$$\begin{aligned} &\overline{\omega}_1^2 = d(y) \wedge d(z), && \overline{\omega}_2^2 = d(z) \wedge d(x), && \overline{\omega}_3^2 = -\beta d(y) \wedge d(x) \\ &\omega_1^2, = \beta d(y) \wedge d(z), && \omega_2^2 = d(z) \wedge d(x), && \omega_3^2 = -d(y) \wedge d(x), \\ &\omega_1^1 = \overline{\omega}_1^1 = d(x), && \omega_2^1 = \overline{\omega}_2^1 = d(y), && \omega_3^1 = \overline{\omega}_3^1 = d(z). \end{aligned}$$

In $k=1$, for $\omega' = d(y)a + d(z)b + d(x)c \in \Omega^1(\mathcal{A})$,

$$\begin{aligned}
&\omega_1^1 \pi_\omega(\overline{\varpi}_1^2 \wedge \omega') + \omega_2^1 \pi_\omega(\overline{\varpi}_2^2 \wedge \omega') + \omega_3^1 \pi_\omega(\overline{\varpi}_3^2 \wedge \omega') \\
&= d(x)\pi_\omega(d(y) \wedge d(z) \wedge (d(y)a + d(z)b + d(x)c)) \\
&+ d(y)\pi_\omega(d(z) \wedge d(x) \wedge (d(y)a + d(z)b + d(x)c)) \\
&+ d(z)\pi_\omega((-\beta d(y) \wedge d(x)) \wedge (d(y)a + d(z)b + d(x)c)) \\
&= d(x)\pi_\omega(d(y) \wedge d(z) \wedge d(x)c) \\
&+ d(y)\pi_\omega(d(z) \wedge d(x) \wedge d(y)a) \\
&+ d(z)\pi_\omega((-\beta d(y) \wedge d(x)) \wedge d(z)b) \\
&= d(x)\pi_\omega(d(y) \wedge d(z) \wedge d(x)c) \\
&+ d(y)\pi_\omega(d(y) \wedge d(z) \wedge d(x)a) \\
&+ d(z)\pi_\omega(d(y) \wedge d(z) \wedge d(x)b) \\
&= d(x)c + d(y)a + d(z)b = \omega'.
\end{aligned}$$

$$\begin{aligned}
&\nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_1^2))\overline{\varpi}_1^1 + \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_2^2))\overline{\varpi}_2^1 + \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_3^2))\overline{\varpi}_3^1 \\
&= \nu_\omega^{-1}(\pi_\omega((d(y)a + d(z)b + d(x)c) \wedge \beta d(y) \wedge d(z)))d(x) \\
&+ \nu_\omega^{-1}(\pi_\omega((d(y)a + d(z)b + d(x)c) \wedge d(z) \wedge d(x)))d(y) \\
&+ \nu_\omega^{-1}(\pi_\omega((d(y)a + d(z)b + d(x)c) \wedge -d(y) \wedge d(x)))d(z) \\
&= \nu_\omega^{-1}(\pi_\omega(d(x)c \wedge \beta d(y) \wedge d(z)))d(x) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(y)a \wedge d(z) \wedge d(x)))d(y) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(z)b \wedge -d(y) \wedge d(x)))d(z) \\
&= \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z) \wedge d(x)(\nu_z \circ \nu_y)(c)))d(x) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z) \wedge d(x)(\nu_x \circ \nu_z)(a)))d(y) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z) \wedge d(x)(\nu_x \circ \nu_y)(b)))d(z) \\
&= \nu_\omega^{-1}((\nu_z \circ \nu_y)(c)))d(x) + \nu_\omega^{-1}((\nu_x \circ \nu_z)(a)))d(y) + \nu_\omega^{-1}((\nu_x \circ \nu_y)(b)))d(z) \\
&= \nu_x^{-1}(c)d(x) + \nu_y^{-1}(a)d(y) + \nu_z^{-1}(b)d(z) = d(x)c + d(y)a + d(z)b = \omega'.
\end{aligned}$$

If $k=2$, for $\omega' = d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c \in \Omega^2(\mathcal{A})$,

$$\begin{aligned}
&\omega_1^2 \pi_\omega(\overline{\varpi}_1^1 \wedge \omega') + \omega_2^2 \pi_\omega(\overline{\varpi}_2^1 \wedge \omega') + \omega_3^2 \pi_\omega(\overline{\varpi}_3^1 \wedge \omega') \\
&= \beta d(y) \wedge d(z)\pi_\omega(d(x) \wedge (d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c)) \\
&+ d(z) \wedge d(x)\pi_\omega(d(y) \wedge (d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c)) \\
&- d(y) \wedge d(x)\pi_\omega(d(z) \wedge (d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c) \\
&= \beta d(y) \wedge d(z)\pi_\omega(d(x) \wedge d(y) \wedge d(z)a) \\
&+ d(z) \wedge d(x)\pi_\omega(d(y) \wedge d(z) \wedge d(x)c) \\
&- d(y) \wedge d(x)\pi_\omega(d(z) \wedge d(y) \wedge d(x)b) \\
&= d(y) \wedge d(z)\pi_\omega(d(y) \wedge d(z) \wedge d(x)a) \\
&+ d(z) \wedge d(x)\pi_\omega(d(y) \wedge d(z) \wedge d(x)c) \\
&+ d(y) \wedge d(x)\pi_\omega(d(y) \wedge d(z) \wedge d(x)b) \\
&= d(y) \wedge d(z)a + d(z) \wedge d(x)c + d(y) \wedge d(x)b = \omega'.
\end{aligned}$$

$$
\begin{aligned}
&\nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_1^1))\overline{\omega}_1^2 + \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_2^1))\overline{\omega}_2^2 + \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_3^1))\overline{\omega}_3^2 \\
&= \nu_\omega^{-1}(\pi_\omega((d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c) \wedge d(x)))d(y) \wedge d(z) \\
&+ \nu_\omega^{-1}(\pi_\omega((d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c) \wedge d(y)))d(z) \wedge d(x) \\
&- \beta\nu_\omega^{-1}(\pi_\omega((d(y) \wedge d(z)a + d(y) \wedge d(x)b + d(z) \wedge d(x)c) \wedge d(z)))d(y) \wedge d(x) \\
&= \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z)a \wedge d(x)))d(y) \wedge d(z) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(z) \wedge d(x)c \wedge d(y)))d(z) \wedge d(x) \\
&- \beta\nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(x)b \wedge d(z)))d(y) \wedge d(x) \\
&= \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z) \wedge d(x)\nu_x(a)))d(y) \wedge d(z) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z) \wedge d(x)\nu_y(c)))d(z) \wedge d(x) \\
&+ \nu_\omega^{-1}(\pi_\omega(d(y) \wedge d(z) \wedge d(x)\nu_z(b)))d(y) \wedge d(x) \\
&= \nu_\omega^{-1}(\nu_x(a))d(y) \wedge d(z) \\
&+ \nu_\omega^{-1}(\nu_y(c))d(z) \wedge d(x) \\
&+ \nu_\omega^{-1}(\nu_z(b))d(y) \wedge d(x) \\
&= d(y) \wedge d(z)a + d(z) \wedge d(x)c + d(y) \wedge d(x)b = \omega'.
\end{aligned}
$$

Hence, by Lemma 1.3.31, we obtain that the differential calculus $\Omega(\mathcal{A})$ is integrable.

Then, since $\operatorname{GKdim}(\mathcal{A}) = 3$ and $\Omega(\mathcal{A})$ is a 3-dimensional integrable and connect differential calculus, we get that $\mathcal{A}$ is a differentially smooth algebra.

Example 1.3.41. The Ore extension $\mathcal{A} = \mathbb{C}[y,z][x;\sigma,\delta]$, such that $\sigma(y) = y$, $\delta(y) = 0$ $\sigma(z) = \beta^{-1}z$ and $0 \neq \delta(z) \in \mathbb{C}y$, with $\beta \in \mathbb{C}$, it is not differentially smooth. The problem is that, since $xz = \beta^{-1}zx - cy$, with $c \in \mathbb{C} \setminus \{0\}$, if we define a differential calculus $(\Omega(\mathcal{A}), d)$, we need that,

$$
\begin{aligned}
d(xz) &= \beta^{-1}d(zx) - cd(y) \\
d(x)z + xd(z) &= \beta^{-1}d(z)x + \beta^{-1}zd(x) - cd(y).
\end{aligned}
$$

Hence, since $cd(y) = \beta^{-1}d(z)x + \beta^{-1}zd(x) - (d(x)z + xd(z))$, by the density condition of $\Omega^1(\mathcal{A})$, we have that $d(x)$ and $d(z)$ form a set of generators of $\Omega^1(\mathcal{A})$, and therefore, $\Omega^3(\mathcal{A}) = 0$. This means that, there is no exist a $\operatorname{GKdim}(\mathcal{A}) = 3$-dimensional calculus, and therefore the differentially smoothness of $\mathcal{A}$ is impossible.

If $\mathbb{C}[x,y][z;\sigma,\delta]$ is an Ore extension of derivation type, i.e., $\sigma = id_{\mathbb{C}[x,y]}$, in this work we obtained the following result.

Theorem 1.3.42. *If $\mathcal{A} = \mathbb{C}[x,y][z; id_{\mathbb{C}[x,y]}, \delta]$ is an Ore extension, $\delta = f(x)\partial_x + g(y)\partial_y$, where $f(x) \in \mathbb{C}[x] \subset \mathbb{C}[x,y]$, $g(y) \in \mathbb{C}[y] \subset \mathbb{C}[x,y]$, and ∂_x and ∂_y are the usual partial derivations, then $\mathcal{A}$ is differentially smooth.*

Proof. Following the ideas in the proof of Theorem 1.3.32 and Example 1.3.40 in the case of $R = \mathbb{C}[x,y]$, we have the following:

1. The following assignments determine $\mathbb{C}$-algebra automorphisms ν_x, ν_y and ν_z of $\mathcal{A}$,

$$\begin{aligned} &\nu_x(x) = x, && \nu_x(y) = y, && \nu_x(z) = z + \partial_x(f(x)),\\ &\nu_y(x) = x, && \nu_y(y) = y, && \nu_y(z) = z + \partial_y(g(y)),\\ &\nu_z(x) = x, && \nu_z(y) = y, && \nu_z(z) = z. \end{aligned}$$

   These automorphisms satisfy $[\nu_x, \nu_y] = [\nu_x, \nu_z] = [\nu_y, \nu_z] = 0$.

2. We define the free right $\mathcal{A}$-module $\Omega^1(\mathcal{A}) = d(x)\mathcal{A} \bigoplus d(y)\mathcal{A} \bigoplus d(z)\mathcal{A}$, and also we define the left structure in the generators as $ad(w) = d(w)\nu_w(a)$, for all $a \in \mathcal{A}$ and $w \in \{x, y, z\}$.

   If we extend the assignments $w \mapsto d(w)$, for $w \in \{x, y, z\}$, to a $\mathbb{C}$-derivative over the algebra $\mathcal{A}$, $d : \mathcal{A} \to \Omega^1(\mathcal{A})$, since $\nu_w(w) = w$, we have $d(w^k) = d(w)kw^{k-1}$, for all $k \in \mathbb{N}$ and $w \in \{x, y, z\}$. By Leibniz's rules,

$$\begin{aligned} d(x^iy^jz^k) &= d(x)ix^{i-1}y^jz^k + x^id(y)jy^{j-1}z^k + x^iy^jd(z)kz^{k-1}\\ &= d(x)ix^{i-1}y^jz^k + d(y)jx^iy^{j-1}z^k + d(z)kx^iy^jz^{k-1}, \end{aligned} \tag{1.3.10}$$

   where the last equation is due to the fact that $\nu_y(x) = x, \nu_z(x) = x$ and $\nu_z(y) = y$. Therefore, if we define the maps $\partial_x, \partial_y, \partial_z : \mathcal{A} \to \mathcal{A}$, such that

$$d(a) = d(x)\partial_x(a) + d(y)\partial_y(a) + d(z)\partial_z(a), \quad \text{for all } a \in \mathcal{A},$$

   then, by the right free structure of $\Omega^1(\mathcal{A})$, $d(a) = 0$ if and only if $\partial_x(a) = \partial_y(a) = \partial_z(a) = 0$. By relations (1.3.10), we have that ∂_w is a left ν_w-skew derivation, for $w \in \{x, y, z\}$, and that over the PBW basis $\{x^iy^jz^k : i, j, k \in \mathbb{N}\}$, the map ∂_w acts as the usual partial derivation respect w. With this, we conclude that for $a \in \mathcal{A}$, $\partial_x(a) = \partial_y(a) = \partial_z(a) = 0$ if and only if $a \in \mathbb{C}$, which means that $\operatorname{Ker}(d) = \mathbb{C}$.

3. Extending the differential graded structure to $\Omega^2(\mathcal{A}) = d(y) \wedge d(z)\mathcal{A} \bigoplus d(x) \wedge d(z)\mathcal{A} \bigoplus d(x) \wedge d(y)\mathcal{A}$, we have that $d(w) \wedge d(w') = -d(w') \wedge d(w)$, for $w, w' \in \{x, y, z\}$. Since the automorphisms ν_w commute between them, the relations $d(w) \wedge d(w') = -d(w') \wedge d(w)$ do not disturb the structure of free right $\mathcal{A}$-module of $\Omega^2(\mathcal{A})$.

4. For $\Omega^3(\mathcal{A}) = d(x) \wedge d(y) \wedge d(z)\mathcal{A}$, we define $\nu_\omega = \nu_x \circ \nu_y \circ \nu_z$ where $\omega := d(x) \wedge d(y) \wedge d(z)$ is the volume form. If we define the assignments,

$$\begin{aligned} &\omega_1^1 = \overline{\omega}_1^1 = d(x), && \omega_2^1 = \overline{\omega}_2^1 = d(y), && \omega_3^1 = \overline{\omega}_3^1 = d(z)\\ &\omega_1^2 = \overline{\omega}_1^2 = d(y) \wedge d(z), && \omega_2^2 = \overline{\omega}_2^2 = -d(x) \wedge d(z), && \omega_3^2 = \overline{\omega}_3^2 = d(x) \wedge d(y), \end{aligned}$$

   by Lemma 1.3.31, we have that $(\Omega(\mathcal{A}), d)$ is integrable.

We have shown that $(\Omega(\mathcal{A}), d)$ is a 3-dimensional, connect and integrable differential calculus of $\mathcal{A}$, and since $\operatorname{GKdim}(\mathcal{A}) = 3$, from [Rey13], Theorem 14, then $\mathcal{A}$ is differentially smooth. □

The following algebras will be useful in Section 2.3.2.

Example 1.3.43. By Theorem 1.3.42, the algebra $\mathbb{C}[y, z][x, \sigma, \delta]$, where $\sigma(y) = y$, $\sigma(z) = z$, $\delta(y) = b$ and $\delta(z) = 0$ is differentially smooth.

Example 1.3.44. The algebra $\mathbb{C}[x,y][z,\sigma,\delta]$, where $\sigma(x)=x$, $\sigma(y)=y-a$, $\delta(x)=x$ and $\delta(y)=0$.

We close this section with the following examples for which the differentially smoothness was obtained in the realization of this work.

Example 1.3.45. Let $\mathcal{A}$ be the $\mathbb{C}$-algebra generated by x,y and z, with relations

$$zy-yz=ay,\quad yx-\beta xy=0,\quad xz-zx=x,\quad \text{with } a\in\mathbb{C}.$$

For this algebra, we can conclude that $\mathcal{A}$ is differentially smooth as in proof of Theorem 1.3.42 with the following tools:

1. Consider the automorphisms $\nu_x,\nu_y,\nu_z:\mathcal{A}\to\mathcal{A}$, that commuting each other and these are defined by,

$$\begin{aligned}
&\nu_x(x)=x, && \nu_x(y)=\beta y, && \nu_x(z)=z-1,\\
&\nu_y(x)=\beta^{-1}x, && \nu_y(y)=y, && \nu_y(z)=z+a,\\
&\nu_z(x)=x, && \nu_z(y)=y, && \nu_z(z)=z.
\end{aligned}$$

2. We define $\Omega^1(\mathcal{A})=d(x)\mathcal{A}\bigoplus d(y)\mathcal{A}\bigoplus d(z)\mathcal{A}$, where $ad(w):=d(w)\nu_w(a)$, for all $a\in\mathcal{A}$ and $w\in\{x,y,z\}$. Since $\nu_w(w)=w$, we have that $d(w^i)=d(w)iw^{i-1}$, for $w\in\{x,y,z\}$ and $i\in\mathbb{N}$. Also, we define $d:\mathcal{A}\to\Omega^1(\mathcal{A})$, and since $\nu_y(x)=\beta^{-1}x$, $\nu_z(x)=x$ and $\nu_z(y)=y$, we define the maps $\partial_x,\partial_y,\partial_z:\mathcal{A}\to\mathcal{A}$ as the right proyections of d over the right basis $d(x),d(y),d(z)$, where

$$\begin{aligned}
d(x^iy^jz^k)&=d(x)\partial_x(x^iy^jz^k)+d(y)\partial_y(x^iy^jz^k)+d(z)\partial_z(x^iy^jz^k)\\
&=d(x)[ix^{i-1}y^jz^k]+d(y)[\beta^{-i}jx^iy^{j-1}z^k]+d(z)[kx^iy^jz^{k-1}].
\end{aligned}$$

   Since $\mathrm{Ker}(d)=\mathbb{C}$, the 3-dimensional differential calculus $(\Omega(\mathcal{A}),d)$ is connect.

3. We obtain that the volume form $\omega:=d(x)\wedge d(y)\wedge d(z)\in\Omega^3(\mathcal{A})=\omega\mathcal{A}$, with $v_\omega=\nu_z\circ\nu_y\circ\nu_x$, is an integral form by Lemma 1.3.31 applied to the choices,

$$\begin{aligned}
&\omega_1^1=\overline{\omega}_1^1=d(x), && \omega_2^1=\overline{\omega}_2^1=d(y), && \omega_3^1=\overline{\omega}_3^1=d(z)\\
&\omega_1^2=d(y)\wedge d(z), && \omega_2^2=-\beta d(x)\wedge d(z), && \omega_3^2=d(x)\wedge d(y),\\
&\overline{\omega}_1^2=\beta d(y)\wedge d(z), && \overline{\omega}_2^2=-d(x)\wedge d(z), && \overline{\omega}_3^2=d(x)\wedge d(y).
\end{aligned}$$

   Then, $(\Omega(\mathcal{A}),d)$ is integrable, which completes the proof of the differentially smoothness of $\mathcal{A}$.

Example 1.3.46. Consider the $\mathbb{C}$-algebra $\mathcal{A}$ generated by x,y and z under the relations,

$$yz-\alpha zy=Az,\quad zx-\beta xz=b,\quad xy-\alpha yx=Cx,$$

in the following two particular cases: (a) when $\alpha=A=C=1$; (b) when $\alpha\in\mathbb{C}\setminus\{0\}$ is of free choice and $A=C=0$. Following the technique of Theorem 1.3.42 and Example 1.3.35, if we consider in each case the following objects we guarantee the differentially smoothness of $\mathcal{A}$.

(a)
1. The automorphisms ν_x, ν_y and ν_z defined by the assignments

$$\begin{aligned} &\nu_x(x) = \beta^{-1}x, && \nu_x(y) = y-1, && \nu_x(z) = \beta z,\\ &\nu_y(x) = x, && \nu_y(y) = y, && \nu_y(z) = z,\\ &\nu_z(x) = \beta^{-1}x, && \nu_z(y) = y+1, && \nu_z(z) = \beta z. \end{aligned}$$

2. To guarantee the integrableness of the differential calculus with volume form $d(x)\wedge d(y)\wedge d(z)$:

$$\begin{aligned} &\omega_1^1 = \overline{\omega}_1^1 = d(x), && \omega_2^1 = \overline{\omega}_2^1 = d(y), && \omega_3^1 = \overline{\omega}_3^1 = d(z)\\ &\omega_1^2 = d(y)\wedge d(z), && \omega_2^2 = -d(x)\wedge d(z), && \omega_3^2 = \beta^{-1}d(x)\wedge d(y),\\ &\overline{\omega}_1^2 = \beta^{-1}d(y)\wedge d(z), && \overline{\omega}_2^2 = -d(x)\wedge d(z), && \overline{\omega}_3^2 = d(x)\wedge d(y). \end{aligned}$$

(b)
1. The automorphisms $\nu_x,\ \nu_y$ and ν_z defined by the assignments

$$\begin{aligned} &\nu_x(x) = \beta^{-1}x, && \nu_x(y) = \alpha^{-1}y, && \nu_x(z) = \beta z,\\ &\nu_y(x) = \alpha x, && \nu_y(y) = y, && \nu_y(z) = \alpha^{-1}z,\\ &\nu_z(x) = \beta^{-1}x, && \nu_z(y) = \alpha y, && \nu_z(z) = \beta z. \end{aligned}$$

2. To guarantee the integrableness of the differential calculus with volume form $\omega := d(x)\wedge d(y)\wedge d(z)$:

$$\begin{aligned} &\omega_1^1 = \overline{\omega}_1^1 = d(x), && \omega_2^1 = \overline{\omega}_2^1 = d(y), && \omega_3^1 = \overline{\omega}_3^1 = d(z)\\ &\omega_1^2 = d(y)\wedge d(z), && \omega_2^2 = -\alpha d(x)\wedge d(z), && \omega_3^2 = \alpha\beta^{-1}d(x)\wedge d(y),\\ &\overline{\omega}_1^2 = \alpha\beta^{-1}d(y)\wedge d(z), && \overline{\omega}_2^2 = -\alpha d(x)\wedge d(z), && \overline{\omega}_3^2 = d(x)\wedge d(y). \end{aligned}$$

1.4 Cycles

In this section, we present a transcription of some pages of [Con85], fulling details that are omitted in the original manuscript and we point out some understood facts. In this treatment, we found another construction of a differential-integral calculus called *cycle*, which is key on noncommutative differential geometry. In Proposition 1.4.7, we find a strong relation between cycles and traces operators.

Definition 1.4.1 ([Con85], p. 97)**.** A *cycle* of dimension n is a triple $(\Omega, d, \int)$, where Ω is a graded $\mathbb{C}$-algebra, d is a graded derivation of degree 1[11] with $d^2 = 0$, i.e., (Ω, d) is a differential calculus and $\int : \Omega^n \to \mathbb{C}$ is a closed[12] graded trace.

Remark 1.4.2. This triple fulfill characteristics of a graded differential calculi is as follows:

---

[11] d is a graded derivation, if $d = \sum d_i$, where d_i is a homogeneous linear map of grade $|d_i|$ such that for all $a \in \operatorname{Dom}(d_i)$ homogeneous and any $b \in \operatorname{Dom}(d_i)$, we have that $d(ab) = d(a)b + (-1)^{|a||d_i|}ad(b)$. If d is of degree 1, which means that $d\Omega^i \subset \Omega^{i+1}$, we have that $d(ab) = d(a)b + (-1)^{|a|}ad(b)$.

[12] Closed means that $\int \circ d = 0$, and graded trace means that $\int$ vanish on supercommutators, see [Pit09], p. 233.

1. $(\Omega^i)(\Omega^j) \subset \Omega^{i+j}$, for all $i, j \in \{0, ..., n\}$, $i + j \leq n$. This by definition of graded algebra and Ω being a graded algebra.
2. $d\Omega^i \subset \Omega^{i+1}$, $d(\omega\omega') = d(\omega)\omega' + (-1)^{\deg\omega}\omega d(\omega')$, because d is a graded linear map of degree 1, and $d^2 = 0$ by hypothesis.
3. $\int d\omega = 0$, for all $\omega \in \Omega^{n-1}$, by definition of closed, and $\int \omega'\omega = (-1)^{\deg\omega\deg\omega'} \int \omega\omega'$, by definition of graded trace.

From two cycles, we can form new cycles with the direct sum and the tensor product.

1. If Ω, Ω' are two cycles of dimension n, their sum $\Omega \oplus \Omega'$ is defined by the homogeneous component $(\Omega'')^i = \Omega^i \oplus \Omega'^i$, the component product $(\omega_1, \omega_1')(\omega_2, \omega_2') = (\omega_1\omega_2, \omega_1'\omega_2')$. If $(\omega_1, \omega_1') \in (\Omega'')^i$ and $(\omega_2, \omega_2') \in (\Omega'')^j$ as $\omega_k\omega_{k+1} \in \Omega^{i+j}$, we have that Ω'' is a graded algebra. If we define $d(\omega, \omega') = (d\omega, d\omega')$ and $\int(\omega, \omega') = \int \omega + \int \omega'$, we obtain a graded derivation of degree 1 and a closed graded trace. Then $(\Omega \oplus \Omega', d, \int)$ is a cycle of dimension n.
2. If Ω, Ω' are two cycles of dimension n and n', respectively, their tensor product $\Omega'' = \Omega \otimes \Omega'$ is the cycle of dimension $n + n'$ which as a differential graded algebra, it is the tensor product of (Ω, d) by (Ω', d') and $\int(\omega \otimes \omega') = (-1)^{nn'} \int \omega \int' \omega'$, for all $\omega \in \Omega^n$ and $\omega' \in \Omega'^{n'}$. In this way, we have $(\Omega'')^i = \bigoplus_{k+j=i}(\Omega^k \otimes \Omega'^j)$, $d'' = d \otimes d'$ (see Remark 1.3.21).

For the next example, we need the following definition.

Definition 1.4.3 ([dR84], p. 34 and 79)**.** A *closed current* of degree p and of dimension $\dim(V) - p$ on a smooth manifold V is a linear continuous[13] map $T : \Omega^{\dim(V)-p} \to \mathbb{C}$, such that $bT = 0$[14]. If α is a form in $\Omega^{\dim(V)-p}$, the value of T at α is denoted $T(\alpha)$ or $\langle T, \alpha\rangle$.

Example 1.4.4. Let V be a smooth closed[15] manifold, let C be a closed current of dimension $q \leq \dim V$ on V, and let Ω^i, $i \in \{0, ..., q\}$ be the space $C^\infty(V, \Lambda^i T^* V)$ of smooth differential forms of degree i. With the usual product structure and differentiation, $\Omega = \bigoplus_{i=0}^q \Omega^i$ is a differential algebra, on which the equality $\int \omega = \langle C, \omega\rangle$, for $\omega \in \Omega^q$, defines a closed graded trace.

Remark 1.4.5. In Example 1.4.4, the fact that $\int$ is graded, i.e., $\langle C, d\omega\rangle = 0$, for all differential form ω, it is a consequence of the Stokes theorem (see [Mor98], p. 109).

Before to state the key proposition of this section, we recall the construction of the universal algebra $\Omega(A)$ (see [Pit09], p. 216). Even if A is unital, let $\tilde{A} = A \oplus \mathbb{C}$ with the usual sum and the following product: if $(a, \lambda), (b, \mu) \in \tilde{A}$ then $(a, \lambda)(b, \mu) = (\lambda b + \mu a + ab, \lambda\mu)$. With this structure, $\tilde{A}$ is an associative unital algebra[16] with unit $(0, 1)$. The elements $(a, \lambda) \in \tilde{A}$ will be denoted $a + \lambda$. With this notation, the product $(a + \lambda)(b + \mu) =$

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[13] This continuity is given in terms of local uniformly convergence of forms, see [dR84], p. 34 for details.

[14] bT is the boundary of a current, such that, for all $\omega \in \mathrm{Dom}T$, $bT(\omega) = T(d\omega)$ (see [dR84], p. 45).

[15] A smooth manifold is *closed* if it is compact and without boundary.

[16] $\tilde{A}$ is a $\mathbb{C}-$algebra, but also $\tilde{A}$ has structure of $A-$algebra saying that $(a+\lambda)b = ab+\lambda b = (a+\lambda)(b+0)$, for all $a + \lambda \in \tilde{A}$ and $b \in A$, i.e., we make the identification $b = b + 0 \in \tilde{A}$.

$ab + a\lambda + b\mu + \lambda\mu$ coincides with a distribution law. For each $n \in \mathbb{N}$, $n \geq 1$, let $\Omega^n(A)$ be the linear space $\Omega^n(A) = \tilde{A} \otimes \bigotimes_1^n A$. The differential $d : \Omega^n \to \Omega^{n+1}$ is given by

$$d((a_0 + \lambda_0) \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n \in \Omega^{n+1}. \tag{1.4.1}$$

By construction, $d^2 = 0$. Let us now define the product $\Omega^i \times \Omega^j \to \Omega^{i+j}$. One first defines a right A-module structure on Ω^n by the equality

$$(\tilde{a}_0 \otimes \cdots \otimes a_n)a = \sum_{j=0}^{n} (-1)^{n-j} (\tilde{a}_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a). \tag{1.4.2}$$

This right action of A on Ω^n extends to a unital action of $\tilde{A}$ saying that $\omega(a+\lambda) = \omega a + \omega\lambda$: we know that $(\omega a)b = \omega(ab)$, if $\omega \in \Omega^n$ and $a, b \in A$. Now, we have to prove that $(\omega(a + \lambda))(b + \mu) = \omega((a + \lambda)(b + \mu))$.

- We have

$$\begin{aligned} (\omega(a + \lambda))(b + \mu) &= (\omega a + \omega\lambda)(b + \mu) \\ &= (\omega a + \omega\lambda)b + (\omega a + \omega\lambda)\mu \\ &= (\omega a)b + (\omega a)\mu + (\omega\lambda)b + (\omega\lambda)\mu. \end{aligned} \tag{1.4.3}$$

- On the other hand,

$$\begin{aligned} \omega((a + \lambda)(b + \mu)) &= \omega(ab + a\mu + b\lambda + \lambda\mu) \\ &= \omega(ab) + \omega(a\mu) + \omega(b\lambda) + \omega(\lambda\mu) \end{aligned} \tag{1.4.4}$$

Then, we have that both expressions are equal because Ω^n is an A-module and A and Ω^n are $\mathbb{C}-$algebras.

We define the product: $\Omega^i \times \Omega^j \to \Omega^{i+j}$ by considering

$$\omega(\tilde{b}_0 \otimes b_1 \otimes \cdots \otimes b_j) = (\omega\tilde{b}_0) \otimes b_1 \otimes \cdots \otimes b_j, \quad \forall \omega \in \Omega^i. \tag{1.4.5}$$

More explicitly, using the notation $\tilde{a}_0 da_1 \cdots da_n = \tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n$, we have

$$\begin{aligned} (a_0 da_1 \cdots da_n)(a_{n+1} da_{n+2} \cdots da_m) = \sum_{j=1}^{n} (-1)^{n-j} a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_n da_{n+1} \cdots da_m \\ +(-1)^n a_0 a_1 da_2 \cdots da_m. \end{aligned} \tag{1.4.6}$$

It is immediate that the product is associative. With $\omega = \tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n \in \Omega^n$, for $a \in A$,

$$d(\omega a) = \sum_{j=0}^{n} (-1)^{n-j} 1 \otimes a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a, \tag{1.4.7}$$

$$\begin{aligned}(d\omega)a &= \sum_{j=0}^{n+1}(-1)^{n+1-j}1\otimes a_0\otimes\cdots\otimes a_{j-1}a_j\otimes\cdots\otimes a\\ &= (-1)^{n-1}\omega da + d(\omega a).\end{aligned} \tag{1.4.8}$$

If $\Omega(A)=\bigoplus_{n\in\mathbb{N}}\Omega^n(A)$, we obtain that $(\Omega(A),d)$ is a differential graded algebra.

Remark 1.4.6. The previous construction of the universal differential calculus $\Omega(A)$ is quite different from that one we present in Section 1.1.3, but, by the universal property, both are isomorphic graded algebras.

Proposition 1.4.7 ([Con85], p. 98)**.** *Let τ be an $(n+1)-$linear functional on A. Then the following conditions are equivalent:*

1. *There exists an n-dimensional cycle $(\Omega,d,\int)$ and a homomorphism $\rho: A\to\Omega^0$ such that:*
$$\tau(a_0,...,a_n)=\int\rho(a_0)d(\rho(a_1))\cdots d(\rho(a_n))\ \forall a_0,...,a_n\in A. \tag{1.4.9}$$

2. *There exists a closed graded trace T of dimension n on $\Omega(A)$ such that*
$$\tau(a_0,...,a_n)=T(a_0da_1\cdots da_n)\qquad\forall a_0,...,a_n\in A. \tag{1.4.10}$$

3. *One has $\tau(a_1,...,a_n,a_0)=(-1)^n\tau(a_0,...,a_n)$, for $a_0,...,a_n\in A$, and for $a_0,...,a_{n+1}$,*
$$\sum_{i=0}^{n}(-1)^i\tau(a_0,...,a_ia_{i+1},...,a_{n+1})+(-1)^{n+1}\tau(a_{n+1}a_0,...,a_n)=0. \tag{1.4.11}$$

Proof. The equality $\tilde{a}_0da_1\cdots da_n=\tilde{a}_0\otimes a_1\otimes\cdots\otimes a_n$ shows that $(\Omega(A),d)$ is generated by A. One checks that any homomorphism $A\to\Omega'^0$ of A in a differential graded algebra (Ω',d'), $d'^2=0$, extends to a homomorphism $\bar{\rho}:(\Omega(A),d)\to(\Omega',d')$ with

$$\bar{\rho}(\tilde{a}_0\ da_1\ \cdots\ da_n)=\rho(a_0)\ d'(\rho(a_1))\ d'(\rho(a_2))\ \cdots\ d'(\rho(a_n))+\lambda_0\ d'(\rho(a_1))\ \cdots\ d'(\rho(a_n)), \tag{1.4.12}$$

for $a_i\in A$, $\tilde{a}_0\in\tilde{A}$, $\tilde{a}_0=(a_0,\lambda_0)=a_0+\lambda_0$. Then, we have that 1 implies 2 saying that $T=\int\circ\bar{\rho}$ and 2 implies 1 taking $(\Omega,d)=(\Omega(A))$, $\int=T$ and $\rho:A\to\tilde{A}:a\mapsto a+0$, where $\tilde{A}=\Omega^0$. We have now that 1 and 2 are equivalent. Let us show that 3 implies 2. Given any $(n+1)$-linear functional φ on A, we define $\hat{\varphi}$ as a linear functional on $\Omega^n(A)$ by

$$\hat{\varphi}((a_0+\lambda)\otimes\cdots\otimes a_n)=\varphi(a_o,a_1...,a_n). \tag{1.4.13}$$

By construction, one has $\hat{\varphi}(d\omega)=0$, for all $\omega\in\Omega^{n-1}(A)$ because, if $\omega=(a+\lambda)\otimes\theta$ we have

$$\hat{\varphi}(d\omega)=\hat{\varphi}(d((a+\lambda)\otimes\theta))=\hat{\varphi}((0+1)\otimes a\otimes\theta)=\varphi(0,a,a_1...,a_{n-1})=0. \tag{1.4.14}$$

Now, with τ satisfying 3, let us show that $\hat{\tau}$ is a graded trace, i.e., that $\hat{\tau}$ vanish in commutators. We have to show that

$$\hat{\tau}((a_0da_1\cdots da_k)(a_{k+1}da_{k+2}\cdots da_{n+1}))=(-1)^{k(n-k)}\hat{\tau}((a_{k+1}da_{k+2}\cdots da_{n+1})(a_0da_1\cdots da_k)). \tag{1.4.15}$$

Using the product in $\Omega(A)$, we found that

$$(a_0da_1\cdots da_k)(a_{k+1}da_{k+2}\cdots da_{n+1}) = ((a_0da_1\cdots da_k)a_{k+1})da_{k+2}\cdots da_{n+1}) \tag{1.4.16}$$

$$= \sum_{j=0}^{k}(-1)^{k-j}(a_0\cdots a_{j-1}d(a_ja_{j+1})a_{j+2}\cdots da_kda_{k+1}\cdots da_{n+1}). \tag{1.4.17}$$

Then,

$$\hat{\tau}((a_0da_1\cdots da_k)(a_{k+1}da_{k+2}\cdots da_{n+1})) = \sum_{j=0}^{k}(-1)^{k-j}\tau(a_0,...,(a_ja_{j+1}),...,a_k,a_{k+1},...,a_{n+1}). \tag{1.4.18}$$

Also,

$$\begin{aligned}(a_{k+1}da_{k+2}\cdots da_{n+1})(a_0da_1\cdots da_k) &= ((a_{k+1}da_{k+2}\cdots da_{n+1})a_0)da_1\cdots da_k \\ &= \sum_{j=0}^{n-k}(-1)^{n-k-j}(a_{k+1}\cdots d(a_ja_{j+1})\cdots da_{n+1}da_0\cdots da_k).\end{aligned} \tag{1.4.19}$$

Therefore,

$$\begin{aligned}&(-1)^{k(n-k)}\hat{\tau}((a_{k+1}da_{k+2}\cdots da_{n+1})(a_0da_1\cdots da_k)) \\ &= (-1)^{k(n-k)}\sum_{j=0}^{n-k}(-1)^{n-k-j}\tau(a_{k+1},...,(a_ja_{j+1}),...,a_{n+1},a_0,...,a_k) \\ &= \sum_{j=0}^{n-k}(-1)^{k(n-k)+n-k-j}\tau(a_{k+1},...,(a_ja_{j+1}),...,a_{n+1},a_0,...,a_k).\end{aligned} \tag{1.4.20}$$

Now, we use the permutation λ such that $\lambda(i) = k+1+i$. This permutation has signature $\varepsilon(\lambda) = (-1)^{n(k+1)}$, and since by hypothesis 3 we have that

$$\begin{aligned}&\tau(a_{k+1},...,a_{j-1},(a_ja_{j+1}),a_{j+2},...,a_{n+1},a_0,...,a_k) \\ &= (-1)^n\tau(a_k,a_{k+1},...,a_{j-1},(a_ja_{j+1}),a_{j+2},...,a_{n+1},a_0,...,a_{k-1}),\end{aligned} \tag{1.4.21}$$

then, we have that $\tau^\lambda = \varepsilon(\lambda)\tau$. We also have

$$\begin{aligned}k(n-k)+n-k-\lambda(j) &= k(n-k)+n-k-(k+1+j) \\ &= k(n-k)+n-k-(k+1+j) = kn-k^2+n-2k-1-j.\end{aligned} \tag{1.4.22}$$

We obtain that (1.4.20) turns to

$$\sum_{j=k+1}^{n+1}(-1)^{kn-k^2+n-2k-1-j+n(k+1)}\tau(a_0,...,a_k,a_{k+1},...,(a_ja_{j+1}),...,a_{n+1}). \tag{1.4.23}$$

Like $kn-k^2+n-2k-1-j+n(k+1) \equiv k-j-1(mod2)$, then $(-1)^{kn-k^2+n-2k-1-j+n(k+1)} =$

$(-1)^{k-j-1}$. Therefore,

$$\begin{aligned} &(-1)^{k(n-k)}\hat{\tau}((a_{k+1}da_{k+2}\cdots da_{n+1})(a_0da_1\cdots da_k)) \\ &= \sum_{j=k+1}^{n+1}(-1)^{k-j-1}\tau(a_0,...,a_k,a_{k+1},...,(a_ja_{j+1}),...,a_{n+1}) \\ &= -\sum_{j=k+1}^{n}(-1)^{k-j}\tau(a_0,...,a_k,a_{k+1},...,(a_ja_{j+1}),...,a_{n+1}) \\ &+(-1)^{k-n}\tau(a_{n+1}a_0,...,a_n). \end{aligned} \tag{1.4.24}$$

By hypothesis 3, we have that

$$\sum_{j=0}^{n}(-1)^j\tau(a_0,...,a_ja_{j+1},...,a_{n+1})+(-1)^{n+1}\tau(a_{n+1}a_0,...,a_n)=0. \tag{1.4.25}$$

Multiplying by $(-1)^k$ and clearing the last right side of (1.4.24), we obtain

$$\sum_{j=0}^{k}(-1)^{k-j}\tau(a_0,...,(a_ja_{j+1}),...,a_k,a_{k+1},...,a_{n+1}), \tag{1.4.26}$$

and hence we have proved that

$$\hat{\tau}((a_0da_1\cdots da_k)(a_{k+1}da_{k+2}\cdots da_{n+1}))=(-1)^{k(n-k)}\hat{\tau}((a_{k+1}da_{k+2}\cdots da_{n+1})(a_0da_1\cdots da_k)), \tag{1.4.27}$$

which conclude this part of the proof.

Let us show now that 1 implies 3. We can asume that $A=\Omega^0$ (and $\rho=id_A$). By hypothesis, we have $\tau(a_0,...,a_n)=\int\rho(a_0)d(\rho(a_1))\ ...\ d(\rho(a_n))=\int a_0\ da_1\ ...\ da_n$. Then

$$\begin{aligned} \tau(a_0,...,a_n) &= \int(a_0\ da_1)\ da_2...\ da_n \\ &= (-1)^{n-1}\int da_2...\ da_n\ (a_0\ da_1) \\ &= (-1)^{n}\int(da_2...\ da_n\ da_0)\ a_1 \\ &= (-1)^n\tau(a_1,...,a_n,a_0). \end{aligned} \tag{1.4.28}$$

The first equality is due to the associativity of the product in A. The second and fourth equalities are by definition of graded trace, i.e., by the rule $\int\omega\omega'=(-1)^{\deg\omega\deg\omega'}\int\omega'\omega$. With the aim of showing the third one, since $\int d=0$ we have that[17]

$$\begin{aligned} A &=: \int da_2\cdots\ da_n\ (a_0\ da_1) \\ &= \int\sum_{j=2}^{n}(-1)^{n-2-(j-1)}da_2\cdots d(a_ja_{j+1})da_nda_0da_1+\int(-1)^{n-1}a_2da_3\cdots da_nda_0da_1 \\ &= (-1)^{n-1}\int a_2da_3\cdots da_nda_0da_1 \end{aligned} \tag{1.4.29}$$

---

[17] Note that $da_2\cdots d(a_ja_{j+1})da_nda_0da_1=d((a_2+1)da_3\cdots d(a_ja_{j+1})da_nda_0da_1)$.

By the same way, we can also see that

$$B =: \int (da_2 \cdots \; da_n \; da_0) \; a_1 = (-1)^n \int a_2 da_3 \cdots da_n da_0 da_1. \tag{1.4.30}$$

Therefore, we have $(-1)^{n-1}A = (-1)^n B$, which proves the third equation. If we want to show the second property, we shall only use the equality

$$\int a\omega = (-1)^{\deg(a)\deg(\omega)} \int \omega a = \int \omega a \;\; \text{for } \omega \in \Omega^n,\, a \in A, \tag{1.4.31}$$

because $\deg(a) = 0$. In particular,

$$\int a_{n+1}(a_0 da_1 \cdots da_n) = \int (a_0 da_1 \cdots da_n) a_{n+1}. \tag{1.4.32}$$

Hence, we have to prove that

$$\sum_{i=0}^{n} (-1)^i \int a_0 da_1 ... d(a_i a_{i+1}) ... da_{n+1} + (-1)^{n+1} \int (a_{n+1} a_0) da_1 ... da_n = 0, \tag{1.4.33}$$

and this is by the last affirmation and by the definition of the product on Ω as follows:

$$\begin{aligned} (-1)^{n+1} \int (a_{n+1} a_0) da_1 ... da_n &= (-1)^{n+1} \int (a_0 da_1 ... da_n) a_{n+1} \\ &= (-1)^{n+1} \int \sum_{j=0}^{n} (-1)^{n+i} a_0 da_1 ... d(a_j a_{j+1}) ... da_{n+1} \\ &= (-1) \sum_{j=0}^{n} (-1)^i \int a_0 da_1 ... d(a_j a_{j+1}) ... da_{n+1}. \end{aligned} \tag{1.4.34}$$

□

We consider that Proposition 1.4.7 shows that the trace operators posses very important information about the (noncommutative) geometrical structure of general algebras $\mathcal{A}$.

CHAPTER 2

Noncommutative algebras and their calculus

As we noted in the first chapter, such like in didderential geometry, we know that the skew derivations of an algebra play a key role in the description of their geometry, in particular, they are the main tool for the Brzezinski's calculus. Therefore, we are interested in looking for descriptions and features of the skew derivations for noncommutative algebras. In this chapter, we consider three classes of noncommutative algebras and study objects associated to them with the aim of obtaining differential calculus structures. We present in the first section, the generalized Weyl algebras, and include some examples of them just as the hyperbolic algebras. We also study some skew derivations of the quantum plane, a particular example of these algebras. In the second section, we consider the diffusion algebras. First, we study how these algebras appear from the study on stationary states of probability in a stochastic flow of particles problem. Later, we present two types of diffusion algebras, some properties about them such as a description of their commutation laws in the 3-dimensional case, which allows us to describe, in terms of the PBW basis, the inner *id*-derivations of each generator; we finish with a study of their automorphisms and skew derivations. The third section is dedicate to study the 3-dimensional skew polynomial algebras, their classification, their structures either of Ore extensions or hyperbolic algebras. In this work, in Section 2.3.1 we establish some extended automorphisms in the Ore extensions, this with aim to investigate the skew derivations related to these automorphisms in a future work. All of these algebras are particular cases of a class of noncommutative algebras called *skew PBW extensions*. For these objects, several properties have been studied (Noetherianity, K-theory, ACCP-condition, McCoy's condition, Kothë's conjecture, among other properties), by some people in [LR14], [RS16],[RS17b],[RS18],[RR19] and [RS19a]. In the fourth section, we close the chapter with a brief revision of [Art15], where the author partially characterizes derivations for skew PBW extensions. We remark some mistakes and questions of the treatment presented in that article and propose answers to some of that questions.

2.1 Generalized Weyl algebras

We are going to recall the *generalized Weyl algebras*, which were defined by Bavula [Bav92].

Definition 2.1.1. ([Mei16], p. 123.) **Generalized Weyl algebra.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Fix a unital associative $\mathbb{K}-$algebra R that is a Noetherian domain. Given n nonzero elements $t = (t_1, \dots, t_n)$ in R and n pairwise commuting algebra automorphisms $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathrm{Aut}(R)$ such that $\sigma_i(t_j) = t_j$, for all $i \neq j$, define the corresponding *generalized Weyl algebra* $A = R(\sigma, t)$ of rank n as follows: it is the $\mathbb{K}-$algebra generated over R by $2n$ generators x_-^i, x_+^i, $1 \leq i \leq n$, with relations given by the following:

- $x_-^i x_+^i = t_i$.
- $x_+^i x_-^i = \sigma_i(t_i)$.
- $x_\pm^i r = \sigma_i^{\pm 1}(r) x_\pm^i$, for all $r \in R$.
- $[x_-^i, x_-^j] = [x_+^i, x_+^j] = [x_-^i, x_+^j] = 0$, for all $i \neq j$.

Next, we present some remarkable examples of generalized Weyl algebras.

Example 2.1.2 ([Ros95], p. 46)**. Quantum plane.** If we consider $n = 1$, $R = \mathbb{K}[h]$, $\sigma(h) = qh$ with $q \neq 0$ and $t = h$, we obtain the generalized Weyl algebra $R(\sigma, t) = \mathbb{K}[h](qh, h)$, which is isomorphic to the quantum plane $\mathbb{K}\langle x, y \mid xy = qyx\rangle$.

Example 2.1.3 ([Cou95], p. 8)**. Weyl algebra** $A_n(\mathbb{K})$**.** If we consider $R = \mathbb{K}[h_1, \dots, h_n]$, $\sigma_i(h_j) = h_j - \delta_{ij}$, where δ_{ij} is the Kronecker's delta, and $t_i = h_i$, for all $i, j \in \{1, \dots, n\}$, we obtain that $\mathbb{K}[h_1, \dots, h_n](\sigma, t)$ is isomorphic to the nth-Weyl algebra $\mathbb{K}\langle x, \partial \mid x\partial - \partial x = 1\rangle$.

Example 2.1.4 ([Bav93], p. 88)**. Factor Algebras of** $Usl(2)$**.** Let $U = Usl(2)$ be the universal enveloping algebra of the Lie algebra $sl(2)$ over a field $\mathbb{K}$ of characteristic zero generated by x, y and h, with the relations:

$$[h, x] = x,\ [h, y] = -y,\ [x, y] = 2h.$$

If $C = h(h+1) + yx$ (the *Cashimir element*), we have that the universal enveloping algebra of $sl(2)$, $Usl(2)$, it is isomorphic to the generalized Weyl algebra $\mathbb{K}[h, c](\sigma, t = c - h(h+1))$ where $\sigma : h \mapsto h - 1, c \mapsto c$. Moreover, for any $\lambda \in \mathbb{K}$, the factor algebra $U(\lambda) = U(sl(2))/U(sl(2))(C - \lambda)$ is also a generalized Weyl algebra with $U(\lambda) \cong \mathbb{K}[h](\sigma, a = \lambda - h(h+1))$.

Remark 2.1.5. The usual relations of $sl(2, \mathbb{C})$, and therefore, of $U(sl(2, \mathbb{C}))$, are given by

$$[h', x] = 2x, \quad [h', y] = -2y, \quad [x, y] = h',$$

which are the same relations of the Example 2.1.4, with the identification $h = 2h'$. Also, it is well known that over $\mathbb{C}$, the Lie algebra $sl(2, \mathbb{C})$ is isomorphic to $so(3, \mathbb{C})$, that is, a Lie algebra $\mathbb{C}$-generated by B_{12}, B_{13} and B_{23} with the relations

$$[B_{12}, B_{13}] = -B_{23}, \quad [B_{12}, B_{23}] = B_{13}, \quad [B_{13}, B_{23}] = -B_{12},$$

by using the identification

$$B_{12} \mapsto \frac{i}{2}h, \quad B_{13} \mapsto \frac{1}{2}(x - y) \quad B_{23} \mapsto \frac{-i}{2}(x + y).$$

If we consider another identification, defined by $B_y = -B_{12}$, $B_z = -B_{23}$ and $B_x = B_{13}$, the bracket in $so(3, \mathbb{C})$ turns to

$$[B_x, B_y] = B_z, \quad [B_y, B_z] = B_x, \quad [B_z, B_x] = B_y.$$

These last relations result in commutation laws of some basis for $U(sl(2, \mathbb{C}))$, and all of them are useful in Section 2.3.

Example 2.1.6 ([Bav96a], p. 1985)**. Woronowicz deformation.** V is generated by V_0, V_+ and V_- subject to the following relations:

$$s^2V_0V_+ - s^{-2}V_+V_0 = V_+, \quad s^2V_-V_0 - s^{-2}V_0V_- = V_-, \quad s^{-1}V_+V_- - sV_-V_+ = V_0.$$

The algebra V is isomorphic to the generalized Weyl algebra $\mathbb{K}[u, v](\sigma, t = v)$ where $V_\pm \leftrightarrow x_\pm$, $V_0 \leftrightarrow u$ and $V_-V_+ \leftrightarrow v$, and where $\sigma : u \mapsto s^2(s^2u - 1)$, $v \mapsto s^2v + su$ is the automorphism of the polynomial ring $\mathbb{K}[u, v]$.

Remark 2.1.7. In [Ros95], p. 100, we found an algebra called the **Woronowicz deformation** of $Usl(2)$, which is the $\mathbb{K}$-algebra generated by x, y, z such that

$$xz - \nu^4 zx = (1 + \nu^2)x, \quad xy - \nu^2 yx = \nu z, \quad zy - \nu^4 yz = (1 + \nu^2)x,$$

where $\nu \in \mathbb{K} \setminus \{0\}$ is not a root of unity. We note the similarity of both Woronowicz deformation of Example 2.1.6, however we do not found yet an isomorphism between them.

About generalized Weyl algebras, we found in literature the following facts:

1. Generalized Weyl algebras have a PBW basis ([Bav92], p. 76).
2. The weak dimension is known for some generalized Weyl algebras that have enough flat extensions with a well behavior of their weak dimensions ([Bav96b], Theorem 1.2).
3. The global dimension[1] of generalized Weyl algebras of degree 1 is computed, obtaining that this belongs to $\{1, 2, \infty\}$ ([Bav96b], Theorem 1.6).
4. For some particular generalized Weyl algebras of degree one is established the groups of Hochschild homology and cohomology ([SSAV13], Theorem 1.1, Theorem 1.2).
5. A criterion of simplicity of generalized Weyl algebras with base ring without zero divisors is constructed ([Bav96a], Theorem 4.5).
6. If $\mathcal{A}$ is a simple generalized Weyl algebra of degree n, then $\mathrm{GKdim}(\mathcal{A}) = 2n$ ([Bav96a], Corollary 4.8).
7. The group of automorphism is computed for some generalized Weyl algebras of degree one ([SAV15]).

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[1] The global dimension of $\mathcal{A}$ is the supremum of the minimal lengths of projective resolutions of all $\mathcal{A}$-modules.

Remark 2.1.8. In this work, we note that the Woronowicz deformations of Example 2.1.6 and Remark 2.1.7, both denoted by $\mathcal{A}$, cannot be differentially smooth, because, such as we see in Example 1.3.41, since one of the defining commutation rules in their definition relations need three generators, where one of the generators appears only in one lineal term, if $(\Omega(\mathcal{A}), d)$ is a differential calculus, we conclude that $\Omega^1(\mathcal{A})$ is generated by two elements as an $\mathcal{A}$-bimodule, and therefore $\Omega^3(\mathcal{A}) = \Omega^1(\mathcal{A}) \wedge \Omega^1(\mathcal{A}) \wedge \Omega^1(\mathcal{A}) = 0$. Hence, $(\Omega(\mathcal{A}), d)$ cannot be 3-dimensional, where $\text{GKdim}(\mathcal{A}) = 3$ by [Rey13], Theorem 14. This implies that $\mathcal{A}$ is not differentially smooth.

Remark 2.1.9. In 1995, it was defined by Rosenberg the notion of *hyperbolic ring*, which is the generalized Weyl algebra of rank $n = 1$ (c.f. [Ros95], p. 61). In [Ros95], it was studied the left spectrum[2] of hyperbolic rings and exhibited for particular cases such as the quantum plane, the quantum torus, the first Weyl algebra and the q-differential operators, among others (c.f. [Ros95], p. 64). Rosenberg denotes an hyperbolic ring $R\{\theta, \xi\}$, which in the sense of Definition 2.1.1, means the algebra $R(\sigma, t)$, where $\sigma = \theta$, $t = \xi$, $x = x_+$ and $y = x_-$.

2.1.1 Hyperbolic rings

In [Ros95], p. 63, we can find a way to construct an hyperbolic ring (a generalized Weyl algebra) from another algebra, denoted by $A\langle \vartheta, \rho, u\rangle$. Let us see the details.

Definition 2.1.10 ([Ros95], p.63)**.** The algebra $A\langle \vartheta, \rho, u\rangle$, where ϑ is an automorphism of the ring A, ρ is an invertible of A an $u \in A$, it is the algebra generated by the elements x and y satisfying the relations,

$$xa = \vartheta(a)x, \qquad ya = \vartheta^{-1}(a)y, \qquad \text{for any } a \in A$$
$$xy - \rho yx = u.$$

Remark 2.1.11. From an algebra $A\langle \vartheta, \rho, u\rangle$, we can construct a hyperbolic ring as follows:

1. Fixing $t = xy$, we obtain that t is a central element of A.
2. The algebra generated by A and t is isomorphic to $A[t]$.
3. Defining $\theta(t) = \vartheta(\rho)t + \vartheta(u)$ we obtain an endomorphism $\theta : A[t] \to A[t]$ which is an extension of the automorphism ϑ. This endomorphism θ is an automorphism with inverse given by
$$\theta'(t) = \rho^{-1}(t - u), \qquad \theta'(a) = \vartheta^{-1}(a), \qquad \text{for all } a \in A.$$
4. $A[t]\{\theta, t\}$ is a hyperbolic ring.

Remark 2.1.12. In fact, we can construct an algebra $A\langle \vartheta, \rho, u\rangle$ from a hyperbolic ring $R\{\theta, t\}$, and we obtain a one to one correspondence between some of these two types of algebras (cf. [Ros95], 64).

The Remark 2.1.11 will be useful in the classification of 3-dimensional skew polynomial algebras (see Section 2.3). Now, we study a specific class of hyperbolic algebras that appears in a few documents about differential calculus.

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[2] The left spectrum in this case is the set of left ideals $\mathfrak{p} \subset \mathcal{A}$ such that $((\mathfrak{p} : x) : w) \subseteq \mathfrak{p}$, for some $x \in \mathcal{A} - \mathfrak{p}$ and some finitely generated $\mathbb{Z}$-submodule $w \subset \mathcal{A}$ (see [Ros95], p. 5).

2.1.2 A degree one generalized Weyl algebra

We are going to present a particular class of generalized Weyl algebras that are basic objects in the next sections.

Definition 2.1.13 ([Brz16b], p. 2)**.** Let $\mathbb{K}$ be a field of characteristic 0, $p \in \mathbb{K}[z]$ and $q \in \mathbb{K} \setminus \{0\}$. Define the following algebras:

- $B(p,q)$ as the affine algebra over $\mathbb{K}$ generated by x, y, z, where
$$xz = q^2 zx, \quad yz = q^{-2} zy, \quad xy = q^2 zp(q^2 z), \quad yx = zp(z).$$
- $A(p,q)$ as the affine algebra over $\mathbb{K}$ generated by x_+, x_-, z_+, z_- satisfying the relations
$$\begin{aligned} z_+z_- &= z_-z_+, & x_+z_\pm &= q^{-1}z_\pm x_+, & x_-z_\pm &= qz_\pm x_-, \\ x_+x_- &= p(z_+z_-), & x_-x_+ &= p(q^2z_-z_+). \end{aligned}$$
- We say that any of these algebras is *regular*, if $p(z)$ and $zJ_{q^2}(p(z))$ (see Example 1.1.42) are coprime[3] in $\mathbb{K}[z]$.

Considering $R = \mathbb{K}[z]$, $x_- = y$, $x_+ = x$, $t = zp(z)$ and $\sigma(z) = q^2 z$, we obtain the generalized Weyl algebra structure of $B(p,q)$, and with $R = \mathbb{K}[z_-, z_+]$, $x_- = x_-$, $x_+ = x_+$, $t = p(q^2 z_- z_+)$ and $\sigma(z_\pm) = q^{-1} z_\pm$, the generalized Weyl algebra structure of $B(p,q)$ is found.

Example 2.1.14 ([Brz16b], p. 3)**.** If we consider $\mathbb{K} = \mathbb{C}$, $q \in [-1,1]$ and $p(z) = 1 - q^{-2}z$, we obtain that $A(p,q)$ is the coordinate algebra of the quantum $SU_q(2)$.

Remark 2.1.15. From [Maj95] and [Brz16a], we have that $A(p,q)$ is a strongly $\mathbb{Z}$-graded algebra with coacting Hopf algebra the group algebra of $\mathbb{Z}$, that is $\mathcal{K}[t, t^{-1}] \cong \mathcal{K}[z_+, z_-]$ (it is the coordinate algebra that describes the circle) when $\deg(z_\pm) = \pm 1$ and $\deg(x_\pm) = \pm 1$, and the degree-zero component is equal to $B(p,q)$. Then we can consider to $A(p,q)$ as a circle principal bundle.

In the following example, we present a differential and integral calculus computed for the algebra $\mathcal{A}$ that appears in [Brz16b].

Example 2.1.16. For a regular $\mathcal{A} = A(p,q)$, in [Brz16b], p. 7, it was constructed a left $\mathcal{A}$-module with basis ω_1, ω_2 and ω_3, using the ideas presented in Section 1.2, (see Remark 1.2.12). To do this, since $\mathcal{A}$ is a graded algebra, we consider the automorphism $\sigma_1(a) = \sigma_3(a) = q^{|a|}a$ and $\sigma_2(a) = q^{2|a|}a$, for all $a \in \mathcal{A}$. Now, consider the maps defined by

$$\partial_2(x_+) = \alpha_2 x_+, \quad \partial_2(x_-) = -q^{-2}\alpha_2 x_-, \quad \partial_2(z_+) = \alpha_2 z_+, \quad \partial_2(z_+) = -q^{-2}\alpha_2 z_-,$$

$$\partial_1(x_+) = \partial_1(z_+) = 0, \quad \partial_1(x_-) = \alpha_1 c(z) z_+, \; \partial_1(z_+) = \alpha_1 x_+,$$

---

[3] In a DIP R, two elements $a, b \in R$ are *coprime*, if $gcd(a,b) = 1$.

$$\partial_3(x_-) = \partial_3(z_-) = 0, \quad \partial_3(x_+) = \alpha_3 c(z) z_-, \; \partial_3(z_+) = \alpha_3 x_-,$$

where $\alpha_i \in \mathbb{K} \setminus \{0\}$ and $c(z) := q[p(q^2 z_+ z_-) - p(z_+ z_-)]/(q^2 - 1) z_+ z_-$. The map ∂_i is an σ_i-skew derivation of $\mathcal{A}$. Define

$$w_i a := \sigma_i(a) w_i, \quad \text{for } i = 1, 2, 3,.$$

If $q^2 \neq 1$ or $\deg(p(z)) \leq 1$, we obtain that (Ω, d) is a first order differential calculus on $\mathcal{A}$ (the Brzezinski's differential calculus of (∂_i, σ_i), see Definition 1.2.7), where $\Omega = \mathcal{A}w_1 \oplus \mathcal{A}w_2 \oplus \mathcal{A}w_3$ and

$$d : \mathcal{A} \to \Omega : a \mapsto \sum_{i=1}^{3} \partial_i(a) w_i.$$

The density condition of (Ω, d) is guaranteed by the Bézout identity[4] applied to $p(z)$ and z in $\mathbb{K}[z]$.

A hom-connection $\nabla : \operatorname{Hom}_{\mathcal{A}}(\Omega, \mathcal{A}) \to \mathcal{A}$ is defined by the assignment

$$\xi \mapsto q^{-2} \partial_1(\xi(w_1)) + \partial_2(\xi(w_2)) + q^2 \partial_3(\xi(w_3)),$$

for all $\xi \in \operatorname{Hom}_{\mathcal{A}}(\Omega, \mathcal{A})$. In this case, the cokernel $\mathcal{A}/\operatorname{Im}(\nabla)$ is $n = \deg(p)$-dimensional with basis $v_i := \Lambda(z^i)$, for $i = 0, \dots, n-1$, where $\Lambda : \mathcal{A} \to \mathcal{A}/\operatorname{Im}(\nabla)$ is the canonical map (see Definition 1.2.11). We have that Λ is zero in all the elements of $\mathcal{A} \setminus \mathbb{K}$, and that for all $k \in \mathbb{N}$,

$$\Lambda(z^{n+k}) = \sum_{i=0}^{n-1} \frac{[i+1]}{[n+k+1]} \beta_i^k v_i, \qquad \text{where } [l] := \frac{1 - (q^2)^l}{1 - (q^2)} = 1 + q^2 + \cdots + q^{2l-2}.$$

Also, it is known how to find the values of β_i^k as follows. Let $\overline{p}(z) = lc(p)^{-1} p(z)$ be the monic polynomial associated to $p(z)$. Then, we can write

$$\overline{p}(z) = z^n - \sum_{i=0}^{n-1} \mu_i z^i.$$

In this way,

$$\beta_i^k = \sum_{j=1}^{n} \mu_{n-1} \beta_i^{k-j} + \mu_{i-k},$$

where, if $l < 0$, $\mu_l = \beta_i^l = 0$.

2.1.3 Skew derivations of the quantum plane

Now, we present a description found in [AB18] of some skew right derivations related to an automorphism of a generalized Weyl algebra of degree one, which is a natural extension of his base subalgebra.

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[4] If R is a DIP, the Bézout identity says that for all $a, b \in R$, if $d = gcd(a, b)$, then there exist $x, y \in R$ such that $ax + by = d$.

Given $R(a,\varphi)$ a generalized Weyl algebra of dimension 1, let σ be a ring automorphism of R such that $\sigma\circ\varphi=\varphi\circ\sigma$, and $\sigma(a)=a$. Then, for any central unit μ in R, the map σ extends to the automorphism σ_μ of $R(a,\varphi)$ by $\sigma_\mu(x)=\mu^{-1}x$ and $\sigma_\mu(y)=y\mu=\varphi^{-1}(\mu)y$.

Theorem 2.1.17 ([AB18], p. 7)**.** *Let $R(a,\varphi)$ be a generalized Weyl algebra and let σ be an automorphism of R commuting with φ and fixing a. Let σ_μ be the degree-counting extension of σ of coarseness μ, and consider the following data:*

1. *skew derivations on $R(\alpha_i,\varphi^i\circ\sigma)_{i\in\mathbb{Z}}$, such that, for all $i\in\mathbb{Z}$, $\alpha_i\circ\varphi=\varphi^i(\mu)\varphi\circ\alpha_i$, and there exists $c\in R^R_\sigma$ such that $\alpha_0(a)=a\varphi^{-1}(c)$;*
2. *elements $c_i\in R^R_{\varphi^i\circ\sigma}=\{c\in R:\forall p\in R, pc=c(\varphi^i\circ\sigma)(p)\}$ and $b_i\in(R\setminus R^R_{\varphi^i\circ\sigma})\cup\{0\}$, $i\in\mathbb{Z}$;*
3. *a set I of positive integers such that, for all $r\in R$, the sets $\{i\in I:\alpha_\pm(r)\neq 0\}$ are finite and the sequences $(c_i)_{\pm i\in I}$, $(b_i)_{\pm i\in I}$ are finitely supported.*

Given above data, we define,

$$\begin{aligned}
\partial(r) &= \sum_{m\in I\cup\{0\}}(\alpha_m(r)+b_m\varphi^m\circ\sigma(r)-rb_m)x^m\\
&\quad+\sum_{n\in I}(\alpha_{-n}(r)+b_{-n}\varphi^{-n}\circ\sigma(r)-rb_{-n})y^n,\\
\partial(x) &= \sum_{m\in I\cup\{0\}}(c_m-\varphi(b_m)+\varphi^m(\mu^{-1})b_m)x^{m+1}\\
&\quad+\sum_{n\in I}\varphi(\alpha_{-n}(a)+\varphi^{-n-1}(\mu^{-1})(\varphi^{-1}(b_{-n})\varphi^{-n}(a)-ac_{-n})-b_{-n}a)y^{n-1}\\
\partial(y) &= \sum_{n\in I}(c_{-n}+\varphi^{-n+1}(\mu)b_{-n}-\varphi^{-1}(b_{-n}))y^{n+1}+(\varphi^{-1}(\mu c-b_0)+\varphi^{-1}(\mu)b_0+C_o)y\\
&\quad+\sum_{m\in I}\varphi^{m-1}(\mu)(\alpha_m(a)-\varphi^{-1}(c_m+\varphi^m(\mu^{-1})b_m)a+b_m\varphi^m(a))x^{m-1},
\end{aligned}$$

where $C_0\in R^R_\sigma$ is a solution to the equation $(C_0+\varphi^{-1}(\mu c_0))a=0$. Then ∂ extends to a skew derivation (∂,σ_μ) on $R(a,\varphi)$.

Definition 2.1.18 ([AB18], p. 12)**.** The assignments $\partial(x),\partial(y)$ and $\partial(r)$ in Theorem 2.1.17 are split in the following way:

1. The zero component:

$$\begin{aligned}
\partial_0(r) &= \alpha_0(r)+b_0\sigma(r)-rb_0\\
\partial_0(x) &= (c_0-\varphi(b_0)+\mu^{-1}b_0)x\\
\partial_0(y) &= (\varphi^{-1}(\mu c-b_0)+\varphi^{-1}(\mu)b_0+C_o)y.
\end{aligned}$$

2. The positive degree case ($m>0$):

$$\begin{aligned}
\partial_m(r) &= (\alpha_m(r)+b_m\varphi^m\circ\sigma(r)-rb_m)x^m\\
\partial_m(x) &= (c_m-\varphi(b_m)+\varphi^m(\mu^{-1})b_m)x^{m+1}\\
\partial_m(y) &= \varphi^{m-1}(\mu)(\alpha_m(a)-\varphi^{-1}(c_m+\varphi^m(\mu^{-1})b_m)a+b_m\varphi^m(a))x^{m-1}.
\end{aligned}$$

3. The negative degree case ($n > 0$):

$$\begin{aligned}
&\partial_{-n}(r) = (\alpha_{-n}(r) + b_{-n}\varphi^{-n} \circ \sigma(r) - rb_{-n})y^n\\
&\partial_{-n}(x) = \varphi(\alpha_{-n}(a) + \varphi^{-n-1}(\mu^{-1})(\varphi^{-1}(b_{-n})\varphi^{-n}(a) - ac_{-n}) - b_{-n}a)y^{n-1}\\
&\partial_{-n}(y) = (c_{-n} + \varphi^{-n+1}(\mu)b_{-n} - \varphi^{-1}(b_{-n}))y^{n+1}.
\end{aligned}$$

Each ∂_i splits in three components $\partial_i = \partial_i^\alpha + \partial_i^b + \partial_i^c$, where:

- ∂_i^α, (called $\alpha - type$) is obtained from ∂_i with $b_i = c_i = 0$.
- ∂_i^b (called $inner - type$) is obtained from ∂_i with $\alpha_i = c_i = 0$.
- ∂_i^c (called $c - type$) is obtained from ∂_i with $\alpha_i = b_i = 0$.

In this section, we reproduce the first part of the proof of Theorem 2.1.21 presented in [AB18], fulling the details, and whose content is a direct application of Theorem 2.1.17 to obtain a complete description of the σ_μ-twisted skew derivations of the quantum plane; we review the description of the classification, omitting the verification that all these skew derivations follows that forms.

First, we present the following two necessary lemmas.

Lemma 2.1.19. ([AB18], p. 9) *An additive map $\partial_m^\alpha : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of positive standard degree m and such that $\partial_m^\alpha(x) = 0$ if and only if there exists a skew derivation $(\alpha_m, \varphi^m \circ \sigma)$ of R such that $\alpha_m \circ \varphi = \varphi^m(\mu)\varphi \circ \alpha_m$, and, for all $r \in R$,*

$$\partial_m^\alpha(r) = \alpha_m(r)x^m, \qquad \partial_m^\alpha(y) = \varphi^{m-1}(\mu)\alpha_m(a)x^{m-1} = \mu\alpha_m(a)x^{m-1}. \tag{2.1.1}$$

An additive map $\partial_0^\alpha : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of standard degree 0 and such that $\partial_0^\alpha(x) = 0$ if and only if there exists a skew derivation (α_0, σ) of R and $c \in R_\sigma^R$ such that $\alpha_0 \circ \varphi = \mu\varphi \circ \alpha_0$, and $\alpha_0(a) = a\varphi^{-1}(c)$ and, for all $r \in R$,

$$\partial_0^\alpha(r) = \alpha_0(r), \qquad \partial_0^\alpha(y) = \varphi^{m-1}(\mu c)y. \tag{2.1.2}$$

An additive map $\partial_{-n}^\alpha : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of negative standard degree $-n$ and such that $\partial_{-n}^\alpha(y) = 0$ if and only if there exists a skew derivation $(\alpha_{-n}, \varphi^{-n} \circ \sigma)$ of R such that $\alpha_{-n} \circ \varphi = \varphi^{-n}(\mu)\varphi \circ \alpha_{-n}$, and, for all $r \in R$,

$$\partial_{-n}^\alpha(r) = \alpha_{-n}(r)y^m, \qquad \partial_{-n}^\alpha(x) = \varphi(\alpha_{-n}(a))y^{n-1}. \tag{2.1.3}$$

Lemma 2.1.20. ([AB18], p. 11) *An additive map $\partial_i^c : R(a,\varphi) \to R(a,\varphi)$ is a σ_μ-twisted skew derivation of standard degree i and such that $\partial_i^c(R) = 0$ if and only if there exists $c_i \in R_{\varphi^i \circ \sigma}^R$ such that,*

$$\partial_i^c(x) = c_i x^{i+1}, \qquad \partial_i^c(y) = -\varphi^{m-1}(\mu)\varphi^{-1}(c_i)ax^{i-1},$$

if i is positive, or

$$\partial_i^c(x) = -\varphi^i(\mu)\varphi^{-1}(ac_i)x^{-i-1}, \qquad \partial_i^c(y) = c_i y^{-i+1},$$

if i is negative, or

$$\partial_0^c(x) = c_0 x, \qquad \partial_0^c(y) = Cy,$$

where $C \in R_\sigma^R$ is a solution to the equation $(C + \varphi^{-1}(\mu c_0))a = 0$.

Now, we present a classification of skew derivations for the quantum plane in the following theorem that is a consequence of Theorem 2.1.17.

Theorem 2.1.21 ([AB18], p. 24)**.** *Assume that a non-zero $q \in \mathbb{K}$ is not a root of unity, and let A be the quantum plane $\mathbb{K}_q[x, y] := \mathbb{K}\langle x, y\rangle/\langle xy - qyx\rangle$. Set $h = yx$, and let μ be a non-zero element of $\mathbb{K}$.*

1. *for all $f(h) \in \mathbb{K}[h]$, the map ∂ on generators of A given by*

$$\partial(x) = f(h)x, \qquad \partial(y) = -\mu f(q^{-1}h)y, \tag{2.1.4}$$

extends to a skew derivation (∂, σ_μ) of A. These are the only σ_{mu}-derivations such that $\partial(h) = 0$. They are inner if and only if there is no $d \in \{0, ..., deg(f)\}$ such that $\mu = q^{-d}$, and the coefficient f_d in $f(h) = \sum_k f_k h^k$ is not zero.

2. *If there exists $d \in \mathbb{N}$ such that $\nu = q^{-d+1}$, then:*
   - *(a) for all $a(x) \in \mathbb{K}[x]$ and $b(y) \in \mathbb{K}[y]$, the map given by $\partial(x) = h^d a(x)$ and $\partial(y) = h^d b(y)$, extends to a skew derivation (∂, σ_μ) of A. All these derivations are inner if $d \neq 0$, and they are not inner if $d = 0$.*
   - *(b) If $d \geq 1$, then for all $\lambda \in \mathbb{K}^*$, the map given by $\partial(x) = 0$, $\partial(y) = \lambda h^{d-1}y$, extends to a non-inner skew derivation on $\mathbb{K}_q[x, y]$.*

3. *The (combinations of the) above maps together with the inner-type derivations exhaust all σ_μ-skew derivations on A contained in Theorem 2.1.17. Every σ_μ-skew derivation on A is of this type.*

Proof. First, we study all possible σ_μ-skew derivations on A that satisfy the assumptions of Theorem 2.1.17. Since in our case $\sigma = id$, we first determine φ^n-skew derivations of the polynomial algebra $\mathbb{K}[h]$. The action of φ^n on any element of $\mathbb{K}[h]$ results in rescaling the h by q^n ($h \mapsto q^n h$). Thus, any φ^n-skew derivation ∂_n of $\mathbb{K}[h]$ takes the form of a multiple of an appropriate Jackson's derivative (see Example 1.1.42 and Example 1.1.43) (understood as the ordinary derivative in case $n = 0$),

$$\partial_n(f(h)) = a_n(h) f'_{q^n}(h) = a_n(h)\frac{f(q^n h) - f(h)}{(q^n - 1)h}.$$

As $\varphi^n(\mu) = \mu$, for all $n \in \mathbb{Z}$, because $\mu \in \mathbb{K}$, requesting that $\partial_n \circ \varphi = \mu\varphi \circ \partial_n$ (i.e. $\partial_n \circ \varphi = \varphi^n(\mu)\varphi \circ \partial_n$), and evaluating it at h, yields the constraint $qa_n(h) = \mu a_n(qh)$, which has the following solutions: either

1. $a_n(h) = 0$ and there are no restrictions on μ, or else,
2. there exists $d \in \mathbb{N}$ such that $\mu = q^{-d+1}$. In this case $qa_n(h) = q^{-d+1}a_n(qh)$, i.e., $q^d a_n(h) = a_n(qh)$ which implies that there is no zero degree term in $a_n(h)$ and that all terms of $a_n(h)$ are divisible by h^d. Then $a_n(h) = p(h)h^d$, and since $q^d a_n(h) = a_n(qh)$, this implies that $q^d p(h)h^d = p(qh)(qh)^d = q^d p(qh)h^d$, whence $p(h) = p(qh)$, which is equivalent to assert $p(h)$ is constant. Therefore $a_n(h)$ is a scalar multiple of h^d.

Actually, we obtain that $\partial_n \circ \varphi(f(h)) = \mu\varphi \circ \partial_n(f(h))$, for all $f(h) \in \mathbb{K}[h]$: note that for all $j \geq 1$,

$$\begin{aligned}\partial_n \circ \varphi(h^j) = \partial_n(q^j h^j) = q^j \partial_n(h^j) = q^j p h^{d+j-1} &= q^{-d+1} p q^{d+j-1} h^{d+j-1} \\ &= q^{-d+1}\varphi(p h^{d+j-1}) = q^{-d+1}\varphi(\partial_n(h^j)).\end{aligned}$$

Therefore, we obtain by $\mathbb{K}$-linearity, that $\partial_n \circ \varphi(f(h)) = \mu\varphi \circ \partial_n(f(h))$, for all $f(h) \in \mathbb{K}[h]$, and $\partial_n \circ \varphi = \mu\varphi \circ \partial_n$, as we desired.

These skew derivations provide us with only choices of maps α_i in Theorem 2.1.17. In the case $a_n(h) = 0$, all elementary α-type derivations ∂_i^α, $i \in \mathbb{Z}\setminus\{0\}$ are trivial, and we are thus left with ∂_0. To do this, first, we declare the values of $b_0, c_0 \in \mathbb{K}[h]$. We are free to select any $c_0 = f(h) \in \mathbb{K}[h] = \mathbb{K}[h]^{\mathbb{K}[h]}_{\varphi^0 \circ id}$ and $b_0 \in \{0\} = \mathbb{K}[h] \setminus (\mathbb{K}[h]^{\mathbb{K}[h]}_{\varphi^0 \circ id}) \cup \{0\}$. Since $a = h$, we have that $\alpha_0(a) = 0 = a\varphi^{-1}(c)$ then $c = 0$, and as $(C_0 + \varphi^{-1}(\mu c_0))h = (C_0 + \mu\varphi^{-1}(f(h)))h = 0$ we have that $C_0 = -\mu\varphi^{-1}(f(h))$. Therefore,

$$\begin{aligned}\partial_0(r) &= \alpha_0(r) + b_0\sigma(r) - r b_0 = \alpha_0(r) \\ \partial_0(x) &= (c_0 - \varphi(b_0) + \mu^{-1} b_0)x = f(h)x \\ \partial_0(y) &= (\varphi^{-1}(\mu c - b_0) + \varphi^{-1}(\mu) b_0 + C_o)y = -\mu\varphi^{-1}(f(h))y = -\mu f(q^{-1}h)y.\end{aligned}$$

Then, by the Theorem 2.1.17, we obtain the skew derivation of the first part of the statement *1.*

In the case of $a_n(h) = ph^d$ with $p \in \mathbb{K}$ and $d \in \mathbb{N}$, we have that $c \in \mathbb{K}[h]$ such that $\alpha_0(h) = ph^d = h\varphi^{-1}(c)$. Then, if $d \geq 1$, we have that $c = p\varphi(h^{d-1}) = pq^{d-1}h^{d-1}$, and if $d = 0$, then $c = p = 0$. Since for all $i \neq 0$, $\mathbb{K}[h]^{\mathbb{K}[h]}_{\varphi^i} = \{0\}$ because $\mathbb{K}[h]$ is Abelian, we have that $c_i = 0$, for all $i \in \mathbb{Z} \setminus \{0\}$ and in the case $i = 0$, we obtain that $b_0 = 0 \in \{0\} = \mathbb{K}[h] \setminus (\mathbb{K}[h]^{\mathbb{K}[h]}_{\varphi^i}) \cup \{0\}$. Then,

$$\begin{aligned}\partial_m^\alpha(r) &= \alpha_m(r)x^m = p\, r'_{q^m(h)} h^d x^m, \\ \partial_m^\alpha(x) &= 0, \\ \partial_m^\alpha(y) &= \varphi^{m-1}(\mu)(\alpha_m(a))x^{m-1} = \mu p h^d x^{m-1}, \\ \partial_{-n}^\alpha(r) &= (\alpha_{-n}(r))y^n = p\, r'_{q^{-n}}(h) h^d y^n, \\ \partial_{-n}^\alpha(x) &= \varphi(\alpha_{-n}(a))y^{n-1} = \varphi(ph^d)y^{n-1} = pq^{-d}h^d y^{n-1}, \\ \partial_{-n}^\alpha(y) &= 0.\end{aligned}$$

Hence, by Lemma 2.1.19, we obtain that in this case ∂_i^α is an α-type elementary σ_μ-twisted skew derivation, for all $i \in \mathbb{Z} \setminus \{0\}$. Considering the σ_μ-twisted skew derivation $\partial = \partial_m^\alpha + \partial_{-n}^\alpha$, for any pair $m, -n > 0$, we obtain the first part of the statement *2.(a)*. If $d \geq 1$, then the monomial ph^d contains factor h, and so gives rise to the α-type weight zero elementary derivation on A,

$$\begin{aligned}\partial_0^\alpha(r) &= \alpha_0(r) \\ \partial_0^\alpha(x) &= 0 \\ \partial_0^\alpha(y) &= (\varphi^{-1}(\mu c) + C_o)y = (\mu\varphi^{-1}(c) + 0)y = ph^{d-1}y,\end{aligned}$$

because $C_0 = -\varphi^{-1}(\mu c_0) = 0$. By Lemma 2.1.19, we obtain that $\partial = \partial_0^\alpha$ and $\lambda = p$ establish the first part of statement *2* (b).

Since in this case $c_i = 0$, for all $i \in \mathbb{Z} \setminus \{0\}$, then there are no non-trivial elementary c-derivations of non-zero weight; in the weight zero case, since $b_0 = 0$, taking $c_0 = f(h)$ and as $0 = a\varphi^{-1}(c)$, we obtain

$$\begin{aligned}
&\partial_0^c(r) = 0\\
&\partial_0^c(x) = (c_0)x\\
&\partial_0^c(y) = (\varphi^{-1}(\mu c - b_0) + C_o)y = (0 - \mu f(q^{-1}h))y = -\mu f(q^{-1}h)y,
\end{aligned}$$

because $C_0 = -\varphi^{-1}(\mu c_0) = -\mu f(q^{-1}h)$ and $c = 0$. These derivations are the only derivations which vanish on $\mathbb{K}[h]$ by 2.1.20 (it is sufficient that vanish on h). This proves the second part of statement *1*. Skew derivations (2.1.4) are inner, provided there exists $j(h) \in \mathbb{K}[h]$ such that $f(h) = \mu^{-1}j(h) - j(qh)$; this, because in that case,

$$\begin{aligned}
\partial(x) &= f(h)x = \mu^{-1}j(h)x - j(qh)x = j(h)\mu^{-1}x - \varphi(j(h))x\\
&= j(h)\sigma_\mu(x) - xj(h) = \partial_{j(h)}(x)\\
\partial(y) &= -\mu f(q^{-1}h)y = -\mu[\mu^{-1}j(q^{-1}h) - j(h)]y = \mu j(h)y - j(q^{-1}h)y\\
&= j(h)\sigma_\mu(y) - \varphi^{-1}(j(h))y = j(h)\sigma_\mu(y) - yj(h) = \partial_{j(h)}(y),
\end{aligned}$$

and then $\partial = \partial_{j(h)}$, but we need to satisfy the next conditions. In order to guarantee the existence of $j(h)$, since $f(h) = \mu^{-1}j(h) - j(qh)$, if $f(h) = \sum_{k=0}^{\deg(f)} f_k h^k$ and $j(h) = \sum_{k=0}^{\deg(f)} j_k h^k$ we have

$$\sum_{k=0}^{\deg(f)} f_k h^k = \sum_{k=0}^{\deg(f)} (\mu^{-1} - q^k) j_k h^k.$$

Therefore, if $f_{k\neq 0}$, we need that $\mu^{-1} \neq q^k$, and we obtain that $j_k = f_k/(\mu^{-1} - q^k)$. This proves the third part of statement *1*. For the derivations defined in *2.* (a) since the polynomials b and a are mutually independent, we can treat the cases $a = 0$ and $b = 0$ separately. In the case $a = 0$, we define

$$j(y) = q^{-d+1} \sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1} - 1} y^{n+1},$$

where $b(y) = \sum_n b_n y^n$. If $d > 0$, then

$$\begin{aligned}
&q^{d-1}h^{d-1}j(y)x - xh^{d-1}j(y)\\
&= q^{d-1}h^{d-1}\left[q^{-d+1}\sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1}-1} y^{n+1}\right]x - xh^{d-1}\left[q^{-d+1}\sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1}-1} y^{n+1}\right]\\
&= h^{d-1}\left[\sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1}-1} q^{-n-1}xy^{n+1}\right] - q^{d-1}h^{d-1}x\left[q^{-d+1}\sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1}-1} y^{n+1}\right]\\
&= h^{d}\left[\sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1}-1} q^{-n-1}y^{n}\right] - h^{d}\left[\sum_{n=0}^{\deg(b)} \frac{b_n}{q^{-n-1}-1} y^{n}\right] = h^d \sum_{k=0}^{\deg(b)} b_n y^n
\end{aligned}$$

$$
\begin{aligned}
&q^{-d+1}h^{d-1}j(y)y - yh^{d-1}j(y)\\
&= q^{-d+1}h^{d-1}\left[q^{-d+1}\sum_{n=0}^{\deg(b)}\frac{b_n}{q^{-n-1}-1}y^{n+1}\right]y - yh^{d-1}\left[q^{-d+1}\sum_{n=0}^{\deg(b)}\frac{b_n}{q^{-n-1}-1}y^{n+1}\right]\\
&= q^{-2d+2}h^{d-1}\left[\sum_{n=0}^{\deg(b)}\frac{b_n}{q^{-n-1}-1}y^{n+1}\right]y - q^{1-d}h^{d-1}y\left[q^{-d+1}\sum_{n=0}^{\deg(b)}\frac{b_n}{q^{-n-1}-1}y^{n+1}\right]\\
&= \left[\sum_{n=0}^{\deg(b)}\frac{q^{-2d+2}h^{d-1}b_n}{q^{-n-1}-1}y^{n+2}\right] - \left[\sum_{n=0}^{\deg(b)}\frac{q^{2-2d}h^{d-1}b_n}{q^{-n-1}-1}y^{n+2}\right] = 0.
\end{aligned}
$$

Thus, as $\partial_{h^{d-1}j(h)}(x) = q^{d-1}h^{d-1}j(y)x - xh^{d-1}j(y) = h^d b(y) = \partial(x)$ and $\partial_{h^{d-1}j(h)}(y) = q^{-d+1}h^{d-1}j(y)y - yh^{d-1}j(y) = 0 = \partial(y)$, we obtain that ∂ is the inner derivation $\partial_{h^{d-1}j(h)}$ in case $d \leq 1$. If $d = 0$, i.e., $\partial(x) = b(y)$, then all combinations of x with polynomials in y produce a polynomial in h of degree at least one, hence they cannot be equal to $b(y)$. The other case, $b(y) = 0$, it is dealt in a similar way. This proves the second statement in *2.* (*a*).

Finally, we look at derivations *2.* (*b*). Since $\partial(x) = 0$, if $j(h) \in \mathbb{K}[h]$ such that $\partial = \partial_{j(h)}$, we obtain that $0 = \partial(x) = \partial_{j(h)}(x) = j(h)\sigma_\mu(x) - xj(h) = j(h)\mu^{-1}x - \varphi(j(h))x = [j(h)\mu^{-1} - j(qh)]x$, whence $j(h)\mu^{-1} - j(qh) = 0$, which is equivalent to $q^{d-1}j(h) = j(qh)$, and this implies that $j(h) = j_{d-1}h^{d-1}$ with $j_{d-1} \in \mathbb{K}$. In this case, however, we would have,

$$
\begin{aligned}
\partial(y) &= j(h)\sigma_\mu(y) - yj(h)\\
&= j(h)\varphi^{-1}(\mu)y - \varphi^{-1}(j(h))y\\
&= j_{d-1}h^{d-1}\mu y - \varphi^{-1}(j_{d-1}h^{d-1})y\\
&= j_{d-1}h^{d-1}\mu y - \mu j_{d-1}h^{d-1}y = 0.
\end{aligned}
$$

Therefore $\lambda h^{d-1}y = \partial(y) = 0$, which is an absurd. In this way, $\partial(x) = 0$ and $\partial(y) = \lambda h^{d-1}y$ with $d \leq 1$ and $\lambda \in \mathbb{K}^*$ defines a σ_μ-twisted skew derivation which is non-inner. This completes the proof of statement *2.* (*b*). □

Now, we present, without proof the following proposition that allows us determine orthogonal pairs of skew derivations (see Remark 1.2.13) of generalized Weyl algebras of degree one.

Proposition 2.1.22 ([AB18], p. 19)**.** *Let $R(a,\varphi)$ be a generalized Weyl algebra and let σ, $\overline{\sigma}$ be automorphisms of R commuting with φ and fixing a. Let $\sigma_\mu, \overline{\sigma}_{\overline{\mu}}$ be their degree-counting extensions with respective coarseness $\mu, \overline{\mu}$. Choose a positive integer N such that a is coprime with $\varphi^i(a)$, for all $i \in \{1, 2, ..., 2N-1\}$, fix $m, n \in \{0, 1, ..., N\}$ and consider the following data:*

1. *A $\sigma \circ \varphi^{m+1}$-skew derivation α of R such that*
    - *(a) $\alpha(a)$ is a central element of R,*
    - *(b) $\alpha(a)$ is coprime with $\varphi^j(a)$, where $j \in \{-m-1, -m, ..., 0, m+1, m+2, ..., 2m\}$, and with $\varphi^{-m}(\alpha(a))$,*

(c) $\alpha \circ \varphi = \varphi^{m+1}(\mu)\varphi \circ \alpha$.

2. *A* $\overline{\sigma} \circ \varphi^{-n-1}$*-skew derivation* $\overline{\alpha}$ *of* R *such that*

   (a) $\overline{\alpha}$ *is in the centre of* R,

   (b) $\varphi^{n+1}(\overline{\alpha})$ *is coprime with* $\varphi^{j}(a)$, *where* $j \in \{-n-1, -n, ..., 0, n+1, n+2, ..., 2n\}$ *and with* $\varphi(\overline{\alpha}(a))$,

   (c) $\overline{\alpha} \circ \varphi = \varphi^{-n-1}(\mu)\varphi \circ \overline{\alpha}$.

Then the elementary α*-type skew derivations (see Definition 2.1.18)* ∂ *of* σ_μ *and* $\overline{\partial}$ *of* $\overline{\sigma}_{\overline{\mu}}$ *of* $R(a, \varphi)$ *associated to* α *and* $\overline{\alpha}$, *form an orthogonal pair.*

Remark 2.1.23. In [AB18], p. 29, we can find that Proposition 2.1.22 is applied to obtain pairs of orthogonal skew derivation on the quantum plane. The only skew derivations that allow the application of Proposition 2.1.22 are the skew derivations *2* (*a*) of Theorem 2.1.21, and then, with $\varphi^i(a) = q^i h$, the only skew derivations that we obtain in this case are such that $\partial(x) = 0$, $\partial(y) = c$, $\overline{\partial}(x) = \overline{c}$, $\overline{\partial}(y) = 0$.

2.2 Diffusion algebras calculi

In this section, we study the diffusion algebras and present some results about them. First we track step by step the origin of the definition of these algebras in the study of stationary states of probability in stochastic flow of particles, and we present their deduction using probability and matricial language. Next, we give the two types of diffusion algebras, we list some facts we can found in literature about them and present a classification in the 3-dimensional case of diffusion algebras of type 1. Also, we describe the commutation laws in each class of this classification in terms of the PBW basis, and we describe the inner derivations of the generators. We finish this section by studying conditions of their derivatives and graded preserving automorphisms.

2.2.1 Origin of diffusion algebras

In [DDM92], it was studied the *stochastic flow of particles*[5] (of the same type/species) through a *discrete linear lattice*[6]. The different cases between a cell i and the next cell $i+1$, this in a time t to the time $t+1$, are expressed by the following tables:

	τ_i	τ_{i+1}
t	0	0
$t+1$	0	0

	τ_i	τ_{i+1}
t	1	0
$t+1$	0	1

	τ_i	τ_{i+1}
t	0	1
$t+1$	0	1

	τ_i	τ_{i+1}
t	1	1
$t+1$	1	1

(2.2.1)

Here, $\tau_i(t) = 1$, if in the cell i there is a particle at the time t and $\tau_i(t) = 0$, if it is empty. The tables information can be summarized with the equations $\tau_i(t+1) = \tau_i(t)\tau_{i+1}(t)$ and

---

[5] The particle's motion is not described by a evolution expression but using a probability map on the possibles configurations on the space.

[6] This is the space where the particle's motion happens, a totally ordered set that has a discrete topology.

$\tau_{i+1}(t+1)=\tau_{i+1}(t)+(1-\tau_{i+1}(t))\tau_i(t)$; this corresponds to a Markov process[7] with the only not quiet transition is from $(1,0)$ to $(0,1)$. This case corresponds to a current toward right.

If the lattice length is N (N cells) and in the time t it has the occupations $(\tau_1(t),...,\tau_N(t))$, we have that the probability $P_N(\tau_1,...,\tau_N)$ is a function of t. If the first cell is empty, a particle can enter to the lattice on this cell with a probability α, and if there is a particle at the last cell, it can leave the lattice with a probability β. The interest of this treatment is to find a probability function P_N such that $\frac{d}{dt}P_N(\tau_1,\tau_2,...,\tau_N)=0$; this is called the *stationary state of probability.* In [DDM92], it is affirmed that for the stationary state we have the following:

$$\begin{aligned}
&P_N(\tau_1,\tau_2,...,\tau_N)\\
&=\frac{1-\alpha}{N+1}P_N(\tau_1,\tau_2,...,\tau_N)+\frac{\alpha}{N+1}\tau_1[P_N(0,\tau_2,...,\tau_N)+P_N(1,\tau_2,...,\tau_N)]\\
&+\frac{1}{N+1}[P_N(\tau_1,\tau_2,...,\tau_N)+(\tau_2-\tau_1)P_N(1,0,\tau_3,...,\tau_N)]\\
&+\cdots+\frac{1}{N+1}[P_N(\tau_1,\tau_2,...,\tau_N)+(\tau_N-\tau_{N-1})P_N(\tau_1,\tau_2,...,\tau_{N-2},1,0)]\\
&+\frac{1-\beta}{N+1}P_N(\tau_1,\tau_2,...,\tau_N)+\frac{\beta}{N+1}(1-\tau_N)[P_N(\tau_1,\tau_2,...,\tau_{N-1},0)+P_N(\tau_1,\tau_2,...,\tau_{N-1},1)].
\end{aligned}\tag{2.2.2}$$

In this work, we illustrate this with the following brief example.

Example 2.2.1. If $N=2$, we have that (2.2.2) turns to

$$(\alpha+\beta)P_2(\tau_1,\tau_2)=\alpha\tau_1[P_2(0,\tau_2)+P_2(1,\tau_2)]+(\tau_2-\tau_1)P_2(1,0)+\beta(1-\tau_2)[P_2(\tau_1,0)+P_2(\tau_1,1)].$$

Therefore, for each element of $\{0,1\}\times\{0,1\}$, we obtain that,

$$\begin{aligned}
(0,0):&\quad \alpha P_2(0,0)=\beta P_2(0,1),\\
(1,0):&\quad P_2(1,0)=\alpha P_2(0,0)+\beta P_2(1,1),\\
(0,1):&\quad (\alpha+\beta)P_2(0,1)=P_2(1,0),\\
(1,1):&\quad \beta P_2(1,1)=\alpha P_2(0,1),
\end{aligned}$$

which is a homogeneous linear system of equations, with solution set

$$\langle(\beta^2/\alpha^2,(\beta^2/\alpha)+\beta,\beta/\alpha,1)\rangle=\langle(\beta^2,\beta^2\alpha+\beta\alpha^2,\beta\alpha,\alpha^2)\rangle,$$

obtaining that $P_2(0,0)=\beta^2$, $P_2(1,0)=\beta^2\alpha+\beta\alpha^2$, $P_2(0,1)=\beta\alpha$ and $P_2(0,0)=\alpha^2$ define the stationary state of probability.

Also, in [DDM92] there is an inductive construction of P_N[8] as follows:

We can define $f_1(1)=\alpha$ and $f_1(0)=\beta$, and for $N\geq 2$:

$$\begin{aligned}
f_N(\tau_1,\tau_2,...,\tau_N)&=\alpha\tau_N f_{N-1}(\tau_1,\tau_2...,\tau_{N-1})\\
&+\alpha\beta(1-\tau_N)\tau_{N-1}[f_{N-1}(\tau_1,\tau_2...,\tau_{N-2},1)+f_{N-1}(\tau_1,\tau_2...,\tau_{N-2},0)]\\
&+\cdots\\
&+\alpha\beta(1-\tau_N)(1-\tau_{N-1})\cdots(1-\tau_2)\tau_1[f_{N-1}(1,\tau_2...,\tau_{N-1})+f_{N-1}(0,\tau_2...,\tau_{N-1})]\\
&+\beta(1-\tau_N)(1-\tau_{N-1})\cdots(1-\tau_1)f_{N-1}(\tau_1,\tau_2...,\tau_{N-1})
\end{aligned}$$

---

[7] A Markov process is a stochastic process such that the behavior of the immediate future only depends on the actual situation and not on the past.

[8] In the paper [DDM92], the authors consider $f_N=P_N/Z_N$ but we are going to explain using only P_N.

Later, they defined $Z_N = \sum_{(\tau_i)_{i\leq N}\in\{0,1\}^N} f_N(\tau_1, ..., \tau_N)$ and finally,

$$P_N(\tau_1, ..., \tau_N) = f_N(\tau_1, ..., \tau_N)/Z_N.$$

Following the inductive formula of f_N in the case of $N = 3$, we obtain

$(\tau_i)_{i\leq 3}$	$f_3(\tau_1, \tau_2, \tau_3)$
$(0,0,0)$	β^3
$(1,0,0)$	$\alpha^2\beta^2(\alpha+\beta)+\alpha\beta^3$
$(0,1,0)$	$\alpha^2\beta^2+\alpha\beta^3$
$(0,0,1)$	$\alpha\beta^2$
$(1,1,0)$	$\alpha^3\beta+\alpha^2\beta^2(\alpha+\beta)$
$(1,0,1)$	$\alpha^2\beta(\alpha+\beta)$
$(0,1,1)$	$\alpha^2\beta$
$(1,1,1)$	α^3

In [DEHP93], the authors gave a matrix formulation of the problem. First, they summarized the base information in the following three intensity matrices:

$$h_1 = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix}, \quad h_N = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \text{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $h = (h_{a,b;c,d})$ such that $(a,b),(c,d) \in \{0,1\}\times\{0,1\}$, and if $(a,b) \neq (c,d)$, $h_{a,b;c,d}$ is the probability that being in the state (a,b) in the positions i and $i+1$ at the time t, we obtain for the same cells (c,d) in $t+1$, following (2.2.1). They gave an equivalent formula to (2.2.2) as follows:

$$\begin{aligned} \frac{d}{dt}P_N(\tau_1, \tau_2, ..., \tau_N) = & \sum_{\sigma_i}(h_1)_{\tau_1,\sigma_1}P_N(\sigma_1, \tau_2, ..., \tau_N) \\ & + \sum_{i=1}^{N-1}\sum_{\sigma_i,\sigma_{i+1}}(h)_{\tau_i,\tau_{i+1};\sigma_i,\sigma_{i+1}}P_N(\tau_1, ..., \sigma_i, \sigma_{i+1}, ..., \tau_N) \\ & + \sum_{\sigma_N}(h_N)_{\tau_N;\sigma_N}P_N(\tau_1, \tau_2, ..., \sigma_N). \end{aligned} \tag{2.2.3}$$

Later, they gave an infinite-dimensional solution (in [DEHP93] the authors talk about the finite dimensional case) for this equation defining:

$$f_N(\tau_1, ..., \tau_N) = \langle W|\prod_{i=1}^{N}\tau_i D + (1-\tau_i)E|V\rangle, \tag{2.2.4}$$

representing with D that there is a particle in the i^{th} position and with E that this position is empty, where

$$D = \begin{pmatrix} 1/\beta & 1/\beta & 1/\beta & 1/\beta & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.2.5}$$

$$\langle W| = (1, 1/\alpha, (1/\alpha)^2, (1/\alpha)^3, (1/\alpha)^4, ...), \qquad |V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{2.2.6}$$

For instance, if $N = 3$ we have for $(1,1,0)$

$$f_3(1,1,0) = \langle W|DDE|V\rangle = \langle W|(DD)(E|V\rangle),$$

and therefore,

$$\begin{aligned} f_3(1,1,0) &= (1, 1/\alpha, (1/\alpha)^2, ...) \begin{pmatrix} 1/\beta^2 & 1/\beta^2 + 1/\beta & 1/\beta^2 + 2/\beta & \cdots \\ 0 & 1 & 2 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \\ &= 1/\beta^2 + 1/\beta + 1/\alpha. \end{aligned}$$

In the same way, we obtain

$(\tau_i)_{i\le 2}$	$f_2(\tau_1, \tau_2)$
$(0,0)$	$1/\alpha^2$
$(1,0)$	$1/\alpha + 1/\beta$
$(0,1)$	$1/\alpha\beta$
$(1,1)$	$1/\beta^2$

$(\tau_i)_{i\le 3}$	$f_3(\tau_1, \tau_2, \tau_3)$
$(0,0,0)$	$(1/\alpha)^3$
$(1,0,0)$	$1/\beta + 1/\alpha + 1/\alpha^2$
$(0,1,0)$	$1/(\alpha\beta) + 1/\alpha^2$
$(0,0,1)$	$1/\alpha^2\beta$
$(1,1,0)$	$1/\beta^2 + 1/\beta + 1/\alpha$
$(1,0,1)$	$1/\beta^2 + 1/(\beta\alpha)$
$(0,1,1)$	$1/(\beta^2\alpha)$
$(1,1,1)$	$1/\beta^3$

We confirm that, for example for $(1,1,0)$, the non zero terms of (2.2.3) give

$$\begin{aligned}
(h_1)_{1;0}f_3(0,1,0) &+ (h)_{1,0;1,0}f_3(1,1,0) + (h_N)_{0;1}f_3(1,1,1)\\
&= \alpha(1/(\alpha\beta)+1/\alpha^2) + (-1)(1/\beta^2+1/\beta+1/\alpha) + \beta(1/\beta^3)\\
&= 1/\beta + 1/\alpha - 1/\beta^2 - 1/\beta - 1/\alpha + 1/\beta^2\\
&= 0.
\end{aligned}$$

With the aim of obtaining $\frac{d}{dt}P_N(\tau_1,...,\tau_N)=0$, in [DEHP93] it was shown that there exist elements $x_{\tau_i}\in\mathbb{R}$ for $1\le i\le N$, such that

$$\begin{aligned}
\sum_{\sigma_1}(h_1)_{\tau_1;\sigma_1}P_N(\sigma_1,\tau_2,...,\tau_N) &= x_{\tau_1}P_{N-1}(\tau_2,...,\tau_N)\\
\sum_{\sigma_i,\sigma_{i+1}}(h)_{\tau_i,\tau_{i+1};\sigma_i,\sigma_{i+1}}P_N(\tau_1,...,\sigma_i,\sigma_{i+1},...,\tau_N) &= -x_{\tau_i}P_{N-1}(\tau_1,...,\tau_{i-1},\tau_{i+1},...,\tau_N)\\
&\quad + x_{\tau_{i+1}}P_{N-1}(\tau_1,...,\tau_i,\tau_{i+2},...,\tau_N)\\
\sum_{\sigma_N}(h_N)_{\tau_N;\sigma_N}P_N(\tau_1,...,\tau_{N-1},\sigma_N) &= -x_{\tau_N}P_{N-1}(\tau_1,...,\tau_{N-1}).
\end{aligned}$$

For instance, in the case of $(\tau_{i\le 3})=(1,1,0)$ we have

$$\begin{aligned}
1/\alpha+1/\beta = \sum_{\sigma_1}(h_1)_{1;\sigma_1}f_3(\sigma_1,1,0) &= x_{\tau_1}f_2(1,0) = x_{\tau_1}(1/\alpha+1/\beta) \tag{2.2.7}\\
i=1: \qquad 0 = \sum_{\sigma_1,\sigma_2}(h)_{1,1;\sigma_1,\sigma_2}f_3(\sigma_1,\sigma_2,0) &= -x_{\tau_1}f_2(\tau_2,\tau_3) + x_{\tau_2}f_2(\tau_1,\tau_3)\\
&= -x_{\tau_1}f_2(1,0) + x_{\tau_2}f_2(1,0) \tag{2.2.8}\\
&= -x_{\tau_1}(1/\alpha+1/\beta) + x_{\tau_2}(1/\alpha+1/\beta)\\
i=2: \qquad -(1/\beta^2+1/\beta+1/\alpha) &= \sum_{\sigma_2,\sigma_3}(h)_{1,0;\sigma_2,\sigma_3}f_3(1,\sigma_2,\sigma_3)\\
&= -x_{\tau_2}f_2(\tau_2,\tau_3) + x_{\tau_3}f_2(\tau_1,\tau_2) \tag{2.2.9}\\
&= -x_{\tau_2}f_2(1,0) + x_{\tau_3}f_2(1,1)\\
&= -x_{\tau_2}(1/\alpha+1/\beta) + x_{\tau_3}(1/\beta^2)\\
1/\beta^2 = \beta(1/\beta^3) &= \sum_{\sigma_3}(h_3)_{0;\sigma_N}f_3(1,1,\sigma_N)\\
&= -x_{\tau_3}f_2(1,1) \tag{2.2.10}\\
&= -x_{\tau_3}1/\beta^2.
\end{aligned}$$

By (2.2.7) and (2.2.10), $x_{\tau_1}=-x_{\tau_3}=1$ and therefore by (2.2.8) and (2.2.9), we have that $x_{\tau_2}=1$.

If we consider $N=2$ and we establish the previous conditions for $(\tau_1,\tau_2)=(1,0)$ with the matrix formulation, we obtain that the above three equations are given by

$$\begin{aligned}
\alpha\langle W|EE|V\rangle &= x_{\tau_1}\langle W|E|V\rangle\\
-\langle W|DE|V\rangle &= -x_{\tau_1}\langle W|E|V\rangle + x_{\tau_2}\langle W|D|V\rangle\\
\beta\langle W|DD|V\rangle &= -x_{\tau_2}\langle W|D|V\rangle
\end{aligned}$$

We note that sufficient conditions for these last three equations are the following:

$$\begin{aligned}\alpha\langle W|E &= x_{\tau_1}\langle W|\\ DE &= x_{\tau_1}E - x_{\tau_2}D\\ \beta D|V\rangle &= -x_{\tau_2}|V\rangle.\end{aligned}$$

These conditions are not only sufficient but required. By a redefinition of D and E with a scalar product, we obtain that these matrices satisfy the equation:

$$DE = E - D.$$

In [DEHP93], there is board the partial asymmetric exclusion process, which let to the particles on the $2 \leq i \leq N-1$ position jump to the left site with a probability q, jump to the right site with a probability p, and now a particle can leave the system in the first cell with a probability γ and enter in the last cell with a probability δ. They summarized this information in the intensity matrices

$$h_1 = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \; h_N = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the same matrix formulation for f_N from the equation (2.2.4), we obtain the following conditions over the matrices D and E:

$$\begin{aligned} pDE - qED &= D + E,\\ (\beta D - \delta E)|V\rangle &= |V\rangle,\\ \langle W|(\alpha E - \gamma D) &= \langle W|.\end{aligned}$$

The authors discuted a condition for the finite dimentional case and gave a proposal to D, E $|V\rangle$ and $\langle W|$ when $\gamma = \delta = 0$ (see [DEHP93], p. 1510).

In [DJLS93] we found, such it was promised in [DEHP93], that this problem can be expanded to the study of a stochastic flow on a linear lattice of two different types of particles, and in this case we have that τ_i can take the values 0 (if the position i is empty), 1 (if in the position i there is a first-type of particles) or 2 (if in the position i there is a first-type of particles). In this situation, we can also obtain a matrix formulation to obtain the probability stationary state using matrix D and E and boundary states $\langle W|$ and $|V\rangle$. Considering $A = DE - ED$, we define

$$f_N(\tau_1, ..., \tau_N) = \langle W|\prod_{i=1}^{N} X_i|V\rangle, \tag{2.2.11}$$

where $X_i = E$ if $\tau_i = 0$, $X_i = D$ if $\tau_i = 1$ and $X_i = A$ if $\tau_i = 2$. With this formulation, we obtain that $DE = D + E$, $A^2 = A$, $DA = A$, $AE = A$ and about f_N,

$f_N(\tau_1, ..., \tau_N) = tr(\prod_{i=1}^{N} X_i)$ when $tr(A) < \infty$.

Using an auxiliary vector space $V_M = \oplus_{j \leq m} V_j$ (where V_j is an infinite dimensional vector space that is spanned by analogous of all different configurations $(\tau_1, ..., \tau_j)$ with $\tau_k \in \{1, ..., m\}$) there is found in [KS97] a way to obtain the stationary state of probability for the one dimensional stochastic model that is totally asymmetric and generalizing the situation to a set of particles of m different types. The initial information is as before in three intensity matrices Γ, L and R (this notation for this matrix is the notation of [ADR98]), where Γ is the matrix of rates in the positions $2 \leq i \leq m-1$, and L, R are the boundary rates matrix. In [KS97] it was obtained the stationary state of probability using $2m$ operators of the space V_M, D_α and X_α with $\alpha \in \{1, ..., m\}$, such that

$$\Gamma^{\alpha\beta}_{\gamma\delta} D_\alpha D_\beta = D_\gamma X_\delta - X_\gamma D_\delta, \qquad \text{for all } \gamma, \delta = 0, 1, ..., N-1. \tag{2.2.12}$$

For an example of how the ideas of [KS97] can be applied, see [HSP96]. In equation (2.2.12), if we consider γ and δ as α and β we obtain

$$\Gamma^{\alpha\beta}_{\alpha\beta} D_\alpha D_\beta = D_\alpha X_\beta - X_\alpha D_\beta, \qquad \Gamma^{\beta\alpha}_{\beta\alpha} D_\beta D_\alpha = D_\beta X_\alpha - X_\beta D_\alpha$$

Therefore, we have

$$\Gamma^{\alpha\beta}_{\alpha\beta} D_\alpha D_\beta - \Gamma^{\beta\alpha}_{\beta\alpha} D_\beta D_\alpha = \{X_\beta, D_\alpha\} - \{X_\alpha, D_\beta\}, \tag{2.2.13}$$

where $\{X_\alpha, D_\beta\} = X_\alpha D_\beta + D_\beta X_\alpha$ is the anti commutator. In [IPR01], *the diffusion algebras* were defined, which are algebras generated by m elements $D'_\alpha s$ having a PBW-basis and satisfies (2.2.13), under the feature that for all α, $X_\alpha = x_\alpha * U$, where x_α is a number and U is an object which behave like an identity for the $D'_\alpha s$, i. e., $D_\alpha U = U D_\alpha = D_\alpha$ (in this way we say that X_α is a $c-$number). Therefore, we obtain that (2.2.13) turns to

$$\Gamma^{\alpha\beta}_{\alpha\beta} D_\alpha D_\beta - \Gamma^{\beta\alpha}_{\beta\alpha} D_\beta D_\alpha = 2x_\beta D_\alpha - 2x_\alpha D_\beta \tag{2.2.14}$$

It is important to recall that by the assumption of X_α as $c-$numbers and the PBW condition, the interpretation of $\Gamma^{\alpha\beta}_{\alpha\beta}$ as probability information possible would be meaningless. Though these algebras were formally called in [IPR01], there have previous works on it (see [ADR98]).

2.2.2 Diffusion algebras definition

Definition 2.2.2 ([IPR01], p. 5817)**.** The *diffusion algebras type 1* are affine algebras[9] that are generated by n variables $\{D_1, ..., D_n\}$ over a field $\mathbb{K}$ and admit a linear PBW basis of ordered monomials of the form $D^{k_1}_{\alpha_1} D^{k_2}_{\alpha_2} \cdots D^{k_n}_{\alpha_n}$ with $k_j \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, and there exist $x_1, ..., x_n \in \mathbb{K}$ such that, for all $1 \leq i < j \leq n$, there exist $\lambda_{ij} \in \mathbb{K}^*$ such that

$$\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j \tag{2.2.15}$$

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[9] Algebras that are rings of polynomials but with rules of commutations of products.

The diffusion algebras type 1 of n generators are constructed from the splice of diffusion algebras type 1 of 3 generators which are classified on 4 families and 9 classes (not isomorphism classes) of algebras that are obtained when we try to guarantee the PBW basis of the algebra using the diamond lemma (see in [IPR01] and [PT02]), as the following results shows.

Theorem 2.2.3 ([PT02], p. 3270)**.** *Consider $\mathcal{D}$ a diffusion algebra type 1 of 3 generators D_1, D_2 and D_3 with $1 < 2 < 3$. Then $\mathcal{D}$ belongs to some of the following classes of diffusion algebras type 1:*

1. *Class A_I: $\lambda_{12} = \lambda_{21} = \lambda_{13} = \lambda_{31} = \lambda_{23} = \lambda_{32} \neq 0$, $x_1, x_2, x_3 \in \mathbb{K} \setminus \{0\}$.*
2. *Class A_{II}: $\lambda_{\alpha\beta} = g_\alpha - g_\beta$ for all $\alpha, \beta \in \{1, 2, 3\}$ with $\alpha < \beta$, where $g_1 = \lambda_{12} + \lambda_{23} - \lambda_{13}$, $g_2 = \lambda_{23} - \lambda_{13}$ and $g_3 = -\lambda_{13}$; $\lambda_{\alpha\beta} \neq 0$ if $\alpha < \beta$ and $\lambda_{\alpha\beta} = 0$ if $\alpha > \beta$; $x_1, x_2, x_3 \in \mathbb{K} \setminus \{0\}$.*
3. *Class B_I: $\lambda_{12} = \lambda_{23} \neq 0$, $\lambda_{21} = \lambda_{32}$, $\lambda_{\alpha\beta} - \lambda_{\beta\alpha} = \Lambda$ for all $\alpha, \beta \in \{1, 2, 3\}$ with $\alpha < \beta$ and for a fix $\Lambda \in \mathbb{K}$, $x_1, x_3 \in \mathbb{K} \setminus \{0\}$, $x_2 = 0$.*
4. *Class B_{II}: $\lambda_{32} = \lambda_{21} = 0$, $\lambda_{12}, \lambda_{13}, \lambda_{23} \in \mathbb{K} \setminus \{0\}$, $x_2 = 0$.*
5. *Class B_{III}: $\lambda_{31} = \lambda_{32} = 0$, $\lambda_{12} \neq 0$, $\lambda_{12} - \lambda_{21} = \lambda_{13} - \lambda_{23}$, $\lambda_{13} \notin \{0, \lambda_{12} - \lambda_{21}\}$, $x_3 = 0$.*
6. *Class B_{IV}: $\lambda_{21} = \lambda_{31} = 0$, $x_1 = 0$, $\lambda_{13} - \lambda_{12} = \lambda_{23} - \lambda_{32}$, $\lambda_{13} \notin \{0, \lambda_{13} - \lambda_{12}\}$.*
7. *Class C_I: $\lambda_{12} - \lambda_{21} = \lambda_{13} - \lambda_{31}$, $\lambda_{12}, \lambda_{13}, \lambda_{23} \in \mathbb{K} \setminus \{0\}$ $x_2 = x_3 = 0$.*
8. *Class C_{II}: $x_2 = x_3 = 0$, $\lambda_{32} = 0$, $\lambda_{12}, \lambda_{23}, \lambda_{13} \in \mathbb{K} \setminus \{0\}$.*
9. *Class D: $x_1 = x_2 = x_3 = 0$, $\lambda_{\alpha\beta} \neq 0$, for all $\alpha, \beta \in \{1, 2, 3\}$ with $\alpha < \beta$.*

The following results are established in [Hin05].

1. A diffusion algebra on $n \geq 2$ is left Noetherian if and only if we have $\lambda_{ji} \neq 0$, for all $i < j$ (see [Hin05], Proposition 2.5.5).
2. A diffusion algebra is left Noetherian if and only if it is a domain (see [Hin05], Lemma 2.1.5).
3. Noetherian diffusion algebras have a PBW basis of standard monomials in whatever order of the indeterminates (see [Hin05], Lemma 2.1.7).
4. n-generator diffusion algebra is a skew polynomial ring of over its $(n-1)$-generator diffusion subalgebra generated by $x_2, \ldots, x_n$ (see [Hin05], Remark 2.2.2).

A diffusion algebra $\mathcal{D}$ of type 1 has the following properties:

1. Since $\mathbb{C}$ is simple, it is a Noetherian ring and therefore $\mathcal{D}$ is a Noetherian ring when $\lambda_{ji} \neq 0$ for all $i < j$. In this case, since $\mathbb{C}$ is regular, then $\mathcal{D}$ is left regular (see [LR14], Corollary 2.6).

2. Since $\mathbb{C}$ is a domain, then $\mathcal{D}$ is a domain (see [LR14], Proposition 4.1).
3. Since $\mathbb{C}$ is a Noetherian regular ring, when $\mathcal{D}$ is such that $\lambda_{ij} \neq 0$, for all pair $i-j$, the Quillen's K-theory of $\mathcal{D}$ is the same of $\mathbb{C}$ (see [LR14], Theorem 5.1).
4. The cyclic homology (also known as homology of Connes) $HC_i(\mathcal{D}) \cong HC_i(\mathbb{C})$, for all $i \geq \dim(G(\mathcal{D}))$, where $\dim(G(\mathcal{D}))$ is the Hochschild dimension of the canonical graduation (see [RS16], Proposition 12).
5. Since $\mathbb{C}$ is a commutative domain, $\mathcal{D}$ is an ACCPL-domain (an algebra satisfying the ascending chain condition of principal left ideals) (see [RS18], Theorem 1).
6. Since the elements of $\mathbb{C}$ commutes with the D's variables, $\mathbb{C}$ is Σ-rigid, therefore $\mathbb{C}$ is (Σ, Δ)-compatible and $\mathcal{D}$ is reduced[10] (see [RS19b], Proposition 5).
7. Since $\mathbb{C}$ is commutative, it is reversible[11]. Also, since $\mathbb{C}$ is (Σ, Δ)-compatible, we obtain that $\mathbb{C}$ is (Σ, Δ)-skew McCoy (see [RR19], Theorem 3.9).
8. Since $\mathbb{C}$ is of endomorphism type and $\mathbb{C}$ is (Σ, Δ)-skew McCoy, then it is corroborated that $\mathbb{C}$ is Dedekind finite[12] (see [RR19], Theorem 3.14).
9. Since $\mathbb{C}$ is (Σ, Δ)-compatible and Σ-skew Armendariz (because $\mathbb{C}$ is Σ-rigid), we get that $\mathcal{D}$ satisfies the Kothë's conjecture[13] (see [RS19a], Proposition 3.19).
10. If $\mathcal{D}$ satisfies that $\lambda_{ij} \neq 0$, for all pair i, j, then $\mathrm{GKdim}(\mathcal{D}) = n$ (see [Rey13], Theorem 14).

Remark 2.2.4. We present this observation to prevent possible mistakes when we try to find automorphisms in diffusion algebras. From the diffusion algebras type 1 of class D, in the following theorem (that we present without proof) we find examples of automorphisms of diffusion algebras that cannot be graded automorphisms.

Theorem 2.2.5 ([VR15], p. 1879)**.** *If $\mathcal{D}$ is a diffusion algebra of type 1 and class D with three generators such that $\lambda_{ij} = 1$ and $\lambda_{ji} = q$, for all $i, j \in \{1, 2, 3\}$ and $i < j$, with q not a root of the unity and if $\sigma : \mathcal{D} \to \mathcal{D}$ is an automorphism then $\sigma(D_1) = \alpha_1 D_1$, $\sigma(D_2) = \alpha_2 D_2 + \beta D_1 D_3$ and $\sigma(D_3) = \alpha_3 D_3$ with $\alpha_i, \beta \in \mathbb{C}$.*

In order to obtain a graded structure that remains the commutative law form of the previous diffusion algebras, it was defined a second class of diffusion algebras as follows.

Definition 2.2.6 ([LFG$^+$20], p. 28)**.** The *diffusion algebras type 2* are affine algebras that are generated by $2n$ variables $\{D_1, ..., D_n, x_1, ..., x_n\}$ over a field $\mathbb{K}$ that admits a linear *PBW* basis of ordered monomials of the form $B_{\alpha_1}^{k_1} B_{\alpha_2}^{k_2} \cdots B_{\alpha_n}^{k_n}$ with $B_{\alpha_i} \in \{D_1, ..., D_n, x_1, ..., x_n\}$, for all $i \leq 2n$, $k_j \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, such that, for all $1 \leq i < j \leq n$, there exist $\lambda_{ij} \in \mathbb{K}^*$ such that

$$\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j \tag{2.2.16}$$

---

[10] An algebra with no non-zero nilpotent elements is called *reduced*.

[11] R is *reversible*, if for any pair $a, b \in R$, $ab = 0$ imply $ba = 0$.

[12] R is *Dedekind finite*, if for any pair $a, b \in R$, $ab = 1$, imply $ba = 1$.

[13] The Kothë's conjecture say that if a ring B has no nonzero nil two sided ideals, then B has no nonzero nil one-sided ideals.

If we consider $\deg(D_i) = \deg(x_i) = 1$, for all $1 \leq i \leq n$, we obtain that $\mathcal{D}$ is a $\mathbb{Z}$-graded algebra. Also, we note that due to the fact of x_i is a central element of $\mathcal{D}$ for all $i \leq n$, we just need that the monomials $D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_n}^{k_n}$ with $k_j \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ be a linear PBW basis where the coefficients come from the ring $\mathbb{K}[x_1, ..., x_n]$.

A diffusion algebra $\mathcal{D}$ of type 2 has the following properties:

1. Since $\mathbb{C}[x_1, ..., x_n]$ is a Noetherian ring (by the Hilbert's basis theorem) then, $\mathcal{D}$ is a Noetherian ring. Hence, using that $\mathbb{C}[x_1, ..., x_n]$ is regular (by the Hilbert's Syzygy theorem), then $\mathcal{D}$ is left regular (see [LR14], Corollary 2.6).
2. Since $\mathbb{C}[x_1, ..., x_n]$ is a domain, then $\mathcal{D}$ is a domain ([LR14], Proposition 4.1).
3. Since $\mathbb{C}[x_1, ..., x_n]$ is a Noetherian regular ring, the Quillen's K-theory of $\mathcal{D}$ is the same of $\mathbb{C}[x_1, ..., x_n]$ (see [LR14], Theorem 5.1).
4. The cyclic homology (also known in literature as homology of Connes) $HC_i(\mathcal{D}) \cong HC_i(\mathbb{C}[x_1, ..., x_n])$, for all $i \geq \dim(G(\mathcal{D}))$, where $\dim(G(\mathcal{D}))$ is the Hochschild dimension of the canonical graduation of $\mathcal{D}$ (see [RS16], Proposition 12).
5. Since $\mathbb{C}[x_1, ..., x_n]$ is a commutative domain, $\mathcal{D}$ is an ACCPL-domain (satisfying the ascending chain condition of principal left ideals) (see [RS18], Theorem 1).
6. Since the elements of $\mathbb{C}[x_1, ..., x_n]$ commute with the D's variables, $\mathbb{C}[x_1, ..., x_n]$ is Σ-rigid. Therefore $\mathbb{C}[x_1, ..., x_n]$ is (Σ, Δ)-compatible and $\mathcal{D}$ is reduced (see [RS19b], Proposition 5).
7. Since $\mathbb{C}[x_1, ..., x_n]$ is commutative, it is reversible. Also, since $\mathbb{C}[x_1, ..., x_n]$ is (Σ, Δ)-compatible, we obtain that $\mathbb{C}[x_1, ..., x_n]$ is (Σ, Δ)-skew McCoy (see [RR19], Theorem 3.9).
8. Since $\mathcal{D}$ is a skew PBW extension of $\mathbb{C}$ of endomorphism type and $\mathbb{C}[x_1, ..., x_n]$ is (Σ, Δ)-skew McCoy, then $\mathbb{C}[x_1, ..., x_n]$ is Dedekind finite (see [RR19], Theorem 3.14).
9. Since $\mathbb{C}[x_1, ..., x_n]$ is (Σ, Δ)-compatible and Σ-skew Armendariz (because $\mathbb{C}$ is Σ-rigid), we get that $\mathcal{D}$ satisfies the Kothë's conjecture (see [RS19a], Proposition 3.19).
10. If $\mathcal{D}$ satisfies that $\lambda_{ij} \neq 0$, for all pair i, j, then [Rey13], Theorem 14 guarantees that

$$\mathrm{GKdim}(\mathcal{D}) = \sigma(\mathbb{C})(D_1, ..., D_n) = \mathrm{GKdim}(\mathbb{C}[x_1, ..., x_n]) + n = 2n.$$

We obtain in this work how we can commute a power of a generator with another generator (see, Proposition 2.2.10). This is with the purpose to rewrite any element of a diffusion algebra in terms of his PBW basis $\{D_n^{\alpha_n} \cdots D_1^{\alpha_1}\}$. In the following, we consider $i < j$. First, we present a definition and a lemma.

Definition 2.2.7. If D_i and D_j are generators of a diffusion algebra $\mathcal{D}$ (as in Definition 2.2.2 or Definition 2.2.6), for all pair $k, n \in \mathbb{N}$ such that $k < n$, we define the numbers $P_k^n, Q_k^n \in \mathbb{C}$,

$$P_k^n = \sum_{t=1}^{k} \binom{n-k+t-1}{n-k} \lambda_{ji}^{t-1} \lambda_{ij}^{k-t}, \qquad Q_k^n = \binom{n}{k-1} \lambda_{ji}^{k-1}.$$

Remark 2.2.8. It is relevant for us highlight the appearance of the Pascal's triangle. This because the numbers

$$\binom{n-k+t-1}{n-k} \in \mathbb{Q},$$

are the entries of the k-diagonal of the triangle formed with the first $n+1$ floors of the Pascal's triangle.

Lemma 2.2.9. *For all $k, n \in \mathbb{N}$ such that $k \leq n$, then:*

1. $P_k^{n+1} = P_{k-1}^n \lambda_{ij} + Q_k^n$ *and* $P_{n+1}^{n+1} = P_n^n \lambda_{ij} + \lambda_{ji}^n$.
2. $Q_k^{n+1} = Q_{k-1}^n \lambda_{ji} + Q_k^n$ *and* $Q_{n+1}^{n+1} = Q_n^n \lambda_{ji} + \lambda_{ji}^n$.
3. $P_1^n = Q_1^n = 1$.

Proof. 1. As it is known, for all $a, b \in \mathbb{N}$, $\binom{a+b}{a} = \binom{a+b}{b}$, whence

$$\begin{aligned} P_{k-1}^n \lambda_{ij} + Q_k^n &= \sum_{t=1}^{k-1} \binom{n-k+1+t-1}{n-k+1} \lambda_{ji}^{t-1} \lambda_{ij}^{k-1-t+1} + \binom{n}{k-1} \lambda_{ji}^{k-1} \\ &= \sum_{t=1}^{k-1} \binom{n+1-k+t-1}{n+1-k} \lambda_{ji}^{t-1} \lambda_{ij}^{k-t} + \binom{n+1-k+k-1}{k-1} \lambda_{ji}^{k-1} \\ &= \sum_{t=1}^{k} \binom{n+1-k+t-1}{n+1-k} \lambda_{ji}^{t-1} \lambda_{ij}^{k-t} = P_k^{n+1}, \end{aligned}$$

and

$$P_n^n \lambda_{ij} + \lambda_{ji}^n = \sum_{t=1}^{n} \lambda_{ji}^{t-1} \lambda_{ij}^{n-t+1} + \lambda_{ji}^n = P_{n+1}^{n+1}.$$

2. Since $\binom{n}{k-2} + \binom{n}{k-1} = \binom{n+1}{k-1}$, for all $n, k \in \mathbb{N}$, then

$$\begin{aligned} Q_{k-1}^n \lambda_{ji} + Q_k^n &= \binom{n}{k-2} \lambda_{ji}^{k-2} \lambda_{ji} + \binom{n}{k-1} \lambda_{ji}^{k-1} \\ &= \left(\binom{n}{k-2} + \binom{n}{k-1} \right) \lambda_{ji}^{k-1} = \binom{n+1}{k-1} \lambda_{ji}^{k-1} = Q_k^{n+1}, \end{aligned}$$

and also we have that

$$Q_n^n \lambda_{ji} + \lambda_{ji}^n = \left(\binom{n}{n-1} + \binom{n}{n} \right) \lambda_{ji}^n = \binom{n+1}{n} \lambda_{ji}^n = Q_{n+1}^{n+1}.$$

3. Follows by a short computation.

□

Proposition 2.2.10. *Let D_i and D_j be generators of the diffusion algebra $\mathcal{D}$. Then, for all natural number $n \geq 1$, we have*

$$\lambda_{ij}^n D_i^n D_j = \lambda_{ji}^n D_j D_i^n + \sum_{k=1}^{n} (-1)^{k+n} \; P_k^n \; x_i^{n-k} x_j \; D_i^k + (-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; D_j D_i^{k-1}, \tag{2.2.17}$$

where the numbers P_k^n and Q_k^n were established in Definition 2.2.7.

Proof. We prove this by induction. For $n = 1$, we see that equation (2.2.17) is

$$\lambda_{ij} D_i D_j = \lambda_{ji} D_j D_i + x_j D_i - x_i D_j,$$

which follows from equation (2.2.15). Now we suppose the assertion holds for a fix $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\lambda_{ij}^{n+1} D_i^{n+1} D_j &= \lambda_{ij} D_i \left(\lambda_{ji}^n D_j D_i^n + \sum_{k=1}^{n} (-1)^{k+n} \; P_k^n \; x_i^{n-k} x_j \; D_i^k + (-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; D_j D_i^{k-1} \right) \\
&= \lambda_{ji}^n (\lambda_{ji} D_j D_i + x_j D_i - x_i D_j) D_i^n + \sum_{k=1}^{n} \Big[(-1)^{k+n} \; P_k^n \; x_i^{n-k} x_j \; \lambda_{ij} D_i^{k+1} \\
&+ (-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; (\lambda_{ji} D_j D_i + x_j D_i - x_i D_j) D_i^{k-1} \Big] \\
&= \lambda_{ji}^{n+1} D_j D_i^{n+1} + \lambda_{ji}^n x_j D_i^{n+1} - \lambda_{ji}^n x_i D_j D_i^n \\
&+ \sum_{k=1}^{n} \Big[(-1)^{k+n} \; P_k^n \; x_i^{n-k} x_j \; \lambda_{ij} D_i^{k+1} + (-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; \lambda_{ji} D_j D_i^k \\
&+ (-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; x_j D_i^k + (-1)^{n+k} \; Q_k^n \; x_i^{n-k+2} \; D_j D_i^{k-1} \Big]
\end{aligned}$$

Now, by associative law of the sum and separating the term $k = n$ of the first sum and the term $k = 1$ of the second, we obtain

$$
\begin{aligned}
\lambda_{ij}^{n+1} D_i^{n+1} D_j &= \lambda_{ji}^{n+1} D_j D_i^{n+1} + \lambda_{ji}^{n} x_j D_i^{n+1} - \lambda_{ji}^{n} x_i D_j D_i^{n} \\
&+ P_n^n \; x_j \; \lambda_{ij} D_i^{n+1} - \; Q_n^n \; x_i \; \lambda_{ji} D_j D_i^n \\
&+ \sum_{k=1}^{n-1} \left[(-1)^{k+n} \; P_k^n \; x_i^{n-k} x_j \; \lambda_{ij} D_i^{k+1} + (-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; \lambda_{ji} D_j D_i^k \right] \\
&+ \sum_{k=2}^{n} \left[(-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; x_j D_i^k + (-1)^{n+k} \; Q_k^n \; x_i^{n-k+2} \; D_j D_i^{k-1} \right] \\
&+ (-1)^n \; Q_1^n \; x_i^n \; x_j D_i + (-1)^{n+1} \; Q_1^n \; x_i^{n+1} \; D_j \\
&= \lambda_{ji}^{n+1} D_j D_i^{n+1} + (\lambda_{ji}^n + P_n^n \lambda_{ij}) x_j D_i^{n+1} - (\lambda_{ji}^n + \lambda_{ji} Q_n^n) x_i D_j D_i^n \\
&+ \sum_{k=2}^{n} \left[(-1)^{k-1+n} \; P_{k-1}^n \; x_i^{n-k+1} x_j \; \lambda_{ij} D_i^k + (-1)^{n+k-2} \; Q_{k-1}^n \; x_i^{n-k} \; \lambda_{ji} D_j D_i^{k-1} \right] \\
&+ \sum_{k=2}^{n} \left[(-1)^{n+k-1} \; Q_k^n \; x_i^{n-k+1} \; x_j D_i^k + (-1)^{n+k} \; Q_k^n \; x_i^{n-k+2} \; D_j D_i^{k-1} \right] \\
&+ (-1)^n \; P_1^{n+1} \; x_i^n \; x_j D_i + (-1)^{n+1} \; Q_1^{n+1} \; x_i^{n+1} \; D_j \\
&= \lambda_{ji}^{n+1} D_j D_i^{n+1} + P_{n+1}^{n+1} x_j D_i^{n+1} - Q_{n+1}^{n+1} x_i D_j D_i^n \\
&+ \sum_{k=2}^{n} \left[(-1)^{k-1+n} \; (P_{k-1}^n \lambda_{ij} + Q_k^n) \; x_i^{n-k+1} x_j \; D_i^k \right. \\
&+ \left. (-1)^{n+k} \; (Q_{k-1}^n \lambda_{ji} + Q_k^n) \; x_i^{n-k} \; D_j D_i^{k-1} \right] \\
&+ (-1)^n \; P_1^{n+1} \; x_i^n \; x_j D_i + (-1)^{n+1} \; Q_1^{n+1} \; x_i^{n+1} \; D_j \\
&= \lambda_{ji}^{n+1} D_j D_i^{n+1} + P_{n+1}^{n+1} x_j D_i^{n+1} - Q_{n+1}^{n+1} x_i D_j D_i^n \\
&+ \sum_{k=2}^{n} \left[(-1)^{k-1+n} \; P_k^{n+1} \; x_i^{n-k+1} x_j \; D_i^k + (-1)^{n+k} \; Q_k^n \; x_i^{n-k} \; D_j D_i^{k-1} \right] \\
&+ (-1)^n \; P_1^{n+1} \; x_i^n \; x_j D_i + (-1)^{n+1} \; Q_1^{n+1} \; x_i^{n+1} \; D_j,
\end{aligned}
$$

where the second equality is due to a substitution on the index of the first sum and Lemma 2.2.9 in the last term, and both last equations are due to the distributivity and parts 1-2 of Lemma 2.2.9. This proves that the assumption is true for $n + 1$. □

Remark 2.2.11. As in diffusion algebras of type 2 the generators x_i's are central elements, in these algebras the Proposition 2.2.10 still holds.

Proposition 2.2.12. *Let D_i and D_j be generators of the diffusion algebra $\mathcal{D}$. Then, for all natural number $n \geq 1$, we have*

$$\lambda_{ij}^n D_i D_j^n = \lambda_{ji}^n D_j^n D_i + \sum_{k=1}^{n} Q_k^n \; x_j^{n-k+1} \; D_j D_i^{k-1} - P_k^n \; x_j^{n-k} x_i \; D_i^k,$$

where the numbers P_k^n and Q_k^n were established in Definition 2.2.7.

Proof. This proof is analogous to the proof of Proposition 2.2.10. □

Remark 2.2.13. For the inner derivations of diffusion algebras with two generators, $\partial_1, \partial_2 : \mathcal{D} \to \mathcal{D}$ defined by $\partial_k(a) = D_k a - a D_k$ with $k = 1, 2$n, we have

$$\begin{aligned}\lambda_{12}^n \partial_2(D_1^n) &= \lambda_{12}^n D_1^n D_2 - \lambda_{12}^n D_2 D_1^n \\ &= (\lambda_{21}^n - \lambda_{12}^n) D_2^n D_1 + \sum_{k=1}^{n} (-1)^{k+n}\ P_k^n\ x_1^{n-k} x_2\ D_1^k + (-1)^{n+k-1}\ Q_k^n\ x_1^{n-k+1}\ D_2 D_1^{k-1}.\end{aligned}$$

Therefore, in the basic element $D_2^m D_1^n$, we have

$$\begin{aligned}\lambda_{12}^n \partial_2(D_2^m D_1^n) &= D_2^m \lambda_{1j}^n \partial_2(D_1^n) \\ &= D_2^m [(\lambda_{21}^n - \lambda_{12}^n) D_2^n D_1 \\ &+ \sum_{k=1}^{n} (-1)^{k+n}\ P_k^n\ x_1^{n-k} x_2\ D_1^k + (-1)^{n+k-1}\ Q_k^n\ x_1^{n-k+1}\ D_2 D_1^{k-1}] \\ &= (\lambda_{21}^n - \lambda_{12}^n) D_2^{m+n} D_1 \\ &+ \sum_{k=1}^{n} (-1)^{k+n}\ P_k^n\ x_1^{n-k} x_2\ D_2^m D_1^k + (-1)^{n+k-1}\ Q_k^n\ x_1^{n-k+1}\ D_2^{m+1} D_1^{k-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda_{12}^m \partial_1(D_2^m) &= \lambda_{12}^m D_2^m D_1 - \lambda_{12}^m D_1 D_2^m \\ &= (\lambda_{12}^m - \lambda_{21}^m) D_2^m D_1 - \sum_{k=1}^{m} \Big[Q_k^m\ x_2^{m-k+1}\ D_2 D_1^{k-1} - P_k^m\ x_2^{m-k} x_1\ D_1^k \Big],\end{aligned}$$

whence, we have that in the basis element $D_j^m D_i^n$,

$$\begin{aligned}\lambda_{12}^m \partial_1(D_2^m D_1^n) &= \lambda_{12}^m \partial_1(D_2^m) D_1^n \\ &= (\lambda_{12}^m - \lambda_{21}^m) D_2^m D_1^{n+1} - \sum_{k=1}^{m} \Big[Q_k^m\ x_2^{m-k+1}\ D_2 D_1^{n+k-1} - P_k^m\ x_2^{m-k} x_1\ D_1^{k+n} \Big].\end{aligned}$$

With all facts above in mind, we can now construct a Brzezinski's calculus (see Definition 1.2.7) over a 2-dimensional diffusion algebra of type 1.

If $\mathcal{D}$ is generated by D_1 and D_2, first, we consider the automorphism $\sigma \in M_2(\mathrm{End}(\mathcal{D}))$, given by the identity matrix with diagonal entries $id_{\mathcal{D}}$, and the multiderivation (∂, σ), where $\partial = (\partial_1, \partial_2)$. Later, we consider $\Omega^1(\mathcal{D})$ the $\mathcal{D}$-module generated by ω_1 and ω_2. We have that right $\mathcal{D}$ action is given by

$$\omega_1 a = a\omega_1, \quad \omega_2 a = a\omega_2$$

Then, we define $\nabla : \mathrm{Hom}_{\mathcal{D}}(\Omega^1(\mathcal{D}), \mathcal{D}) \to \mathcal{D}$ as $\nabla(f) = \partial_1(f(\omega_1)) + \partial_2(f(\omega_2))$. If $f(\omega_1) =$

$D_2^{m_1}D_1^{n_1}$ and $f(\omega_2) = D_2^{m_2}D_1^{n_2}$, then

$$\begin{aligned}\lambda_{12}^{m_1+n_2}\nabla(f) &= \lambda_{12}^{m_1+n_2}\partial_1(D_2^{m_1}D_1^{n_1}) + \lambda_{12}^{m_1+n_2}\partial_2(D_2^{m_2}D_1^{n_2})\\ &= \lambda_{12}^{n_2}\Bigg[(\lambda_{12}^{m_1} - \lambda_{21}^{m_1})D_2^{m_1}D_1^{n_1+1} - \sum_{k=1}^{m_1}\Bigg[Q_k^{m_1}\ x_2^{m_1-k+1}\ D_2D_1^{n_1+k-1}\\ &- P_k^{m_1}\ x_2^{m_1-k}x_1\ D_1^{k+n_1}\Bigg]\Bigg]\\ &+ \lambda_{12}^{m_1}\Bigg[(\lambda_{21}^{n_2} - \lambda_{12}^{n_2})D_2^{m_2+n_2}D_1\\ &+ \sum_{k=1}^{n_2}(-1)^{k+n_2}\ P_k^{n_2}\ x_1^{n_2-k}x_2\ D_2^{m_2}D_1^{k} + (-1)^{n_2+k-1}\ Q_k^{n_2}\ x_1^{n_2-k+1}\ D_2^{m_2+1}D_1^{k-1}\Bigg].\end{aligned}$$

Example 2.2.14. If $\mathcal{D}$ is a quantum plane, i.e., if $x_1 = 0$ and $x_2 = 0$, then

$$\begin{aligned}\lambda_{12}^{m_1+n_2}\nabla(f) &= \lambda_{12}^{m_1+n_2}\partial_1(D_2^{m_1}D_1^{n_1}) + \lambda_{12}^{m_1+n_2}\partial_2(D_2^{m_2}D_1^{n_2})\\ &= \lambda_{12}^{n_2}(\lambda_{12}^{m_1} - \lambda_{21}^{m_1})D_2^{m_1}D_1^{n_1+1} + \lambda_{12}^{m_1}(\lambda_{21}^{n_2} - \lambda_{12}^{n_2})D_2^{m_2+n_2}D_1.\end{aligned}$$

Now, in each class of 3-generators diffusion algebra of type 1 (see Theorem 2.2.3), we establish the commutation law of Proposition 2.2.10 as follows.

1. Class A_I: In this class, we obtain that for all $n, k \in \mathbb{N}$ such that $k \leq n$,

$$P_k^n = \sum_{t=1}^{k}\binom{n-k+t-1}{n-k}\lambda^{k-1}, \qquad Q_k^n = \binom{n}{k-1}\lambda^{k-1}.$$

We have the following property.

Proposition 2.2.15. *For all pair $n, k \in \mathbb{N}$ such that $k \leq n$,*

$$\sum_{t=1}^{k}\binom{n-k+t-1}{n-k} = \binom{n}{k-1}.$$

Proof. We prove this by induction on n. The case $n = 1$ follows by a short verification. If we suppose that is true for n and all $k \leq n$, then we have

$$\begin{aligned}\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1}\\ &= \sum_{t=1}^{k+1}\binom{n-(k+1)+t-1}{n-(k+1)} + \sum_{t=1}^{k}\binom{n-k+t-1}{n-k}\\ &= \sum_{t=1}^{k+1}\binom{n-k-1+t-1}{n-k-1} + \sum_{t=1}^{k}\binom{n-k+t-1}{n-k}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{k} \binom{n-k+t-1}{n-k-1} + \sum_{t=1}^{k} \binom{n-k+t-1}{n-k} \\
&= 1 + \sum_{t=1}^{k} \binom{n-k+t-1}{n-k-1} + \binom{n-k+t-1}{n-k} \\
&= \sum_{t=0}^{k} \binom{n-k+t}{n-k} = \sum_{t=1}^{k+1} \binom{(n+1)-(k+1)+t-1}{(n+1)-(k+1)},
\end{aligned}
$$

which proves that the assumption is true for $n+1$, and then it is true for all pair $n, k \in \mathbb{N}$ such that $k \leq n$. □

Remark 2.2.16. By the Proposition 2.2.15, we conclude that if $\lambda_{ij} = \lambda_{ji}$ then $P_k^n = Q_k^n$ for all $n, k \in \mathbb{N}$. Therefore, Propositions 2.2.10 and 2.2.12 for diffusion algebras of class A_I are given by

$$
\begin{aligned}
\lambda^n D_i^n D_j &= \lambda^n D_j D_i^n + \sum_{k=1}^{n} \binom{n}{k-1} \lambda^{k-1} \Big[(-1)^{k+n} \;\; x_i^{n-k} x_j \; D_i^k + (-1)^{n+k-1} \;\; x_i^{n-k+1} \; D_j D_i^{k-1} \Big], \\
\lambda^n D_i D_j^n &= \lambda^n D_j^n D_i + \sum_{k=1}^{n} \binom{n}{k-1} \lambda^{k-1} \Big[\; x_j^{n-k+1} \; D_j D_i^{k-1} - x_j^{n-k} x_i \; D_i^k \Big],
\end{aligned}
$$

respectively.

2. Class A_{II}: Since $\lambda_{\beta\alpha} = 0$ for $\alpha < \beta$, we obtain that in each pair of generators D_α and D_β, $Q_k^n = 0$ and $P_k^n = \lambda_{\alpha\beta}^{k-1}$, for all $k \neq 1$. Therefore, Propositions 2.2.10 and 2.2.12 for diffusion algebras of class A_{II} are given by

$$
\begin{aligned}
\lambda_{\alpha\beta}^n D_\alpha^n D_\beta &= \lambda_{\beta\alpha}^n D_\beta D_\alpha^n + \sum_{k=2}^{n} \Big[(-1)^{k+n} \; \lambda_{\alpha\beta}^{k-1} \; x_\alpha^{n-k} x_\beta \; D_\alpha^k \Big] \\
&\quad + (-1)^{n+1} \; x_\alpha^{n-1} x_\beta \; D_\alpha + (-1)^n \; x_\alpha^n \; D_\beta, \\
\lambda_{\alpha\beta}^n D_\alpha D_\beta^n &= \lambda_{\beta\alpha}^n D_\beta^n D_\alpha + \; x_\beta^n \; D_\beta - \sum_{k=1}^{n} \lambda_{\alpha\beta}^{k-1} \; x_\beta^{n-k} x_\alpha \; D_\alpha^k,
\end{aligned}
$$

respectively.

3. Class B_I: Since $x_2 = 0$, then

$$
\begin{aligned}
\lambda_{12}^n D_1^n D_2 &= \lambda_{21}^n D_2 D_1^n + \sum_{k=1}^{n} (-1)^{n+k-1} \; Q_k^n \; x_1^{n-k+1} \; D_2 D_1^{k-1}, \\
\lambda_{12}^n D_1 D_2^n &= \lambda_{21}^n D_2^n D_1 - P_n^n x_1 \; D_1^n, \\
\lambda_{23}^n D_2^n D_3 &= \lambda_{32}^n D_3 D_2^n + \sum_{k=1}^{n} (-1)^{k+n} \; P_k^n \; x_2^{n-k} x_3 \; D_2^k, \\
\lambda_{23}^n D_2 D_3^n &= \lambda_{32}^n D_3^n D_2 + \sum_{k=1}^{n} Q_k^n \; x_3^{n-k+1} \; D_3 D_j^{k-1},
\end{aligned}
$$

where $\lambda_{\beta\alpha} = \lambda_{\alpha\beta} - \Lambda$ for $\alpha, \beta \in \{1, 2, 3\}$ and $\alpha < \beta$ with a fix $\Lambda \in \mathbb{K} \setminus \{0\}$. In the case of D_1 and D_3, both Propositions 2.2.10 and 2.2.12 have no a shorter form.

4. Class B_{II}: In these algebras, for the pairs D_1, D_2 and D_2, D_3, we have that the numbers are $P_k^n = \lambda_{ij}^{k-1}$ or $P_k^n = \lambda_{jm}^{k-1}$, for all $n, k \in \mathbb{N}$ and $Q_k^n = 0$ for all $k > 1$; this because $\lambda_{21} = \lambda_{32} = 0$. Then, as $x_2 = 0$ too,

$$\begin{aligned}
\lambda_{12}^n D_1^n D_2 &= (-1)^n x_1^n \; D_2,\\
\lambda_{12}^n D_1 D_2^n &= -\lambda_{12}^{n-1} x_1 \; D_1^n,\\
\lambda_{23}^n D_2^n D_3 &= \lambda_{23}^{n-1} x_3 \; D_2^n,\\
\lambda_{23}^n D_2 D_3^n &= x_3^n \; D_3.
\end{aligned}$$

5. Class B_{III}: Since $\lambda_{32} = \lambda_{31} = x_3 = 0$, we obtain that

$$\begin{aligned}
\lambda_{13}^n D_1^n D_3 &= (-1)^n x_1^n \; D_3,\\
\lambda_{13}^n D_1 D_3^n &= -x_1 \; D_1^n,\\
\lambda_{23}^n D_2^n D_3 &= (-1)^n \; x_2^n \; D_3,\\
\lambda_{23}^n D_2 D_3^n &= -\lambda_{23}^{n-1} \; x_2 \; D_2^n.
\end{aligned}$$

6. Class B_{IV}: $\lambda_{21} = \lambda_{31} = 0$, $x_1 = 0$, $\lambda_{13} - \lambda_{12} = \lambda_{23} - \lambda_{32}$, $\lambda_{13} \notin \{0, \lambda_{13} - \lambda_{12}\}$.

7. Class C_I: Since $x_2 = x_3 = 0$,

$$\begin{aligned}
\lambda_{12}^n D_1^n D_2 &= \lambda_{21}^n D_2 D_1^n + \sum_{k=1}^{n} (-1)^{n+k-1} \; Q_k^n \; x_1^{n-k+1} \; D_2 D_1^{k-1},\\
\lambda_{12}^n D_1 D_2^n &= \lambda_{21}^n D_2^n D_1 - P_n^n \; x_1 \; D_1^n,\\
\lambda_{13}^n D_1^n D_3 &= \lambda_{31}^n D_3 D_1^n + \sum_{k=1}^{n} (-1)^{n+k-1} \; Q_k^n \; x_1^{n-k+1} \; D_3 D_1^{k-1},\\
\lambda_{13}^n D_1 D_3^n &= \lambda_{31}^n D_3^n D_1 - P_n^n x_1 \; D_1^n,\\
\lambda_{23}^n D_2^n D_3 &= \lambda_{32}^n D_3 D_2^n,\\
\lambda_{32}^n D_2 D_3^n &= \lambda_{32}^n D_3^n D_2.
\end{aligned}$$

8. Class C_{II}: $x_2 = x_3 = 0$, $\lambda_{32} = 0$, $\lambda_{12}, \lambda_{23}, \lambda_{13} \in \mathbb{K} \setminus \{0\}$. Since $\lambda_{32} = 0$, we have that for the pair D_2 and D_3, $Q_k^n = 0$, for all $k > 1$, and for all $k \leq n$, we have $P_k^n = \lambda_{23}^{k-1}$. Then

$$\begin{aligned}
\lambda_{12}^n D_1^n D_2 &= \lambda_{21}^n D_2 D_1^n + \sum_{k=1}^{n} (-1)^{n+k-1} \; Q_k^n \; x_1^{n-k+1} \; D_2 D_1^{k-1},\\
\lambda_{12}^n D_1 D_2^n &= \lambda_{21}^n D_2^n D_1 - P_n^n \; x_1 \; D_1^n,\\
\lambda_{13}^n D_1^n D_3 &= \lambda_{31}^n D_3 D_1^n + \sum_{k=1}^{n} (-1)^{n+k-1} \; Q_k^n \; x_1^{n-k+1} \; D_3 D_1^{k-1},\\
\lambda_{13}^n D_1 D_3^n &= \lambda_{31}^n D_3^n D_1 - P_n^n x_1 \; D_1^n,\\
\lambda_{23}^n D_2^n D_3 &= \lambda_{32}^n D_3 D_2^n = 0,\\
\lambda_{23}^n D_2 D_3^n &= \lambda_{32}^n D_3^n D_2 = 0.
\end{aligned}$$

9. Class D: $x_1 = x_2 = x_3 = 0$, $\lambda_{\alpha\beta} \neq 0$, for all $\alpha, \beta \in \{i, j, k\}$ with $\alpha < \beta$,

$$\begin{aligned}
\lambda_{12}^n D_1^n D_2 &= \lambda_{21}^n D_2 D_1^n, \\
\lambda_{12}^n D_1 D_2^n &= \lambda_{21}^n D_2^n D_1, \\
\lambda_{13}^n D_1^n D_3 &= \lambda_{31}^n D_3 D_1^n, \\
\lambda_{13}^n D_1 D_3^n &= \lambda_{31}^n D_3^n D_1, \\
\lambda_{23}^n D_2^n D_3 &= \lambda_{32}^n D_3 D_2^n, \\
\lambda_{23}^n D_2 D_3^n &= \lambda_{32}^n D_3^n D_2,
\end{aligned}$$

$$\begin{aligned}
\lambda_{12}^{n^2+m} D_1^m D_2^n &= \sum_{p=1}^{n} \lambda_{21}^{(p-1)n} \lambda_{12}^{n^2-pn+m} \sum_{k=1}^{n} Q_k^n x_2^{n-k+1} D_1^{m-p} D_2 D_1^{k+p-2} - P_k^n x_2^{n-k} x_1 D_1^{m+k-1} \\
&= \sum_{p=1}^{n} \sum_{k=1}^{n} \lambda_{21}^{(p-1)n+m-p} \lambda_{12}^{n^2-pn+p} Q_k^n x_2^{n-k+1} D_2 D_1^{m+k-2} \\
&+ \sum_{p=1}^{n} \sum_{k=1}^{n} \sum_{s=1}^{m-p} \lambda_{21}^{(p-1)n} \lambda_{12}^{n^2-pn+p} Q_k^n x_2^{n-k+1} (-1)^{s+m-p} P_s^{m-p} x_1^{m-p-s} x_2 D_1^{s+k+p-2} \\
&+ \sum_{p=1}^{n} \sum_{k=1}^{n} \sum_{s=1}^{m-p} \lambda_{21}^{(p-1)n} \lambda_{12}^{n^2-pn+p} Q_k^n x_2^{n-k+1} (-1)^{m-p+s-1} D_2 D_1^{s+k+m-3} \\
&- \sum_{p=1}^{n} \sum_{k=1}^{n} \lambda_{21}^{(p-1)n} \lambda_{12}^{n^2-pn} P_k^n x_2^{n-k} x_1 D_1^{m+k-1}.
\end{aligned}$$

Remark 2.2.17. We want to highlight that the diffusion algebra $\mathcal{D}$ generated by $D_1, \ldots, D_n$ is an Ore extension of the subalgebra R generated by $D_2, \ldots, D_n$ (see [Hin05], p. 26) with the endomorphism σ and the σ-skew derivation δ are defined by:

$$\begin{aligned}
\sigma(D_j) &= \lambda_{j1}\lambda_{1j}^{-1} D_j - \lambda_{1j}^{-1} x_j, & \delta(D_j) &= -\lambda_{1j}^{-1} x_1 D_j, & &\text{for all } j \geq 2 \\
\sigma(r) &= r, & \delta(r) &= 0, & &\text{for all } r \in R.
\end{aligned}$$

Next, we are going to show particular behaviors of automorphisms and skew derivations of diffusion algebras.

2.2.3 Derivations in case $n = 2$ of type 1

In this work, we found necessary conditions needed to get a derivation in some examples of diffusion algebras. Let $\mathcal{D}$ be a diffusion algebra of type 1 generated by D_1 and D_2. If $\partial : \mathcal{D} \to \mathcal{D}$ is a derivation, we know that $\{D_2^i D_1^j : i, j \in \mathbb{N}\}$ is a $\mathbb{C}$-basis of $\mathcal{D}$. If we consider in the deg-lex monomial order $(\alpha_2, \alpha_1) \in \mathbb{N}^2$ as the maximum of the monomials that appears in $\partial(D_1)$ and $\partial(D_2)$, let $A = (a_{ij}), B = (b_{ij}) \in M_{\alpha_2 \times \alpha_1}(\mathbb{C})$ matrices such that

$$\partial(D_1) = \sum_{i=1}^{\alpha_2} \sum_{j=1}^{\alpha_1} A_{ij} D_2^i D_1^j, \qquad \partial(D_2) = \sum_{i=1}^{\alpha_2} \sum_{j=1}^{\alpha_1} B_{ij} D_2^i D_1^j. \tag{2.2.18}$$

Then, ∂ evaluated on the equation $\lambda_{12}D_1D_2 - \lambda_{21}D_2D_1 - x_2D_1 + x_1D_2 = 0$, in terms of the PBW basis, it is given by

$$
\begin{aligned}
&\sum_{i=2}^{\alpha_2+1}\sum_{k=1}^{\alpha_1}(-1)^k A_{(i-1)k}\lambda_{12}^{1-k}Q_1^k x_1^k D_2^i + \sum_{k=2}^{\alpha_1}(-1)^{k+1}A_{\alpha_2 k}\lambda_{12}^{1-k}Q_2^k x_1^{k-1}D_2^{\alpha_2+1}D_1\\
&+\sum_{i=2}^{\alpha_2}\left[\sum_{k=1}^{\alpha_1}(-1)^{k+1}A_{ik}\lambda_{12}^{1-k}P_1^k x_1^{k-1}x_2 + \sum_{k=2}^{\alpha_1}(-1)^{k+1}A_{(i-1)k}\lambda_{12}^{1-k}Q_2^k x_1^{k-1} + (x_1B_{i1} - x_2A_{i1})\right]D_2^iD_1\\
&+\sum_{j=1}^{\alpha_1}\left[x_1B_{1j} - x_2A_{1j} + \sum_{k=j}^{\alpha_1}(-1)^{k+j}A_{1k}\lambda_{12}^{1-k}P_j^k x_1^{k-j}x_2 + \sum_{\substack{i+p\geq j+1\\ i\leq\alpha_2,\ p\leq\alpha_1}} B_{ip}\lambda_{12}^{1-i}Q_{j+1-p}^i x_2^{i+p-j}\right]D_2D_1^j\\
&+\sum_{j=\alpha_1+1}^{\alpha_1+\alpha_2-1}\ \sum_{\substack{i+p\geq j+1\\ i\leq\alpha_2,\ p\leq\alpha_1}} B_{ip}\lambda_{12}^{1-i}Q_{j+1-p}^i x_2^{i+p-j}D_2D_1^j\\
&+\sum_{i=2}^{\alpha_2-1}(\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^i - \lambda_{21})B_{i\alpha_1}D_2^iD_1^{\alpha_1+1}\\
&+\sum_{i=2}^{\alpha_2}\sum_{j=2}^{\alpha_1-1}\Bigg[(\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^j - \lambda_{21})A_{(i-1)j} + (\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^i - \lambda_{21})B_{i(j-1)} + (x_1B_{ij} - x_2A_{ij})\\
&+\left(\sum_{k=j}^{\alpha_1}(-1)^{k+j}A_{ik}\lambda_{12}^{1-k}P_j^k x_1^{k-j}x_2\right) + \left(\sum_{k=j+1}^{\alpha_1}(-1)^{k+j}A_{(i-1)k}\lambda_{12}^{1-k}Q_{j+1}^k x_1^{k-j}\right)\Bigg]D_2^iD_1^j\\
&+\sum_{i=2}^{\alpha_2}\Bigg[(\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^{\alpha_1} - \lambda_{21})A_{(i-1)\alpha_1} + (\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^i - \lambda_{21})B_{i(\alpha_1-1)}\\
&+x_1B_{i\alpha_1} + A_{i\alpha_1}x_2(\lambda_{12}^{1-\alpha_1}P_{\alpha_1}^{\alpha_1} - 1)\Bigg]D_2^iD_1^{\alpha_1}\\
&+\sum_{j=2}^{\alpha_1-1}\left[(\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^j - \lambda_{21})A_{\alpha_2 j} + \sum_{k=j+1}^{\alpha_1}(-1)^{j+k}A_{\alpha_2 k}\lambda_{12}^{1-k}Q_{j+1}^k x_i^{k-j}\right]D_2^{\alpha_2+1}D_1^j\\
&+(\lambda_{12}(\frac{\lambda_{21}}{\lambda_{12}})^{\alpha_1} - \lambda_{21})A_{\alpha_2\alpha_1}D_2^{\alpha_2+1}D_1^{\alpha_1} = 0.
\end{aligned}
$$

Remark 2.2.18. We note that if $A_{\alpha_2\alpha_2} \neq 0$ then, since the coefficient of $D_2^{\alpha_2+1}D_1^{\alpha_1}$ must be vanish, we obtain that $\lambda_{21} = 0$ or $\frac{\lambda_{21}}{\lambda_{12}}$ is an $(\alpha_1 - 1)$-root of the unity.

Now, following the ideas of proof of Theorem 1.3.32, as in Example 1.3.40, in this work we obtain for the case of diffusion algebras of type 1 with two generators, that these have a restricted scope, as follows.

Remark 2.2.19. Since $D_iD_j - a_{ij}D_jD_i = x_jD_i - x_iD_j$, where $a_{ij} = \lambda_{ij}/\lambda_{ji} \neq 0$, then if we try to define $d : \mathcal{D} \to \Omega^1(\mathcal{D})$, in the sense and with notation of Section 1.3,

$$
\begin{aligned}
d(D_iD_j) - a_{ij}d(D_jD_i) &= d(x_jD_i) - d(x_iD_j)\\
d(D_i)D_j + D_id(D_j) - a_{ij}d(D_j)D_i - a_{ij}D_jd(D_i) &= x_jd(D_i) - x_id(D_j)\\
d(D_i)D_j + d(D_j)\nu_{D_j}(D_i) - d(D_j)a_{ij}D_i - d(D_i)\nu_{D_i}(a_{ij}D_j) &= x_jd(D_i) - x_id(D_j)\\
d(D_i)[D_j - \nu_{D_i}(a_{ij}D_j) - x_j] + d(D_j)[\nu_{D_j}(D_i) - a_{ij}D_i + x_i] = 0.&
\end{aligned}
$$

Therefore, $\nu_{D_i}(D_j) = a_{ij}^{-1}(D_j - x_j)$ and $\nu_{D_j}(D_i) = a_{ij}D_i - x_i$. Now to define an automorphism ν_{D_i}, we need that it respects the equation $D_iD_j - a_{ij}D_jD_i = x_jD_i - x_iD_j$. Then

$$\begin{aligned}\nu_{D_i}(D_i)\nu_{D_i}(D_j) - a_{ij}\nu_{D_i}(D_j)\nu_{D_i}(D_i) &= x_j\nu_{D_i}(D_i) - x_i\nu_{D_i}(D_j)\\ \nu_{D_i}(D_i)a_{ij}^{-1}(D_j - x_j) - a_{ij}a_{ij}^{-1}(D_j - x_j)\nu_{D_i}(D_i) &= x_j\nu_{D_i}(D_i) - x_ia_{ij}^{-1}(D_j - x_j).\end{aligned}$$

By inspection, the only way that we found to solve it, was $\nu_{D_i}(D_i) = a_{ij}^{-1}(D_i - x_i)$, and with this,

$$\begin{aligned}(a_{ij} - 1)(x_jD_i - x_iD_j) &= (a_{ij} - 1)(D_ix_j + D_jx_i) + (1 - a_{ij})x_ix_j\\ -2(a_{ij} - 1)x_iD_j &= (1 - a_{ij})x_ix_j\\ 2(a_{ij} - 1)x_iD_j &= (a_{ij} - 1)x_ix_j,\end{aligned}$$

then, $a_{ij} = 1$. With $a_{ij} = 1$, we get

$$\nu_{D_i}(D_j) = \nu_{D_j}(D_j) = D_j - x_j, \quad \nu_{D_i}(D_i) = \nu_{D_j}(D_i) = D_i - x_i.$$

The definition of the differential calculus does not have problem, neither its dimensionality. But, since $D_\alpha d(D_\alpha) = d(D_\alpha)(D_\alpha - x_\alpha)$, the general value of $d(D_\alpha^k)$ as a right multiple of $d(D_\alpha)$ will have a expression similar to the form of Proposition 2.2.10. With this behavior, we could confront problems on the connectivity of the differential calculus, issue that we will consider in the future.

2.2.4 Universal differential calculus of a diffusion algebra

Now, we proceed to construct the universal differential graded algebra of a diffusion algebra of type and degree 2. We consider the $\mathbb{K}$-map $\mu : \mathcal{D} \otimes_{\mathbb{K}} \mathcal{D} \to \mathcal{D} : a \otimes b \mapsto ab$ and define $\Omega\mathcal{D} = \mathrm{Ker}(\mu)$. As it was mentioned in [BM93], p. 595, we have that all the first order differential graded calculus are obtained as a quotient of $\Omega\mathcal{D}$; so, we are going to study this algebra. We have that $span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \mathcal{D}\}) \subset \Omega\mathcal{A}$ because $\mu(1 \otimes a - a \otimes 1) = a - a = 0$ and $e \otimes f - f \otimes e \in \Omega$ if $e \in \{x_1, x_2\}$ and $f \in \{D_1, D_2\}$. In fact, in this case we have that $e \otimes f - f \otimes e \in span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \mathcal{D}\})$ as follows:

$$\begin{aligned}d(D_1x_1) - D_1d(x_1) - d(x_1)D_1 &= (1 \otimes D_1x_1 - D_1x_1 \otimes 1) - D_1(1 \otimes x_1 - x_1 \otimes 1) - (1 \otimes x_1 - x_1 \otimes 1)D_1\\ &= 1 \otimes D_1x_1 - D_1x_1 \otimes 1 - D_1 \otimes x_1 + D_1x_1 \otimes 1 - 1 \otimes x_1D_1 + x_1 \otimes D_1\\ &= x_1 \otimes D_1 - D_1 \otimes x_1,\end{aligned}$$

where in the last equation we have used the relation $x_iD_i = D_ix_i$. In the same way, we have

$$x_2 \otimes D_2 - D_2 \otimes x_2 = d(D_2x_2) - D_2d(x_2) - d(x_2)D_2,$$

$$x_1 \otimes D_2 - D_2 \otimes x_1 = d(D_2x_1) - D_2d(x_1) - d(x_1)D_2,$$

$$x_2 \otimes D_1 - D_1 \otimes x_2 = d(D_1x_2) - D_1d(x_2) - d(x_2)D_1.$$

Also, if we say that $B := \lambda_{12}D_1 \otimes D_2 - \lambda_{21}D_2 \otimes D_1 - x_2 \otimes D_1 + x_1 \otimes D_2$, then

$$\begin{aligned}
&d(D_2)\lambda_{21}D_1 + x_1 d(D_2) - [d(D_1)\lambda_{12}D_2 + x_2 d(D_1)] \\
&= (1 \otimes D_2 - D_2 \otimes 1)\lambda_{21}D_1 + x_1(1 \otimes D_2 - D_2 \otimes 1) \\
&- [(1 \otimes D_1 - D_1 \otimes 1)\lambda_{12}D_2 + x_2(1 \otimes D_1 - D_1 \otimes 1)] \\
&= 1 \otimes \lambda_{21}D_2D_1 - D_2 \otimes \lambda_{21}D_1 + x_1 \otimes D_2 - x_1D_2 \otimes 1 \\
&- 1 \otimes \lambda_{12}D_1D_2 + D_1 \otimes \lambda_{12}D_2 - x_2 \otimes D_1 + x_2D_1 \otimes 1] \\
&= D_1 \otimes \lambda_{12}D_2 - D_2 \otimes \lambda_{21}D_1 - x_2 \otimes D_1 + x_1 \otimes D_2 \\
&+ 1 \otimes \lambda_{21}D_2D_1 - 1 \otimes \lambda_{12}D_1D_2 + x_2D_1 \otimes 1 - x_1D_2 \otimes 1 \\
&= B + 1 \otimes (\lambda_{21}D_2D_1 - \lambda_{12}D_1D_2) + (x_2D_1 - x_1D_2) \otimes 1 \\
&= B - 1 \otimes (x_2D_1 - x_1D_2) + (x_2D_1 - x_1D_2) \otimes 1 \\
&= B - d(x_2D_1 - x_1D_2),
\end{aligned}$$

where the fifth equality is due by the relation (2.2.16) in $\mathcal{D}$. Hence, since we already proved that

$$B = d(x_2D_1 - x_1D_2) + d(D_2)\lambda_{21}D_1 + x_1 d(D_2) - d(D_1)\lambda_{12}D_2 - x_2 d(D_1),$$

then $B \in span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \mathcal{D}\})$. Also, we have the following proposition.

Proposition 2.2.20. *$\{1 \otimes a - a \otimes 1 : a \in \mathcal{D}\} \subseteq span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_1, D_2, x_1, x_2\}\})$ i.e., we have that $span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \mathcal{D}\}) = span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_1, D_2, x_1, x_2\}\})$.*

Proof. We proceed by induction on k, the length of the monomial terms.

- $k = 2$: Let $y_1 \cdot y_2 \in \mathcal{D}$ where $y_1, y_2 \in \{D_i, D_j, x_i, x_j\}$. By the Leibniz's rule,

  $$1 \otimes y_1y_2 - y_1y_2 \otimes 1 = (1 \otimes y_1 - y_1 \otimes 1)y_2 + y_1(1 \otimes y_2 - y_2 \otimes 1).$$

- Suppose that for every $\overline{x} \in \mathcal{D}$ monomial of lenght $k-1$, $d(\overline{x})$ is an element of $span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_i, D_j, x_i, x_j\}\})$. Let be $y_1y_2\cdots y_k \in \mathcal{D}$ where $y_i^{\alpha_i} \in \{D_i, D_j, x_i, x_j\}$, for all $1 \leq i \leq k$. Then

  $$\begin{aligned}
  1 \otimes (y_1y_2\cdots y_k) - (y_1y_2\cdots y_k) \otimes 1 &= (1 \otimes y_1y_2\cdots y_{k-1} - y_1y_2\cdots y_{k-1} \otimes 1)y_k \\
  &+ y_1y_2\cdots y_{k-1}(1 \otimes y_k - y_k \otimes 1) \\
  &= d(y_1y_2\cdots y_{k-1})y_1 + y_1y_2\cdots y_{k-1}d(y_k).
  \end{aligned}$$

  Since $y_1y_2\cdots y_{k-1}$ has length $k-1$, we have that $d(y_1y_2\cdots y_{k-1}) \in span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_i, D_j, x_i, x_j\}\})$, and then we obtain that $1 \otimes (y_1y_2\cdots y_k) - (y_1y_2\cdots y_k) \otimes 1 \in span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_i, D_j, x_i, x_j\}\})$.

Since every monomial of $\mathcal{D}$ is an element of $span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_i, D_j, x_i, x_j\}\})$, by the $\mathcal{K}$-linearity of d we have that $\{1 \otimes a - a \otimes 1 : a \in \mathcal{D}\} \subseteq span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_i, D_j, x_i, x_j\}\})$. $\square$

Then, we consider the first order differential calculus over $\mathcal{A}$, $(\Omega^1(\mathcal{D}), d)$, where $d : \mathcal{D} \to \Omega^1(\mathcal{D}) : a \mapsto 1 \otimes a - a \otimes 1$ with $\Omega(\mathcal{D}) = span_{\mathcal{D}-\mathcal{D}}(\{1 \otimes a - a \otimes 1 : a \in \{D_i, D_j, x_i, x_j\}\}) \subseteq \mathcal{D} \otimes_{\mathcal{K}} \mathcal{D}$. This is a $\mathcal{D}$-bimodule finitely generated by the set $\{d(D_i), d(D_j), d(x_i), d(x_j)\}$.

Finally, we want to remark some words about the differentially smoothness (see Section 1.3.2) of this class of algebras.

Remark 2.2.21. In this work, we note that the diffusion algebras $\mathcal{D}$ of type 2 with $\{D_i, D_j,\ x_i, x_j\}$ as generators cannot be differentially smooth. This because if we define $(\Omega(\mathcal{D}), d)$, a first order differential calculus on $\mathcal{D}$, since d must respect the relations of $\mathcal{D}$, then by Leibniz's rule

$$\lambda_{ij} d(D_i) D_j + \lambda_{ij} D_i d(D_j) - \lambda_{ji} d(D_j) D_i - \lambda_{ji} D_j d(D_i) = d(x_j) D_i + x_j d(D_i) - d(x_i) D_j - x_i d(D_j),$$

whence, by the density condition, $\Omega^1(\mathcal{D})$ is generated as a $\mathcal{D}$-bimodule by $\{d(x_i), d(D_i), d(D_j)\}$, and hence $\Omega^4(\mathcal{D}) = 0$. This denies the existence of a differential calculus 4-dimensional, where $\text{GKdim}(\mathcal{D}) = 4$ by [Rey13], Theorem 14, and therefore, it shows that it is no possible the differentially smoothness of $\mathcal{D}$.

2.2.5 Automorphisms and skew derivations in case $n = 2$ of type 2

In this work, this section is dedicated to search automorphisms and skew derivations, by using a computational way, over diffusion algebra of type 2 in case $n = 2$, that is, a diffusion algebra of four generators $\{D_1, D_2, x_1, x_2\}$. All results of this section were obtained in the realization of this work and they have been submitted for publication.

First, to obtain an automorphism $\sigma : \mathcal{D} \to \mathcal{D}$, we define on the generators in a way that the equality (2.2.19) holds (with $\sigma(1) = 1$):

$$\lambda_{12} \sigma(D_1) \sigma(D_2) - \lambda_{21} \sigma(D_2) \sigma(D_1) = \sigma(x_2) \sigma(D_1) - \sigma(x_1) \sigma(D_2). \tag{2.2.19}$$

Later, we find right σ-skew derivations. With this in mind, we only have to fix ∂ on the generators and verify that if we apply linear and Leibniz's properties to (2.2.16), the equality still holds. We obtain:

$$\begin{aligned}
\partial(\lambda_{12} D_1 D_2 - \lambda_{21} D_2 D_1) &= \partial(x_2 D_1 - x_1 D_2) \\
\lambda_{12} \partial(D_1 D_2) - \lambda_{21} \partial(D_2 D_1) &= \partial(x_2 D_1) - \partial(x_1 D_2) \\
\lambda_{12} \partial(D_1) \sigma(D_2) + \lambda_{12} D_1 \partial(D_2) - \lambda_{21} \partial(D_2) \sigma(D_1) - \lambda_{21} D_2 \partial(D_1) & \\
= \partial(x_2) \sigma(D_1) + x_2 \partial(D_1) - \partial(x_1) \sigma(D_2) - x_1 \partial(D_2). &
\end{aligned} \tag{2.2.20}$$

Remark 2.2.22. We note that in (2.2.20) there is no interest in the values $\sigma(x_i), \sigma(x_j) \in \mathcal{D}$.

If we consider $\sigma_1 = id_{\mathcal{D}}$ (which satisfies (2.2.19) clearly) in (2.2.20), we obtain:

$$\lambda_{12} \partial(D_1) D_2 + \lambda_{12} D_1 \partial(D_2) - \lambda_{21} \partial(D_2) D_1 - \lambda_{21} D_2 \partial(D_1) = \partial(x_2) D_1 + x_2 \partial(D_1) - \partial(x_1) D_2 - x_1 \partial(D_2).$$

This is equivalent to say that

$$[\lambda_{12} \partial(D_1) + \partial(x_1)] D_2 + [\lambda_{12} D_1 + x_1] \partial(D_2) - [\lambda_{21} \partial(D_2) + \partial(x_2)] D_1 - [\lambda_{21} D_2 + x_2] \partial(D_1) = 0.$$

We also have to guarantee for $a \in \{D_1, D_2, x_1, x_2\}$ and $b \in \{x_1, x_2\}$ that

$$\partial(ab) = \partial(ba).$$

We found the following solutions (but we do not denied the existence other different solutions) of these last equations:

- $\partial(D_1) = D_1$, $\partial(D_2) = -D_2$, $\partial(x_1) = x_1$ and $\partial(x_2) = -x_2$.
- $\partial(D_1) = x_1$, $\partial(D_2) = x_2$, $\partial(x_1) = (\lambda_{21} - \lambda_{12})x_1$ and $\partial(x_2) = (\lambda_{12} - \lambda_{21})x_2$.

With these skew derivations, we have that $\sigma^{-1} \circ \partial \circ \sigma = \partial$. For the next proposition, we recall that $\mathcal{D} = \oplus_{n\in\mathbb{N}}\mathcal{D}_n$ is an $\mathbb{N}$-graded algebra with the natural degree $\deg(D_1) = \deg(D_2) = \deg(x_1) = \deg(x_2) = 1$. The following proposition was obtained in this work.

Proposition 2.2.23. *If $\partial : \mathcal{D} \to \mathcal{D}$ is a $id_{\mathcal{D}}$-derivation, then the elements $\partial(D_1)$, $\partial(D_2)$, $\partial(x_1)$ and $\partial(x_2)$ have no zero degree terms.*

Proof. If there exist $a, b, c, d \in \mathbb{C}$ such that these are the zero degree terms of $\partial(D_1)$, $\partial(D_2)$, $\partial(x_1)$ and $\partial(x_2)$, respectively, due to the $\mathbb{N}$-graded structure of $\mathcal{D}$, in the zero terms of (2.2.20) we must have that

$$\lambda_{12}aD_2 + \lambda_{12}D_1b - \lambda_{21}bD_1 - \lambda_{21}D_2a = dD_1 + x_2a - cD_2 - x_1b,$$

which is equivalent to

$$[(\lambda_{12} - \lambda_{21})b - d]D_1 + [(\lambda_{12} - \lambda_{21})a + c]D_2 - ax_2 + bx_1 = 0.$$

where, from the independence of the generators in the one degree $\mathbb{C}$-subspace, we obtain that $a = b = c = d = 0$. ☐

The following proposition is a corollary of the previous proposition, and it was obtained in this work.

Proposition 2.2.24. *If $\partial : \mathcal{D} \to \mathcal{D}$ is a $id_{\mathcal{D}}$-derivation, then $\partial(\mathcal{D}) \cap \mathbb{C} = \emptyset$.*

Now, we consider the general case. Let $\mathrm{Aut}_L(\mathcal{D})$ be the set of $\mathbb{C}$-algebras automorphisms σ such that $\sigma(\{D_1, D_2, x_1, x_2\}) \subseteq \mathcal{D}_1$. Let $\sigma \in \mathrm{Aut}_L(\mathcal{D})$ and $A_\alpha, B_\alpha, S_\alpha, H_\alpha \in \mathbb{C}$ for $\alpha \in \{Di, Dj, x_1, x_2, k\}$ be the coeficients of $\sigma(D_1)$, $\sigma(D_2)$, $\sigma(x_1)$ and $\sigma(x_2)$. Then:

$$\begin{aligned}
\sigma(D_1) &= A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + A_k, \qquad (2.2.21)\\
\sigma(D_2) &= B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + B_k,\\
\sigma(x_1) &= S_{D1}D_1 + S_{D2}D_2 + S_{x1}x_1 + S_{x2}x_2 + S_k,\\
\sigma(x_2) &= H_{D1}D_1 + H_{D2}D_2 + H_{x1}x_1 + H_{x2}x_2 + H_k.
\end{aligned}$$

Hence, in this work we got the following important fact:

Proposition 2.2.25. *If $\sigma : \mathcal{D} \to \mathcal{D}$ is an automorphism defined as in (2.2.21), then we have that* $\det(A) \neq 0$, *where:*

$$A = \begin{pmatrix} A_{D1} & B_{D1} & S_{D1} & H_{D1} \\ A_{D2} & B_{D2} & S_{D2} & H_{D2} \\ A_{x1} & B_{x1} & S_{x1} & H_{x1} \\ A_{x2} & B_{x2} & S_{x2} & H_{x2} \end{pmatrix}. \tag{2.2.22}$$

Proof. Suppose that $\det(A) = 0$. Then the set $\{v_1, v_2, v_3, v_4\}$ of columns of A[14] is linear dependent, i.e., there exist $\alpha_i \in \mathbb{C}$ with $i \in \{1,2,3,4\}$ such that $v_{\beta(1)} = \alpha_2 v_{\beta(2)} + \alpha_3 v_{\beta(3)} + \alpha_4 v_{\beta(4)}$, where β is a permutation of $\{1,2,3,4\}$. Then,

$$\sigma(v_{\beta(1)} + \epsilon) = \alpha_2 \sigma(v_{\beta(2)}) + \alpha_3 \sigma(v_{\beta(3)}) + \alpha_4 \sigma(v_{\beta(4)}),$$

where $\epsilon = \alpha_2 k_{\beta(2)} + \alpha_3 k_{\beta(3)} + \alpha_4 k_{\beta(4)} - k_{\beta(1)}$, with $k_1 = A_k$, $k_2 = B_k$, $k_3 = S_k$ and $k_4 = H_k$. Since $v_{\beta(1)} + \epsilon \neq \alpha_2 v_{\beta(2)} + \alpha_3 v_{\beta(3)} + \alpha_4 v_{\beta(4)}$, we obtain that σ is not injective, which it is not the case. □

With (2.2.21) we have the following terms:

$$\begin{aligned}
&\lambda_{12}\sigma(D_1)\sigma(D_2) \\
&= \lambda_{12}(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + A_k)(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + B_k) \\
&= \lambda_{12}A_{D1}B_{D1}D_1^2 + A_{D1}B_{D2}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + \lambda_{12}A_{D1}B_{x1}D_1x_1 + \lambda_{12}A_{D1}B_{x2}D_1x_2 + \lambda_{12}A_{D1}B_kD_1 \\
&+ \lambda_{12}A_{D2}B_{D1}D_2D_1 + \lambda_{12}A_{D2}B_{D2}D_2^2 + \lambda_{12}A_{D2}B_{x1}D_2x_1 + \lambda_{12}A_{D2}B_{x2}D_2x_2 + \lambda_{12}A_{D2}B_kD_2 \\
&+ \lambda_{12}A_{x1}B_{D1}x_1D_1 + \lambda_{12}A_{x1}B_{D2}x_1D_2 + \lambda_{12}A_{x1}B_{x1}x_1^2 + \lambda_{12}A_{x1}B_{x2}x_1x_2 + \lambda_{12}A_{x1}B_kx_1 \\
&+ \lambda_{12}A_{x2}B_{D1}x_2D_1 + \lambda_{12}A_{x2}B_{D2}x_2D_2 + \lambda_{12}A_{x2}B_{x1}x_2x_1 + \lambda_{12}A_{x2}B_{x2}x_2^2 + \lambda_{12}A_{x2}B_kx_2 \\
&+ \lambda_{12}A_kB_{D1}D_1 + \lambda_{12}A_kB_{D2}D_2 + \lambda_{12}A_kB_{x1}x_1 + \lambda_{12}A_kB_{x2}x_2 + \lambda_{12}A_kB_k,
\end{aligned}$$

$$\begin{aligned}
&\lambda_{21}\sigma(D_2)\sigma(D_1) \\
&= \lambda_{12}(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + B_k)(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + A_k) \\
&= \lambda_{21}B_{D1}A_{D1}D_1^2 + \lambda_{21}B_{D1}A_{D2}\lambda_{12}^{-1}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + \lambda_{21}B_{D1}A_{x1}D_1x_1 + \lambda_{21}B_{D1}A_{x2}D_1x_2 + \lambda_{21}B_{D1}A_kD_1 \\
&+ \lambda_{21}B_{D2}A_{D1}D_2D_1 + \lambda_{21}B_{D2}A_{D2}D_2^2 + \lambda_{21}B_{D2}A_{x1}D_2x_1 + \lambda_{21}B_{D2}A_{x2}D_2x_2 + \lambda_{21}B_{D2}A_kD_2 \\
&+ \lambda_{21}B_{x1}A_{D1}x_1D_1 + \lambda_{21}B_{x1}A_{D2}x_1D_2 + \lambda_{21}B_{x1}A_{x1}x_1^2 + \lambda_{21}B_{x1}A_{x2}x_1x_2 + \lambda_{21}B_{x1}A_kx_1 \\
&+ \lambda_{21}B_{x2}A_{D1}x_2D_1 + \lambda_{21}B_{x2}A_{D2}x_2D_2 + \lambda_{21}B_{x2}A_{x1}x_2x_1 + \lambda_{21}B_{x2}A_{x2}x_2^2 + \lambda_{21}B_{x2}A_kx_2 \\
&+ \lambda_{21}B_kA_{D1}D_1 + \lambda_{21}B_kA_{D2}D_2 + \lambda_{21}B_kA_{x1}x_1 + \lambda_{21}B_kA_{x2}x_2 + \lambda_{21}B_kA_k,
\end{aligned}$$

$$\begin{aligned}
&\sigma(x_2)\sigma(D_1) \\
&= (H_{D1}D_1 + H_{D2}D_2 + H_{x1}x_1 + H_{x2}x_2 + H_k)(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + A_k) \\
&= H_{D1}A_{D1}D_1^2 + H_{D1}A_{D2}\lambda_{12}^{-1}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + H_{D1}A_{x1}D_1x_1 + H_{D1}A_{x2}D_1x_2 + H_{D1}A_kD_1 \\
&+ H_{D2}A_{D1}D_2D_1 + H_{D2}A_{D2}D_2^2 + H_{D2}A_{x1}D_2x_1 + H_{D2}A_{x2}D_2x_2 + H_{D2}A_kD_2 \\
&+ H_{x1}A_{D1}x_1D_1 + H_{x1}A_{D2}x_1D_2 + H_{x1}A_{x1}x_1^2 + H_{x1}A_{x2}x_1x_2 + H_{x1}A_kx_1 \\
&+ H_{x2}A_{D1}x_2D_1 + H_{x2}A_{D2}x_2D_2 + H_{x2}A_{x1}x_2x_1 + H_{x2}A_{x2}x_2^2 + H_{x2}A_kx_2 \\
&+ H_kA_{D1}D_1 + H_kA_{D2}D_2 + H_kA_{x1}x_1 + H_kA_{x2}x_2 + H_kA_k,
\end{aligned}$$

$$\begin{aligned}
&\sigma(x_1)\sigma(D_2) \\
&= (S_{D1}D_1 + S_{D2}D_2 + S_{x1}x_1 + S_{x2}x_2 + S_k)(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + B_k) \\
&= S_{D1}B_{D1}D_1^2 + S_{D1}B_{D2}\lambda_{12}^{-1}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + S_{D1}B_{x1}D_1x_1 + S_{D1}B_{x2}D_1x_2 + S_{D1}B_kD_1 \\
&+ S_{D2}B_{D1}D_2D_1 + S_{D2}B_{D2}D_2^2 + S_{D2}B_{x1}D_2x_1 + S_{D2}B_{x2}D_2x_2 + S_{D2}B_kD_2 \\
&+ S_{x1}B_{D1}x_1D_1 + S_{x1}B_{D2}x_1D_2 + S_{x1}B_{x1}x_1^2 + S_{x1}B_{x2}x_1x_2 + S_{x1}B_kx_1 \\
&+ S_{x2}B_{D1}x_2D_1 + S_{x2}B_{D2}x_2D_2 + S_{x2}B_{x1}x_2x_1 + S_{x2}B_{x2}x_2^2 + S_{x2}B_kx_2 \\
&+ S_kB_{D1}D_1 + S_kB_{D2}D_2 + S_kB_{x1}x_1 + S_kB_{x2}x_2 + S_kB_k.
\end{aligned}$$

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[14] If we say that D_1 is the first generator a_1, D_2 is a_2, x_1 is a_3 and x_2 is a_4, we mean that A_i is the vector of one degree coefficients of $\sigma(a_i)$.

With these terms and the equation (2.2.19), in this work we obtain the following proposition.

Proposition 2.2.26. *If* $\sigma : \mathcal{D} \to \mathcal{D}$ *is an automorphism defined as in (2.2.21), such that* $\det(Z_\sigma) = 0$ *then* $A_k = B_k = S_k = H_k = 0$*, where*

$$Z_\sigma = \begin{pmatrix} \lambda_{12}A_{D1} + S_{D1} & A_{D2} \\ \lambda_{21}B_{D1} + H_{D1} & B_{D2} \end{pmatrix}.$$

Proof. As we must guarantee that (2.2.19) is satisfied, we need that:

$$\lambda_{12}\sigma(D_1)\sigma(D_2) - \lambda_{21}\sigma(D_2)\sigma(D_1) - \sigma(x_2)\sigma(D_1) + \sigma(x_1)\sigma(D_2) = 0. \tag{2.2.23}$$

Therefore, we ought to have that the coefficients of the one degree terms $D_1, D_2.x_1$ and x_2 must be zero (this due to 4 $\mathcal{D}$ is a quadratic algebra). With this, we vanish the coefficients of D_1, D_2, x_1 and x_2 in (2.2.23), i.e., we obtain the following equations:

$$\begin{aligned}
D_1 :\quad & A_{D1}B_{D2}x_2 - \lambda_{21}B_{D1}A_{D2}\lambda_{12}^{-1}x_2 - H_{D1}A_{D2}\lambda_{12}^{-1}x_2 + S_{D1}B_{D2}\lambda_{12}^{-1}x_2 \\
& + \lambda_{12}A_{D1}B_k + \lambda_{12}A_kB_{D1} - \lambda_{21}B_{D1}A_k - \lambda_{21}B_kA_{D1} - H_{D1}A_k - H_kA_{D1} + S_{D1}B_k + S_kB_{D1} = 0 \\
D_2 :\quad & - A_{D1}B_{D2}x_1 + \lambda_{21}B_{D1}A_{D2}\lambda_{12}^{-1}x_1 + H_{D1}A_{D2}\lambda_{12}^{-1}x_1 - S_{D1}B_{D2}\lambda_{12}^{-1}x_1 \\
& \lambda_{12}A_{D2}B_k + \lambda_{12}A_kB_{D2} - \lambda_{21}B_{D2}A_k - \lambda_{21}B_kA_{D2} - H_{D2}A_k - H_kA_{D2} + S_{D2}B_k + S_kB_{D2} = 0 \\
x_1 :\quad & \lambda_{12}A_{x1}B_k + \lambda_{12}A_kB_{x1} - \lambda_{21}B_{x1}A_k - \lambda_{21}B_kA_{x1} - H_{x1}A_k - H_kA_{x1} + S_{x1}B_k + S_kB_{x1} = 0 \\
x_2 :\quad & \lambda_{12}A_{x2}B_k + \lambda_{12}A_kB_{x2} - \lambda_{21}B_{x2}A_k - \lambda_{21}B_kA_{x2} - H_{x2}A_k - H_kA_{x2} + S_{x2}B_k + S_kB_{x2} = 0 \\
k :\quad & \lambda_{12}A_kB_k - \lambda_{21}B_kA_k - H_kA_k + S_kB_k = 0.
\end{aligned}$$

By the algebraic properties in $\mathbb{C}$ and the fact that $\det(Z_\sigma) = 0$, we obtain the equations:

$$\begin{aligned}
D_1 :\quad & [(\lambda_{12} - \lambda_{21})B_{D1} - H_{D1}]A_k + [(\lambda_{12} - \lambda_{21})A_{D1} + S_{D1}]B_k + B_{D1}S_k - A_{D1}H_k = 0, \\
D_2 :\quad & [(\lambda_{12} - \lambda_{21})B_{D2} - H_{D2}]A_k + [(\lambda_{12} - \lambda_{21})A_{D2} + S_{D2}]B_k + B_{D2}S_k - A_{D2}H_k = 0, \\
x_1 :\quad & [(\lambda_{12} - \lambda_{21})B_{x1} - H_{x1}]A_k + [(\lambda_{12} - \lambda_{21})A_{x1} + S_{x1}]B_k + B_{x1}S_k - A_{x1}H_k = 0, \\
x_2 :\quad & [(\lambda_{12} - \lambda_{21})B_{x2} - H_{x2}]A_k + [(\lambda_{12} - \lambda_{21})A_{x2} + S_{x2}]B_k + B_{x2}S_k - A_{x2}H_k = 0, \\
k :\quad & (\lambda_{12} - \lambda_{21})A_kB_k + S_kB_k - H_kA_k = 0,
\end{aligned}$$

where, the equations obtained in $D_1,\ D_2\ x_1$ and x_2 are the linear system $\Gamma\overline{x} = 0$ with

$$\Gamma = \begin{pmatrix} [(\lambda_{12} - \lambda_{21})B_{D1} - H_{D1}] & [(\lambda_{12} - \lambda_{21})A_{D1} + S_{D1}] & B_{D1} & -A_{D1} \\ [(\lambda_{12} - \lambda_{21})B_{D2} - H_{D2}] & [(\lambda_{12} - \lambda_{21})A_{D2} + S_{D2}] & B_{D2} & -A_{D2} \\ [(\lambda_{12} - \lambda_{21})B_{x1} - H_{x1}] & [(\lambda_{12} - \lambda_{21})A_{x1} + S_{x1}] & B_{x1} & -A_{x1} \\ [(\lambda_{12} - \lambda_{21})B_{x2} - H_{x2}] & [(\lambda_{12} - \lambda_{21})A_{x2} + S_{x2}] & B_{x2} & -A_{x2} \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} A_k \\ B_k \\ S_k \\ H_k \end{pmatrix}.$$

As the first column of Γ is a linear combination of the third column, the column of B's, and a column of H's, like the second column is a combination of the column of A's and a vector of S's we obtain that $\det(\Gamma) = \det(A)$ (see (2.2.22)). In this way, by the Proposition 2.2.25, $\det(A) \neq 0$, therefore we obtain that the system $\Gamma\overline{x} = 0$ has a unique solution $\overline{x} = 0$. □

Now, for a $\sigma : \mathcal{D} \to \mathcal{D}$ defined by (2.2.21), and for a $\partial : \mathcal{D} \to \mathcal{D}$ a σ-derivation defined on its basic elements as follows,

$$\partial(D_1) = A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + a_k, \tag{2.2.24}$$

$$\begin{aligned}
\partial(D_2) &= B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + b_k,\\
\partial(x_1) &= c_{Di}D_1 + c_{Dj}D_2 + c_{xi}x_1 + c_{xj}x_2 + c_k,\\
\partial(x_2) &= d_{Di}D_1 + d_{Dj}D_2 + d_{xi}x_1 + d_{xj}x_2 + d_k.
\end{aligned}$$

We can check that

$$\begin{aligned}
&\lambda_{12}\partial(D_1)\sigma(D_2)\\
&= \lambda_{12}(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + a_k)(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + B_k)\\
&= \lambda_{12}A_{D1}B_{D1}D_1^2 + A_{D1}B_{D2}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + \lambda_{12}A_{D1}B_{x1}D_1x_1 + \lambda_{12}A_{D1}B_{x2}D_1x_2 + \lambda_{12}A_{D1}B_kD_1\\
&+ \lambda_{12}A_{D2}B_{D1}D_2D_1 + \lambda_{12}A_{D2}B_{D2}D_2^2 + \lambda_{12}A_{D2}B_{x1}D_2x_1 + \lambda_{12}A_{D2}B_{x2}D_2x_2 + \lambda_{12}A_{D2}B_kD_2\\
&+ \lambda_{12}A_{x1}B_{D1}x_1D_1 + \lambda_{12}A_{x1}B_{D2}x_1D_2 + \lambda_{12}A_{x1}B_{x1}x_1^2 + \lambda_{12}A_{x1}B_{x2}x_1x_2 + \lambda_{12}A_{x1}B_kx_1\\
&+ \lambda_{12}A_{x2}B_{D1}x_2D_1 + \lambda_{12}A_{x2}B_{D2}x_2D_2 + \lambda_{12}A_{x2}B_{x1}x_2x_1 + \lambda_{12}A_{x2}B_{x2}x_2^2 + \lambda_{12}A_{x2}B_kx_2\\
&+ \lambda_{12}a_kB_{D1}D_1 + \lambda_{12}a_kB_{D2}D_2 + \lambda_{12}a_kB_{x1}x_1 + \lambda_{12}a_kB_{x2}x_2 + \lambda_{12}a_kB_k,
\end{aligned}$$

$$\begin{aligned}
&\lambda_{12}D_1\partial(D_2)\\
&= \lambda_{12}D_1(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + b_k)\\
&= \lambda_{12}B_{D1}D_1^2 + B_{D2}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + \lambda_{12}B_{x1}D_1x_1 + \lambda_{12}B_{x2}D_1x_2 + \lambda_{12}b_kD_1,
\end{aligned}$$

$$\begin{aligned}
&\lambda_{21}\partial(D_2)\sigma(D_1)\\
&= \lambda_{21}(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + b_k)(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + A_k)\\
&= \lambda_{21}B_{D1}A_{D1}D_1^2 + \lambda_{21}B_{D1}A_{D2}\lambda_{12}^{-1}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + \lambda_{21}B_{D1}A_{x1}D_1x_1 + \lambda_{21}B_{D1}A_{x2}D_1x_2 + \lambda_{21}B_{D1}A_kD_1\\
&+ \lambda_{21}B_{D2}A_{D1}D_2D_1 + \lambda_{21}B_{D2}A_{D2}D_2^2 + \lambda_{21}B_{D2}A_{x1}D_2x_1 + \lambda_{21}B_{D2}A_{x2}D_2x_2 + \lambda_{21}B_{D2}A_kD_2\\
&+ \lambda_{21}B_{x1}A_{D1}x_1D_1 + \lambda_{21}B_{x1}A_{D2}x_1D_2 + \lambda_{21}B_{x1}A_{x1}x_1^2 + \lambda_{21}B_{x1}A_{x2}x_1x_2 + \lambda_{21}B_{x1}A_kx_1\\
&+ \lambda_{21}B_{x2}A_{D1}x_2D_1 + \lambda_{21}B_{x2}A_{D2}x_2D_2 + \lambda_{21}B_{x2}A_{x1}x_2x_1 + \lambda_{21}B_{x2}A_{x2}x_2^2 + \lambda_{21}B_{x2}A_kx_2\\
&+ \lambda_{21}b_kA_{D1}D_1 + \lambda_{21}b_kA_{D2}D_2 + \lambda_{21}b_kA_{x1}x_1 + \lambda_{21}b_kA_{x2}x_2 + \lambda_{21}b_kA_k,
\end{aligned}$$

$$\begin{aligned}
&\lambda_{21}D_2\partial(D_1)\\
&= \lambda_{21}D_2(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + a_k)\\
&= \lambda_{21}A_{D1}D_2D_1 + \lambda_{21}A_{D2}D_2^2 + \lambda_{21}A_{x1}D_2x_1 + \lambda_{21}A_{x2}D_2x_2 + \lambda_{21}a_kD_2,
\end{aligned}$$

$$\begin{aligned}
&\partial(x_2)\sigma(D_1)\\
&= (d_{Di}D_1 + d_{Dj}D_2 + d_{xi}x_1 + d_{xj}x_2 + d_k)(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + A_k)\\
&= d_{Di}A_{D1}D_1^2 + d_{Di}A_{D2}\lambda_{12}^{-1}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + d_{Di}A_{x1}D_1x_1 + d_{Di}A_{x2}D_1x_2 + d_{Di}A_kD_1\\
&+ d_{Dj}A_{D1}D_2D_1 + d_{Dj}A_{D2}D_2^2 + d_{Dj}A_{x1}D_2x_1 + d_{Dj}A_{x2}D_2x_2 + d_{Dj}A_kD_2\\
&+ d_{xi}A_{D1}x_1D_1 + d_{xi}A_{D2}x_1D_2 + d_{xi}A_{x1}x_1^2 + d_{xi}A_{x2}x_1x_2 + d_{xi}A_kx_1\\
&+ B_{x2}A_{D1}x_2D_1 + d_{xj}A_{D2}x_2D_2 + d_{xj}A_{x1}x_2x_1 + d_{xj}A_{x2}x_2^2 + d_{xj}A_kx_2\\
&+ d_kA_{D1}D_1 + d_kA_{D2}D_2 + d_kA_{x1}x_1 + d_kA_{x2}x_2 + d_kA_k,
\end{aligned}$$

$$\begin{aligned}
&x_2\partial(D_1)\\
&= x_2(A_{D1}D_1 + A_{D2}D_2 + A_{x1}x_1 + A_{x2}x_2 + a_k)\\
&= A_{D1}x_2D_1 + A_{D2}x_2D_2 + A_{x1}x_2x_1 + A_{x2}x_2^2 + a_kx_2,
\end{aligned}$$

$$
\begin{aligned}
&\partial(x_1)\sigma(D_2)\\
&= (c_{Di}D_1 + c_{Dj}D_2 + c_{xi}x_1 + c_{xj}x_2 + c_k)(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + B_k)\\
&= c_{Di}B_{D1}D_1^2 + c_{Di}B_{D2}(\lambda_{21}D_2D_1 + x_2D_1 - x_1D_2) + c_{Di}B_{x1}D_1x_1 + c_{Di}B_{x2}D_1x_2 + c_{Di}B_kD_1\\
&+ c_{Dj}B_{D1}D_2D_1 + c_{Dj}B_{D2}D_2^2 + c_{Dj}B_{x1}D_2x_1 + c_{Dj}B_{x2}D_2x_2 + c_{Dj}B_kD_2\\
&+ c_{xi}B_{D1}x_1D_1 + c_{xi}B_{D2}x_1D_2 + c_{xi}B_{x1}x_1^2 + c_{xi}B_{x2}x_1x_2 + c_{xi}B_kx_1\\
&+ c_{xj}B_{D1}x_2D_1 + c_{xj}B_{D2}x_2D_2 + c_{xj}B_{x1}x_2x_1 + c_{xj}B_{x2}x_2^2 + c_{xj}B_kx_2\\
&+ c_kB_{D1}D_1 + c_kB_{D2}D_2 + c_kB_{x1}x_1 + c_kB_{x2}x_2 + c_kB_k,
\end{aligned}
$$

$$
\begin{aligned}
&x_1\partial(D_2)\\
&= x_1(B_{D1}D_1 + B_{D2}D_2 + B_{x1}x_1 + B_{x2}x_2 + b_k)\\
&= B_{D1}x_1D_1 + B_{D2}x_1D_2 + B_{x1}x_1^2 + B_{x2}x_1x_2 + b_kx_1.
\end{aligned}
$$

With these previous terms, we obtain the following proposition.

Proposition 2.2.27. *If $\sigma : \mathcal{D} \to \mathcal{D}$ is an automorphism as defined in (2.2.21) such that $\operatorname{span}_{\mathbb{C}}(S, H) = \operatorname{span}_{\mathbb{C}}(L_1, L_2)$ and $\partial : \mathcal{D} \to \mathcal{D}$ is a σ-derivation such that $\det(Z_\sigma) = 0$ and $\det(Z_{\sigma,\partial}) = \lambda_{12}B_{D2}$, then the elements $\partial(D_1)$, $\partial(D_2)$, $\partial(x_1)$ and $\partial(x_2)$ have no zero degree terms, where,*

$$
Z_{\sigma,\partial} = \begin{pmatrix} A_{D2} & \lambda_{12}A_{D1} + \lambda_{12}c_{Di} \\ B_{D2} & \lambda_{21}B_{D1} + d_{Di} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \lambda_{12} & -A_{D1} & B_{D1} \\ -\lambda_{21} & 0 & -A_{D2} & B_{D2} \\ 0 & 1 & -A_{x1} & B_{x1} \\ 1 & 0 & -A_{x2} & B_{x2} \end{pmatrix},
$$

$$
L_1 = \begin{pmatrix} 0 \\ -\lambda_{21} \\ 0 \\ 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \lambda_{12} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_{D1} \\ S_{D2} \\ S_{x1} \\ S_{x2} \end{pmatrix}, \quad H = \begin{pmatrix} H_{D1} \\ H_{D2} \\ H_{x1} \\ H_{x2} \end{pmatrix}.
$$

Proof. From the one degree terms of (2.2.20), we obtain the equations:

$$
\begin{aligned}
D_1 :\ & A_{D1}B_{D2}x_2 + B_{D2}x_2 - \lambda_{21}B_{D1}A_{D2}\lambda_{12}^{-1}x_2 - d_{Di}A_{D2}\lambda_{12}^{-1}x_2 + c_{Di}B_{D2}x_2\\
& \lambda_{12}A_{D1}B_k + \lambda_{12}a_kB_{D1} + \lambda_{12}b_k - \lambda_{21}B_{D2}A_k - \lambda_{21}b_kA_{D1} - d_{Di}A_k - d_kA_{D1} + c_{Di}B_k + c_kB_{D1} = 0\\
D_2 :\ & - A_{D1}B_{D2}x_1 - B_{D2}x_1 + \lambda_{21}B_{D1}A_{D2}\lambda_{12}^{-1}x_1 + d_{Di}A_{D2}\lambda_{12}^{-1}x_1 - c_{Di}B_{D2}x_1\\
& \lambda_{12}A_{D2}B_k + \lambda_{12}a_kB_{D2} - \lambda_{21}b_kA_{D2} - \lambda_{21}B_{D2}A_k - \lambda_{21}a_k - d_{Dj}A_k - d_kA_{D2} + c_{Dj}B_k + c_kB_{D2} = 0\\
x_1 :\ & \lambda_{12}A_{x1}B_k + \lambda_{12}a_kB_{x1} - \lambda_{21}b_kA_{x1} - \lambda_{21}B_{x1}A_k - d_{xi}A_k - d_kA_{x1} + c_{xi}B_k + c_kB_{x1} + b_k = 0\\
x_2 :\ & \lambda_{12}A_{x2}B_k + \lambda_{12}a_kB_{x2} - \lambda_{21}b_kA_{x2} - \lambda_{21}B_{x2}A_k - d_{xj}A_k - d_kA_{x2} + c_{xj}B_k + c_kB_{x2} + a_k = 0
\end{aligned}
$$

Now, since $\det(Z_{\sigma,\partial}) = \lambda_{12}B_{D1}$ and $\det(A) \neq 0$, then by Proposition 2.2.26, we have that $A_k = B_k = 0$. Therefore, the previous equations turns to

$$
\begin{aligned}
D_1 :\quad & \lambda_{12}a_kB_{D1} - \lambda_{21}b_kA_{D1} - d_kA_{D1} + c_kB_{D1} = -\lambda_{12}b_k,\\
D_2 :\quad & \lambda_{12}a_kB_{D2} - \lambda_{21}b_kA_{D2} - d_kA_{D2} + c_kB_{D2} = \lambda_{21}a_k,\\
x_1 :\quad & \lambda_{12}a_kB_{x1} - \lambda_{21}b_kA_{x1} - d_kA_{x1} + c_kB_{x1} = -b_k,\\
x_2 :\quad & \lambda_{12}a_kB_{x2} - \lambda_{21}b_kA_{x2} - d_kA_{x2} + c_kB_{x2} = -a_k,
\end{aligned}
$$

which is a linear system of equations on the variables a_k, b_k, c_k and d_k, i.e., it is a system $\Theta\overline{y} = 0$, where

$$\Theta = \begin{pmatrix} \lambda_{12}B_{D1} & \lambda_{12} - \lambda_{21}A_{D1} & -A_{D1} & B_{D1} \\ \lambda_{12}B_{D2} - \lambda_{21} & -\lambda_{21}A_{D2} & -A_{D2} & B_{D2} \\ \lambda_{12}B_{x1} & 1 - \lambda_{21}A_{x1} & -A_{x1} & B_{x1} \\ \lambda_{12}B_{x2} + 1 & -\lambda_{21}A_{x2} & -A_{x2} & B_{x2} \end{pmatrix}, \quad \overline{y} = \begin{pmatrix} a_k \\ b_k \\ d_k \\ c_k \end{pmatrix}.$$

In other words,

$$\begin{aligned} \det(\Theta) &= \det\begin{pmatrix} \lambda_{12}B_{D1} & \lambda_{12} - \lambda_{21}A_{D1} & -A_{D1} & B_{D1} \\ \lambda_{12}B_{D2} & -\lambda_{21}A_{D2} & -A_{D2} & B_{D2} \\ \lambda_{12}B_{x1} & 1 - \lambda_{21}A_{x1} & -A_{x1} & B_{x1} \\ \lambda_{12}B_{x2} & -\lambda_{21}A_{x2} & -A_{x2} & B_{x2} \end{pmatrix} \\ &+ \det\begin{pmatrix} 0 & \lambda_{12} - \lambda_{21}A_{D1} & -A_{D1} & B_{D1} \\ -\lambda_{21} & -\lambda_{21}A_{D2} & -A_{D2} & B_{D2} \\ 0 & 1 - \lambda_{21}A_{x1} & -A_{x1} & B_{x1} \\ 1 & -\lambda_{21}A_{x2} & -A_{x2} & B_{x2} \end{pmatrix} \\ &= \det\begin{pmatrix} 0 & \lambda_{12} - \lambda_{21}A_{D1} & -A_{D1} & B_{D1} \\ -\lambda_{21} & -\lambda_{21}A_{D2} & -A_{D2} & B_{D2} \\ 0 & 1 - \lambda_{21}A_{x1} & -A_{x1} & B_{x1} \\ 1 & -\lambda_{21}A_{x2} & -A_{x2} & B_{x2} \end{pmatrix} \\ &= \det\begin{pmatrix} 0 & -\lambda_{21}A_{D1} & -A_{D1} & B_{D1} \\ -\lambda_{21} & -\lambda_{21}A_{D2} & -A_{D2} & B_{D2} \\ 0 & -\lambda_{21}A_{x1} & -A_{x1} & B_{x1} \\ 1 & -\lambda_{21}A_{x2} & -A_{x2} & B_{x2} \end{pmatrix} \\ &+ \det\begin{pmatrix} 0 & \lambda_{12} & -A_{D1} & B_{D1} \\ -\lambda_{21} & 0 & -A_{D2} & B_{D2} \\ 0 & 1 & -A_{x1} & B_{x1} \\ 1 & 0 & -A_{x2} & B_{x2} \end{pmatrix} \\ &= \det\begin{pmatrix} 0 & \lambda_{12} & -A_{D1} & B_{D1} \\ -\lambda_{21} & 0 & -A_{D2} & B_{D2} \\ 0 & 1 & -A_{x1} & B_{x1} \\ 1 & 0 & -A_{x2} & B_{x2} \end{pmatrix} = \det(L) \end{aligned}$$

Due to the fact that $\operatorname{span}_{\mathbb{C}}(S, H) = \operatorname{span}_{\mathbb{C}}(L_1, L_2)$, we have that the matrices A^T and L^T are row equivalent, and therefore $\det(L) \neq 0$, because $\det(A) \neq 0$. Since $\det(A) = \det(L)$, $\det(\Theta) \neq 0$ which implies that the unique solution to the homogeneous linear system $\Theta\overline{y} = 0$ is the trivial solution $\overline{y} = 0$, i.e., $a_k = b_k = c_k = d_k = 0$. □

Corollary 2.2.28. *If $\sigma : \mathcal{D} \to \mathcal{D}$ is an automorphism defined as in* (2.2.21) *such that the linear components satisfies* $\operatorname{span}_{\mathbb{C}}(S, H) = \operatorname{span}_{\mathbb{C}}(L_1, L_2)$ *as in* Proposition 2.2.27 *and if $\partial : \mathcal{D} \to \mathcal{D}$ is a σ-derivation such that* $\det(Z_\sigma) = 0$ *and* $\det(Z_{\sigma,\partial}) = \lambda_{12}B_{D1}$*, then $\partial(\mathcal{D}) \cap \mathbb{C} = \emptyset$.*

Remark 2.2.29. As a future task, we can search skew derivations for these graded automorphism, in order to construct differential calculi following Remark 1.2.12, but by the

Corollary 2.2.28, we can not guarantee the density conditions (see Remark 1.2.13). Therefore, when we work in diffusion algebras of type 2 with 2 generators, we prefer to choose skew derivations of graded automorphism such that $\text{span}_{\mathbb{C}}(S, H) \neq \text{span}_{\mathbb{C}}(L_1, L_2)$.

In this section, we applied linear algebra to obtain all results that we found, but we do not used the PBW basis of diffusion algebras. In the next section, we will see how we can use the PBW basis to describe automorphisms in other algebras.

2.3 Skew polynomial algebras

In this section, we study the *3-dimensional skew-polynomial algebras* and show that there exists a classification of them as generalized Weyl algebras or skew polynomial rings. Further, from a theorem of classification, we mention in each case if it is or not a 3-dimensional diffusion algebra type 1 or a generalized Weyl algebra.

Definition 2.3.1 ([Ros95], Definition C4.3)**.** A 3-dimensional algebra $\mathcal{A}$ is defined by the relations:

$$yz - \alpha zy = \lambda, \quad zx - \beta xz = \mu, \quad xy - \gamma yx = \nu, \tag{2.3.1}$$

where $\lambda, \mu, \nu \in \mathbb{K}x + \mathbb{K}y + \mathbb{K}z + \mathbb{K}$, and $\alpha, \beta, \gamma \in \mathbb{K}^*$. $\mathcal{A}$ is called a *3-dimensional skew polynomial* $\mathbb{K}$*-algebra*, if the set $\{x^i y^j z^k : i, j, k \geq 0\}$ forms a basis of the algebra (i.e., the PBW condition).

In the literature, we found the following facts about 3-dimensional skew polynomial algebras $\mathcal{A}$.

1. Relations that guarantee the PBW basis of an algebra satisfying equations (2.3.1) were established in ([RS17b], Section 5).
2. Since $\mathbb{C}$ is Noetherian, then $\mathcal{A}$ is left Noetherian ([LR14], Theorem 2.2).
3. By the left Noetherianity of $\mathbb{C}$, $\mathcal{A}$ is left regular ([LR14], Corollary 2.6).
4. Since $\mathbb{C}$ is Noetherian, regular and PSF[15], then $\mathcal{A}$ is PSF ([LR14], Corollary 2.8).
5. $\mathcal{A}$ is a domain ([LR14], Proposition 4.1).
6. The Quillen's K-theory of $\mathcal{A}$ and $\mathbb{C}$ is the same ([LR14], Theorem 5.1).
7. Since $\mathbb{C}$ is ACCPL-domain, $\mathcal{A}$ is an ACCPL-domain ([RS18], Theorem 1).
8. By the commutativity of $\mathbb{C}$, $\mathcal{A}$ is an ACCPR-domain ([RS18], Corollary 1).
9. $\mathcal{A}$ has no non-zero nilpotent elements ([RS19b], Proposition 5).
10. Since $\mathbb{C}$ is right zip, then $\mathcal{A}$ is right zip ([RR19], Corollary 4.3).
11. $\mathcal{A}$ satisfies the Kothë's conjecture ([RS19a], Proposition 3.19).

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[15] B is a PSF ring, if every finitely generated projective B-module is stably free.

The following theorem gives a full classification of 3-dimensional skew polynomial algebras.

Theorem 2.3.2 ([Ros95], Theorem C4.3.1)**.** *Up to isomorphism, a 3-dimensional skew polynomial $\mathbb{K}$-algebra $\mathcal{A}$ is given by the following relations:*

1. *If $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by*

$$yz - \alpha zy = 0, \quad zx - \beta xz = 0, \quad xy - \gamma yx = 0.$$

2. *If $|\{\alpha, \beta, \gamma\}| = 2$ and if $\beta \neq \alpha = \gamma = 1$, $\mathcal{A}$ is defined by one of the following rules:*

(a)

$$yz - zy = z \qquad zx - \beta xz = y \qquad xy - yx = x$$

(b)

$$yz - zy = z \qquad zx - \beta xz = b \qquad xy - yx = x$$

(c)

$$yz - zy = 0 \qquad zx - \beta xz = y \qquad xy - yx = 0$$

(d)

$$yz - zy = 0 \qquad zx - \beta xz = b \qquad xy - yx = 0$$

(e)

$$yz - zy = az \qquad zx - \beta xz = 0 \qquad xy - yx = x$$

(f)

$$yz - zy = z \qquad zx - \beta xz = 0 \qquad xy - yx = 0$$

Here $a, b \in \mathbb{K}$ are arbitrary; all nonzero values of b yield isomorphic algebras.

3. *If $\alpha = \beta = \gamma \neq 1$, and if $\beta \neq \alpha = \gamma \neq 1$, then*

(a)

$$yz - \alpha zy = 0 \qquad zx - \beta xz = y + b \qquad xy - \alpha yx = 0$$

(b)

$$yz - \alpha zy = 0 \qquad zx - \beta xz = b \qquad xy - \alpha yx = 0$$

Here $a, b \in \mathbb{K}$ is arbitrary; all nonzero values of b yield isomorphic algebras.

4. *If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is isomorphic to one of the following:*

$$yz - \alpha zy = a_1 x + b_1$$

$$zx - \alpha xz = a_2 y + b_2$$

$$xy - \alpha yx = a_3 z + b_3$$

If $a_i = 0$, then all nonzero values of b_i yield isomorphic algebras.

5. *if $\alpha = \beta = \gamma = 1$, then $\mathcal{A}$ is isomorphic to one of the following algebras:*

(a)
$$yz - zy = x$$
$$zx - xz = y$$
$$xy - yx = z$$

(b)
$$yz - zy = 0$$
$$zx - xz = 0$$
$$xy - yx = z$$

(c)
$$yz - zy = 0$$
$$zx - xz = 0$$
$$xy - yx = b$$

(d)
$$yz - zy = -y$$
$$zx - xz = x + y$$
$$xy - yx = 0$$

(e)
$$yz - zy = az$$
$$zx - xz = x$$
$$xy - yx = 0$$

Here $a, b \in \mathbb{K}$ are arbitrary; all nonzero values of b yield isomorphic algebras.

The following examples are not diffusion algebras of type 1:

2. (a) & (c): Due to $zx - \beta xz = y$, and as $y \neq r_1x + r_2z$ for all $r_1, r_2 \in \mathbb{K}$, this algebra is not a diffusion algebra of type 1 because (2.2.15), for $D_i = z$ and $D_j = x$.

2. (b) & (d): Due to $zx - \beta xz = b$, and as $b \neq r_1x + r_2z$ for all $r_1, r_2 \in \mathbb{K}$, this algebra is not a diffusion algebra of type 1 because (2.2.15), for $D_i = z$ and $D_j = x$.
(f): There is no possible that only one of the right terms of the equations be different of zero.

3. Due to $zx - \beta xz = \phi y + b$ with $\phi \in \{0, 1\}$ and as $\phi y + b \neq r_1x + r_2z$, for all $r_1, r_2 \in \mathbb{K}$, this algebra is not a diffusion algebra of type 1 because (2.2.15) for $D_i = z$ and $D_j = x$.

4. The only way for these algebras are diffusion algebras is when if $a_i = b_i = 0$, for all $i = 1, 2, 3$.

5. (a) & (b) Due to $xy - \beta yx = z$, and as $z \neq r_1y + r_2x$ for all $r_1, r_2 \in \mathbb{K}$, this algebra is not a diffusion algebra of type 1 because (2.2.15), for $D_i = x$ and $D_j = y$.
(d) Due to $zx - xz = x + y$ and as $x + y \neq r_1z + r_2x$ for all $r_1, r_2 \in \mathbb{K}$, this algebra is not a diffusion algebra of type 1 because (2.2.15), for $D_i = z$ and $D_j = x$.
(e) If $D_i = z$, $D_j = x$, $D_k = y$, by $zx - xz = x$ we have that x_i must be equal to -1, but in $yz - zy = az$ we can deduce that $x_i = 0$. Therefore, this algebra is not a diffusion algebra of type 1. (c) If $b \neq 0$, due to $xy - yx = b$ this is not a diffusion algebra of type 1, by the fact that $b \neq r_1x + r_2y$, for all $r_1, r_2 \in \mathbb{K}$.

Now, we have that the following algebras of the list are diffusion algebras of type 1:

1. This is a diffusion algebra type 1 of class D where $D_i = y$, $D_j = z$, $D_k = x$, $\lambda_{ij} = \lambda_{jk} = \lambda_{ik} = 1$, $\lambda_{ji} = \alpha$, $\lambda_{kj} = \beta$, $\lambda_{ki} = \gamma$, $x_i = x_j = x_k = 0$.

2. (e) This is a diffusion algebra of type 1 of class C_I under the condition $a = 1$, where $D_i = y$, $D_j = z$, $D_k = x$, $x_i = -a$, $x_i = -1$, $\Lambda = 0$, $\lambda_{jk} = 1$, $\lambda_{kj} = \beta$, $\lambda_{ik} = \lambda_{ki} = x_i = -1$, $\lambda_{ij} = \lambda_{ji} = 1$, $x_i = a$. If $a \neq 1$, this algebra is not a diffusion algebra of type 1.

Remark 2.3.3. We want to highlight the fact that by Definition 2.3.1, we conclude that the diffusion algebras of type 1 of 3 generators (see Definition 2.2.2) such that all $\lambda_{ji} \neq 0$, with $i < j$, are skew polynomial algebras. Therefore the different classes listed in Theorem 2.2.3, must be also distributed in the classification of skew polynomial algebras of Theorem 2.3.2: in the cases of diffusion algebras of class C_I and D are skew polynomial algebras of type *2*(e) and *1*, respectively. But the cases of diffusion algebras of classes A_I and B_I, not belong to any of the list of Theorem 2.3.2 in a first time. This mean that for view classes A_I and B_I as skew polynomial algebras, we need first make an identification (establish an automorphism of algebras) with, apparently, *2*(a), *2*(b), *2*(e) *5*(a), *5*(d) or *5*(e). The rest of 3-degree diffusion algebras of type 1 are not skew polynomial algebras because these need some λ equals zero.

Also, the following algebras are hyperbolics, i.e., generalized Weyl algebras of dimension 3 (see Remark 2.1.9 and Remark 2.1.11): *2.* $(a), (b)$ *3.* (a), (b) *5.* (a). and their structure is given by:

- *2.*(a) $A = \mathbb{K}[y]$, $\vartheta(f(y)) = f(y-1)$, then $\mathcal{A} = \langle \vartheta, \beta, y\rangle$; the corresponding hyperbolic ring is $R\{\theta, \xi\}$ with $R = \mathbb{K}[y, \xi]$, $\theta(f(y,\xi)) = f(y-1, \beta\xi + y - 1)$ ([Ros95], p. 102).
- *2.*(b) $A = \mathbb{K}[y]$, $\vartheta(f(y)) = f(y-1)$, then $\mathcal{A} = \langle \vartheta, \beta, b\rangle$; the corresponding hyperbolic ring is $R\{\theta, \xi\}$ with $R = \mathbb{K}[y, \xi]$, $\theta(f(y,\xi)) = f(y-1, \beta\xi + b)$ ([Ros95], p. 104).
- *3.*(a) $A = \mathbb{K}[y]$, $\vartheta(f(y)) = f(\alpha y)$, $u = y + b$, then $\mathcal{A} = A\langle \vartheta, \beta, u\rangle$. The corresponding hyperbolic ring is $R\{\theta, \xi\}$, where $R = \mathbb{K}[y, \xi]$, $\theta(f(y,\xi)) = f(\alpha^{-1}y, \beta\xi + \alpha^{-1}y + b)$. ([Ros95], p. 105).
- *3.*(b) $\mathcal{A} = A\langle \vartheta, \beta, b\rangle$, where $A = \mathbb{K}[y]$, $\vartheta(f(y)) = f(\alpha y)$; the corresponding hyperbolic ring is $R\{\theta, \xi\}$, where $R = \mathbb{K}[y, \xi]$ and $\theta(f(y,\xi)) = f(\alpha^{-1}y, \beta\xi + b)$ ([Ros95], p. 108).
- *5.*(a) This is the algebra $U(sl(2))$, under the identification of Remark 2.1.5 ([Ros95], p. 108).

Another 3-dimensional skew polynomial algebras are classified as Ore extensions in the next subsection. Nevertheless, *Dispin algebra* (in Theorem 2.3.2, the algebra of type *2*(a) with $\beta = -1$, see [Ros95], p. 99) is 3-dimensional but not an Ore extension.

2.3.1 Extended automorphisms of skew polynomial algebras

In this section, we study by a computational way, conditions over a special case of automorphisms in some of the 3-dimensional skew polynomial algebras appearing in Theorem 2.3.2.

We found in this work that the following commutation works.

Proposition 2.3.4. *If* $R[x; \sigma, \delta]$ *is an Ore extension and* $\sigma \circ \delta = \delta \circ \sigma$*, then for all* $n \in \mathbb{N}$ *and all* $r \in R$*,*

$$x^n r = \sum_{i=0}^{n} \binom{n}{i} \sigma^i \circ \delta^{n-i}(r)\ x^i.$$

Proof. The case $n = 1$ is the definition of $R[x; \sigma, \delta]$. If we suppose it holds for $n \in \mathbb{N}$, then

$$\begin{aligned}
x^{n+1}r &= x\left(\sum_{i=0}^{n}\binom{n}{i}\sigma^i \circ \delta^{n-i}(r)\ x^i\right)\\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\sigma \circ \sigma^i \circ \delta^{n-i}(r)x + \delta \circ \sigma^i \circ \delta^{n-i}(r)\right)\ x^i\\
&= \sum_{i=0}^{n}\binom{n}{i}\sigma^{i+1} \circ \delta^{n-i}(r)x^{i+1} + \sum_{i=0}^{n}\binom{n}{i}\sigma^i \circ \delta^{n+1-i}(r)\ x^i\\
&= \sum_{i=1}^{n+1}\binom{n}{i-1}\sigma^i \circ \delta^{n+1-i}(r)x^i + \sum_{i=0}^{n}\binom{n}{i}\sigma^i \circ \delta^{n+1-i}(r)\ x^i\\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}\sigma^i \circ \delta^{n+1-i}(r)x^i,
\end{aligned}$$

where the third equal is due to $\sigma \circ \delta = \delta \circ \sigma$. □

Remark 2.3.5. Now, if we want to extend the automorphism $\sigma : R \to R$ to $R[x; \sigma, \delta]$, with $[\sigma, \delta] = 0$, then we need to define $\sigma(x) = \sum_{i=0}^{m} w_i x^i$, with $m \in \mathbb{N}$ and $w_i \in R$, for all $i \leq m$. Then $\sigma(xr - \sigma(r)x - \delta(r)) = 0$, and so

$$\begin{aligned}
0 &= \sum_{i=0}^{m} w_i x^i \sigma(r) - \sigma^2(r)\left(\sum_{i=0}^{m} w_i x^i\right) - \sigma(\delta(r))\\
&= \left[\sum_{i=0}^{m}\sum_{j=0}^{i}\binom{i}{j} w_i \sigma^j \circ \delta^{i-j}(\sigma(r))\ x^j\right] - \sigma^2(r)\left(\sum_{i=0}^{m} w_i x^i\right) - \sigma(\delta(r))
\end{aligned}$$

$$\begin{aligned}
&= \left[\sum_{j=0}^{m}\sum_{i=j}^{m}\binom{i}{j} w_i \sigma^j \circ \delta^{i-j}(\sigma(r))\ x^j\right] - \sigma^2(r)\left(\sum_{j=0}^{m} w_j x^j\right) - \sigma(\delta(r))\\
&= \sum_{j=0}^{m}\left[\sum_{i=j}^{m}\binom{i}{j} w_i \sigma^j \circ \delta^{i-j}(\sigma(r)) - \sigma^2(r) w_j\right] x^j - \sigma(\delta(r)).
\end{aligned}$$

Therefore, the equations that define an extended automorphism $\sigma : R \to R$ to $R[x; \sigma, \delta]$ are given by

$$\begin{aligned}
\left[\sum_{i=j}^{m}\binom{i}{j} w_i \sigma^j \circ \delta^{i-j}(\sigma(r)) - \sigma^2(r) w_j\right] &= 0, \quad \forall j \geq 1,\\
\left[\sum_{i=0}^{m} w_i \delta^i(\sigma(r)) - \sigma^2(r) w_0\right] - \sigma(\delta(r)) &= 0.
\end{aligned}$$

Remark 2.3.6. In Theorem 2.3.2 we have that the following 3-dimensional skew polynomial algebras are Ore extensions (see Definition 1.3.25), where their respective Ore structures are given in the following way:

1. $R = \mathbb{C}[x]$, $R[z;\sigma_1,\delta_1][y;\sigma_2,\delta_2]$, $\sigma_1(x) = \beta x$, $\delta_1(x) = 0$, $\sigma_2(z) = \alpha z$, $\sigma_2(x) = \gamma^{-1}x$, $\delta_2(x) = \delta_2(z) = 0$.

2. (*c*) $R = \mathbb{C}[y,z]$, $R[x;\sigma_1,\delta_1]$, $\sigma_1(y) = y$, $\delta_1(y) = 0$, $\sigma_1(z) = \beta^{-1}z$, $\delta_1(z) = -\beta^{-1}y$.

2. (*d*) $R = \mathbb{C}[y,z]$, $R[x;\sigma_1,\delta_1]$, $\sigma_1(y) = y$, $\delta_1(y) = 0$, $\sigma_1(z) = \beta^{-1}z$, $\delta_1(z) = -\beta^{-1}b$.

2. (*e*) $R = \mathbb{C}[x]$, $R[z;\sigma_1,\delta_1][y;\sigma_2,\delta_2]$, $\sigma_1(x) = \beta x$, $\delta_1(x) = 0$, $\sigma_2(z) = z$, $\delta_2(z) = az$, $\sigma_2(x) = x$, $\delta_2(x) = -x$.

2. (*f*) $R = \mathbb{C}[z]$, $R[y;\sigma_1,\delta_1][x;\sigma_2,\delta_2]$, $\sigma_1(z) = z$, $\delta_1(z) = z$, $\sigma_2(y) = y$, $\delta_2(y) = 0$, $\sigma_2(z) = \beta^{-1}z$, $\delta_2(z) = 0$.

4. $R = \mathbb{C}[x]$, $R[y;\sigma_1,\delta_1][z;\sigma_2,\delta_2]$, $\sigma_1(x) = \alpha^{-1}x$, $\delta_1(x) = -\alpha^{-1}(a_3x + b_3)$, $\sigma_2(y) = \alpha^{-1}y$, $\delta_2(y) = -\alpha^{-1}(a_1x_1 + b_1)$, $\sigma_2(x) = \alpha x$, $\delta_2(x) = a_2x + b_2$.

5. (*b*) $R = \mathbb{C}[y,z]$, $R[x;\sigma_1,\delta_1]$, $\sigma_1(z) = z$, $\delta_1(z) = 0$, $\sigma_1(y) = y$, $\delta_1(y) = z$.

5. (*c*) $R = \mathbb{C}[y,z]$, $R[x;\sigma_1,\delta_1]$, $\sigma_1(y) = y$, $\delta_1(y) = b$, $\sigma_1(z) = z$, $\delta_1(z) = 0$.

5. (*d*) $R = \mathbb{C}[x,y]$, $R[z;\sigma_1,\delta_1]$, $\sigma_1(x) = x$, $\delta_1(x) = x + y$, $\sigma_1(y) = y$, $\delta_1(y) = y$.

5. (*e*) $R = \mathbb{C}[x]$, $R[z;\sigma_1,\delta_1][y;\sigma_2,\delta_2]$, $\sigma_1(x) = x$, $\delta_1(x) = x$, $\sigma_2(z) = z$, $\delta_2(z) = az$, $\sigma_2(x) = x$, $\delta_2(x) = 0$.

Remark 2.3.7. From these Ore extensions, there are some of these which satisfies $\sigma \circ \delta = \delta \circ \sigma$: *1.*, *2.*(*e*), *2.*(*f*), *5.*(*b*), *5.*(*c*), *5.*(*d*) and *5.*(*e*). Therefore, if we want an automorphism of these structures such that the restriction of it to R, in each case it is also an automorphism of R, then by the Remark 2.3.5, we obtain that

$$\left[\sum_{i=j+1}^{m} \binom{i}{j} w_i \sigma^j \circ \delta^{i-j}(\sigma(r))\right] + w_j\sigma^{j+1}(r) - \sigma^2(r)w_j = 0, \quad \forall j \geq 1,$$

$$\left[\sum_{i=0}^{m} w_i \delta^i(\sigma(r))\right] - \sigma^2(r)w_0 - \sigma(\delta(r)) = 0. \tag{2.3.2}$$

For the algebras mentioned in Theorem 2.3.7, we show the explicit conditions to get the extended automorphism.

Case *1.* In this case, as $R = \mathbb{C}[x][z;\sigma_1,\delta_1]$, we get in $R[y;\sigma_2,\delta_2]$ with $\sigma_2(y) = \sum_{i=0}^{m} w_iy^i$ with $w_i \in R$, then since $\delta_2 = 0$, we obtain that

$$w_0\sigma_2(r) - \sigma_2^2(r)w_0 = 0 \ \text{ and } \ w_j\sigma_2^j(r) - \sigma_2^2(r)w_j = 0, \quad m \geq j > 0.$$

Let us consider for $k, j \in \mathbb{N}$ and $\theta \in \mathbb{C}$ the polynomial $P_{kj}^{\theta}(x) = x^k - \theta^{j-1}$. We note that if $\eta \in \mathbb{C}$ is a root of $P_{kj}^{\theta}(x)$, then for all $n \in \mathbb{N}$, η is also a root of $P_{(nk)(n(j-1)+1)}^{\theta}(x)$. In this way, we have the following result obtained in the realization of this work.

Proposition 2.3.8. *If σ is an algebra endomorphism of a 3-dimensional skew polynomial algebra of type 1 such that $\sigma|_{R[z;\sigma_1,\delta_1]} = \sigma_2$, then*

1. If $P_{k(j+1)}^{\gamma}(\beta) \neq 0$, for all $k \in \mathbb{N}$, then $w_j = 0$.

2. *If $P^{\gamma}_{kj}(\beta) = 0$, for some $k \in \mathbb{N}$, and β is not a root of unity (neither γ) then $w_j = g(x)z^k$. In this case, for all $n \in \mathbb{N}$, $w_{n(j-1)+1} = f(x)z^{nk}$.*
   - (a) *If $P^{\alpha}_{ij}(\beta) \neq 0$, for all $i \in \mathbb{N}$, then $g(x) = 0$ and therefore $w_j = 0$.*
   - (b) *If there exists, $e \in \mathbb{N}$ such that $P^{\alpha}_{ej}(\beta) = 0$ then $g(x) = gx^e$ with $g \in \mathbb{C}$ and $\alpha^{j-1} = \beta^e$, and $w_0 = 0$.*

Proof. For $r = x$, if $w_j = \sum_{k=0}^{p} g_k(x)z^k$ with $g_k(x) \in \mathbb{C}[x]$, we get for $j > 0$,

$$\begin{aligned} 0 = w_j\sigma_2^{j+1}(x) - \sigma_2^2(x)w_j &= \left(\sum_{k=0}^{p} g_k(x)z^k\right)\gamma^{-(j+1)}x - \gamma^{-2}x\left(\sum_{k=0}^{p} g_k(x)z^k\right) \\ &= \left(\sum_{k=0}^{p} g_k(x)\gamma^{-(j+1)}\beta^k xz^k - g_k(x)\gamma^{-2}xz^k\right) \\ &= \sum_{k=0}^{p} g_k(x)[\gamma^{-(j+1)}\beta^k - \gamma^{-2}]xz^k. \end{aligned}$$

If $\gamma^{-(j+1)}\beta^k - \gamma^{-2} \neq 0$, for all $k \in \{0, ..., p\}$, then $w_j = 0$. If $\gamma^{-(j+1)}\beta^k - \gamma^{-2} = 0$, for some $k \in \{0, ..., p\}$, called k_j, and if γ is not a root of unity (and therefore neither β), then $w_j = g_{k_j}(x)z^{k_j}$. In this case, since $\beta^{k_j} = \gamma^{j-1}$, then for all $n \in \mathbb{N}$, then $\beta^{nk_j} = \gamma^{(n(j-1)+1)-1}$. If $w_j = g(x)z^{k_j}$, then if $g(x) = \sum_{i=0} a_i x^i$ with $a_i \in \mathbb{C}$, we get for $r = z$,

$$\begin{aligned} 0 = w_j\sigma_2^{j+1}(z) - \sigma_2^2(z)w_j &= g(x)z^{k_j}\alpha^{j+1}z - \alpha^2 z g(x)z^{k_j} \\ &= [g(x)\alpha^{j+1} - \alpha^2 g(\beta x)]z^{k_j+1}, \end{aligned}$$

then $g(x)\alpha^{j-1} = g(\beta x)$, thus $g(x) = gx^e$ with $g \in \mathbb{C}$ and $\alpha^{j-1} = \beta^e$. Now, for $j = 0$, since $w_0\sigma_2(r) - \sigma_2^2(r)w_0 = 0$, with $r = x$, we get

$$\begin{aligned} 0 &= w_0(x,z)\gamma^{-1}x - \gamma^{-2}xw_0(x,z) \\ &= \gamma^{-1}xw_0(x,\beta z) - \gamma^{-2}xw_0(x,z), \end{aligned}$$

which implies that $\gamma w_0(x, \beta z) = w_0(x, z)$, then $w_0(x, z) = g_0 x^{e_0} z^{k_0}$, with $k_0 \in \mathbb{N}$. Since $\gamma w_0(x, \beta z) = w_0(x, z)$, if $g_0 \neq 0$ we obtain that $\beta^{-k_0} = \gamma$, and since $\beta^{k_j} = \gamma^{j-1}$ and γ is not a root of unity, we get $-k_0(j-1) = k_j \geq 0$, which means that $0 < j \leq 1$, but $j \neq 1$, because in that case $\beta^k = 1$. Therefore, we have that $g_0 = 0$ and $w_0 = 0$. □

Corollary 2.3.9. *If σ is an endomorphism such as it is described in* Proposition 2.3.8, *such that it is an automorphisnm, then $\sigma(y) = gy$, with $g \in \mathbb{C}$.*

Proof. If σ' is the inverse of an extended automorphism as in Proposition 2.3.8, then $\sigma'|_R = \sigma_2^{-1}$. If $\sigma'(y) = \sum_{j=0}^{p'} w'_j y^j$. Then as σ' has to respect $\sigma'(xr - \sigma(r)x - \delta(r)) = 0$, for all $r \in R$, which means,

$$0 = \sigma'(yr - \sigma_2(r)y - \delta_2(r)) = \left(\sum_{j=0}^{p'} w'_j y^j\right)\sigma_2^{-1}(r) - r\left(\sum_{j=0}^{p'} w'_j y^j\right).$$

If $r = x$, we get,

$$0 = \left(\sum_{j=0}^{p'} w'_j y^j\right)\sigma_2^{-1}(x) - x\left(\sum_{j=0}^{p'} w'_j y^j\right) = \left(\sum_{j=0}^{p'} \gamma w'_j y^j x\right) - x\left(\sum_{j=0}^{p'} w'_j y^j\right) = \sum_{j=0}^{p'}(\gamma^{1-j} w'_j x - x w'_j) y^j.$$

Then $(\gamma^{1-j}w'_jx - xw'_j) = 0$, for all $0 \leq j \leq p'$, $w'_j \neq 0$, then $w'_j = g'_j x^{e'_j} z^{k'_j}$ and $\gamma^{1-j}\beta^{k'_j} = 1$ whence $\gamma^{j-1} = \beta^{k'_j}$. Since γ is not a root of unity, $k'_j = k_j$. Therefore, $\sigma'(y) = g'_j x^{e'_j} z^{k_j} y^j$. Hence

$$\sigma(\sigma'(y)) = \sigma(g'_j x^{e'_j} z^{k_j} y^j) = \sigma(g_j x^{e'_j} z^{k_j})(g x^e z^{k_j} y^j)^j,$$

then, if we want $\sigma(\sigma'(y)) = y$, then we must have that $j = 1$, and therefore $k_j = 0$. With this, since $\alpha^{j-1} = \beta^e$, we must have $e = 0$. □

Case 2.(e): In this case, we have that $\sigma_2 = id$, for all $r(x, z) = \sum_{s=0}^{p}\sum_{t=0}^{q} c_{st}x^s z^t \in R[z; \sigma_1, \delta_1]$, so

$$\delta_2(r(x, z)) = \sum_{s=0}^{p}\sum_{t=0}^{q} c_{st}(ta - s)x^s z^t,$$

and the equations (2.3.2) turns to

$$\left[\sum_{i=j}^{m}\binom{i}{j} w_i \delta^{i-j}(\sigma(r))\right] - r w_j = 0, \quad \forall j \geq 1,$$

$$\left[\sum_{i=0}^{m} w_i \delta^i(r) - r w_0\right] - \delta(r) = 0 \quad j = 0.$$

In this work we obtain the next result.

Proposition 2.3.10. *If $\sigma : R[z; \sigma_1, \delta_1][y; \sigma_1, \delta_1] \to R[z; \sigma_1, \delta_1][y; \sigma_1, \delta_1]$ is an algebra endomorphism of a 3-dimensional skew polynomial algebra of type 2(e) such that $\sigma|_{R[z, \sigma_1, \delta_1]} = \sigma_2 = id$, then $\sigma(y) = y + c$, with $c \in \mathbb{C}$.*

Proof. For $j = m$, if $w_m = \sum_{k=0}^{p} g_k(x) z^k$ with $g_k(x) \in \mathbb{C}[x]$ we obtain that

$$0 = w_m r - r w_m = \sum_{k=0}^{p} g_k(x) z^k x - x g_k(x) z^k = \sum_{k=0}^{p} x g_k(x)(\beta^k - 1) z^k.$$

Since β is not a root of unity, $g_k(x) = 0$, for all $k \geq 1$, which means that $w_m \in \mathbb{C}$. If we suppose, that $w_t = 0$, for all $n + 1 < t \leq m$ with $n \geq 1$, and $w_{n+1} \in \mathbb{C}$, then, for $j = n$, we get that $n w_{n+1}\delta_2(r) + w_n r - r w_n = 0$, for all $r \in R[z, \sigma_1, \delta_1]$. If we choose $r = x$, saying that $w_n = \sum_{k=0}^{p} f_k(x) z^k$ with $f_k(x) \in \mathbb{C}[x]$, we obtain

$$0 = n w_{n+1}\delta_2(x) + w_n x - x w_n = n w_{n+1}(-x) + \sum_{k=0}^{p} x f_k(x)(\beta^k - 1) z^k,$$

then, $f_k(x) = 0$, for all $k \geq 1$, and $n w_{n+1}(-x) = 0$ which implies that $w_{n+1} = 0$. With this, we obtain that $w_n \in \mathbb{C}$. Therefore, $w_2 = \cdots = w_m = 0$ and $w_1 \in \mathbb{C}$. Thus, in $j = 0$, we obtain that $w_1\delta_2(r)+w_0r-rw_0-\delta_2(r)=0$, for all $r\in R[z;\sigma_1,\delta_1]$. If $w_0=\sum_{k=0}^{p}h_k(x)z^k$ with $h_k(x)\in\mathbb{C}[x]$ then, for $r=x$, we get

$$0=w_1\delta_2(x)+w_0x-xw_0-\delta_2(x)=(1-w_1)x+\sum_{k=0}^{p}xh_k(x)(\beta^k-1)z^k,$$

which implies that $w_1=1$ and $h_k(x)=0$, for all $k\geq 1$, i.e., $w_0\in\mathbb{C}$.

□

Case *2.(f)* : Since $\delta_2=0$ and $\sigma_2(r(y,z))=r(y,\beta^{-1}z)$, for all $r(y,z)\in R[y;\sigma_1,\delta_1]$, in this work we obtain the following in this case.

Proposition 2.3.11. *There is no exist $\sigma:R[y;\sigma_1,\delta_1][x;\sigma_2,\delta_2]\to R[y;\sigma_1,\delta_1][x;\sigma_2,\delta_2]$ a non trivial ring endomorphism of a skew polynomial algebra of type of type 2(f), where β is not a root of unity, such that $\sigma|_{R[y;\sigma_1,\delta_1]}=\sigma_2$.*

Proof. In expression (2.3.2), for all $r(y,z)\in R[y;\sigma_1,\delta_1]$,

$$w_jr(y,\beta^{-(j+1)}z)-r(y,\beta^{-2}z)w_j=0,\qquad \text{for }\ 0<j\leq m.$$

From definition of Ore extension, we have that, for all $g(z)\in R$, $yg(z)=g(z)y+\delta_1(g(z))$. Then, if $w_j=\sum_{i=0}^{p}w_{ji}(z)y^i$ and we choose $r(y,z)=y$, we obtain that

$$\begin{aligned}w_jy-yw_j&=\Big(\sum_{i=0}^{p}w_{ji}(z)y^i\Big)y-y\Big(\sum_{i=0}^{p}w_{ji}(z)y^i\Big)\\&=\sum_{i=0}^{p}w_{ji}(z)y^{i+1}-(w_{ji}(z)y+\delta_1(w_{ji}(z)))y^i\\&=-\sum_{i=0}^{p}\delta_1(w_{ji}(z))y^i=0,\end{aligned}$$

then we get that $\delta_1(w_{ji}(z))=0$, and so, $w_{ji}(z)=w_{ji}\in\mathbb{C}$. Now, again by Proposition 2.3.2 we have

$$y^iz=\sum_{k=0}^{i}\binom{i}{k}zy^k.$$

Then, since $w_jr(y,\beta^{-(j+1)}z)-r(y,\beta^{-2}z)w_j=0$, for all $0\leq j\leq m$, if we choose $r(x,z)=z$, we get

$$\begin{aligned}0=w_j\beta^{-(j+1)}z-\beta^{-2}zw_j&=\Big(\sum_{i=0}^{p}w_{ji}\beta^{-(j+1)}y^iz\Big)-\Big(\sum_{i=0}^{p}\beta^{-2}zw_{ji}y^i\Big)\\&=\sum_{i=0}^{p}w_{ji}\beta^{-(j+1)}\Big(\sum_{k=0}^{i}\binom{i}{k}zy^k\Big)-\sum_{i=0}^{p}\beta^{-2}zw_{ji}y^i\\&=\sum_{k=0}^{p}\Big[\sum_{i=k}^{p}w_{ji}\beta^{-(j+1)}\binom{i}{k}zy^k)\Big]-\sum_{i=0}^{p}\beta^{-2}zw_{ji}y^i\\&=\sum_{k=0}^{p}\Big[\Big(\sum_{i=k}^{p}w_{ji}\beta^{-(j+1)}\binom{i}{k}z\Big)-\beta^{-2}zw_{jk}\Big]y^k,\end{aligned}$$

and then, we obtain that, for all $0 \leq k \leq p$,

$$\left(\sum_{i=k}^{p} w_{ji}\beta^{-(j+1)}\binom{i}{k}z\right) - \beta^{-2}zw_{jk} = 0.$$

In particular, for $k = p$, we obtain that $w_{jp}(\beta^{-(j+1)} - \beta^{-2}) = 0$, and then $w_{jp} = 0$. With this, if we suppose that $w_{jk} = 0$, for all $n < k \leq p$, then for $k = n$, we obtain that $i = n$ and $w_{jn}(\beta^{-(j+1)} - \beta^{-2}) = 0$, which implies that $w_{jn} = 0$. Therefore $w_j = 0$. □

Case *5.(b)* : Since $\sigma_1 = id_{\mathbb{C}[y,z]}$ and $R = \mathbb{C}[y,z]$ is a domain, we have the following proposition obtaining during the realization of this work.

Proposition 2.3.12. *If $\sigma : R[x,\sigma_1,\delta_1] \to R[x,\sigma_1,\delta_1]$ is an algebra endomorphism of a 3-dimensional skew polynomial algebra of type 5(b) such that $\sigma|_R = \sigma_1 = id_{\mathbb{C}[y,z]}$ then $\sigma(x) = x + g(z,y)$ with $g(z,y) \in \mathbb{C}[z,y]$.*

Proof. Since the equations (2.3.2) in this case are given by

$$\left[\sum_{i=j}^{m}\binom{i}{j}w_i\delta^{i-j}(r)\right] - rw_j = 0, \quad \forall j \geq 1,$$

$$\left[\sum_{i=0}^{m} w_i\delta^{i}(r)\right] - rw_0 - \delta(r) = 0 \quad j = 0,$$

with $j = m-1$, and using that R is commutative, we obtain that $mw_m\delta_1(r) = 0$, for all $r \in R$, and by the fact that R is a domain, $w_m = 0$. If we suppose that $w_k = 0$, for all $n < k \leq m$, with $n \geq 2$, then $nw_m\delta_1(r) = 0$, for all $r \in R$, so $w_n = 0$. This fact implies that $w_2 = \cdots = w_m = 0$. For $j = 0$, by the commutativity of R, we get that $w_1\delta_1(r) = \delta_1(r)$, for all $r \in R$, whence $w_1 = 1$ and $w_0 \in R$ is free.

□

Case *5.(c)* : We have that $R = \mathbb{C}[y,z]$. Since for all $r = \sum_{t=0}^{q} f_t(z)y^t \in R$ we have $\delta_1(\sum_{t=0}^{q} f_t(z)y^t) = \sum_{t=1}^{q} f_t(z)tby^{t-1}$, then

$$\delta_1^s\left(\sum_{t=0}^{q} f_t(z)y^t\right) = \sum_{t=s}^{q} f_t(z)P_s^t b^s y^{t-s},$$

where $P_s^t = \prod_{k=t-s}^{t} k$. Just like the case of *5.(b)*, since R is a domain and $\sigma_1 = id_R$, we have the following proposition; its proof it is similar to the proof of Proposition 2.3.12.

Proposition 2.3.13. *If $\sigma : R[x;\sigma_1,\delta_1] \to R[x;\sigma_1,\delta_1]$ is a ring endomorphism of an algebra of type 5(c), such that $\sigma|_R = \sigma_1$, then $\sigma(x) = x + r(y,z)$, where $r(y,z) \in \mathbb{C}[y,z]$.*

Case *5(d)*: Since in this case, R is a domain, $\sigma_1 = id_{\mathbb{C}[x,y]}$ and $\delta \neq 0$, in the realization of this document, we obtain an analogous to Proposition 2.3.12 as follows.

Proposition 2.3.14. *If $\sigma : R[z;\sigma_1,\delta_1] \to R[z;\sigma_1,\delta_1]$ is a ring endomorphism of an algebra of type 5(d), such that $\sigma|_R = \sigma_1$, then $\sigma(z) = z + r(x,y)$, where $r(x,y) \in \mathbb{C}[x,y]$.*

We have also that in this case, for all $r(x,y) = \sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij}x^iy^j \in \mathbb{C}[x,y]$, the derivation δ_1 takes the value

$$\delta_1(r) = \sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij}(i+j)x^iy^j + c_{ij}ix^{i-1}y^{j+1}.$$

Case *5(e)*: Since $\sigma_2 = id_{R[z,\sigma_1,\delta_1]}$, we obtain the following from equations (2.3.2):

$$\begin{aligned} &\left[\sum_{i=j}^{m}\binom{i}{j}w_i\delta^{i-j}(r) - rw_j\right] = 0, \quad \forall j \geq 1, \\ &\left[\sum_{i=0}^{m} w_i\delta^i(r) - rw_0\right] - \delta(r) = 0, \end{aligned}$$

and with this, we obtain the following proposition.

Proposition 2.3.15. *If $\sigma : R[z;\sigma_1,\delta_1][y;\sigma_2,\delta_2] \to R[z;\sigma_1,\delta_1][y;\sigma_2,\delta_2]$ is an algebra endomorphism of a skew polynomial algebra of type 5(e) such that $\sigma|_{R[z;\sigma_1,\delta_1]} = \sigma_2 = id$, then $\sigma(y) = y + c$ with $c \in \mathbb{C}$.*

Proof. In the case $j = m-1$, we obtain that $mw_m\delta_2(r) + w_{m-1}r - rw_{m-1} = 0$, for all $r \in R[z;\sigma_1,\delta_1]$. If we choose $r = x$, since $\delta_2(x) = 0$ and saying that $w_{m-1} = \sum_{k=0}^{q} b_k(x)z^k$, we obtain that

$$\begin{aligned} w_{m-1}x - xw_{m-1} &= \left(\sum_{k=0}^{q} b_k(x)z^kx\right) - \left(\sum_{k=0}^{q} xb_k(x)z^k\right) \\ &= \left(\sum_{k=0}^{q}\sum_{i=0}^{k}\binom{k}{i}b_k(x)xz^i\right) - \left(\sum_{k=0}^{q} xb_k(x)z^k\right) \\ &= \left(\sum_{i=0}^{q}\sum_{k=i}^{q}\binom{k}{i}b_k(x)xz^i\right) - \left(\sum_{i=0}^{q} xb_i(x)z^i\right), \end{aligned}$$

where the second equation is due to Proposition 2.3.4. Then, for each $i \in \{0,...,q-1\}$, we get

$$\sum_{k=i+1}^{q}\binom{k}{i}b_k(x) = 0.$$

Then, with $i = q-1$, we obtain that $b_q(x) = 0$; with $i = q-2$, we get $b_{q-1}(x) = 0$, and following in that way we obtain that $b_1 = \cdots = b_m = 0$ and $w_{m-1} = b_0(x) \in \mathbb{C}[x]$. Then, we have that $mw_m\delta_2(r) + b_o(x)r - rb_0(x) = 0$, for all $r \in R[z;\sigma_1,\delta_1]$. If $r = z$ we get

$$\begin{aligned} mw_m\delta_2(z) + b_0(x)z - zb_0(x) &= mw_maz + b_o(x)z - (b_0(x)z + \delta_1(b_0(x)) \\ &= mw_maz - \delta_1(b_0(x)) = 0, \end{aligned}$$

and so $mw_maz = \delta_1(b_0(x)) \in R$, which implies that $w_m = 0$. With this $\delta_1(b_0(x)) = 0$, and hence $w_{m-1} = b_0(x) \in \mathbb{C}$.

If we suppose that $w_k = 0$, for all $n+1 < k \leq m$ with $n \geq 1$ and $w_{n+1} \in \mathbb{C}$, we obtain in $j = n$ that,

$$(n+1)w_{n+1}\delta_2(r) + w_n r - r w_n = 0.$$

With $r = x$ we obtain $w_n x - x w_n = 0$, and so, like in a previous argument, that $w_n = b(x) \in \mathbb{C}[x]$. Since $(n+1)w_{n+1}\delta_2(r) + b(x)r - rb(x) = 0$, for all $r \in R[z; \sigma_1, \delta_1]$, for $r = z$, we obtain $(n+1)w_{n+1}\delta_2(r) = \delta_1(b(x))$, which implies that $w_{n+1} = 0$ and $w_n = b(x) \in \mathbb{C}$. Then $w_2 = \cdots = w_m = 0$ and $w_1 \in \mathbb{C}$. This shows that with $j = 0$,

$$w_1\delta_2(r) + w_0 r - r w_0 - \delta_2(r) = 0.$$

If $r = x$, we obtain that $w_0 r - r w_0 = 0$ and therefore $w_0 = w_0(x) \in \mathbb{C}[x]$. In the case $r = z$,

$$\begin{aligned} w_1\delta_2(r) + w_0 r - r w_0 - \delta_2(r) &= (w_1 - 1)az + w_0(x)z - z w_0(x) \\ &= (w_1 - 1)az - \delta_1(w_0(x)), \end{aligned}$$

and this let us conclude that $w_1 = 1$ and $w_0 \in \mathbb{C}$. □

Now, the equations to find the possible values of the extended σ, in the case of Ore extensions such that the automorphism and the skew derivation are not commutative, we mean the algebras of type *2(c)*, *2(d)* and *4* are

$$0 = \sum_{i=0}^{m} w_i x^i \sigma(r) - \sigma^2(r)\left(\sum_{i=0}^{m} w_i x^i\right) - \sigma(\delta(r)). \tag{2.3.3}$$

In this way, in each case, we get the following.

Case *2(c)*: Since $R = \mathbb{C}[y, z]$, $xy = yx$ and $xz = \beta^{-1}zx - \beta^{-1}y$, then, in this work, we obtain the following two propositions.

Proposition 2.3.16. *In the 3-dimensional skew polynomial algebra of type 2.(c), it is satisfied that*

$$x^i z = \beta^{-i} z x^i - \theta_i y x^{i-1}, \text{ where } \theta_i = \frac{\beta^{-(i+1)} - \beta^{-1}}{\beta^{-1} - 1}.$$

Proof. We proceed by induction. The definition of the algebras shows that the proposition holds for $n = 1$. If we suppose true for i, we get

$$\begin{aligned} x^{i+1} z &= x(\beta^{-i} z x^i - \theta_i y x^{i-1}) \\ &= \beta^{-i}(\beta^{-1} z x - \beta^{-1} y) x^i - \theta_i y x^i \\ &= \beta^{-(i+1)} z x^{i+1} - (\beta^{-(i+1)} + \theta_i) y x^i \\ &= \beta^{-(i+1)} z x^{i+1} - \theta_{i+1} y x^i, \end{aligned}$$

which shows it is true for $i+1$ too, concluding the proof. □

Proposition 2.3.17. *If $\sigma : R[x;\sigma_1,\delta_1] \to R[x;\sigma_1,\delta_1]$ is an algebra endomorphism of a 3-dimensional skew polynomial algebra of type 2.(c), with not a root of unity, such that $\sigma|_R = \sigma_1$, then there exists $q(y,z) \in R$ such that*

$$\sigma(x) = yq(y,z) + [(\beta-1)zq(y,z) - \beta]x.$$

Proof. In equations 2.3.3, with $r = z$, we get that

$$\begin{aligned}
0 &= \sum_{i=0}^{m} w_i x^i \sigma(z) - \sigma^2(z)(\sum_{i=0}^{m} w_i x^i) - \sigma(\delta(z)) = \sum_{i=0}^{m} w_i x^i \beta^{-1} z - \beta^{-2} z(\sum_{i=0}^{m} w_i x^i) + \beta^{-1}\sigma(y) \\
&= \sum_{i=0}^{m} w_i \beta^{-1}(\beta^{-i} z x^i - \theta_i y x^{i-1}) - (\sum_{i=0}^{m} \beta^{-2} z w_i x^i) + \beta^{-1} y \\
&= \sum_{i=0}^{m} w_i(\beta^{-(i+1)} - \beta^{-2}) z x^i - \sum_{i=0}^{m} w_i \beta^{-1}\theta_i y x^{i-1} + \beta^{-1} y \\
&= \sum_{i=0}^{m} w_i(\beta^{-(i+1)} - \beta^{-2}) z x^i - \sum_{i=0}^{m-1} w_{i+1} \beta^{-1}\theta_{i+1} y x^{i} + \beta^{-1} y \\
&= w_m(\beta^{-(m+1)} - \beta^{-2}) z x^m + \sum_{i=0}^{m-1} [w_i(\beta^{-(i+1)} - \beta^{-2}) z - w_{i+1}\beta^{-1}\theta_{i+1} y] x^i + \beta^{-1} y \\
&= w_m(\beta^{-(m+1)} - \beta^{-2}) z x^m + w_0(\beta^{-1} - \beta^{-2}) z - w_1 \beta^{-1}\theta_1 y + \beta^{-1} y \\
&- w_2 \beta^{-1}\theta_2 y x + \sum_{i=2}^{m-1} [w_i(\beta^{-(i+1)} - \beta^{-2}) z - w_{i+1}\beta^{-1}\theta_{i+1} y] x^i.
\end{aligned}$$

Since $\{x^l : l \in \mathbb{N}\}$ is an R-base, we get, by the fact β is not a root of unity, that $w_m = 0$. Then $w_{m-1}(\beta^{-m} - \beta^{-2})z = 0$, and so $w_{m-1} = 0$, again because β is not a root of unity. In the same way, we obtain that $w_2 = w_3 = \cdots = w_m = 0$. Also, we have

$$0 = w_0(\beta^{-1} - \beta^{-2})z - (w_1\theta_1 + 1)\beta^{-1} y.$$

Then, since $w_0(1-\beta^{-1})z = (w_1\theta_1 + 1)y$, if $w_0 = \sum_{i=0}^{p} f_k(z) y^k$ and $w_1 = \sum_{i=0}^{p'} g_k(z) y^k$ then

$$\sum_{i=0}^{p} (1-\beta^{-1}) z f_k(z) y^k = \sum_{i=0}^{p'} \theta_1 g_k(z) y^{k+1} + y = \left[\sum_{i=2}^{p'+1} \theta_1 g_{k-1}(z) y^k\right] + (\theta_1 g_0(z) + 1) y.$$

From this we get that $f_0(z) = 0$, $(1-\beta^{-1}) z f_1(z) = \theta_1 g_0(z) + 1$ which is equivalent to $(\beta - 1) z f_1(Z) - \beta = g_0(z)$, also we deduce $p = p' + 1$ and $(\beta - 1) z f_{k+1}(Z) = g_k(z)$, for $k = 1, ..., p-1$. With this,

$$w_0 = \sum_{i=1}^{p} f_k(z) y^k, \qquad w_0 = \sum_{i=0}^{p-1} (\beta - 1) z f_{k+1}(z) y^k - \beta.$$

With these descriptions of the coefficients w_i's, we get that the automorphism satisfies

$$\begin{aligned}\sigma(x) &= \sum_{i=1}^{p} f_k(z)y^k + \left[\sum_{i=0}^{p-1}(\beta-1)zf_{k+1}(z)y^k - \beta\right]x\\ &= \left[\sum_{i=0}^{p-1} f_{k+1}(z)y^k\right]y + \left[(\beta-1)z\sum_{i=0}^{p-1} f_{k+1}(z)y^k - \beta\right]x.\end{aligned}$$

Taking $q(y,z) = \sum_{i=0}^{p-1} f_{k+1}(z)y^k \in \mathbb{C}[y,z]$, we conclude the proof. □

Corollary 2.3.18. *There exists an unique automorphism of* $R[x;\sigma_1,\delta_1]$ *such that* $\sigma|_R = \sigma_1$*, and it is the automorphism defined by* $\sigma(x) = -\beta x$*.*

Case *2(d)*: Since $R = \mathbb{C}[y,z]$, $xy = yx$ and $xz = \beta^{-1}zx - \beta^{-1}b$, then we have the following result obtained during the realization of this work.

Proposition 2.3.19. *In the 3-dimensional skew polynomial algebra of type 2.(c), it is satisfied that*

$$x^i z = \beta^{-i}zx^i - \theta_i b x^{i-1}, \qquad \textit{where } \theta_i = \frac{\beta^{-(i+1)} - \beta^{-1}}{\beta^{-1}-1}.$$

Proof. It is analogous to the proof of Proposition 2.3.16. □

Proposition 2.3.20. *If* $\sigma : R[x;\sigma_1,\delta_1] \to R[x;\sigma_1,\delta_1]$ *is an algebra endomorphism of a 3-dimensional skew polynomial algebra of type 2.(d), with not a root of unity, such that* $\sigma|_R = \sigma_1$*, then*

1. *if* $b \neq 0$*, then there exists* $q(y,z) \in R$ *such that*

$$\sigma(x) = q(y,z) + [(\beta-1)zb^{-1}q(y,z) - \beta]x.$$

2. *if* $b = 0$*, then there exists* $q(y,z) \in R$ *such that,* $\sigma(x) = q(y,z)x$*.*

Proof. Such as in the proof of Proposition 2.3.17, we get that $w_2 = w_3 = \cdots = w_m = 0$, and in this case we obtain that $w_0(1-\beta^{-1})z = (w_1\theta_1 + 1)b$, which is equivalent, when $b \neq 0$, to $w_0(\beta-1)zb^{-1} - \beta = w_1$ □

Corollary 2.3.21. *If* $b \neq 0$*, then there exists an unique automorphism of* $R[x;\sigma_1,\delta_1]$ *such that* $\sigma|_R = \sigma_1$*, and it is the automorphism defined by* $\sigma(x) = -\beta x$*.*

Case *4*: Since $R = \mathbb{C}[x]$, $R[y;\sigma_1,\delta_1][z;\sigma_2,\delta_2]$, $\sigma_1(x) = \alpha^{-1}x$, $\delta_1(x) = -\alpha^{-1}(a_3x + b_3)$, $\sigma_2(y) = \alpha^{-1}y$, $\delta_2(y) = -\alpha^{-1}(a_1x_1 + b_1)$, $\sigma_2(x) = \alpha x$, $\delta_2(x) = a_2x + b_2$. In this work we obtain the following result.

Proposition 2.3.22. *In this skew polynomial algebra of type 4, we obtain the commutation rule*

$$\begin{aligned} z^{n+1}x &= \alpha^{n+1}xz^{n+1} + \sum_{i=0}^{n}\left[\binom{n+1}{i}\alpha^i a_2^{n+1-i}x + b_2 P_i^n a_2^{n-i}\right]z^i \\ &= x\left[\sum_{i=0}^{n+1}\binom{n+1}{i}(\alpha z)^i a_2^{n+1-i}\right] + b_2\left[\sum_{i=0}^{n} P_i^n a_2^{n-i} z^i\right], \end{aligned} \tag{2.3.4}$$

$$\alpha^{n+1}y^{n+1}x = xy^{n+1} + \sum_{i=0}^{n}\left[\binom{n+1}{i}a_3^{n+1-i}x + T_i^n a_3^{n-i}b_3\right](-1)^{n+1-k}y^i, \tag{2.3.5}$$

where $P_i^n, T_i^n \in \mathbb{C}$ *are defined as*

$$P_i^n = \sum_{k=0}^{i}\binom{n+k-i}{n-i}\alpha^k, \qquad T_i^n = \sum_{k=0}^{i}\binom{n+k-i}{n-i}\alpha^{i-k}.$$

Proof. In case of equation (2.3.4), we obtain by a short computation that

$$(\alpha z + a_2)\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha z)^i a_2^{n-i}\right) = \sum_{i=0}^{n+1}\binom{n+1}{i}(\alpha z)^i a_2^{n+1-i}, \tag{2.3.6}$$

$$\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha z)^i a_2^{n-i}\right) + z\left(\sum_{i=0}^{n-1} a_2^{n-1-i}P_i^{n-1}z^i\right) = \sum_{i=0}^{n} P_i^n a_2^{n-i} z^i. \tag{2.3.7}$$

Then, we proceed by induction. For $n = 0$, it is due to the definition of the algebra. If we suppose that is true for $n-1$,

$$\begin{aligned} z^{n+1}x &= zx\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha z)^i a_2^{n-i}\right) + zb_2\left(\sum_{i=0}^{n-1}P_i^{n-1}a_2^{n-1-i}z^i\right) \\ &= (\alpha xz + a_2 x)\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha z)^i a_2^{n-i}\right) + b_2\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha z)^i a_2^{n-i}\right) + zb_2\left(\sum_{i=0}^{n-1}P_i^{n-1}a_2^{n-1-i}z^i\right) \\ &= x\left[\sum_{i=0}^{n+1}\binom{n+1}{i}(\alpha z)^i a_2^{n+1-i}\right] + b_2\left[\sum_{i=0}^{n}P_i^n a_2^{n-i}z^i\right], \end{aligned}$$

where, the last equation is due to equations (2.3.6) and (2.3.7), finishing the proof of expression (2.3.4). Now, in case of equation (2.3.5), for $n = 0$, we get $\alpha yx = xy - a_3 x - b_3$

which is true by definition of the algebra. If we suppose for $n-1$, then

$$\begin{aligned}
(\alpha yx)^{n+1} &= \alpha y(xy^n + \sum_{i=0}^{n-1}\left[\binom{n}{i}a_3^{n-i}x + T_i^{n-1}a_3^{n-1-i}b_3\right](-1)^{n-i}y^i) \\
&= (xy - a_3x - b_3)y^n + \sum_{i=0}^{n-1}\left[\binom{n}{i}a_3^{n-i}(xy - a_3x - b_3) + \alpha T_i^{n-1}a_3^{n-1-i}b_3y\right](-1)^{n-i}y^i) \\
&= xy^{n+1} - a_3xy^n - b_3y^n + \sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i}a_3^{n-i}xy^{i+1} + \sum_{i=0}^{n-1}(-1)^{n+1-i}\binom{n}{i}a_3^{n-i}a_3xy^i \\
&+ \sum_{i=0}^{n-1}(-1)^{n+1-i}\binom{n}{i}a_3^{n-i}b_3y^i + \sum_{i=0}^{n-1}(-1)^{n-i}\alpha T_i^{n-1}a_3^{n-1-i}b_3y^{i+1} \\
&= xy^{n+1} - a_3(n+1)xy^n + (-1)^{n+1}a_3^{n+1}x + (-1)^{n+1}a_3^nb_3 - (1+\alpha T_{n-1}^{n-1})b_3y^n \\
&+ \sum_{i=0}^{n-2}(-1)^{n-i}\binom{n}{i}a_3^{n-i}xy^{i+1} + \sum_{i=1}^{n-1}(-1)^{n+1-i}\binom{n}{i}a_3^{n+1-i}xy^i \\
&+ \sum_{i=1}^{n-1}(-1)^{n+1-i}\binom{n}{i}a_3^{n-i}b_3y^i + \sum_{i=0}^{n-2}(-1)^{n-i}\alpha T_i^{n-1}a_3^{n-1-i}b_3y^{i+1} \\
&= xy^{n+1} - a_3(n+1)xy^n + (-1)^{n+1}a_3^{n+1}x + (-1)^{n+1}a_3^nb_3 - (1+\alpha T_{n-1}^{n-1})b_3y^n \\
&+ \sum_{i=1}^{n-1}(-1)^{n+1-i}\left[\binom{n}{i-1} + \binom{n}{i}\right]a_3^{n+1-i}xy^i + \sum_{i=1}^{n-1}(-1)^{n+1-i}\left[\binom{n}{i} + \alpha T_i^{n-1}\right]a_3^{n-i}b_3y^i \\
&= xy^{n+1} + \sum_{i=0}^{n}(-1)^{n+1-i}\binom{n+1}{i}a_3^{n+1-i}xy^i + \sum_{i=0}^{n}(-1)^{n+1-i}T_i^na_3^{n-i}b_3y^i,
\end{aligned}$$

where $\binom{n}{i} + \alpha T_{i-1}^{n-1} = T_i^n$, for all $i \leq n$. □

Remark 2.3.23. In this work, we found complications to solve by inspection the question of this section in these algebras of type *4*, but we want to show the following advance in the description of the endomorphisms of these algebras of type *4*.

With $r = x$ in equation (2.3.3), we obtain,

$$\begin{aligned}
0 &= \sum_{i=0}^{m} w_i z^i \alpha x - \alpha^2 x(\sum_{i=0}^{m} w_i z^i) - \sigma(a_2 x + b_2) \\
&= \sum_{i=0}^{m-1}[w_{i+1}\alpha z^{i+1}x - \alpha^2 x w_{i+1} z^{i+1}] + w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2 \\
&= \sum_{i=0}^{m-1}[w_{i+1}\alpha\Bigg(\alpha^{i+1}xz^{i+1} + \sum_{t=0}^{i}\Bigg[\binom{i+1}{t}\alpha^t a_2^{i+1-t}x + b_2 P_t^i a_2^{i-t}\Bigg]z^t\Bigg) - \alpha^2 x w_{i+1} z^{i+1}] \\
&+ w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2 \\
&= \sum_{t=0}^{m-1} w_{t+1}\alpha^{t+2} x z^{t+1} + \sum_{t=0}^{m-1}\sum_{i=t}^{m-1}\Bigg(w_{i+1}\alpha\Bigg[\binom{i+1}{t}\alpha^t a_2^{i+1-t}x + b_2 P_t^i a_2^{i-t}\Bigg]z^t\Bigg) - \sum_{t=0}^{m-1}\alpha^2 x w_{t+1} z^{t+1} \\
&+ w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2 \\
&= \sum_{t=0}^{m-1}(w_{t+1}\alpha^{t+2}x - \alpha^2 x w_{t+1})z^{t+1} + \sum_{t=0}^{m-1}\sum_{i=t}^{m-1}\Bigg(w_{i+1}\alpha\Bigg[\binom{i+1}{t}\alpha^t a_2^{i+1-t}x + b_2 P_t^i a_2^{i-t}\Bigg]z^t\Bigg) \\
&+ w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2 \\
&= \sum_{t=1}^{m-1}(w_t\alpha^{t+1}x - \alpha^2 x w_t)z^t + \sum_{t=0}^{m-1}\sum_{i=t}^{m-1}\Bigg(w_{i+1}\alpha\Bigg[\binom{i+1}{t}\alpha^t a_2^{i+1-t}x + b_2 P_t^i a_2^{i-t}\Bigg]z^t\Bigg) \\
&+ (w_m\alpha^{m+1}x - \alpha^2 x w_m)z^m + w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2 \\
&= \sum_{t=1}^{m-1}\Bigg((w_t\alpha^{t+1}x - \alpha^2 x w_t) + \sum_{i=t}^{m-1} w_{i+1}\alpha\Bigg[\binom{i+1}{t}\alpha^t a_2^{i+1-t}x + b_2 P_t^i a_2^{i-t}\Bigg]\Bigg)z^t \\
&+ (w_m\alpha^{m+1}x - \alpha^2 x w_m)z^m + w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2
\end{aligned}$$

Then,

$$(w_m\alpha^{m+1}x - \alpha^2 x w_m) = 0,$$

$$(w_t\alpha^t x - \alpha x w_t) + \sum_{i=t}^{m-1} w_{i+1}\Bigg[\binom{i+1}{t}\alpha^t a_2^{i+1-t}x + b_2 P_t^i a_2^{i-t}\Bigg] = 0, \qquad \text{for all } 1 \leq t \leq m-1, \tag{2.3.8}$$

$$w_0 \alpha x - \alpha^2 x w_0 - \alpha a_2 x - b_2 = 0.$$

If $w_m = \sum_{n=0}^{p} f_n(x)y^n$,

$$\begin{aligned}
0 &= (w_m\alpha^{m+1}x - \alpha^2 x w_m)\\
&= \alpha^{m+1}f_0(x)x + \left(\sum_{n=0}^{p-1}\alpha^{m+1}f_{n+1}(x)y^{n+1}x\right) - \left(\sum_{n=0}^{p}\alpha^2 x f_n(x)y^n\right)\\
&= \left(\sum_{n=0}^{p-1}\alpha^{m+1-(n+1)}f_{n+1}(x)\left(xy^{n+1} + \sum_{i=0}^{n}\left[\binom{n+1}{i}a_3^{n+1-i}x + T_i^n a_3^{n-i}b_3\right](-1)^{n+1-k}y^i\right)\right)\\
&+ \alpha^{m+1}f_0(x)x - \left(\sum_{n=0}^{p}\alpha^2 x f_n(x)y^n\right)\\
&= \left(\sum_{i=0}^{p-1}\alpha^{m-i}f_{i+1}(x)xy^{i+1} + \sum_{i=0}^{p-1}\sum_{n=i}^{p-1}\alpha^{m-n}f_{n+1}(x)\left[\binom{n+1}{i}a_3^{n+1-i}x + T_i^n a_3^{n-i}b_3\right](-1)^{n+1-k}y^i\right)\right)\\
&+ \alpha^{m+1}f_0(x)x - \left(\sum_{i=0}^{p}\alpha^2 x f_i(x)y^i\right)\\
&= \sum_{i=1}^{p-1}\left(\alpha^{m+1-i}f_i(x)x - \alpha^2 x f_i(x) + \sum_{n=i}^{p-1}\alpha^{m-n}f_{n+1}(x)\left[\binom{n+1}{i}a_3^{n+1-i}x + T_i^n a_3^{n-i}b_3\right](-1)^{n+1-i}\right)y^i\\
&+ \alpha^{m+1-p}f_p(x)xy^p + \alpha^{m+1}f_0(x)x - \alpha^2 x f_p(x)y^p - \alpha^2 x f_0(x)\\
&+ \sum_{n=0}^{p-1}\alpha^{m-n}f_{n+1}(x)\left[a_3^{n+1}x + T_0^n a_3^n b_3\right](-1)^{n+1}.
\end{aligned}$$

Since $\{y^k : k \in \mathbb{N}\}$ is a left-basis of $R[y;\sigma_1,\delta_1]$, we get, for all $1 \leq i \leq p-1$, that

$$\begin{aligned}
\alpha^{m+1-i}f_i(x)x - \alpha^2 x f_i(x) + \sum_{n=i}^{p-1}\alpha^{m-n}f_{n+1}(x)\left[\binom{n+1}{i}a_3^{n+1-i}x + T_i^n a_3^{n-i}b_3\right](-1)^{n+1-i} &= 0,\\
(\alpha^{m+1} - \alpha^2)xf_0(x) + \sum_{n=0}^{p-1}\alpha^{m-n}f_{n+1}(x)\left[a_3^{n+1}x + T_0^n a_3^n b_3\right](-1)^{n+1} &= 0,\\
(\alpha^{m+1-p} - \alpha^2)xf_p(x)y^p &= 0.
\end{aligned}$$

Then, we have the following cases:

1. If $m > p+1$, since α is not a root of unity, we obtain $f_p(x) = 0$, and if $i = p-1$, we get

   $$\begin{aligned}
   0 &= \alpha^{m+2-p}f_{p-1}(x)x - \alpha^2 x f_{p-1}(x) + \alpha^{m-(p-1)}f_p(x)\left[\binom{p}{p-1}a_3x + T_{p-1}^{p-1}b_3\right](-1)^p\\
   &= (\alpha^{m+2-p} - \alpha^2)xf_{p-1}(x).
   \end{aligned}$$

   Again, since α is not a root of unity, and $m \neq p$, $f_{p-1}(x) = 0$. If $m = p$, then

   $$0 = (\alpha^2 - \alpha^2)xf_{p-1}(x) = -\alpha f_p(x)\left[pa_3x + T_{p-1}^{p-1}b_3\right](-1)^p,$$

   which implies that $f_p(x) = 0$. Following in this way, we get that $f_i(x) = 0$, for all $i = 1, ..., p$. Therefore, for $i = 0$, we conclude, $(\alpha^{m+1} - \alpha^2)xf_0(x) = 0$, which means that $f_0(x) = 0$. We conclude that $w_m = 0$. Then, in expression (2.3.8), for $t = m-1$,

   $$(w_{m-1}\alpha^{m-1}x - \alpha x w_{m-1}) = (w_{m-1}\alpha^{m-1}x - \alpha x w_{m-1}) + w_m\left[\binom{m}{m-1}\alpha^{m-1}a_2x + b_2P_{m-1}^{m-1}\right] = 0,$$

We note that this is equivalent to condition on w_m, whence $w_{m-1} = 0$. In the same way, we conclude, by expression (2.3.8), that $w_2 = \cdots = w_{m-1} = 0$, and, for $t = 1$, by expression 2.3.8, $w_1 x = x w_1$, and, for $t = 0$, w_0 is such that $w_0 x - \alpha x w_0 = a_2 x + \alpha^{-1} b_2$.

2. If $m = p+1$, since $(\alpha^{m+1-p} - \alpha^2) x f_p(x) y^p = 0 x f_p(x) y^p = 0$, we get that $f_p(x) \in \mathbb{C}[x]$ can be chosen freely. Hence, for $i = p-1$, we get

$$0 = \alpha^{m+2-p} f_{p-1}(x) x - \alpha^2 x f_{p-1}(x) + \alpha^{m-(p-1)} f_p(x) \left[\binom{p}{p-1} a_3 x + T_{p-1}^{p-1} b_3 \right] (-1)^p,$$

which is equivalent to

$$(1-\alpha) x f_{p-1}(x) = f_p(x) \left[p a_3 x + T_{p-1}^{p-1} b_3 \right] (-1)^p.$$

.

For $i = p-2$, we get,

$$\begin{aligned} 0 &= (\alpha^{m+1-(p-2)} - \alpha^2) x f_{p-2}(x) - \alpha^{m-(p-2)} f_{p-1}(x) \left[\binom{p-1}{p-2} a_3 x + T_{p-2}^{p-1} b_3 \right] \\ &\quad + \alpha^{m-p-1} f_p(x) \left[\binom{p}{p-2} a_3^{p-(p-2)} x + T_{p-2}^{p-1} a_3 b_3 \right] (-1)^2 \\ &= (\alpha^2 - 1) x f_{p-2}(x) - \alpha f_{p-1}(x) \left[(p-1) a_3 x + T_{p-2}^{p-1} b_3 \right] \\ &\quad + f_p(x) \left[p(p-2) 2^{-1} a_3^2 x + T_{p-2}^{p-1} a_3 b_3 \right]. \end{aligned}$$

Then,

$$(1-\alpha^2) x f_{p-2}(x) = -\alpha f_{p-1}(x) \left[(p-1) a_3 x + T_{p-2}^{p-1} b_3 \right] + f_p(x) \left[p(p-2) 2^{-1} a_3^2 x + T_{p-2}^{p-1} a_3 b_3 \right].$$

Multiplying by $(1-\alpha)x$,

$$\begin{aligned} (1-\alpha)(1-\alpha^2) x^2 f_{p-2}(x) &= -\alpha (1-\alpha) f_{p-1}(x) \left[(p-1) a_3 x + T_{p-2}^{p-1} b_3 \right] \\ &\quad + (1-\alpha) x f_p(x) \left[p(p-2) 2^{-1} a_3^2 x + T_{p-2}^{p-1} a_3 b_3 \right] \\ &= -\alpha f_p(x) \left[p a_3 x + T_{p-1}^{p-1} b_3 \right] (-1)^p \left[(p-1) a_3 x + T_{p-2}^{p-1} b_3 \right] \\ &\quad + (1-\alpha) x f_p(x) \left[p(p-2) 2^{-1} a_3^2 x + T_{p-2}^{p-1} a_3 b_3 \right], \end{aligned}$$

which means that

$$\begin{aligned} (1-\alpha)(1-\alpha^2) x^2 f_{p-2}(x) &= f_p(x) \Bigg(-\alpha \left[p a_3 x + T_{p-1}^{p-1} b_3 \right] (-1)^p \left[(p-1) a_3 x + T_{p-2}^{p-1} b_3 \right] \\ &\quad + (1-\alpha) x \left[p(p-2) 2^{-1} a_3^2 x + T_{p-2}^{p-1} a_3 b_3 \right] \Bigg). \end{aligned}$$

In this way,

$$
\begin{aligned}
&\left((-1)^{p+1}\alpha\left[pa_3x+T_{p-1}^{p-1}b_3\right]\left[(p-1)a_3x+T_{p-2}^{p-1}b_3\right]+(1-\alpha)x\left[p(p-2)2^{-1}a_3^2x+T_{p-2}^{p-1}a_3b_3\right]\right)\\
&=(-1)^{p+1}\alpha\left(\left[pa_3x(p-1)a_3x+T_{p-1}^{p-1}b_3(p-1)a_3x\right]+\left[pa_3xT_{p-2}^{p-1}b_3+T_{p-1}^{p-1}b_3T_{p-2}^{p-1}b_3\right]\right)\\
&+\left[(1-\alpha)xp(p-2)2^{-1}a_3^2x+(1-\alpha)xT_{p-2}^{p-1}a_3b_3\right]\\
&=(-1)^{p+1}\alpha\left(\left[p(p-1)a_3^2x^2+(p-1)T_{p-1}^{p-1}b_3a_3x\right]+\left[pT_{p-2}^{p-1}b_3a_3x+T_{p-1}^{p-1}T_{p-2}^{p-1}b_3^2\right]\right)\\
&+\left[(1-\alpha)p(p-2)2^{-1}a_3^2x^2+(1-\alpha)T_{p-2}^{p-1}a_3b_3x\right]\\
&=(-1)^{p+1}\alpha p(p-1)a_3^2x^2+(-1)^{p+1}\alpha\left[(p-1)T_{p-1}^{p-1}+pT_{p-2}^{p-1}\right]b_3a_3x+(-1)^{p+1}\alpha T_{p-1}^{p-1}T_{p-2}^{p-1}b_3^2\\
&+(1-\alpha)p(p-2)2^{-1}a_3^2x^2+(1-\alpha)T_{p-2}^{p-1}a_3b_3x\\
&=\left[(-1)^{p+1}\alpha(p-1)+(1-\alpha)(p-2)2^{-1}\right]pa_3^2x^2\\
&+\left[(-1)^{p+1}\alpha\left[(p-1)T_{p-1}^{p-1}+pT_{p-2}^{p-1}\right]+(1-\alpha)T_{p-2}^{p-1}\right]b_3a_3x+(-1)^{p+1}\alpha T_{p-1}^{p-1}T_{p-2}^{p-1}b_3^2
\end{aligned}
$$

Now, we determine which of these extended endomorphisms are, in fact, automorphisms. The following observation is useful in this way.

Remark 2.3.24. If $\sigma : R[t,\sigma_1,\delta_1][s,\sigma_2,\delta_2] \to R[t,\sigma_1,\delta_1][s,\sigma_2,\delta_2]$ is an extended endomorphism of σ_2, such that $\sigma(s) = s+g$ with $g \in R[t,\sigma_1,\delta_1]$, then the extended endomorphism of σ_2, defined by $\sigma'(s) = s-g$, is the inverse of σ. Therefore σ is an automorphism.

Remark 2.3.25. From the several descriptions, that we have obtained in this work, we obtain the following facts:

1. The only extended automorphism is $\sigma(x) = gx$, with $g \in \mathbb{C}$, by the Corollary 2.3.9.

2.(c) The only extended automorphism is $\sigma(x) = -\beta x$ by the Corollary 2.3.18.

2.(d) The only extended automorphism is $\sigma(x) = -\beta x$ by the Corollary 2.3.21.

2.(e) All extended endomorphisms are automorphisms with $\sigma(y) = y+c$ by Remark 2.3.24.

2.(f) There is no exists extended automorphism in this case.

5.(b) All extended endomorphisms are automorphisms with $\sigma(x) = x+g(z,y)$ by Remark 2.3.24.

5.(c) All extended endomorphisms are automorphisms with $\sigma(x) = x+g(z,y)$ by Remark 2.3.24.

5.(d) All extended endomorphisms are automorphisms with $\sigma(x) = x+g(z,y)$ by Remark 2.3.24.

5.(e) All extended endomorphisms are automorphisms with $\sigma(y) = y+c$ by Remark 2.3.24.

Since we have now descriptions of extended automorphisms, we can search skew derivations of these automorphisms with the aim of constructing Brzezinski's differential calculi, in each case. Let us clarify that in iterated Ore extensions cases, we could also extended in two steps the automorphisms σ_1, in the same way as we did it with σ_2.

2.3.2 Smoothness of 3-dimensional skew polynomial algebras

In this section, we establish some comments about the verification techniques of smoothness that we studied in the first chapter applied to Ore extensions and other algebras. All of them are related with 3-dimensional skew polynomial algebras.

Remark 2.3.26. The 3-dimensional skew polynomial algebra of type *1* is differentially smooth (see Example 1.3.29). This is obtained using the commutative differential calculus that we obtain in the Brzezinski's calculus of Remark 1.2.12, when we define in the base ring $\mathbb{C}[x]$, $d : \mathbb{C}[x] \to \Omega^1(\mathbb{C}[x])$, $d(a) = d(x)\partial(a)$ using a derivation $\partial = a\partial_x$, where ∂_x is the usual partial derivation by x, and $a \in \mathbb{C}$.

Remark 2.3.27. For iterated algebras of type *2*(e), *2*(f) and *4*, the respective subalgebras obtained in the first iterations are subalgebras differentially smooth (see Examples 1.3.35, 1.3.37 and 1.3.36).

Remark 2.3.28. In this work, we showed that the 3-dimensional skew polynomial algebras of type *2*(f) are differentially smooth algebras (see Example 1.3.38).

Now, we mention the following observation obtained in this work.

Remark 2.3.29. In order to apply the Theorem 1.3.27 to deduce the differentially smoothness of the subalgebra $R[w; \sigma_1, \delta_1]$ in the algebras of type *1* and *2*(e), where $R = \mathbb{C}[x]$, the Brzezinski's calculus constructed for R in Example 1.3.15 cannot be used, this because it satisfies the conditions needed only in the case that $\partial = aJ_{q^{-1}}$ with $q = 1$, i.e., when $\Omega^1\mathbb{C}[x] = 0$, which contradicts the dimensional condition of Definition 1.3.14: the hypothesis of Theorem 1.3.27 ask for an extended degree-preserving automorphism σ of ΩR such that $\sigma|_R = \sigma_1$, such that $\sigma \circ d = d \circ \sigma$. For $x \in \mathbb{C}[x]$, we have

$$d \circ \sigma(x) = d(\beta x) = \beta\partial(x)\omega = \beta a J_{q^{-1}}(x)\omega = \beta a\omega.$$

Since, $\sigma \circ d(x) = \sigma(\partial(x)\omega) = \sigma(a\omega) = \sigma(a)\sigma(\omega) = a\sigma(\omega)$, we need that $\sigma(\omega) = \beta\omega$. With this we observe,

$$\begin{aligned} d \circ \sigma(x^2) = \beta^2[d(x)\sigma'(x) + xd(x)] &= \beta^2[aJ_{q^{-1}}(x)\omega\sigma'(x) + xaJ_{q^{-1}}(x)\omega] \\ &= \beta^2[ax\omega + xa\omega] \\ &= \beta^2 2ax\omega, \end{aligned}$$

and $\sigma \circ d(x^2) = \sigma(aJ_{q^{-1}}(x^2)\omega) = \sigma(aJ_{q^{-1}}(x^2))\sigma(\omega) = a(q^{-1}+1)\beta^2 x\omega$. Then, we also need that $q^{-1} + 1 = 2$, that is equivalent to $q = 1$.

Therefore, to apply the Theorem 1.3.27 in this case using the strategy of Remark 1.2.12, we need to find a φ-skew derivation of $\mathbb{C}[x]$, with φ being an automorphism that does not send x to a multiple of it.

We finish highlight the restricted scope of Theorems 1.3.27 and 1.3.28, because these theorems need that Ore extensions satisfy that their derivations being zero. For establish the differential smoothness of the Ore algebras treated in this section, we need a way to heritage the smoothness of the base ring R, to the Ore extension $R[w;\sigma,\delta]$, even when $\delta \neq 0$. With this in mind, in chapter 1 we obtain the following examples.

Remark 2.3.30. In the case of the algebra of type *2*(d), we obtain in this work that it is differentially smooth (see Example 1.3.40).

Remark 2.3.31. Although the algebras of types *2*(c) and *2*(d) looks similar, we cannot deduce the differentially smoothness of *2*(c) as we obtained for *2*(d). Actually, algebras of type 2(c) are no differentially smooth (see Example 1.3.41).

Remark 2.3.32. We want to say that, following the ideas of Example 1.3.41, Remarks 2.1.8 and 2.2.21, we can discard the differentially smoothness of 3-dimensional skew polynomial algebras of type *2*(a)*,* *2*(c)*,* *3*(a)*,* *4,* *5*(a)*,* *5*(b) and *5*(d).

Remark 2.3.33. In this work, we obtained the differentially smoothness of the algebras of type *5*(c) (see Example 1.3.43).

Remark 2.3.34. In this work, we have established the differentially smoothness of algebras of type *2*(e) (see Example 1.3.45).

Remark 2.3.35. During the realization of this work, we guarantee the differential smoothness of the generalized Weyl algebras of type *2*(b) and *3*(b) (see Example 1.3.46).

Remark 2.3.36. However for the algebra *5(e)*, we cannot conclude his differentially smoothness by this way, because applying d to $zx - x(z+1) = 0$,

$$\begin{aligned} d(z)x + zd(x) - d(x)(z+1) - xd(z) &= 0 \\ d(z)x + d(x)\nu_x(z) - d(x)(z+1) - d(z)\nu_z(x) &= 0 \\ d(z)[x - \nu_z(x)] + d(x)[\nu_x(z) - (z+1)] &= 0 \end{aligned}$$

By the right free structure of Ω^1, we must have that $\nu_z(x) = x$ and $\nu_x(z) = z+1$. In the same way, with $xy - yx = 0$ we need $\nu_x(y) = y$ and $\nu_y(x) = x$. Hence, ν_x respect $zy - (y+a)z = 0$ if and only if $a = 0$, i.e., if $a \neq 0$, we cannot define a left structure on $\Omega^1(\mathcal{A})$ such that allows exist a well defined derivation $d : \mathcal{A} \to \Omega^1(\mathcal{A})$, which is our main trouble about the differentially smoothness of *5(e)*.

We close the section summarizing some of the previous facts about 3-dimensional skew polynomial algebras in Table 2.1.

S-P-A	DF-A-Type 1	GWA	Ore	Diff. Smooth
1.	$\surd$	$\times$	$\surd$	$\surd$
2.(a).	$\times$	$\surd$	$\times$	$\times$
2.(b).	$\times$	$\surd$	$\times$	$\surd$
2.(c).	$\times$	$\times$	$\surd$	$\times$
2.(d).	$\times$	$\times$	$\surd$	$\surd$
2.(e).	$\surd$	$\times$	$\surd$	$\surd$
2.(f).	$\times$	$\times$	$\surd$	$\surd$
3.(a).	$\times$	$\surd$	$\times$	$\times$
3.(b).	$\times$	$\surd$	$\times$	$\surd$
4.	$\times$	$\times$	$\surd$	$\times$
5.(a).	$\times$	$\surd$	$\times$	$\times$
5.(b).	$\times$	$\times$	$\surd$	$\times$
5.(c).	$\times$	$\times$	$\surd$	$\surd$
5.(d).	$\times$	$\times$	$\surd$	$\times$
5.(e).	$\times$	$\times$	$\surd$	$--$

Table 2.1: 3-dimensional skew polynomial algebras.

2.3.3 Smoothness of some n-dimensional skew polynomial algebras

From the Examples 1.3.35, 1.3.40, 1.3.43 and 1.3.46, we note that the needed objects are automorphisms ν_{x_i} of $\mathcal{A}$ for each generator x_i, such that

1. these automorphisms must be allows define a differential $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ such that under the left action $ad(x_i) = d(x_i)\nu_{x_i}(a)$, for $a \in \mathcal{A}$ and $1 \leq i \leq n$.
2. for any pair of generators x_i and x_j, we need that $[\nu_{x_i}, \nu_{x_j}] = 0$.

In a general way, if $\mathcal{A}$ is an algebra generated by a set $\{x_1, ..., x_n\}$, under the conditions that, for all $1 \leq i < j \leq n$

$$x_i x_j - a_{ij} x_j x_i = b_{ij} x_i + c_{ij} x_j + e_{ij}, \quad \text{for all } a_{ij}, b_{ij}, c_{ij}, e_{ij} \in \mathbb{C},\ a_{ij} \neq 0, \tag{2.3.9}$$

in the way of defining d we need the conditions, for $i < j$, that $\nu_{x_i}(x_j) = a_{ij}^{-1} x_j - a_{ij}^{-1} b_{ij}$, $\nu_{x_j}(x_i) = a_{ij} x_i - c_{ij}$. Then, for any x_k, if $j > k$, we have that $\nu_{x_j}(x_k) = a_{kj} x_k - c_{kj}$, and if $j < k$, $\nu_{x_j}(x_k) = a_{jk}^{-1} x_k - a_{jk}^{-1} b_{jk}$. In order to establish if ν's automorphisms are well defined, we do the following.

1. If $j < k$, we have $\nu_{x_k}(x_j) = a_{jk} x_j - c_{jk}$, since $x_j x_k - a_{jk} x_k x_j = b_{jk} x_j + c_{jk} x_k + e_{jk}$, when we apply ν_{x_k} we get,

$$\begin{aligned} &(a_{jk} x_j - c_{jk})\nu_{x_k}(x_k) - a_{jk}\nu_{x_k}(x_k)(a_{jk} x_j - c_{jk}) \\ &= b_{jk}(a_{jk} x_j - c_{jk}) + c_{jk}\nu_{x_k}(x_k) + e_{jk}. \end{aligned}$$

If $\nu_{x_k}(x_k) = a_{kk} x_k - b_{kk}$, we get the expresion,

$$\begin{aligned} &(2a_{jk} a_{kk} c_{jk} - 2c_{jk} a_{kk}) x_k \\ &+ (a_{jk} b_{kk}(a_{jk} - 1) + a_{jk} b_{jk}(a_{kk} - 1)) x_j \\ &= (e_{jk} - b_{jk} c_{jk} - 2c_{jk} b_{kk}) - (a_{jk} a_{kk} e_{jk} - a_{jk} b_{kk} c_{jk}). \end{aligned}$$

Hence,

$$\begin{aligned} a_{jk}c_{jk} - c_{jk} = (a_{jk}-1)c_{jk} &= 0, \\ b_{kk}(a_{jk}-1) + b_{jk}(a_{kk}-1) &= 0, \\ (e_{jk} - b_{jk}c_{jk} - 2c_{jk}b_{kk}) - (a_{jk}a_{kk}e_{jk} - a_{jk}b_{kk}c_{jk}) &= 0. \end{aligned} \tag{2.3.10}$$

By the first equation or $a_{jk} = 1$ or $c_{jk} = 0$.

(a) If $a_{jk} = 1$, then in the second equation $b_{jk} = 0$ or $a_{kk} = 1$.
   i. If $b_{jk} = 0$, then the third equation turns to $(1 - a_{kk})e_{jk} - c_{jk}b_{kk} = 0$.
   ii. If $a_{kk} = 1$, then the third equation turns to $c_{jk}(b_{jk} + b_{kk}) = 0$.

(b) If $a_{jk} \neq 1$, then $c_{jk} = 0$, therefore the third equation turns to $e_{jk} - a_{jk}a_{kk}e_{jk} = e_{jk}(1 - a_{jk}a_{kk}) = 0$. Hence, if $e_{jk} \neq 0$, then $a_{kk} = a_{jk}^{-1}$.

2. Now, if $k < j$, we have $\nu_{x_k}(x_j) = a_{kj}^{-1}x_j - a_{kj}^{-1}b_{kj}$, since $x_kx_j - a_{kj}x_jx_k = b_{kj}x_k + c_{kj}x_j + e_{kj}$, when we apply ν_{x_k} we get,

$$\begin{aligned} &\nu_{x_k}(x_k)(a_{kj}^{-1}x_j - a_{kj}^{-1}b_{kj}) - a_{kj}(a_{kj}^{-1}x_j - a_{kj}^{-1}b_{kj})\nu_{x_k}(x_k) \\ &= b_{kj}\nu_{x_k}(x_k) + c_{kj}(a_{kj}^{-1}x_j - a_{kj}^{-1}b_{kj}) + e_{kj}. \end{aligned}$$

If $\nu_{x_k}(x_k) = a_{kk}x_k + b_{kk}$,

$$\begin{aligned} &(a_{kk}a_{kj}^{-1}c_{kj} + b_{kk}a_{kj}^{-1} - a_{kj}a_{kj}^{-1}b_{kk} - c_{kj}a_{kj}^{-1})x_j \\ &+ (a_{kk}a_{kj}^{-1}b_{kj} + a_{kj}a_{kj}^{-1}b_{kj}a_{kk} - a_{kk}a_{kj}^{-1}b_{kj} - b_{kj}a_{kk})x_k \\ &= -c_{kj}a_{kj}^{-1}b_{kj} + e_{kj} + b_{kj}b_{kk} - a_{kj}a_{kj}^{-1}b_{kj}b_{kk} - a_{kk}a_{kj}^{-1}e_{kj} + b_{kk}a_{kj}^{-1}b_{kj}, \end{aligned}$$

which is equivalent to the equations

$$\begin{aligned} (a_{kk}-1)c_{kj} + b_{kk}(1 - a_{kj}) &= 0, \\ e_{kj}(a_{kj} - a_{kk}) + (b_{kk} - c_{kj})b_{kj} &= 0. \end{aligned} \tag{2.3.11}$$

Now, if we apply ν_{x_k} to a relation that do not involve x_k, we mean to the relation between x_j and x_t, without lost of generality, with $i < t$, i.e., $x_jx_t - a_{jt}x_tx_j = b_{jt}x_j + c_{jt}x_t + e_{jt}$, if we apply ν_{x_k}, we get the following cases.

1. If $j < t < k$, we have that $\nu_{x_k}(x_j) = a_{jk}x_j - c_{jk}$ and $\nu_{x_k}(x_t) = a_{tk}x_t - c_{tk}$, we get

$$\begin{aligned} &(a_{jt}a_{tk}c_{jk} - c_{jk}a_{tk} - c_{jt}a_{tk} + a_{jk}a_{tk}c_{jt})x_t \\ &+ (a_{jt}c_{tk}a_{jk} - a_{jk}c_{tk} - b_{jt}a_{jk} + a_{jk}a_{tk}b_{jt})x_j \\ &= e_{jt} - b_{jt}c_{jk} - c_{jt}c_{tk} - c_{jk}c_{tk} + a_{jt}c_{tk}c_{jk} - a_{jk}a_{tk}e_{jt}, \end{aligned}$$

which is equivalent to,

$$\begin{aligned} (a_{jt}-1)c_{jk} + (a_{jk}-1)c_{jt} &= 0, \\ (a_{jt}-1)c_{tk} + (a_{tk}-1)b_{jt} &= 0, \\ e_{jt} - b_{jt}c_{jk} - c_{jt}c_{tk} - c_{jk}c_{tk} + a_{jt}c_{tk}c_{jk} - a_{jk}a_{tk}e_{jt} &= 0. \end{aligned} \tag{2.3.12}$$

2. If $j < k < t$, we have that $\nu_{x_k}(x_j) = a_{jk}x_j - c_{jk}$ and $\nu_{x_k}(x_t) = a_{kt}^{-1}x_t - a_{kt}^{-1}b_{kt}$, then
$$\begin{aligned}&(a_{jt}a_{kt}^{-1}c_{jk} - c_{jk}a_{kt}^{-1} - c_{jt}a_{kt}^{-1} + a_{jk}a_{kt}^{-1}c_{jt})x_t\\ &+ (a_{jt}a_{kt}^{-1}b_{kt}a_{jk} - b_{jt}a_{jk} - a_{jk}a_{kt}^{-1}b_{kt} + a_{jk}a_{kt}^{-1}b_{jt})x_j\\ &= e_{jt} - b_{jt}c_{jk} - c_{jt}a_{kt}^{-1}b_{kt} - c_{jk}a_{kt}^{-1}b_{kt} + a_{jt}a_{kt}^{-1}b_{kt}c_{jk} - a_{jk}a_{kt}^{-1}e_{jt},\end{aligned}$$
which is equivalent to the equations,
$$\begin{aligned}(a_{jt}-1)c_{jk} + (a_{jk}-1)c_{jt} &= 0,\\ (a_{jt}-1)b_{kt} + b_{jt}(1-a_{kt}) &= 0,\\ a_{kt}e_{jt} - a_{kt}b_{jt}c_{jk} - c_{jt}b_{kt} - c_{jk}b_{kt} + a_{jt}b_{kt}c_{jk} - a_{jk}e_{jt} &= 0.\end{aligned} \tag{2.3.13}$$
3. If $k < j < t$, we have that $\nu_{x_k}(x_j) = a_{kj}^{-1}x_j - a_{kj}^{-1}b_{kj}$ and $\nu_{x_k}(x_t) = a_{kt}^{-1}x_t - a_{kt}^{-1}b_{kt}$
$$\begin{aligned}&(a_{jt}a_{kt}^{-1}b_{kt}a_{kj}^{-1} - a_{kj}^{-1}a_{kt}^{-1}b_{kt} - b_{jt}a_{kj}^{-1} + a_{kj}^{-1}a_{kt}^{-1}b_{jt})x_j\\ &+ (a_{jt}a_{kt}^{-1}a_{kj}^{-1}b_{kj} - a_{kj}^{-1}b_{kj}a_{kt}^{-1} - c_{jt}a_{kt}^{-1} + a_{kj}^{-1}a_{kt}^{-1}c_{jt})x_t\\ &= e_{jt} - b_{jt}a_{kj}^{-1}b_{kj} - c_{jt}a_{kt}^{-1}b_{kt} - a_{kj}^{-1}b_{kj}a_{kt}^{-1}b_{kt} + a_{jt}a_{kt}^{-1}b_{kt}a_{kj}^{-1}b_{kj} - a_{kj}^{-1}a_{kt}^{-1}e_{jt},\end{aligned}$$
which is equivalent to the equations,
$$\begin{aligned}(a_{jt}-1)b_{kt} + (1-a_{kt})b_{jt} &= 0,\\ b_{kj}(a_{jt}-1) + (1-a_{kj})c_{jt} &= 0,\\ (a_{kj}a_{kt}-1)e_{jt} - b_{jt}b_{kj}a_{kt} - a_{kj}c_{jt}b_{kt} + (a_{jt}-1)b_{kt}b_{kj} &= 0.\end{aligned} \tag{2.3.14}$$

Additionally, if we want that ν's automorphisms commutes, since for any x_k, if $j > k$, we have that $\nu_{x_j}(x_k) = a_{kj}x_k - c_{kj}$, if $j < k$, $\nu_{x_j}(x_k) = a_{jk}^{-1}x_k - a_{jk}^{-1}b_{jk}$, and $\nu_{x_k}(x_k) = a_{kk}x_k + b_{kk}$, we need the *commutativity equations*:

1. If $j, t > k$, that $c_{kj}(a_{kt}-1) = c_{kt}(a_{kj}-1)$.
2. If $j > k > t$ that $c_{kj}(a_{tk}^{-1}-1) = a_{tk}^{-1}b_{tk}(a_{kj}-1)$.
3. If $j, t < k$, that $a_{jk}^{-1}b_{jk}(a_{tk}^{-1}-1) = a_{tk}^{-1}b_{tk}(a_{jk}^{-1}-1)$.
4. If $k < j$, that $-c_{kj}(a_{kk}-1) = b_{kk}(a_{kj}-1)$.
5. If $j < k$, that $b_{kk}(a_{jk}^{-1}-1) = -a_{jk}^{-1}b_{jk}(a_{kk}-1)$.

Remark 2.3.37. Now, we have to guarantee that the differential calculus $(\Omega(\mathcal{A}), d)$, where $\Omega^1(\mathcal{A}) = \bigoplus_{j=i}^{n} d(x_j)\mathcal{A}$ and $\Omega^i(A) = \bigwedge_{j=1}^{i}\Omega^1(\mathcal{A})$, is a differential connected calculus. Since $\nu_{x_k}(x_k) = a_{kk}x_k + b_{kk}$ we have that
$$\begin{aligned}d(x_k^i) &= \sum_{j=1}^{i} x_k^{j-1}d(x_k)x_k^{i-j} = \sum_{j=1}^{i} d(x_k)\nu_{x_k}(x_k^{j-1})x_k^{i-j}\\ &= d(x_k)\left(\sum_{j=1}^{i}(a_{kk}x_k + b_{kk})^{j-1}x_k^{i-j}\right)\\ &= d(x_k)\left(\sum_{j=1}^{i}\sum_{t=0}^{j-1}\binom{j-1}{t}a_{kk}^{t}x_k^{i+t-j}b_{kk}^{j-1-t}\right).\end{aligned}$$

With this we have that $d(a) = \sum_{i=1}^{n} d(x_i)\overline{\partial}_i(a)$, where

$$\overline{\partial}_i(x_1^{l_1}\cdots x_n^{l_n}) = \nu_{x_i}(x_1^{l_1})\cdots\nu_{x_i}(x_{i-1}^{l_{i-1}})\left(\sum_{j=1}^{l_i}\sum_{t=0}^{j-1}\binom{j-1}{t}a_{kk}^{t}x_k^{l_i+t-j}b_{kk}^{j-1-t}\right)x_{i+1}^{l_{i+1}}\cdots x_n^{l_n}.$$

Then, for $i = 1$, we have that for $\sum_{r\in\Gamma_r}\alpha_r x_1^{l_{1r}}\cdots x_n^{l_{nr}} \in \mathcal{A}$,

$$\overline{\partial}_1(\sum_r \alpha_r x_1^{l_{1r}}\cdots x_n^{l_{nr}}) = \sum_r \alpha_r\left(\sum_{j=1}^{l_{1r}}\sum_{t=0}^{j-1}\binom{j-1}{t}a_{11}^{t}x_1^{l_{1r}+t-j}b_{11}^{j-1-t}\right)x_2^{l_{2r}}\cdots x_n^{l_{nr}} = 0,$$

if and only if $\sum_{r'\in\Gamma_r}\alpha_r\left(\sum_{j=1}^{l_{1r}}\sum_{t=0}^{j-1}\binom{j-1}{t}a_{11}^{t}x_1^{l_{1r}+t-j}b_{11}^{j-1-t}\right) = 0$, where $r'\in\Gamma_r$ if and only if $x_2^{l_{2r}}\cdots x_n^{l_{nr}} = x_2^{l_{2r'}}\cdots x_n^{l_{nr'}}$, i.e., we can re write this as $\overline{\partial}_1(\sum_r \alpha_r x_1^{l_{1r}}\cdots x_n^{l_{nr}}) = 0$ if and only if $\overline{\partial}_1(p_r(x_1)x_2^{l_{2r}}\cdots x_n^{l_{nr}}) = 0$, for all r. If $p_r(x_1) = \sum_{m=0}^{l_{1r}} q_m x_1^m$, with $q_m \in \mathbb{C}$, then

$$\begin{aligned}
0 &= \overline{\partial}_1(p_r(x)x_2^{l_{2r}}\cdots x_n^{l_{nr}})\\
&= \sum_{m=0}^{l_{1r}} q_m\left(\sum_{j=1}^{m}(a_1x_1+b_{11})^{j-1}x_1^{m-j}\right)x_2^{l_{2r}}\cdots x_n^{l_{nr}}\\
&= \sum_{m=0}^{l_{1r}} q_m\left(\sum_{j=1}^{m}\sum_{t=0}^{j-1}\binom{j-1}{t}a_{11}^{t}b_{11}^{j-1-t}x_1^{m+t-j}\right)x_2^{l_{2r}}\cdots x_n^{l_{nr}}
\end{aligned}$$

Since in the last expression, $m = l_{1r}, j = 1$ and $t = 0$ is the unique way to obtain the highest exponent of x_1, we have that $q_{l_{1r}}\binom{1-1}{0}a_{11}^{t}b_{11}^{1-1-0} = q_{l_{1r}} = 0$, therefore p_r does not have no constant term, i.e., $p_r(x_1) \in \mathbb{C}$, for all r. In other words we have that $a \in \mathrm{Ker}(\overline{\partial}_1) = \mathrm{gen}_{\mathbb{C}}\{x_2, ..., x_n\}$. Following in this fashion, for each $1 \leq k \leq n$, $\mathrm{Ker}(\overline{\partial}_1)\cap\cdots\cap\mathrm{Ker}(\overline{\partial}_k) = \mathrm{gen}_{\mathbb{C}}\{x_{k+1}, ..., x_n\}$, therefore $\mathrm{Ker}(d) = \mathrm{Ker}(\overline{\partial}_1)\cap\cdots\cap\mathrm{Ker}(\overline{\partial}_n) = \mathbb{C}$.

Remark 2.3.38. In order to apply the Lemma 1.3.31 to obtain that this is an integrable differential calculus with volume form $\omega = d(x_1)\wedge\cdots\wedge d(x_n)$ and automorphism $\nu_\omega = \nu_{x_1}\circ\cdots\circ\nu_{x_n}$, we define for $1 \leq k \leq n$, and for each injective crescent maps $\varphi_i : \{1, ..., k\} \to \{1, ..., n\}$ and $\overline{\varphi_i} : \{1, ..., (n-k)\} \to \{1, ..., n\}\setminus \mathrm{Im}(\varphi_i)$, that determine a n-permutations $\varphi_i\_\overline{\varphi}_i$ such that $\varphi_i\_\overline{\varphi}_i(p) = \varphi_i(p)$ if $p \leq k$, and $\varphi_i\_\overline{\varphi}_i(p) = \overline{\varphi}_i(p-k)$ if $k < p \leq n$, and analogously, $\overline{\varphi}_i\_\varphi_i$, the elements $\omega_i^k, \overline{\omega}_i^k \in \Omega^k(\mathcal{A})$ as,

$$\begin{aligned}
\omega_i^k &= A_{ik}d(x_{\varphi_i(1)})\wedge\cdots\wedge d(x_{\varphi_i(k)}),\\
\overline{\omega}_i^{n-k} &= \overline{A}_{i(n-k)}d(x_{\overline{\varphi}_i(1)})\wedge\cdots\wedge d(x_{\overline{\varphi}_i(n-k)}),
\end{aligned}$$

where $A_{ik}, \overline{A}_{ik} \in \mathbb{C}$ are defined, in aim to obtain $\overline{\omega}_i^{n-k}\wedge\omega_i^k = \omega$, as follows: if $i < j$, we have that $x_j d(x_i) = -d(x_i)\nu_{x_i}(x_j) = -d(x_i)(a_{ij}^{-1}x_j - a_{ij}^{-1}b_{ij})$, therefore, applying d, $d(x_i)\wedge d(x_j) = -a_{ij}d(x_j)\wedge d(x_i)$. Hence,

1. If $\varphi_i(1) = 1$, then

$$A_{ik} = \prod_{s=1}^{n-k}\prod_{t=1}^{\overline{\varphi}_i(s)-1}(-1)a_{t\overline{\varphi}_i(s)}, \qquad \overline{A}_{i(n-k)} = \prod_{s=1}^{n-k-1}\prod_{t=s+1}^{n-k}(-1)a^{-1}_{\overline{\varphi}_i(s)\overline{\varphi}_i(t)}.$$

2. If $\overline{\varphi}_i(1) = 1$, then

$$A_{ik} = \prod_{s=1}^{k-1}\prod_{t=s+1}^{k}(-1)a^{-1}_{\varphi_i(s)\varphi_i(t)}, \qquad \overline{A}_{i(n-k)} = \prod_{s=1}^{k}\prod_{t=\varphi_i(s)+1}^{\overline{\varphi}_i(n-k)}(-1)a_{\overline{\varphi}_i(s)t}.$$

Noting, $\varphi_i(p) := i_p$, for $1 \leq p \leq k$, since the set $\{d(x_{i_1})\wedge\cdots\wedge d(x_{i_k}) \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ forms a right (and left) base of $\Omega^k(\mathcal{A})$, for an arbitrary $\omega' = \sum_{1\leq i_1<\cdots<i_k\leq n} d(x_{i_1}) \wedge \cdots \wedge d(x_{i_k})a_{i_1\cdots i_k} \in \Omega^k(\mathcal{A})$, where $a_{i_1\cdots i_k} \in \mathcal{A}$, we have

$$\begin{aligned}\omega_i^k\pi_\omega(\overline{\omega}_i^{n-k}\wedge\omega') &= \omega_i^k\pi_\omega(\overline{\omega}_i^{n-k}\wedge[d(x_{i_1})\wedge\cdots\wedge d(x_{i_k})a_{i_1\cdots i_k}])\\ &= d(x_{i_1})\wedge\cdots\wedge d(x_{i_k})\pi_\omega(\overline{\omega}_i^{n-k}\wedge\omega_i^k a_{i_1\cdots i_k})\\ &= d(x_{i_1})\wedge\cdots\wedge d(x_{i_k})a_{i_1\cdots i_k},\end{aligned}$$

$$\begin{aligned}\nu_\omega^{-1}(\pi_\omega(\omega'\wedge\omega_i^{n-k}))\overline{\omega}_i^k &= \nu_\omega^{-1}(\pi_\omega(d(x_{i_1})\wedge\cdots\wedge d(x_{i_k})a_{i_1\cdots i_k}\wedge\omega_i^{n-k}))\overline{\omega}_i^k\\ &= \nu_\omega^{-1}(\pi_\omega(\overline{\omega}_i^k\wedge\omega_i^{n-k}b_{i_1\cdots i_k}))d(x_{i_1})\wedge\cdots\wedge d(x_{i_k})\\ &= \nu^{-1}_{x_{\varphi_i(1)}}\circ\cdots\circ\nu^{-1}_{x_{\varphi_i(k)}}(a_{i_1\cdots i_k})d(x_{i_1})\wedge\cdots\wedge d(x_{i_k}),\end{aligned}$$

where $b_{i_1\cdots i_k} := \nu_{x_{\overline{\varphi}_i(1)}}\circ\cdots\circ\nu_{x_{\overline{\varphi}_i(n-k)}}(a_{i_1\cdots i_k})$. With this we get that equations (1.3.4) hold, and then, by Lemma 1.3.31, $\Omega(\mathcal{A})$ is an integrable differential calculus.

Since the algebra $\mathcal{A}$ defined by relations (2.3.9) is a bijective skew PBW algebra, we have, from [Rey13], Theorem 14, that $\text{GKdim}(\mathcal{A}) = n$, which completes the proof of the following theorem, which is obtained in the realization of this work.

Theorem 2.3.39. *If $\mathcal{A}$ is a PBW algebra generated by $x_1, \ldots, x_n$ under the relations, for all $1 \leq i < j \leq n$*

$$x_ix_j - a_{ij}x_jx_i = b_{ij}x_i + c_{ij}x_j + e_{ij}, \quad \text{where } a_{ij}, b_{ij}, c_{ij}, e_{ij} \in \mathbb{C},\ a_{ij} \neq 0,$$

such that (2.3.10), (2.3.11), (2.3.12), (2.3.13), (2.3.14) and commutativity equations hold, then $\mathcal{A}$ is differentially smooth.

We close this section highlighting that by Theorem 2.3.39, obtained in this work, we have guaranteed the differential smoothness of a huge amount of algebras, within which we can mention:

1. The algebra of linear partial differential operators (see [LR14], p. 1212).
2. The algebra of linear partial q-dilation operators (see [LR14], p. 1213).
3. The additive analogue of Weyl algebra (see [LR14], p. 1214).
4. The multiplicative analogue of the Weyl algebra (see [LR14], p. 1214).

2.4 Review of Artamonov's paper

As we can see in the previous sections, the search of explicit skew derivations, or even before, the quest of automorphisms in an explicit form could be a longer and tedious work. The task is more extensive if we do the job for each algebra, playing with their specific features. Therefore, working with algebras that generalize our study objects, and to try to resolve the problem in a general situation, it is an useful and interesting work. Because of that, we are interested in Artamonov's paper [Art15], titled *Derivations of skew PBW extensions*, where these skew PBW extensions (see Definition 2.4.5 above) are algebras that generalize some of the generalized Weyl algebras (Section 2.1), diffusion algebras (Section 2.2) and skew polynomial algebras (Section 2.3). Then, any treatment about derivations (*id*-skew derivations) of skew PBW extensions it is of very interest for us in order to obtain (commutative) differential calculus, in particular Brzezinski's calculus (see Section 1.2) and with this determine differentially smoothness of skew PBW extensions (see Section 1.3.2).

In this short section, we review the first pages of the article [Art15], where there is treatment about a classification of derivations of *skew PBW extensions* (which includes the diffusion algebras and the skew polynomial algebras), and where we found a mistake in a crucial result for the theory. We close the section with a list of doubts and answers about the content of this treatment.

For any automorphism γ of an associative ring A and an element $a \in A$, we can define an inner γ-derivation $\mathrm{ad}_\gamma a$ as $(\mathrm{ad}_\gamma a)x = ax - \gamma(x)a$; it is satisfied that $\mathrm{ad}_\gamma a$ is a left γ-skew derivation (see Example 1.1.40). In particular, if $\gamma = \varepsilon$ is an identical automorphism, then we write $\mathrm{ad}_\varepsilon = \mathrm{ad}$.

Let R be a subring of A containing unit element, and $x_1, ..., x_n \in A$ such that some endomorphisms $\sigma_1, ..., \sigma_n$ of R the skew derivations $\mathrm{ad}_{\sigma_i} x_i$ map R into itself. Denote by $\partial_i = \mathrm{ad}_{\sigma_i} x_i|_R$ the induced σ_i-derivation of R.

Proposition 2.4.1 ([Art15], p.2). *Suppose that ∂ is a derivation of A mapping R into itself and commuting with each σ_i. Then $(\mathrm{ad}_{\sigma_i}\partial(x_i))r = [\partial, \partial_i](r) \in R$ for any $r \in R$.*

Proof. By assumptions, we have

$$\begin{aligned}\partial\partial_i(r) &= \partial(x_i r - \sigma_i(r)x_i) = \partial(x_i)r + x_i\partial(r) - (\partial\sigma_i(r))x_i - \sigma_i(r)\partial(x_i)\\ &= [\partial(x_i)r - \sigma_i(r)\partial(x_i)] + [x_i\partial(r) - (\partial\sigma_i(r))x_i]\\ &= (\mathrm{ad}_{\sigma_i}\partial(x_i))r + (\mathrm{ad}_{\sigma_i}x_i)(\partial r)\\ &= (\mathrm{ad}_{\sigma_i}\partial(x_i))r + \partial_i(\partial r) \in R,\end{aligned}$$

where the third equation is due to the fact that $\partial\sigma_i = \sigma_i\partial$, for all $i = 1, ..., n$, and the last equation follows from $\partial r \in R$, and so $(\mathrm{ad}_{\sigma_i}x_i)(\partial r) = (\mathrm{ad}_{\sigma_i}x_i)|_R(\partial r) = \partial_i(\partial r)$. Hence $(\mathrm{ad}_{\sigma_i}\partial(x_i))r = -\partial_i\partial(r) + \partial\partial_i(r) = [\partial, \partial_i](r) \in R$. □

Theorem 2.4.2. *Let $R, A, x_1, ..., x_n, \sigma_i, \partial_i$ be as above. Then, for any $r \in R$, we have*

$$\begin{aligned}(x_1 \cdots x_n)r = \sum_{t\geq 0} \sum_{1\leq i_1<\cdots<i_t\leq n} &(\sigma_1 \cdots \sigma_{i_1-1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots\\ &\cdots \sigma_{i_t-1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_n(r))x_1 \cdots \hat{x}_{i_1} \cdots \hat{x}_{i_t} \cdots x_n\end{aligned}$$

Here, $\hat{x}_{i_j}$ means that the factor x_{i_j} is omitted.

Proof. We shall proceed by induction on n. The case $n = 1$ follows from the definition of $\partial_1 = (\mathrm{ad}_{\sigma_1} x_1)|_R$: we have that $\partial_1(r) = x_1 r - \sigma_1(r)x_1$, which means that $x_1 r = \partial_1(r) + \sigma_1(r)x_1$. Note that for $t = 0$ we obtain $\sigma_1(r)x_1$, and for $t = 1$ the correspondent term is $\partial_1(r)$ which omits the factor x_1.

Now, suppose that theorem is proved for $n-1$. As $x_n r = \sigma_n(r)x_n + \partial_n(r)$ by definition of ∂_n, we have that

$$\begin{aligned}
x_1 \cdots x_n r &= x_1 \cdots x_{n-1} \cdot (x_n r) \\
&= x_1 \cdots x_{n-1} \cdot \sigma_n(r) x_n + x_1 \cdots x_{n-1} \cdot \partial_n(r) \\
&= \Bigg[\sum_{t \geq 0} \sum_{1 \leq i_1 < \cdots < i_t \leq n-1} (\sigma_1 \cdots \sigma_{i_1 - 1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \\
&\quad \cdots \sigma_{i_t - 1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_{n-1}(r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_{n-1}\Bigg] x_n \\
&\quad + \Bigg[\sum_{t \geq 0} \sum_{1 \leq i_1 < \cdots < i_t \leq n-1} (\sigma_1 \cdots \sigma_{i_1 - 1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \\
&\quad \cdots \sigma_{i_t - 1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_{n-1}(\partial_n r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_{n-1}\Bigg] \\
&= \sum_{t \geq 0} \sum_{1 \leq i_1 < \cdots < i_t \leq n} (\sigma_1 \cdots \sigma_{i_1 - 1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \\
&\quad \cdots \sigma_{i_t - 1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_n(r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_n.
\end{aligned}$$

This because for $t = n$, we have that all terms that omit the factor x_n are exactly

$$\Bigg[\sum_{t \geq 0} \sum_{1 \leq i_1 < \cdots < i_t \leq n-1} (\sigma_1 \cdots \sigma_{i_1-1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \sigma_{i_t-1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_{n-1}(\partial_n r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_{n-1}\Bigg]$$

□

In this work we indude the proof of the following corollary, which is presented in [Art15] without proof.

Corollary 2.4.3. *Under assumptions of Theorem 2.4.2, we have*

$$\begin{aligned}
\mathrm{ad}_{\sigma_1 \cdots \sigma_n}(x_1 \cdots x_n) r = \sum_{t \geq 1} \sum_{1 \leq i_1 < \cdots < i_t \leq n} & (\sigma_1 \cdots \sigma_{i_1 - 1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \\
& \cdots \sigma_{i_t - 1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_n(r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_n
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
\mathrm{ad}_{\sigma_1 \cdots \sigma_n}(x_1 \cdots x_n) r &= (x_1 \cdots x_n) r - \sigma_1 \cdots \sigma_n(r)(x_1 \cdots x_n) \\
&= \sum_{t \geq 0} \sum_{1 \leq i_1 < \cdots < i_t \leq n} (\sigma_1 \cdots \sigma_{i_1 - 1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \\
&\quad \cdots \sigma_{i_t - 1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_n(r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_n - (\sigma_1 \cdots \sigma_n)(r)(x_1 \cdots x_n) \\
&= \sum_{t \geq 1} \sum_{1 \leq i_1 < \cdots < i_t \leq n} (\sigma_1 \cdots \sigma_{i_1 - 1} \cdot \partial_{i_1} \cdot \sigma_{i_1+1} \cdots \\
&\quad \cdots \sigma_{i_t - 1} \cdot \partial_{i_t} \cdot \sigma_{i_t+1} \cdots \sigma_n(r)) x_1 \cdots \hat{x_{i_1}} \cdots \hat{x_{i_t}} \cdots x_n,
\end{aligned}$$

where the first equal is due to the definition of ad, the second is by the Theorem 2.4.2, and the last is because with $t = 0$, the correspond term in Theorem 2.4.2 is precisely $(\sigma_1 \cdots \sigma_n)(r)(x_1 \cdots x_n)$. □

Remark 2.4.4. Theorem 2.4.2 and Corollary 2.4.3 hold for $(x_1^{l_1} \cdots x_n^{l_n})r$ and also for $\mathrm{ad}_{\sigma_1^{l_1}\cdots\sigma_n^{l_n}}(x_1^{l_1} \cdots x_n^{l_n})r$, respectively.

Definition 2.4.5 ([Art15], p. 3)**.** An associative unital ring A is a *skew PBW-extension* of its subring R, if the following conditions are satisfied:

1. there exist a finite set of elements $x_1, ..., x_n \in A$ such that A is a left R-free module with the basis $Mon(A) = \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha = (\alpha_1, ..., \alpha_n) \in (\mathbb{N} \cup \{0\})^n\}$.
2. there exist endomorphisms $\sigma_1, ..., \sigma_n$ of R such that $(\mathrm{ad}_{\sigma_i} x_i)r = \partial_i(r) \in R$ for each $r \in R$ and $1 \leq i \leq n$. Note that this is the same that say $x_i r = \sigma_i(r)x_i + \partial_i(r)$, for ∂_i a σ_i-derivation of R.
3. for any indices $1 \leq i, j \leq n$, there exist elements $c_{ij}, r_{ij}, r_{ji}^t \in R$ such that $c_{ij}c_{ji} = 1$ and $x_i x_j - c_{ij}x_j x_i = \sum_t r_{ij}^t x_t + r_{ij}$.

In this work we want to remark that there exist results about skew derivations over skew PBW extensions, as the following theorem shows.

Theorem 2.4.6 ([RS17a], p. 20)**.** *If the endomorphisms and skew derivations of Definition 2.4.5 satisfy* $[\sigma_i, \partial_j] = [\partial_i, \partial_j] = 0$, *and* $\partial_k(c_{ij}) = \partial_k(r_{ij}^t) = 0$, *for all* $1 \leq 1, j, k, t \leq n$, *then the assignments*

$$f = a_0 + a_1 X_1 + \cdots + a_m X_m \mapsto \overline{\sigma}_k(f) = \sigma_k(a_0) + \sigma_k(a_1)X_1 + \cdots + \sigma_k(a_m)X_m,$$
$$f = a_0 + a_1 X_1 + \cdots + a_m X_m \mapsto \overline{\delta}_k(f) = \delta_k(a_0) + \delta_k(a_1)X_1 + \cdots + \delta_k(a_m)X_m,$$

define an injective endomorphism of A, $\overline{\sigma}_k$, *and a* $\overline{\sigma}_k$*-skew derivation of* A, $\overline{\delta}_k$, *for all* $k = 1, ..., n$.

By Theorem 2.4.6, for diffusion algebras, generalized Weyl algebras and skew polynomial algebras of degree n, we can guarantee n skew derivations, which in case of $\mathrm{GKdim}(A) = n$, it is a sufficient quantity of skew derivations to construct a Brzezinski's calculus (see Remark 1.2.12).

Now, we continue with the content of [Art15]. Let us consider the degree-lexicographic order on monomials of A, which is denoted by $>_{d-l}$.

In this work, we found a counterexample to the Lemma 3.1 of [Art15], which is established in the following proposition.

Proposition 2.4.7. *The following lemma is false: "Let* $m_1, ..., m_n$ *be non-negative integers such that* $(l_1, ..., l_n) >_{d-l} (m_1, ..., m_n) >_{d-l} (l_1, ..., l_{n-1}, l_n - 1)$, *then* $m_1 = l_1$ *,...,* $m_{n-1} = l_{n-1}, m_n < l_n$*."*

Proof. Consider $n = 3$, $(l_1, l_2, l_3) = (1, 4, 1)$ and $(m_1, m_2, m_3) = (1, 2, 3)$. Then $(l_1, l_2, l_3 - 1) = (1, 4, 0)$, therefore $(1, 4, 1) >_{d-l} (1, 2, 3) >_{d-l} (1, 4, 0)$ but $3 \not< 1$. □

Proposition 2.4.8 ([Art15], p. 4)**.** *Let ∂ be a derivation of A commuting with each ∂_i such that R is stable under ∂. Suppose that $\alpha x_1^{l_1}\cdots x_{n-1}^{l_{n-1}}x_n^{l_n}$, with $\alpha \in R\setminus\{0\}$ is the leading term of ∂x_m such that $l_1+\cdots+l_n>1$. Suppose that $\sigma_1,\dots,\sigma_n$ are automorphisms of R. If $l_j>0$, then*

$$\sum_{\substack{s_1+s_2=l_j-1\\ s_1,s_2\leq 0}} \sigma_n^{s_1}\partial_j\sigma_n^{s_j}=0. \tag{2.4.1}$$

If R is torsion-free $\mathbb{Z}$-module, then $\partial_j=0$.

Proof. Without loss of generality, we can assume that $j=n$. By using Proposition 2.4.7, we consider the coefficient in $x_1^{l_1}\cdots x_{n-1}^{l_{n-1}}x_n^{l_n-1}$ in ∂x_m. By Corollary 2.4.3 and Proposition 2.4.7, it has the form

$$\sum_{\substack{s_1+s_2=l_n-1\\ s_1,s_2\geq 0}} \alpha\sigma_1^{l_1}\cdots\sigma_{n-1}^{l_{n-1}}\sigma_n^{s_1}\partial_n\sigma_n^{s_j}. \tag{2.4.2}$$

By the Proposition 2.4.1, for any $r\in R$, we have $\partial(x_m)r-\sigma_i(r)\partial(x_m)=\mathrm{ad}_{\sigma_m}\partial(x_m)(r)\in R$. If $(l_1,\dots,l_{n-1},l_n-1\neq(0,\dots,0))$, then the expression (2.4.2) is equal to zero. In this case, we obtain (2.4.1), since each σ_i is bijective.

If σ_n commutes with ∂_n, then (2.4.1) can be written as $l_n\sigma_n^{l_n-1}\partial_n=0$. By assumption, we obtain $\partial_n=0$. ☐

Questions:

1. For $r\in R$, $\mathrm{ad}_{\sigma_1^{l_1}\cdots\sigma_n^{l_n}}(x_1^{l_1}\cdots x_n^{l_n})(r)\in R$? Apparently, this is used to say that the terms not belonging to R (using the left basis) are zero, and therefore (2.4.1) holds.

2. Is $\mathrm{ad}_{\sigma_m}(\partial(x_m))$ related with (2.4.1)?

3. We know that ∂ commutes with the derivations ∂_i. Does ∂ commutes with the automorphism σ_i?

   Since $\partial\partial_i=\partial_i\partial$, then $\partial\partial_i(r)=\partial_i\partial(r)$, for all $r\in R$. Thus, $\partial(\mathrm{ad}_{\sigma_i}(x_i)(r))=(\mathrm{ad}_{\sigma_i}(x_i))(\partial(r))$,

   $$\partial(x_ir-\sigma_i(r)x_i)=x_i\partial(r)-\sigma_i(\partial(r))x_i.$$

   Therefore,

   $$\partial(x_i)r+x_i\partial(r)-\partial(\sigma_i(r))x_i-\sigma_i(r)\partial(x_i)=x_i\partial(r)-\sigma_i(\partial(r))x_i,$$

   and then,

   $$\mathrm{ad}_{\sigma_i}(\partial(x_i))(r)=\partial(x_i)r-\sigma_i(r)\partial(x_i)=\partial(\sigma_i(r))x_i-\sigma_i(\partial(r))x_i=[\partial(\sigma_i(r))-\sigma_i(\partial(r))]x_i.$$

   Finally, we have that if ∂ commutes with ∂_i, then $\mathrm{ad}_{\sigma_i}(\partial(x_i))(r)=[\partial,\sigma_i](r)x_i$, where this is the expression of $\mathrm{ad}_{\sigma_i}(\partial(x_i))(r)$ in the monomial bases x^α as left R-module.

   But, from $[\partial,\partial_i]=0$, we can not conclude $[\partial,\sigma_i]=0$. A well argument to obtain $[\partial,\sigma_i]=0$ could be $\mathrm{ad}_{\sigma_i}(\partial(x_i)(r)\in R$, which by Proposition 2.4.1, is related to $[\partial,\sigma_i]=0$.

4. Does $\mathrm{ad}_{\sigma_1^{l_1}\cdots\sigma_n^{l_n}}(x_1^{l_1}\cdots x_n^{l_n})$ commutes with ∂?

5. When ∂_n commutes with σ_n?

   Since $\partial_n\sigma_n = \sigma_n\partial_n$, then $\partial_n\sigma_n(r) = \sigma_n\partial_n(r)$, for all $r \in R$. Therefore, in terms of the adjoint, $(\mathrm{ad}_{\sigma_n}(x_n))(\sigma_n(r)) = \sigma_n(\mathrm{ad}_{\sigma_n}(x_n)(r))$. We get that

   $$x_n\sigma_n(r) - \sigma^2(r)x_n = \sigma_n(x_n)\sigma_n(r) - \sigma_n^2(r)\sigma_n(x_n).$$

   With this, we obtain $[x_n - \sigma_n(x_n)]\sigma_n(r) = \sigma_n^2(r)[x_n - \sigma_n(x_n)]$.

6. If R is torsion free, does σ_i commute with ∂_n?

More questions and some answers are the following:

1. It is necessary that ∂ commutes with the automorphisms σ_i. With this, Proposition 2.4.1 holds (as is mentioned in the proof), and so $[\partial, \partial_i](r) = \mathrm{ad}_{\sigma_i}(\partial(x_i))(r) \in R$. Hence, we focus on the coefficient of $x_1^{l_1}\cdots x_1^{l_n-1}$ in the expression of $\mathrm{ad}_{\sigma_i}(\partial(x_i))(r) = \partial(x_m)r - \sigma_m\partial(x_m)$, where, if $\partial(x_m) = \sum a_\beta x^\beta + \alpha x_1^{l_1}\cdots x_n^{l_n}$, then,

   $$\partial(x_m)r = \sum a_\beta x^\beta r + \alpha x_1^{l_1}\cdots x_n^{l_n} r,$$

   where the second term, by Proposition 2.4.2, has coefficient of $x_1^{l_1}\cdots x_1^{l_n-1}$ in the sum 2.4.1; we are forgetting the coefficient of $x_1^{l_1}\cdots x_1^{l_n-1}$ in $-\sigma_m(r)\partial(x_m)$. Considering everything, we obtain that the coefficient of $x_1^{l_1}\cdots x_1^{l_n-1}$ in $\mathrm{ad}_{\sigma_i}(\partial(x_i))(r)$ is given by

   $$\left[\sum_{s_1+s_2=l_n-1;s_1,s_2\leq 0} \alpha\sigma_1^{l_1}\cdots\sigma_{n-1}^{l_{n-1}}\sigma_n^{s_1}\partial_n\sigma_n^{s_j}(r)\right] - \sigma_m(r)a_{(l_1,\ldots,l_n-1)}.$$

   We have to say that we are omitting the possibility that there exist more multiples of $x_1^{l_1}\cdots x_1^{l_n-1}$ in $\sum a_\beta x^\beta r$ (we really think that these terms do not appear).

2. We have that $[\partial_i, \sigma_i](r) \in R$ because, by definition, $\partial_i : R \to R$, (we still have the doubt that if R is $\mathbb{Z}$-free, then we can conclude that the commutator is zero).

3. $\mathrm{ad}_\alpha(x)(r)$ is a skew derivation in the argument r, but in the argument x is only $\mathbb{Z}$-linear. If we want the scalar product, R is necessarily commutative.

4. The Lemma 3.1 of [Art15] stated in Proposition 2.4.7 is true with the following conclusion:

   Let $m_1, \ldots, m_n$ be non-negative integers such that

   $$(l_1, \ldots, l_n) >_{d-l} (m_1, \ldots, m_n) >_{d-l} (l_1, \ldots, l_{n-1}, l_n - 1).$$

   Then, $(m_1, \ldots, m_n)$ satisfies some of the following conditions:

   (a) $m_1 + \cdots + m_n = l_1 + \cdots + l_n$ and there exists $1 \leq k \leq n$ such that $m_i = l_i$, for all $i < k$ and $m_k < l_k$.

   (b) $m_1 + \cdots + m_n = l_1 + \cdots + l_n - 1$ and there exists $1 \leq k \leq n$ such that $m_i = l_i$, for all $i < k$ and $m_k \geq l_k$.

Conclusions

The generalizations of the differential geometry to the non commutative context are strong lines of study and there exists a lot of literature about them. In the theory that we call *Brzezinski's calculus*, we found a good treatment of differential structures that we can relate to any affine algebra and that posses a constructive formality that results interesting for us. The raw material of this theory is the set of skew derivations of the algebra. All the objects and properties of them in Brzezinski's calculus constructions, are features about the automorphisms and skew derivations over these rings.

We observe that for non-commutative algebras, the search of automorphisms could be result counter intuitive, because there exist automorphisms of affine algebras that do not respect the usual notion of graduation. We mean that could not let stable some generating subspace of the algebra, like the Theorem 2.2.5 shows us.

In the Brzezinski's works, one of the immediately problem that arises is to guarantee that the density condition of this calculus is satisfied, as the Remark 1.2.13 points out. In the Example 2.1.16, the density condition was a consequence of the choice of the polynomial $p(z)$ and the Bézout theorem of $\mathbb{K}[z]$, but in the search of these structures for diffusion algebras, for instance, we found this difficulty. Then, we must set criteria to density condition for the Brzezinski's differential calculus of Definition 1.2.7, before to continue the construction of the Brzezinski's calculus of a particular algebra $\mathcal{A}$.

In the work of establishing the commutation laws (and describe the inner derivations of the generators), in terms of the PBW basis, for powers of variables in diffusion algebras and in some 3-dimensional skew polynomial algebras, we found that the combinatorial techniques, in particular the knowing of the Pascal triangle, is very important and surprising, as we pointed in Remark 2.2.15.

The task of determine skew derivations for a class of algebras simultaneously could bring problems as a hard language or possible mistakes as we observe in Section 2.4, or poor impact as we get in Proposition 1.1.44. We also note that the technique of work with particular algebras (with few generators) are the study objects of effective works, as [Brz16b] and [AB18], which we studied in Sections 2.1.1 and 2.1.3 respectively, where the principal work tool was the direct compute in terms of generators and relations.

For the 3-dimensional skew polynomial algebras that are Ore extensions $R[x, \sigma, \delta]$, where $[\sigma, \delta] = 0$, we conclude in Section 2.3.1 that the automorphism σ' such that $\sigma'|_R = \sigma$, in general, leaves subspace $Rx + R$ invariant, and that in one of these algebras, σ' can not be defined.

Future work

In the case of skew polynomial algebras of dimension 3, its smoothness study could be completed with the smoothness of iterated Ore extensions and generalized Weyl algebras. Since the diffusion algebras of both types are affine algebras constructed from $\mathbb{C}$ or $\mathbb{C}[x_1, ..., x_n]$, by the fact that these commutative algebras are differentialy smooth, we wish to know if there is a way to inherit the smoothness of diffusion algebras of type 1. In a wider way, we know that these algebras are iterated Ore extensions of $\mathbb{C}$ or $\mathbb{C}[x]$ with the skew derivation $\delta \neq 0$. By this way, we wish to explore how the smoothness of the ring base R is related with the smoothness of an Ore extension $R[w; \sigma, \delta]$ in the general case that $R \neq \mathbb{C}[x]$ and $\delta \neq 0$. Also, in [Brz16b] we found an example of first order differential calculus over a particular class of generalized Weyl algebras $A(p, q)$ studied in Section 2.1.2. Until this moment, there is no exist sufficient or necessary conditions that guarantee the integrability and the differentially smoothness of these algebras. These works can give us a guide to establish the smoothness of generalized Weyl algebras in general using the Brzeziński's calculus.

With this in mind, we consider, as a future work, to search good objects that let the inherit of smoothness of bases subalgebras. To do this, we will start researching compatibility properties in automorphisms of algebras, not only of extended type as in Section 2.3.1, but in a general way, and to optimize features of the skew derivations that guarantee the density condition of the Brzeziński's differential calculus (Ω, d), the connectivity of d, the flat behavior of the hom-connection ∇ and the integrability of the complex of integral forms $(\mathcal{IA}, \nabla)$.

In the general frame work of skew PBW extensions, we want investigate conditions for the skew derivations established in Theorem 2.4.6, so we can deduce the differentially smoothness using the construction of the Brzeziński's calculus. Second, we want to start the search of general characterizations of skew derivations of skew PBW extensions with the aim of establishing its Brzeziński's calculus. In order to do that, we have to understand and to correct the propositions and proofs presented in [Art15].

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