On the noncommutative differential geometry of semi-graded Artin-Schelter regular algebras

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On the noncommutative differential geometry of semi-graded Artin-Schelter regular algebras

Título en español

Sobre la geometría diferencial no conmutativa de las álgebras Artin-Schelter regulares semi-graduadas

Abstract: In this thesis we investigate the noncommutative differential geometry of families of semi-graded rings or semi-graded Artin-Schelter regular algebras. With this end, we present some definitions and preliminaries on these algebraic structures and on the smooth geometry of noncommutative algebras. We also present some differences between Artin-Schelter regular algebras and semi-graded Artin-Schelter regular algebras. Next, we concentrate our research on the smooth geometry of bi-quadratic algebras on three generators having PBW basis, double extension regular algebras of type (14641), diffusion algebras and skew Poincaré-Birkhoff-Witt extensions. Finally, we establish some related ideas with the aim of extending the Serre duality theorem for noncommutative projective schemes on graded algebras to the setting of semi-graded algebras.

Resumen: En esta tesis investigamos la geometría diferencial no conmutativa de familias de anillos semi-graduados o álgebras regulares Artin-Schelter semi-graduadas. Para ello, presentamos algunas definiciones y preliminares sobre estas estructuras algebraicas y sobre la geometría suave de álgebras no conmutativas. También presentamos algunas diferencias entre las álgebras regulares de Artin-Schelter y las álgebras regulares Artin-Schelter semi-graduadas. A continuación, concentramos nuestra investigación en la geometría suave de álgebras bicuadráticas en tres generadores que tienen base PBW, extensiones dobles de álgebras regulares de tipo (14641), álgebras de difusión y extensiones de Poincaré-Birkhoff-Witt torcidas. Finalmente, establecemos algunas ideas relacionadas con el objetivo de extender el teorema de dualidad de Serre para esquemas proyectivos no conmutativos en álgebras graduadas al entorno de álgebras semi-graduadas.

Keywords: Noncommutative differential geometry, smooth geometry, differential smoothness, differential calculus, integrable calculus, Artin-Schelter regular algebra, noncommutative projective space, semi-graded ring, Serre duality.

Palabras clave: Geometría diferencial no conmutativa, geometría suave, suavidad diferencial, cálculo diferencial, cálculo integral, álgebra regular Artin-Schelter, espacio proyectivo no conmutativo, anillo semi-graduado, dualidad de Serre.

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Dedicatory

Dedicated to my grandma Odilia who has always made me dream

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Contents

Contents I

Introduction III

1. Definitions and preliminaries 1

1.1 Artin-Schelter regular algebras . . . 1

1.2 Semi-graded rings . . . 5

1.2.1 Skew polynomial rings . . . 8

1.2.2 3-dimensional skew polynomial algebras . . . 11

1.2.3 Generalized Weyl algebras and down-up algebras . . . 12

1.2.4 Other families of quantum algebras . . . 14

1.3 Semi-graded Artin-Schelter regular algebras . . . 17

1.3.1 Examples . . . 18

1.4 Noncommutative differential geometry of algebras . . . 20

1.4.1 Hom-connections (Divergences) and differential calculi . . . 20

1.4.2 Differential smoothness of algebras . . . 21

2. Smooth geometry of bi-quadratic algebras on three generators having PBW basis 31

2.1 Bi-quadratic algebras with PBW basis . . . 31

2.2 Differential and integral calculus . . . 35

2.2.1 Bi-quadratic algebras on two generators . . . 35

2.2.2 Bi-quadratic algebras on three generators . . . 38

2.3 Future work . . . 46

3. Smooth geometry of double extension regular algebras of type (14641) 51

3.1 Double Ore extensions and extension regular algebras of type (14641) . . . 51

3.1.1 Double extensions of $\Bbbk_Q[x_1, x_2]$ . . . 57
3.2 Differential smoothness . . . 62
3.3 Future work . . . 69

4. Smooth geometry of diffusion algebras 75
4.1 Diffusion algebras . . . 75
4.1.1 Diffusion algebras on three generators . . . 77
4.1.2 Diffusion algebras on N generators . . . 79
List of diffusion algebras on N generators . . . 80
4.2 Differential smoothness of diffusion algebras . . . 84
4.2.1 Diffusion algebras on four generators . . . 84
4.2.2 Diffusion algebras on five generators . . . 84
4.2.3 Differential smoothness of diffusion algebras with N generators . . . 84

5. Smooth geometry of skew Poincaré-Birkhoff-Witt extensions 102
5.1 SPBW extensions . . . 103
5.2 Differential smoothness of SPBW extensions . . . 107
5.2.1 Extension of automorphisms and derivations . . . 108
5.3 Differential smoothness of SPBW extensions over $\Bbbk[t]$ . . . 118
5.3.1 SPBW extensions in two indeterminates . . . 119
5.3.2 SPBW extensions in three indeterminates . . . 128
5.3.3 SPBW extensions in n indeterminates . . . 135
5.4 Differential smoothness of SPBW extensions over $\Bbbk[t_1, t_2]$ . . . 139
5.4.1 SPBW extensions in two indeterminates . . . 139
5.4.2 SPBW extensions in three indeterminates . . . 152
5.4.3 SPBW extensions in n indeterminates . . . 159
5.5 Future work . . . 166

6. Serre duality for noncommutative projective schemes 168
6.1 Serre duality theorem for $\mathbb{N}$-graded algebras . . . 168
6.2 Noncommutative projective schemes on semi-graded rings . . . 170

Introduction

The *Noncommutative differential geometry* of our interest in this thesis corresponds to the *smooth geometry* defined by Brzeziński and Sitarz [BS17]. Briefly, they defined one notion of *smoothness of algebras*, termed *differential smoothness* due to the use of differential graded algebras of a specified dimension that admits a noncommutative version of the Hodge star isomorphism, which considers the existence of a top form in a differential calculus over an algebra together with a string version of the Poincaré duality realized as an isomorphism between complexes of differential and integral forms. This new notion of smoothness is different and more constructive than others homological smoothness defined previously in the literature [Gro64, Sch86, CQ95, SZ94, VdB10]. "The idea behind the *differential smoothness* of algebras is rooted in the observation that a classical smooth orientable manifold, in addition to de Rham complex of differential forms, admits also the complex of *integral forms* isomorphic to the de Rham complex [Man97, Section 4.5]. The de Rham differential can be understood as a special left connection, while the boundary operator in the complex of integral forms is an example of a *right connection*" [BS17, p. 413].

Several authors (e.g. [Brz15, Brz16, BEKL10, BL18, BS17, DV88, DVKM90, Kar15, KL14, RS22]) have characterized the differential smoothness of algebras such as the quantum two - and three - spheres, disc, plane, the noncommutative torus, the coordinate algebras of the quantum group $\mathsf{SU}_q(2)$, the noncommutative pillow algebra, the quantum cone algebras, the quantum polynomial algebras, Hopf algebra domains of Gelfand-Kirillov dimension two that are not PI, families of Ore extensions, some 3-dimensional skew polynomial algebras, diffusion algebras in three generators, and noncommutative coordinate algebras of deformations of several examples of classical orbifolds such as the pillow orbifold, singular cones and lens spaces. An interesting fact is that some of these algebras are also homologically smooth in the Van den Bergh's sense.

With respect to our objects of interest, the *semi-graded Artin-Schelter regular algebras* were defined by Lezama [Lez21] as a generalization of the *Artin-Schelter regular algebras* introduced by Artin and Schelter [AS87]. The latter have been considered in the literature as *noncommutative analogues of commutative polynomial rings* due to its important role in *noncommutative algebraic geometry* (see the excellent treatments carried out by Bellami et al. [BRS+16] and Rogalski [Rog24] for more details). The key difference between *semi-graded Artin-Schelter regular algebras* and *Artin-Schelter regular algebras* is that the former are defined on *semi-graded rings* (these also were defined by Lezama [LL17]) which are precisely an extension of the classical $\mathbb{Z}$-graded rings.

Having in mind the facts above, our purpose in this thesis is to investigate the smooth geometry of the semi-graded Artin-Schelter regular algebras. In this way, we contribute to the research on the noncommutative (algebraic and differential) geometry of these algebras that has been realized by some researchers [Art15, AT24, Cha22, CRR24, CR24a, CR24b, FGL$^+$20, Gal15, GL10, GL17, GS20, HHR20a, HKA17, Lez20, Lez21, LG19, LWZ19, LR20, NR24, Rey19, RS16, Rub20, SRS23, TRS20]. Due to the Gorenstein condition in the definitions of Artin-Schelter regular algebras and semi-graded Artin-Schelter regular algebras, we establish some related ideas with the aim of extending the Serre duality theorem for noncommutative projective schemes on graded algebras to the setting of semi-graded algebras.

On the structure of the thesis, this is not a classical monograph, but rather it is based on a collection of papers. Chapter 1 presents definitions and preliminaries of Artin-Schelter regular algebras, semi-graded rings, semi-graded Artin-Schelter regular algebras and noncommutative differential geometry of algebras in order to set up notation and terminology for the entire document. With the aim of showing their generality in areas such as ring theory and noncommutative geometry, we include a non-exhaustive list of noncommutative algebras that are particular examples of semi-graded rings or semi-graded Artin-Schelter regular algebras. Chapters 2, 3, 4 and 5 contain the original results of the thesis. There, we investigate the smooth geometry in the sense of Brzeziński of bi-quadratic algebras on three generators having PBW basis, double extension regular algebras of type (14641), diffusion algebras and skew Poincaré-Birkhoff-Witt extensions, respectively. Finally, in Chapter 6 we present some ideas concerning the Serre duality for noncommutative projective schemes in the setting of semi-graded rings.

Notation

Symbol	Meaning
$\mathbb{N}$	The set of natural numbers including zero
$\mathbb{Z}$	The ring of integer numbers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
R	Associative (not necessarily commutative) ring with identity
R^*	The set of non-zero elements of the ring R
K	Commutative ring with identity
$\Bbbk$	Field
$Z(R)$	The center of R
$\mathrm{Aut}(R)$	The set of automorphisms of R
$\mathbb{A}^n(R)$	n-Affine space over R
$\mathbb{P}^n(R)$	n-Projective space over R
$\dim_{\Bbbk}(V)$	Dimension of V as a $\Bbbk$-vectorial space
$M_{r\times c}(R)$	Algebra of $r \times c$ matrices with entries in R
$L_n(R)$	The set of lower triangular half-matrices of size $n \times n$ with entries from R
$(M)_{ij}$	Entry of a matrix M in the position i, j
$R\{x_1, \ldots, x_n\}$	Free algebra in the indeterminates $x_1, \ldots, x_n$ on R
$R[x_1, \ldots, x_n]$	Commutative ring of polynomials in the indeterminates $x_1, \ldots, x_n$ on R

The word ring means an associative ring with identity not necessarily commutative. All vector spaces and algebras (always associative and with unit) are over a fixed field $\Bbbk$.

Statement of contributions

The original results obtained in this thesis (Chapters 2, 3, 4, 5 and 6) correspond to the following publications and preprints containing original results.

- **Rubiano, A.** and Reyes, A. Smooth geometry of bi-quadratic algebras on three generators with PBW basis (2024) [Manuscript submitted for publication].

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- **Rubiano, A.** and Reyes, A. A note on the differential smoothness of skew PBW extensions (2024) [Manuscript submitted for publication].

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- **Rubiano, A.** and Reyes, A. Differential smoothness of skew PBW extensions over commutative polynomial rings II (2025) [Manuscript submitted for publication].

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- **Rubiano, A.**, Chacón, A. and Reyes, A. Serre Duality Theorem for Semi-graded Artin-Schelter regular algebras (2024). Preprint.

Conferences

The topics of this thesis have been socialized in the following events.

- **XXIV LCA.** July 22 – July 26, 2024. Speaker at the XXIV Latin American Colloquium on Algebra at the Pontificia Universidad Católica de Chile in Santiago de Chile; academic contribution entitled: "Noncommutative differential geometry of quantum planes".
- **V EMC.** July 02 – July 05, 2024. Speaker at the V Caribbean Mathematical Meeting at the UTB in Cartagena, Colombia; academic contribution entitled: "Noncommutative differential geometry of quantum planes".
- **XXIII CCM.** June 05 – June 09, 2023. Speaker at the XXIII Colombian Congress of Mathematics at the UPTC in Tunja, Colombia; academic contribution entitled: "Noncommutative differential geometry of quantum planes".
- **ENEMAT.** November 02 – November 04, 2022. Speaker at the National Meeting of Students in Mathematics at the San Joaquín campus of the Pontificia Universidad Católica de Chile, in Santiago de Chile; academic contribution entitled: "On differential smoothness in quantized algebras".
- **USaCH.** October 31 – November 04, 2022. Exhibition entitled "On differential smoothness in quantized algebras" in the Postgraduate Mathematics Seminar.

CHAPTER 1

Definitions and preliminaries

In this chapter we present the algebraic structures of interest in this thesis: the *semi-graded Artin-Schelter regular algebras*.

With this aim, in Section 1.1 we start by recalling definitions and preliminaries on *Artin-Schelter regular algebras* introduced by Artin and Schelter [AS87]. Next, Section 1.2 contains definitions and some key properties of *semi-graded rings*, *finitely semi-graded rings*, *finitely semi-graded algebras* and modules over these rings. We also present a non-exhaustive list of examples of semi-graded rings with the aim of illustrating its generality. In Section 1.3 we consider the notion of *semi-graded Artin-Schelter regular algebra*, and present some examples that illustrate the differences between *Artin-Schelter regular algebras* and *semi-graded Artin-Schelter regular algebra*.

1.1 Artin-Schelter regular algebras

In the setting of noncommutative algebras appearing in noncommutative geometry, *Artin-Schelter regular algebras* introduced by Artin and F. Schelter [AS87] as a class of three-dimensional graded algebras may be regarded as noncommutative versions of the polynomial ring in three indeterminates. These algebras have been extensively studied (for more details, see the excellent surveys carried out by Bellamy et al. [BRS+16] and Rogalski [Rog24]).

Definition 1.1. A $\Bbbk$-algebra A is said *finitely graded* if the following conditions hold:

(i) A is $\mathbb{N}$-graded, i.e., $A = \bigoplus_{n\geq 0} A_n$.

(ii) A is *connected*, that is, $A_0 = \Bbbk$.

(iii) A is finitely generated as $\Bbbk$-algebra.

By definition, $A_{\geq 1} = \bigoplus_{n\geq 1} A_n$.

Let A be a finitely graded $\Bbbk$-algebra and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a $\mathbb{Z}$-graded A-module which is finitely generated. Then:

(i) For every $n \in \mathbb{Z}$, we have that $\dim_{\Bbbk} M_n < \infty$.

(ii) The *Hilbert series* of A is defined by

$$h_A(t) := \sum_{n=0}^{\infty} (\dim_{\Bbbk} A_n)\, t^n.$$

Similarly, the *Hilbert series* of M is the formal Laurent series

$$h_M(t) = \sum_{n\in\mathbb{Z}} (\dim_{\Bbbk} M_n)\, t^n.$$

(iii) The *Gelfand-Kirillov dimension* of A is defined by

$$\mathrm{GKdim}(A) = \sup_V \overline{\lim_{n\to\infty}} \log_n \dim_{\Bbbk} V^n, \tag{1.1.1}$$

where V ranges over all *frames* of A and $V^n := {}_{\Bbbk}\langle v_1 \cdots v_n \mid v_i \in V\rangle$ (recall that a *frame* of A is a finite-dimensional $\Bbbk$-subspace of A such that $1 \in V$; since A is a $\Bbbk$-algebra, then $\Bbbk \to A$, and so $\Bbbk$ is a frame of A of dimension one).

If $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is an $\mathbb{N}$-graded A-module and $i \in \mathbb{Z}$, then the graded module $M(i)$ defined by $M(i)_n := M_{i+n}$ is called a *shift* of M, i.e.

$$M(i) = \bigoplus_{n\in\mathbb{Z}} M(i)_n = \bigoplus_{n\in\mathbb{Z}} M_{i+n}.$$

Definition 1.2. An *Artin-Schelter* (AS, for short) *regular algebra* is an $\mathbb{N}$-graded algebra $A = \bigoplus_{n\geq 0} A_n$ over $\Bbbk$ which is *connected* (that is, $A_0 = \Bbbk$) and satisfies the following three conditions:

(i) A has finite global dimension d, that is, $\mathrm{gld}(A) = d < \infty$.

(ii) A has polynomial growth, i.e., $\mathrm{GKdim}(A) < \infty$.

(iii) (*Gorenstein condition*) A is *Gorenstein* in the sense that

$$\mathrm{Ext}^i_A({}_A\Bbbk,\ {}_AA) \cong \begin{cases} 0, & \text{if } i \neq d, \\ \Bbbk(l)_A, & \text{if } i = d, \end{cases}$$

for some shift $l \in \mathbb{Z}$. Equivalently,

$$\mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong \begin{cases} 0, & \text{if } i \neq d, \\ {}_A\Bbbk(l), & \text{if } i = d, \end{cases}$$

for some shift $l \in \mathbb{Z}$ [Rog24].

The following algebras show the importance of conditions in Definition 1.2.

Example 1.1 ([Rog24, Examples 1.6 and 1.7]).

- Let $A = \Bbbk\{x, y\}/\langle xy\rangle$. Then A has a $\Bbbk$-basis consisting of words $\{x^i y^j \mid i, j \geq 0\}$, as occurs for a polynomial ring in two indeterminates $\Bbbk[x, y]$. Notice that A has polynomial growth and the global dimension of the algebra is two. However, A is not a domain, non-Noetherian, and A is not Gorenstein, so condition (iii) above fails.
- Let $A = \Bbbk\{x, y, z\}/\langle x^2 + y^2 + z^2\rangle$. Then A is a graded domain with global dimension two satisfying the Gorenstein property, having exponential growth, but A is non-Noetherian.

Regular algebras of global dimension at most 2 are easy to classify and they share properties similar to those of commutative polynomial rings. More exactly:

Proposition 1.1 ([Rog24, Examples 1.8 and 1.9]). (1) *If A is a $\Bbbk$-algebra which is regular of global dimension one, then A is the commutative polynomial ring in one indeterminate, i.e. $A \cong \Bbbk[x]$.*

(2) *Let A be a regular algebra of global dimension two. Then either $A \cong \mathcal{O}_q(\Bbbk) = \Bbbk\{x, y\}/\langle yx - qxy\rangle$, for some $q \in \Bbbk^*$* (Manin's plane, *also known as the* Quantum plane) *or $A \cong \mathcal{J}(\Bbbk) = \Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$* (Jordan's plane [Jor01]). *Recall that these both algebras are not isomorphic* [Shi05, Theorem 1.4].

Artin and Schelter's work with the subsequent work of Artin, Tate, and Van den Bergh [AS87, ATdB07, ATVdB91, Ste96] allowed a complete classification of AS regular algebras of global dimension 3 since these algebras have geometric interpretations related to point schemes in projective space. Proposition 1.2 recalls this classification.

Proposition 1.2 ([BRS+16, Lemma 2.3.1]). *Let A be an AS regular algebra of global dimension 3 which is generated in degree 1. Then exactly one of the following holds:*

(1) *$A \cong \Bbbk\{t_1, t_2, t_3\}/\langle f_1, f_2, f_3\rangle$, where the f_i have degree 2 for $i = 1, 2, 3$, and $h_A(t) = \frac{1}{(1-t)^3}$.*

(2) *$A \cong \Bbbk\{t_1, t_2\}/\langle f_1, f_2\rangle$, where the f_i have degree 3 for $i = 1, 2$, and $h_A(t) = \frac{1}{(1-t)^2(1-t^2)}$.*

Notice that there are 13 types of non-isomorphic algebras for Proposition 1.2 (seven for the first case and six for the second one). Based on the degree in which the relations occur, we say that A is a *quadratic* regular algebra in the first case of Proposition 1.2, and that A is *cubic* in the second case.

Remark 1. (i) From their definition, AS regular algebras are $\mathbb{N}$-graded connected algebras. In his PhD thesis, Gaddis [Gad13] extended the ideas developed for Artin-Schelter regular algebra to algebras that are not necessarily graded. Briefly, given an algebra A which is presented by generators and (not-necessarily homogeneous) relations, Gaddis [Gad13] homogenized the relations to produce a new, graded algebra, $H(A)$, called the *homogenization* of A. In case $H(A)$ is regular, Gaddis said

that A is *essentially regular* [Gad13, Definition 2.3.1]. In other words, an algebra is *essentially regular of dimension* d *if its homogenization is regular of dimension* $d+1$. These algebras are also referred to as *central extensions of a regular algebra* in [CS07, BSdB96]. This is due to the fact that $H(A)$ is regular if and only if $G(A)$ is regular [Gad13, Proposition 2.3.7]. Therefore, the study of regular homogenizations is equivalent to the study of PBW deformations of regular algebras. Examples of this kind of algebras include the quantum Weyl algebras and the enveloping algebra of the two-dimensional nonabelian Lie algebra. Notice that if A is Noetherian, then it has finite global and GK dimension [Gad16, Corollary 2.11] resembling AS regularity properties.

The following classification of two-generated algebras gives us a full classification of PBW deformations of AS regular algebras in global dimension two.

Proposition 1.3 ([Gad13, Theorem 4.0.7]; [Gad15, Theorem 1.1]). *Suppose* $A \cong \Bbbk\{x,y\}/\langle f\rangle$ *where* f *is a polynomial of degree two. Then* A *is isomorphic to one of the following algebras:*

- $\mathcal{O}_q(\Bbbk)$, $f = xy - qyx$ $(q \in \Bbbk^*)$;
- $A_1^q(\Bbbk)$, $f = xy - qyx - 1$ $(q \in \Bbbk^*)$ (*Quantum Weyl algebra*);
- $\mathcal{J}(\Bbbk)$, $f = yx - xy - y^2$;
- $\mathcal{J}_1(\Bbbk)$, $f = yx - xy + y^2 + 1$ (*Deformed Jordan plane*);
- $U(\Bbbk)$, $f = yx - xy + y$ (*Universal enveloping algebra of a 2-d solvable Lie algebra*);
- $\Bbbk[x]$, $f = x^2 + y$;
- $R_{x^2}(\Bbbk)$, $f = x^2$;
- $R_{x^2-1}(\Bbbk)$, $f = x^2 - 1$;
- $R_{yx}(\Bbbk)$, $f = yx$;
- $\mathcal{S}(\Bbbk)$, $f = yx - 1$.

Furthermore, the above algebras are pairwise nonisomorphic, except

$$\mathcal{O}_q(\Bbbk) \cong \mathcal{O}_{q^{-1}}(\Bbbk) \quad \text{and} \quad A_1^q(\Bbbk) \cong A_1^{q^{-1}}(\Bbbk).$$

If A is a PBW deformation of an AS regular algebra of dimension two, then A is isomorphic to one of the following: $\mathcal{O}_q(\Bbbk), A_1^q(\Bbbk), \mathcal{J}(\Bbbk), \mathcal{J}_1(\Bbbk), U(\Bbbk)$ [Gad16, Corollary 2.13]. Notice that the list in Proposition 1.3 slightly contradicts that given by Smith [Smi92] since $\mathcal{S}(\Bbbk)$ and $\mathcal{J}_1(\Bbbk)$ both have GK-dimension two.

(ii) With the purpose of giving new examples of AS regular algebras, Lu et al. [LZ14, LSZ15] defined the $\mathbb{Z}^s$-graded AS regular algebras and some results for the classification of AS regular algebras of dimension five with two generators were proven.

The classification of AS regular algebras in higher global dimensions is an open and active problem in the field. For instance, there has been an extensive research about AS regular algebras of dimension four. Some examples of AS regular algebras of dimension four include

graded regular Clifford and *skew-Clifford algebras* [CV10]; *homogenization of the universal enveloping algebra of* $\mathfrak{sl}(2)$, $\mathsf{U}(\mathfrak{sl}(2))$ [LBS93]; *deformations of the Sklyanin algebra* [Sta94]; *skew polynomial rings in four variables*, the *quantum* 2×2 *matrix algebra* [Van94], AS *regular algebras with finitely many points*, and AS *regular algebras containing a commutative quadric* [Rub20, SV99, SV06, VVR97, VRV00, VRVW98]. Chapter 3 is dedicated to one of the most important family of Artin-Schelter regular algebras of dimension four: *double Ore extensions*, and in particular, *double extension regular algebras of type* (*14641*).

1.2 Semi-graded rings

Lezama and Latorre [LL17] introduced the *semi-graded rings* as a generalization of $\mathbb{N}$-graded rings and several families of noncommutative rings of polynomial type non-$\mathbb{N}$-graded (not in a trivial way). These rings extend several kinds of noncommutative rings of polynomial type such as

- Ore extensions [Ore31, Ore33],
- ambiskew polynomial rings [Jor00],
- 3-dimensional skew polynomial rings [BS90, RS22, Ros95],
- bi-quadratic algebras on 3 generators with PBW bases [Bav23] (Chapter 2),
- double Ore extensions [ZZ08, ZZ09] (Chapter 3), and
- algebras appearing in mathematical physics, the diffusion algebras [IPR01, RS22] (Chapter 4),
- families of differential operators generalizing Weyl algebras and universal enveloping algebras of finite dimensional Lie algebras, the PBW extensions [BG88] (Chapter 5),
- skew PBW extensions [GL10, LR14] (Chapter 5), and
- other algebras having PBW bases [GM98, GM05, NR24].

A detailed list of examples of semi-graded rings and its relationships with other algebras can be found in Fajardo et al. [FGL$^+$20].

Definition 1.3 ([LL17, Definition 2.1])**.** Let R be a ring. R is said to be *semi-graded* (SG) if there exists a collection $\{\mathsf{R}_n\}_{n\in\mathbb{Z}}$ of subgroups of the additive group R^+ such that the following conditions hold:

(i) $\mathsf{R} = \bigoplus_{n\in\mathbb{Z}} \mathsf{R}_n$.

(ii) For every $m, n \in \mathbb{Z}$ we have that $\mathsf{R}_m\mathsf{R}_n \subseteq \bigoplus_{k\leq m+n} \mathsf{R}_k$.

(iii) $1 \in \mathsf{R}_0$.

The collection $\{R_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* R and the elements of R_n are said to be *homogeneous of degree* n. We say that R is *positively semi-graded* if $R_n = 0$ for every $n < 0$. If R and S are semi-graded rings and $f : R \to S$ is a ring homomorphism then f is called *homogeneous* if $f(R_n) \subseteq S_n$ for every $n \in \mathbb{Z}$.

Definitions 1.4 and 1.5 present the notion of finitely semi-graded ring and finitely semi-graded algebra, respectively.

Definition 1.4 ([LL17, Definition 2.4])**.** A ring R is called *finitely semi-graded* (FSG) if it satisfies the following conditions:

(i) R is SG.

(ii) There exist finitely many elements $x_1, \ldots, x_n \in R$ such that the subring generated by R_0 and $x_1, \ldots, x_n$ coincides with R.

(iii) For every $n \geq 0$, we have that R_n is a free R_0-module of finite dimension.

Definition 1.5 ([LG19, Definition 10])**.** A $\Bbbk$-algebra R is said to be *finitely semi-graded* (FSG) if the following conditions hold:

(i) R is an FSG ring with semi-graduation given by $R = \bigoplus_{n\geq 0} R_n$.

(ii) For every $m, n \geq 1$, we have that $R_m R_n \subseteq \bigoplus_{k\leq m+n} R_k$.

(iii) R is connected, i.e. $R_0 = \Bbbk$.

(iv) R is generated in degree 1.

From Definition 1.5 it follows that if R is a FSG $\Bbbk$-algebra, then $R_{\geq 1} := \bigoplus_{n\geq 1} R_n$ is a maximal ideal of R.

It is clear that $\mathbb{Z}$-graded rings are SG, while finitely graded $\Bbbk$-algebras, diffusion algebras [IPR01] (Chapter 4), PBW extensions [BG88] and skew PBW extensions [GL10] (Chapter 5), 3-dimensional skew polynomial rings [BS90], down-up algebras [Ben99, BR98] are examples of FSG rings.

Now, we present some results on modules over semi-graded rings.

Definition 1.6 ([LL17, Definition 2.1])**.** Let R be an SG ring and let M be an R-module. We say that M is *semi-graded* (SG) if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups M_n of the additive group M^+ such that the following conditions hold:

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) For every $m, n \in \mathbb{Z}$ we have that $R_m M_n \subseteq \bigoplus_{k\leq m+n} M_k$.

The collection $\{M_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* M and we say that the elements of M_n are *homogeneous of degree* n.

M is said to be *positively semi-graded* if $M_n = 0$ for every $n < 0$. Let $f : M \to N$ be a homomorphism of R-modules where M and N are semi-graded R-modules. We say that f is *homogeneous* if $f(M_n) \subseteq N_n$ for every $n \in \mathbb{Z}$.

Definition 1.7 ([LL17, Definition 2.3])**.** Let R be an SG ring, M an SG R-module, and N a submodule of M. We say that N is a *semi-graded* (SG) *submodule of* M if $N = \bigoplus_{n\in\mathbb{Z}} N_n$ where $N_n = M_n \cap N$. In this case, N is an SG R-module.

Proposition 1.4 ([LL17, Proposition 2.6])**.** *If R is an SG ring, M is an SG R-module, and N is a submodule of M, then the following conditions are equivalent:*

(1) *N is a semi-graded submodule of M.*

(2) *For every $z \in N$ the homogeneous components of z belong to N.*

(3) *M/N is an SG R-module with semi-graduation given by*

$$(M/N)_n = (M_n + N)/N, \ n \in \mathbb{Z}.$$

Remark 2. Let R be an SG ring and M be an SG R-module. Then:

(i) If N is an SG submodule of M then the canonical map $M \to M/N$ is a homogeneous homomorphism.

(ii) If $\{M_i\}_{i\in I}$ is a family of SG submodules of M, then $\bigcap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are SG submodules of M.

Let N be a subset of M. We define the SG *submodule generated by* N as the intersection of all SG submodules of M containing N and we denote it as $\langle N\rangle^{SG}$. If $N = \{n_1, \dots, n_l\}$ then we write $\langle N\rangle^{SG} = \langle n_1, \dots, n_l\rangle^{SG}$. We say that M is a *finitely generated* SG R-module if there exist finitely elements $m_1, \dots, m_t$ such that $M = \langle m_1, \dots, m_t\rangle^{SG}$. If M is simultaneously a module over different kinds of rings and there is risk of confusion, we write $\langle -\rangle^{SG}_R$ to indicate the ring R we are considering. If N is an SG submodule of M, the notion of *finitely generated* SG submodule is defined in the natural way.

Example 1.2. Consider the first Weyl algebra $A_1(\Bbbk) = \Bbbk\{x, y\}/\langle yx - xy - 1\rangle$ over $\Bbbk$. By using that $yx = xy + 1$, the graded condition for multiplication does not hold since we do not obtain a product of pure degree one, so $A_1(\Bbbk)$ is not an $\mathbb{N}$-graded ring. On the other hand, the canonical semi-graduation of $A_1(\Bbbk)$ [Lez21, Example 4.5] is given by

$$A_1(\Bbbk) = \Bbbk \oplus {}_{\Bbbk}\langle x, y\rangle \oplus {}_{\Bbbk}\langle x^2, xy, x^2\rangle \oplus \cdots \tag{1.2.1}$$

Then:

(i) $A_1(\Bbbk)y$ is an SG submodule of $A_1(\Bbbk)$, whence $\langle y\rangle^{SG} = A_1(\Bbbk)y$.

(ii) $1 \notin A_1(\Bbbk)x$, but due to the relation $yx = xy + 1$ it follows that $1 \in \langle x\rangle^{SG}$.

Different properties of modules over families of semi-graded rings have been investigated in [FGL+20, GL17, HR23, HR24, Lez20, LG19, Lez21, LR20, NRR20, NR24, Rey19].

Next, we present a non exhaustive list of noncommutative rings that are examples of SG rings.

1.2.1 Skew polynomial rings

Ore [Ore31, Ore33] introduced a kind of noncommutative polynomial rings (related with that, Noether and Schmeidler [NS20] were interested in some kind of differential operator rings) which has become one of most basic and useful constructions in ring theory and noncommutative algebra. For σ an endomorphism of a ring R, a left σ-*derivation* on R is any additive map $\delta : R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$, for all $r, s \in R$ (a *right* σ-derivation is any additive map $\delta : R \to R$ satisfying the rule $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$). If σ is the identity map on R, then σ-derivations are just ordinary derivations. Notice that

$$\delta(1) = \delta(1 \cdot 1) = \sigma(1)\delta(1) + \delta(1)1 = 2\delta(1) \quad \text{whence} \quad \delta(1) = 0.$$

The pair (σ, δ) is called a *quasi-derivation on* R [BGTV03, Definition 3.1].

For any pair of maps $\sigma, \delta : R \to R$, the following assertions are equivalent [BGTV03, Proposition 3.2]:

(1) (σ, δ) is a quasi-derivation;

(2) the map

$$\begin{aligned} \phi : R &\to M_{2\times 2}(R) \\ r &\mapsto \begin{bmatrix} \sigma(r) & \delta(r) \\ 0 & r \end{bmatrix} \end{aligned}$$

is a ring homomorphism.

Definition 1.8 ([GJ04, p. 34])**.** Let R be a ring, σ a ring endomorphism of R and δ a σ-derivation on R. We will write $R[x; \sigma, \delta]$ provided

(i) $R[x; \sigma, \delta]$ is a ring, containing R as a subring;

(ii) x is an element of $R[x; \sigma, \delta]$;

(iii) $R[x; \sigma, \delta]$ is a free left R-module with basis $\{1, x, x^2, \dots\}$;

(iv) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

Such a ring $R[x; \sigma, \delta]$ is called a *skew polynomial ring over* R or an *Ore extension of* R *of mixed type*. If σ is an injective map of R then we call it an *Ore extension of injective type*, while if σ is the identity of R then we write $R[x; \delta]$ and call it an *Ore extension of derivation type*. On the other hand, if δ is the zero map then we write $R[x; \sigma]$ which is known as an *Ore extension of endomorphism type*. The construction of Ore extensions can be iterated.

Indeed, if we consider $S_1 = R[x_1; \sigma_1, \delta_1]$ for a quasi-derivation (σ_1, δ_1) on R, and we assume that (σ_2, δ_2) is a quasi-derivation on S_1, then one may associate to it the Ore extension

$$S_2 = S_1[x_2; \sigma_2, \delta_2] = R[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2].$$

Repeating this procedure, one obtains the so-called *iterated Ore extensions*.

As one can appreciate in the literature, a lot of papers and books have been published concerning ring-theoretical, homological, geometrical properties and applications of these extensions (e.g. [BG02, BGTV03, FGL+20, FLP+24, Goo94, GJ04, MRS01, Li02, Ros95, Sei10] and references therein).

Ore extensions are one of the most important techniques to define noncommutative algebras. Let us see some examples:

(i) The Jordan plane $\mathcal{J}(\Bbbk) = \Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$ can be expressed as $\Bbbk[y][x; \delta_2]$ with $\delta_2(y) = -y^2$.

(ii) Díaz and Pariguan [DP09] defined the q-*meromorphic Weyl algebra* (also known as the q-*skew Jordan's plane*) $\mathcal{J}_q(\Bbbk)$ as the unital associative algebra with generators x, y and defining relation $qyx - xy - y^2 = 0$ for an element $q \in \Bbbk^*$. This algebra can be written as the Ore extension $\Bbbk[y][x; \sigma_2, \delta_2]$ with $\sigma_2(y) = qy$ and $\delta_2(y) = -y^2$. Lopes noted that using the generator $x' := x + (q-1)^{-1}y$ instead of x we find that the q-skew Jordan's plane is the quantum plane in disguise [Lop24, Example 3.1].

(iii) About the family of Weyl algebras $A_n(\Bbbk)$ in the literature it is common to find characterizations of these algebras as rings of differential operators. Following Coutinho [Cou95], the nth *Weyl algebra* $A_n(\Bbbk)$ over $\Bbbk$ is the $\Bbbk$-algebra generated by the $2n$ indeterminates $x_1, \ldots, x_n, y_1, \ldots, y_n$ where

$$\begin{aligned} &x_j x_i = x_i x_j, \quad y_j y_i = y_i y_j, && 1 \leq i < j \leq n, \\ &y_j x_i = x_i y_j + \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker's delta}, && 1 \leq i, j \leq n. \end{aligned}$$

From the relations defining Weyl algebras it follows that these cannot be expressed as skew polynomial rings of automorphism type (since the algebra is simple) but skew polynomial rings with non-trivial derivations.

(iv) Based on Wess and Zumino's paper [WZ91], for an element $q \in \Bbbk \setminus \{0\}$, Giaquinto and Zhang [GZ95] defined the $\Bbbk$-algebra $A_1^q(\Bbbk)$ generated by two indeterminates x and y subject to the relation $xy - qyx = 1$, which is known as a *quantized Weyl algebra* over $\Bbbk$. Note that $A_1^q(\Bbbk) = A_1(\Bbbk) = \Bbbk[y][x; d/dy]$, when $q = 1$. If $q \neq 1$, then $A_1^q(\Bbbk) = \Bbbk[y][x; \sigma, \delta]$, where σ is the $\Bbbk$-algebra automorphism given by $\sigma(f(y)) = f(qy)$, and δ is the q-*difference operator* (also known as *Eulerian derivative*)

$$\delta(f(y)) = \frac{f(qy) - f(y)}{qy - y} = \frac{\alpha(f) - f}{\alpha(y) - y},$$

as it can be seen in [GJ04, Exercise 2N].

(v) A generalization of $A_1^q(\Bbbk)$ is given by the *Additive analogue of the Weyl algebra* $A_n(q_1,\dots,q_n)$. This algebra was introduced in Quantum Physics by Kuryshkin [Kur80] and studied by Jannussis et al. [JBS81] (see [Li02] for more details). For non-zero elements $q_1,\dots,q_n \in \Bbbk$, this algebra is generated by the indeterminates $x_1,\dots,x_n$ and $y_1,\dots,y_n$ satisfying the relations

$$x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, \quad \text{for every } 1 \le i,j \le n, \quad \text{and}$$
$$y_ix_j = x_jy_i, \quad \text{for all } i \ne j, \quad \text{and} \quad y_ix_i = q_ix_iy_i + 1, \quad \text{for} \quad 1 \le i \le n.$$

(vi) The *Multiplicative analogue of the Weyl algebra* (see Jategaonkar [Jat84] and McConnell and Pettit [MP88] for more details) is the algebra $\mathcal{O}_n(\lambda_{ij})$ generated over a field $\Bbbk$ by the indeterminates $x_1,\dots,x_n$ subject to the relations

$$x_jx_i = \lambda_{ij}x_ix_j, \quad \text{for } 1 \le i < j \le n \text{ and } \lambda_{ij} \in \Bbbk^*.$$

As it can be seen, $\mathcal{O}_n(\lambda_{ij})$ is isomorphic to the iterated skew polynomial ring

$$\Bbbk[x_1][x_2;\sigma_2]\cdots[x_n;\sigma_n] \quad \text{where } x_jx_i = \sigma_j(x_i)x_j = \lambda_{ij}x_ix_j \text{ for } 1 \le i < j \le n.$$

If $n = 2$, then $\mathcal{O}_2(\lambda_{12})$ is the *quantum plane* in the sense of Manin [Man18]. If $\lambda_{ij} = q^{-2} \ne 0$ for some $q \in \Bbbk^*$ and all $1 \le i < j \le n$, then $\mathcal{O}_{\lambda_{ij}}$ becomes the well-known coordinate ring of the so called *quantum affine* n*-space* [Smi92].

(vii) Another deformation of Weyl algebras was introduced by Giaquinto and Zhang [GZ95] with the aim of studying the Jordan Hecke symmetry as a quantization of the usual second Weyl algebra. By definition, the *quantum Weyl algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the indeterminates $x_1, x_2, \partial_1, \partial_2$, with relations (depending on parameters $a, b \in \Bbbk$)

$$\begin{aligned} &x_1x_2 = x_2x_1 + ax_1^2, && \partial_2\partial_1 = \partial_1\partial_2 + b\partial_2^2 \\ &\partial_1x_1 = 1 + x_1\partial_1 + ax_1\partial_2, && \partial_1x_2 = -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2 \\ &\partial_2x_1 = x_1\partial_2, && \partial_2x_2 = 1 - bx_1\partial_2 + x_2\partial_2. \end{aligned}$$

Note that if $a = b = 0$, then $A_2(J_{0,0})$ is precisely the second Weyl algebra $A_2(\Bbbk)$.

(viii) By definition, the q-*Heisenberg algebra* $\mathbf{H}_n(q)$ is the $\Bbbk$-algebra generated over $\Bbbk$ by the indeterminates x_i, y_i, z_i, for $1 \le i \le n$, subject to the relations

$$\begin{aligned} x_ix_j = x_jx_i, \quad y_iy_j = y_jy_i, \quad z_jz_i = z_iz_j, && \qquad 1 \le i < j \le n, \\ x_iz_i - qz_ix_i = z_iy_i - qy_iz_i = x_iy_i - q^{-1}y_ix_i + z_i = 0, && \qquad 1 \le i \le n, \\ x_iy_j = y_jx_i, \quad x_iz_j = z_jx_i, \quad y_iz_j = z_jy_i, && \qquad i \ne j. \end{aligned}$$

It can be seen that $\mathbf{H}_n(q)$ can be expressed as an iterated skew polynomial ring.

(ix) The *single parameter quantum matrix algebra* $O_p(M_n(\Bbbk))$ for $p \in \Bbbk^*$ has generating set $\{x_{ij} \mid 1 \le i,j \le n\}$, subject to the relations given by

$$x_{ij}x_{lm} = \begin{cases} px_{lm}x_{ij}, & i > l, j = m, \\ px_{lm}x_{ij}, & i = l, j > m, \\ x_{lm}x_{ij}, & i > l, j < m, \\ x_{lm}x_{ij} + (p - p^{-1})x_{im}x_{lj}, & i > l, j > m. \end{cases}$$

It is straightforward to show that

$$O_p(M_2(\Bbbk)) \cong \Bbbk[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}][x_{22};\sigma_{22},\delta_{22}]$$

with the maps given by

$$\begin{aligned}
&\sigma_{12}(x_{11}) = px_{11}, && \sigma_{21}(x_{11}) = px_{11},\\
&\sigma_{22}(x_{21}) = px_{21}, && \sigma_{22}(x_{12}) = px_{12},\\
&\delta_{22}(x_{11}) = (p-p^{-1})x_{21}x_{12}.
\end{aligned}$$

Otherwise, the value of σ_{ij} is the identity map, and for $1 \le i,j \le 2$, the maps δ_{ij} are zero [GJ04, Exercise 2V]

(x) Jordan [Jor95] introduced a certain class of iterated Ore extensions called *ambiskew polynomial rings*. These structures have been studied by Jordan et al. [Jor00, JW96] at various levels of generality that contain different examples of noncommutative algebras. Next, we recall briefly its definition.

Consider a commutative $\Bbbk$-algebra B, a $\Bbbk$-automorphism of B, and elements $c \in B$ and $p \in \Bbbk^*$. Let S be the Ore extension $B[x;\sigma^{-1}]$ and extend σ to S by setting $\sigma(x) = px$. By [Coh85, p. 41], there is a σ-derivation δ of S such that $\delta(B) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $R = R(B,\sigma,c,p)$ is the Ore extension $S[y;\sigma,\delta]$, whence the following relations hold:

$$yx - pxy = c, \quad \text{and, for all } b \in B, \quad xb = \sigma^{-1}(b)x \quad \text{and} \quad yb = \sigma(b)y. \tag{1.2.2}$$

Equivalently, R can be presented as $R = B[y;\sigma][x;\sigma^{-1},\delta']$ with $\sigma(y) = p^{-1}y$, $\delta'(B) = 0$, and $\delta'(y) = -p^{-1}c$, so that $xy - p^{-1}yx = -p^{-1}c$. If we consider the relation $xb = \sigma^{-1}(b)x$ as $bx = x\sigma(b)$, then we can see that the definition involves twists from both sides using σ; this is the reason for the name of the objects.

1.2.2 3-dimensional skew polynomial algebras

Another kind of noncommutative rings which includes the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$, the Dispin algebra $U(\mathfrak{osp}(1,2))$ and the Woronowicz's algebra $W_\nu(\mathfrak{sl}_2(\Bbbk))$ [Wor87], is the family of *3-dimensional skew polynomial algebras*. These algebras were introduced by Bell and Smith [BS90] and are very important in noncommutative algebraic geometry and noncommutative differential geometry (e.g., [Red96a, Red99, RS22, RS17a, Ros95] and references therein).

Definition 1.9 ([Ros95, Definition C4.3]). *A 3-dimensional skew polynomial algebra* A is a $\Bbbk$-algebra generated by the indeterminates x,y,z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that

(i) $\lambda,\mu,\nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha,\beta,\gamma \in \Bbbk \setminus \{0\}$;

(ii) standard monomials $\{x^i y^j z^l \mid i,j,l \ge 0\}$ are a $\Bbbk$-basis of the algebra.

Proposition 1.5 ([Ros95, Theorem C.4.3.1]). *If A is a 3-dimensional skew polynomial algebra, then A is one of the following algebras:*

(1) *if* $|\{\alpha,\beta,\gamma\}| = 3$, *then* A *is given by the relations* $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$.

(2) *if* $|\{\alpha,\beta,\gamma\}| = 2$ *and* $\beta \neq \alpha = \gamma = 1$, *then* A *is one of the following algebras:*

(i) $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$;
(ii) $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$;
(iii) $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$;
(iv) $yz - zy = 0$, $zx - \beta xz = b$, $xy - yx = 0$;
(v) $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$;
(vi) $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$,

where a, b *are any elements of* $\Bbbk$. *All non-zero values of* b *give isomorphic algebras.*

(3) *If* $|\{\alpha,\beta,\gamma\}| = 2$ *and* $\beta \neq \alpha = \gamma \neq 1$, *then* A *is one of the following algebras:*

(i) $yz - \alpha zy = 0$, $zx - \beta xz = y + b$, $xy - \alpha yx = 0$;
(ii) $yz - \alpha zy = 0$, $zx - \beta xz = b$, $xy - \alpha yx = 0$.

In this case, b *is an arbitrary element of* $\Bbbk$. *Again, any non-zero values of* b *give isomorphic algebras.*

(4) *If* $\alpha = \beta = \gamma \neq 1$, *then* A *is the algebra defined by the relations* $yz - \alpha zy = a_1x + b_1$, $zx - \alpha xz = a_2y + b_2$, $xy - \alpha yx = a_3z + b_3$. *If* $a_i = 0$ $(i = 1,2,3)$, *then all non-zero values of* b_i *give isomorphic algebras.*

(5) *If* $\alpha = \beta = \gamma = 1$, *then* A *is isomorphic to one of the following algebras:*

(i) $yz - zy = x$, $zx - xz = y$, $xy - yx = z$;
(ii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = z$;
(iii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = b$;
(iv) $yz - zy = -y$, $zx - xz = x + y$, $xy - yx = 0$;
(v) $yz - zy = az$, $zx - xz = z$, $xy - yx = 0$;

Parameters $a, b \in \Bbbk$ *are arbitrary, and all non-zero values of* b *generate isomorphic algebras.*

1.2.3 Generalized Weyl algebras and down-up algebras

Other algebraic structures that illustrate the results obtained in this thesis are the *generalized Weyl algebras* and *down-up algebras*. We briefly present the definitions and some relations between these algebras (see [Jor95, Jor00, JW96] for a detailed description).

Given an automorphism σ and a central element a of a ring R, Bavula [Bav92] defined the *generalized Weyl algebra* $R(\sigma, a)$ as the ring extension of R generated by the indeterminates

X^- and X^+ subject to the relations $X^-X^+ = a$, $X^+X^- = \sigma(a)$, and, for all $b \in R$, $X^+b = \sigma(b)X^+$, $X^-\sigma(b) = bX^-$. This family of algebras includes the classical Weyl algebras, primitive quotients of $U(\mathfrak{sl}_2)$, and ambiskew polynomial rings. Generalized Weyl algebras have been extensively studied in the literature by various authors (see [Bav92, Bav23, Jor00], and references therein).

On the other hand, the *down-up algebras* $A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$, were defined by Benkart and Roby [Ben99, BR98] as generalizations of algebras generated by a pair of operators, precisely, the "down" and "up" operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered set P. More exactly, consider a partially ordered set $(P, \prec)$ and let $\mathbb{C}P$ be the complex vector space with basis P. If for an element p of P, the sets $\{x \in P \mid x \succ p\}$ and $\{x \in P \mid x \prec p\}$ are finite, then we can define the "down" operator d and the "up" operator u in $\mathrm{End}_{\mathbb{C}}\, \mathbb{C}P$ as $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$, respectively (for partially ordered sets in general, one needs to complete $\mathbb{C}P$ to define d and u). For any $\alpha, \beta, \gamma \in \mathbb{C}$, the *down-up algebra* is the $\mathbb{C}$-algebra generated by d and u subject to the relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$ and $du^2 = \alpha udu + \beta u^2 d + \gamma u$. A partially ordered set P is called (q, r)-*differential* if there exist $q, r \in \mathbb{C}$ such that the down and up operators for P satisfy both relations, and $\alpha = q(q+1)$, $\beta = -q^3$, and $\gamma = r$. From [BR98], we know that for $0 \neq \lambda \in \mathbb{C}$, $A(\alpha, \beta, \gamma) \simeq A(\alpha, \beta, \lambda\gamma)$. This means that when $\gamma \neq 0$, no problem if we assume $\gamma = 1$. For more details about the combinatorial origins of down-up algebras, see [Ben99, Section 1].

Remarkable examples of down-up algebras include the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and some of its deformations introduced by Witten [Wit90] and Woronowicz [Wor87]. Related to the theoretical properties of these algebras, Kirkman et al. [KMP99] proved that a down-up algebra $A(\alpha, \beta, \gamma)$ is Noetherian if and only if β is non-zero. As a matter of fact, they showed that $A(\alpha, \beta, \gamma)$ is a generalized Weyl algebra and that $A(\alpha, \beta, \gamma)$ has a filtration for which the associated graded ring is an iterated Ore extension over $\mathbb{C}$.

Following [Ben99, p. 32], if $\mathfrak{g}$ is a 3-dimensional Lie algebra over $\mathbb{C}$ with basis $x, y, [x, y]$ such that $[x, [x, y]] = \gamma x$ and $[[x, y], y] = \gamma y$, then in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ these relations are given by $x^2y - 2xyx + yx^2 = \gamma x$ and $xy^2 - 2yxy + y^2x = \gamma y$. Notice that $U(\mathfrak{g})$ is a homomorphic algebra of the down-up algebra $A(2, -1, \gamma)$ via the mapping $\phi : A(2, -1, \gamma) \to U(\mathfrak{g})$, $d \mapsto x, u \mapsto y$, and the mapping $\psi : \mathfrak{g} \to A(2, -1, \gamma)$, $x \mapsto d, y \mapsto u$, $[x, y] \mapsto du - ud$, extends by the universal property of $U(\mathfrak{g})$ to an algebra homomorphism $\psi : U(\mathfrak{g}) \to A(2, -1, \gamma)$ which is the inverse of ψ. Hence, $U(\mathfrak{g})$ is isomorphic to $A(2, -1, \gamma)$.

It is straightforward to see that $U(\mathfrak{sl}_2(\mathbb{C})) \cong A(2, -1, -2)$. Also, for the Heisenberg Lie algebra $\mathfrak{h}$ with basis x, y, z where $[x, y] = z$ and $[z, x] = [z, y] = 0$, $U(\mathfrak{h}) \cong A(2, -1, 0)$.

Now, with the aim of providing an explanation of the existence of quantum groups, Witten [Wit90, Wit91] introduced a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$. By definition, Witten's deformation is a unital associative algebra over a field $\Bbbk$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\underline{\xi} = (\xi_1, \ldots, \xi_7)$ of elements of $\Bbbk$. This algebra, denoted by $W(\underline{\xi})$, is generated by the indeterminates x, y, z subject to the defining relations $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4$, and $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$. From [Ben99, Section 2], we know that a Witten's deformation

algebra $W(\xi)$ with

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4, \tag{1.2.3}$$

is isomorphic to one down-up algebra. Notice that any down-up algebra $A(\alpha, \beta, \gamma)$ with not both α and β equal to 0 is isomorphic to a Witten deformation algebra $W(\underline{\xi})$ whose parameters satisfy (1.2.3).

Since algebras $W(\underline{\xi})$ are filtered, Le Bruyn [LB94, LB95] studied the algebras $W(\underline{\xi})$ whose associated graded algebras are Auslander regular. He determined a 3-parameter family of deformation algebras which are said to be *conformal* $\mathfrak{sl}_2$ *algebras* that are generated by the indeterminates x, y, z over a field $\Bbbk$ subject to the relations given by $zx - axz = x$, $zy - ayz = y$, and $yx - cxy = bz^2 + z$. In the case $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with these three defining relations is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1+ac), \beta = -ac^{-1}$ and $\gamma = -c^{-1}$. Notice that if $c = b = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}, \beta = 0$, and $\gamma = -a^{-1}$. As one can check, conformal $\mathfrak{sl}_2$ algebras are not Ore extensions.

Kulkarni [Kul99] showed that under certain assumptions on the parameters, a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2(\Bbbk)$ algebra or to an iterated Ore extension. More exactly, following [Kul99, Theorem 3.0.3] if $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $W(\underline{\xi})$ is isomorphic to one of the following algebras: (i) a conformal $\mathfrak{sl}_2$ algebra with generators x, y, z and relations given above or (ii) an iterated Ore extension whose generators satisfy

- $xz - zx = x$, $zy - yz = \zeta y$, $yx - \eta xy = 0$, or
- $xw = \theta wx$, $wy = \kappa yw$, $yx = \lambda xy$, for parameters $\zeta, \eta, \theta, \kappa, \lambda \in \Bbbk$.

Notice that iterated Ore extensions above are defined in the following way: (i) the Witten deformation algebra is isomorphic to $\Bbbk[z][y; \sigma_1][x; \sigma_2]$ where σ_1 is the automorphism of $\Bbbk[z]$ defined as $\sigma_1(z) = z - \zeta$, with $zy - yz = \zeta y$; σ_2 is the automorphism of $\Bbbk[z][y, \sigma_1]$ defined as $\sigma_2(y) = \eta^{-1}y$, $\sigma_2(z) = z + 1$, which satisfies $xz - zx = x$ and $yx - \eta xy = 0$. (ii) The Witten deformation algebra is isomorphic to $\Bbbk[w][y; \sigma_1][x; \sigma_2]$ where σ_1 is the automorphism of $\Bbbk[w]$ defined as $\sigma_1(w) = \kappa^{-1}w$ with $wy = \kappa yw$, and σ_2 is the automorphism of $\Bbbk[w][y; \sigma_1]$ defined as $\sigma_2(w) = \theta w$, $\sigma_2(y) = \lambda^{-1}y$ such that $wy = \kappa yw$ and $yx = \lambda xy$.

Notice that universal enveloping algebras above are PBW extensions over K in the sense of Bell and Goodearl [BG88] (note that these authors presented another examples of enveloping rings related to enveloping universal algebras).

1.2.4 Other families of quantum algebras

We recall the definitions of some examples of noncommutative rings known in the literature as *quantum algebras* or *quantized algebras*.

(i) Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \ldots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The *homogenized enveloping algebra of* $\mathfrak{g}$ is $\mathcal{A}(\mathfrak{g}) := T(\mathfrak{g} \oplus \Bbbk z)/\langle R \rangle$,

where $T(\mathfrak{g}\oplus\Bbbk z)$ denotes the tensor algebra, z is a new indeterminate, and R is spanned by the union of sets $\{z\otimes x - x\otimes z \mid x\in\mathfrak{g}\}$ and $\{x\otimes y - y\otimes x - [x,y]\otimes z \mid x,y\in\mathfrak{g}\}$.

(ii) From [GJ04, p. 41], for q an element of $\Bbbk$ with $q\neq\pm 1$, the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ corresponding to the choice of q is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ presented by the generators E, F, K, K^{-1} and the relations $KK^{-1} = K^{-1}K = 1$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, $KE = q^2EK$, and $KF = q^{-2}FK$. From [GJ04, Exercise 2T], we know that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][K^{\pm 1};\sigma_1][F;\sigma_2,\delta_2]$ [GJ04, Exercise 2T], so that this algebra is not of automorphism type.

(iii) The *Lie-deformed Heisenberg* is the free $\mathbb{C}$-algebra defined by the commutation relations

$$\begin{aligned} q_j(1+i\lambda_{jk})p_k - p_k(1-i\lambda_{jk})q_j &= i\hbar\delta_{jk},\\ [q_j,q_k] = [p_j,p_k] &= 0,\quad j,k=1,2,3, \end{aligned}$$

where q_j, p_j are the position and momentum operators, and $\lambda_{jk} = \lambda_k\delta_{jk}$, with λ_k real parameters. If $\lambda_{jk} = 0$, then one recovers the usual Heisenberg algebra.

(iv) With the aim of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi [Hay90] considered the A_q^- algebra by using the free algebra $\mathbf{U}$. Following Berger [Ber92, Example 2.7.7], this $\Bbbk$-algebra $\mathbf{U}$ is generated by the indeterminates $\omega_1,\dots,\omega_n,\psi_1,\dots,\psi_n$, and $\psi_1^*,\dots,\psi_n^*$, subject to the relations

$$\begin{aligned} \psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* = 0, &\qquad 1\le i<j\le n,\\ \omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* = 0, &\qquad 1\le i,j\le n,\\ \psi_i^*\psi_i - q^2\psi_i\psi_i^* = -q^2\omega_i^2, &\qquad 1\le i\le n. \end{aligned}$$

(v) The *Non-Hermitian realization of a Lie deformed* defined by Jannussis et al. [JLM95] is an important example of a non-canonical Heisenberg algebra considering the case of non-Hermitian (i.e., $\hbar = 1$) operators A_j, B_k, where the following relations are satisfied:

$$\begin{aligned} A_j(1+i\lambda_{jk})B_k - B_k(1-i\lambda_{jk})A_j &= i\delta_{jk},\\ [A_j,B_k] &= 0\quad (j\neq k),\\ [A_j,A_k] = [B_j,B_k] &= 0, \end{aligned}$$

and,

$$\begin{aligned} A_j^+(1+i\lambda_{jk})B_k^+ - B_k^+(1-i\lambda_{jk})A_j^+ &= i\delta_{jk},\\ [A_j^+,B_k^+] &= 0\ (j\neq k),\\ [A_j^+,A_k^+] = [B_j^+,B_k^+] &= 0, \end{aligned} \tag{1.2.4}$$

with $A_j\neq A_j^+$, $B_k\neq B_k^+$ $(j,k=1,2,3)$. If the operators A_j, B_k are in the form $A_j = f_j(N_j+1)a_j$, $B_k = a_k^+ f_k(N_k+1)$, where a_j, a_j^+ are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator ($N_j =$

$a_j^+ a_j$, $\langle N_j \mid n_j \rangle = \langle n_j | n_j \rangle$), and the structure functions $f_j(N_j + 1)$ complex, then it is showed that A_j and B_k are given by

$$A_j = \sqrt{\frac{i}{1+i\lambda_j}} \left(\frac{[(1-i\lambda_j)/(1+i\lambda_j)]^{N_j+1} - 1}{(1-i\lambda_j)/(1+i\lambda_j) - 1} \frac{1}{N_j+1} \right)^{\frac{1}{2}} a_j,$$
$$B_k = \sqrt{\frac{i}{1+i\lambda_k}} a_k^+ \left(\frac{[(1-i\lambda_k)/(1+i\lambda_k)]^{N_k+1} - 1}{(1-i\lambda_k)/(1+i\lambda_k) - 1} \frac{1}{N_k+1} \right)^{\frac{1}{2}}.$$

(vi) Following Havlíček et al. [HKP00, p. 79], the $\mathbb{C}$-algebra $U_q'(\mathfrak{so}_3)$ is generated by the indeterminates I_1, I_2, and I_3, subject to the relations given by

$$I_2 I_1 - q I_1 I_2 = -q^{\frac{1}{2}} I_3, \quad I_3 I_1 - q^{-1} I_1 I_3 = q^{-\frac{1}{2}} I_2, \quad I_3 I_2 - q I_2 I_3 = -q^{\frac{1}{2}} I_1,$$

where q is a non-zero element of $\mathbb{C}$. It is straightforward to show that $U_q'(\mathfrak{so}_3)$ cannot be expressed as an iterated Ore extension.

(vii) Zhedanov [Zhe91, Section 1] introduced the *Askey-Wilson algebra* $AW(3)$ as the algebra generated by three operators K_0, K_1, and K_2, that satisfy the commutation relations

$$[K_0, K_1]_\omega = K_2, [K_2, K_0]_\omega = BK_0 + C_1 K_1 + D_1, \text{ and } [K_1, K_2]_\omega = BK_1 + C_0 K_0 + D_0,$$

where B, C_0, C_1, D_0, and D_1 are the structure constants of the algebra, which Zhedanov assumes are real, and the q-commutator $[-,-]_\omega$ is given by $[\square, \triangle]_\omega := e^\omega \square\triangle - e^{-\omega}\triangle\square$, where $\omega \in \mathbb{R}$. Notice that in the limit $\omega \to 0$, the algebra $AW(3)$ becomes an ordinary Lie algebra with three generators (D_0 and D_1 are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as

$$e^\omega K_0 K_1 - e^{-\omega} K_1 K_0 = K_2,$$
$$e^\omega K_2 K_0 - e^{-\omega} K_0 K_2 = BK_0 + C_1 K_1 + D_1,$$
$$e^\omega K_1 K_2 - e^{-\omega} K_2 K_1 = BK_1 + C_0 K_0 + D_0.$$

According to these relations that define the algebra, it follows that $AW(3)$ cannot be expressed as an iterated Ore extension.

(viii) With the purpose of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [Gre53] and Greenberg and Messiah [GM65] introduced by means of generators and relations the *parafermionic* and *parabosonic algebras*. For the completeness of the thesis, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [KD09]. Let $[\square, \triangle] := \square\triangle - \triangle\square$ and $\{\square, \triangle\} := \square\triangle + \triangle\square$.

Consider the $\Bbbk$-vector space V_F freely generated by the elements f_i^+, f_j^-, with $i, j = 1, \ldots, n$. If $T(V_F)$ is the tensor algebra of V_F and I_F is the two-sided ideal I_F generated by the elements $[[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon - \eta)^2 \delta_{jk} f_i^\xi + \frac{1}{2}(\varepsilon - \xi)^2 \delta_{ik} f_j^\eta$, for all values of $\xi, \eta, \varepsilon =$

± 1, and $i, j, k = 1, \ldots, n$, then the *parafermionic algebra* in $2n$ generators $P_F^{(n)}$ (n parafermions) is the quotient algebra of $T(V_F)$ with the ideal I_F, that is,

$$P_F^{(n)} = \frac{T(V_F)}{\left\langle [[f_i^{\xi}, f_j^{\eta}], f_k^{\varepsilon}] - \frac{1}{2}(\varepsilon - \eta)^2 \delta_{jk} f_i^{\xi} + \frac{1}{2}(\varepsilon - \xi)^2 \delta_{ik} f_j^{\eta} \mid \xi, \eta, \varepsilon = \pm 1, i, j, k = 1, \ldots, n \right\rangle}.$$

It is well-known (e.g., [KD09, Section 18.2]) that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$, i.e., $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$.

Similarly, if V_B denotes the $\Bbbk$-vector space freely generated by the elements b_i^+, b_j^-, $i, j = 1, \ldots, n$, $T(V_B)$ is the tensor algebra of V_B, and I_B is the two-sided ideal of $T(V_B)$ generated by the elements $[\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon - \eta)\delta_{jk} b_i^{\xi} - (\varepsilon - \xi)\delta_{ik} b_j^{\eta}$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j = 1, \ldots, n$, then the *parabosonic algebra* $P_B^{(n)}$ in $2n$ generators (n parabosons) is defined as the quotient algebra $P_B^{(n)}/I_B$, that is,

$$P_B^{(n)} = \frac{T(V_B)}{\left\langle [\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon - \eta)\delta_{jk} b_i^{\xi} - (\varepsilon - \xi)\delta_{ik} b_j^{\eta} \mid \xi, \eta, \varepsilon = \pm 1, i, j = 1, \ldots, n \right\rangle}.$$

It is known that the parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0, n)$, that is, $P_B^{(n)} \cong U(B(0, n))$. For more details about *parafermionic* and *parabosonic algebras*, see [KD09, Proposition 18.2], and references therein.

In the next chapters we present other families of algebras that are semi-graded rings.

1.3 Semi-graded Artin-Schelter regular algebras

Definition 1.10 ([Lez21, Definition 4.3])**.** Let A be a $\Bbbk$-algebra. A is said to be a left *semi-graded Artin-Schelter* (SAS for short) *regular algebra* if the following conditions hold:

(i) A is an FSG ring with semi-graduation $A = \bigoplus_{p \geq 0} A_p$.

(ii) A is connected, i.e., $A_0 = \Bbbk$.

(iii) A has left finite global dimension d.

(iv) $\operatorname{Ext}^i_A({}_AA/A_{\geq 1}, {}_AA) \cong \begin{cases} 0, & \text{if } i \neq d, \\ (A/A_{\geq 1})_A, & \text{if } i = d. \end{cases}$

Remark 3. Note that if $\Bbbk \cap A_{\geq 1} \neq 0$, then $A/A_{\geq 1} = 0$. In the case $\Bbbk \cap A_{\geq 1} = 0$, we get that $A/A_{\geq 1} \cong \Bbbk$. This isomorphism of $\Bbbk$-algebras is induced by the canonical projection $\epsilon : A \to \Bbbk$ ($A_{\geq 1}$ is semi-graded). As a matter of fact, ${}_A(A/A_{\geq 1}) \cong_A \Bbbk$ and $(A/A_{\geq 1})_A \cong \Bbbk_A$.

Definition 1.10 extends (except for the GKdim) the classical notion of Artin-Schelter regular algebra since in such case $A/A_{\geq 1} \cong \Bbbk$, $\operatorname{lgld}(A) = \operatorname{pd}({}_A\Bbbk) = \operatorname{pd}(\Bbbk_A) = \operatorname{rgld}(A) = \operatorname{gld}(A)$, that is, we guarantee the equality of projective dimensions and left (right) global dimensions. This means that every Artin-Schelter regular algebra is an SAS regular algebra [Lez21, Remark 4.4]. From now on we just say SAS regular algebra.

1.3.1 Examples

Let us see some illustrative examples of SAS regular algebras in comparison with AS regular algebras.

(i) [Lez21, Example 4.5] The *first Weyl algebra* $A_1(\Bbbk)$ is not AS but it is an SAS regular algebra. Since that the canonical semi-graduation of $A_1(\Bbbk)$ is given by (1.2.1), $\operatorname{gld}(A_1(\Bbbk)) \leq 2$ [LR14, Theorem 4.2] and $A_1(\Bbbk)_{\geq 1} = A_1(\Bbbk)$, the condition (iv) in Definition 1.10 holds. A similar argument can be used to show that the nth *Weyl algebra* $A_n(\Bbbk)$ Weyl is also an SAS regular algebra.

(ii) [Lez21, Example 4.6] The *quantum deformation* $A_1^q(\Bbbk)$ *of the first Weyl algebra* is not AS but it is an SAS regular algebra. Recall that $A_1^q(\Bbbk) = \Bbbk\{x,y\}/\langle yx - qxy - 1, q \in \Bbbk^*\rangle$, which is precisely the *Additive analogue of the Weyl algebra in two indeterminates* or the *linear algebra of* q*-differential operators in two indeterminates*. Since the semi-graduation of $A_1^q(\Bbbk)$ is the same for $A_1(\Bbbk)$, $\operatorname{gld}(A_1^q(\Bbbk)) \leq 2$ [LR14, Theorem 4.2], and $A_1^q(\Bbbk)_{\geq 1} = A_1^q(\Bbbk)$, the condition (iv) in Definition 1.10 is satisfied. Therefore, $A_1^q(\Bbbk)$ is an SAS regular algebra.

(iii) [Lez21, Example 4.7] Consider the algebra $\mathcal{J}_1 := \Bbbk\{x,y\}/\langle yx - xy + y^2 + 1\rangle$. In his PhD Thesis, Gaddis proved that $\operatorname{gld}(\mathcal{J}_1) = 2$ [Gad13, Corollary 2.3.14], and since the semi-graduation of $\mathcal{J}_1$ is the same as in (1.2.1), it follows that $(\mathcal{J}_1)_{\geq 1} = \mathcal{J}_1$. Thus, $\mathcal{J}_1$ is an SAS regular algebra but not AS.

(iv) [Lez21, Example 4.8] Consider the Dispin algebra $U(\mathfrak{osp}(1,2))$. Recall that this $\Bbbk$-algebra is generated by the indeterminates x_1, x_2 and x_3 subject to the relations

$$x_1x_2 - x_2x_1 = 1, \quad x_3x_1 + x_1x_3 = x_2 \quad \text{and} \quad x_2x_3 - x_3x_2 = x_3.$$

Since $U(\mathfrak{osp}(1,2))$ is not finitely graded, then $U(\mathfrak{osp}(1,2))$ is not an AS regular algebra. Let us see that $U(\mathfrak{osp}(1,2))$ is an SAS regular algebra. Note that $\operatorname{gld}(U(\mathfrak{osp}(1,2)) = 3$ [LR14, Theorem 4.2], and

$$U(\mathfrak{osp}(1,2)) = \Bbbk \oplus {}_{\Bbbk}\langle x_1, x_2, x_3\rangle \oplus {}_{\Bbbk}\langle x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\rangle \oplus \cdots$$

$U(\mathfrak{osp}(1,2))_{\geq 1} = \bigoplus_{p\geq 1} U(\mathfrak{osp}(1,2))_p$ coincides with the two-sided ideal of $U(\mathfrak{osp}(1,2))$ generated by x_1, x_2, x_3, and

$${}_{U(\mathfrak{osp}(1,2))}(U(\mathfrak{osp}(1,2))/U(\mathfrak{osp}(1,2))_{\geq 1}) \cong_{U(\mathfrak{osp}(1,2))} \Bbbk,$$

where the structure of left $U(\mathfrak{osp}(1,2))$-ideal module for $\Bbbk$ is given by the canonical projection $\epsilon : U(\mathfrak{osp}(1,2)) \to \Bbbk$. The same is true for

$$(U(\mathfrak{osp}(1,2))/U(\mathfrak{osp}(1,2))_{\geq 1})_{U(\mathfrak{osp}(1,2))} \cong \Bbbk_{U(\mathfrak{osp}(1,2))}.$$

It can be found a free resolution of ${}_{U(\mathfrak{osp}(1,2))}\Bbbk$ (let $A := U(\mathfrak{osp}(1,2))$ given by

$$0 \xrightarrow{\phi_2=\begin{bmatrix}-x_3 & x_2 & x_1\end{bmatrix}} A^3 \xrightarrow{\phi_1=\begin{bmatrix}1+x_2 & -x_1 & 0\\ x_3 & -1 & x_1\\ 0 & x_3 & 1-x_2\end{bmatrix}} A^3 \xrightarrow{\phi_0=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}} A \xrightarrow{\epsilon} \Bbbk \to 0.$$

If we apply $\mathrm{Hom}_A(-, {}_A A)$ then we obtain the complex of right A-modules given by

$$0 \to \mathrm{Hom}_A(\Bbbk, A) \xrightarrow{\epsilon^*} \mathrm{Hom}_A(A, A) \xrightarrow{\phi_0^*} \mathrm{Hom}_A(A^3, A) \xrightarrow{\phi_1^*} \mathrm{Hom}_A(A^3, A) \xrightarrow{\phi_2^*} \mathrm{Hom}_A(A, A) \to 0.$$

After some computations [Lez21, p. 367], this complex is reduced to

$$0 \xrightarrow{\quad} A^3 \xrightarrow{\phi^*=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} A^3 \xrightarrow{\phi_1^*=\begin{bmatrix} 1+x_2 & -x_1 & 0 \\ x_3 & -1 & x_1 \\ 0 & x_3 & 1-x_2 \end{bmatrix}} A^3 \xrightarrow{\phi_2^*=\begin{bmatrix} -x_3 & x_2 & x_1 \end{bmatrix}} A \to 0.$$

Thus, $\mathrm{Ext}^0_A(\Bbbk, A) = \mathrm{Hom}_A(\Bbbk, A) = 0$, $\mathrm{Ext}^1_A(\Bbbk, A) = 0 = \mathrm{Ext}^2_A(\Bbbk, A)$, and $\mathrm{Ext}^3_A(\Bbbk, A) = A/\mathrm{Im}(\phi_2^*) = A/A_{\geq 1} \cong \Bbbk_A$. Therefore, $U(\mathfrak{osp}(1,2))$ is an SAS regular algebra.

Example 1.3 ([Lez21, Example 4.9])**.** The following algebras are SAS regular algebras with global dimension three.

(i) Let $U(\mathfrak{sl}(2,\Bbbk))$ be the universal enveloping algebra of $\mathfrak{sl}(2,\Bbbk)$. By definition, this $\Bbbk$-algebra is generated by the indeterminates x, y, z subject to the relations

$$xy - yx = z, \quad xz - zx = -2x \quad \text{and} \quad yz - zy = 2y.$$

(ii) Let $U(\mathfrak{so}(3,\Bbbk))$ be the universal enveloping algebra of $\mathfrak{so}(3,\Bbbk)$, that is, it is the $\Bbbk$-algebra generated by the indeterminates x, y, z subject to the relations

$$xy - yx = z, \quad xz - zx = -y \quad \text{and} \quad yz - zy = x.$$

(iii) The quantum universal enveloping algebra $U'(\mathfrak{so}(3,\Bbbk))$, with $q \in \Bbbk^*$, is defined as the $\Bbbk$-algebra generated by x, y, z and relations

$$yx - qxy = -q^{1/2}z, \quad zx - q^{-1}xz = q^{-1/2}y \quad \text{and} \quad zy - qyz = -q^{1/2}x.$$

(iv) The *Woronowicz algebra* $W_\nu(\mathfrak{sl}(2,\Bbbk))$, where $\nu \in \Bbbk - \{0\}$ is not a root of unity, is the $\Bbbk$-algebra generated by the indeterminates x, y, z and relations given by

$$xz - \nu^4 zx = (1+\nu^2)x, \quad xy - \nu^2 yx = \nu z \quad \text{and} \quad zy - \nu^4 yz = (1+\nu^2)y.$$

(v) The three 3-dimensional skew polynomial algebras (Section 1.2.2) given by

$$\begin{array}{lll} yz - zy = 0, & zx - \beta xz = y, & xy - yx = 0, \\ yz - zy = z, & zx - \beta xz = 0, & xy - yx = x, \quad \text{and} \\ yz - zy = 0, & zx - xz = 0, & xy - yx = z. \end{array}$$

Remark 4. Of course, there algebras that are not SAS. The following three 3-dimensional skew polynomial algebras illustrate this situation.

(i) $yz - zy = z$, $zx - \beta xz = 0$ and $xy - yx = 0$;

(ii) $yz - zy = -y$, $zx - xz = x + y$, and $xy - yx = 0$; and

(iii) $yz - zy = z$, $zx - xz = z$, and $xy - yx = 0$.

Remark 5. (i) [Lez21, Example 4.11] The $\Bbbk$-algebra $\Bbbk\{x, y\}/\langle yx - xy + x\rangle$ is essentially regular of global dimension two [Gad13, Corollary 2.3.14]. Nevertheless, as it is clear, this algebra is not AS, and in fact, it is not SAS.

(ii) For the $\Bbbk$-algebra B given by $\Bbbk\{x, y\}/\langle xy - 1\rangle$ [Gad13, Theorem 4.0.7], we know that it has a semi-graduation as in Example 1.3.1 (i), $B/B_{\geq 1} = 0$, and its global dimension is one [Gad13, Proposition 4.1.1]. However, $G(B) \cong R_{yx} = \Bbbk\{x, y\}/\langle yx\rangle$, and by [Gad13, Corollary 2.3.14], B is not essentially regular.

(iii) The $\Bbbk$-algebra $R_{yx} = \Bbbk\{x, y\}/\langle yx\rangle$ is neither SAS nor essentially regular.

1.4 Noncommutative differential geometry of algebras

The first examples of noncommutative differential geometry are quantum planes. For this case, Wess and Zumino in [WZ91] used matrix calculations to show differentiability and integrability in the general case of quantum planes. In this section, we review the key ingredients of the noncommutative differential algebra in the Brzeziński and Sitarz's sense.

1.4.1 Hom-connections (Divergences) and differential calculi

The *theory of connections* in noncommutative geometry is well-known (for more details, see the beautiful treatments presented by Connes [Con94] or Giachetta et al. [GMS05]). Briefly, one considers a differential graded algebra $\Omega A = \bigoplus_{n=0} \Omega^n A$ over a $\Bbbk$-algebra $A = \Omega^0 A$ with $\Bbbk$ a field, and then defines a *connection* in a left A-module M as a linear map $\nabla^0 : M \to \Omega^1 A \otimes_A M$ that satisfies the Leibniz's rule $\nabla^0(am) = da \otimes_A m + a\nabla^0(m)$ for all $m \in M$ and $a \in A$. As it can be seen, this is a noncommutative definition obtained by a replacement of commutative algebras of functions on a manifold X, and their modules of sections of a vector bundle over X (in the classical definition of a connection), by noncommutative algebras and their general one-sided modules. Just as Brzeziński said, "this captures very well the classical context in which connections appear and brings it successfully to the realm of noncommutative geometry" [Brz08, p. 557].

Brzeziński in his paper noted that, on the algebraic side, this definition of connection seems to be only a half of a more general picture. In the first place, a noncommutative connection is defined by using the tensor functor, and as is well-known, this functor has a right adjoint, the *hom-functor*, so it is natural to ask whether it is possible to introduce connection-like objects defined with the use of the hom-functor. In the second place, the vector space dual to M is a right A-module and a left connection in the above sense does not induce a right connection on the dual of M, so having in mind the adjointness properties between tensor and hom functors, the induced map necessarily involves the hom-functor.

Motivated by all these facts, Brzeziński [Brz08] showed that there is a natural and potentially rich theory of connnection-like objects defined as maps on the spaces of morphisms of modules. Due to the role of spaces of homomorphisms, these objects are

termed *hom-connections* (also are called *divergences* due to that if $\mathcal{A}$ is an algebra of functions on the Euclidean space $\mathbb{R}^n$ and $\Omega^1(\mathcal{A})$ is the standard module of one-form, then we obtain the classical divergence of the elementary vector calculus [Brz11, p. 892]). As a matter of fact, he proved that hom-connections arise naturally from (strong) connections in *noncommutative principal bundles*, and that every left connection on a bimodule (in the sense of Cuntz and Quillen [CQ95]) gives rise to a hom-connection. Brzeziński also studied the induction procedure of hom-connections via differentiable bimodules (and hence, via maps of differential graded algebras), and proved that any hom-connection can be extended to *higher forms*. He introduced the notion of *curvature* and showed that a consecutive application of hom-connections can be expressed in terms of the curvature, which leads to a chain complex associated to a *flat* (i.e. curvature-zero) *hom-connection* (this chain complex and its homology can be considered as dual complements of the cochain complex associated to a connection and the twisted cohomology, which is crucial in the theory of noncommutative differential fibrations [BB05]).

Two years later, Brzeziński et al. [BEKL10] presented a construction of *differential calculi* which admits hom-connections. This construction is based on the use of *twisted multi-derivations*, where the constructed first-order calculus $\Omega^1(\mathcal{A})$ is free as a left and right $\mathcal{A}$-module; $\Omega^1(\mathcal{A})$ should be understood as a module of sections on the cotangent bundle over a manifold represented by $\mathcal{A}$, and hence their construction corresponds to parallelizable manifolds or to an algebra of functions on a local chart. One year later, Brzeziński asserted that "one should expect $\Omega^1(\mathcal{A})$ to be a finitely generated and projective module over $\mathcal{A}$ (thus corresponding to sections of a non-trivial vector bundle by the Serre-Swan theorem)" [Brz11, p. 885]. In his paper, he extended the construction in [BEKL10] to finitely generated and projective modules.

1.4.2 Differential smoothness of algebras

Related to differential calculi, we have the *smoothness of algebras*. Briefly, and just as Brzeziński and Lomp said [BL18, Section 1], "the study of this smoothness goes back at least to Grothendieck's EGA [Gro64]. The concept of a *formally smooth commutative* (*topological*) *algebra* introduced by him was extended to the noncommutative setting by Schelter [Sch86]. An algebra is *formally smooth* if and only if the kernel of the multiplication map is projective as a bimodule. As argued by Schelter himself, this notion arose as a replacement of a far too general definition based on the finiteness of the global dimension; Cuntz and Quillen [CQ95] called these algebras *quasi-free*". Precisely, the notion of smoothness based on the finiteness of this dimension was refined by Stafford and Zhang [SZ94], where a Noetherian algebra is said to be *smooth* provided that it has a finite global dimension equal to the homological dimension of all its simple modules. In the homological setting, Van den Bergh [VdB10] called an algebra *homologically smooth* if it admits a finite resolution by finitely generated projective bimodules. The characterization of this kind of smoothness for the noncommutative pillow, the quantum teardrops, and quantum homogeneous spaces was made by Brzeziński [Brz08, Brz14] and Krähmer [Kra12], respectively.

Brzeziński and Sitarz [BS17] defined other notion of smoothness of algebras, termed *differential smoothness* due to the use of differential graded algebras of a specified dimension that admits a noncommutative version of the Hodge star isomorphism, which considers

the existence of a top form in a differential calculus over an algebra together with a string version of the Poincaré duality realized as an isomorphism between complexes of differential and integral forms. This new notion of smoothness is different and more constructive than the homological smoothness mentioned above. "The idea behind the *differential smoothness* of algebras is rooted in the observation that a classical smooth orientable manifold, in addition to de Rham complex of differential forms, admits also the complex of *integral forms* isomorphic to the de Rham complex [Man97, Section 4.5]. The de Rham differential can be understood as a special left connection, while the boundary operator in the complex of integral forms is an example of a *right connection*" [BS17, p. 413].

Several authors (e.g. [Brz15, Brz16, BEKL10, BL18, BS17, DV88, DVKM90, Kar15, KL14, RS22]) have characterized the differential smoothness of algebras such as the quantum two - and three - spheres, disc, plane, the noncommutative torus, the coordinate algebras of the quantum group $\mathsf{SU}_q(2)$, the noncommutative pillow algebra, the quantum cone algebras, the quantum polynomial algebras, Hopf algebra domains of Gelfand-Kirillov dimension two that are not PI, families of Ore extensions, some 3-dimensional skew polynomial algebras, diffusion algebras in three generators, and noncommutative coordinate algebras of deformations of several examples of classical orbifolds such as the pillow orbifold, singular cones and lens spaces. An interesting fact is that some of these algebras are also homologically smooth in the Van den Bergh's sense.

Next, we recall the preliminaries on differential smoothness of algebras and bi-quadratic algebras with PBW basis that are necessary for the rest of the thesis. We follow Brzeziński and Sitarz's presentation on differential smoothness carried out in [BS17, Section 2] (c.f. [Brz08, Brz14]).

Definition 1.11 ([BS17, Section 2.1]). (i) By a *differential graded algebra* we mean a non-negatively graded algebra Ω with the product denoted by $\wedge$ together with a degree-one linear map $\mathrm{d} : \Omega^\bullet \to \Omega^{\bullet+1}$ that satisfies the graded Leibniz's rule and is such that $\mathrm{d} \circ \mathrm{d} = 0$.

(ii) A differential graded algebra (Ω, d) is a *calculus over an algebra* A if $\Omega^0 A = A$ and $\Omega^n A = A\ \mathrm{d}A \wedge \mathrm{d}A \wedge \cdots \wedge \mathrm{d}A$ ($\mathrm{d}A$ appears n-times) for all $n \in \mathbb{N}$ (this last is called the *density condition*). We write $(\Omega A, \mathrm{d})$ with

$$\Omega A = \bigoplus_{n \in \mathbb{N}} \Omega^n A.$$

By using the Leibniz's rule, it follows that $\Omega^n A = \mathrm{d}A \wedge \mathrm{d}A \wedge \cdots \wedge \mathrm{d}A\ A$. A differential calculus ΩA is said to be *connected* if $\ker(\mathrm{d}\,|_{\Omega^0 A}) = \Bbbk$.

(iii) A calculus $(\Omega A, \mathrm{d})$ is said to have *dimension* n if $\Omega^n A \neq 0$ and $\Omega^m A = 0$ for all $m > n$. An n-dimensional calculus ΩA *admits a volume form* if $\Omega^n A$ is isomorphic to A as a left and right A-module.

The existence of a right A-module isomorphism means that there is a free generator, say ω, of $\Omega^n A$ (as a right A-module), i.e. $\omega \in \Omega^n A$, such that all elements of $\Omega^n A$ can be uniquely expressed as ωa with $a \in A$. If ω is also a free generator of $\Omega^n A$ as a left A-module, this is said to be a *volume form* on ΩA.

The right A-module isomorphism $\Omega^n A \to A$ corresponding to a volume form ω is denoted by π_ω, i.e.

$$\pi_\omega(\omega a) = a, \quad \text{for all } a \in A. \tag{1.4.1}$$

By using that $\Omega^n A$ is also isomorphic to A as a left A-module, any free generator ω induces an algebra endomorphism ν_ω of A by the formula

$$a\omega = \omega\nu_\omega(a). \tag{1.4.2}$$

Note that if ω is a volume form, then ν_ω is an algebra automorphism.

Now, we proceed to recall the key ingredients of the *integral calculus* on A as dual to its differential calculus. For more details, see Brzezinski et al. [Brz08, BEKL10].

Let $(\Omega A, d)$ be a differential calculus on A. The space of n-forms $\Omega^n A$ is an A-bimodule. Consider $\mathcal{I}_n A$ the right dual of $\Omega^n A$, the space of all right A-linear maps $\Omega^n A \to A$, that is, $\mathcal{I}_n A := \mathrm{Hom}_A(\Omega^n(A), A)$. Notice that each of the $\mathcal{I}_n A$ is an A-bimodule with the actions

$$(a \cdot \phi \cdot b)(\omega) = a\phi(b\omega), \quad \text{for all } \phi \in \mathcal{I}_n A,\ \omega \in \Omega^n A \text{ and } a, b \in A.$$

The direct sum of all the $\mathcal{I}_n A$, that is, $\mathcal{I}A = \bigoplus_n \mathcal{I}_n A$, is a right ΩA-module with action given by

$$(\phi \cdot \omega)(\omega') = \phi(\omega \wedge \omega'), \quad \text{for all } \phi \in \mathcal{I}_{n+m} A,\ \omega \in \Omega^n A \text{ and } \omega' \in \Omega^m A. \tag{1.4.3}$$

Definition 1.12 ([Brz08, Definition 2.1])**.** A *divergence* (also called *hom-connection*) on A is a linear map $\nabla : \mathcal{I}_1 A \to A$ such that

$$\nabla(\phi \cdot a) = \nabla(\phi)a + \phi(da), \quad \text{for all } \phi \in \mathcal{I}_1 A \text{ and } a \in A. \tag{1.4.4}$$

Note that a divergence can be extended to the whole of $\mathcal{I}A$,

$$\nabla_n : \mathcal{I}_{n+1} A \to \mathcal{I}_n A,$$

by considering

$$\nabla_n(\phi)(\omega) = \nabla(\phi \cdot \omega) + (-1)^{n+1}\phi(d\omega), \quad \text{for all } \phi \in \mathcal{I}_{n+1}(A) \text{ and } \omega \in \Omega^n A. \tag{1.4.5}$$

By putting together (1.4.4) and (1.4.5), we get the Leibniz's rule

$$\nabla_n(\phi \cdot \omega) = \nabla_{m+n}(\phi) \cdot \omega + (-1)^{m+n}\phi \cdot d\omega, \tag{1.4.6}$$

for all elements $\phi \in \mathcal{I}_{m+n+1} A$ and $\omega \in \Omega^m A$ [Brz08, Lemma 3.2]. In the case $n = 0$, if $\mathrm{Hom}_A(A, M)$ is canonically identified with M, then ∇_0 reduces to the classical Leibniz's rule.

Definition 1.13 ([Brz08, Definition 3.4])**.** The right A-module map

$$F = \nabla_0 \circ \nabla_1 : \mathrm{Hom}_A(\Omega^2 A, M) \to M$$

is called a *curvature* of a hom-connection (M, ∇_0). (M, ∇_0) is said to be *flat* if its curvature is the zero map, that is, if $\nabla \circ \nabla_1 = 0$. This condition implies that $\nabla_n \circ \nabla_{n+1} = 0$ for all $n \in \mathbb{N}$.

$\mathcal{I}A$ together with the ∇_n form a chain complex called the *complex of integral forms* over A. The cokernel map of ∇, that is, $\Lambda : A \to \mathrm{Coker}\nabla = A/\mathrm{Im}\nabla$ is said to be the *integral on* A *associated to* $\mathcal{I}A$.

Given a left A-module X with action $a \cdot x$, for all $a \in A$, $x \in X$, and an algebra automorphism ν of A, the notation ${}^{\nu}X$ stands for X with the A-module structure twisted by ν, i.e. with the A-action $a \otimes x \mapsto \nu(a) \cdot x$.

The following definition of an *integrable differential calculus* seeks to portray a version of Hodge star isomorphisms between the complex of differential forms of a differentiable manifold and a complex of dual modules of it [Brz15, p. 112].

Definition 1.14 ([BS17, Definition 2.1])**.** An n-dimensional differential calculus $(\Omega A, d)$ is said to be *integrable* if $(\Omega A, d)$ admits a complex of integral forms $(\mathcal{I}A, \nabla)$ for which there exist an algebra automorphism ν of A and A-bimodule isomorphisms

$$\Theta_k : \Omega^k A \to {}^{\nu}\mathcal{I}_{n-k}A, \quad k = 0, \dots, n$$

rendering commmutative the following diagram:

$$\begin{array}{ccccccccccc}
A & \xrightarrow{d} & \Omega^1 A & \xrightarrow{d} & \Omega^2 A & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1} A & \xrightarrow{d} & \Omega^n A \\
\downarrow{\scriptstyle \Theta_0} & & \downarrow{\scriptstyle \Theta_1} & & \downarrow{\scriptstyle \Theta_2} & & & & \downarrow{\scriptstyle \Theta_{n-1}} & & \downarrow{\scriptstyle \Theta_n} \\
{}^{\nu}\mathcal{I}_n A & \xrightarrow[\nabla_{n-1}]{} & {}^{\nu}\mathcal{I}_{n-1} A & \xrightarrow[\nabla_{n-2}]{} & {}^{\nu}\mathcal{I}_{n-2} A & \xrightarrow[\nabla_{n-3}]{} & \cdots & \xrightarrow[\nabla_1]{} & {}^{\nu}\mathcal{I}_1 A & \xrightarrow[\nabla]{} & {}^{\nu}A
\end{array}$$

The n-form $\omega := \Theta_n^{-1}(1) \in \Omega^n A$ is called an *integrating volume form.*

The algebra of complex matrices $M_n(\mathbb{C})$ with the n-dimensional calculus generated by derivations presented by Dubois-Violette et al. [DV88, DVKM90], the quantum group $SU_q(2)$ with the three-dimensional left covariant calculus developed by Woronowicz [Wor87] and the quantum standard sphere with the restriction of the above calculus, are examples of algebras admitting integrable calculi. For more details on the subject, see Brzeziński et al. [BEKL10].

The following proposition shows that the integrability of a differential calculus can be defined without explicit reference to integral forms. This allows us to guarantee the integrability by considering the existence of finitely generator elements that allow to determine left and right components of any homogeneous element of $\Omega(A)$.

Proposition 1.6 ([BS17, Theorem 2.2])**.** *Let* $(\Omega A, d)$ *be an* n*-dimensional differential calculus over an algebra* A*. The following assertions are equivalent:*

(1) $(\Omega A, d)$ *is an integrable differential calculus.*

(2) *There exists an algebra automorphism* ν *of* A *and* A*-bimodule isomorphisms*

$$\Theta_k : \Omega^k A \to {}^{\nu}\mathcal{I}_{n-k}A, \quad k = 0, \dots, n$$

such that, for all $\omega' \in \Omega^k A$ *and* $\omega'' \in \Omega^m A$,

$$\Theta_{k+m}(\omega' \wedge \omega'') = (-1)^{(n-1)m}\Theta_k(\omega') \cdot \omega''.$$

(3) *There exists an algebra automorphism* ν *of* A *and an* A*-bimodule map* $\vartheta : \Omega^n A \to {}^{\nu}A$ *such that all left multiplication maps*

$$\begin{aligned} \ell^k_\vartheta : \Omega^k A &\to \mathcal{I}_{n-k}A, \\ \omega' &\mapsto \vartheta \cdot \omega', \quad k = 0, 1, \dots, n, \end{aligned}$$

where the actions $\cdot$ *are defined by (1.4.3), are bijective.*

(4) $(\Omega A, d)$ *has a volume form* ω *such that all left multiplication maps*

$$\begin{aligned} \ell^k_{\pi_\omega} : \Omega^k A &\to \mathcal{I}_{n-k}A, \\ \omega' &\mapsto \pi_\omega \cdot \omega', \quad k = 0, 1, \dots, n-1, \end{aligned}$$

where π_ω *is defined by (1.4.1), are bijective.*

A volume form $\omega \in \Omega^n A$ is an *integrating form* if and only if it satisfies condition (4) of Proposition 1.6 [BS17, Remark 2.3].

The most interesting cases of differential calculi are those where $\Omega^k A$ are finitely generated and projective right or left (or both) A-modules [Brz11].

Proposition 1.7. (1) [BS17, Lemma 2.6] *Let* $(\Omega A, d)$ *be an integrable and* n*-dimensional calculus over* A *with integrating form* ω*. Then* $\Omega^k A$ *is a finitely generated projective right* A*-module if there exist a finite number of forms* $\omega_i \in \Omega^k A$ *and* $\overline{\omega}_i \in \Omega^{n-k}A$ *such that, for all* $\omega' \in \Omega^k A$*, we have that*

$$\omega' = \sum_i \omega_i \pi_\omega(\overline{\omega}_i \wedge \omega').$$

(2) [BS17, Lemma 2.7] *Let* $(\Omega A, d)$ *be an* n*-dimensional calculus over* A *admitting a volume form* ω*. Assume that for all* $k = 1, \dots, n-1$*, there exists a finite number of forms* $\omega^k_i, \overline{\omega}^k_i \in \Omega^k(A)$ *such that for all* $\omega' \in \Omega^k A$*, we have that*

$$\omega' = \sum_i \omega^k_i \pi_\omega(\overline{\omega}^{n-k}_i \wedge \omega') = \sum_i \nu^{-1}_\omega(\pi_\omega(\omega' \wedge \omega^{n-k}_i))\overline{\omega}^k_i,$$

where π_ω *and* ν_ω *are defined by (1.4.1) and (1.4.2), respectively. Then* ω *is an integral form and all the* $\Omega^k A$ *are finitely generated and projective as left and right* A*-modules.*

Brzeziński and Sitarz [BS17, p. 421] asserted that to connect the integrability of the differential graded algebra $(\Omega A, d)$ with the algebra A, it is necessary to relate the dimension of the differential calculus ΩA with that of A, and since we are dealing with algebras that are deformations of coordinate algebras of affine varieties, the *Gelfand-Kirillov dimension* introduced by Gelfand and Kirillov [GK66a, GK66b] seems to be the best suited. Briefly, given an affine $\Bbbk$-algebra A, the *Gelfand-Kirillov dimension of* A, denoted by $\mathrm{GKdim}(A)$, is given by

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log n},$$

where V is a finite-dimensional subspace of A that generates A as an algebra. This definition is independent of choice of V. If A is not affine, then its Gelfand-Kirillov dimension is defined to be the supremum of the Gelfand-Kirillov dimensions of all affine subalgebras of A. An affine domain of Gelfand-Kirillov dimension zero is precisely a division ring that is finite-dimensional over its center. In the case of an affine domain of Gelfand-Kirillov dimension one over $\Bbbk$, this is precisely a finite module over its center, and thus polynomial identity. In some sense, this dimensions measures the deviation of the algebra A from finite dimensionality. For more details about this dimension, see the excellent treatment developed by Krause and Lenagan [KL00].

After preliminaries above, we arrive to the key notion of this section.

Definition 1.15 ([BS17, Definition 2.4])**.** An affine algebra A with integer Gelfand-Kirillov dimension n is said to be *differentially smooth* if it admits an n-dimensional connected integrable differential calculus $(\Omega A, d)$.

From Definition 1.15 it follows that a differentially smooth algebra comes equipped with a well-behaved differential structure and with the precise concept of integration [BL18, p. 2414].

Example 1.4. As we said in the Introduction, several examples of noncommutative algebras have been proved to be differentially smooth (e.g. [Brz15, BEKL10, BL18, BS17, Kar15, KL14, RS22]).

Let us briefly see at some of these that are related to Ore extensions.

(i) The polynomial algebra $\Bbbk[x_1, \ldots, x_n]$ has Gelfand-Kirillov dimension n and the usual exterior algebra is an n-dimensional integrable calculus, which guarantees that $\Bbbk[x_1, \ldots, x_n]$ is differentially smooth.

(ii) Brzeziński et al. [BEKL10] proved that the coordinate algebras of the quantum group $SU_q(2)$ and the standard quantum Podleś are differentially smooth.

Example 1.5. Brzeziński [Brz15] characterized the differential smoothness of skew polynomial rings of the form $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ where $\sigma_{q,r}(t) = qt + r$, with $q, r \in \Bbbk,\ q \neq 0$, and the $\sigma_{q,r}$–derivation $\delta_{p(t)}$ is defined as

$$\delta_{p(t)}(f(t)) = \frac{f(\sigma_{q,r}(t)) - f(t)}{\sigma_{q,r}(t) - t} p(t),$$

for an element $p(t) \in \Bbbk[t]$.

Note that for any elements $f(t), g(t) \in \Bbbk[t]$, we have that

$$\delta_p(f(t)g(t)) \ = \frac{(fg)(\sigma_{q,r}(t)) - (fg)(t)}{\sigma_p(t) - t} p(t),$$

and

$$\begin{aligned}\sigma_{q,r}(f(t))\delta_p(g(t)) + \delta(f(t))g(t) &= \sigma_{q,r}(f(t))\frac{g(\sigma_p(t)) - g(t)}{\sigma_p(t) - t}p(t) + \frac{f(\sigma_p(t)) - f(t)}{\sigma_p(t) - t}p(t)g(t) \\ &= \left(\frac{\sigma_{q,r}(f(t))g(\sigma_{q,r}(t)) - \sigma_{q,r}(f(t))g(t) + f(\sigma_{q,r}(t))g(t) - f(t)g(t)}{\sigma_{q,r}(t) - t}\right) p(t).\end{aligned}$$

Hence, if $f(t) = \sum_{i=0}^{n} k_i t^i$, then $\sigma_{q,r}(f(t)) = \sum_{i=0}^{n} k_i(\sigma_{q,r}(t))^n = f(\sigma_{q,r}(t))$, and so

$$\begin{aligned}\sigma_{q,r}(f(t))\delta_p(g(t)) + \delta(f(t))g(t) &= \left(\frac{f(\sigma_{q,r}(t))g(\sigma_{q,r}(t)) - f(\sigma_{q,r}(t))g(t) + f(\sigma_{q,r}(t))g(t) - f(t)g(t)}{\sigma_{q,r}(t) - t}\right) p(t) \\ &= \left(\frac{f(\sigma_{q,r}(t))g(\sigma_{q,r}(t)) - f(t)g(t)}{\sigma_{q,r}(t) - t}\right) p(t) \\ &= \frac{(fg)(\sigma_{q,r}(t)) - (fg)(t)}{\sigma_{q,r}(t) - t}p(t) \\ &= \delta_p(f(t)g(t)),\end{aligned}$$

which shows that the map δ_p is a $\sigma_{q,r}$-derivation on $\Bbbk[t]$. $\delta_{p(t)}(f(t))$ is a suitable limit when $q = 1$ and $r = 0$, that is, when $\sigma_{q,r}$ is the identity map of $\Bbbk[t]$. In this way, the defining relation of the Ore extension $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ is given by

$$xt = qtx + rx + p(t).$$

For the maps

$$\nu_t(t) = t, \quad \nu_t(x) = qx + p'(t) \quad \text{and} \quad \nu_x(t) = \sigma_{q,r}^{-1}(t), \quad \nu_x(x) = x, \tag{1.4.7}$$

where $p'(t)$ is the classical t-derivative of $p(t)$, Brzeziński [Brz15, Lemma 3.1] showed that all of them simultaneously extend to algebra automorphisms ν_t and ν_x of $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ only in the following three cases:

(a) $q = 1, r = 0$ with no restriction on $p(t)$;

(b) $q = 1, r \neq 0$ and $p(t) = c$, $c \in \Bbbk$;

(c) $q \neq 1, p(t) = c\left(t + \frac{r}{q-1}\right)$, $c \in \Bbbk$ with no restriction on r.

In any of the cases (a) - (c) we have that $\nu_x \circ \nu_t = \nu_t \circ \nu_x$. If the Ore extension $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ satisfies one of these three conditions, then it is differentially smooth [Brz15, Proposition 3.3].

Due to Brzeziński's result on $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$, we get that the algebras

- The *polynomial commutative ring* $\Bbbk[x_1, x_2]$;

- The *Weyl algebra* $A_1(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_1x_2 - x_2x_1 - 1\rangle$;
- The *universal enveloping algebra of the Lie algebra* $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$, that is, $U(\mathfrak{n}_2) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - x_1x_2 - x_1\rangle$, and
- *Manin's plane* $\mathcal{O}_q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - qx_1x_2\rangle$, where $q \in \Bbbk \setminus \{0, 1\}$, and
- *Jordan's plane* $\mathcal{J}(\Bbbk)$ with the relation given by $xt = tx + t^2$,

are all differentially smooth.

Example 1.6. Related to Ore extensions, and under some mild and geometrically natural assumptions, Brzeziński and Lomp proved that if R and S are algebras with integrable calculi $(\Omega R, d_R)$ and $(\Omega S, d_S)$, ΩR is a finitely generated projective right R-module and Ω is a finitely generated projective right S-module, then $(\Omega R \otimes \Omega S, d)$ is an integrable differential calculus for $R \otimes S$ [BL18, Proposition 3.1]. With this, if R and S are differentially smooth algebras with respect to calculi which are finitely generated projective as right modules $\mathrm{GKdim}(R \otimes S) = \mathrm{GKdim}(R) + \mathrm{GKdim}(S)$, then the tensor product algebra $R \otimes S$ is differentially smooth [BL18, Corollary 3.2].

They also proved that if R is an algebra with an integrable differential calculus $(\Omega R, d)$ such that ΩR is a finitely generated right R-module, for any automorphism σ of R that extends to a degree-preserving automorphism of ΩR, which commutes with d, there exists an integrable differential calculus $(\Omega A, d)$ on the skew polynomial ring $R[x; \sigma]$ and the Laurent skew polynomial ring $R[x^{\pm 1}; \sigma]$. If R is differentially smooth with respect to $(\Omega R, d)$ and $\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + 1$, then A is also differentially smooth [BL18, Theorem 4.1]. The following examples illustrates this situation.

For any non-zero $q \in \Bbbk^*$, let A_q be the algebra generated by the indeterminates x, y, z and relations $xy = yx,\ xz = qzy,\ yz = zx$, and let B_q be the algebra generated by x, y and invertible z subject to the same relations. Since $A_q = \Bbbk[x, y][z; \sigma]$ and $B_q = \Bbbk[x, y][z^{\pm 1}; \sigma]$ with $\sigma(x) = y$ and $\sigma(y) = qx$, then A_q and B_q are differentially smooth [BL18, Example 4.6]. Let us see the details.

Note that the polynomial algebra $\Bbbk[x, y]$ is differentially smooth with the usual commutative differential calculus $\Omega(\Bbbk[x, y])$, i.e.,

$$\begin{aligned} x\,dx &= dx\,x, & x\,dy &= dy\,x, & y\,dx &= dx\,y, \\ y\,dy &= dy\,y, & dx\,dy &= -\,dy\,dx, & (dx)^2 &= (dy)^2 = 0. \end{aligned}$$

The automorphism σ extends to an automorphism of $\Omega(\Bbbk[x, y])$ by requesting it commute with d, that is, $\sigma(dx) = dy$ and $\sigma(dy) = q\,dx$.

Note that $\Omega(\Bbbk[x, y])$ is finitely generated as a right $\Bbbk[x, y]$-module and

$$\mathrm{GKdim}(A_q) = \mathrm{GKdim}(B_q) = 3 = \mathrm{GKdim}(\Bbbk[x, y]) + 1,$$

whence A_q and B_q are differentially smooth.

By using a similar reasoning [BL18, Corollary 4.9], it can be seen that the coordinate ring of the so called *quantum affine* $\mathfrak{n}$*-space*, that is, the algebra generated by the indeterminates $x_1, \ldots, x_n$ subject to the relations $x_jx_i = q_{ij}x_ix_j$ for all $i < j$ with $q_{ij} \in \Bbbk^*$, is differentially smooth (c.f. [KL14, Corollary 6 and Theorem 9]).

Example 1.7. Reyes and Sarmiento [RS22] showed that some 3-dimensional skew polynomial algebras (Section 1.2.2) are differentially smooth.

Let A be the $\Bbbk$-algebra generated by the indeterminates x, y and z subject to the relations given by

$$yz - z(\alpha y + a) = 0, \quad zx - \beta xz = b, \quad xy - (\gamma y + d)x = 0,$$

where $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$, $b \in \Bbbk$, such that (i) $d = a = 0$ or (ii) $\alpha = \gamma = 1$. If $\mathrm{GKdim}(A) = 3$, then A is differentially smooth [RS22, Theorem 2.7].

Suppose that for some generators x_i, x_j and x_k of an algebra A generated by the indeterminates $x_1, \ldots, x_n$ one has the relationships given by $x_i x_j - a x_j x_i = b x_i + c x_j + f x_k + e$. If we have a first order differential calculus (Ω^1, d) and d is a well defined derivation $d : \mathcal{A} \to \Omega^1$, we get that

$$d(x_i)x_j + x_i d(x_j) - a\, d(x_j)x_i - a x_j d(x_i) = b\, d(x_i) + c\, d(x_j) + f\, d(x_k) + e,$$

whence $d(x_k)$ is generated in the $\mathcal{A}$-bimodule by the elements $d(x_i)$ and $d(x_j)$. Since Ω^1 is generated as $\mathcal{A}$-bimodule by $d(\mathcal{A})$, then Ω^1 is generated by $n-1$ elements, and hence $\Omega^n = \bigwedge_{i=1}^{n} \Omega^1 = 0$. Besides, if $\mathrm{GKdim}(\mathcal{A}) = n$, we get that $\mathcal{A}$ cannot be differentially smooth because there is no a differential graded calculus of dimension n [RS22, Remark 2.8].

By using the first result, the following 3-dimensional skew polynomial $\Bbbk$-algebras are differentially smooth: (1), (2)(b), (2)(d), (2)(e), (2)(f), (3)(b) and (5)(c). On the other hand, due to the second result the algebras (2)(a), (2)(c), (3)(a), (4) with $|a_1|+|a_2|+|a_3| > 0$, (5)(a), (5)(b) and (5)(d), are no differentially smooth.

Example 1.8 ([BS17, Example 2.5])**.** As expected, there are examples of algebras that are not differentially smooth. Consider the commutative algebra $A = \mathbb{C}[x, y]/\langle xy \rangle$. A proof by contradiction shows that for this algebra there are no one-dimensional connected integrable calculi over A, so it cannot be differentially smooth. Let us see the details.

Following [BS17, Example 2.5], since $xy = yx = 0$, the algebra A has the basis the elements $1, x^n, y^n$, $n = 1, 2, \ldots$ which shows that $\mathrm{GKdim}(A) = 1$. By contradiction, suppose there is one-dimensional connected integrable calculus ΩA and let $\Theta_1 : \Omega^1 A \to {}^{\nu}A$, where ν is an algebra automorphism on A, be the required bimodule isomorphism. The Leibniz rule together with the equalities $xy = yx = 0$ imply that

$$x\ dy = -dx\ y, \quad \text{and} \quad y\ dx = -dy\ x.$$

If we apply Θ_1 to these identities, then we obtain that

$$\nu(x)\theta_y = -\theta_x y, \quad \text{and} \quad \nu(y)\theta_x = -\theta_y x, \tag{1.4.8}$$

where $\theta_x := \Theta_1(dx)$ and $\theta_y := \Theta_1(dy)$.

The algebra A admits two types of automorphisms $\nu_i : A \to A$, $i = 1, 2$,

$$\nu_1(x) = ax, \ \nu_1(y) = by, \quad \text{and} \quad \nu_2(x) = ay, \ \nu_2(y) = bx, \quad a, b \in \Bbbk^*.$$

For $\nu = \nu_1$, expression (1.4.8) implies that

$$ax\theta_y = -\theta_x y, \quad by\theta_x = -\theta_y x,$$

where the only solutions are $\theta_y = yp(y)$ and $\theta_x = xq(x)$, with $p(x)$ and $q(x)$ polynomials of $\Bbbk[x]$. In this way, the image under Θ_1 of any one-form must be a polynomial without a scalar term, so 1 cannot lie in $\Theta_1(\Omega^1 A)$, and hence Θ_1 is not surjective. On the other hand, in the case of the automorphism ν_1, by using (1.4.8) we get that

$$ay\theta_y = -\theta_x y, \quad bx\theta_x = -\theta_y x.$$

By using the commutativity of the algebra, we can assert that

$$y(a\theta_y + \theta_x) = 0 = x(b\theta_x + \theta_y),$$

so we proceed to solve it. Note that if $ab \neq 1$ then both θ_x and θ_y are polynomials without any scalar terms, and by using the same reasoning above, Θ_1 is not surjective which contradicts the assumption that Θ_1 is an isomorphism of bimodules. In the case that $ab = 1$, it follows that the only solution is $\theta_y = c$ and $\theta_x = -ac$, for some $c \in \Bbbk$. Hence, $d(ay + x) = 0$, but this is a contradiction since we supposed that the differential calculus is connected.

We conclude that there are no one-dimensional connected integrable calculi over A, that is, A is not differentially smooth.

Remark 6. The algebra $A = \Bbbk\{x, y\}/\langle xy \rangle$ is neither SAS regular algebra (Section 1.3.1) nor differentially smooth.

CHAPTER 2

Smooth geometry of bi-quadratic algebras on three generators having PBW basis

In this chapter we characterize the differential smoothness for the *bi-quadratic algebras on three generators with PBW basis* defined recently by Bavula [Bav23, BK23].

The chapter is organized as follows. In Section 2.1 we consider the necessary preliminaries on bi-quadratic algebras with PBW basis. Section 2.2 contains the original results of the chapter. We extend Brzeziński's ideas developed for skew polynomial rings of the commutative polynomial ring $\Bbbk[t]$ [Brz15] (see Section 2.2.1; Proposition 2.3 shows that the quantized Weyl algebra $A_1^q(\Bbbk)$ is differentially smooth) to the setting of bi-quadratic algebras on three generators with PBW basis. More exactly, in Theorem 2.4 and Proposition 2.6 we formulate sufficient conditions to guarantee that a bi-quadratic algebra on three generators with PBW basis is differentially smooth, while Theorem 2.5 presents sufficient conditions on the impossibility of this fact. Following Bavula's classification of these algebras presented in his paper [Bav23], we organize our results using different tables to improve its presentation. In Section 2.3 we say a few words about possible future work related to the topic.

2.1 Bi-quadratic algebras with PBW basis

For a natural number $n \geq 2$, a family $M = (m_{ij})_{i>j}$ of elements m_{ij} belonging to R $(1 \leq j < i \leq n)$ is called a *lower triangular half-matrix* with coefficients in R. The set of all such matrices is denoted by $L_n(R)$.

Definition 2.1 ([Bav23, Section 1])**.** If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is an n-tuple of commuting endomorphisms of R, $\delta = (\delta_1, \ldots, \delta_n)$ is an n-tuple of σ-endomorphisms of R (that is, δ_i is a σ_i-derivation of R for $i = 1, \ldots, n$), $Q = (q_{ij}) \in L_n(Z(R))$, $\mathbb{A} := (a_{ij,k})$ where $a_{ij,k} \in R$, $1 \leq j < i \leq n$ and $k = 1, \ldots, n$, and $\mathbb{B} := (b_{ij}) \in L_n(R)$, the *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1, \ldots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ is a ring generated by the ring R and elements

$x_1, \dots, x_n$ subject to the defining relations

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \quad \text{for } i = 1, \dots, n, \text{ and every } r \in R, \tag{2.1.1}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij}, \quad \text{for all } j < i. \tag{2.1.2}$$

If $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$ for $i = 1, \dots, n$, the ring A is called the *bi-quadratic algebra* (*BQA*) and is denoted by $A = R[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$. A has *PBW basis* if $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x^\alpha$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Remark 7. As we can see from its definition, many of these are *generalized Weyl algebras* and *diskew polynomial rings* (see Bavula [Bav20] for more details). Bi-quadratic algebras with PBW basis are related to noncommutative algebras of polynomial type defined and investigated previously by different authors (e.g. Apel [Ape88], Bell et al. [BG88, BS90], Bueso et al. [BGTV03], Hinchcliffe [Hin05], Levandovskyy [Lev05], Lezama et al. [FGL+20, GL10, LR14, RS17b], Li [Li02], McConnell and Robson [MRS01], Pyatov et al. [IPR01, PT02], Redman [Red96b, Red99], Rosenberg [Ros95] and Seiler [Sei10]).

On the set W_n of all words in the alphabet $\{x_1, \dots, x_n\}$, Bavula considered the *degree-by-lexicographic ordering* where $x_1 < \cdots < x_n$. More exactly, $x_{i_1} \cdots x_{i_s} < x_{j_1} \cdots x_{j_t}$ if either $s < t$ or $s = t$, $i_1 = j_1, \dots, i_k = j_k$ and $i_{k+1} < j_{k+1}$ for some k such that $1 \le k < s$. Hence, if $A = R[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$ is a quadratic algebra where $n \ge 3$, then for each triple $i, j, k \in \{1, \dots, n\}$ such that $i < j < k$, there are exactly two different ways to simplify the product $x_k x_j x_i$ with respect to this order given by

$$x_k x_j x_i = q_{kj} q_{ki} q_{ji} x_i x_j x_k + \sum_{|\alpha| \le 2} c_{k,j,i,\alpha} x^\alpha,$$

$$x_k x_j x_i = q_{kj} q_{ki} q_{ji} x_i x_j x_k + \sum_{|\alpha| \le 2} c'_{k,j,i,\alpha} x^\alpha,$$

where in the first (resp. second) equality we start to simplify the product with the relation $x_k x_j = q_{kj} x_j x_k + \cdots$ (resp. $x_j x_i = q_{ji} x_i x_j + \cdots$) [Bav23, p. 696]. Thus, the defining relations (2.1.2) are consistent (i.e. $A \neq 0$) and $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x^\alpha$ if and only if for all triples $i, j, k \in \{1, \dots, n\}$ such that $i < j < k$, we have that $c_{k,j,i,\alpha} = c'_{k,j,i,\alpha}$. If this is the case, then for all $\sigma \in S_n$ (S_n denotes the *symmetric group of order* n) we have that $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x_\sigma^\alpha$ where $x_\sigma^\alpha = x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(n)}^{\alpha_n}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ [Bav23, Theorem 1.1].

Proposition 2.1 ([Bav23, Theorem 2.1])**.** *The classification of bi-quadratic algebras on two generators over a field $\Bbbk$ is as follows: if $q \in \Bbbk^*$ and $a, b, c \in \Bbbk$, then the algebra*

$$A = \Bbbk[x_1, x_2; q, a, b, c] := \Bbbk\{x_1, x_2\}/\langle x_2 x_1 - q x_1 x_2 - a x_1 - b x_2 - c \rangle$$

is a bi-quadratic algebra on two generators [Bav23, p. 704]*. The algebra $A = \Bbbk[x_1][x_2; \sigma, \delta]$ is a* skew polynomial ring *of $\Bbbk[x_1]$ where $\sigma(x_1) = q x_1 + b$ and $\delta(x_1) = a x_1 + c$.*

In this way, the algebra A *is a Noetherian domain with PBW basis,* $A = \bigoplus_{i,j\in\mathbb{N}} \Bbbk x_1^i x_2^j$ *and the scalars* a, b, c *are arbitrary.*

Up to isomorphism, there are only five bi-quadratic algebras of $\Bbbk$ *on two generators:*

(1) *The* polynomial algebra $\Bbbk[x_1, x_2]$*;*

(2) *The* Weyl algebra $A_1(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_1x_2 - x_2x_1 - 1\rangle$*;*

(3) *The* universal enveloping algebra of the Lie algebra $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$*, that is,* $U(\mathfrak{n}_2) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - x_1x_2 - x_1\rangle$*;*

(4) *The* quantum plane (Manin's plane) $\mathcal{O}_q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - qx_1x_2\rangle$*, where* $q \in \Bbbk \setminus \{0, 1\}$*;*

(5) *The* quantum Weyl algebra $A_1^q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - qx_1x_2 - 1\rangle$*, where* $q \in \Bbbk \setminus \{0, 1\}$.

Proposition 2.2 gives necessary and sufficient conditions for the bi-quadratic algebras on three-generators have a PBW basis. As we will see in Section 2.2.2, this is key to characterize the differential smoothness of these algebras.

Proposition 2.2 ([Bav23, Theorem 1.3])**.** *If* $A = \Bbbk[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$ *is a bi-quadratic algebra where* $Q = (q_1, q_2, q_3) \in \Bbbk^{*3}$*,*

$$\mathbb{A} = \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{bmatrix} \quad \text{and} \quad \mathbb{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The algebra A *is generated over* $\Bbbk$ *by the elements* x_1, x_2 *and* x_3 *subject to the defining relations*

$$x_2x_1 - q_1x_1x_2 = ax_1 + bx_2 + cx_3 + b_1, \tag{2.1.3}$$
$$x_3x_1 - q_2x_1x_3 = \alpha x_1 + \beta x_2 + \gamma x_3 + b_2, \tag{2.1.4}$$
$$x_3x_2 - q_3x_2x_3 = \lambda x_1 + \mu x_2 + \nu x_3 + b_3. \tag{2.1.5}$$

The relations (2.1.3)*,* (2.1.4) *and* (2.1.5) *are consistent and* $A = \bigoplus_{\alpha\in\mathbb{N}^3} \Bbbk x^\alpha$ *where*

$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ *if and only if the following conditions hold:*

$$(1-q_3)\alpha = (1-q_2)\mu, \tag{2.1.6}$$
$$(1-q_3)a = (1-q_1)\nu, \tag{2.1.7}$$
$$(1-q_2)b = (1-q_1)\gamma, \tag{2.1.8}$$
$$(1-q_1q_2)\lambda = 0, \tag{2.1.9}$$
$$(q_1-q_3)\beta = 0, \tag{2.1.10}$$
$$(1-q_2q_3)c = 0, \tag{2.1.11}$$
$$((1-q_3)\alpha-\mu)a + (b+q_1\gamma)\lambda - \nu\alpha + (q_1q_2-1)b_3 = 0, \tag{2.1.12}$$
$$(a-\nu)\beta + q_1\gamma\mu - q_3\alpha b + (q_1-q_3)b_2 = 0, \tag{2.1.13}$$
$$(a+(q_1-1)\nu)\gamma + b\nu - (\mu+q_3\alpha)c + (1-q_2q_3)b_1 = 0, \quad \text{and} \tag{2.1.14}$$
$$-(\mu+q_3\alpha)b_1 + (a-\nu)b_2 + (b+q_1\gamma)b_3 = 0. \tag{2.1.15}$$

Furthermore, if $A = \bigoplus_{\alpha\in\mathbb{N}^3} \Bbbk x^\alpha$ *where* $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$ *then* $A = \bigoplus_{\alpha\in\mathbb{N}^3} \Bbbk x_\sigma^\alpha$ *for all* $\sigma \in S_3$ *where* $x_\sigma^\alpha = x_{\sigma(1)}^{\alpha_1}x_{\sigma(2)}^{\alpha_2}x_{\sigma(3)}^{\alpha_3}$.

Example 2.1. Let us see some examples of bi-quadratic algebras on three generators.

(a) The universal enveloping algebra of any 3-dimensional Lie algebra.

(b) The 3-*dimensional quantum space* $\mathbb{A}^3_{q_1,q_2,q_3} := \Bbbk[x_1,x_2,x_3; Q, \mathbb{A}=0, \mathbb{B}=0]$.

(c) Following Havlíček et al. [HKP00, p. 79], the $\Bbbk$-algebra $U_q'(\mathfrak{so}_3)$ is generated by the indeterminates I_1, I_2, and I_3 subject to the relations given by

$$\begin{aligned} I_2I_1 - qI_1I_2 &= -q^{\frac{1}{2}}I_3, \\ I_3I_1 - q^{-1}I_1I_3 &= q^{-\frac{1}{2}}I_2, \quad \text{and} \\ I_3I_2 - qI_2I_3 &= -q^{\frac{1}{2}}I_1, \quad \text{for } q \in \Bbbk \setminus \{0,\pm 1\}. \end{aligned}$$

(d) Zhedanov [Zhe91, Section 1] introduced the *Askey-Wilson algebra* $AW(3)$ as the $\mathbb{R}$-algebra generated by three operators K_0, K_1, and K_2, that satisfy the commutation relations

$$\begin{aligned} [K_0,K_1]_\omega &= K_2, \\ [K_2,K_0]_\omega &= BK_0 + C_1K_1 + D_1, \quad \text{and} \\ [K_1,K_2]_\omega &= BK_1 + C_0K_0 + D_0, \end{aligned}$$

where B, C_0, C_1, D_0, and D_1 are elements of $\mathbb{R}$ that represent the structure constants of the algebra, and the q-commutator $[-,-]_\omega$ is given by $[\square,\triangle]_\omega := e^\omega\square\triangle - e^{-\omega}\triangle\square$, where $\omega \in \mathbb{R}$. Notice that in the limit $\omega \to 0$, the algebra $AW(3)$ becomes an ordinary Lie algebra with three generators (D_0 and D_1 are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl

type). The relations defining the algebra can be written as

$$\begin{aligned} e^{\omega}K_0K_1 - e^{-\omega}K_1K_0 &= K_2, \\ e^{\omega}K_2K_0 - e^{-\omega}K_0K_2 &= BK_0 + C_1K_1 + D_1, \quad \text{and} \\ e^{\omega}K_1K_2 - e^{-\omega}K_2K_1 &= BK_1 + C_0K_0 + D_0. \end{aligned}$$

With the aim of classifying bi-quadratic algebras on three generators, Bavula [Bav23, BK23] considered $Q = (q_1, q_2, q_3) \in \Bbbk^{*3}$ in Proposition 2.2 into the following four cases:

- $q_1 = q_2 = q_3 = 1$ (Lie type);
- $q_1 \neq 1,\ q_2 = q_3 = 1$;
- $q_1 \neq 1,\ q_2 \neq 1,\ q_3 = 1$, and
- $q_1 \neq 1,\ q_2 \neq 1,\ q_3 \neq 1$.

We will also consider these cases in the characterization of the differential smoothness of these algebras presented in the next section. As it can be seen from Bavula's papers, there are exactly 44 types (up to isomorphism and considering $\sqrt{\Bbbk} \subseteq \Bbbk$ and $\sqrt[3]{\Bbbk} \subseteq \Bbbk$) of non-isomorphic algebras of bi-quadratic algebras on three generators (see Tables 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 for the details of each algebra).

2.2 Differential and integral calculus

In this section we investigate the differential smoothness of bi-quadratic algebras on three generators. Before, we say a few words about the smoothness of the bi-quadratic algebras on two generators.

2.2.1 Bi-quadratic algebras on two generators

Brzeziński [Brz15] characterized the differential smoothness of skew polynomial rings of the form $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ where $\sigma_{q,r}(t) = qt + r$, with $q, r \in \Bbbk,\ q \neq 0$, and the $\sigma_{q,r}$−derivation $\delta_{p(t)}$ is defined as

$$\delta_{p(t)}(f(t)) = \frac{f(\sigma_{q,r}(t)) - f(t)}{\sigma_{q,r}(t) - t} p(t), \tag{2.2.1}$$

for an element $p(t) \in \Bbbk[t]$. $\delta_{p(t)}(f(t))$ is a suitable limit when $q = 1$ and $r = 0$, that is, when $\sigma_{q,r}$ is the identity map of $\Bbbk[t]$.

For the maps

$$\nu_t(t) = t, \quad \nu_t(x) = qx + p'(t) \quad \text{and} \quad \nu_x(t) = \sigma_{q,r}^{-1}(t), \quad \nu_x(x) = x, \tag{2.2.2}$$

where $p'(t)$ is the classical t-derivative of $p(t)$, Brzeziński [Brz15, Lemma 3.1] showed that all of them simultaneously extend to algebra automorphisms ν_t and ν_x of the skew polynomial ring $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ only in the following three cases:

(a) $q = 1, r = 0$ with no restriction on $p(t)$;

(b) $q = 1, r \neq 0$ and $p(t) = c$, $c \in \Bbbk$;

(c) $q \neq 1, p(t) = c\left(t + \frac{r}{q-1}\right)$, $c \in \Bbbk$ with no restriction on r.

In any of the cases (a) - (c) we have that $\nu_x \circ \nu_t = \nu_t \circ \nu_x$. If the Ore extension $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$ satisfies one of these three conditions, then it is differentially smooth [Brz15, Proposition 3.3].

As we saw in Proposition 2.1, up to isomorphism, there are only five bi-quadratic algebras on two generators over $\Bbbk$, so that having in mind Brzeziński's result on the differential smoothness of $\Bbbk[t][x; \sigma_{q,r}, \delta_{p(t)}]$, we get that the algebras $\Bbbk[x_1, x_2]$, $A_1(\Bbbk)$ and $U(\mathfrak{n}_2)$ belong to the case (a), while $\mathcal{O}_q(\Bbbk)$ belongs to the case (c), whence these four algebras are differentially smooth.

Remark 8. $A_1(\Bbbk)$ is an SAS regular algebra (not an AS regular algebra, Section 1.3.1) that is differentially smooth.

With respect to the quantum Weyl algebra $A_1^q(\Bbbk)$, this cannot be expressed by using this kind of skew polynomial rings so that its differential smoothness should be investigated by considering other and alternative approach. This is the content of the following proposition.

Proposition 2.3. *The quantum Weyl algebra $A_1^q(\Bbbk)$ is differentially smooth.*

Proof. We know that $\mathrm{GKdim}(A_1^q(\Bbbk)) = 2$ [GZ95, p. 867]. Consider the maps ν_{x_1} and ν_{x_2} given by

$$\nu_{x_1}(x_1) = q^{-1}x_1, \qquad \nu_{x_1}(x_2) = qx_2, \quad \text{and} \tag{2.2.3}$$

$$\nu_{x_2}(x_1) = q^{-1}x_1, \qquad \nu_{x_2}(x_2) = qx_2. \tag{2.2.4}$$

The map ν_{x_1} can be extended to an algebra homomorphism of $A_1^q(\Bbbk)$ if and only if the definitions of $\nu_{x_1}(x_1)$ and $\nu_{x_1}(x_2)$ respect relation $x_2x_1 = qx_1x_2 + 1$, i.e.

$$\nu_{x_1}(x_2)\nu_{x_1}(x_1) = q\nu_{x_1}(x_1)\nu_{x_1}(x_2) + \nu_{x_1}(1).$$

In this way, $qx_1x_2 + qq^{-1} = qx_1x_2 + 1$, which always holds. The case ν_{x_2} is similar:

$$\nu_{x_2}(x_2)\nu_{x_2}(x_1) = q\nu_{x_2}(x_1)\nu_{x_2}(x_2) + \nu_{x_2}(1),$$

whence $qx_1x_2 + qq^{-1} = qx_1x_2 + 1$. Now, since we need to guarantee that

$$\nu_{x_1} \circ \nu_{x_2} = \nu_{x_2} \circ \nu_{x_1}, \tag{2.2.5}$$

then we impose that

$$\begin{aligned}
\nu_{x_1} \circ \nu_{x_2}(x_1) &= \nu_{x_1}(q^{-1}x_1) = q^{-2}x_1,\\
\nu_{x_2} \circ \nu_{x_1}(x_1) &= \nu_{x_2}(q^{-1}x_1) = q^{-2}x_1,\\
\nu_{x_1} \circ \nu_{x_2}(x_2) &= \nu_{x_1}(qx_2) = q^{2}x_2, \quad \text{and}\\
\nu_{x_2} \circ \nu_{x_1}(x_2) &= \nu_{x_2}(qx_2) = q^{2}x_2.
\end{aligned}$$

As it is clear, composition $\nu_{x_1} \circ \nu_{x_2} = \nu_{x_2} \circ \nu_{x_1}$ holds.

Consider $\Omega^1(A_1^q(\Bbbk))$ a free right $A_1^q(\Bbbk)$-module of rank two with generators dx_1 and dx_2. For all $a \in A_1^q(\Bbbk)$ define a left $A_1^q(\Bbbk)$-module structure by

$$a dx_1 = dx_1 \nu_{x_1}(a) \quad \text{and} \quad a dx_2 = dx_2 \nu_{x_2}(a). \tag{2.2.6}$$

The relations in $\Omega^1(A_1^q(\Bbbk))$ are given by

$$\begin{aligned} x_1 dx_1 &= q^{-1} dx_1 x_1, \\ x_1 dx_2 &= q^{-1} dx_2 x_1, \\ x_2 dx_1 &= q dx_1 x_2, \quad \text{and} \\ x_2 dx_2 &= q dx_2 x_2. \end{aligned} \tag{2.2.7}$$

Since we want to extend the correspondences

$$x_1 \mapsto dx_1 \quad \text{and} \quad x_2 \mapsto dx_2$$

to a map $d : A_1^q(\Bbbk) \to \Omega^1(A_1^q(\Bbbk))$ satisfying the Leibniz's rule, from the relation $x_2x_1 = qx_1x_2 + 1$ we get that $dx_2x_1 + x_2dx_1 = q dx_1x_2 + qx_1dx_2$.

Define $\Bbbk$-linear maps

$$\partial_{x_1}, \partial_{x_2} : A_1^q(\Bbbk) \to A_1^q(\Bbbk)$$

such that

$$d(a) = dx_1 \partial_{x_1}(a) + dx_2 \partial_{x_2}(a) \quad \text{for all } a \in A_1^q(\Bbbk).$$

Having in mind that dx_1 and dx_2 are free generators of the right $A_1^q(\Bbbk)$-module $\Omega^1(A_1^q(\Bbbk))$, these maps are well-defined. Then $d(a) = 0$ if and only if $\partial_{x_1}(a) = \partial_{x_2}(a) = 0$. By using the two relations in (2.2.6) and the definitions of the maps ν_{x_1} and ν_{x_2}, it follows that

$$\partial_{x_1}(x_1^l x_2^s) = l x_1^{l-1} x_2^s \quad \text{and} \quad \partial_{x_2}(x_1^l x_2^s) = s q^{-l} x_1^l x_2^{s-1}.$$

Thus, $d(a) = 0$ if and only if a is a scalar multiple of the identity. Hence, $(\Omega(A_1^q(\Bbbk)), d)$ is connected where $\Omega(A_1^q(\Bbbk)) = \Omega^0(A_1^q(\Bbbk)) \oplus \Omega^1(A_1^q(\Bbbk))$.

Since $\Omega^2(A_1^q(\Bbbk)) = \omega A_1^q(\Bbbk) \cong A_1^q(\Bbbk)$ as a right and left $A_1^q(\Bbbk)$-module, with $\omega = dx_1 \wedge dx_2$, where $\nu_\omega = \nu_{x_1} \circ \nu_{x_2}$, we have that ω is a volume form of $A_1^q(\Bbbk)$. From Proposition 1.7 (2) we get that ω is an integral form by setting

$$\omega_1^1 = dx_1, \quad \omega_2^1 = dx_2, \quad \bar{\omega}_1^1 = -q^{-1} dx_2, \quad \text{and} \quad \bar{\omega}_2^1 = dx_1.$$

By Proposition 1.7 (2), we consider the expression $\omega' := dx_1\alpha + dx_2\beta$ with $\alpha, \beta \in \Bbbk$, to obtain that

$$\begin{aligned} \sum_{i=1}^{2} \omega_i^1 \pi_\omega(\bar{\omega}_i^1 \wedge \omega') &= dx_1 \pi_\omega(-\alpha q^{-1} dx_2 \wedge dx_1) + dx_2 \pi_\omega(\beta dx_1 \wedge dx_2) \\ &= d_1 x \alpha + dx_2 \beta \\ &= \omega'. \end{aligned}$$

As we saw above, all elements of $\Omega^1(A_q^1(\Bbbk))$ can be generated by ω_i^1 and $\bar{\omega}_i^1$ for $i = 1, 2$, so Proposition 1.7 (2) guarantees that ω is an integral form. Proposition 1.6 shows that $(\Omega(A_1^q(\Bbbk)), d)$ is an integrable differential calculus of degree 2, whence $A_1^q(\Bbbk)$ is differentially smooth. □

Remark 9. $A_1^q(\Bbbk)$ is an SAS regular algebra (not an AS regular algebra, Section 1.3.1) that is differentially smooth.

2.2.2 Bi-quadratic algebras on three generators

Throughout this section, A denotes a bi-quadratic algebra on three generators x_1, x_2 and x_3 subject to the relations (2.1.3), (2.1.4) and (2.1.5) in Proposition 2.2. Since these are a subclass of the *skew PBW extensions* introduced by Gallego and Lezama [GL10], it follows from [Rey13, Theorems 14 and 18] that its Gelfand-Kirillov dimension is three. This fact is key in Theorems 2.4 and 2.5.

The following theorem is the first important result of the chapter. We extend Brzeziński's ideas [Brz15] mentioned in Section 2.2.1.

Theorem 2.4. *If the conditions*

$$c = \beta = \lambda = 0, \tag{2.2.8}$$
$$b_1(q_1 - 1) - ab = 0, \tag{2.2.9}$$
$$b_2(q_2 - 1) - \alpha\gamma = 0, \tag{2.2.10}$$
$$b_3(q_3 - 1) - \mu\nu = 0, \quad \text{and} \tag{2.2.11}$$
$$\mu a = 0 \tag{2.2.12}$$

hold, then A is differentially smooth.

Proof. Consider the following automorphisms:

$$\nu_{x_1}(x_1) = x_1, \quad \nu_{x_1}(x_2) = q_1x_2 + a, \quad \nu_{x_1}(x_3) = q_2x_3 + \alpha, \tag{2.2.13}$$
$$\nu_{x_2}(x_1) = q_1^{-1}x_1 - bq_1^{-1}, \quad \nu_{x_2}(x_2) = x_2, \quad \nu_{x_2}(x_3) = q_3x_3 + \mu, \tag{2.2.14}$$
$$\nu_{x_3}(x_1) = q_2^{-1}x_1 - \gamma q_2^{-1}, \quad \nu_{x_3}(x_2) = q_3^{-1}x_2 - \nu q_3^{-1}, \quad \nu_{x_3}(x_3) = x_3. \tag{2.2.15}$$

The map ν_{x_1} can be extended to an algebra homomorphism of A if and only if the definitions of $\nu_{x_1}(x_1)$, $\nu_{x_1}(x_2)$ and $\nu_{x_1}(x_3)$ respect relations (2.1.3), (2.1.4) and (2.1.5), i.e.

$$\begin{aligned}
&\nu_{x_1}(x_2)\nu_{x_1}(x_1) - q_1\nu_{x_1}(x_1)\nu_{x_1}(x_2) = a\nu_{x_1}(x_1) + b\nu_{x_1}(x_2) + b_1,\\
&\nu_{x_1}(x_3)\nu_{x_1}(x_1) - q_2\nu_{x_1}(x_1)\nu_{x_1}(x_3) = \alpha\nu_{x_1}(x_1) + \gamma\nu_{x_1}(x_3) + b_2, \quad \text{and}\\
&\nu_{x_1}(x_3)\nu_{x_1}(x_2) - q_3\nu_{x_1}(x_2)\nu_{x_1}(x_3) = \mu\nu_{x_1}(x_2) + \nu\nu_{x_1}(x_3) + b_3.
\end{aligned}$$

In this way, we obtain the equations

$$\begin{aligned} b_1(q_1-1)-ab &= 0,\\ b_2(q_2-1)-\alpha\gamma &= 0,\\ \mu(q_2-1) &= \alpha(q_3-1),\\ \nu(q_1-1) &= a(q_3-1), \quad \text{and}\\ b_3(q_1q_2-1) &= \alpha(a(q_3-1)+\nu). \end{aligned}$$

By putting together these five relations with those appearing in Proposition 2.2 we get three new and additional relations given by

$$b_1(q_1-1)-ab=0, \quad b_2(q_2-1)-\alpha\gamma=0 \quad \text{and} \quad \mu a=0. \tag{2.2.16}$$

Similarly, the map ν_{x_2} can be extended to an algebra homomorphism of A if and only if the equalities

$$\begin{aligned} b_1(q_1-1)-ab &= 0,\\ \mu(q_2-1) &= \alpha(q_3-1),\\ b(q_2-1) &= \gamma(q_1-1),\\ b_2(q_1-q_3)+q_1\gamma\mu+b(\mu-q_2\mu-\alpha) &= 0, \quad \text{and}\\ b_3(q_3-1)-\nu\mu &= 0 \end{aligned}$$

are satisfied. These relations together with Proposition 2.2 and equations (2.2.16) generate the new relation

$$b_3(q_3-1)-\nu\mu=0. \tag{2.2.17}$$

By considering the extension of the map ν_{x_3} to an algebra homomorphism of A, we get the equations

$$\begin{aligned} b_3(q_3-1)-\mu\nu &= 0,\\ \nu(q_1-1) &= a(q_3-1),\\ b(q_2-1) &= \gamma(q_1-1),\\ b_1(1-q_2q_3)+\nu\mu(1-q_1)+q_3a\gamma+q_2b\nu &= 0, \quad \text{and}\\ b_2(q_2-1)-\alpha\gamma &= 0. \end{aligned}$$

These relations with all obtained above do not generate a new relation, so we can extend the maps ν_{x_1}, ν_{x_2} and ν_{x_3} to an algebra homomorphism if and only if

$$b_1(q_1-1)-ab=0, \tag{2.2.18}$$
$$b_2(q_2-1)-\alpha\gamma=0, \tag{2.2.19}$$
$$b_3(q_3-1)-\mu\nu=0, \quad \text{and} \tag{2.2.20}$$
$$\mu a=0. \tag{2.2.21}$$

Since we need to guarantee that

$$\nu_{x_1} \circ \nu_{x_2} = \nu_{x_2} \circ \nu_{x_1}, \tag{2.2.22}$$
$$\nu_{x_1} \circ \nu_{x_3} = \nu_{x_3} \circ \nu_{x_1}, \quad \text{and} \tag{2.2.23}$$
$$\nu_{x_3} \circ \nu_{x_2} = \nu_{x_2} \circ \nu_{x_3}, \tag{2.2.24}$$

it is enough to satisfy these equalities for the generators x_1, x_2 and x_3, i.e.

$$\nu_{x_1} \circ \nu_{x_2}(x_1) = q_1^{-1}x_1 - bq_1^{-1}, \tag{2.2.25}$$
$$\nu_{x_2} \circ \nu_{x_1}(x_1) = q_1^{-1}x_1 - bq_1^{-1}, \tag{2.2.26}$$
$$\nu_{x_1} \circ \nu_{x_2}(x_2) = q_1x_2 + a, \tag{2.2.27}$$
$$\nu_{x_2} \circ \nu_{x_1}(x_2) = q_1x_2 + a, \tag{2.2.28}$$
$$\nu_{x_1} \circ \nu_{x_2}(x_3) = q_2q_3x_3 + q_3\alpha + \mu, \quad \text{and} \tag{2.2.29}$$
$$\nu_{x_2} \circ \nu_{x_1}(x_3) = q_2q_3x_3 + q_2\mu + \alpha. \tag{2.2.30}$$

As we can see, composition $\nu_{x_1} \circ \nu_{x_2} = \nu_{x_2} \circ \nu_{x_1}$ is always satisfied, so we only consider the relations (5.3.53) and (3.2.8) which hold when $\alpha(q_3 - 1) = \mu(q_2 - 1)$; this is precisely the relation (2.1.6) in Proposition 2.2.

Next,

$$\nu_{x_1} \circ \nu_{x_3}(x_1) = q_2^{-1}x_1 - \gamma q_2^{-1}, \tag{2.2.31}$$
$$\nu_{x_3} \circ \nu_{x_1}(x_1) = q_2^{-1}x_1 - \gamma q_2^{-1}, \tag{2.2.32}$$
$$\nu_{x_1} \circ \nu_{x_3}(x_2) = q_1q_3^{-1}x_2 + q_3^{-1}a - \nu q_3^{-1}, \tag{2.2.33}$$
$$\nu_{x_3} \circ \nu_{x_1}(x_2) = q_1q_3^{-1}x_2 - \nu q_1q_3^{-1} + a, \tag{2.2.34}$$
$$\nu_{x_1} \circ \nu_{x_3}(x_3) = q_2x_3 + \alpha, \quad \text{and} \tag{2.2.35}$$
$$\nu_{x_3} \circ \nu_{x_1}(x_3) = q_2x_3 + \alpha. \tag{2.2.36}$$

Again, the compositions shown in (5.3.20) and (5.3.21), and compositions (3.2.13) and (5.3.23) are the same. Relations (3.2.11) and (5.3.22) coincide when $a(q_3 - 1) = \nu(q_1 - 1)$ which occurs is satisfied due to Proposition 2.2.

Finally,

$$\nu_{x_3} \circ \nu_{x_2}(x_1) = q_1^{-1}q_2^{-1}x_1 - q_1^{-1}q_2^{-1}\gamma - bq_1^{-1}, \tag{2.2.37}$$
$$\nu_{x_2} \circ \nu_{x_3}(x_1) = q_1^{-1}q_2^{-1}x_1 - q_1^{-1}q_2^{-1}b - \gamma q_2^{-1}, \tag{2.2.38}$$
$$\nu_{x_3} \circ \nu_{x_2}(x_2) = q_3^{-1}x_2 - \nu q_3^{-1}, \tag{2.2.39}$$
$$\nu_{x_2} \circ \nu_{x_3}(x_2) = q_3^{-1}x_2 - \nu q_3^{-1}, \tag{2.2.40}$$
$$\nu_{x_3} \circ \nu_{x_2}(x_3) = q_3x_3 + \mu, \quad \text{and} \tag{2.2.41}$$
$$\nu_{x_2} \circ \nu_{x_3}(x_3) = q_3x_3 + \mu. \tag{2.2.42}$$

Relations (3.2.17) and (5.3.27) are equal, and (3.2.19) and (5.3.29) coincide. Note that relations (3.2.15) and (5.3.25) are the same if $\gamma(q_1 - 1) = b(q_2 - 1)$, which is the case by (2.1.8) in Proposition 2.2.

Consider $\Omega^1 A$ a free right A-module of rank three with generators dx_1, dx_2 and dx_3. For all $p \in A$ define a left A-module structure by

$$\begin{aligned} p\,dx_1 &= dx_1 \nu_{x_1}(p), \\ p\,dx_2 &= dx_2 \nu_{x_2}(p), \text{ and} \\ p\,dx_3 &= dx_3 \nu_{x_3}(p). \end{aligned} \tag{2.2.43}$$

The relations in $\Omega^1 A$ are given by

$$\begin{aligned} x_1 dx_1 &= dx_1 x_1, \\ x_1 dx_2 &= q_1^{-1} dx_2 x_1 - b q_1^{-1} dx_2, \\ x_1 dx_3 &= q_2^{-1} dx_3 x_1 - \gamma q_2^{-1} dx_3, \end{aligned} \tag{2.2.44}$$

$$\begin{aligned} x_2 dx_1 &= q_1 dx_1 x_2 + a dx_1, \\ x_2 dx_2 &= dx_2 x_2, \\ x_2 dx_3 &= q_3^{-1} dx_3 x_2 - \nu q_3^{-1} dx_3, \end{aligned} \tag{2.2.45}$$

$$\begin{aligned} x_3 dx_1 &= q_2 dx_1 x_3 + \alpha dx_1, \\ x_3 dx_2 &= q_3 dx_2 x_3 + \mu dx_2, \text{ and} \\ x_3 dx_3 &= dx_3 x_3. \end{aligned} \tag{2.2.46}$$

We want to extend the correspondences

$$x_1 \mapsto dx_1, \quad x_2 \mapsto dx_2 \quad \text{and} \quad x_3 \mapsto dx_3$$

to a map $d : A \to \Omega^1 A$ satisfying the Leibniz's rule. This is possible if it is compatible with the nontrivial relations (2.1.3), (2.1.4) and (2.1.5), i.e. if the equalities

$$\begin{aligned} dx_2 x_1 + x_2 dx_1 &= q_1 dx_1 x_2 + q_1 x_1 dx_2 + a dx_1 + b dx_2, \\ dx_3 x_1 + x_3 dx_1 &= q_2 dx_1 x_3 + q_2 x_1 dx_3 + \alpha dx_1 + \gamma dx_3, \text{ and} \\ dx_3 x_2 + x_3 dx_2 &= q_3 dx_2 x_3 + q_3 x_2 dx_3 + \mu dx_2 + \nu dx_3 \end{aligned}$$

hold. Note that $d(b_i) = 0$ for $i = 1, 2, 3$.

Define $\Bbbk$-linear maps

$$\partial_{x_1}, \partial_{x_2}, \partial_{x_3} : A \to A$$

such that

$$d(a) = dx_1 \partial_{x_1}(a) + dx_2 \partial_{x_2}(a) + dx_3 \partial_{x_3}(a), \quad \text{for all } a \in A.$$

Since dx_1, dx_2 and dx_3 are free generators of the right A-module $\Omega^1 A$, these maps are well-defined. Note that $d(a) = 0$ if and only if $\partial_{x_1}(a) = \partial_{x_2}(a) = \partial_{x_3}(a) = 0$. By using the three relations in (5.3.30) and the definitions of the maps ν_{x_1}, ν_{x_2} and ν_{x_3}, we get that

$$\begin{aligned} \partial_{x_1}(x_1^k x_2^l x_3^s) &= k x_1^{k-1} x_2^l x_3^s, \\ \partial_{x_2}(x_1^k x_2^l x_3^s) &= l q_1^{-k} (x_1 - b)^k x_2^{l-1} x_3^s, \quad \text{and} \\ \partial_{x_3}(x_1^k x_2^l x_3^s) &= s k q_2^{-k} q_3^{-l} (x_1 - \gamma)^k (x_2 - \nu)^l x_3^{s-1}. \end{aligned} \tag{2.2.47}$$

Thus $\mathrm{d}(a) = 0$ if and only if a is a scalar multiple of the identity. This shows that $(\Omega A, \mathrm{d})$ is connected where $\Omega A = \Omega^0 A \oplus \Omega^1 A \oplus \Omega^2 A$.

The universal extension of d to higher forms compatible with (5.3.31), (5.3.32) and (5.3.33) gives the following rules for $\Omega^2 A$:

$$dx_2 \wedge dx_1 = -q_1 dx_1 \wedge dx_2, \tag{2.2.48}$$
$$dx_3 \wedge dx_1 = -q_2 dx_1 \wedge dx_3, \quad \text{and} \tag{2.2.49}$$
$$dx_3 \wedge dx_2 = -q_3 dx_2 \wedge dx_3. \tag{2.2.50}$$

Since the automorphisms ν_{x_1}, ν_{x_2} and ν_{x_3} commute with each other, there are no additional relationships to the previous ones, so we get the expression

$$\Omega^2 A = dx_1 \wedge dx_2 A \oplus dx_1 \wedge dx_3 A \oplus dx_2 \wedge dx_3 A.$$

Since $\Omega^3 A = \omega A \cong A$ as a right and left A-module, with $\omega = dx_1 \wedge dx_2 \wedge dx_3$, where $\nu_\omega = \nu_{x_1} \circ \nu_{x_2} \circ \nu_{x_3}$, we have that ω is a volume form of A. From Proposition 1.7 (2) we get that ω is an integral form by setting

$$\begin{aligned}
\omega_1^1 &= \bar{\omega}_1^1 = dx_1, \\
\omega_2^1 &= \bar{\omega}_2^1 = dx_2, \\
\omega_3^1 &= \bar{\omega}_3^1 = dx_3, \\
\omega_1^2 &= dx_2 \wedge dx_3, \\
\omega_2^2 &= -q_1^{-1} dx_1 \wedge dx_3, \\
\omega_3^2 &= q_2^{-1} q_3^{-1} dx_1 \wedge dx_3, \\
\bar{\omega}_1^2 &= q_1^{-1} q_2^{-1} dx_2 \wedge dx_3, \\
\bar{\omega}_2^2 &= -q_3^{-1} dt \wedge dx_2, \quad \text{and} \\
\bar{\omega}_3^2 &= dx_1 \wedge dx_2.
\end{aligned}$$

By Proposition 1.7 (2), we consider the expression $\omega' := dx_1 a + dx_2 b + dx_3 c$ with $a, b, c \in \Bbbk$, to obtain the equality

$$\begin{aligned}
\sum_{i=1}^{3} \omega_i^1 \pi_\omega(\bar{\omega}_i^2 \wedge \omega') &= dx_1 \pi_\omega(q_1^{-1} q_2^{-1} a\, dx_2 \wedge dx_3 \wedge dx_1) \\
&\quad + dx_2 \pi_\omega(-q_3^{-1} b\, dx_1 \wedge dx_3 \wedge dx_2) \\
&\quad + dx_3 \pi_\omega(c\, dx_1 \wedge dx_2 \wedge dx_3) \\
&= dx_1 a + dx_2 b + dx_3 c \\
&= \omega'.
\end{aligned}$$

On the other hand, if $\omega'' := dx_1 \wedge dx_2 a + dx_1 \wedge dx_3 b + dx_2 \wedge dx_3 c$ where $a, b, c \in \Bbbk$,

we get that

$$\begin{aligned}\sum_{i=1}^{3}\omega_i^2\pi_\omega(\bar{\omega}_i^1\wedge\omega'') &= dx_2\wedge dx_3\pi_\omega(cdx_1\wedge dx_2\wedge dx_3)\\ &\quad - q_1^{-1}dx_1\wedge dx_3\pi_\omega(bdx_2\wedge dx_1\wedge dx_3)\\ &\quad + q_2^{-1}q_3^{-1}dx_1\wedge dx_2\pi_\omega(adx_3\wedge dx_1\wedge dx_2)\\ &= dx_1\wedge dx_2a + dx_1\wedge dx_3b + dx_2\wedge dx_3\\ &= \omega''.\end{aligned}$$

As we have seen above, all elements of different degrees can be generated by ω_i^j and $\bar{\omega}_i^{3-j}$ for $j = 1, 2$ and $i = 1, 2, 3$, so Proposition 1.7 (2) guarantees that ω is an integral form. Proposition 1.6 shows that $(\Omega A, d)$ is an integrable differential calculus of degree n, and since $\mathrm{GKdim}(A)$ is also n, it follows that A is differentially smooth. □

Following Bavula's classification presented in [Bav23], Theorem 2.4 shows that there are twenty two bi-quadratic algebras on three generators that are differentially smooth. On the other hand, Theorem 2.5 shows that twenty one bi-quadratic algebras cannot be differentially smooth. This is the second important result of the chapter.

Theorem 2.5. *If one of the conditions $c \neq 0$, $\beta \neq 0$ or $\lambda \neq 0$ holds, then A is not differentially smooth.*

Proof. By contradiction. Suppose that A has a first order differential calculus $(\Omega A, d)$. Without loss of generality, we consider the case $c \neq 0$. Since d must be compatible with the relation (2.1.3), then we get that

$$dx_2x_1 + x_2dx_1 = q_1dx_1x_2 + q_1x_1dx_2 + adx_1 + bdx_2 + cdx_3,$$

whence dx_3 is generated by dx_1 and dx_2. This means that Ω^1A is generated by two elements and $\Omega^3A = \Omega^1A\wedge\Omega^1A\wedge\Omega^1A = 0$, i.e. there is no third-order calculus. Since $\mathrm{GKdim}(A) = 3$, we conclude that A cannot be differentially smooth. □

Tables 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 contain the explicit relations of bi-quadratic algebras on three-generators. There, by Theorems 2.4 and 2.5, the symbols ✓ and ⋆ denote a positive and a negative answer, respectively, on its differential smoothness.

In the case of algebra D_1 (Table 2.2), Theorems 2.4 and 2.5 cannot be applied. We consider other approach to investigate its differential smoothness as the following proposition shows.

Proposition 2.6. *The algebra D_1 is differentially smooth.*

Proof. Let

$$\nu_{x_1}(x_1) = q_1^{-1}x_1, \qquad \nu_{x_1}(x_2) = q_1x_2, \qquad \nu_{x_1}(x_3) = x_3 + 1, \tag{2.2.51}$$
$$\nu_{x_2}(x_1) = q_1^{-1}x_1, \qquad \nu_{x_2}(x_2) = q_1x_2, \qquad \nu_{x_2}(x_3) = x_3 - 1, \tag{2.2.52}$$
$$\nu_{x_3}(x_1) = x_1, \qquad \nu_{x_3}(x_2) = x_2, \qquad \nu_{x_3}(x_3) = x_3. \tag{2.2.53}$$

It is straightforward to show that the maps ν_{x_1}, ν_{x_2} and ν_{x_3} can be extended to algebra homomorphisms of D_1; these maps respect the relations (2.1.3), (2.1.4) and (2.1.5) for D_1. Furthermore, these maps commute with each other.

Consider $\Omega^1(D_1)$ a free right D_1-module of rank three with generators dx_1, dx_2 and dx_3. For all $p \in D_1$ define a left D_1-module structure by

$$\begin{aligned} pdx_1 &= dx_1\nu_{x_1}(p), \\ pdx_2 &= dx_2\nu_{x_2}(p), \text{ and} \\ pdx_3 &= dx_3\nu_{x_3}(p). \end{aligned} \tag{2.2.54}$$

The relations in $\Omega^1(D_1)$ are given by

$$\begin{aligned} x_1dx_1 &= q_1^{-1}dx_1x_1, \\ x_1dx_2 &= q_1^{-1}dx_2x_1, \\ x_1dx_3 &= dx_3x_1, \end{aligned} \tag{2.2.55}$$

$$\begin{aligned} x_2dx_1 &= q_1dx_1x_2, \\ x_2dx_2 &= q_1dx_2x_2, \\ x_2dx_3 &= dx_3x_2, \end{aligned} \tag{2.2.56}$$

$$\begin{aligned} x_3dx_1 &= dx_1(x_3 + 1), \\ x_3dx_2 &= dx_2(x_3 - 1), \text{ and} \\ x_3dx_3 &= dx_3x_3. \end{aligned} \tag{2.2.57}$$

We want to extend the correspondences

$$x_1 \mapsto dx_1, \quad x_2 \mapsto dx_2 \quad \text{and} \quad x_3 \mapsto dx_3$$

to a map $d : A \to \Omega^1(D_1)$ satisfying the Leibniz's rule. This is possible if it is compatible with the nontrivial relations (2.1.3), (2.1.4) and (2.1.5), i.e. if the equalities

$$\begin{aligned} dx_2x_1 + x_2dx_1 &= q_1dx_1x_2 + q_1x_1dx_2 + adx_1 + bdx_2, \\ dx_3x_1 + x_3dx_1 &= q_2dx_1x_3 + q_2x_1dx_3 + \alpha dx_1 + \gamma dx_3, \text{ and} \\ dx_3x_2 + x_3dx_2 &= q_3dx_2x_3 + q_3x_2dx_3 + \mu dx_2 + \nu dx_3 \end{aligned}$$

hold. Note that $d(b_i) = 0$ for $i = 1, 2, 3$.

Define $\Bbbk$-linear maps

$$\partial_{x_1}, \partial_{x_2}, \partial_{x_3} : D_1 \to D_1$$

such that

$$d(a) = dx_1\partial_{x_1}(a) + dx_2\partial_{x_2}(a) + dx_3\partial_{x_3}(a), \quad \text{for all } a \in A.$$

Since dx_1, dx_2 and dx_3 are free generators of the right D_1-module $\Omega^1(D_1)$, these maps are well-defined. Note that $d(a) = 0$ if and only if $\partial_{x_1}(a) = \partial_{x_2}(a) = \partial_{x_3}(a) = 0$. By using the three relations in (2.2.54) and the definitions of the maps ν_{x_1}, ν_{x_2} and ν_{x_3}, we get that

$$\begin{aligned}
\partial_{x_1}(x_1^k x_2^l x_3^s) &= k x_1^{k-1} x_2^l x_3^s, \\
\partial_{x_2}(x_1^k x_2^l x_3^s) &= l q_1^{-k} x_1^k x_2^{l-1} x_3^s, \quad \text{and} \\
\partial_{x_3}(x_1^k x_2^l x_3^s) &= s x_1^k x_2^l x_3^{s-1}.
\end{aligned} \tag{2.2.58}$$

Thus $d(a) = 0$ if and only if a is a scalar multiple of the identity. This shows that $(\Omega D_1, d)$ is connected where $\Omega(D_1) = \Omega^0(D_1) \oplus \Omega^1(D_1) \oplus \Omega^2(D_1)$.

The universal extension of d to higher forms compatible with (2.2.55), (2.2.56) and (2.2.57) gives the following rules for $\Omega^2(D_1)$:

$$dx_2 \wedge dx_1 = -q_1 dx_1 \wedge dx_2, \tag{2.2.59}$$
$$dx_3 \wedge dx_1 = -dx_1 \wedge dx_3, \quad \text{and} \tag{2.2.60}$$
$$dx_3 \wedge dx_2 = -dx_2 \wedge dx_3. \tag{2.2.61}$$

By using that the automorphisms ν_{x_1}, ν_{x_2} and ν_{x_3} commute with each other, there are no additional relationships to the previous ones, so we get the expression

$$\Omega^2(D_1) = dx_1 \wedge dx_2(D_1) \oplus dx_1 \wedge dx_3(D_1) \oplus dx_2 \wedge dx_3(D_1).$$

Since $\Omega^3(D_1) = \omega D_1 \cong D_1$ as a right and left A-module with $\omega = dx_1 \wedge dx_2 \wedge dx_3$, where $\nu_\omega = \nu_{x_1} \circ \nu_{x_2} \circ \nu_{x_3}$, we have that ω is a volume form of D_1. From Proposition 1.7 (2) we get that ω is an integral form by setting

$$\begin{aligned}
\omega_1^1 &= \bar{\omega}_1^1 = dx_1, \\
\omega_2^1 &= \bar{\omega}_2^1 = dx_2, \\
\omega_3^1 &= \bar{\omega}_3^1 = dx_3, \\
\omega_1^2 &= dx_2 \wedge dx_3, \\
\omega_2^2 &= -q_1^{-1} dx_1 \wedge dx_3, \\
\omega_3^2 &= dx_1 \wedge dx_3, \\
\bar{\omega}_1^2 &= q_1^{-1} dx_2 \wedge dx_3, \\
\bar{\omega}_2^2 &= -dx_1 \wedge dx_2, \quad \text{and} \\
\bar{\omega}_3^2 &= dx_1 \wedge dx_2.
\end{aligned}$$

By Proposition 1.7 (2), we consider the expression $\omega' := dx_1 a + dx_2 b + dx_3 c$ with

$a, b, c \in \Bbbk$ and obtain the equality

$$\begin{aligned}\sum_{i=1}^{3} \omega_i^1 \pi_\omega(\bar{\omega}_i^2 \wedge \omega') &= dx_1 \pi_\omega(q_1^{-1} a dx_2 \wedge dx_3 \wedge dx_1) \\ &\quad + dx_2 \pi_\omega(-b dx_1 \wedge dx_3 \wedge dx_2) \\ &\quad + dx_3 \pi_\omega(c dx_1 \wedge dx_2 \wedge dx_3) \\ &= dx_1 a + dx_2 b + dx_3 c \\ &= \omega'.\end{aligned}$$

On the other hand, if $\omega'' := dx_1 \wedge dx_2 a + dx_1 \wedge dx_3 b + dx_2 \wedge dx_3 c$ where $a, b, c \in \Bbbk$, we get that

$$\begin{aligned}\sum_{i=1}^{3} \omega_i^2 \pi_\omega(\bar{\omega}_i^1 \wedge \omega'') &= dx_2 \wedge dx_3 \pi_\omega(c dx_1 \wedge dx_2 \wedge dx_3) \\ &\quad - q_1^{-1} dx_1 \wedge dx_3 \pi_\omega(b dx_2 \wedge dx_1 \wedge dx_3) \\ &\quad + dx_1 \wedge dx_2 \pi_\omega(a dx_3 \wedge dx_1 \wedge dx_2) \\ &= dx_1 \wedge dx_2 a + dx_1 \wedge dx_3 b + dx_2 \wedge dx_3 \\ &= \omega''.\end{aligned}$$

As we have seen above, all elements of different degrees can be generated by ω_i^j and $\bar{\omega}_i^{3-j}$ for $j = 1, 2$ and $i = 1, 2, 3$, so Proposition 1.7 (2) guarantees that ω is an integral form. Proposition 1.6 shows that $(\Omega(D_1), d)$ is an integrable differential calculus of degree n, and since $\mathrm{GKdim}(D_1) = n$, it follows that A is differentially smooth. □

Remark 10. It is important to note that if one algebra is not differential smooth, this does not mean that the algebra does not possess a differential calculus. The universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ is an illustration of this fact as it was shown by Beggs and Majid [BM20]. By using Hopf algebras, they proved the existence of a first-order differential calculus and other properties of Riemannian quantum geometry that are different from the differential smoothness shown in this thesis.

2.3 Future work

As expected, a natural task is to investigate the differential smoothness of bi-quadratic algebras on n generators. Precisely, Bavula [Bav23, p. 699] asserted that a construction of bi-quadratic algebras on four generators was introduced by Zhang and Zhang [ZZ08, ZZ09] with their class of *double Ore extensions*. As one can appreciate in the literature, these extensions are of great importance in the noncommutative algebraic geometry, and more exactly, in the classification of *Artin-Schelter regular algebras* introduced by Artin and Schelter [AS87] (see the excellent surveys on these algebras carried out by Bellamy et al. [BRS+16] and Rogalski [Rog24]). The characterization of the differential smoothness of double Ore extensions is one of our immediate objectives.

On the other hand, since bi-quadratic algebras are related to other families of noncommutative rings of polynomial type such as those mentioned in the Introduction, a second natural task is to investigate the extension of the theory developed here to these families of algebras with the aim of studying its differential smoothness.

Table 2.1: Bi-quadratic algebras on three generators of Lie type [Bav23, Theorem 1.4]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 = q_2 = q_3 = 1$	$\Bbbk[x_1, x_2, x_3]$	$\mathbb{A} = 0_{3\times 3}$, $\mathbb{B} = 0_{3\times 1}$	✓
	$U(\mathfrak{sl}_2(\Bbbk))$	$\mathbb{A} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$	⋆
	$U(\mathcal{H}_3)$	$\mathbb{A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$	⋆
	$U(\mathcal{N})/(c-1)$	$\mathbb{A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$	⋆
	$\Bbbk\langle x_1, x_2, x_3 \mid [x_1, x_2] = x_2 \rangle$	$\mathbb{A} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$	✓
	$U(\mathcal{M})/(c-1)$	$\mathbb{A} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$	✓

Table 2.2: Bi-quadratic algebras on three generators [Bav23, Theorem 1.5]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 \neq 1$ and $q_2 = q_3 = 1$	A_1	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \mu & 0 \end{bmatrix}$, $\mu \neq -1$, $\mathbb{B} = 0_{3\times 1}$	✓
	A_2	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$	✓
	B_1	$\mathbb{A} = 0_{3\times 3}$, $\mathbb{B} = 0_{3\times 1}$	✓
	B_2	$\mathbb{A} = 0_{3\times 3}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	✓
	B_3	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$	⋆
	B_4	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	⋆
	C_1	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$	✓
	C_2	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	⋆
	D_1	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	✓
	D_2	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}$, $b_1 \neq 1$	⋆

Table 2.3: Bi-quadratic algebras on three generators [Bav23, Theorem 1.6]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 \neq 1$, $q_2 \neq 1$ and $q_3 = 1$	$\mathbb{A}^3_{(q_1,q_2,1)}$	$\mathbb{A} = 0_{3\times 3}$, $\mathbb{B} = 0_{3\times 1}$, $q_1 q_2 \neq 1$	✓
	E_1	$\mathbb{A} = 0_{3\times 3}$, $\mathbb{B} = 0_{3\times 1}$, $q_2 = q_1^{-1}$	✓
	E_2	$\mathbb{A} = 0_{3\times 3}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $q_2 = q_1^{-1}$	✓
	E_3	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times 1}$, $q_2 = q_1^{-1}$	⋆
	E_4	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $q_2 = q_1^{-1}$	⋆

Table 2.4: Quantum bi-quadratic algebras on three generators [Bav23, Theorem 1.7]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 - q_3 = 0$ and $1 - q_1q_2 = 0$	F_1	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $b_1, b_2, b_3 \in \Bbbk$	$\star$
	F_2	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$, $b_1, b_2 \in \Bbbk$	$\star$
	F_3	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}$, $b_1, b_2 \in \Bbbk$	$\star$
	F_4	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}$, $b_1 \in \Bbbk$	$\star$
	F_5	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ 1 \\ 0 \end{bmatrix}$, $b_1 \in \Bbbk$	$\star$
	F_6	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} b_1 \\ 1 \\ 1 \end{bmatrix}$, $b_1 \in \Bbbk$	$\star$
	F_7	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = 0_{3\times1}$	✓
	F_8	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	✓
	F_9	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	✓
	F_{10}	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	✓

Table 2.5: Bi-quadratic algebras on three generators [Bav23, Theorem 1.8]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 - q_3 = 0$ and $1 - q_1q_2 \neq 0$	G_1	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = 0_{3\times1}$	✓
	G_2	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	✓
	G_3	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times1}$	$\star$
	G_4	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\star$

Table 2.6: Bi-quadratic algebras on three generators [Bav23, Theorem 1.9]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 - q_3 \neq 0$, $1 - q_1q_2 = 0$ and $1 - q_2q_3 \neq 0$	H_1	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = 0_{3\times1}$	✓
	H_2	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	✓
	H_3	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times1}$	$\star$
	H_4	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\star$

Table 2.7: Bi-quadratic algebras on three generators [Bav23, Theorem 1.10]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$ and $1 - q_2q_3 = 0$	I_1	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = 0_{3\times1}$	✓
	I_2	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	✓
	I_3	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = 0_{3\times1}$	⋆
	I_4	$\mathbb{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	⋆

Table 2.8: Bi-quadratic algebras on three generators [Bav23, Theorem 1.11]

Conditions	Bi-quadratic algebra	Matrix form (Proposition 2.2)	Differential smoothness
$q_1 - q_3 \neq 0$, $1 - q_1q_2 \neq 0$ and $1 - q_2q_3 \neq 0$	$\mathbb{A}^3_Q$	$\mathbb{A} = 0_{3\times3}$, $\mathbb{B} = 0_{3\times1}$	✓

CHAPTER 3

Smooth geometry of double extension regular algebras of type (14641)

At the end of Chapter 2, we said that a natural task is to investigate the differential smoothness of bi-quadratic algebras on n generators. Since Bavula [Bav23, p. 699] asserted that a construction of bi-quadratic algebras on four generators was introduced by Zhang and Zhang [ZZ08, ZZ09] with their class of *double Ore extensions*, and having in mind the interest on the differential smoothness of Ore extensions and Zhang and Zhang's assertion about the few properties known to be preserved under double extensions, it is the purpose of this chapter to investigate the differential smoothness of the double extension regular algebras of type (14641).

The chapter is organized as follows. In Section 3.1 we review the key facts on double Ore extensions and and extension regular algebras of type (14641) in order to set up notation and render this chapter self-contained. The corresponding list of these algebras is given in Tables 3.1, 3.2, 3.3 and 3.4. In Section 3.2 we prove our main result, Theorem 3.7, which says that none double extension of type (14641) is differentially smooth. Examples 3.2, 3.3 and 3.4 show interesting facts on the differential smoothness of double extensions of differentially smooth algebras; these allow us to answer Question 1. In Section 3.3 we say some words about a future research.

3.1 Double Ore extensions and extension regular algebras of type (14641)

The classification of noncommutative projective 3-spaces or quantum $\mathbb{P}^3$s corresponds to the classification of Artin-Schelter regular algebras of global dimension four. With the aim of presenting new examples of Artin-Schelter regular algebras generated by four elements of degree one (Artin-Schelter regular algebras of dimension one and two that are generated by elements of degree one are well-known [Rog24, Examples 1.8 and 1.9], while of dimension three have been classified by Artin, Schelter, Tate and Van den Bergh [AS87, ATdB07, ATVdB91, Ste96]), Zhang and Zhang [ZZ08, ZZ09] introduced algebra extensions which they called *double Ore extensions* (or *double extensions* for short) as

a natural generalization of the Ore extensions. In fact, from the definition of double extensions it is possible to appreciate some similarities to that of a two-step iterated Ore extensions. Nevertheless, there are no inclusions between the classes of all double extensions of an algebra and of all length two iterated Ore extensions of the same algebra [ZZ08, Example 4.2 and Proposition 0.5(c)]; Carvalho et al. [CLM11, Theorems 2.2 and 2.4] formulated necessary and sufficient conditions for a double extension to be presented as two-step iterated Ore extensions.

As Zhang and Zhang asserted, rather than Ore extensions very few properties are known to be preserved under double extensions since some techniques used for Ore extensions are invalid for double extensions [ZZ08, Section 4]: "double extensions seem much more difficult to study than Ore extensions [...] Many new and complicated constraints have to be posted in constructing a double extension. General ring-theoretic properties of double extensions are not known" [ZZ08, Section 0]. Some authors have contributed to the research on double extensions and its relations with algebraic structures such as Poisson, Hopf, Koszul and Calabi-Yau algebras (e.g. [GS20, Li22, LOW20, LOWY18, JLZ15, RR24, LRS17, ZvOZ17]).

As we said in the Introduction, in their paper [ZZ09] Zhang and Zhang considered regular algebras B of dimension four that are generated in degree one. By Lu et al. [LPWZ07], it is known that B is generated by either two, or three, or four elements and the projective resolution of the trivial module $\Bbbk_B$ is given in [LPWZ07, Proposition 1.4] ($\Bbbk$ is an algebraically closed field). In the case that B is generated by four elements, the projective resolution of the trivial module $\Bbbk_B$ is of the form

$$0 \to B(-4) \to B(-3)^{\oplus 4} \to B(-2)^{\oplus 6} \to B(-1)^{\oplus 4} \to B \to \Bbbk_B \to 0. \tag{3.1.1}$$

Due to the form of this resolution, Zhang and Zhang said that such an algebra is *of type* (14641). They classified all double extensions $R_P[y_1, y_2; \sigma]$ (with $\delta = 0$ and $\tau = (0, 0, 0)$) of type (14641). Having in mind that Ore extensions and normal extensions of regular algebras of dimension three were studied by Le Bruyn et al. [BSdB96], they omitted some of these from their classification, and hence their "partial" classification consists of 26 families of regular algebras of type (14641) which are labeled by $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{Z}$. $\mathcal{LIST}$ denotes the class of all algebras in the families from $\mathbb{A}$ to $\mathbb{Z}$. In this class many of the double extensions $R_P[y_1, y_2; \sigma]$ are still Ore extensions; however, there might be non-zero δ and τ such that $R_P[y_1, y_2; \sigma]$ (with the same (P, σ)) is not an Ore extension. Zhang and Zhang's classification is basically the classification of (P, σ) so that $R_P[y_1, y_2; \sigma, \delta, \tau]$ is not an Ore extension for possible (δ, τ) [ZZ09, p. 374]. All details about this terminology and notions are given below.

We recall the definition of a double extension introduced by Zhang and Zhang [ZZ08]. Since some typos ocurred in their papers [ZZ08, p. 2674] and [ZZ09, p. 379] concerning the relations that the data of a double extension must satisfy, we follow the corrections presented by Carvalho et al. [CLM11].

Definition 3.1 ([ZZ08, Definition 1.3]; [CLM11, Definition 1.1])**.** Let R be a subalgebra of a $\Bbbk$-algebra B.

(a) B is called a *right double extension* of R if the following conditions hold:

(i) B is generated by R and two new indeterminates y_1 and y_2;

(ii) y_1 and y_2 satisfy the relation

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0, \tag{3.1.2}$$

for some $p_{12}, p_{11} \in \Bbbk$ and $\tau_1, \tau_2, \tau_0 \in R$;

(iii) B is a free left R-module with basis $\left\{y_1^i y_2^j \mid i, j \geq 0\right\}$;

(iv) $y_1R + y_2R + R \subseteq Ry_1 + Ry_2 + R$.

(b) A right double extension B of R is called a *double extension* of R if

(i) $p_{12} \neq 0$;

(ii) B is a free right R-module with basis $\left\{y_2^i y_1^j \mid i, j \geq 0\right\}$;

(iii) $y_1R + y_2R + R = Ry_1 + Ry_2 + R$.

Remark 11. It is well-known that two-step iterated Ore extensions $R[y_1; \sigma_1, \delta_1][y_2; \sigma_2, \delta_2]$ are free left R-modules with a basis $\left\{y_1^{n_1}y_2^{n_2}\right\}_{n_1,n_2\geq 0}$ [ZZ08, Lemma 1.5]. It can be seen that in general these extensions are not a (right) double extension in the sense of Definition 3.1 (a) because this kind of extensions might not have a relation of the form (3.1.2). However, if σ_2 is chosen properly so that (3.1.2) holds for these extensions, then all of them become a (right) double extension [ZZ08, p. 2671]. In this way, many double extensions are iterated Ore extensions and the condition in Definition 3.1(a)(iii) is reasonable. As Zhang and Zhang [ZZ08, p. 2671] said, "Our definition of a double extension is neither most general nor ideal, but it works very well in [ZZ09]".

Condition (a)(iv) from Definition 3.1 is equivalent to the existence of two maps

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : R \to M_{2\times 2}(R) \quad \text{and} \quad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} : R \to M_{2\times 1}(R),$$

such that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} r = \sigma(r) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(r) \quad \text{for all } r \in R. \tag{3.1.3}$$

If B is a right double extension of R, we write $B = R_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = (p_{12}, p_{11})$ with elements belonging to $\Bbbk$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq R$, and σ, δ are as above. P is called a *parameter* and τ a *tail*, while the set $\{P, \sigma, \delta, \tau\}$ is said to be the *DE-data*. One of the particular cases of the double extensions is presented by Zhang and Zhang [ZZ08, Convention 1.6.(c)] as a *trimmed double extension*, for which δ is the zero map and $\tau = \{0, 0, 0\}$. We use the short notation $R_p[y_1, y_2; \sigma]$ to denote this subclass of extensions. This kind of double extensions will be important later.

For a right double extension $R_p[y_1, y_2; \sigma, \delta, \tau]$, all maps σ_{ij} and δ_i are endomorphisms of the $\Bbbk$-vector space R. From [ZZ08, Lemma 1.7] we know that σ must be a homomorphism of algebras, and δ is a σ-derivation in the sense that δ is $\Bbbk$-linear and satisfes $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$ for all $r, r' \in R$. It is straightforward to see that if the matrix $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is triangular, then both σ_{11} and σ_{22} are algebra homomorphisms.

In the case that $\tau \subseteq \Bbbk$, the subalgebra of $R_P[y_1, y_2; \sigma, \delta, \tau]$ generated by y_1 and y_2 is the double extension $\Bbbk_P[y_1, y_2; \sigma', \delta', \tau']$, where $\sigma' = \sigma|_\Bbbk$ is the canonical embedding of $\Bbbk$ in $M_{2\times 2}(\Bbbk)$ and $\delta' = \delta|_\Bbbk = 0$ is the zero map. Carvalho et al. [CLM11, Proposition 1.2] proved that the latter is always an iterated Ore extension.

The next result characterizes double extensions.

Proposition 3.1 ([ZZ08, Lemma 1.10 and Proposition 1.11]; [CLM11, Proposition 1.5])**.** *Given a $\Bbbk$-algebra R, let σ be a homomorphism from R to $M_{2\times 2}(R)$, δ a σ-derivation from R to $M_{2\times 1}(R)$, $P = (p_{12}, p_{11})$ a set of elements of $\Bbbk$, and $\tau = \{\tau_0, \tau_1, \tau_2\}$ a set of elements of R. Then, the associative $\Bbbk$-algebra B generated by R, y_1 and y_2 subject to the relations (3.1.2) and (3.1.3), is a right double extension if and only if the maps σ_{ij} and ρ_k, $i \in \{1,2\}, j, k \in \{0,1,2\}$, satisfy the six relations (3.1.4) - (3.1.9) below, where $\sigma_{i0} = \delta_i$ and ρ_k is a right multiplication by τ_k:*

$$\sigma_{21}\sigma_{11} + p_{11}\sigma_{22}\sigma_{11} = p_{11}\sigma_{11}^2 + p_{11}^2\sigma_{12}\sigma_{11} + p_{12}\sigma_{11}\sigma_{21} + p_{11}p_{12}\sigma_{12}\sigma_{21}, \tag{3.1.4}$$

$$\sigma_{21}\sigma_{12} + p_{12}\sigma_{22}\sigma_{11} = p_{11}\sigma_{11}\sigma_{12} + p_{11}p_{12}\sigma_{12}\sigma_{11} + p_{12}\sigma_{11}\sigma_{22} + p_{12}^2\sigma_{12}\sigma_{21}, \tag{3.1.5}$$

$$\sigma_{22}\sigma_{12} = p_{11}\sigma_{12}^2 + p_{12}\sigma_{12}\sigma_{22}, \tag{3.1.6}$$

$$\begin{aligned}\sigma_{20}\sigma_{11} + \sigma_{21}\sigma_{10} + \underline{\rho_1\sigma_{22}\sigma_{11}} &= p_{11}(\sigma_{10}\sigma_{11} + \sigma_{11}\sigma_{10} + \rho_1\sigma_{12}\sigma_{11}) \\ &\quad + p_{12}(\sigma_{10}\sigma_{21} + \sigma_{11}\sigma_{20} + \rho_1\sigma_{12}\sigma_{21}) + \tau_1\sigma_{11} + \tau_2\sigma_{21},\end{aligned} \tag{3.1.7}$$

$$\begin{aligned}\sigma_{20}\sigma_{12} + \sigma_{22}\sigma_{10} + \underline{\rho_2\sigma_{22}\sigma_{11}} &= p_{11}(\sigma_{10}\sigma_{12} + \sigma_{12}\sigma_{10} + \rho_2\sigma_{12}\sigma_{11}) \\ &\quad + p_{12}(\sigma_{10}\sigma_{22} + \sigma_{12}\sigma_{20} + \rho_2\sigma_{12}\sigma_{21}) + \tau_1\sigma_{12} + \tau_2\sigma_{22},\end{aligned} \tag{3.1.8}$$

$$\begin{aligned}\sigma_{20}\sigma_{10} + \underline{\rho_0\sigma_{22}\sigma_{11}} &= p_{11}(\sigma_{10}^2 + \rho_0\sigma_{12}\sigma_{11}) + p_{12}(\sigma_{10}\sigma_{20} + \rho_0\sigma_{12}\sigma_{21}) \\ &\quad + \tau_1\sigma_{10} + \tau_2\sigma_{10} + \tau_0 \mathrm{id}_R.\end{aligned} \tag{3.1.9}$$

Remark 12. (i) [CLM11, Remark 1.6] Proposition 3.1 implies the uniqueness, up to isomorphism, of a right double extension of R, with given σ, δ, P and τ, provided such an extension exists. Indeed, assume $\overline{B} = R_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of R. Then, by [ZZ08, Lemmas 1.7 and 1.10(b)], the data σ, δ, P and τ satisfy the conditions of Proposition 3.1. Let B be as in this proposition. Then, there is an algebra homomorphism from B to $\overline{B}$ which restricts to the identity on R and maps $y_i \in B$ to the corresponding element $y_i \in \overline{B}$, $i = 1, 2$. Since B is a free left R-module with basis $\left\{ y_1^i y_2^j \mid i, j \geq 0 \right\}$ and the same holds for $\overline{B}$, this map is an isomorphism, thus proving uniqueness.

(ii) [CLM11, p. 2842] Let $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension and suppose that $p_{12} \neq 1$. Then, from the ideas above, by choosing adequate generators $\overline{y}_i$ and (possibly) modifying the data σ, δ, τ one can assume that $p_{11} = 0$. If $\overline{B} = R_P[\overline{y}_1, \overline{y}_2; \overline{\sigma}, \overline{\delta}, \overline{\tau}]$ is a right double extension with $p_{11} = 0$, then $\overline{B}$ has a natural filtration, given by setting $\deg R = 0$ and $\deg \overline{y}_1 = \deg \overline{y}_2 = 1$. It can be seen that in view of relations (3.1.2) and (3.1.3), that the associated graded algebra $G(B)$ is isomorphic to $R_P[\overline{y}_1, \overline{y}_2; \overline{\sigma}, \{0, 0, 0\}]$.

Next, we recall the key results formulated by Carvalho et al. [CLM11] about relations between double extensions and two-step iterated Ore extensions (c.f. Zhang and Zhang [ZZ09, Proposition 3.6]).

Proposition 3.2. (1) [CLM11, Theorem 2.2] *Let* R, B *be* $\Bbbk$*-algebras such that* B *is an extension of* R. *Assume* $P = \{p_{12}, p_{11}\} \subseteq \Bbbk$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq R$, σ *is an algebra homomorphism from* R *to* $M_2(R)$ *and* δ *is a* σ*-derivation from* R *to* $M_{2\times 1}(R)$.

(a) *The following conditions are equivalent:*
- $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ *is a right double extension of* R *which can be presented as an iterated Ore extension* $R[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$.
- $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ *is right double extension of* R *with* $\sigma_{12} = 0$.
- $R[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ *is and iterated Ore extensions such that*

$$\sigma_2(R) \subseteq (R), \quad \sigma_2(y_1) = p_{12}y_1 + \tau_2,$$
$$d_2(R) \subseteq Ry_1 + R, \quad d_2(y_1) = p_{11}y_1^2 + \tau_1 y_1 + \tau_0,$$

for some $p_{ij} \in \Bbbk$ *and* $\tau_i \in R$. *The maps* σ, δ, σ_i *and* δ_i, $i = 1, 2$, *are related by*

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_{21} & \sigma_2|_R \end{bmatrix}, \quad \delta(a) = \begin{bmatrix} d_1(a) \\ d_2(r) - \sigma_{21}(a)y_1 \end{bmatrix}, \quad \text{for all } a \in R.$$

(b) *If one of the equivalent statements from (1) holds, then* B *is a double extension of* R *if and only if* $\sigma_1 = \sigma_{11}$ *and* $\sigma_2|_R = \sigma_{22}$ *are automorphisms of* R *and* $p_{12} \neq 0$.

(2) [CLM11, Theorem 2.4] *Let* $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ *be a right double extension of the* $\Bbbk$*-algebra* R, *where* $P = \{p_{12}, p_{11}\} \in \Bbbk$, $\tau = \{\tau_2, \tau_1, \tau_0\} \subseteq R$, $\sigma : R \to M_2(R)$ *is an algebra homomorphism and* $\delta : R \to M_{2\times 1}(R)$ *is a* σ*-derivation. Then,* B *can be presented as an iterated Ore extension* $R[y_2; \sigma_2', d_2'][y_1; \sigma_1', d_1']$ *if and only if* $\sigma_{21} = 0$, $p_{12} \neq 0$ *and* $p_{11} = 0$. *In this case,* B *is a double extension if and only if* $\sigma_2' = \sigma_{22}$ *and* $\sigma_1'|_R = \sigma_{11}$ *are automorphisms of* R.

Example 3.1 ([ZZ08, Example 4.1])**.** Let $R = \Bbbk[x]$. As expected, there are so many different right double extensions of R and if we assume that $\deg x = \deg y_1 = \deg y_2 = 1$, then all connected graded double extensions $\Bbbk[x]_P[y_1, y_2; \sigma, \delta, \tau]$ are regular algebras of global dimension three investigated by Artin and Schelter [AS87]. Let us see some particular situations to illustrate that the DE-data can be of different kinds.

From relations (3.1.2) and (3.1.3) we have that

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + axy_1 + bxy_2 + cx^2,$$
$$y_1x = dxy_1 + exy_2 + fx^2, \quad \text{and}$$
$$y_2x = gxy_1 + hxy_2 + ix^2.$$

The six relations appearing in Proposition 3.1 are equivalent to the fact that the overlap between the above three relations can be resolved. By using that $p_{12} \neq 0$, we may choose P to be $(p, 0)$ or $(1, 1)$, while the invertibility of σ is equivalent to the condition $dh - ge \neq 0$.

Next, we present the list of defining relations of different possibilities for the algebra B.

(i) **Algebra** $B^1 := B^1(p, a, b, c)$, where $a, b, c, p \in \Bbbk$ and $p \neq 0,\ b \neq 0, 1$.

$$y_2y_1 = py_1y_2 + \frac{bc}{1-b}(pb-1)xy_1 + ax^2, \tag{3.1.10}$$
$$y_1x = bxy_1, \quad \text{and} \tag{3.1.11}$$
$$y_2x = b^{-1}xy_2 + cx^2. \tag{3.1.12}$$

The homomorphism σ is determined by $\sigma(x) = \begin{bmatrix} bx & 0 \\ 0 & b^{-1}x \end{bmatrix}$. In this case $\sigma_{12} = \sigma_{21} = 0$ and σ_{11} and σ_{22} are algebra automorphisms. The derivation is determined by $\delta(x) = \begin{bmatrix} 0 & cx^2 \end{bmatrix}$. The parameter P is $(p, 0)$ and the tail is τ is $\left\{\frac{bc}{1-b}(pb-1)x, 0, ax^2\right\}$. By using the linear transformation $y_2 \to y_2 + \frac{bc}{b-1}x$ we get that $c = 0$. Thus, $B^1(1, p, a, b, c) \cong B^1(1, p, a, b, 0)$.

(ii) **Algebra** $B^2 := B^2(a, b, c)$, where $a, b, c \in \Bbbk$ and $b \neq 0$.

$$\begin{aligned} y_2y_1 &= -y_1y_2 + ax^2, \\ y_1x &= b^{-1}xy_2 + cx^2, \quad \text{and} \\ y_2x &= bxy_1 + (-bc)x^2. \end{aligned}$$

The homomorphism σ is determined by $\sigma(x) = \begin{bmatrix} 0 & b^{-1}x \\ bx & 0 \end{bmatrix}$. It can be seen that $\sigma(x^n) = \begin{bmatrix} 0 & b^{-1}x^n \\ bx^n & 0 \end{bmatrix}$ when n is odd and $\sigma(x^n) = \begin{bmatrix} x^n & 0 \\ 0 & x^n \end{bmatrix}$ when n is even. Note that in this case σ_{11} and σ_{22} are not algebra homomorphisms of A. The derivation is determined by $\delta(x) = \begin{bmatrix} cx^2 & -bcx^2 \end{bmatrix}$. The parameter P is $(-1, 0)$ and the tail is τ is $\left\{0, 0, ax^2\right\}$. Hence, $B^2(a, b, c) \cong B^2(ab^{-1}, 1, c)$.

Note that for most (a, b, c) (for example, $(a, b, c) = (1, 1, 0)$), the algebra $B^2(a, b, c)$ is not an Ore extension of any regular algebra of global dimension two, so $B^2(a, b, c)$ is not an iterated Ore extension of $\Bbbk[x]$ with $\operatorname{char}(\Bbbk) = 0$. Carvalho et al. [CLM11, Proposition 2.7] showed that this is not so in case the characteristic of the base field is two.

(iii) **Algebra** $B^3 := B^3(a)$, where $a \in \Bbbk^*$.

$$\begin{aligned} y_2y_1 &= y_1y_2, \\ y_1x &= axy_1 + xy_2, \quad \text{and} \\ y_2x &= axy_2. \end{aligned}$$

The homomorphism σ is determined by $\sigma(x) = \begin{bmatrix} ax & x \\ 0 & ax \end{bmatrix}$ and the derivation is zero. The parameter P is $(1, 0)$ and the tail τ is $\{0, 0, 0\}$.

(iii) **Algebra** $B^4 := B^4(a, b, c)$, where $a, b, c \in \Bbbk$ and $b \neq -1$.

$$\begin{aligned} y_2y_1 &= y_1y_2 + y_1^2 + axy_1 + \frac{b}{1+b}xy_2 + cx^2, \\ y_1x &= xy_1 + bx^2, \quad \text{and} \\ y_2x &= (2 + 2b^{-1})xy_1 + xy_2. \end{aligned}$$

The homomorphism σ is determined by $\sigma(x) = \begin{bmatrix} x & 0 \\ (2+2b^{-1})x & x \end{bmatrix}$. The derivation is determined by $\delta(x) = \begin{bmatrix} bx^2 \\ 0 \end{bmatrix}$. The parameter P is $(1,1)$ and the tail τ is $\{ax, \frac{b}{1+b}x, cx^2\}$.

If we denote by B any of the algebras B^i, then it can be seen that the element x is normalizing in B and $B/\langle x\rangle$ is a regular algebra of global dimension two. For this reason, these algebras are called *normal extensions*. Except for B^2, these algebras are also iterated Ore extensions of $\Bbbk[x]$. A detailed classification and general properties of these algebras can be found in [AS87, ATdB07, ATVdB91].

As we said in the Introduction, Zhang and Zhang [ZZ09] were interested only in regular algebras B of dimension four that are generated in degree one, and in the case that B is generated by four elements, the projective resolution of the trivial module $\Bbbk_B$ is of the form (3.1.1), that is, algebras *of type* (14641). The next proposition shows explicitly the relation of these algebras with Ore extensions.

Proposition 3.3 ([ZZ09, Theorem 0.1])**.** *Let B be a connected graded algebra generated by four elements of degree one. If B is a double extension $R_P[y_1, y_2; \sigma, \tau]$ where R is an Artin-Schelter regular algebra of dimension two, then the following assertions hold:*

(1) *B is a strongly Noetherian, Auslander regular and Cohen-Macaulay domain.*

(2) *B is of type (14641). As a consequence, B is Koszul.*

(3) *If B is not isomorphic to an Ore extension of an Artin-Schelter regular algebra of dimension three, then the trimmed double extension $R_P[y_1, y_2; \sigma]$ is isomorphic to one of 26 families.*

In Tables 3.1, 3.2, 3.3 and 3.4 we present the detailed list of the 26 families following the labels $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{Z}$ used in [ZZ09].

3.1.1 Double extensions of $\Bbbk_Q[x_1, x_2]$

Since the base field $\Bbbk$ is algebraically closed, R is isomorphic to $\Bbbk_Q[x_1, x_2] = \mathcal{O}_q(\Bbbk)$ with the relation $x_2x_1 = qx_1x_2$ (note that $Q = (q, 0)$, the *Manin's plane*), or $\Bbbk_J[x_1, x_2] = \mathcal{J}(\Bbbk)$ with the relation $x_2x_1 = x_1x_2 + x_1^2$ (here, $Q = J = (1,1)$, the *Jordan's plane*) [Shi05, Theorem 1.4] or [ZZ09, Lemma 2.4]. Manin's plane and Jordan's plane are the only regular algebras of global dimension two [Rog24, Examples 1.8 and 1.9].

In general, we will write $\Bbbk_Q[x_1, x_2]$, with $Q = (q_{12}, q_{11})$, and

$$\Bbbk_Q[x_1, x_2] = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - q_{11}x_1^2 - q_{12}x_1x_2\rangle,$$

and for the computation we set Q to be either $(1,1)$ or $(q, 0)$. Zhang and Zhang [ZZ09, Section 3] classified regular domains of dimension four of the form $R_P[y_1, y_2; \sigma]$ up

to isomorphism (equivalently, to classify (P, σ)) up to some equivalence relation. They were interested in double extensions that are not iterated Ore extensions. Throughout this section we consider R as just said.

Let $\sigma : R \to M_2(R)$ be a graded algebra homomorphism. Let

$$\sigma_{ij}(x_s) = \sum_{t=1}^{2} a_{ijst}x_t, \quad \text{for all } i, j, s = 1, 2 \text{ and } a_{ijst} \in \Bbbk. \tag{3.1.13}$$

Using (3.1.13), the relation (3.1.3) of the algebra $R_P[y_1, y_2; \sigma]$ can be expressed as follows (note that in this case $\delta = 0$). If $r = x_1$ and x_2 in (3.1.3), we get the following relations:

$$\begin{aligned}
y_1x_1 &= \sigma_{11}(x_1)y_1 + \sigma_{12}(x_1)y_2 \\
&= a_{1111}x_1y_1 + a_{1112}x_2y_1 + a_{1211}x_1y_2 + a_{1212}x_2y_2, && (3.1.14)\\
y_1x_2 &= \sigma_{11}(x_2)y_1 + \sigma_{12}(x_2)y_2 \\
&= a_{1121}x_1y_1 + a_{1122}x_2y_1 + a_{1221}x_1y_2 + a_{1222}x_2y_2, && (3.1.15)\\
y_2x_1 &= \sigma_{21}(x_1)y_1 + \sigma_{22}(x_1)y_2 \\
&= a_{2111}x_1y_1 + a_{2112}x_2y_1 + a_{2211}x_1y_2 + a_{2212}x_2y_2, \quad \text{and} && (3.1.16)\\
y_2x_2 &= \sigma_{21}(x_2)y_1 + \sigma_{22}(x_2)y_2 \\
&= a_{2121}x_1y_1 + a_{2122}x_2y_1 + a_{2221}x_1y_2 + a_{2222}x_2y_2. && (3.1.17)
\end{aligned}$$

These four relations are called *mixing relations* between x_i and y_i. The double extension $R_P[y_1, y_2; \sigma]$ also has two non-mixing relations given by

$$x_2x_1 = q_{12}x_1x_2 + q_{11}x_1^2, \quad \text{and} \tag{3.1.18}$$
$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2. \tag{3.1.19}$$

If we consider the matrices

$$\Sigma_{ij} := \begin{bmatrix} a_{ij11} & a_{ij12} \\ a_{ij21} & a_{ij22} \end{bmatrix} \quad \text{and} \quad \Sigma := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} a_{1111} & a_{1112} & a_{1211} & a_{1212} \\ a_{1121} & a_{1122} & a_{1221} & a_{1222} \\ a_{2111} & a_{2112} & a_{2211} & a_{2212} \\ a_{2121} & a_{2122} & a_{2221} & a_{2222} \end{bmatrix}, \tag{3.1.20}$$

since it is assumed that σ is a graded algebra homomorphism, then σ is uniquely determined by Σ. Another matrix related to Σ is given in the following way. Let

$$M_{ij} := \begin{bmatrix} a_{11ij} & a_{12ij} \\ a_{21ij} & a_{22ij} \end{bmatrix} \quad \text{and} \quad M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{3.1.21}$$

As we can see, the matrix M is obtained by re-arranging the entries of Σ and Σ is invertible if and only if M is invertible.

Remark 13 ([ZZ09, p. 388-390]). Using that σ is an algebra homomorphism, for elements i, j, f, g we have that

$$\begin{aligned}
\sigma_{ij}(x_f x_g) &= \sum_{p=1}^{2} \sigma_{ip}(x_f)\sigma_{pj}(x_g) \\
&= \sum_{p,s,t=1}^{2} (a_{ipfs}x_s)(a_{pjgt}x_t) \\
&= \sum_{p,s,t=1}^{2} (a_{ipfs}a_{pjgt})x_s x_t \\
&= \left(\sum_{p=1}^{2} a_{ipf1}a_{pjg1}\right) x_1^2 + \left(\sum_{p=1}^{2} a_{ipf1}a_{pjg2}\right) x_1x_2 \\
&\quad + \left(\sum_{p=1}^{2} a_{ipf2}a_{pjg1}\right) x_2x_2 + \left(\sum_{p=1}^{2} a_{ipf2}a_{pjg2}\right) x_2^2.
\end{aligned}$$

Since $x_2x_1 = q_{11}x_1^2 + q_{12}x_1x_2$ in R, we obtain that

$$\begin{aligned}
\sigma_{ij}(x_f x_g) = {} & [(a_{i1f1}a_{1jg1} + a_{i2f_1}a_{2jg1}) + q_{11}(a_{i1f2}a_{1jg1} + a_{i2f2}a_{2jg1})]\, x_1^2 \\
& + [(a_{i1f1}a_{1jg2} + a_{i2f1}a_{2jg2}) + q_{12}(a_{i1f2}a_{1jg1} + a_{i2f2}a_{2jg1})]\, x_1x_1 \\
& + (a_{i1f2}a_{1jg2} + a_{i2f2}a_{2jg2})\, x_2^2. \qquad (3.1.22)
\end{aligned}$$

Using the same relation and that each σ_{ij} is a $\Bbbk$-linear map, it follows that

$$\sigma_{ij}(x_2x_1) = q_{11}\sigma_{ij}(x_1x_1) + q_{12}\sigma_{ij}(x_1x_2) \quad \text{for all } i, j = 1, 2. \qquad (3.1.23)$$

By (3.1.22) the left-hand and the right-hand sides of (3.1.23) can be expressed as polynomials of x_1 and x_2. If we compare the coefficients of x_1^2, x_1x_2 and x_2^2, respectively, then we obtain the following identities: the coefficients of x_1^2 of (3.1.23) give rise to a constraint between coefficients

$$\begin{aligned}
& (a_{i121}a_{1j11} + a_{i221}a_{2j11}) + q_{11}(a_{i122}a_{1j11} + a_{i222}a_{2j11}) \\
& \quad = q_{11}[(a_{i111}a_{1j11} + a_{i211}a_{2j11}) + q_{11}(a_{i112}a_{1j11} + a_{i212}a_{2j11})] \\
& \qquad + q_{12}[(a_{i111}a_{1j21} + a_{i211}a_{2j21}) + q_{11}(a_{i112}a_{1j21} + a_{i212}a_{2j21})]. \qquad (3.1.24)
\end{aligned}$$

With respect to the coefficients of x_1x_2 of (3.1.23), we get that

$$\begin{aligned}
& (a_{i121}a_{1j12} + a_{i221}a_{2j12}) + q_{12}(a_{i122}a_{1j11} + a_{i222}a_{2j11}) \\
& \quad = q_{11}[(a_{i111}a_{1j12} + a_{i211}a_{2j12}) + q_{12}(a_{i112}a_{1j11} + a_{i212}a_{2j11})] \\
& \qquad + q_{12}[(a_{i111}a_{1j22} + a_{i211}a_{2j22}) + q_{12}(a_{i112}a_{1j21} + a_{i212}a_{2j21})]. \qquad (3.1.25)
\end{aligned}$$

The coefficients of x_2^2 of (3.1.23) satisfy the equality

$$\begin{aligned}
& (a_{i122}a_{1j12} + a_{i222}a_{2j12}) \\
& \quad = q_{11}(a_{i112}a_{1j12} + a_{i212}a_{2j12}) + q_{12}(a_{i112}a_{1j22} + a_{i212}a_{2j22}). \qquad (3.1.26)
\end{aligned}$$

Applying relations (3.1.2), (3.1.3) and all of them appearing in Proposition 3.1 to the elements $r = x_1$ and x_2, we obtain more relations between the elements a_{ijkl}. For $i, f, g, s, t = 1, 2$,

$$\begin{aligned}\sigma_{fg}(\sigma_{st}(x_i)) = \sigma_{fg}\left(\sum_{w=1}^{2} a_{stiw}x_w\right) &= \sum_{w=1}^{2} a_{stiw}\sigma_{fg}(x_w)\\ \sum_{w=1}^{2} a_{stiw}\sum_{j=1}^{2} a_{fgwj}x_j &= \sum_{j=1}^{2}\left(a_{sti1}a_{fg1j} + a_{sti2}a_{fg2j}\right)x_j\\ = \left(a_{sti1}a_{fg11} + a_{sti2}a_{fg21}\right)x_1 &+ \left(a_{sti1}a_{fg12} + a_{sti2}a_{fg22}\right)x_2. \qquad (3.1.27)\end{aligned}$$

(Since $P = (p_{12}, p_{11})$, Zhang and Zhang will consider $P = (1, 1)$ or $(p, 0)$ in some computations). By (3.1.27), the expressions (3.1.2), (3.1.3) and all of them appearing in Proposition 3.1, when applied to x_i are equivalent to the following constraints:

$$\begin{aligned}&\left(a_{11i1}a_{211j} + a_{11i2}a_{212j}\right) + p_{11}\left(a_{11i2}a_{222j}\right)\\ &\quad = p_{11}\left(a_{11i1}a_{111j} + a_{11i2}a_{112j}\right) + p_{11}^2\left(a_{11i1}a_{121j} + a_{11i2}a_{122j}\right)\\ &\qquad + p_{12}\left(a_{21i1}a_{111j} + a_{21i2}a_{112j}\right) + p_{11}p_{12}\left(a_{21i1}a_{121j} + a_{21i2}a_{122j}\right), \qquad (3.1.28)\\ &\left(a_{12i1}a_{211j} + a_{12i2}a_{212j}\right) + p_{12}\left(a_{11i1}a_{221j} + a_{11i2}a_{222j}\right)\\ &\quad = p_{11}\left(a_{22i1}a_{111j} + a_{12i2}a_{112j}\right) + p_{11}p_{12}\left(a_{11i1}a_{121j} + a_{11i2}a_{122j}\right)\\ &\qquad + p_{12}\left(a_{22i1}a_{111j} + a_{22i2}a_{112j}\right) + p_{12}^2\left(a_{21i1}a_{121j} + a_{21i2}a_{122j}\right), \qquad (3.1.29)\\ &\left(a_{12i1}a_{221}j + a_{12i2}a_{222j}\right)\\ &\quad p_{11}\left(a_{12i1}a_{121j} + a_{12i2}a_{122j}\right) + p_{12}\left(a_{22i1}a_{121j} + a_{22i2}a_{122j}\right). \qquad (3.1.30)\end{aligned}$$

As we can see, there is a symmetry between the first three C-constraints ((3.1.24), (3.1.25), (3.1.26)) and the last three C-constraints ((3.1.28), (3.1.29), (3.1.30)). The reasoning above is the content of the following proposition.

Proposition 3.4 ([ZZ09, Proposition 3.1]). (1) *Let* $R_P[y_1, y_2; \sigma]$ *be a right double extension and suppose the data* $\{\Sigma, P, Q\}$ *are defined as above corresponding to the* $\Bbbk$*-linear basis* $\{x_1, x_2, y_1, y_2\}$. *Then the six equations* (3.1.24), (3.1.25), (3.1.26), (3.1.28), (3.1.29) *and* (3.1.30) *are satisfied. Further,* $\det \Sigma \neq 0$ *if and only if* $R_P[y_1, y_2; \sigma]$ *is a double extension.*

(2) *Suppose a matrix* Σ *as in and parameter sets* $P = (p_{12}, p_{11})$ *and* $Q = (q_{12}, q_{11})$ *with* $p_{12}q_{12} \neq 0$ *are given. If the six equations hold and* $\det \Sigma \neq 0$, *then the six relations define a double extension* $R_P[y_1, y_2; \sigma]$.

The *System C* is the system of the six equations (3.1.24), (3.1.25), (3.1.26), (3.1.28), (3.1.29) and (3.1.30) together with $\det \Sigma \neq 0$. A *solution to System C* or a *C-solution* is a matrix Σ with entries a_{ijst} satisfying System C.

Proposition 3.5. *Let* Σ *be a C-solution and let* $B = (\Bbbk_Q[x_1, x_2])_p[y_1, y_2; \sigma]$ *where* σ *is determined by* Σ. *Let* $0 \neq h \in \Bbbk$.

(1) B *is a* $\mathbb{Z}^2$*-graded algebra. Let* $\gamma : x_i \mapsto x_i,\ y_i \mapsto hy_i$. *Then* γ *extends to a graded automorphism of* B.

(2) $h\Sigma$ *is a* C*-solution. Let* σ' *be the algebra automorphism determined by* $h\Sigma$. *Then* $B' := (\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma']$ *is a graded twist of* B *by* γ *in the sense of Zhang [Zha96].*

In general, the algebra B and its twist B^γ are not isomorphic as algebras. Nevertheless, both algebras have common properties since the category of graded B-mpdules is equivalent to the category of graded B^γ-modules [Zha96, Theorem 1.1].

Definition 3.2 ([ZZ09, Definition 3.4])**.** (i) Σ and Σ' are *twist equivalent* if we have that $\Sigma' = h\Sigma$ for some $0 \neq h \in \Bbbk$. In this case, Σ' is called a *twist* of Σ. Σ can be replace by its twists (without changing Q and P) to obtain another double extension [ZZ09, Lemma 3.3(b)].

(ii) (Σ, Q, P) and (Σ', Q', P') are *linearly equivalent* if there is a graded algebra isomorphism from $(\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ to $(\Bbbk_{Q'}[x_1', x_2'])'_P[y_1', y_2'; \sigma']$ mapping $\Bbbk x_1 + \Bbbk x_2 \mapsto \Bbbk x_1' + \Bbbk x_2'$ and $\Bbbk y_1 + \Bbbk y_2 \mapsto \Bbbk y_1' + \Bbbk y_2'$. Using this isomorphism we can pull back x_i' and y_i' to the algebra $(\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma]$; then we can assume that $\{x_1', x_2'\}$ (resp. $\{y_1', y_2'\}$) is another basis of $\{x_1, x_2\}$ (resp. $\{y_1, y_2\}$). In case $Q = Q'$ and $P = P'$, then we just say that Σ and Σ' are *linearly equivalent*.

(iii) (Σ, Q, P) and (Σ', Q', P') are *equivalent* if (Σ, Q, P) and $(h\Sigma', Q', P')$ are linearly equivalent for some $0 \neq h \in \Bbbk$.

It is easy to show that twist equivalence and linear equivalence are equivalence relations. Also, the equivalence between (Σ, Q, P) and (Σ', Q', P') is an equivalence relation.

Zhang and Zhang [ZZ09] classified $R_P[y_1, y_2; \sigma]$ up to isomorphism (or even up to twist) by classifying Σ up to (linear) equivalence.

Proposition 3.6. (1) [ZZ09, Proposition 3.6] *Let* B *be a double extension given by* $(\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma, \tau, \delta]$.

(a) *If* $\Sigma_{12} = 0$, *then* B *is an iterated Ore extension.*

(b) *If* $\Sigma_{21} = 0$ *and* $p_{11} = 0$, *then* B *is an iterated Ore extension.*

(2) [ZZ09, Proposition 3.7] *Let* $B = (\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ *be a trimmed double extension where* σ *is determined by the matrix* Σ.

(a) *Considering* $R' : \Bbbk_P[y_1, y_2]$ *as the subring and* $\{x_1, x_2\}$ *as the set of generators over* R', B *is a double extension* $(\Bbbk_P[y_1, y_2])_Q[x_1, x_2; \alpha]$ *where* α *is determined by the matrix* M^{-1}.

(b) *If* $M_{12} = 0$, *then* B *is an iterated Ore extension of* $\Bbbk_P[y_1, y_2]$.

(c) *If* $M_{21} = 0$ *and* $q_{11} = 0$, *then* B *is an iterated Ore extension of* $\Bbbk_P[y_1, y_2]$.

Remark 14. In the classification of double extensions, there are some of them that satisfy some condition of Proposition 3.6. In fact, Zhang and Zhang comment that the algebra

$\mathbb{A}$ satisfies $M_{12} = 0$, and therefore it is an iterated Ore extension. However, under the choice of τ and δ, the extension is not an iterated Ore extension for any regular algebra of dimension 2 [ZZ09, p. 395].

By Proposition 3.6, we can see at the Tables 3.1, 3.2, 3.3 and 3.4 that:

- All algebras $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{Z}$ satisfy condition (2) (a).
- Algebra $\mathbb{N}$ satisfies condition (1)(a) when $f = 0$.
- Condition (2)(b) is satisfied by algebras $\mathbb{D}$, $\mathbb{G}$, $\mathbb{H}$, $\mathbb{K}$, $\mathbb{L}$, $\mathbb{Q}$, $\mathbb{X}$ and $\mathbb{Y}$.
- Algebra $\mathbb{M}$ satisfies condition (2)(c) when $f = 0$.
- Algebra $\mathbb{V}$ also satisfies (2)(c).
- The other algebras do not satisfy any of the assumptions (except (2)(a)) of Proposition 3.6.

3.2 Differential smoothness

This section contains the important result of the chapter.

Following an argument similar to the one presented by Brzeziński and Sitarz [BS17, Example 2.5], we obtain the following.

Theorem 3.7. *Double extension regular algebras of type (14641) are not differentially smooth.*

Proof. Consider a double extension B generated by the set of indeterminates $B := \{z_+, z_-, z'_+, z'_-\}$. The proof is by contradiction. We divide it into two parts.

(1) Suppose that for the elements z'_+ and z_+ the commutation rule

$$z'_+ z_+ = \alpha z_+ z'_+ + \beta z_- z'_+ + \gamma z_+ z'_- + \delta z_- z'_-, \quad \text{where } \alpha, \beta, \gamma, \delta \in \Bbbk^*, \tag{3.2.1}$$

is satisfied. Suppose that B has a first order differential calculus ΩA with $d : A \to \Omega^1 A$ a derivation. By applying d to (3.2.1) we get that

$$0 = d(z'_+ z_+) - d(\alpha z_+ z'_+ + \beta z_- z'_+ + \gamma z_+ z'_- + \delta z_- z'_-).$$

Since d is $\Bbbk$-linear, using the Leibniz's rule it follows that

$$\begin{aligned} 0 = {} & dz'_+ z_+ + z'_+ dz_+ - \alpha dz_+ z'_+ - \alpha z_+ dz'_+ - \beta dz_- z'_+ \\ & - \beta z_- dz'_+ - \gamma dz_+ z'_- - \gamma z_+ dz'_- - \delta dz_- z'_- - \delta z_- dz'_-. \end{aligned}$$

Using Property 1.4.2, the action of the module is written using automorphisms ν_{z_+}, $\nu_{z'_+}$ and $\nu_{z'_-}$

$$\begin{aligned}0 = {} & dz'_+z_+ + dz_+\nu_{z_+}(z'_+) - \alpha dz_+z'_+ - \alpha dz'_+\nu_{z'_+}(z_+) - \beta dz_-z'_+ \\ & - \beta dz'_+\nu_{z'_+}(z_-) - \gamma dz_+z'_- - \gamma dz'_-\nu_{z'_-}(z_+) - \delta dz_-z'_- - \delta dz'_-\nu_{z'_-}(z_-)\end{aligned}$$

The terms that multiply the different differentials are put together to obtain

$$\begin{aligned}0 \ = {} & dz'_+(z_+ - \alpha\nu_{z'_+}(z_+) - \beta\nu_{z'_+}(z_-)) + dz_+(\nu_{z_+}(z'_+) - \alpha z'_+ - \gamma z'_-) \\ & - dz_-(\beta z'_+ + \delta z'_-) - dz'_-(\gamma\nu_{z'_-}(z_+) + \delta\nu_{z'_-}(z_-)).\end{aligned}$$

Since the differentials are generating elements for $\Omega^1 A$ and all of them are equal to zero, the elements that multiply them are also zero. We obtain that

$$\begin{aligned}\nu_{z'_+}(\alpha z_+ + \beta z_-) &= z_+, \\ \nu_{z_+}(z'_+) &= \alpha z'_+ + \gamma z'_-, \\ \beta z'_+ + \delta z'_- &= 0, \quad \text{and} \\ \nu_{z'_-}(\gamma z_+ + \delta z_-) &= 0,\end{aligned}$$

which are satisfied only when $\beta = \gamma = \delta = 0$. This contradicts our initial assumption.

(2) Suppose that in the algebra B is satisfied the quadratic relation

$$\sum_{t,w\in B} c_{t,w}tw = 0, \quad \text{where } c_{t,w} \in \Bbbk, \tag{3.2.2}$$

Without loss of generality, notice that if the indeterminate z_+ appears only once in the list of pairs (t, w), call $(\bar{t}, \bar{w})$, then there is no first order calculus. More exactly, consider the set $B' = B^2 \setminus \{(\bar{t}, \bar{w})\}$. By applying the differential d to the expression (3.2.2) we get that

$$0 = d\left(\sum_{t,w\in B} c_{t,k}tk\right).$$

Since d is $\Bbbk$-linear, using the Leibniz's rule and emphasizing leaving aside the terms $\bar{t}\bar{w}$, it follows that

$$\begin{aligned}0 &= \sum_{(t,w)\in B'} c_{t,w}d(tw) + c_{\bar{t},\bar{w}}d(\bar{t}\bar{w}) \\ &= \sum_{(t,w)\in B'} c_{t,w}(dtw + tdw) + c_{\bar{t},\bar{w}}(d\bar{t}\bar{w} + \bar{t}d\bar{w})\end{aligned}$$

Again, using Property 1.4.2, the action of the module is written using automorphisms ν_w, and $\nu_{\bar{w}}$

$$0 = \left(\sum_{(t,w)\in B'} c_{t,w}(dtw + dw\nu_w(t))\right) + d\bar{t}c_{\bar{t},\bar{w}}\bar{w} + d\bar{w}c_{\bar{t},\bar{w}}\nu_{\bar{w}}(\bar{t}).$$

The generator z_+ does not appear as any component of the pairs $(t, w) \in B'$. In the case that $\bar{t} = z_+$, we obtain that $c_{\bar{t},\bar{w}}\bar{w} = 0$, whence $\bar{w} = 0$, which cannot be happen. Now, if $\bar{w} = z_+$, then $c_{\bar{t},\bar{w}}\nu_{\bar{w}}(\bar{t}) = 0$, and therefore $\nu_{\bar{w}}(\bar{t}) = 0$, which contradicts that ν_{z_+} is an automorphism.

□

From Remark 14 and Theorem 3.7 we obtain the following facts:

- Double extensions satisfying relation (3.2.1) are $\mathbb{C}$, $\mathbb{F}$, $\mathbb{I}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$ (see Tables 3.1, 3.2, 3.3 and 3.4).
- Double extensions that satisfy the commutation rule (3.2.2) are $\mathbb{A}$, $\mathbb{B}$, $\mathbb{D}$, $\mathbb{E}$, $\mathbb{G}$, $\mathbb{H}$, $\mathbb{J}$, $\mathbb{K}$, $\mathbb{L}$, $\mathbb{M}$, $\mathbb{N}$, $\mathbb{O}$, $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{V}$, $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ (see Tables 3.1, 3.2, 3.3 and 3.4).

Let us see three illustrative examples on the differential smoothness of double extensions of differentially smooth algebras. These allow us to give an answer to the following question:

Question 1. Let B be a double extension of R. If R is differentially smooth, then is B differentially smooth?

Example 3.2 ([ZZ09, Subcase 4.1.1])**.** Let $B = (\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma, \delta, \tau]$ be the right double extension generated by x_1, x_2, y_1, y_2 subject to the relations

$$\begin{aligned} x_2x_1 &= x_1x_2 + x_1^2, & y_2y_1 &= y_1y_2 + y_1^2,\\ y_1x_1 &= fx_1y_1, & y_1x_2 &= gx_1y_1 + fx_2y_1,\\ y_2x_1 &= hx_1y_1 + fx_1y_2, & y_2x_2 &= mx_1y_1 + hx_2y_1 + gx_1y_2 + fx_2y_2, \end{aligned}$$

where $f, g, h, m \in \Bbbk$ and $f \neq 0$. Note that in the relations

$$y_1x_1 = \sigma_{11}(x_1)y_1 + \sigma_{12}(x_1)y_2 \quad \text{and} \quad y_1x_2 = \sigma_{11}(x_2)y_1 + \sigma_{12}(x_2)y_2,$$

we have that $\sigma_{12} = 0$. In this case $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of $R = \Bbbk_Q[x_1, x_2]$. By Proposition 3.2 (1), B can be presented as the two-step iterated Ore extension $\Bbbk_Q[x_1, x_2][y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$. If we take the relation $y_2x_1 = hx_1y_1 + fx_1y_2$, we can see that it satisfies the condition (3.2.2), and thus B is not differentially smooth.

Example 3.3 ([ZZ08, Example 4.2])**.** Consider $h \in \Bbbk^*$. Let $B(h)$ denote the graded algebra generated by x_1, x_2, y_1 and y_2 subject to the conditions

$$\begin{aligned} x_2x_1 &= -x_1x_2,\\ y_2y_1 &= -y_1y_2,\\ y_1x_1 &= h(x_1y_1 + x_2y_1 + x_1y_2),\\ y_1x_2 &= h(x_1y_2),\\ y_2x_1 &= h(x_2y_1), \quad \text{and}\\ y_2x_2 &= h(-x_2y_1 - x_1y_2 + x_2y_2). \end{aligned}$$

Following the notation in [ZZ09], the set of the last four relations is associated to the matrix

$$\Sigma = h\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

$B(h)$ is a quadratic connected graded algebra where $\deg x_1 = \deg x_2 = \deg y_1 = \deg y_2 = 1$. In fact, from [ZZ08, Lemma 4.5] we know that by considering $P = (-1, 0), \delta = 0, \tau = (0, 0, 0)$ and the algebra homomorphism

$$\begin{aligned} \sigma : A &\to M_2(A) \\ x_1 &\mapsto h\begin{bmatrix} x_1 + x_2 & x_1 \\ x_2 & 0 \end{bmatrix}, \\ x_2 &\mapsto h\begin{bmatrix} 0 & x_1 \\ -x_2 & -x_1 + x_2 \end{bmatrix}, \end{aligned}$$

then the data $\{P, \sigma, \delta, \tau\}$ satisfy the relations (3.1.4) - (3.1.9) for the generating set $\{x_1, x_2\}$, and hence $B(h)$ is a graded double extension of the Manin's plane $\mathcal{O}_{-1}(\Bbbk) = A = \Bbbk_{-1}[x_1, x_2] = \Bbbk\{x_1, x_2\}/\langle x_1x_2 + x_2x_1\rangle$. Note that if we write $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} \sigma_{11}(x_1) & \sigma_{12}(x_1) \\ \sigma_{21}(x_1) & \sigma_{22}(x_1) \end{bmatrix} &= h\begin{bmatrix} x_1 + x_2 & x_1 \\ x_2 & 0 \end{bmatrix}, \quad \text{and} \\ \begin{bmatrix} \sigma_{11}(x_1^2) & \sigma_{12}(x_1^2) \\ \sigma_{21}(x_1^2) & \sigma_{22}(x_1^2) \end{bmatrix} &= h^2\begin{bmatrix} (x_1 + x_2)^2 + x_1x_2 & (x_1 + x_2)x_1 \\ x_2(x_1 + x_2) & x_2x_1 \end{bmatrix}, \end{aligned}$$

which shows that $\sigma_{ij}(x_1^2) \neq \sigma_{ij}(x_1)^2$ for each pair σ_{ij}, and so each one of them is not an algebra homomorphism.

$B(h)$ is a remarkable algebra because it is neither an Ore extension nor a normal extension of an Artin-Schelter regular algebra (generated by three elements in degree one) of global dimension three.

Theorem 3.7 shows that $B(h)$ is not differentially smooth: in the relation $y_1x_1 - h(x_1y_1 + x_2y_1 + x_1y_2) = 0$ the indeterminate x_2 only appears once, and so $B(h)$ satisfies (3.2.2).

Example 3.4. Consider Example 3.1. Although Theorem 3.7 is formulated for polynomial extensions on four indeterminates, since the relations

$$\begin{aligned} y_1x &= b^{-1}xy_2 + cx^2, \\ y_1x &= axy_1 + xy_2, \quad \text{and} \\ y_2x &= (2 + 2b^{-1})xy_1 + xy_2 \end{aligned}$$

define the algebras B^2, B^3 and B^4, respectively, and all of them satisfy the condition (3.2.2), it follows that these three double extensions of $\Bbbk[t]$ are not differentially smooth.

The case of algebra B^1 is different and interesting: this algebra is differentially smooth! Let us see the details.

We know that by using the linear transformation $y_2 \to y_2 + \frac{bc}{b-1}x$ we get that $c = 0$, whence $B^1(1, p, a, b, c) \cong B^1(1, p, a, b, 0)$. Let $p := b^{-2}$.

Consider the maps of B^1 given by

$$\begin{aligned}
&\nu_x(x) = x, && \nu_x(y_1) = by_1, && \nu_x(y_2) = b^{-1}y_2,\\
&\nu_{y_1}(x) = b^{-1}x, && \nu_{y_1}(y_1) = y_1, && \nu_{y_1}(y_2) = b^{-2}y_2,\\
&\nu_{y_2}(x) = bx, && \nu_{y_2}(y_1) = b^2y_1, && \nu_{y_2}(y_2) = y_2.
\end{aligned}$$

The map ν_x can be extended to an algebra homomorphism of B^1 if and only if the definitions of $\nu_x(x)$, $\nu_x(y_1)$ and $\nu_x(y_2)$ respect relations (3.1.10), (3.1.11) and (3.1.12), i.e.

$$\begin{aligned}
\nu_x(y_2)\nu_x(y_1) - p\nu_x(y_1)\nu_x(y_2) &= \frac{bc}{1-b}(pb-1)\nu_x(x)\nu_x(y_1) + a\nu_x(x^2),\\
\nu_x(y_1)\nu_x(x) - b\nu_x(x)\nu_x(y_1) &= 0, \quad \text{and}\\
\nu_x(y_2)\nu_x(x) - b^{-1}\nu_x(x)\nu_x(y_2) &= c\nu_x(x^2).
\end{aligned}$$

We obtain the equation

$$a^{-1}b\frac{bc}{1-b}(pb-1)(b-1) = 0,$$

which due to the conditions on the constants defining the algebra only holds when $c = 0$.

Similarly, the map ν_{y_1} can be extended to an algebra homomorphism of B^1 if and only if the equality $p - b^{-2} = 0$ is satisfied.

By considering the extension of the map ν_{y_2} to an algebra homomorphism of B^1, we obtain again the condition $p - b^{-2} = 0$.

Since we need to guarantee that

$$\begin{aligned}
\nu_x \circ \nu_{y_1} &= \nu_{y_1} \circ \nu_x,\\
\nu_x \circ \nu_{y_2} &= \nu_{y_2} \circ \nu_x, \quad \text{and}\\
\nu_{y_2} \circ \nu_{y_1} &= \nu_{y_1} \circ \nu_{y_2},
\end{aligned}$$

it is enough to satisfy these equalities for the generators x, y_1 and y_2, that is,

$$\nu_x \circ \nu_{y_1}(x) = b^{-1}x, \tag{3.2.3}$$
$$\nu_{y_1} \circ \nu_x(x) = b^{-1}x, \tag{3.2.4}$$
$$\nu_x \circ \nu_{y_1}(y_1) = by_1, \tag{3.2.5}$$
$$\nu_{y_1} \circ \nu_x(y_1) = by_1, \tag{3.2.6}$$
$$\nu_x \circ \nu_{y_1}(y_2) = b^{-3}y_2, \quad \text{and} \tag{3.2.7}$$
$$\nu_{y_1} \circ \nu_x(y_2) = qb^{-3}y_2. \tag{3.2.8}$$

As it is clear, composition $\nu_x \circ \nu_{y_1} = \nu_{y_1} \circ \nu_x$ is always satisfied. Now,

$$\nu_x \circ \nu_{y_2}(x) = bx, \tag{3.2.9}$$
$$\nu_{y_2} \circ \nu_x(x) = bx, \tag{3.2.10}$$
$$\nu_x \circ \nu_{y_2}(y_1) = b^3 y_1, \tag{3.2.11}$$
$$\nu_{y_2} \circ \nu_x(y_1) = b^3 y_1, \tag{3.2.12}$$
$$\nu_x \circ \nu_{y_2}(y_2) = b^{-1} y_2, \quad \text{and} \tag{3.2.13}$$
$$\nu_{y_2} \circ \nu_x(y_2) = b^{-1} y_2. \tag{3.2.14}$$

Again, we can see that composition $\nu_x \circ \nu_{y_2} = \nu_{y_2} \circ \nu_x$ is always satisfied.

Finally,

$$\nu_{y_2} \circ \nu_{y_1}(x) = x, \tag{3.2.15}$$
$$\nu_{y_1} \circ \nu_{y_2}(x) = x, \tag{3.2.16}$$
$$\nu_{y_2} \circ \nu_{y_1}(y_1) = b^2 y_1, \tag{3.2.17}$$
$$\nu_{y_1} \circ \nu_{y_2}(y_1) = b^2 y_1, \tag{3.2.18}$$
$$\nu_{y_2} \circ \nu_{y_1}(y_2) = b^{-2} y_2, \quad \text{and} \tag{3.2.19}$$
$$\nu_{y_1} \circ \nu_{y_2}(y_2) = b^{-2} y_2. \tag{3.2.20}$$

Hence the composition $\nu_{y_2} \circ \nu_{y_1} = \nu_{y_1} \circ \nu_{y_2}$ holds.

Consider $\Omega^1 B^1$ a free right B^1-module of rank three with generators dx, dy_1 and dy_2. For all $q \in B^1$ define a left B^1-module structure by

$$\begin{aligned} q\,dx &= dx\,\nu_x(q), \\ q\,dy_1 &= dy_1\,\nu_{y_1}(q), \text{ and} \\ q\,dy_2 &= dy_2\,\nu_{y_2}(q). \end{aligned} \tag{3.2.21}$$

The relations in $\Omega^1 B^1$ are given by

$$\begin{aligned} x\,dx &= dx\,x, \\ x\,dy_1 &= b^{-1} dy_1\,x, \\ x\,dy_2 &= b\,dy_2\,x, \end{aligned} \tag{3.2.22}$$

$$\begin{aligned} y_1\,dx &= b\,dx\,y_1, \\ y_1\,dy_1 &= dy_1\,y_1, \\ y_1\,dy_2 &= b^2 dy_2\,y_1, \end{aligned} \tag{3.2.23}$$

$$\begin{aligned} y_2\,dx &= b^{-1} dx\,y_2, \\ y_2\,dy_1 &= b^{-2} dy_1\,y_2, \text{ and} \\ y_2\,dy_2 &= dy_2\,y_2. \end{aligned} \tag{3.2.24}$$

We want to extend the correspondences

$$x \mapsto dx, \quad y_1 \mapsto dy_1 \quad \text{and} \quad y_2 \mapsto dy_2$$

to a map $d : B^1 \to \Omega^1 B^1$ satisfying the Leibniz's rule. This is possible if it is compatible with the nontrivial relations (3.1.10), (3.1.11) and (3.1.12), i.e. if the equalities

$$\begin{aligned} dy_2y_1 + y_2dy_1 &= pdy_1y_2 + py_1dy_2 + adxx + axdx, \\ dy_1x + y_1dx &= bdxy_1 + bxdy_1, \quad \text{and} \\ dy_2x + y_2dx &= b^{-1}dxy_2 + b^{-1}xdy_2. \end{aligned}$$

hold.

Define $\Bbbk$-linear maps

$$\partial_x, \partial_{y_1}, \partial_{y_2} : B^1 \to B^1$$

such that

$$d(q) = dx\partial_x(q) + dy_1\partial_{y_1}(q) + dy_2\partial_{y_2}(q), \quad \text{for all } q \in B^1.$$

Since dx, dy_1 and dy_2 are free generators of the right B^1-module $\Omega^1 B^1$, these maps are well-defined. Note that $d(q) = 0$ if and only if $\partial_x(q) = \partial_{y_1}(q) = \partial_{y_2}(q) = 0$. By using the three relations in (5.3.30) and the definitions of the maps ν_x, ν_{y_1} and ν_{y_2}, we get that

$$\begin{aligned} \partial_x(x^k y_1^l y_2^s) &= kx^{k-1} y_1^l y_2^s, \\ \partial_{y_1}(x^k y_1^l y_2^s) &= lb^{-k} x^k y_1^{l-1} y_2^s, \quad \text{and} \\ \partial_{y_2}(x^k y_1^l y_2^s) &= sb^{k+2l} x^k y_1^l y_2^{s-1}. \end{aligned} \tag{3.2.25}$$

Thus $d(q) = 0$ if and only if q is a scalar multiple of the identity. This shows that $(\Omega B^1, d)$ is connected where $\Omega B^1 = \Omega^0 B^1 \oplus \Omega^1 B^1 \oplus \Omega^2 B^1$.

The universal extension of d to higher forms compatible with (5.3.31), (5.3.32) and (5.3.33) gives the following rules for $\Omega^2 B^1$:

$$dy_1 \wedge dx = -bdx \wedge dy_1, \tag{3.2.26}$$

$$dy_2 \wedge dx = -b^{-1}dx \wedge dy_2, \quad \text{and} \tag{3.2.27}$$

$$dy_2 \wedge dy_1 = -b^{-2}dy_1 \wedge dy_2. \tag{3.2.28}$$

Since the automorphisms ν_x, ν_{y_1} and ν_{y_2} commute with each other, there are no additional relationships to the previous ones, so we get the expression

$$\Omega^2 B^1 = dx \wedge dy_1 B^1 \oplus dx \wedge dy_2 B^1 \oplus dy_1 \wedge dy_2 B^1.$$

Since $\Omega^3 B^1 = \omega B^1 \cong B^1$ as a right and left B^1-module, with $\omega = dx \wedge dy_1 \wedge dy_2$, where $\nu_\omega = \nu_x \circ \nu_{y_1} \circ \nu_{y_2}$, we have that ω is a volume form of B^1. From Proposition 1.7

(2) we get that ω is an integral form by setting

$$\begin{aligned}
\omega_1^1 &= \bar{\omega}_1^1 = dx,\\
\omega_2^1 &= \bar{\omega}_2^1 = dy_1,\\
\omega_3^1 &= \bar{\omega}_3^1 = dy_2,\\
\omega_1^2 &= dy_1 \wedge dy_2,\\
\omega_2^2 &= b^3 dx \wedge dy_1,\\
\omega_3^2 &= -b^{-1} dx \wedge dy_2,\\
\bar{\omega}_1^2 &= dy_1 \wedge dy_2,\\
\bar{\omega}_2^2 &= -b^{-1} dx \wedge dy_2, \quad \text{and}\\
\bar{\omega}_3^2 &= dx \wedge dy_1.
\end{aligned}$$

By Proposition 1.7 (2), we consider the expression $\omega' := dx\alpha + dy_1\beta + dy_2\gamma$ with $\alpha, \beta, \gamma \in \Bbbk$, to obtain the equality

$$\begin{aligned}
\sum_{i=1}^{3} \omega_i^1 \pi_\omega(\bar{\omega}_i^2 \wedge \omega') &= dx\pi_\omega(\alpha dy_1 \wedge dy_2 \wedge dx) + dy_1\pi_\omega(-\beta b^2 dt \wedge dy_2 \wedge dy_1)\\
&\quad + dy_2\pi_\omega(\gamma dx \wedge dy_1 \wedge dy_2) = dx\alpha + dy_1\beta + dy_2\gamma = \omega'.
\end{aligned}$$

On the other hand, if $\omega'' := dx \wedge dy_1\alpha + dx \wedge dy_2\beta + dy_1 \wedge dy_2\gamma$ where $\alpha, \beta, \gamma \in \Bbbk$, we get that

$$\begin{aligned}
\sum_{i=1}^{3} \omega_i^2 \pi_\omega(\bar{\omega}_i^1 \wedge \omega'') &= dy_1 \wedge dy_2\pi_\omega(\gamma dx \wedge dy_1 \wedge dy_2)\\
&\quad + b^3 dx \wedge dy_1\pi_\omega(\alpha dy_2 \wedge dx \wedge dy_1)\\
&\quad - b^{-1} dx \wedge dy_2\pi_\omega(\beta dy_1 \wedge dx \wedge dy_2)\\
&= dx \wedge dy_1\alpha + dx \wedge dy_2\beta + dy_1 \wedge dy_2\gamma\\
&= \omega''.
\end{aligned}$$

As we have seen above, all elements of different degrees can be generated by ω_i^j and $\bar{\omega}_i^{3-j}$ for $j = 1, 2$ and $i = 1, 2, 3$, so Proposition 1.7 (2) guarantees that ω is an integral form. Proposition 1.6 shows that $(\Omega B^1, d)$ is an integrable differential calculus of degree 3, and since $\mathrm{GKdim}(B^1) = 3$, it follows that B^1 is differentially smooth.

Answer to the Question 1. False. Consider the Manin's plane $\mathcal{O}_q(\Bbbk)$. As we know from Example 1.6, this algebra is differentially smooth, and since that $\mathrm{GKdim}(\mathcal{O}_q(\Bbbk)) = 2$. However, in Example 3.3 we saw that $B(h)$ is a double extension of $\mathcal{O}_q(\Bbbk)$ with $\mathrm{GKdim}(B(h)) = 4$, and $B(h)$ is not differentially smooth.

3.3 Future work

In Examples 3.2, 3.3 and 3.4 we saw some interesting facts on the differential smoothness of double extensions of differentially smooth algebras. As we said in the Introduction,

Ore extensions and normal extensions of regular algebras of dimension three were studied by Le Bruyn et al. [BSdB96], and all of them were not considered by Zhang and Zhang [ZZ09] in their classification of 26 families of regular algebras of type (14641). We think that a natural task is to investigate the differential smoothnes of the algebras appearing in [BSdB96].

Since $\Bbbk[x]$, $\mathcal{O}_q(\Bbbk)$ and $\mathcal{J}(\Bbbk)$ are differentially smooth (Example 1.5), and that Artin-Schelter regular algebras of dimension three generated by elements of degree one have been classified by Artin, Schelter, Tate and Van den Bergh [AS87, ATdB07, ATVdB91]), and that the characterization of these algebras requires greater mathematical techniques that have not been considered at the time when this chapter was written, the study of the differential smoothness of these algebras will be one of our immediate tasks.

Table 3.1: Double extensions

Double extension	Relations defining the double extension	Σ_{ij}	M_{ij}	Data (Σ, M, P, Q)	Conditions
A	$x_2x_1 = x_1x_2, \quad y_2y_1 = y_1y_2 + y_1^2,$ $y_1x_1 = x_1y_1, \quad y_1x_2 = x_2y_1 + x_1y_2,$ $y_2x_1 = x_1y_2, \quad y_2x_2 = -2x_2y_1 - x_1y_2 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}, \quad P = (1, 1), \quad Q = (1, 0)$	
B	$x_2x_1 = px_1x_2, \quad y_2y_1 = py_1y_2,$ $y_1x_1 = x_2y_2, \quad y_1x_2 = x_1y_2,$ $y_2x_1 = -x_2y_1, \quad y_2x_2 = x_1y_1$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P = (p, 0), \quad Q = (p, 0)$	$p^2 = -1$
C	$x_2x_1 = px_1x_2, \quad y_2y_1 = py_1y_2,$ $y_1x_1 = -x_1y_1 + p^2x_2y_1 + x_1y_2 - px_2y_2, \quad y_1x_2 = -px_1y_1 + x_2y_1 + x_1y_2 - px_2y_2,$ $y_2x_1 = -px_1y_1 - 2p^2x_2y_1 + px_1y_2 - px_2y_2, \quad y_2x_2 = -px_1y_1 + p^2x_2y_1 + x_1y_2 - x_2y_2$	$\Sigma_{11} = \begin{bmatrix} -1 & p^2 \\ -p & 1 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 1 & -p \\ 1 & -p \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} -p & -2p^2 \\ -p & p^2 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} p & -p \\ 1 & -1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} -1 & 1 \\ -p & p \end{bmatrix}, \quad M_{12} = \begin{bmatrix} p^2 & -p \\ -2p^2 & -p \end{bmatrix}, \quad M_{21} = \begin{bmatrix} -p & 1 \\ -p & 1 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 1 & -p \\ p^2 & -1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} -1 & p^2 & 1 & -p \\ -p & 1 & 1 & -p \\ -p & -2p^2 & p & -p \\ -p & p^2 & 1 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 1 & p^2 & -p \\ -p & p & -2p^2 & -p \\ -p & 1 & 1 & -p \\ -p & 1 & p^2 & -1 \end{bmatrix}, \quad P = (p, 0), \quad Q = (p, 0)$	$p^2 + p + 1 = 0$
D	$x_2x_1 = -x_1x_2, \quad y_2y_1 = py_1y_2,$ $y_1x_1 = -px_1y_1, \quad y_1x_2 = -p^2x_2y_1 + x_1y_2,$ $y_2x_1 = px_1y_2, \quad y_2x_2 = x_1y_1 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} -p & 0 \\ 0 & -p^2 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} -p & 0 \\ 0 & p \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} -p^2 & 0 \\ 0 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} -p & 0 & 0 & 0 \\ 0 & -p^2 & 1 & 0 \\ 0 & 0 & p & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 1 & -p^2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad P = (p, 0), \quad Q = (-1, 0)$	$p \in \{-1, 1\}$
E	$x_2x_1 = -x_1x_2, \quad y_2y_1 = py_1y_2,$ $y_1x_1 = x_1y_2 + x_2y_2, \quad y_1x_2 = x_1y_2 - x_2y_2,$ $y_2x_1 = -x_1y_1 + x_2y_1, \quad y_2x_2 = x_1y_1 + x_2y_1$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad P = (p, 0), \quad Q = (-1, 0)$	$p^2 = -1$
F	$x_2x_1 = -x_1x_2, \quad y_2y_1 = py_1y_2,$ $y_1x_1 = -x_1y_1 - px_2y_1 + x_1y_2 - x_2y_2, \quad y_1x_2 = -px_1y_1 + x_2y_1 + x_1y_2 + x_2y_2,$ $y_2x_1 = -px_1y_1 + px_2y_1 + px_1y_2 + x_2y_2, \quad y_2x_2 = -px_1y_1 - px_2y_1 + x_1y_2 - px_2y_2$	$\Sigma_{11} = \begin{bmatrix} -1 & -p \\ -p & 1 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} -p & p \\ -p & -p \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} p & 1 \\ 1 & -p \end{bmatrix}$	$M_{11} = \begin{bmatrix} +1 & 1 \\ -p & p \end{bmatrix}, \quad M_{12} = \begin{bmatrix} -p & -1 \\ p & 1 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} -p & 1 \\ -p & 1 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 1 & 1 \\ -p & -p \end{bmatrix}$	$\Sigma = \begin{bmatrix} -1 & -p & 1 & -1 \\ -p & 1 & 1 & 1 \\ -p & p & p & 1 \\ -p & -p & 1 & -p \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 1 & -p & -1 \\ -p & p & p & 1 \\ -p & 1 & 1 & 1 \\ -p & 1 & -p & -p \end{bmatrix}, \quad P = (p, 0), \quad Q = (-1, 0)$	$p^2 = -1$
G	$x_2x_1 = x_1x_2, \quad y_2y_1 = py_1y_2,$ $y_1x_1 = px_1y_1, \quad y_1x_2 = px_1y_1 + p^2x_2y_1 + x_1y_2,$ $y_2x_1 = px_1y_2, \quad y_2x_2 = fx_1y_1 - x_1y_2 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} p & 0 \\ p & p^2 \end{bmatrix}, \quad \Sigma_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}, \quad \Sigma_{22} = \begin{bmatrix} p & 0 \\ -1 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} p & 1 \\ f & -1 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} p^2 & 0 \\ 0 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} p & 0 & 0 & 0 \\ p & p^2 & 1 & 0 \\ 0 & 0 & p & 0 \\ f & 0 & -1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ p & 1 & p^2 & 0 \\ f & -1 & 0 & 1 \end{bmatrix}, \quad P = (p, 0), \quad Q = (1, 0)$	$p \neq 0, \pm 1$ and $f \neq 0$

Table 3.2: Double extensions

Double extension	Relations defining the double extension	Σ_{ij}	M_{ij}	Data $\{\Sigma, M, P, Q\}$	Conditions
H	$x_2x_1 = x_1x_2 + x_1^2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = x_1y_2$, $y_1x_2 = fx_1y_2 + x_2y_2$, $y_2x_1 = x_1y_1$, $y_2x_2 = fx_1y_1 + x_2y_1$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & f & 1 \\ 1 & 0 & 0 & 0 \\ f & 1 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f & 0 & 1 \\ f & 0 & 1 & 0 \end{bmatrix}$, $P = (-1, 0)$, $Q = (1, 1)$	$f \neq 0$
I	$x_2x_1 = qx_1x_2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = -qx_1y_1 - qx_2y_1 + x_1y_2 - qx_2y_2$, $y_1x_2 = x_1y_1 + x_2y_1 + x_1y_2 - qx_2y_2$, $y_2x_1 = x_1y_1 + qx_2y_1 + qx_1y_2 - qx_2y_2$, $y_2x_2 = -x_1y_1 - qx_2y_1 + x_1y_2 - x_2y_2$	$\Sigma_{11} = \begin{bmatrix} -q & -q \\ 1 & 1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & -q \\ 1 & -q \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & q \\ -1 & -q \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} q & -q \\ 1 & -1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} -q & 1 \\ 1 & q \end{bmatrix}$, $M_{12} = \begin{bmatrix} -q & -q \\ q & -q \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & -q \\ -q & -1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} -q & -q & 1 & -q \\ 1 & 1 & 1 & -q \\ 1 & q & q & -q \\ -1 & -q & 1 & -1 \end{bmatrix}$, $M = \begin{bmatrix} -q & 1 & -q & -q \\ 1 & q & q & -q \\ 1 & 1 & 1 & -q \\ -1 & 1 & -q & -1 \end{bmatrix}$, $P = (-1, 0)$, $Q = (q, 0)$	$q^2 = -1$
J	$x_2x_1 = qx_1x_2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = x_2y_1 + x_2y_2$, $y_1x_2 = -x_1y_1 + x_1y_2$, $y_2x_1 = x_2y_1 - x_2y_2$, $y_2x_2 = x_1y_1 + x_1y_2$	$\Sigma_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $M_{21} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $P = (-1, 0)$, $Q = (q, 0)$	$q^2 = -1$
K	$x_2x_1 = qx_1x_2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = x_1y_1$, $y_1x_2 = x_2y_2$, $y_2x_1 = x_1y_2$, $y_2x_2 = fx_2y_1$	$\Sigma_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 1 \\ f & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & f & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 \end{bmatrix}$, $P = (-1, 0)$, $Q = (q, 0)$	$q \in \{-1, 1\}$ and $f \neq 0$
L	$x_2x_1 = qx_1x_2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = fx_1y_2$, $y_1x_2 = x_2y_2$, $y_2x_1 = fx_1y_1$, $y_2x_2 = x_2y_1$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \\ f & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & f & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $P = (-1, 0)$, $Q = (q, 0)$	$q \in \{-1, 1\}$ and $f \neq 0$
M	$x_2x_1 = -x_1x_2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = x_2y_1 + x_1y_2$, $y_1x_2 = fx_1y_1 - x_2y_2$, $y_2x_1 = x_1y_1 - x_2y_2$, $y_2x_2 = -x_2y_1 - fx_1y_2$	$\Sigma_{11} = \begin{bmatrix} 0 & 1 \\ f & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & -1 \\ -f & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $M_{21} = \begin{bmatrix} f & 0 \\ 0 & -f \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 1 & 1 & 0 \\ f & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -f & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ f & 0 & 0 & -1 \\ 0 & -f & -1 & 0 \end{bmatrix}$, $P = (-1, 0)$, $Q = (-1, 0)$	$f \neq 1$
N	$x_2x_1 = -x_1x_2$, $y_2y_1 = -y_1y_2$, $y_1x_1 = -gx_2y_1 + fx_2y_2$, $y_1x_2 = gx_1y_1 + fx_1y_2$, $y_2x_1 = fx_2y_1 - gx_2y_2$, $y_2x_2 = fx_1y_1 + gx_1y_2$	$\Sigma_{11} = \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} -g & f \\ f & -g \end{bmatrix}$, $M_{21} = \begin{bmatrix} g & f \\ f & g \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & -g & 0 & f \\ g & 0 & f & 0 \\ 0 & f & 0 & -g \\ f & 0 & g & 0 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & -g & f \\ 0 & 0 & f & -g \\ g & f & 0 & 0 \\ f & g & 0 & 0 \end{bmatrix}$, $P = (-1, 0)$, $Q = (-1, 0)$	$f^2 \neq g^2$

Table 3.3: Double extensions

Double extension	Relations defining the double extension	Σ_{ij}	M_{ij}	Data (Σ, M, P, Q)	Conditions
$\mathbb{O}$	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_1y_1 + fx_2y_2, \quad y_1x_2 = -x_2y_1 + x_1y_2,$ $y_2x_1 = fx_2y_1 - x_1y_2, \quad y_2x_2 = x_1y_1 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 1 & 0 & 0 & f \\ 0 & -1 & 1 & 0 \\ 0 & f & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 & f \\ 0 & -1 & f & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	$f \neq -1$
$\mathbb{P}$	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_1y_2 + fx_2y_2, \quad y_1x_2 = x_1y_2 + x_2y_2,$ $y_2x_1 = x_1y_1 - fx_2y_1, \quad y_2x_2 = -x_1y_1 + x_2y_1$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & f \\ 1 & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & -f \\ -1 & 1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & 1 & f \\ 0 & 0 & 1 & 1 \\ 1 & -f & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 0 & f \\ 1 & 0 & -f & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	$f \neq -1$
$\mathbb{Q}$	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_1y_2, \quad y_1x_2 = x_1y_1 + x_2y_1 + x_1y_2,$ $y_2x_1 = -x_1y_1, \quad y_2x_2 = x_1y_1 - x_1y_2 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	
$\mathbb{R}$	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_1y_1 + x_2y_1 + x_1y_2, \quad y_1x_2 = x_1y_2,$ $y_2x_1 = x_2y_1, \quad y_2x_2 = -x_2y_1 - x_1y_2 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	
$\mathbb{S}$	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = -x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2, \quad y_1x_2 = x_1y_1 - x_2y_1 + x_1y_2 + x_2y_2,$ $y_2x_1 = x_1y_1 + x_2y_1 - x_1y_2 + x_2y_2, \quad y_2x_2 = x_1y_1 + x_2y_1 + x_1y_2 - x_2y_2$	$\Sigma_{11} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	
$\mathbb{T}$	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = -x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2, \quad y_1x_2 = x_1y_1 - x_2y_1 + x_1y_2 + x_2y_2,$ $y_2x_1 = x_1y_1 + x_2y_1 + x_1y_2 - x_2y_2, \quad y_2x_2 = x_1y_1 + x_2y_1 - x_1y_2 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	

Table 3.4: Double extensions

Double extension	Relations defining the double extension	Σ_{ij}	M_{ij}	Data $\{\Sigma, M, P, Q\}$	Conditions
U	$x_2x_1 = -x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = -x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2, \quad y_1x_2 = x_1y_1 + x_2y_1 + x_1y_2 - x_2y_2,$ $y_2x_1 = x_1y_1 + x_2y_1 - x_1y_2 + x_2y_2, \quad y_2x_2 = x_1y_1 - x_2y_1 + x_1y_2 + x_2y_2$	$\Sigma_{11} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (-1, 0)$	
V	$x_2x_1 = x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_2y_1 + x_1y_2, \quad y_1x_2 = x_2y_1,$ $y_2x_1 = -x_1y_1 + x_2y_1, \quad y_2x_2 = x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (1, 0)$	
W	$x_2x_1 = x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = fx_2y_1 + x_1y_2, \quad y_1x_2 = x_1y_1 - x_2y_2,$ $y_2x_1 = x_1y_1 + fx_2y_2, \quad y_2x_2 = -x_2y_1 + x_1y_2$	$\Sigma_{11} = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & f & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & f \\ 0 & -1 & 1 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & f & 0 \\ 1 & 0 & 0 & f \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$, $P = (-1, 0), \quad Q = (1, 0)$	$f \neq -1$
X	$x_2x_1 = x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_1y_2, \quad y_1x_2 = x_1y_2 + x_2y_2,$ $y_2x_1 = x_1y_1, \quad y_2x_2 = x_1y_1 + x_2y_1$	$\Sigma_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $P = (-1, 0), \quad Q = (1, 0)$	
Y	$x_2x_1 = x_1x_2, \quad y_2y_1 = -y_1y_2,$ $y_1x_1 = x_1y_1, \quad y_1x_2 = fx_1y_1 - x_2y_1 + x_1y_2,$ $y_2x_1 = x_1y_2, \quad y_2x_2 = x_1y_1 + fx_1y_2 - x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 1 & 0 \\ f & -1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} 1 & 0 \\ f & -1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} f & 1 \\ 1 & f \end{bmatrix}$, $M_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ f & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & f & -1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f & 1 & -1 & 0 \\ 1 & f & 0 & -1 \end{bmatrix}$, $P = (-1, 0), \quad Q = (1, 0)$	f is general
Z	$x_2x_1 = -x_1x_2, \quad y_2y_1 = y_1y_2,$ $y_1x_1 = x_1y_1 + x_2y_2, \quad y_1x_2 = x_2y_1 + x_1y_2,$ $y_2x_1 = fx_2y_1 - x_1y_2, \quad y_2x_2 = fx_1y_1 - x_2y_2$	$\Sigma_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Sigma_{21} = \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}$, $\Sigma_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 1 \\ f & 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0 & 1 \\ f & 0 \end{bmatrix}$, $M_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & f & -1 & 0 \\ f & 0 & 0 & -1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & f & 0 \\ 0 & 1 & 1 & 0 \\ f & 0 & 0 & -1 \end{bmatrix}$, $P = (1, 0), \quad Q = (-1, 0)$	$f(1 + f) \neq 0$ is general

Remark 15. Zhang and Zhang [ZZ09, Subcase 4.4.4] formulated the relations of the algebra $\mathbb{Z}$. However, these contain one typo since the coefficients of the relations do not match the entries of the matrix Σ. Our version of these relations is presented in Table 3.4. They wrote the relations $y_1x_1 = x_1y_2 + x_2y_2$ and $y_2x_2 = fx_1y_2 - x_2y_2$, and the typo concerns that they considered the first factor x_1y_2 when it should be x_1y_1 according to matrix Σ.

CHAPTER 4

Smooth geometry of diffusion algebras

In this chapter we investigate the differential smoothness of *diffusion algebras* introduced by Isaev et al. [IPR01] in the context of one-dimensional stochastic processes with exclusion in statistical mechanics.

The chapter is organized as follows. Section 4.1 contains the key facts on diffusion algebras in order to set up notation and render this chapter self-contained. We adopt the terminology and notation presented by Pyatov and Twarock [PT02] who presented a construction formalism for these algebras from the mathematical point of view, and proved the results formulated in [IPR01]. Sections 4.1.1 and 4.1.2 recall the classification of diffusion algebras with three and n generators, respectively. In Section 4.2 we present the original results of the chapter. Tables 4.1 and 4.2 present the diffusion algebras on four generators while Tables 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11 and 4.12 contain diffusion algebras on five generators. Our key results are Theorems 4.3 and 4.4 since these describe explicitly those diffusion algebras that are differentially smooth, and also those families of algebras which are not, respectively. This chapter continues with the research on the differential smoothness of diffusion algebras on four and more generators started by Reyes and Sarmiento [RS22] with diffusion algebras on three generators.

4.1 Diffusion algebras

Just as Pyatov and Twarock [PT02] said, "Diffusion algebras play a key role in the understanding of one-dimensional stochastic processes. In the case of N species of particles with only nearest-neighbor interactions with exclusion on a one-dimensional lattice, diffusion algebras are useful tools in finding expressions for the probability distribution of the stationary state of these processes. Following the idea of matrix product states, the latter are given in terms of monomials built from the generators of a quadratic algebra" [PT02, p. 3268].

Hinchcliffe in his PhD Thesis [Hin05] and different researchers have investigated several ring, theoretical and homological properties of diffusion algebras [FGL+20, HHR20b, Lev05, RR21, Twa02].

Let us recall the details of Pyatov and Twarock's treatment [PT02].

Let α, β be two elements belonging to the set $I_N := \{1, \dots, N\}$ with $\alpha < \beta$. Consider quadratic relations of the form

$$g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha = x_\beta D_\alpha - x_\alpha D_\beta, \tag{4.1.1}$$

with $g_{\alpha\beta} \in \mathbb{R} \setminus \{0\}$, $g_{\beta\alpha} \in \mathbb{R}$, and $x_\alpha, x_\beta \in \mathbb{C}$.

Definition 4.1 ([PT02, Definition 1.1])**.** An algebra with set of generators given by $\{D_\alpha \mid \alpha \in I_N\}$ and relations of type (5.1.1) is called *diffusion algebra*, if it admits a linear PBW-basis of ordered monomials of the form

$$D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_N}^{k_N}, \quad \text{with} \quad k_j \in \mathbb{N} \quad \text{and} \quad \alpha_1 > \alpha_2 > \cdots > \alpha_N. \tag{4.1.2}$$

Due to physical reasons only relations with positive coefficients $g_{\alpha\beta} \in \mathbb{R}_{>0}$ and $g_{\beta\alpha} \in \mathbb{R}_{\geq 0}$ $(\alpha < \beta)$ are relevant because they are interpreted as hopping rates in stochastic models [PT02, p. 3268].

Note that the requirement of having a PBW basis (5.1.2) implies conditions on the coefficients $g_{\alpha\beta}$ and x_α in (5.1.1) according the the *Diamond Lemma* in ring theory formulated by Bergman [Ber78]. This means that we have a criterion to verify under which conditions the relations in (5.1.1) are of PBW type: this is the case precisely if each subset of three generators $\{D_\alpha, D_\beta, D_\gamma\}$ with ordering $\alpha < \beta < \gamma$ is reduction unique with respect to the ordering, that is if the two ways of reducing the monomial $D_\alpha D_\beta D_\gamma$ to the monomial $D_\gamma D_\beta D_\alpha$ lead to the same result when expressed in the PBW basis (5.1.2).

Just as Pyatov and Twarock [PT02, p. 3269] asserted, the task of deriving all diffusion algebras with N generators reduces to the following two steps:

(1) Find all diffusion algebras with three generators.

(2) Find all algebras with N generators such that each subset of three generators coincides with one of the cases listed before.

As it can be seen, the step (1) is equivalent to find those coefficients $g_{\alpha\beta}$ and x_α in (5.1.1) for which a set $\{D_\gamma, D_\beta, D_\alpha\}$ of three generators is reduction unique in the above sense. The list of diffusion algebras of three generators is given in Section 4.1.1. The second step is a combinatorial problem: it requires one to combine in a consistent way the three generators algebras listed before to algebras with N generators for general $N > 3$.

Pyatov and Twarock considered a constructive method to approach the second step, the so-called *blending procedure* (Section 4.1.2) which is an inductive procedure for the construction of diffusion algebras: "It uses the three generators cases and augments them to larger units by attaching further generators in accordance with the requirements of the diamond lemma, then giving a prescription of how theser larger building blocks may be glued together in order to obtain a general diffusion algebra of N generators" [PT02, p. 3269].

4.1.1 Diffusion algebras on three generators

Consider a set $\{D_\alpha, D_\beta, D_\gamma\}$ of three generators with an ordering induced by the ordering of the index set $\alpha < \beta < \gamma$ and relations as in (5.1.1). Since $g_{\alpha\beta} \neq 0$ for all $\alpha, \beta \in I_3 = \{\alpha, \beta, \gamma, \}$ with $\alpha < \beta$ by assumption, we get the relations of the algebra given by

$$\begin{aligned} D_\alpha D_\beta &= q_{\beta\alpha} D_\beta D_\alpha + x_\beta^{\alpha\beta} D_\alpha - x_\alpha^{\alpha\beta} D_\beta, \\ D_\alpha D_\gamma &= q_{\gamma\alpha} D_\gamma D_\alpha + x_\gamma^{\alpha\gamma} D_\alpha - x_\alpha^{\alpha\gamma} D_\gamma, \quad \text{and} \\ D_\beta D_\gamma &= q_{\gamma\beta} D_\gamma D_\beta + x_\gamma^{\beta\gamma} D_\beta - x_\beta^{\beta\gamma} D_\gamma, \end{aligned} \tag{4.1.3}$$

(4.1.4)

where $q_{ij} := \frac{g_{ji}}{g_{ij}}$, $x_\gamma^{ij} := \frac{x_\gamma}{g_{ij}}$ for $i, j, k \in \{\alpha, \beta, \gamma\}$ with $i < j$. Using (4.1.3), any monomial can be expressed in terms of the PBW basis (5.1.2), and this is well defined if we apply (4.1.3) in different orders and obtain the same result, that is, if the reductions

$$D_\alpha D_\beta D_\gamma \to D_\beta D_\alpha D_\gamma \to D_\beta D_\gamma D_\alpha \to D_\gamma D_\beta D_\alpha \tag{4.1.5}$$

and

$$D_\alpha D_\beta D_\gamma \to D_\alpha D_\gamma D_\beta \to D_\gamma D_\alpha D_\beta \to D_\gamma D_\beta D_\alpha \tag{4.1.6}$$

using (4.1.3) coincide when expressed in the PBW basis. These equalities lead to restrictions on the coefficients $g_{\alpha\beta}$ and x_α in (5.1.1) given by a set of six equations and their solutions determine all diffusion algebras of three generators.

Next, we recall the list of six equations. We assume $\alpha < \beta < \gamma$ and $x_\beta \neq 0$ for $j \in \{\alpha, \beta, \gamma\}$.

(1) *The case of* A_I:

$$\begin{aligned} gD_\alpha D_\beta - gD_\beta D_\alpha &= x_\beta D_\alpha - x_\alpha D_\beta, \\ gD_\alpha D_\gamma - gD_\gamma D_\alpha &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ gD_\beta D_\gamma - gD_\gamma D_\beta &= x_\gamma D_\beta - x_\beta D_\gamma, \end{aligned}$$

where $g \neq 0$.

(2) *The case of* A_{II}:

$$\begin{aligned} g_{\alpha\beta} D_\alpha D_\beta &= x_\beta D_\alpha - x_\alpha D_\beta, \\ g_{\alpha\gamma} D_\alpha D_\gamma &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ g_{\beta\gamma} D_\beta D_\gamma &= x_\gamma D_\beta - x_\beta D_\gamma, \end{aligned}$$

where $g_{ij} := g_i - g_j$ with $g_i \neq g_j$ for all $i, j \in \{\alpha, \beta, \gamma\}$ with $i < j$.

(3) *The case of* $\mathrm{B}^{(1)}$:

$$\begin{aligned} g_\beta D_\alpha D_\beta - (g_\beta - \Lambda) D_\beta D_\alpha &= -x_\alpha D_\beta, \\ g D_\alpha D_\gamma - (g - \Lambda) D_\gamma D_\alpha &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ g_\beta D_\beta D_\gamma - (g_\beta - \Lambda) D_\gamma D_\beta &= x_\gamma D_\beta, \end{aligned}$$

where $g, g_\beta \neq 0$.

(4) *The case of* $\mathrm{B}^{(2)}$:

$$\begin{aligned} g_{\alpha\beta} D_\alpha D_\beta &= -x_\alpha D_\beta, \\ g_{\alpha\gamma} D_\alpha D_\gamma - g_{\gamma\alpha} D_\gamma D_\alpha &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ g_{\beta\gamma} D_\beta D_\gamma &= x_\gamma D_\beta, \end{aligned}$$

where $g_{\alpha\beta}, g_{\alpha\gamma}, g_{\beta\gamma} \neq 0$.

(5) *The case of* $\mathrm{B}^{(3)}$:

$$\begin{aligned} g D_\alpha D_\beta - (g - \Lambda) D_\beta D_\alpha &= x_\beta D_\alpha - x_\alpha D_\beta, \\ g_\gamma D_\alpha D_\gamma &= -x_\alpha D_\gamma, \quad \text{and} \\ (g_\gamma - \Lambda) D_\beta D_\gamma &= -x_\beta D_\gamma, \end{aligned}$$

where $g \neq 0$ and $g_\gamma \neq 0, \Lambda$.

(6) *The case of* $\mathrm{B}^{(4)}$:

$$\begin{aligned} (g_\alpha - \Lambda) D_\alpha D_\beta &= x_\beta D_\alpha, \\ g_\alpha D_\alpha D_\gamma &= x_\gamma D_\alpha, \quad \text{and} \\ g D_\beta D_\gamma - (g - \Lambda) D_\gamma D_\beta &= x_\gamma D_\beta - x_\beta D_\gamma, \end{aligned}$$

where $g \neq 0$ and $g_\alpha \neq 0, \Lambda$.

(7) *The case of* $\mathrm{C}^{(1)}$:

$$\begin{aligned} g_\beta D_\alpha D_\beta - (g_\beta - \Lambda) D_\beta D_\alpha &= -x_\alpha D_\beta, \\ g_\gamma D_\alpha D_\gamma - (g_\gamma - \Lambda) D_\gamma D_\alpha &= -x_\alpha D_\gamma, \quad \text{and} \\ g_{\beta\gamma} D_\beta D_\gamma - g_{\gamma\beta} D_\gamma D_\beta &= 0, \end{aligned}$$

where $g_\beta, g_\gamma, g_{\beta,\gamma} \neq 0$.

(8) *The case of* $\mathrm{C}^{(2)}$:

$$\begin{aligned} g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha &= -x_\alpha D_\beta, \\ g_{\alpha\gamma} D_\alpha D_\gamma - g_{\gamma\alpha} D_\gamma D_\alpha &= -x_\alpha D_\gamma, \quad \text{and} \\ D_\beta D_\gamma &= 0, \end{aligned}$$

where $g_{\alpha\beta}, g_{\alpha\gamma} \neq 0$.

(9) *The case of* D: With $q_{ji} := \frac{g_{ji}}{g_{ij}}$, where $i, j \in \{\alpha, \beta, \gamma\}$ (recall that $g_{ij} \neq 0$, for $i < j$), we have that

$$\begin{aligned} D_\alpha D_\beta - q_{\beta\alpha} D_\beta D_\alpha &= 0, \\ D_\alpha D_\gamma - q_{\gamma\alpha} D_\gamma D_\alpha &= 0, \quad \text{and} \\ D_\beta D_\gamma - q_{\gamma\beta} D_\gamma D_\beta &= 0. \end{aligned}$$

The division into algebras A, B, C and D reflects the number of coefficients $x_j, j \in \{\alpha, \beta, \gamma\}$ being zero in the expression (5.1.1). As it was shown above, for these algebras, none, one, two, or all three of the coefficients x_i vanish, respectively.

4.1.2 Diffusion algebras on N generators

Consider the following decomposition of the index set $I_N = \{1, \ldots, N\}$:

$$I_N = I \cup R, \quad \text{with} \quad I := \{\alpha \in I_N \mid x_\alpha \neq 0\} \text{ and } R := \{\alpha \in I_N \mid x_\alpha = 0\}. \tag{4.1.7}$$

Definition 4.2. 1. [PT02, Definition 3.1] Normal orderings of two generators D_α and D_β is defined as

$$(D_\alpha D_\beta) := \begin{cases} D_\alpha D_\beta, & \text{if } \alpha < \beta \\ D_\beta D_\alpha, & \text{if } \beta < \alpha. \end{cases} \tag{4.1.8}$$

2. [PT02, Definition 3.2] For $\alpha < \beta$, consider the following notation:

$$[D_\alpha, D_\beta]_{q_{\beta\alpha}} := D_\alpha D_\beta - q_{\beta\alpha} D_\beta D_\alpha, \tag{4.1.9}$$

where the index at the commutator is referring to the coefficients $q_{\beta\alpha}$ in terms of which the commutator is defined.

Considering notation in Definition 4.2, the set R is subdivided into nonintersecting and nonempty subsets

$$R := R_1 \cup R_2 \cup \cdots \cup R_{M_R} \tag{4.1.10}$$

according to the following requirements:

- Relations between generators from the sets R_a and R_b for $a \neq b$ are given by

$$(D_{r_1} D_{r_2}) =: 0, \quad \text{for all } r_1 \in R_a \text{ and } r_2 \in R_b. \tag{4.1.11}$$

- Relations within a set R_a such that $|R_a| \geq 2$ are given by

$$[D_{r_1}, D_{r_2}]_{q_{r_2 r_1}} = 0, \quad \text{for all } r_1, r_2 \in R_a \text{ with } r_1 < r_2, \tag{4.1.12}$$

where the coefficients in (4.1.12) are subject to the condition opposite to (4.1.11), that is: for any subdivision $R_a = R' \cup R''$ into two nonintersecting and nonempty parts R' and R'',

$$\text{there exists } r_1 \in R' \text{ and } r_2 \in R'' \text{ such that } g_{r_1 r_2} g_{r_2 r_1} \neq 0. \tag{4.1.13}$$

This means that for any pair of indices $r, s \in R_\alpha$ there exists a finite sequence $\{r_k \in R_\alpha \mid k = 1, \dots, N\}$ such that $r_1 = r, r_N = s$ and

$$\prod_{k=1}^{N-1} g_{r_k r_{k+1}} g_{r_{k+1} r_k} \neq 0. \tag{4.1.14}$$

Relations (4.1.13) and (4.1.14) may be represented graphically via a *connectivity condition* on an ordered graph the vertices of which are labeled by the indices $r \in R_\alpha$ and the edges connect only those vertices $r_1 < r_2$ for which the condition $q_{r_2 r_1} \neq 0$ is satisfied.

Furthermore, for $|I| \geq 2$ the set R is split into two sets S and T as follows:

For any $R_\alpha \subset R$, let

$$R_\alpha := \begin{cases} S_\alpha, & \text{if there exist } r \in R_\alpha \text{ and } i \in I \text{ with } g_{ir} g_{ri} \neq 0, \\ T_\alpha & \text{otherwise.} \end{cases} \tag{4.1.15}$$

Suppose that the M_R sets R_α in (4.1.10) split into M_S sets S_α and M_T sets T_α. Then $M_R = M_S + M_T$. Number these sets as S_α for $\alpha = 1, \dots, M_S$, and T_α for $\alpha = 1, \dots, M_T$, and introduce

$$S := \bigcup_{\alpha=1}^{M_S} S_\alpha \quad \text{and} \quad T := \bigcup_{\alpha=1}^{M_T} T_\alpha. \tag{4.1.16}$$

The decomposition of the set S into subsets S_α has been used in the definition of the set S, and this will not be consider. However, the structure of the set T is crucial and needs further refinement.

For every $T_\alpha \subset T$, let

$$T_\alpha := \begin{cases} T_\alpha^\bullet, & \text{if } \exists\ i, j \in I \text{ with } i < j \text{ such that } T_\alpha \subset \{i+1, i+2, \dots, j-1\} \text{ and} \\ & \qquad I \cap \{i+1, i+2, \dots, j-1\} = \emptyset, \\ T_\alpha^\circ, & \text{otherwise.} \end{cases} \tag{4.1.17}$$

In this way, we write

$$T = \{T_\alpha^\bullet \mid \alpha = 1, \dots, M_T^\bullet\} \bigcup \{T_\alpha^\circ \mid \alpha = 1, \dots, M_T^\circ\} \quad \text{with } M_T = M_T^\bullet + M_T^\circ.$$

List of diffusion algebras on N generators

From now on, the expression "generators of a set I, S, T, or R" means the generators indexed by elements from the corresponding set.

Definition 4.3 ([PT02, Definition 3.3])**.** A set of three generators $\{D_x, D_y, D_z\}$ with $x \in X, y \in Y$ and $z \in Z$, where X, Y and Z are any of the sets I, R, S and T or any set in their decomposition will be called a *triplet* (*of type*) $\{X, Y, Z\}$.

As it can be seen, any triplet of type $\{I, I, I\}$ in a diffusion algebra of $N \geq 3$ generators gives rise to a diffusion algebra of type A_I or A_{II}, any triple of type $\{I, I, R\}$ to a diffusion algebra of type $B^{(1)}, B^{(2)}, B^{(3)}$ or $B^{(4)}$, any triplet of type $\{I, R, R\}$ to a diffusion algebra of type $C^{(1)}$ or $C^{(2)}$, and any triplet of type $\{R, R, R\}$ to a diffusion algebra of type D.

Proposition 4.1 ([PT02, Lemma 4.3]). *For any diffusion algebra* (1) *with* $N \geq 3$ *generators, the following statements hold:*

(1) *If* $|I| \geq 3$*, then all subalgebras corresponding to triplets of type* $\{I, I, I\}$ *are of the same type, which is either* A_I *(that is,* $g_{ij} = g$ *for all* $i, j \in I$*) or* A_{II} *(that is,* $g_{ji} = 0$*,* $g_{ij} = g_i - g_j$*,* $g_i \neq g_j$ *for all* $i < j$*).*

(2) *If* $|I| \geq 3$ *and all subalgebras corresponding to triplets* $\{I, I, I\}$ *are of type* A_I*, then for any* $s \in S$ *and for all* $i \in I$*, we have that*

$$g_{is} = g_{si} = g_s. \tag{4.1.18}$$

(3) *If* $|I| \geq 3$ *and all subalgebras corresponding to triplets* $\{I, I, I\}$ *are of type* A_{II}*, then* $S = \emptyset$*.*

(4) *Let* $|I| \geq 2$*. For any* $i \in I$ *and every* $t, t' \in T_a$ *(*T_a *means both* T_a° *and* $T_a^\bullet$*) with* $t < i$ *and* $t' > i$*, the coefficients* g_{ti} *and* $g_{it'}$ *depend only on the index* a *of the set* T_a *and not on the individual indices* t *or* t'*. If* $t, t' \in T_a^\circ$ *one furthermore has* $g_{ti} = -g_{it'}$*.*

For any $i < j$ *and every* $t, t' \in T_a$*:* $t < i$ *and* $t' > j$*,*

$$g_{ti} + \Lambda_{ij} = g_{tj} \quad \text{and} \quad g_{it'} = g_{jt'} + \Lambda_{ij}, \quad \text{where } \Lambda_{ij} := g_{ij} - g_{ji}. \tag{4.1.19}$$

(5) *Let* $|I| = 1$*. Denote the only index in* I *as* $\mathbf{i}$ *in order to stress that it is not a running index. For all* r in R_a *one has*

$$g_{ir} - g_{ri} = \Lambda_a. \tag{4.1.20}$$

Note that both the left- and the right-hand sides of Relation (4.1.20) *depend only on the index* a *of the set* R_a *and not on the individual index* r*.*

Diffusion algebras with N generators are listed as five families of algebras: A_I, A_{II}, B, C and D. As in the case of $N = 3$ the number of nonzero coefficients x_α, or equivalently, the cardinality of the set I is used as a criterion for separating diffusion algebras into families of the types $A(|I| \geq 3)$, $B(|I| \geq 2)$, $C(|I| = 1)$ or $D(|I| = 0)$. Type A algebras are separated further into two families A_I and A_{II} depending on the number of nonzero coefficients g_{ij} with indices i, j in the set I.

Next, we will see that different algebras in the families are obtained in dependence on the choice of the decomposition of the set $I_N = \{1, 2, \ldots, N\}$ into *ordered subsets* I, S, T_a° $(a = 1, \ldots, M_T^\circ), T_b^\bullet$ $(b = 1, \ldots, M_T^\bullet)$ (or R_a, $a = 1, \ldots, M_R$ for $N_I = 1$) as well as on the choice of coefficients in their defining relations. Next, we consider a notation for diffusion algebras where the corresponding decomposition of the set I_N is given explicitly as

argument to the family symbol. The subscript indices a and b in our notation are treated as running ones so that, e.g.

$$A_I(I, S, T^\circ_a, T^\bullet_b) \equiv A_I(I, S, T^\circ_a, \dots, T^\circ_{M^\circ_T}, T^\bullet_b, \dots, T^\bullet_{M^\bullet_T}),$$

where $I_N = I \cup S \cup \left(\bigcup_{a=1}^{M^\circ_T} T^\circ_a\right) \cup \left(\bigcup_{b=1}^{M^\bullet_T} T^\bullet_a\right)$, and I, S, T°_a and $T^\bullet_b$ are mutually nonintersecting ordered subsets in I_N. Notice that the values of the coefficients $g_{\alpha\beta}$ are not shown explicitly in these notations, so that this notation displays connective components in a variety of diffusion algebras rather than the particular algebras.

Next, relations in (4.1.21) - (4.1.25) below are to be complemented by relations (4.1.11), (4.1.12) for the elements of the subset R together with the conditions (4.1.13) or (4.1.14) on the coefficients involved.

Proposition 4.2 ([PT02, Theorem 3.5]). *The following list contains all possible diffusion algebras with N generators:*

(1) *Diffusion algebras of type $A_I(I, S, T^\circ_a, T^\bullet_b)$ with $|I| \geq 3$:*

$$\begin{aligned}
gD_iD_j - gD_jD_i &= x_jD_i - x_iD_j, \quad \text{for all } i, j \in I,\\
g_sD_sD_i - g_sD_iD_s &= x_iD_s, \quad \text{for all } s \in S, i \in I,\\
g^\circ_a : D_iD_t &:= -x_iD_t, \quad \text{for all } a, t \in T^\circ_a,\ i \in I,\\
g^+_bD_iD_t &= -x_iD_t, \quad \text{for all } b, t \in T^\bullet_b, \text{ and every } i < t,\\
g^-_bD_tD_i &= x_iD_t, \quad \text{for all } b, t \in T^\bullet_b, \text{ and every } i > t,
\end{aligned} \tag{4.1.21}$$

where $g, g_s, g^\circ_a, g^\pm_b \neq 0$.

(2) *Diffusion algebras of type $A_{II}(I, T^\circ_a, T^\bullet_b)$, $|I| \geq 3$:*

$$\begin{aligned}
(g_i - g_j)D_iD_j &= x_jD_i - x_iD_j, \quad \text{for all } i < j,\\
(g_i + g^\circ_a) : D_iD_t &:= -x_iD_t, \quad \text{for all } a, t \in T^\circ_a,\ i < t,\\
(g_i + g^+_a)D_iD_t &= -x_iD_t, \quad \text{for all } b, t \in T^\bullet_b,\ i < t,\\
(g^-_b - g_i) : D_tD_i &:= x_iD_t, \quad \text{for all } b, t \in T^\bullet_b,\ i > t,
\end{aligned} \tag{4.1.22}$$

where $g_i \neq g_j$ for $i \neq j$ and $g_i \notin \{g^\circ_a, \pm g^\pm_b\}$.

(3) *Diffusion algebras of type $B(I = \{\mathbf{i}, \mathbf{j}\}, S, T^\circ_a, T^\bullet_b)$: We use the notation $\mathbf{i}$ and $\mathbf{j}$ with $\mathbf{i} < \mathbf{j}$ for the two elements of the set I to emphasize that they are not running indices. Note that $\mathbf{i} < t < \mathbf{j}$ for all $t \in T^\bullet_b$ in this case.*

$$\begin{aligned}
gD_iD_j - (g - \Lambda)D_jD_i &= x_jD_i - x_iD_j,\\
g_sD_iD_s - (g_s - \Lambda)D_sD_i &= -x_iD_s, \quad \text{for all } s \in S,\\
g_sD_sD_j - (g_s - \Lambda)D_jD_s &= x_jD_s, \quad \text{for all } s \in S,\\
g^\circ_a : D_iD_t &:= -x_iD_t, \quad \text{for all } t \in T^\circ_a,\\
(g^\circ_a - \Lambda) : D_jD_t &:= -x_jD_t, \quad \text{for all } t \in T^\circ_a,\\
g^+_bD_iD_t &= -x_iD_t, \quad \text{for all } t \in T^\bullet_b,\\
g^-_bD_tD_j &= x_jD_t, \quad \text{for all } t \in T^\bullet_b,
\end{aligned} \tag{4.1.23}$$

where $g \neq 0$, $g_s \neq 0$ *for all* s *and* $g_s \neq \Lambda$ *for* s *such that either* $s < \mathbf{i}$ *or* $s > \mathbf{j}$, $g_a^\circ \notin \{0, \Lambda\}$ *and* $g_b^\pm \neq 0$.

(4) *Diffusion algebras of type* $C(I = \{\mathbf{i}\}, R_a)$*: The only element of* I *is denoted by* $\mathbf{i}$*, and hence*

$$g_r D_{\mathbf{i}} D_r - (g_r - \Lambda_a) D_r D_{\mathbf{i}} = -x_{\mathbf{i}} D_r, \quad \text{for all } r \in R_a, \tag{4.1.24}$$

where $g_r \neq 0$ *for* $r < \mathbf{i}$*, and* $g_r \neq \Lambda_a$ *for* $r > \mathbf{i}$.

(5) *Diffusion algebras of type* $D(R)$*:*

$$D_r D_s - q_{sr} D_s D_r = 0, \quad \text{for all } r, s \in R \text{ with } r < s. \tag{4.1.25}$$

Remark 16. (i) Hinchcliffe in his PhD thesis [Hin05, Definition 2.1.1] considered the following notation for diffusion algebras. Let R be the algebra generated by n indeterminates $x_1, x_2, \ldots, x_n$ over $\mathbb{C}$ subject to relations

$$a_{ij} x_i x_j - b_{ij} x_j x_i = r_j x_i - r_i x_j$$

whenever $i < j$, for some parameters $a_{ij} \in \mathbb{C} \setminus \{0\}$, for all $i < j$ and $b_{ij}, r_i \in \mathbb{C}$, for all $i < j$. He defined the *standard monomials* to be those of the form $x_n^{i_n} x_{n-1}^{i_{n-1}} \cdots x_2^{i_2} x_1^{i_1}$. R is called a *diffusion algebra* if it admits a *PBW basis of these standard monomials*. In other words, R is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for R. If all the elements $q_{ij} := \frac{b_{ij}}{a_{ij}}$'s are non-zero, then the diffusion algebras have a PBW basis in any order of the indeterminates [Hin05, Remark 2.1.6].

From his definition, a diffusion algebra generated by n indeterminates has Gelfand-Kirillov dimension n since because of the PBW basis, the vector subspace consisting of elements of total degree at most l is isomorphic to that of a commutative polynomial ring in n indeterminates. Notice that a diffusion algebra in one indeterminate is precisely a commutative polynomial ring in one indeterminate. A diffusion algebra with $x_t = 0$, for all $t = 1, \ldots, n$, is a *multiparameter quantum affine* $n-$*space*.

(ii) Fajardo et al. [FGL+20, Section 2.4] studied ring-theoretical properties of a graded version of these algebras. The *diffusion algebras type 2* are affine algebras $\mathcal{D}$ generated by $2n$ variables $\{D_1, \ldots, D_n, x_1, \ldots, x_n\}$ over $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $B_{\alpha_1}^{k_1} B_{\alpha_2}^{k_2} \cdots B_{\alpha_n}^{k_n}$ with $B_{\alpha_i} \in \{D_1, \ldots, D_n, x_1, \ldots, x_n\}$, for all $i \leq 2n$, $k_j \in \mathbb{N}$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, such that for all $1 \leq i < j \leq n$, there exist elements $\lambda_{ij} \in \Bbbk^*$ satisfying the relations

$$\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j. \tag{4.1.26}$$

Following Krebs and Sandow [KS97], the relations (4.1.26) are consequence of subtracting (quadratic) operator relations of the type

$$\Gamma_{\gamma\delta}^{\alpha\beta} D_\alpha D_\beta = D_\gamma X_\delta - X_\gamma D_\delta, \quad \text{for all } \gamma, \delta = 0, 1, \ldots, n-1,$$

where $\Gamma_{\gamma\delta}^{\alpha\beta} \in \Bbbk$, and D_i's and X_j's are operators of a particular vector space such that not necessarily $[D_i, X_j] = 0$ holds [KS97, p. 3168].

4.2 Differential smoothness of diffusion algebras

Throughout this section, $\mathcal{D}$ denotes a diffusion algebra on $N \geq 3$ generators.

4.2.1 Diffusion algebras on four generators

For the four-generator construction, the explicit algebras are given in Tables 4.1 and 4.2. It is worth noting that several of these algebras appear to be isomorphic upon some index change. However, isomorphism would have to be shown explicitly, and what is desired here is to obtain all possible combinations of algebras, even if some are isomorphic.

4.2.2 Diffusion algebras on five generators

As in Subsection 4.2.1, all possible diffusion algebras on five generators are shown. These can be found in Tables 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11 and 4.12.

4.2.3 Differential smoothness of diffusion algebras with N generators

If $\mathcal{D}$ is a diffusion algebra on N generators, then $\mathrm{GKdim}(\mathcal{D}) = N$ [Rey13, Theorems 4.14 and 4.18]. Our aim in this section is to find an integral calculus of degree N that guarantees the differentiable smoothness of $\mathcal{D}$.

Theorem 4.3. *Let $\mathcal{D}$ be a diffusion algebra on $N \geq 3$ generators. If $\mathcal{D}$ satisfies any one of the following properties:*

(i) $|I| = L$ *and* $|S| = N - L$, *for* $3 \leq L \leq N$;

(ii) $|I| = 1$, $|S| = N - 1$ *and* $g_s = \mathcal{G} \notin \{0, \Lambda_a\}$ *for all* $s \in S$;

(iii) $|I| = 2$, $|S| = N - 2$ *and* $g_s = \mathcal{G} \notin \{0, \Lambda\}$ *for all* $s \in S$; *or*

(iii) $|I| = 0$,

then $\mathcal{D}$ is diferentially smooth.

Proof. From Section 1.4.2 we know that we must to consider $\Omega^1(\mathcal{D})$, a free right $\mathcal{D}$-module of rank N with generators dD_i, $1 \leq i \leq N$. Define a left $\mathcal{D}$-module structure by

$$p\,dD_a = dD_a \nu_{D_a}(p), \quad \text{for } a \in I \cup S,\ p \in \mathcal{D}, \tag{4.2.1}$$

where ν_{D_a} are algebra automorphisms of $\mathcal{D}$.

(i) Consider the following maps:

$$\nu_{D_i}(D_j) = D_j - g^{-1}x_j, \text{ for } i,j \in I, \tag{4.2.2}$$
$$\nu_{D_i}(D_s) = D_s, \text{ for } i \in I,\ s \in S \tag{4.2.3}$$
$$\nu_{D_s}(D_i) = D_i - g_s^{-1}x_i, \text{ for } i \in I,\ s \in S, \quad \text{and} \tag{4.2.4}$$
$$\nu_{D_s}(D_l) = D_l, \text{ for } s,l \in S. \tag{4.2.5}$$

It can be seen that the maps ν_{D_i} and ν_{D_s} for each $i \in I$ and $s \in S$ can be extended to algebra homomorphisms of $\mathcal{D}$. As a matter of fact, these maps respect the relations (4.1.21) for $\mathcal{D}$ and commute with each other.

Consider $\Omega^1(\mathcal{D})$ a free right $\mathcal{D}$-module of rank N with generators dD_i, dD_s for all $i \in I$ and $s \in S$. For every element $p \in \mathcal{D}$ define a left $\mathcal{D}$-module structure by

$$pdD_a = dD_a\nu_{D_a}(p), \quad \text{with } a \in I \cup S. \tag{4.2.6}$$

The relations in $\Omega^1(\mathcal{D})$ are given by

$$\begin{aligned} D_i dD_j &= dD_j(D_i - g^{-1}x_i), \quad i,j \in I, \\ D_i dD_s &= dD_s(D_i - g_s^{-1}x_i), \quad i \in I, s \in S, \\ D_s dD_i &= dD_i D_s, \quad i \in I, s \in S, \text{ and} \\ D_s dD_l &= dD_l D_s, \quad s,l \in S. \end{aligned} \tag{4.2.7}$$

We want to extend the correspondences

$$D_a \mapsto dD_a, \quad \text{for every } a \in I \cup S,$$

to a map $d : \mathcal{D} \to \Omega^1(\mathcal{D})$ satisfying the Leibniz's rule. This is possible if it is compatible with the nontrivial relations (4.1.21), i.e. if the following two equalities

$$\begin{aligned} gdD_iD_j + gD_idD_j - gdD_jD_i - gD_jdD_i &= x_jdD_i - x_idD_j, \quad \text{for } i,j \in I, \text{ and} \\ g_sdD_sD_i + g_sD_sdD_i - g_sdD_iD_s - g_sD_idD_s &= x_idD_s, \quad \text{for } i \in I,\ s \in S, \end{aligned}$$

hold.

Define $\Bbbk$-linear maps

$$\partial_{D_a} : \mathcal{D} \to \mathcal{D}, \quad \text{for every } a \in I \cup S,$$

such that

$$d(p) = \sum_{a \in I \cup S} dD_a\partial_{D_a}(p), \quad \text{for all } p \in \mathcal{D}.$$

Since dD_a with $a \in I \cup S$ are free generators of the right $\mathcal{D}$-module $\Omega^1(\mathcal{D})$, these maps are well-defined. Note that $d(p) = 0$ if and only if $\partial_{D_a}(p) = 0$ for each $a \in I \cup S$.

By using the relations in (4.2.7) and the definitions of the maps ν_{D_a} with $a \in I \cup S$, we get that

$$\partial_{D_a}(D_1^{k_1} \cdots D_N^{k_N}) = k_a(D_1 - g_a^{-1}x_1)^{k_1} \cdots (D_{a-1} - g_a^{-1}x_{a-1})^{k_{a-1}} D_a^{k_a-1} \cdots D_N^{k_N},$$

where $g_a^{-1} = g^{-1}$ if $a \in I$, and $g_a^{-1} = g_s^{-1}$ if $r = s \in S$. Thus, $d(p) = 0$ if and only if p is a scalar multiple of the identity. This shows that $(\Omega\mathcal{D}, d)$ is connected with $\Omega(\mathcal{D}) = \bigoplus_{k=0}^{n-1} \Omega^k(\mathcal{D})$.

The universal extension of d to higher forms compatible with (4.2.6) gives the following rules for $\Omega^k(\mathcal{D})$ $(2 \leq k \leq N-1)$:

$$\bigwedge_{r=1}^{k} dD_{q(r)} = (-1)^{\sharp} \bigwedge_{r=1}^{k} dD_{p(r)}, \tag{4.2.8}$$

where $q : \{1, \ldots, k\} \to \{1, \ldots, N\}$ is an injective map, $p : \{1, \ldots, k\} \to \mathrm{Im}(q)$ is an increasing injective map and $\sharp$ is the number of 2-permutation needed to transform q into p.

By using that the automorphisms ν_{D_i} and ν_{D_s}, for every $i \in I$ and $s \in S$, commute with each other, there are no additional relationships to the previous ones. In this way,

$$\begin{aligned}\Omega^{N-1}(\mathcal{D}) = \ & [dD_1 \wedge dD_2 \wedge \cdots \wedge dD_{N-1} \oplus dD_1 \wedge dD_3 \wedge \cdots \wedge dD_N \\ & \oplus \cdots \oplus \ dD_2 \wedge \cdots \wedge dD_N]\,\mathcal{D}.\end{aligned}$$

Since $\Omega^N(\mathcal{D}) = \omega\mathcal{D} \cong \mathcal{D}$ as a right and left $\mathcal{D}$-module with

$$\omega = dD_1 \wedge \cdots \wedge dD_N \quad \text{and} \quad \nu_\omega = \nu_{D_1} \circ \cdots \circ \nu_{D_N}$$

we have that ω is a volume form of $\mathcal{D}$. From Proposition 1.7 (2) we obtain that ω is an integral form by setting

$$\omega_i^j = \bigwedge_{k=1}^{j} dD_{p_{i,j}(k)}, \quad \text{for } 1 \leq i \leq \binom{N}{j}, \quad \text{and}$$

$$\bar{\omega}_i^{N-j} = (-1)^{\sharp_i} \bigwedge_{k=j+1}^{n} dD_{\bar{p}_{i,j}(k)}, \quad \text{for } 1 \leq i \leq \binom{N}{j},$$

where $p_{i,j} : \{1, \ldots, j\} \to \{1, \ldots, N\}$ is an increasing injective map, $\bar{p}_{i,j} : \{j+1, \ldots, N\} \to (\mathrm{Im}(p_{i,j}))^c$ is also an increasing injective map and $\sharp_{i,j}$ is the number of 2-permutation needed to transform $\{\bar{p}_{i,j}(j+1), \ldots, \bar{p}_{i,j}(N), p_{i,j}(1), \ldots, p_{i,j}(j)\}$ into $\{1, \ldots, N\}$.

Let $\omega' \in \Omega^j(\mathcal{D})$. Then:

$$\omega' = \sum_{i=1}^{\binom{N}{j}} \bigwedge_{k=1}^{j} dD_{p_{i,j}(k)} a_i, \quad \text{with } a_i \in \Bbbk.$$

This implies that we have the equalities given by

$$\begin{aligned}\sum_{i=1}^{\binom{N}{j}} \omega_i^j \pi_\omega(\bar{\omega}_i^{N-j} \wedge \omega') &= \sum_{i=1}^{\binom{N}{j}} \bigwedge_{k=1}^{j} dD_{p_{i,j}(k)} \pi_\omega \left(a_i (-1)^{\sharp_{i,j}} \bigwedge_{k=j+1}^{N} dD_{\bar{p}_{i,j}(k)} \wedge \bigwedge_{k=1}^{j} dD_{p_{i,j}(k)} \right) \\ &= \sum_{i=1}^{\binom{N}{j}} \bigwedge_{k=1}^{j} dx_{p_{i,j}(k)} a_i \\ &= \omega'.\end{aligned}$$

By Proposition 1.7 (2) it follows that $\mathcal{D}$ is differentially smooth.

(ii) Consider the maps given by

$$\begin{aligned}\nu_{D_i}(D_i) &= D_i, \\ \nu_{D_i}(D_s) &= \mathcal{G}(\mathcal{G} - \Lambda_a)^{-1} D_s, \qquad (4.2.9) \\ \nu_{D_s}(D_i) &= \mathcal{G}^{-1}((\mathcal{G} - \Lambda_a)D_i - x_i), \\ \nu_{D_s}(D_l) &= D_l, \quad \text{for } s, l \in S. \qquad (4.2.10)\end{aligned}$$

It is straightforward to show that the maps ν_{D_i} and ν_{D_s} for each $s \in S$ can be extended to algebra homomorphisms of $\mathcal{D}$ and respect the relations (4.1.24) for $\mathcal{D}$. Again, these maps commute with each other.

Consider $\Omega^1(\mathcal{D})$ a free right $\mathcal{D}$-module of rank N with generators dD_i for every dD_s and $s \in S$. For all $p \in \mathcal{D}$ define a left $\mathcal{D}$-module structure by

$$p dD_a = dD_a \nu_{D_a}(p), \text{for every } a \in \{i\} \cup S. \qquad (4.2.11)$$

The relations in $\Omega^1(\mathcal{D})$ are given by

$$\begin{aligned}D_i dD_s &= dD_s \mathcal{G}^{-1}((\mathcal{G} - \Lambda_a)D_i - x_i), \quad s \in S, \\ D_s dD_i &= dD_i \mathcal{G}(\mathcal{G} - \Lambda_a)^{-1} D_s, \quad s \in S, \text{ and} \\ D_s dD_l &= dD_l D_s, \quad s, l \in S. \qquad (4.2.12)\end{aligned}$$

We want to extend the correspondences

$$D_a \mapsto dD_a, \quad \text{with } a \in \{i\} \cup S$$

to a map $d : \mathcal{D} \to \Omega^1(\mathcal{D})$ satisfying the Leibniz's rule. By using the relations (4.1.24) the equality

$$\mathcal{G} dD_i D_s + \mathcal{G} D_i dD_s - (\mathcal{G} - \Lambda_a) dD_s D_i - (\mathcal{G} - \Lambda_a) D_s dD_i = -x_i dD_s, \quad \text{for all } s \in S,$$

must be satisfied.

Define $\Bbbk$-linear maps

$$\partial_{D_a} : \mathcal{D} \to \mathcal{D}, \quad \text{for each } a \in \{i\} \cup S,$$

such that

$$\mathrm{d}(p) = \sum_{a \in \{\mathbf{i}\} \cup S} \mathrm{d}D_a \partial_{D_a}(p), \quad \text{for all } p \in \mathcal{D}.$$

Since $\mathrm{d}D_a$, $a \in \{\mathbf{i}\} \cup S$, are free generators of the right $\mathcal{D}$-module $\Omega^1(\mathcal{D})$, these maps are well-defined, and $\mathrm{d}(p) = 0$ if and only if $\partial_{D_a}(p) = 0$, $a \in \{\mathbf{i}\} \cup S$. By using the relations in (4.2.12) and the definitions of the maps ν_{D_a}, $a \in \{\mathbf{i}\} \cup S$, we get that

$$\begin{aligned}
\partial_{D_i}(D_1^{k_1} \cdots D_i^{k_i} \cdots D_N^{k_N}) &= \prod_{j=1}^{i-1} \mathcal{G}^{k_j} (\mathcal{G} - \Lambda_a)^{-k_j} k_i D_1^{k_1} \cdots D_i^{k_i - 1} \cdots D_N^{k_N}, \\
\partial_{D_s}(D_1^{k_1} \cdots D_i^{k_i} \cdots D_N^{k_N}) &= k_s D_1^{k_1} \cdots D_s^{k_s - 1} \cdots D_N^{k_N}, \text{ for } s < \mathbf{i}, \\
\partial_{D_s}(D_1^{k_1} \cdots D_i^{k_i} \cdots D_N^{k_N}) &= k_s \mathcal{G}^{-k_i} D_1^{k_1} \cdots ((\mathcal{G} - \Lambda_a) D_i - x_i)^{k_i} \cdots D_s^{k_s - 1} \cdots D_N^{k_N}, \text{ for } s > \mathbf{i},
\end{aligned}$$

Thus $\mathrm{d}(p) = 0$ if and only if p is a scalar multiple of the identity, whence shows $(\Omega\mathcal{D}, \mathrm{d})$ is connected where $\Omega(\mathcal{D}) = \bigoplus_{k=0}^{n-1} \Omega^k(\mathcal{D})$.

From this treatment, the rest of the proof is completely analogous to case (i). Thus, $\mathcal{D}$ is differentially smooth.

(iii) For all elements $s, l \in S$, consider the following maps

$$\begin{aligned}
\nu_{D_i}(D_i) &= g^{-1}((g - \Lambda)D_i - x_i), \\
\nu_{D_i}(D_j) &= (g - \Lambda)^{-1}(gD_j - x_j), \\
\nu_{D_i}(D_s) &= (\mathcal{G} - \Lambda)^{-1}\mathcal{G}D_s, \\
\nu_{D_j}(D_i) &= g^{-1}((g - \Lambda)D_i - x_i), \\
\nu_{D_j}(D_j) &= g^{-1}((g - \Lambda)D_j - x_j), \\
\nu_{D_j}(D_s) &= (\mathcal{G} - \Lambda)^{-1}\mathcal{G}D_s, \\
\nu_{D_s}(D_i) &= \mathcal{G}^{-1}((\mathcal{G} - \Lambda)D_i - x_i), \\
\nu_{D_s}(D_j) &= (\mathcal{G} - \Lambda)^{-1}(\mathcal{G}D_j - x_j), \quad \text{and} \\
\nu_{D_s}(D_l) &= D_l.
\end{aligned}$$

Notice that these morphisms are similar to those corresponding in case (ii), so by using a similar reasoning we can prove that all of them guarantee that $\mathcal{D}$ is differentially smooth.

(iv) In this case, the algebra $\mathcal{D}$ is precisely the *quantum affine* N*-space*, whence its differential smoothness follows from [KL14, Corollary 6 and Theorem 9] or [BL18, Corollary 4.9].

□

Following an argument similar to the one presented by Brzeziński and Sitarz [BS17, Example 2.5], we obtain the following.

Theorem 4.4. *If* $|T| \neq 0$*, then* $\mathcal{D}$ *is not differentially smooth.*

Proof. (i) Suppose that $|T^{\circ}| \geq 1$. We can write the relations relevant to T° in the unique form given by

$$\mathcal{G} : D_i D_t := -x_i D_t, \quad i \in I,\ t \in T_a^{\circ}, \tag{4.2.13}$$

where $\mathcal{G} \in \{g_a^{\circ}, g_i + g_a^{\circ}, g_a^{\circ} - \Lambda\}$ depending on the case of Proposition (4.2).

By applying the differential d to the expression (4.2.13) we get that

$$\mathcal{G} d(: D_i D_t :) = d(-x_i D_t), \quad i \in I,\ t \in T_a^{\circ}.$$

If $: D_i D_t := D_i D_t$, by using that d is $\Bbbk$-linear the Leibniz's rule yields that

$$\mathcal{G} dD_i D_t + \mathcal{G} D_i dD_t + x_i dD_t = 0, \quad i \in I, t \in T_a^{\circ}.$$

By (1.4.2) the action of the module is written using automorphism ν_{D_t}

$$\mathcal{G} dD_i D_t + \mathcal{G} dD_t \nu_{D_t}(D_i) + x_i dD_t = 0, \quad i \in I, t \in T_a^{\circ}.$$

In this case, we have that

$$\mathcal{G} D_t = 0, \quad t \in T_a^{\circ},$$

which occurs only if $\mathcal{G} = 0$. This fact contradicts any of the cases in Proposition (4.2).

On the other hand, if $: D_i D_t := D_t D_i$, using an argument similar to the previous one, we obtain that

$$\mathcal{G} \nu_{D_i}(D_t) = 0, \quad i \in I, t \in T_a^{\circ}.$$

If $\mathcal{G} = 0$, once more again this contradicts any of the cases of the Proposition (4.2). On the other hand, if $\nu_{D_i}(D_t) = 0$, this contradicts the fact that ν_{D_i} is an automorphism.

(ii) Suppose that $|T^{\bullet}| \geq 1$. We can write the relations relevant to $T^{\bullet}$ in the following unique form

$$\mathcal{G} : D_i D_t := \operatorname{sgn}(i - t) x_i D_t, \quad i \in I,\ t \in T_b^{\bullet}, \tag{4.2.14}$$

where $\mathcal{G} \in \{g_b^+, g_b^-, g_i + g_b^+, g_b^- - g_i\}$, depending on the case of Proposition (4.2). Here, the proof is exactly the same as in the expression (4.2.13): we only need to change the sign of $i - t$.

□

Table 4.1: Diffusion algebras with four generators

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_1 - gD_1D_2 = x_2D_1 - x_1D_2$, $gD_3D_1 - gD_1D_3 = x_3D_1 - x_1D_3$, $gD_3D_2 - gD_2D_3 = x_3D_2 - x_2D_3$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_1D_4$	$I = \{1,2,3\}$, $S = \{4\}$	$g, g_4 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_3D_1 - x_1D_3$, $gD_4D_1 - gD_1D_4 = x_4D_1 - x_1D_4$, $gD_4D_3 - gD_3D_4 = x_4D_3 - x_3D_4$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_3D_2 - g_2D_2D_3 = x_3D_2$, $g_2D_4D_2 - g_2D_2D_4 = x_4D_2$	$I = \{1,3,4\}$, $S = \{2\}$	$g, g_2 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_2D_1 - x_1D_2$, $gD_4D_1 - gD_1D_4 = x_4D_1 - x_1D_4$, $gD_4D_3 - gD_3D_4 = x_4D_3 - x_3D_4$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$	$I = \{1,2,4\}$, $S = \{3\}$	$g, g_3 \neq 0$
	$gD_3D_2 - gD_2D_3 = x_3D_2 - x_2D_3$, $gD_4D_2 - gD_2D_4 = x_4D_2 - x_2D_4$, $gD_4D_3 - gD_3D_4 = x_4D_3 - x_3D_4$, $g_1D_2D_1 - g_1D_1D_2 = x_2D_1$, $g_1D_3D_1 - g_1D_1D_3 = x_3D_1$, $g_1D_4D_1 - g_1D_1D_4 = x_4D_1$	$I = \{2,3,4\}$, $S = \{1\}$	$g, g_1 \neq 0$
A_{II}	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2$, $(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3$, $(g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4$, $(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3$, $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4$, $(g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4$	$I = \{1,2,3,4\}$	$g_i \neq g_j$ for $i,j \in I$, $i \neq j$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2$, $(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3$, $(g_2 - g_3)D_3D_2 = x_3D_2 - x_2D_3$, $(g_1 + g_1^\circ)D_1D_4 = -x_1D_4$, $(g_2 + g_1^\circ)D_2D_4 = -x_2D_4$, $(g_3 + g_1^\circ)D_3D_4 = -x_3D_4$	$I = \{1,2,3\}$, $T_a^\circ = \{4\}$	$g_i \neq g_j$ for $i,j \in I$, $i \neq j$, $g_i \neq g_1^\circ$
	$(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3$, $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4$, $(g_3 - g_4)D_4D_3 = x_4D_3 - x_3D_4$, $(g_2 + g_1^\circ)D_1D_2 = -x_2D_1$, $(g_3 + g_1^\circ)D_1D_3 = -x_3D_1$, $(g_4 + g_1^\circ)D_1D_4 = -x_4D_1$	$I = \{2,3,4\}$, $T_a^\circ = \{1\}$	$g_i \neq g_j$ for $i,j \in I$, $i \neq j$, $g_i \neq g_1^\circ$
	$(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3$, $(g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4$, $(g_3 - g_4)D_4D_3 = x_4D_3 - x_3D_4$, $(g_1 + g_1^+)D_1D_2 = -x_1D_2$, $(g_1^- - g_3)D_2D_3 = x_3D_2$, $(g_1^- - g_4)D_2D_4 = x_4D_2$	$I = \{1,3,4\}$, $T_b^\bullet = \{2\}$	$g_i \neq g_j$ for $i,j \in I$, $i \neq j$, $g_i \neq \mp g_1^\pm$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2$, $(g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4$, $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4$, $(g_1 + g_1^+)D_1D_3 = -x_1D_3$, $(g_2 + g_1^-)D_2D_3 = -x_2D_3$, $(g_1^- - g_4)D_3D_4 = x_4D_3$	$I = \{1,2,4\}$, $T_b^\bullet = \{3\}$	$g_i \neq g_j$ for $i,j \in I$, $i \neq j$, $g_i \neq \mp g_1^\pm$
B	$gD_1D_2 - (g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3$, $g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4$, $g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3$, $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4$	$I = \{1,2\}$, $S = \{3,4\}$	$g \neq 0$, $g_s \notin \{0, \Lambda\}$, $s \in S$
	$gD_1D_3 - (g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2$, $g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4$, $g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2$, $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4$	$I = \{1,3\}$, $S = \{2,4\}$	$g \neq 0$, $g_2 \neq 0$, $g_4 \notin \{0, \Lambda\}$
	$gD_1D_4 - (g-\Lambda)D_4D_1 = x_4D_1 - x_1D_4$, $g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3$, $g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2$, $g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3$, $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2$	$I = \{1,4\}$, $S = \{2,3\}$	$g \neq 0$, $g_s \neq 0$, $s \in S$
	$gD_2D_3 - (g-\Lambda)D_3D_2 = x_3D_2 - x_2D_3$, $g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1$, $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4$, $g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1$, $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4$	$I = \{2,3\}$, $S = \{1,4\}$	$g \neq 0$, $g_s \notin \{0, \Lambda\}$, $s \in S$
	$gD_2D_4 - (g-\Lambda)D_4D_2 = x_4D_2 - x_2D_4$, $g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1$, $g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3$, $g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1$, $g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3$	$I = \{2,4\}$, $S = \{1,3\}$	$g \neq 0$, $g_3 \neq 0$, $g_1 \notin \{0, \Lambda\}$
	$gD_3D_4 - (g-\Lambda)D_4D_3 = x_4D_3 - x_3D_4$, $g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1$, $g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2$, $g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1$, $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2$	$I = \{3,4\}$, $S = \{1,2\}$	$g \neq 0$, $g_s \notin \{0, \Lambda\}$, $s \in S$
	$gD_1D_2 - (g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3$, $g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ - \Lambda)D_2D_4 = -x_2D_4$,	$I = \{1,2\}$, $S = \{3\}$, $T_a^\circ = \{4\}$	$g \neq 0$, $g_3, g_a^\circ \notin \{0, \Lambda\}$
	$gD_1D_2 - (g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4$, $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4$, $g_a^\circ D_1D_3 = -x_1D_3$, $(g_a^\circ - \Lambda)D_2D_3 = -x_2D_3$,	$I = \{1,2\}$, $S = \{4\}$, $T_a^\circ = \{3\}$	$g \neq 0$, $g_4, g_a^\circ \notin \{0, \Lambda\}$
	$gD_1D_3 - (g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2$, $g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ - \Lambda)D_3D_4 = -x_3D_4$,	$I = \{1,3\}$, $S = \{2\}$, $T_a^\circ = \{4\}$	$g \neq 0$, $g_2 \neq 0$, $g_a^\circ \notin \{0, \Lambda\}$
	$gD_2D_3 - (g-\Lambda)D_3D_2 = x_3D_2 - x_2D_3$, $g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1$, $g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1$, $g_a^\circ D_2D_4 = -x_2D_4$, $(g_a^\circ - \Lambda)D_3D_4 = -x_3D_4$,	$I = \{2,3\}$, $S = \{1\}$, $T_a^\circ = \{4\}$	$g \neq 0$, $g_1, g_a^\circ \notin \{0, \Lambda\}$
	$gD_2D_3 - (g-\Lambda)D_3D_2 = x_3D_2 - x_2D_3$, $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4$, $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4$, $g_a^\circ D_1D_2 = -x_2D_1$, $(g_a^\circ - \Lambda)D_1D_3 = -x_3D_1$,	$I = \{2,3\}$, $S = \{4\}$, $T_a^\circ = \{1\}$	$g \neq 0$, $g_4, g_a^\circ \notin \{0, \Lambda\}$
	$gD_2D_4 - (g-\Lambda)D_4D_2 = x_4D_2 - x_2D_4$, $g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3$, $g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3$, $g_a^\circ D_1D_2 = -x_2D_1$, $(g_a^\circ - \Lambda)D_1D_4 = -x_4D_1$,	$I = \{2,4\}$, $S = \{3\}$, $T_a^\circ = \{1\}$	$g \neq 0$, $g_3 \neq 0$, $g_a^\circ \notin \{0, \Lambda\}$
	$gD_3D_4 - (g-\Lambda)D_4D_3 = x_4D_3 - x_3D_4$, $g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1$, $g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1$, $g_a^\circ D_2D_3 = -x_3D_2$, $(g_a^\circ - \Lambda)D_2D_4 = -x_4D_2$,	$I = \{3,4\}$, $S = \{1\}$, $T_a^\circ = \{2\}$	$g \neq 0$, $g_1, g_a^\circ \notin \{0, \Lambda\}$
	$gD_3D_4 - (g-\Lambda)D_4D_3 = x_4D_3 - x_3D_4$, $g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2$, $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2$, $g_a^\circ D_1D_3 = -x_3D_1$, $(g_a^\circ - \Lambda)D_1D_4 = -x_4D_1$,	$I = \{3,4\}$, $S = \{2\}$, $T_a^\circ = \{1\}$	$g \neq 0$, $g_2, g_a^\circ \notin \{0, \Lambda\}$
	$gD_1D_3 - (g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4$, $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4$, $g_b^+ D_1D_2 = -x_1D_2$, $g_b^- D_2D_3 = x_2D_3$,	$I = \{1,3\}$, $S = \{4\}$, $T_b^\bullet = \{2\}$	$g \neq 0$, $g_4 \notin \{0, \Lambda\}$, $g_b^\pm \neq 0$

Table 4.2: Diffusion algebras with four generators

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_4D_2-(g_2-\Lambda)D_2D_4=-x_4D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{1,4\}$, $S=\{2\}$, $T_b^\bullet=\{3\}$	$g\neq 0$, $g_2\neq 0$, $g_b^\pm\neq 0$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_4D_3-(g_3-\Lambda)D_3D_4=-x_4D_3,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_4=x_4D_2,$	$I=\{1,4\}$, $S=\{3\}$, $T_b^\bullet=\{2\}$	$g\neq 0$, $g_3\neq 0$, $g_b^\pm\neq 0$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1,$ $g_1D_4D_1-(g_1-\Lambda)D_1D_4=-x_4D_1,\quad g_b^+D_2D_3=-x_2D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{2,4\}$, $S=\{1\}$, $T_b^\bullet=\{3\}$	$g\neq 0$, $g_1\notin\{0,\Lambda\}$, $g_b^\pm\neq 0$
	$gD_1D_2-(g-\Lambda)D_2D_1=x_1D_2-x_2D_1,\quad g_a^\circ D_1D_3=-x_1D_3,$ $g_a^\circ D_1D_4=-x_1D_4,\quad (g_a^\circ-\Lambda)D_2D_3=-x_2D_3,$ $(g_a^\circ-\Lambda)D_2D_4=-x_2D_4,$	$I=\{1,2\}$, $T_a^\circ=\{3,4\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1=x_1D_3-x_3D_1,\quad g_a^\circ D_1D_4=-x_1D_4,$ $(g_a^\circ-\Lambda)D_3D_4=-x_3D_4,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_3=x_3D_2,$	$I=\{1,3\}$, $T_b^\bullet=\{2\}$, $T_a^\circ=\{4\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$, $g_b^\pm\neq 0$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_1D_4-x_4D_1,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_4=x_4D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{1,4\}$, $T_b^\bullet=\{2,3\}$	$g\neq 0$, $g_b^\pm\neq 0$
	$gD_2D_3-(g-\Lambda)D_3D_2=x_2D_3-x_3D_2,\quad g_a^\circ D_1D_2=-x_1D_2,$ $g_a^\circ D_1D_3=-x_1D_3,\quad (g_a^\circ-\Lambda)D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_3D_4=-x_3D_4,$	$I=\{2,3\}$, $T_a^\circ=\{1,4\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_2D_4-x_4D_2,\quad g_a^\circ D_1D_2=-x_2D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,\quad g_b^+D_2D_3=-x_2D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{2,4\}$, $T_a^\circ=\{1\}$, $T_b^\bullet=\{3\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$, $g_b^\pm\neq 0$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_3D_4-x_4D_3,\quad g_a^\circ D_1D_3=-x_3D_1,$ $g_a^\circ D_2D_3=-x_3D_2,\quad (g_a^\circ-\Lambda)D_1D_4=-x_4D_1,$ $(g_a^\circ-\Lambda)D_2D_4=-x_4D_2,$	$I=\{3,4\}$, $T_a^\circ=\{1,2\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
C	$g_2D_1D_2-(g_2-\Lambda_a)D_2D_1=-x_1D_2,\quad g_3D_1D_3-(g_3-\Lambda_a)D_3D_1=-x_1D_3,$ $g_4D_1D_4-(g_4-\Lambda_a)D_4D_1=-x_1D_4,$	$I=\{1\}$, $R_a=\{2,3,4\}$	$g_2,g_3,g_4\neq\Lambda_a$
	$g_1D_2D_1-(g_1-\Lambda_a)D_1D_2=-x_2D_1,\quad g_3D_2D_3-(g_3-\Lambda_a)D_3D_2=-x_2D_3,$ $g_4D_2D_4-(g_4-\Lambda_a)D_4D_2=-x_2D_4,$	$I=\{2\}$, $R_a=\{1,3,4\}$	$g_1\neq 0$, $g_3,g_4\neq\Lambda_a$
	$g_1D_3D_1-(g_1-\Lambda_a)D_1D_3=-x_3D_1,\quad g_2D_3D_2-(g_2-\Lambda_a)D_2D_3=-x_3D_2,$ $g_4D_3D_4-(g_4-\Lambda_a)D_4D_3=-x_3D_4,$	$I=\{3\}$, $R_a=\{1,2,4\}$	$g_1,g_2\neq 0$, $g_4\neq\Lambda_a$
	$g_1D_4D_1-(g_1-\Lambda_a)D_1D_4=-x_4D_1,\quad g_2D_4D_2-(g_2-\Lambda_a)D_2D_4=-x_4D_2,$ $g_3D_4D_3-(g_3-\Lambda_a)D_3D_4=-x_4D_3,$	$I=\{4\}$, $R_a=\{1,2,3\}$	$g_1,g_2,g_3\neq 0$
D	$D_1D_2-q_{21}D_2D_1=0,\quad D_1D_3-q_{31}D_3D_1=0,$ $D_1D_4-q_{41}D_4D_1=0,\quad D_2D_3-q_{32}D_3D_2=0,$ $D_2D_4-q_{42}D_4D_2=0,\quad D_3D_4-q_{43}D_4D_3=0$	$R=\{1,2,3,4\}$	$q_{21},q_{31},q_{41},q_{32},q_{42},q_{43}\neq 0$

Table 4.3: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$	$I = \{1,2,3,4\}$, $S = \{5\}$	$g, g_5 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$	$I = \{1,2,3,5\}$, $S = \{4\}$	$g, g_4 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$	$I = \{1,2,4,5\}$, $S = \{3\}$	$g, g_3 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3$, $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$	$I = \{1,3,4,5\}$, $S = \{2\}$	$g, g_2 \neq 0$
	$gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3$, $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$, $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$	$I = \{2,3,4,5\}$, $S = \{1\}$	$g, g_1 \neq 0$
A_I	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$, $g_a^\circ D_5D_1 = -x_1D_5$, $g_a^\circ D_5D_2 = -x_2D_5$, $g_a^\circ D_5D_3 = -x_3D_5$	$I = \{1,2,3\}$, $S = \{4\}$, $T_a^\circ = \{5\}$	$g, g_4, g_a^\circ \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_a^\circ D_4D_1 = -x_1D_4$, $g_a^\circ D_4D_2 = -x_2D_4$, $g_a^\circ D_4D_3 = -x_3D_4$	$I = \{1,2,3\}$, $S = \{5\}$, $T_a^\circ = \{4\}$	$g, g_5, g_a^\circ \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$	$I = \{1,2,3\}$, $S = \{4,5\}$	$g, g_4, g_5 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$, $g_a^\circ D_5D_1 = -x_1D_5$, $g_a^\circ D_5D_2 = -x_2D_5$, $g_a^\circ D_5D_4 = -x_4D_5$	$I = \{1,2,4\}$, $S = \{3\}$, $T_a^\circ = \{5\}$	$g, g_3, g_a^\circ \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$, $g_b^+ D_1D_3 = -x_1D_3$, $g_b^+ D_2D_3 = -x_2D_3$, $g_b^- D_3D_4 = x_4D_3$	$I = \{1,2,4\}$, $S = \{5\}$, $T_b^\bullet = \{3\}$	$g, g_5, g_b^\pm \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$	$I = \{1,2,4\}$, $S = \{3,5\}$	$g, g_3, g_5 \neq 0$

Table 4.4: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$, $g_b^+D_1D_4 = -x_1D_4$, $g_b^+D_2D_4 = -x_2D_4$, $g_b^-D_4D_5 = x_5D_4$	$I = \{1,2,5\}$, $S = \{3\}$, $T_b^\bullet = \{4\}$	$g, g_3, g_b^\pm \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$, $g_b^+D_1D_3 = -x_1D_3$, $g_b^+D_2D_3 = -x_2D_3$, $g_b^-D_3D_5 = x_5D_3$	$I = \{1,2,5\}$, $S = \{4\}$, $T_b^\bullet = \{3\}$	$g, g_4, g_b^\pm \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$	$I = \{1,2,5\}$, $S = \{3,4\}$	$g, g_3, g_4 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$, $g_a^\circ D_5D_1 = -x_1D_5$, $g_a^\circ D_5D_3 = -x_3D_5$, $g_a^\circ D_5D_4 = -x_4D_5$	$I = \{1,3,4\}$, $S = \{2\}$, $T_a^\circ = \{5\}$	$g, g_2, g_a^\circ \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$, $g_b^+D_1D_2 = -x_1D_2$, $g_b^-D_2D_3 = x_3D_2$, $g_b^-D_2D_4 = x_4D_2$	$I = \{1,3,4\}$, $S = \{5\}$, $T_b^\bullet = \{2\}$	$g, g_5, g_b^\pm \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3$, $g_5D_5D_1 - g_5D_1D_5 = x_1D_5$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$	$I = \{1,3,4\}$, $S = \{2,5\}$	$g, g_2, g_5 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$, $g_b^+D_1D_4 = -x_1D_4$, $g_b^+D_3D_4 = -x_3D_4$, $g_b^-D_4D_5 = x_5D_4$	$I = \{1,3,5\}$, $S = \{2\}$, $T_b^\bullet = \{4\}$	$g, g_2, g_b^\pm \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$, $g_b^+D_1D_2 = -x_1D_2$, $g_b^-D_2D_3 = x_3D_2$, $g_b^-D_2D_5 = x_5D_2$	$I = \{1,3,5\}$, $S = \{4\}$, $T_b^\bullet = \{2\}$	$g, g_4, g_b^\pm \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3$, $g_4D_4D_1 - g_4D_1D_4 = x_1D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$	$I = \{1,3,5\}$, $S = \{4\}$, $T_b^\bullet = \{2\}$	$g, g_4, g_b^\pm \neq 0$
	$gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$, $g_b^+D_1D_3 = -x_1D_3$, $g_b^-D_3D_4 = x_4D_3$, $g_b^-D_3D_5 = x_5D_3$	$I = \{1,4,5\}$, $S = \{2\}$, $T_b^\bullet = \{3\}$	$g, g_2, g_b^\pm \neq 0$
	$gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$, $g_b^+D_1D_2 = -x_1D_2$, $g_b^-D_2D_4 = x_4D_2$, $g_b^-D_2D_5 = x_5D_2$	$I = \{1,4,5\}$, $S = \{3\}$, $T_b^\bullet = \{2\}$	$g, g_3, g_b^\pm \neq 0$
	$gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1$, $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1$, $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $g_3D_3D_1 - g_3D_1D_3 = x_1D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$, $g_2D_2D_1 - g_2D_1D_2 = x_1D_2$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$	$I = \{1,4,5\}$, $S = \{2,3\}$	$g, g_2, g_3 \neq 0$

Table 4.5: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_4 - gD_4D_2 = x_4D_2 - x_2D_4$, $gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$, $g_a^\circ D_4D_5 = -x_4D_5$, $g_a^\circ D_2D_5 = -x_2D_5$, $g_a^\circ D_3D_5 = -x_3D_5$	$I = \{2,3,4\}$, $S = \{1\}$, $T_a^\circ = \{5\}$	$g, g_1, g_a^\circ \neq 0$
	$gD_2D_4 - gD_4D_2 = x_4D_2 - x_2D_4$, $gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_a^\circ D_1D_4 = -x_4D_1$, $g_a^\circ D_1D_2 = -x_2D_1$, $g_a^\circ D_1D_3 = -x_3D_1$	$I = \{2,3,4\}$, $S = \{5\}$, $T_a^\circ = \{1\}$	$g, g_5, g_a^\circ \neq 0$
	$gD_2D_4 - gD_4D_2 = x_4D_2 - x_2D_4$, $gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_5D_5D_4 - g_5D_4D_5 = x_4D_5$, $g_5D_5D_2 - g_5D_2D_5 = x_2D_5$, $g_5D_5D_3 - g_5D_3D_5 = x_3D_5$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$	$I = \{2,3,4\}$, $S = \{1,5\}$	$g, g_1, g_5 \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5$, $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$, $g_b^+D_2D_4 = -x_2D_4$, $g_b^+D_3D_4 = -x_3D_4$, $g_b^-D_4D_5 = x_5D_4$	$I = \{2,3,5\}$, $S = \{1\}$, $T_b^\bullet = \{4\}$	$g, g_1, g_b^\pm \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5$, $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$, $g_a^\circ D_1D_5 = -x_5D_1$, $g_a^\circ D_1D_2 = -x_2D_1$, $g_a^\circ D_1D_3 = -x_3D_1$	$I = \{2,3,5\}$, $S = \{4\}$, $T_a^\circ = \{1\}$	$g, g_4, g_a^\circ \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5$, $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5$, $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2$, $g_4D_4D_5 - g_4D_5D_4 = x_5D_4$, $g_4D_4D_2 - g_4D_2D_4 = x_2D_4$, $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$	$I = \{2,3,5\}$, $S = \{1,4\}$	$g, g_1, g_5 \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5$, $gD_4D_5 - gD_5D_4 = x_5D_4 - x_4D_5$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$, $g_b^+D_2D_3 = -x_2D_3$, $g_b^-D_3D_4 = x_4D_3$, $g_b^-D_3D_5 = x_5D_3$	$I = \{2,4,5\}$, $S = \{1\}$, $T_b^\bullet = \{3\}$	$g, g_1, g_b^\pm \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5$, $gD_4D_5 - gD_5D_4 = x_5D_4 - x_4D_5$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$, $g_a^\circ D_1D_5 = -x_5D_1$, $g_a^\circ D_1D_2 = -x_2D_1$, $g_a^\circ D_1D_4 = -x_4D_1$	$I = \{2,4,5\}$, $S = \{3\}$, $T_a^\circ = \{1\}$	$g, g_5, g_a^\circ \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5$, $gD_4D_5 - gD_5D_4 = x_5D_4 - x_4D_5$, $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2$, $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$, $g_3D_3D_2 - g_3D_2D_3 = x_2D_3$, $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$, $g_1D_1D_2 - g_1D_2D_1 = x_2D_1$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$	$I = \{2,4,5\}$, $S = \{1,3\}$	$g, g_1, g_5 \neq 0$
	$gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4$, $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$, $g_a^\circ D_2D_4 = -x_4D_2$, $g_a^\circ D_2D_5 = -x_5D_2$, $g_a^\circ D_2D_3 = -x_3D_2$	$I = \{3,4,5\}$, $S = \{1\}$, $T_a^\circ = \{2\}$	$g, g_1, g_a^\circ \neq 0$
	$gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4$, $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $g_a^\circ D_1D_4 = -x_4D_1$, $g_a^\circ D_1D_5 = -x_5D_1$, $g_a^\circ D_1D_3 = -x_3D_1$	$I = \{3,4,5\}$, $S = \{2\}$, $T_a^\circ = \{1\}$	$g, g_2, g_a^\circ \neq 0$
	$gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4$, $gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4$, $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5$, $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$, $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$, $g_2D_2D_3 - g_2D_3D_2 = x_3D_2$, $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$, $g_1D_1D_5 - g_1D_5D_1 = x_5D_1$, $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$	$I = \{3,4,5\}$, $S = \{1,2\}$	$g, g_1, g_2 \neq 0$

Table 4.6: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
A_{II}	$(g_1-g_2)D_1D_2=x_2D_1-x_1D_2$, $(g_1-g_3)D_1D_3=x_3D_1-x_1D_3$, $(g_1-g_4)D_1D_4=x_4D_1-x_1D_4$, $(g_1-g_5)D_1D_5=x_5D_1-x_1D_5$, $(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_2-g_4)D_2D_4=x_4D_2-x_2D_4$, $(g_2-g_5)D_2D_5=x_5D_2-x_2D_5$, $(g_3-g_4)D_3D_4=x_4D_3-x_3D_4$, $(g_3-g_5)D_3D_5=x_5D_3-x_3D_5$, $(g_4-g_5)D_4D_5=x_5D_4-x_4D_5$	$I=\{1,2,3,4,5\}$	$g_i\neq g_j$ for $i,j\in I$, $i\neq j$
	$(g_1-g_2)D_1D_2=x_2D_1-x_1D_2$, $(g_1-g_3)D_1D_3=x_3D_1-x_1D_3$, $(g_1-g_4)D_1D_4=x_4D_1-x_1D_4$, $(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_2-g_4)D_2D_4=x_4D_2-x_2D_4$, $(g_3-g_4)D_3D_4=x_4D_3-x_3D_4$, $(g_1+g_a^\circ)D_1D_5=-x_1D_5$, $(g_2+g_a^\circ)D_2D_5=-x_2D_5$, $(g_3+g_a^\circ)D_3D_5=-x_3D_5$, $(g_4+g_a^\circ)D_4D_5=-x_4D_5$	$I=\{1,2,3,4\}$, $T_a^\circ=\{5\}$	$g_i\neq g_j$, $i\neq j$, $g_i\neq g_a^\circ$, for $i,j\in I$
	$(g_1-g_2)D_1D_2=x_2D_1-x_1D_2$, $(g_1-g_3)D_1D_3=x_3D_1-x_1D_3$, $(g_1-g_5)D_1D_5=x_5D_1-x_1D_5$, $(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_2-g_5)D_2D_5=x_5D_2-x_2D_5$, $(g_3-g_5)D_3D_5=x_5D_3-x_3D_5$, $(g_1+g_b^+)D_1D_4=-x_1D_4$, $(g_2+g_b^+)D_2D_4=-x_2D_4$, $(g_3+g_b^+)D_3D_4=-x_3D_4$, $(g_b^--g_5)D_4D_5=x_5D_4$	$I=\{1,2,3,5\}$, $T_b^\bullet=\{4\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{\mp g_b^\pm\}$, for $i,j\in I$
	$(g_1-g_2)D_1D_2=x_2D_1-x_1D_2$, $(g_1-g_4)D_1D_4=x_4D_1-x_1D_4$, $(g_1-g_5)D_1D_5=x_5D_1-x_1D_5$, $(g_2-g_4)D_2D_4=x_4D_2-x_2D_4$, $(g_2-g_5)D_2D_5=x_5D_2-x_2D_5$, $(g_4-g_5)D_4D_5=x_5D_4-x_4D_5$, $(g_1+g_b^+)D_1D_3=-x_1D_3$, $(g_2+g_b^+)D_2D_3=-x_2D_3$, $(g_b^--g_4)D_3D_4=x_4D_3$, $(g_b^--g_5)D_3D_5=x_5D_3$	$I=\{1,2,4,5\}$, $T_b^\bullet=\{3\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{\mp g_b^\pm\}$, for $i,j\in I$
	$(g_1-g_3)D_1D_3=x_3D_1-x_1D_3$, $(g_1-g_4)D_1D_4=x_4D_1-x_1D_4$, $(g_1-g_5)D_1D_5=x_5D_1-x_1D_5$, $(g_3-g_4)D_3D_4=x_4D_3-x_3D_4$, $(g_3-g_5)D_3D_5=x_5D_3-x_3D_5$, $(g_4-g_5)D_4D_5=x_5D_4-x_4D_5$, $(g_1+g_b^+)D_1D_2=-x_1D_2$, $(g_b^--g_3)D_2D_3=x_3D_2$, $(g_b^--g_4)D_2D_4=x_4D_2$, $(g_b^--g_5)D_2D_5=x_5D_2$	$I=\{1,3,4,5\}$, $T_b^\bullet=\{2\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{\mp g_b^\pm\}$, for $i,j\in I$
	$(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_2-g_4)D_2D_4=x_4D_2-x_2D_4$, $(g_2-g_5)D_2D_5=x_5D_2-x_2D_5$, $(g_3-g_4)D_3D_4=x_4D_3-x_3D_4$, $(g_3-g_5)D_3D_5=x_5D_3-x_3D_5$, $(g_4-g_5)D_4D_5=x_5D_4-x_4D_5$, $(g_2+g_a^\circ)D_1D_2=-x_2D_1$, $(g_3+g_a^\circ)D_1D_3=-x_3D_1$, $(g_4+g_a^\circ)D_1D_4=-x_4D_1$, $(g_5+g_a^\circ)D_1D_5=-x_5D_1$	$I=\{2,3,4,5\}$, $T_a^\circ=\{1\}$	$g_i\neq g_j$, $i\neq j$, $g_i\neq g_a^\circ$, for $i,j\in I$
	$(g_1-g_2)D_1D_2=x_2D_1-x_1D_2$, $(g_1-g_3)D_1D_3=x_3D_1-x_1D_3$, $(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_1+g_a^\circ)D_1D_4=-x_1D_4$, $(g_2+g_a^\circ)D_2D_4=-x_2D_4$, $(g_3+g_a^\circ)D_3D_4=-x_3D_4$, $(g_1+g_a^\circ)D_1D_5=-x_1D_5$, $(g_2+g_a^\circ)D_2D_5=-x_2D_5$, $(g_3+g_a^\circ)D_3D_5=-x_3D_5$	$I=\{1,2,3\}$, $T_a^\circ=\{4,5\}$	$g_i\neq g_j$, $i\neq j$, $g_i\neq g_a^\circ$, for $i,j\in I$
	$(g_4-g_2)D_4D_2=x_2D_4-x_4D_2$, $(g_4-g_3)D_4D_3=x_3D_4-x_4D_3$, $(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_4+g_a^\circ)D_1D_4=-x_4D_1$, $(g_2+g_a^\circ)D_1D_2=-x_2D_1$, $(g_3+g_a^\circ)D_1D_3=-x_3D_1$, $(g_4+g_a^\circ)D_4D_5=-x_4D_5$, $(g_2+g_a^\circ)D_2D_5=-x_2D_5$, $(g_3+g_a^\circ)D_3D_5=-x_3D_5$	$I=\{2,3,4\}$, $T_a^\circ=\{1,5\}$	$g_i\neq g_j$, $i\neq j$, $g_i\neq g_a^\circ$, for $i,j\in I$
	$(g_4-g_5)D_4D_5=x_5D_4-x_4D_5$, $(g_4-g_3)D_4D_3=x_3D_4-x_4D_3$, $(g_5-g_3)D_5D_3=x_3D_5-x_5D_3$, $(g_4+g_a^\circ)D_1D_4=-x_4D_1$, $(g_5+g_a^\circ)D_1D_5=-x_5D_1$, $(g_3+g_a^\circ)D_1D_3=-x_3D_1$, $(g_4+g_a^\circ)D_2D_4=-x_4D_2$, $(g_5+g_a^\circ)D_2D_4=-x_5D_2$, $(g_3+g_a^\circ)D_2D_3=-x_3D_2$	$I=\{3,4,5\}$, $T_a^\circ=\{1,2\}$	$g_i\neq g_j$, $i\neq j$, $g_i\neq g_a^\circ$, for $i,j\in I$
	$(g_1-g_2)D_1D_2=x_2D_1-x_1D_2$, $(g_1-g_4)D_1D_4=x_4D_1-x_1D_4$, $(g_2-g_4)D_2D_4=x_4D_2-x_2D_4$, $(g_1+g_b^+)D_1D_3=-x_1D_3$, $(g_2+g_b^+)D_2D_3=-x_2D_3$, $(g_b^--g_4)D_3D_4=x_4D_3$, $(g_1+g_a^\circ)D_1D_5=-x_1D_5$, $(g_2+g_a^\circ)D_2D_5=-x_2D_5$, $(g_4+g_a^\circ)D_4D_5=-x_4D_5$	$I=\{1,2,4\}$, $T_b^\bullet=\{3\}$ $T_a^\circ=\{5\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{g_a^\circ,\mp g_b^\pm\}$, for $i,j\in I$
	$(g_1-g_3)D_1D_3=x_3D_1-x_1D_3$, $(g_1-g_4)D_1D_4=x_4D_1-x_1D_4$, $(g_3-g_4)D_3D_4=x_4D_3-x_3D_4$, $(g_1+g_b^+)D_1D_2=-x_1D_2$, $(g_b^--g_3)D_2D_3=x_3D_2$, $(g_b^--g_4)D_2D_4=x_4D_2$, $(g_1+g_a^\circ)D_1D_5=-x_1D_5$, $(g_3+g_a^\circ)D_3D_5=-x_3D_5$, $(g_4+g_a^\circ)D_4D_5=-x_4D_5$	$I=\{1,2,4\}$, $T_b^\bullet=\{3\}$ $T_a^\circ=\{5\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{g_a^\circ,\mp g_b^\pm\}$, for $i,j\in I$
	$(g_2-g_3)D_2D_3=x_3D_2-x_2D_3$, $(g_2-g_5)D_2D_5=x_5D_2-x_2D_5$, $(g_3-g_5)D_3D_5=x_5D_3-x_3D_5$, $(g_2+g_b^+)D_2D_4=-x_2D_4$, $(g_3+g_b^+)D_3D_4=-x_3D_4$, $(g_b^--g_5)D_4D_5=x_5D_4$, $(g_2+g_a^\circ)D_1D_2=-x_2D_1$, $(g_3+g_a^\circ)D_1D_3=-x_3D_1$, $(g_5+g_a^\circ)D_1D_5=-x_1D_5$	$I=\{2,3,5\}$, $T_b^\bullet=\{4\}$ $T_a^\circ=\{1\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{g_a^\circ,\mp g_b^\pm\}$, for $i,j\in I$
	$(g_2-g_4)D_2D_4=x_4D_2-x_2D_4$, $(g_2-g_5)D_2D_5=x_5D_2-x_2D_5$, $(g_4-g_5)D_4D_5=x_5D_4-x_4D_5$, $(g_2+g_b^+)D_2D_3=-x_2D_3$, $(g_b^--g_4)D_3D_4=x_4D_3$, $(g_b^--g_5)D_3D_5=x_5D_3$, $(g_2+g_a^\circ)D_1D_2=-x_2D_1$, $(g_4+g_a^\circ)D_1D_4=-x_4D_1$, $(g_5+g_a^\circ)D_1D_5=-x_1D_5$	$I=\{2,4,5\}$, $T_b^\bullet=\{3\}$ $T_a^\circ=\{1\}$	$g_i\neq g_j$, $i\neq j$, $g_i\notin\{g_a^\circ,\mp g_b^\pm\}$, for $i,j\in I$

Table 4.7: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
A_{II}	$(g_2-g_4)D_2D_4 = x_4D_2 - x_2D_4$, $(g_2-g_5)D_2D_5 = x_5D_2 - x_2D_5$, $(g_4-g_5)D_4D_5 = x_5D_4 - x_4D_5$, $(g_2+g_b^+)D_2D_3 = -x_2D_3$, $(g_b^- - g_4)D_3D_4 = x_4D_3$, $(g_b^- - g_5)D_3D_5 = x_5D_3$, $(g_2+g_a^\circ)D_1D_2 = -x_2D_1$, $(g_4+g_a^\circ)D_1D_4 = -x_4D_1$, $(g_5+g_a^\circ)D_1D_5 = -x_1D_5$	$I=\{2,4,5\}$, $T_b^\bullet=\{3\}$ $T_a^\circ=\{1\}$	$g_i \neq g_j$, $i \neq j$, $g_i \notin \{g_a^\circ, \mp g_b^\pm\}$, for $i,j \in I$
	$(g_1-g_2)D_1D_2 = x_2D_1 - x_1D_2$, $(g_1-g_5)D_1D_5 = x_5D_1 - x_1D_5$, $(g_2-g_5)D_2D_5 = x_5D_2 - x_2D_5$, $(g_1+g_b^+)D_1D_3 = -x_1D_3$, $(g_2+g_b^+)D_2D_3 = -x_2D_3$, $(g_b^- - g_5)D_3D_5 = x_5D_3$, $(g_1+g_b^+)D_1D_4 = -x_1D_4$, $(g_2+g_b^+)D_2D_4 = -x_2D_4$, $(g_b^- - g_5)D_4D_5 = x_5D_4$	$I=\{1,2,5\}$, $T_b^\bullet=\{3,4\}$	$g_i \neq g_j$, $i \neq j$, $g_i \notin \{\mp g_b^\pm\}$, for $i,j \in I$
	$(g_1-g_3)D_1D_3 = x_3D_1 - x_1D_3$, $(g_1-g_5)D_1D_5 = x_5D_1 - x_1D_5$, $(g_3-g_5)D_3D_5 = x_5D_3 - x_3D_5$, $(g_1+g_b^+)D_1D_2 = -x_1D_2$, $(g_b^- - g_3)D_2D_3 = x_2D_3$, $(g_b^- - g_5)D_2D_5 = x_5D_2$, $(g_1+g_b^+)D_1D_4 = -x_1D_4$, $(g_3+g_b^+)D_3D_4 = -x_3D_4$, $(g_b^- - g_5)D_4D_5 = x_5D_4$	$I=\{1,3,5\}$, $T_b^\bullet=\{2,4\}$	$g_i \neq g_j$, $i \neq j$, $g_i \notin \{\mp g_b^\pm\}$, for $i,j \in I$
	$(g_1-g_4)D_1D_4 = x_4D_1 - x_1D_4$, $(g_1-g_5)D_1D_5 = x_5D_1 - x_1D_5$, $(g_4-g_5)D_4D_5 = x_5D_4 - x_4D_5$, $(g_1+g_b^+)D_1D_2 = -x_1D_2$, $(g_b^- - g_4)D_2D_4 = -x_4D_2$, $(g_b^- - g_5)D_2D_5 = x_5D_2$, $(g_1+g_b^+)D_1D_3 = -x_1D_3$, $(g_b^- - g_4)D_3D_4 = x_4D_3$, $(g_b^- - g_5)D_3D_5 = x_5D_3$	$I=\{1,4,5\}$, $T_b^\bullet=\{2,3\}$	$g_i \neq g_j$, $i \neq j$, $g_i \notin \{\mp g_b^\pm\}$, for $i,j \in I$
B	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_3D_1D_3-(g_3-\Lambda)D_3D_1 = -x_1D_3$, $g_3D_3D_2-(g_3-\Lambda)D_2D_3 = x_2D_3$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_2-(g_4-\Lambda)D_2D_4 = x_2D_4$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_2-(g_5-\Lambda)D_2D_5 = x_2D_5$,	$I=\{1,2\}$, $S=\{3,4,5\}$	$g \neq 0$, $g_s \notin \{0,\Lambda\}, s \in S$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_3D_1D_3-(g_3-\Lambda)D_3D_1 = -x_1D_3$, $g_3D_3D_2-(g_3-\Lambda)D_2D_3 = x_2D_3$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_2-(g_4-\Lambda)D_2D_4 = x_2D_4$, $g_a^\circ D_1D_5 = -x_1D_5$, $(g_a^\circ-\Lambda)D_2D_5 = -x_2D_5$,	$I=\{1,2\}$, $S=\{3,4\}$, $T_a^\circ=\{5\}$	$g \neq 0$, $g_s, g_a^\circ \notin \{0,\Lambda\}$, $s \in S$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_3D_1D_3-(g_3-\Lambda)D_3D_1 = -x_1D_3$, $g_3D_3D_2-(g_3-\Lambda)D_2D_3 = x_2D_3$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_2-(g_5-\Lambda)D_2D_5 = x_2D_5$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ-\Lambda)D_2D_4 = -x_2D_4$,	$I=\{1,2\}$, $S=\{3,5\}$, $T_a^\circ=\{4\}$	$g \neq 0$, $g_s, g_a^\circ \notin \{0,\Lambda\}$, $s \in S$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_2-(g_4-\Lambda)D_2D_4 = x_2D_4$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_2-(g_5-\Lambda)D_2D_5 = x_2D_5$, $g_a^\circ D_1D_3 = -x_1D_3$, $(g_a^\circ-\Lambda)D_2D_3 = -x_2D_3$,	$I=\{1,2\}$, $S=\{4,5\}$, $T_a^\circ=\{3\}$	$g \neq 0$, $g_s, g_a^\circ \notin \{0,\Lambda\}$, $s \in S$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_3D_1D_3-(g_3-\Lambda)D_3D_1 = -x_1D_3$, $g_3D_3D_2-(g_3-\Lambda)D_2D_3 = x_2D_3$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ-\Lambda)D_2D_4 = -x_2D_4$, $g_a^\circ D_1D_5 = -x_1D_5$, $(g_a^\circ-\Lambda)D_2D_5 = -x_2D_5$,	$I=\{1,2\}$, $S=\{3\}$, $T_a^\circ=\{4,5\}$	$g \neq 0$, $g_3, g_a^\circ \notin \{0,\Lambda\}$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_2-(g_4-\Lambda)D_2D_4 = x_2D_4$, $g_a^\circ D_1D_3 = -x_1D_3$, $(g_a^\circ-\Lambda)D_2D_3 = -x_2D_3$, $g_a^\circ D_1D_5 = -x_1D_5$, $(g_a^\circ-\Lambda)D_2D_5 = -x_2D_5$,	$I=\{1,2\}$, $S=\{4\}$, $T_a^\circ=\{3,5\}$	$g \neq 0$, $g_4, g_a^\circ \notin \{0,\Lambda\}$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_2-(g_5-\Lambda)D_2D_5 = x_2D_5$, $g_a^\circ D_1D_3 = -x_1D_3$, $(g_a^\circ-\Lambda)D_2D_3 = -x_2D_3$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ-\Lambda)D_2D_4 = -x_2D_4$,	$I=\{1,2\}$, $S=\{5\}$, $T_a^\circ=\{3,4\}$	$g \neq 0$, $g_5, g_a^\circ \notin \{0,\Lambda\}$
	$gD_1D_2-(g-\Lambda)D_2D_1 = x_2D_1 - x_1D_2$, $g_a^\circ D_1D_3 = -x_1D_3$, $(g_a^\circ-\Lambda)D_2D_3 = -x_2D_3$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ-\Lambda)D_2D_4 = -x_2D_4$, $g_a^\circ D_1D_5 = -x_1D_5$, $(g_a^\circ-\Lambda)D_2D_5 = -x_2D_5$,	$I=\{1,2\}$, $T_a^\circ=\{3,4,5\}$	$g \neq 0$, $g_a^\circ \notin \{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_2D_1D_2-(g_2-\Lambda)D_2D_1 = -x_1D_2$, $g_2D_2D_3-(g_2-\Lambda)D_3D_2 = x_3D_2$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_3-(g_4-\Lambda)D_3D_4 = x_3D_4$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_3-(g_5-\Lambda)D_3D_5 = x_3D_5$,	$I=\{1,3\}$, $S=\{2,4,5\}$	$g, g_2 \neq 0$, $g_4, g_5 \notin \{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_2D_1D_2-(g_2-\Lambda)D_2D_1 = -x_1D_2$, $g_2D_2D_3-(g_2-\Lambda)D_3D_2 = x_3D_2$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_3-(g_4-\Lambda)D_3D_4 = x_3D_4$, $g_a^\circ D_1D_5 = -x_1D_5$, $(g_a^\circ-\Lambda)D_3D_5 = -x_3D_5$,	$I=\{1,3\}$, $S=\{2,4\}$, $T_a^\circ=\{5\}$	$g, g_2 \neq 0$, $g_4, g_a^\circ \notin \{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_2D_1D_2-(g_2-\Lambda)D_2D_1 = -x_1D_2$, $g_2D_2D_3-(g_2-\Lambda)D_3D_2 = x_3D_2$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_3-(g_5-\Lambda)D_3D_5 = x_3D_5$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ-\Lambda)D_3D_4 = -x_3D_4$,	$I=\{1,3\}$, $S=\{2,5\}$, $T_a^\circ=\{4\}$	$g, g_2 \neq 0$, $g_5, g_a^\circ \notin \{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_4D_1D_4-(g_4-\Lambda)D_4D_1 = -x_1D_4$, $g_4D_4D_3-(g_4-\Lambda)D_3D_4 = x_3D_4$, $g_5D_1D_5-(g_5-\Lambda)D_5D_1 = -x_1D_5$, $g_5D_5D_3-(g_5-\Lambda)D_3D_5 = x_3D_5$, $g_b^+D_1D_2 = -x_1D_2$, $g_b^-D_2D_3 = x_3D_2$,	$I=\{1,3\}$, $S=\{4,5\}$, $T_b^\bullet=\{2\}$	$g, g_b^\pm \neq 0$, $g_s \notin \{0,\Lambda\}$, $s \in S$
	$gD_1D_3-(g-\Lambda)D_3D_1 = x_3D_1 - x_1D_3$, $g_2D_1D_2-(g_2-\Lambda)D_2D_1 = -x_1D_2$, $g_2D_2D_3-(g_2-\Lambda)D_3D_2 = x_3D_2$, $g_a^\circ D_1D_4 = -x_1D_4$, $(g_a^\circ-\Lambda)D_3D_4 = -x_3D_4$, $g_a^\circ D_1D_5 = -x_1D_5$, $(g_a^\circ-\Lambda)D_3D_5 = -x_3D_5$,	$I=\{1,3\}$, $S=\{2\}$, $T_a^\circ=\{4,5\}$	$g, g_2 \neq 0$, $g_a^\circ \notin \{0,\Lambda\}$

Table 4.8: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_1D_3-(g-\Lambda)D_3D_1=x_3D_1-x_1D_3,\quad g_4D_1D_4-(g_4-\Lambda)D_4D_1=-x_1D_4,$ $g_4D_4D_3-(g_4-\Lambda)D_3D_4=x_3D_4,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_3=x_3D_2,\quad g_a^\circ D_1D_5=-x_1D_5,$ $(g_a^\circ-\Lambda)D_3D_5=-x_3D_5,$	$I=\{1,3\}$, $S=\{4\}$, $T_b^\bullet=\{2\}$, $T_a^\circ=\{5\}$	$g, g_b^\pm\neq 0$, $g_4, g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1=x_3D_1-x_1D_3,\quad g_5D_1D_5-(g_5-\Lambda)D_5D_1=-x_1D_5,$ $g_5D_5D_3-(g_5-\Lambda)D_3D_5=x_3D_5,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_3=x_3D_2,\quad g_a^\circ D_1D_4=-x_1D_4,$ $(g_a^\circ-\Lambda)D_3D_4=-x_3D_4,$	$I=\{1,3\}$, $S=\{5\}$, $T_b^\bullet=\{2\}$, $T_a^\circ=\{4\}$	$g, g_b^\pm\neq 0$, $g_5, g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_3-(g-\Lambda)D_3D_1=x_3D_1-x_1D_3,\quad g_a^\circ D_1D_5=-x_1D_5,$ $(g_a^\circ-\Lambda)D_3D_5=-x_3D_5,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_3=x_3D_2,\quad g_a^\circ D_1D_4=-x_1D_4,$ $(g_a^\circ-\Lambda)D_3D_4=-x_3D_4,$	$I=\{1,3\}$, $T_b^\bullet=\{2\}$, $T_a^\circ=\{4,5\}$	$g, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_5D_1D_5-(g_5-\Lambda)D_5D_1=-x_1D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,$	$I=\{1,4\}$, $S=\{2,3,5\}$	$g, g_2, g_3\neq 0$, $g_5\notin\{0,\Lambda\}$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_a^\circ D_1D_5=-x_1D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{1,4\}$, $S=\{2,3\}$, $T_a^\circ=\{5\}$	$g, g_s\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_5D_1D_5-(g_5-\Lambda)D_5D_1=-x_1D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{1,4\}$, $S=\{2,5\}$, $T_b^\bullet=\{3\}$	$g, g_2\neq 0$, $g_5, g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_5D_1D_5-(g_5-\Lambda)D_5D_1=-x_1D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_4=x_4D_2,$	$I=\{1,4\}$, $S=\{3,5\}$, $T_b^\bullet=\{2\}$	$g, g_3, g_5, g_b^\pm\neq 0$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_4=x_4D_3,\quad g_a^\circ D_1D_5=-x_1D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{1,4\}$, $S=\{2\}$, $T_b^\bullet=\{3\}$, $T_a^\circ=\{5\}$	$g, g_2, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_4=x_4D_2,\quad g_a^\circ D_1D_5=-x_1D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{1,4\}$, $S=\{3\}$, $T_b^\bullet=\{2\}$, $T_a^\circ=\{5\}$	$g, g_3, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_5D_1D_5-(g_5-\Lambda)D_5D_1=-x_1D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_4=x_4D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{1,4\}$, $S=\{5\}$, $T_b^\bullet=\{2,3\}$	$g, g_b^\pm\neq 0$, $g_5\notin\{0,\Lambda\}$
	$gD_1D_4-(g-\Lambda)D_4D_1=x_4D_1-x_1D_4,\quad g_a^\circ D_1D_5=-x_1D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_4=x_4D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^+D_3D_4=x_4D_3,$	$I=\{1,4\}$, $T_b^\bullet=\{2,3\}$, $T_a^\circ=\{5\}$	$g, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_5-(g_2-\Lambda)D_5D_2=x_5D_2,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3,\quad g_4D_1D_4-(g_4-\Lambda)D_4D_1=-x_1D_4,$ $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4,$	$I=\{1,5\}$, $S=\{2,3,4\}$	$g, g_s\neq 0$, $s\in S$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_5-(g_2-\Lambda)D_5D_2=x_5D_2,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3,\quad g_b^+D_1D_4=-x_1D_4,$ $g_b^-D_4D_5=x_5D_4,$	$I=\{1,5\}$, $S=\{2,3\}$, $T_b^\bullet=\{4\}$	$g, g_s\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_5-(g_2-\Lambda)D_5D_2=x_5D_2,\quad g_4D_1D_4-(g_4-\Lambda)D_4D_1=-x_1D_4,$ $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_5=x_5D_3,$	$I=\{1,5\}$, $S=\{2,4\}$, $T_b^\bullet=\{3\}$	$g, g_s\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3,\quad g_4D_1D_4-(g_4-\Lambda)D_4D_1=-x_1D_4,$ $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_5=x_5D_2,$	$I=\{1,5\}$, $S=\{3,4\}$, $T_b^\bullet=\{2\}$	$g, g_s\neq 0$, $g_b^\pm\neq 0$, $s\in S$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_2D_1D_2-(g_2-\Lambda)D_2D_1=-x_1D_2,$ $g_2D_2D_5-(g_2-\Lambda)D_5D_2=x_5D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_5=x_5D_3,\quad g_b^+D_1D_4=-x_1D_4,$ $g_b^-D_4D_5=x_5D_4,$	$I=\{1,5\}$, $S=\{2\}$, $T_b^\bullet=\{3,4\}$	$g, g_2, g_b^\pm\neq 0$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_3D_1D_3-(g_3-\Lambda)D_3D_1=-x_1D_3,$ $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_5=x_5D_2,\quad g_b^+D_1D_4=-x_1D_4,$ $g_b^-D_4D_5=x_5D_4,$	$I=\{1,5\}$, $S=\{3\}$, $T_b^\bullet=\{2,4\}$	$g, g_3, g_b^\pm\neq 0$

Table 4.9: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_4D_1D_4-(g_4-\Lambda)D_4D_1=-x_1D_4,$ $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_5=x_5D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^-D_3D_5=x_5D_3,$	$I=\{1,5\}$, $S=\{4\}$, $T_b^\bullet=\{2,3\}$	$g,g_4,g_b^\pm\neq 0$
	$gD_1D_5-(g-\Lambda)D_5D_1=x_5D_1-x_1D_5,\quad g_b^+D_1D_4=-x_1D_4,$ $g_b^-D_4D_5=x_5D_4,\quad g_b^+D_1D_2=-x_1D_2,$ $g_b^-D_2D_5=x_5D_2,\quad g_b^+D_1D_3=-x_1D_3,$ $g_b^+D_3D_5=x_5D_3,$	$I=\{1,5\}$, $T_b^\bullet=\{2,3,4\}$	$g,g_b^\pm\neq 0$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1,$ $g_1D_1D_2-(g_1-\Lambda)D_2D_1=x_2D_1,\quad g_4D_3D_4-(g_4-\Lambda)D_4D_3=-x_3D_4,$ $g_4D_4D_2-(g_4-\Lambda)D_2D_4=x_2D_4,\quad g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5,$ $g_5D_5D_2-(g_5-\Lambda)D_2D_5=x_2D_5,$	$I=\{2,3\}$, $S=\{1,4,5\}$	$g\neq 0$, $g_s\notin\{0,\Lambda\}, s\in S$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1,$ $g_1D_1D_2-(g_1-\Lambda)D_2D_1=x_2D_1,\quad g_4D_3D_4-(g_4-\Lambda)D_4D_3=-x_3D_4,$ $g_4D_4D_2-(g_4-\Lambda)D_2D_4=x_2D_4,\quad g_a^\circ D_3D_5=-x_3D_5,$ $(g_a^\circ-\Lambda)D_2D_5=-x_2D_5,$	$I=\{2,3\}$, $S=\{1,4\}$, $T_a^\circ=\{5\}$	$g\neq 0$, $g_s,g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1,$ $g_1D_1D_2-(g_1-\Lambda)D_2D_1=x_2D_1,\quad g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5,$ $g_5D_5D_2-(g_5-\Lambda)D_2D_5=x_2D_5,\quad g_a^\circ D_3D_4=-x_3D_4,$ $(g_a^\circ-\Lambda)D_2D_4=-x_2D_4,$	$I=\{2,3\}$, $S=\{1,5\}$, $T_a^\circ=\{4\}$	$g\neq 0$, $g_s,g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_4D_3D_4-(g_4-\Lambda)D_4D_3=-x_3D_4,$ $g_4D_4D_2-(g_4-\Lambda)D_2D_4=x_2D_4,\quad g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5,$ $g_5D_5D_2-(g_5-\Lambda)D_2D_5=x_2D_5,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_2=-x_2D_1,$	$I=\{2,3\}$, $S=\{4,5\}$, $T_a^\circ=\{1\}$	$g\neq 0$, $g_s,g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1,$ $g_1D_1D_2-(g_1-\Lambda)D_2D_1=x_2D_1,\quad g_a^\circ D_3D_4=-x_3D_4,$ $(g_a^\circ-\Lambda)D_2D_4=-x_2D_4,\quad g_a^\circ D_3D_5=-x_3D_5,$ $(g_a^\circ-\Lambda)D_2D_5=-x_2D_5,$	$I=\{2,3\}$, $S=\{1\}$, $T_a^\circ=\{4,5\}$	$g\neq 0$, $g_1,g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_4D_3D_4-(g_4-\Lambda)D_4D_3=-x_3D_4,$ $g_4D_4D_2-(g_4-\Lambda)D_2D_4=x_2D_4,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_2=-x_2D_1,\quad g_a^\circ D_3D_5=-x_3D_5,$ $(g_a^\circ-\Lambda)D_2D_5=-x_2D_5,$	$I=\{2,3\}$, $S=\{4\}$, $T_a^\circ=\{1,5\}$	$g\neq 0$, $g_4,g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5,$ $g_5D_5D_2-(g_5-\Lambda)D_2D_5=x_2D_5,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_2=-x_2D_1,\quad g_a^\circ D_3D_4=-x_3D_4,$ $(g_a^\circ-\Lambda)D_2D_4=-x_2D_4,$	$I=\{2,3\}$, $S=\{5\}$, $T_a^\circ=\{1,4\}$	$g\neq 0$, $g_5,g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_2-(g-\Lambda)D_2D_3=x_2D_3-x_3D_2,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_2=-x_2D_1,\quad g_a^\circ D_3D_4=-x_3D_4,$ $(g_a^\circ-\Lambda)D_2D_4=-x_2D_4,\quad g_a^\circ D_3D_5=-x_3D_5,$ $(g_a^\circ-\Lambda)D_2D_5=-x_2D_5,$	$I=\{2,3\}$, $T_a^\circ=\{1,4,5\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_5D_2D_5-(g_5-\Lambda)D_5D_2=-x_2D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,$	$I=\{2,4\}$, $S=\{1,3,5\}$	$g,g_3\neq 0$, $g_1,g_5\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_a^\circ D_2D_5=-x_2D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{2,4\}$, $S=\{1,3\}$, $T_a^\circ=\{5\}$	$g,g_3\neq 0$, $g_1,g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_5D_2D_5-(g_5-\Lambda)D_5D_2=-x_2D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,\quad g_b^+D_2D_3=-x_2D_3,$ $g_b^-D_3D_4=x_4D_3,$	$I=\{2,4\}$, $S=\{1,5\}$, $T_b^\bullet=\{3\}$	$g,g_b^\pm\neq 0$, $g_s\notin\{0,\Lambda\}$, $s\in S$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_5D_2D_5-(g_5-\Lambda)D_5D_2=-x_2D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,\quad g_a^\circ D_1D_2=-x_2D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,$	$I=\{2,4\}$, $S=\{3,5\}$, $T_a^\circ=\{1\}$	$g,g_3\neq 0$, $g_5,g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_b^+D_2D_3=-x_2D_3,$ $g_b^-D_3D_4=x_4D_3,\quad g_a^\circ D_2D_5=-x_2D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{2,4\}$, $S=\{1\}$, $T_b^\bullet=\{3\}$, $T_a^\circ=\{5\}$	$g,g_b^\pm\neq 0$, $g_1,g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_a^\circ D_1D_2=-x_2D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,\quad g_a^\circ D_2D_5=-x_2D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{2,4\}$, $S=\{3\}$, $T_a^\circ=\{1,5\}$	$g,g_3\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_5D_2D_5-(g_5-\Lambda)D_5D_2=-x_2D_5,$ $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5,\quad g_b^+D_2D_3=-x_2D_3,$ $g_b^-D_3D_4=x_4D_3,\quad g_a^\circ D_1D_1=-x_2D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,$	$I=\{2,4\}$, $S=\{5\}$, $T_b^\bullet=\{3\}$, $T_a^\circ=\{1\}$	$g,g_b^\pm\neq 0$, $g_5,g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_4-(g-\Lambda)D_4D_2=x_4D_2-x_2D_4,\quad g_a^\circ D_2D_5=-x_2D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,\quad g_b^+D_2D_3=-x_2D_3,$ $g_b^-D_3D_4=x_4D_3,\quad g_a^\circ D_2D_5=-x_2D_5,$ $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5,$	$I=\{2,4\}$, $T_b^\bullet=\{3\}$, $T_a^\circ=\{1,5\}$	$g,g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$

Table 4.10: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1$, $g_1D_1D_5-(g_1-\Lambda)D_5D_1=x_5D_1$, $g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3$, $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3$, $g_4D_2D_4-(g_4-\Lambda)D_4D_2=-x_2D_4$, $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4$,	$I=\{2,5\}$, $S=\{1,3,4\}$	$g, g_3, g_4\neq 0$, $g_1\notin\{0,\Lambda\}$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1$, $g_1D_1D_5-(g_1-\Lambda)D_5D_1=x_5D_1$, $g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3$, $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3$, $g_b^+D_2D_4=-x_2D_4$, $g_b^-D_4D_5=x_5D_4$,	$I=\{2,5\}$, $S=\{1,3\}$, $T_b^\bullet=\{4\}$	$g, g_3, g_b^\pm\neq 0$, $g_1\notin\{0,\Lambda\}$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1$, $g_1D_1D_5-(g_1-\Lambda)D_5D_1=x_5D_1$, $g_4D_2D_4-(g_4-\Lambda)D_4D_2=-x_2D_4$, $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4$, $g_b^+D_2D_3=-x_2D_3$, $g_b^-D_3D_5=x_5D_3$,	$I=\{2,5\}$, $S=\{1,4\}$, $T_b^\bullet=\{3\}$	$g, g_4, g_b^\pm\neq 0$, $g_1\notin\{0,\Lambda\}$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3$, $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3$, $g_4D_2D_4-(g_4-\Lambda)D_4D_2=-x_2D_4$, $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4$, $g_a^\circ D_1D_2=-x_2D_1$, $(g_{a^\circ}-\Lambda)D_1D_5=-x_5D_1$,	$I=\{2,5\}$, $S=\{3,4\}$, $T_a^\circ=\{1\}$	$g, g_s\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_1D_2D_1-(g_1-\Lambda)D_1D_2=-x_2D_1$, $g_1D_1D_5-(g_1-\Lambda)D_5D_1=x_5D_1$, $g_b^+D_2D_3=-x_2D_3$, $g_b^-D_3D_5=x_5D_3$, $g_b^+D_2D_4=-x_2D_4$, $g_b^-D_4D_5=x_5D_4$,	$I=\{2,5\}$, $S=\{1\}$, $T_b^\bullet=\{3,4\}$	$g, g_b^\pm\neq 0$, $g_1\notin\{0,\Lambda\}$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_3D_2D_3-(g_3-\Lambda)D_3D_2=-x_2D_3$, $g_3D_3D_5-(g_3-\Lambda)D_5D_3=x_5D_3$, $g_a^\circ D_1D_2=-x_2D_1$, $(g_a^\circ-\Lambda)D_1D_5=-x_5D_1$, $g_b^+D_2D_4=-x_2D_4$, $g_b^-D_4D_5=x_5D_4$,	$I=\{2,5\}$, $S=\{3\}$, $T_a^\circ=\{1\}$, $T_b^\bullet=\{4\}$	$g, g_3, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_4D_2D_4-(g_4-\Lambda)D_4D_2=-x_2D_4$, $g_4D_4D_5-(g_4-\Lambda)D_5D_4=x_5D_4$, $g_a^\circ D_1D_2=-x_2D_1$, $(g_a^\circ-\Lambda)D_1D_5=-x_5D_1$, $g_b^+D_2D_3=-x_2D_3$, $g_b^-D_3D_5=x_5D_3$,	$I=\{2,5\}$, $S=\{4\}$, $T_a^\circ=\{1\}$, $T_b^\bullet=\{3\}$	$g, g_4, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_2D_5-(g-\Lambda)D_5D_2=x_5D_2-x_2D_5$, $g_b^+D_2D_4=-x_2D_4$, $g_b^-D_4D_5=x_5D_4$, $g_b^+D_2D_3=-x_2D_3$, $g_b^-D_3D_5=x_5D_3$, $g_a^\circ D_1D_2=-x_2D_1$, $(g_a^\circ-\Lambda)D_1D_5=-x_5D_1$,	$I=\{2,5\}$, $S=\{3,4\}$, $T_a^\circ=\{1\}$, $T_b^\bullet=\{3,4\}$	$g, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1$, $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1$, $g_2D_3D_2-(g_2-\Lambda)D_2D_3=-x_3D_2$, $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2$, $g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5$, $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5$,	$I=\{3,4\}$, $S=\{1,2,5\}$	$g\neq 0$, $g_s\notin\{0,\Lambda\}, s\in S$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1$, $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1$, $g_2D_3D_2-(g_2-\Lambda)D_2D_3=-x_3D_2$, $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2$, $g_a^\circ D_3D_5=-x_3D_5$, $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5$,	$I=\{3,4\}$, $S=\{1,2\}$, $T_a^\circ=\{5\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1$, $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1$, $g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5$, $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5$, $g_a^\circ D_3D_2=-x_3D_2$, $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2$,	$I=\{3,4\}$, $S=\{1,5\}$, $T_a^\circ=\{2\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_2D_3D_2-(g_2-\Lambda)D_2D_3=-x_3D_2$, $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2$, $g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5$, $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5$, $g_a^\circ D_1D_3=-x_3D_1$, $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1$,	$I=\{3,4\}$, $S=\{2,5\}$, $T_a^\circ=\{1\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_1D_3D_1-(g_1-\Lambda)D_1D_3=-x_3D_1$, $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1$, $g_a^\circ D_3D_2=-x_3D_2$, $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2$, $g_a^\circ D_3D_5=-x_3D_5$, $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5$,	$I=\{3,4\}$, $S=\{1\}$, $T_a^\circ=\{2,5\}$	$g\neq 0$, $g_1, g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_2D_3D_2-(g_2-\Lambda)D_2D_3=-x_3D_2$, $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2$, $g_a^\circ D_1D_3=-x_3D_1$, $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1$, $g_a^\circ D_3D_5=-x_3D_5$, $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5$,	$I=\{3,4\}$, $S=\{2\}$, $T_a^\circ=\{1,5\}$	$g\neq 0$, $g_2, g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_5D_3D_5-(g_5-\Lambda)D_5D_3=-x_3D_5$, $g_5D_5D_4-(g_5-\Lambda)D_4D_5=x_4D_5$, $g_a^\circ D_1D_3=-x_3D_1$, $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1$, $g_a^\circ D_3D_2=-x_3D_2$, $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2$,	$I=\{3,4\}$, $S=\{5\}$, $T_a^\circ=\{1,2\}$	$g\neq 0$, $g_5, g_a^\circ\notin\{0,\Lambda\}$
	$gD_3D_4-(g-\Lambda)D_4D_3=x_4D_3-x_3D_4$, $g_a^\circ D_1D_3=-x_3D_1$, $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1$, $g_a^\circ D_3D_2=-x_3D_2$, $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2$, $g_a^\circ D_3D_5=-x_3D_5$, $(g_a^\circ-\Lambda)D_4D_5=-x_4D_5$,	$I=\{3,4\}$, $T_a^\circ=\{1,2,5\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3$, $g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2$, $g_2D_2D_3-(g_2-\Lambda)D_3D_2=x_3D_2$, $g_4D_5D_4-(g_4-\Lambda)D_4D_5=-x_5D_4$, $g_4D_4D_3-(g_4-\Lambda)D_3D_4=x_3D_4$, $g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1$, $g_1D_1D_3-(g_1-\Lambda)D_3D_1=x_3D_1$,	$I=\{3,5\}$, $S=\{1,2,4\}$	$g, g_4\neq 0$, $g_1, g_2\notin\{0,\Lambda\}$

Table 4.11: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_3-(g_2-\Lambda)D_3D_2=x_3D_2,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_3-(g_1-\Lambda)D_3D_1=x_3D_1,\quad g_b^+D_3D_4=-x_3D_4,$ $g_b^-D_4D_5=x_5D_4,$	$I=\{3,5\}$, $S=\{1,2\}$, $T_b^\bullet=\{4\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_4D_5D_4-(g_4-\Lambda)D_4D_5=-x_5D_4,$ $g_4D_4D_3-(g_4-\Lambda)D_3D_4=x_3D_4,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_3-(g_1-\Lambda)D_3D_1=x_3D_1,\quad g_a^\circ D_2D_5=-x_5D_2,$ $(g_a^\circ-\Lambda)D_2D_3=-x_3D_2,$	$I=\{3,5\}$, $S=\{1,4\}$, $T_a^\circ=\{2\}$	$g, g_4\neq 0$, $g_1, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_4D_5D_4-(g_4-\Lambda)D_4D_5=-x_5D_4,$ $g_4D_4D_3-(g_4-\Lambda)D_3D_4=x_3D_4,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_3-(g_2-\Lambda)D_3D_2=x_3D_2,\quad g_a^\circ D_1D_5=-x_5D_1,$ $(g_a^\circ-\Lambda)D_1D_3=-x_3D_1,$	$I=\{3,5\}$, $S=\{2,4\}$, $T_a^\circ=\{1\}$	$g, g_4\neq 0$, $g_2, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_3-(g_1-\Lambda)D_3D_1=x_3D_1,\quad g_a^\circ D_2D_3=-x_3D_2,$ $(g_a^\circ-\Lambda)D_2D_5=-x_5D_2,\quad g_b^+D_3D_4=-x_3D_4,$ $g_b^-D_4D_5=x_5D_4,$	$I=\{3,5\}$, $S=\{1\}$, $T_b^\bullet=\{4\}$, $T_a^\circ=\{2\}$	$g, g_b^\pm\neq 0$, $g_1, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_3-(g_2-\Lambda)D_3D_2=x_3D_2,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_5=-x_5D_1,\quad g_b^+D_3D_4=-x_3D_4,$ $g_b^-D_4D_5=x_5D_4,$	$I=\{3,5\}$, $S=\{2\}$, $T_b^\bullet=\{4\}$, $T_a^\circ=\{1\}$	$g, g_b^\pm\neq 0$, $g_2, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_4D_5D_4-(g_4-\Lambda)D_4D_5=-x_5D_4,$ $g_4D_4D_3-(g_4-\Lambda)D_3D_4=x_3D_4,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_5=-x_5D_1,\quad g_a^\circ D_2D_3=-x_3D_2,$ $(g_a^\circ-\Lambda)D_2D_5=-x_5D_2,$	$I=\{3,5\}$, $S=\{4\}$, $T_a^\circ=\{1,2\}$	$g, g_4\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_3-(g-\Lambda)D_3D_5=x_3D_5-x_5D_3,\quad g_b^+D_3D_4=-x_3D_4,$ $g_b^-D_4D_5=x_5D_4,\quad g_a^\circ D_1D_3=-x_3D_1,$ $(g_a^\circ-\Lambda)D_1D_5=-x_5D_1,\quad g_a^\circ D_2D_3=-x_3D_2,$ $(g_a^\circ-\Lambda)D_2D_5=-x_5D_2,$	$I=\{3,5\}$, $S=\{4\}$, $T_b^\bullet=\{4\}$, $T_a^\circ=\{1,2\}$	$g, g_b^\pm\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_3D_5D_3-(g_3-\Lambda)D_3D_5=-x_5D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,$	$I=\{4,5\}$, $S=\{1,2,3\}$	$g\neq 0$, $g_s\notin\{0,\Lambda\}, s\in S$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_a^\circ D_5D_3=-x_5D_3,$ $(g_a^\circ-\Lambda)D_4D_3=-x_4D_3,$	$I=\{4,5\}$, $S=\{1,2\}$, $T_a^\circ=\{3\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_3D_5D_3-(g_3-\Lambda)D_3D_5=-x_5D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_a^\circ D_5D_2=-x_5D_2,$ $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2,$	$I=\{4,5\}$, $S=\{1,3\}$, $T_a^\circ=\{2\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_3D_5D_3-(g_3-\Lambda)D_3D_5=-x_5D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_a^\circ D_1D_5=-x_5D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,$	$I=\{4,5\}$, $S=\{2,3\}$, $T_a^\circ=\{1\}$	$g\neq 0$, $g_s, g_a^\circ\notin\{0,\Lambda\}$, $s\in S$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_1D_5D_1-(g_1-\Lambda)D_1D_5=-x_5D_1,$ $g_1D_1D_4-(g_1-\Lambda)D_4D_1=x_4D_1,\quad g_a^\circ D_5D_2=-x_5D_2,$ $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2,\quad g_a^\circ D_5D_3=-x_5D_3,$ $(g_a^\circ-\Lambda)D_4D_3=-x_4D_3,$	$I=\{4,5\}$, $S=\{1\}$, $T_a^\circ=\{2,3\}$	$g\neq 0$, $g_1, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_2D_5D_2-(g_2-\Lambda)D_2D_5=-x_5D_2,$ $g_2D_2D_4-(g_2-\Lambda)D_4D_2=x_4D_2,\quad g_a^\circ D_1D_5=-x_5D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,\quad g_a^\circ D_5D_3=-x_5D_3,$ $(g_a^\circ-\Lambda)D_4D_3=-x_4D_3,$	$I=\{4,5\}$, $S=\{2\}$, $T_a^\circ=\{1,3\}$	$g\neq 0$, $g_2, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_3D_5D_3-(g_3-\Lambda)D_3D_5=-x_5D_3,$ $g_3D_3D_4-(g_3-\Lambda)D_4D_3=x_4D_3,\quad g_a^\circ D_1D_5=-x_5D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,\quad g_a^\circ D_5D_2=-x_5D_2,$ $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2,$	$I=\{4,5\}$, $S=\{3\}$, $T_a^\circ=\{1,2\}$	$g\neq 0$, $g_3, g_a^\circ\notin\{0,\Lambda\}$
	$gD_5D_4-(g-\Lambda)D_4D_5=x_4D_5-x_5D_4,\quad g_a^\circ D_1D_5=-x_5D_1,$ $(g_a^\circ-\Lambda)D_1D_4=-x_4D_1,\quad g_a^\circ D_5D_2=-x_5D_2,$ $(g_a^\circ-\Lambda)D_4D_2=-x_4D_2,\quad g_a^\circ D_5D_3=-x_5D_3,$ $(g_a^\circ-\Lambda)D_4D_3=-x_4D_3,$	$I=\{4,5\}$, $T_a^\circ=\{1,2,3\}$	$g\neq 0$, $g_a^\circ\notin\{0,\Lambda\}$

Table 4.12: Diffusion algebras with five generators

Diffusion algebra	Relations	Conditions	Restrictions
C	$g_2D_1D_2-(g_2-\Lambda_a)D_2D_1=-x_1D_2,\quad g_3D_1D_3-(g_3-\Lambda_a)D_3D_1=-x_1D_3,$ $g_4D_1D_4-(g_4-\Lambda_a)D_4D_1=-x_1D_4,\quad g_5D_1D_5-(g_5-\Lambda_a)D_5D_1=-x_1D_5,$	$I=\{1\},\ R_a=\{2,3,4,5\}$	$g_2,g_3,g_4,g_5\neq\Lambda_a$
	$g_1D_2D_1-(g_1-\Lambda_a)D_1D_2=-x_2D_1,\quad g_3D_2D_3-(g_3-\Lambda_a)D_3D_2=-x_2D_3,$ $g_4D_2D_4-(g_4-\Lambda_a)D_4D_2=-x_2D_4,\quad g_5D_2D_5-(g_5-\Lambda_a)D_5D_2=-x_2D_5,$	$I=\{2\},\ R_a=\{1,3,4,5\}$	$g_1\neq 0,\ g_3,g_4,g_5\neq\Lambda_a$
	$g_1D_3D_1-(g_1-\Lambda_a)D_1D_3=-x_3D_1,\quad g_2D_3D_2-(g_2-\Lambda_a)D_2D_3=-x_3D_2,$ $g_4D_3D_4-(g_4-\Lambda_a)D_4D_3=-x_3D_4,\quad g_5D_3D_5-(g_5-\Lambda_a)D_5D_3=-x_3D_5,$	$I=\{3\},\ R_a=\{1,2,4,5\}$	$g_1,g_2\neq 0,\ g_4,g_5\neq\Lambda_a$
	$g_1D_4D_1-(g_1-\Lambda_a)D_1D_4=-x_4D_1,\quad g_2D_4D_2-(g_2-\Lambda_a)D_2D_4=-x_4D_2,$ $g_3D_4D_3-(g_3-\Lambda_a)D_3D_4=-x_4D_3,\quad g_5D_4D_5-(g_5-\Lambda_a)D_5D_4=-x_4D_5,$	$I=\{4\},\ R_a=\{1,2,3,5\}$	$g_1,g_2,g_3\neq 0,\ g_5\neq\Lambda$
	$g_1D_5D_1-(g_1-\Lambda_a)D_1D_5=-x_5D_1,\quad g_2D_5D_2-(g_2-\Lambda_a)D_2D_5=-x_5D_2,$ $g_3D_5D_3-(g_3-\Lambda_a)D_3D_5=-x_5D_3,\quad g_4D_5D_4-(g_4-\Lambda_a)D_4D_5=-x_5D_4,$	$I=\{5\},\ R_a=\{1,2,3,4\}$	$g_1,g_2,g_3,g_4\neq 0$
D	$D_1D_2-q_{21}D_2D_1=0,\quad D_1D_3-q_{31}D_3D_1=0,$ $D_1D_4-q_{41}D_4D_1=0,\quad D_1D_5-q_{51}D_5D_1=0,$ $D_2D_3-q_{32}D_3D_2=0,\quad D_2D_4-q_{42}D_4D_2=0,$ $D_2D_5-q_{52}D_5D_2=0,\quad D_3D_4-q_{43}D_4D_3=0$ $D_3D_5-q_{53}D_5D_3=0,\quad D_4D_5-q_{54}D_5D_4=0$	$R=\{1,2,3,4,5\}$	$q_{sr}\neq 0,\ 1\leq r<s\leq 5$

CHAPTER 5

Smooth geometry of skew Poincaré-Birkhoff-Witt extensions

Bell and Goodearl [BG88] defined the *Poincaré-Birkhoff-Witt* (*PBW* for short) *extensions* with the aim of cover several families of generalized operator rings as the enveloping algebra of a finite-dimensional Lie algebra, Weyl algebras, differential operators over Lie algebras, the twisted or smash product differential operator rings and universal enveloping rings [BG88, Section 5]. Different properties of PBW extensions have been studied by some researchers [AM16, MWGZ02, GLZ14, Goo90, MZ96, Mat98, Sei10, TRS20].

With the aim of generalizing Ore extensions of injective type and PBW extensions, Gallego and Lezama [GL10] introduced the notion of *skew PBW* (*SPBW*) *extension*. Over the years several authors have shown that SPBW extensions also generalize families of noncommutative algebras such as *3-dimensional skew polynomial algebras* introduced by Bell and Smith [BS90], *diffusion algebras* defined by Isaev et al. [IPR01, PT02], *ambiskew polynomial rings* introduced by Jordan [Jor00, JW96], *solvable polynomial rings* introduced by Kandri-Rody and Weispfenning [KRW90], *almost normalizing extensions* defined by McConnell and Robson [MRS01], and *skew bi-quadratic algebras with PBW basis* introduced by Bavula [Bav23]. As expected, there are different relations between SPBW extensions and other noncommutative algebras having PBW bases defined in the literature (e.g. [Ape88, Bav20, BGTV03, GT14, Lev05, Li02, MRS01, Ros95, Sei10]).

Having in mind that ring-theoretical and geometrical properties of SPBW extensions (and hence of PBW extensions) have been investigated by different authors [AT24, Art15, GT14, FGL$^+$20, HHR20a, HKA17, LG19, NRR20, RR21, RS20, SRS23, TRS20], our purpose in this chapter is to investigate this smoothness for the SPBW extensions over the commutative polynomial rings $\Bbbk[\mathtt{t}]$ and $\Bbbk[\mathtt{t}_1, \mathtt{t}_2]$ (in a sequel paper we study the differential smoothness in the case of commutative polynomial rings generated on three and more indeterminates). In this way, we contribute to the study of the noncommutative geometry (algebraic and differential) of SPBW extensions that has been carried out by Lezama [Lez20, Lez21, LG19, LL17] and other people [CR24a, NR24, SRS23].

The chapter is organized as follows. In Section 5.1 we recall the definitions and preliminaries on SPBW extensions in order to set up notation and render this chapter

self-contained. Section 5.2.1 contains preliminary facts on the extension of automorphisms and derivations of a ring R to a SPBW extension over R. Next, we present the first original results of the chapter: we characterize the differential smoothness of this kind of SPBW extensions (Theorem 5.8). Next, in Section 5.3 we extend Brzeziński's ideas developed for skew polynomial rings of the commutative polynomial ring $\Bbbk[t]$ [Brz15] (Section 2.2.1) to the setting of SPBW extensions over $\Bbbk[t]$. Due to the length of the non-trivial computations, first we take as toy models the SPBW extensions generated by two and three indeterminates (Sections 5.3.1 and 5.3.2, respectively), while the general case is presented in Section 5.3.3. Theorems 5.11, 5.13 and 5.15 are the key results that establish sufficient conditions to assert that a SPBW extension over $\Bbbk[t]$ is differentially smooth. In Section 5.4, we study the differential smoothness of SPBW extensions over $\Bbbk[t_1, t_2]$. As it can be seen, the computations are highly non-trivial. Just as we did in Section 5.3, we divide our treatment in the case of two, three and n indeterminates (Sections 5.4.1, 5.4.2, and 5.4.3, respectively). The important results in this section are Theorems 5.17, 5.19 and 5.21. In Section 5.5 we say a few words about a future work related to the sequel paper.

5.1 SPBW extensions

Let us start with the *Poincaré-Birkhoff-Witt extensions* defined by Bell and Goodearl [BG88].

Definition 5.1 ([BG88, p. 27])**.** A ring A is a *Poincaré-Birkhoff-Witt* (*PBW*) *extension of* R if the following conditions hold:

(i) A contains R as a proper subring.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a free left R-module with basis $\text{Mon}(A) := \left\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\right\}$.

(iii) For each $r \in R$ and every $1 \leq i \leq n$, we have that $x_i r - r x_i \in R$.

(iv) For every $1 \leq i, j \leq n$, we have that $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$.

We write $A = R\langle x_1, \dots, x_n\rangle$, and R is called *the ring of coefficients* of the extension A.

We present some examples of PBW extensions. Some of these cannot be expressed as iterated skew polynomial rings. These examples are adapted from [BG88, McC74, Pas87, Rin63].

Example 5.1. (i) If R is a $\Bbbk$-algebra containing $\Bbbk$, then $R \otimes_{\Bbbk} U(\mathfrak{g})$ and the *crossed product* $R * U(\mathfrak{g})$ of R by $U(\mathfrak{g})$ are PBW extensions of R.

(ii) If L is a Lie algebra and also a free R-module equipped with a Lie algebra map to derivations on R, then the *differential operator ring* $V(R, L)$ satisfies the property PBW [Rin63, Theorem 3.1], and hence $V(R, L)$ is a PBW extension of R.

(iii) If $\mathfrak{g}$ acts on R by derivations and σ is Lie 2-cocycle, then the *twisted product differential operator ring* $R\#_\sigma U(\mathfrak{g})$ satisfies the property PBW by [McC74, Theorem 2.8], and thus $R\#_\sigma U(\mathfrak{g})$ is a PBW extension of R.

(iv) If R is a $\Bbbk$-algebra and V is a $\Bbbk$-vector space which is also a Lie ring containing R and $\Bbbk$ as Lie ideals, then the *universal enveloping ring* $\mathcal{U}(V, R, \Bbbk)$ satisfies the property PBW [Pas87, Theorem 1.3], and hence it is a PBW extension of R.

Definition 5.2 ([GL10, Definition 1]). Let R and A be rings. We say that A is a *SPBW extension* over R if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist elements $x_1, \dots, x_n \in A \setminus R$ such that A is a left free R-module with basis given by the set $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i, j \leq n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n,$$

i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)} \in R$ with

$$x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{k=1}^{n} r_k^{(i,j)} x_k.$$

We use freely the notation $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ to denote a SPBW extension A over a ring R in the indeterminates $x_1, \dots, x_n$. R is called the *ring of coefficients* of the extension A.

Since $\text{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ in Definition 5.2 are unique. Every element $f \in A \setminus \{0\}$ has a unique representation as $f = \sum_{i=0}^{t} r_i X_i$, with $r_i \in R \setminus \{0\}$ and $X_i \in \text{Mon}(A)$ for $0 \leq i \leq t$ with $X_0 = 1$. When necessary, we use the notation $f = \sum_{i=0}^{t} r_i Y_i$. For $X = x^\alpha \in \text{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. Let $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$ [GL10, Remark 2 and Definition 6].

If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a SPBW extension over R, then for each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$ [GL10, Proposition 3]. We use the notation $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$, and say that the pair (Σ, Δ) is a *system of endomorphisms and Σ-derivations* of R with respect to A. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^\alpha := \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes the classical composition of functions.

Definition 5.3 ([GL10, Definition 4], [LAR15, Definition 2.3 (ii)]). Consider a SPBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ over R.

(i) A is called *quasi-commutative* if the conditions (iii) - (iv) in Definition (5.2) are replaced by the following:

- For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.
- For every $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j} x_i x_j$.

(ii) A is *bijective* if σ_i is bijective, for every $1 \leq i \leq n$, and $d_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(iii) If σ_i is the identity map of R for each $i = 1, \ldots, n$, then we say that A is of *derivation type*. Similarly, if δ_i is zero, for every i, then A is called of *endomorphism type*.

(iv) A is said to be *semi-commutative* if it is quasi-commutative and $x_i r = r x_i$, for each i and every $r \in R$.

Next, we consider some interesting families examples of SPBW extensions.

Example 5.2. (i) SPBW extensions of endomorphism type over a ring are more general than iterated Ore extensions of endomorphism type of the same ring. Let us illustrate the situation with two and three indeterminates.

For the iterated Ore extension of endomorphism type $R[x;\sigma_x][y;\sigma_y]$, if $r \in R$ then we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. Now, if we have $\sigma(R)\langle x, y\rangle$ a SPBW extension of endomorphism type over R, then for any $r \in R$, Definition 5.2 establishes that $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and r_2 belong to R.

If we have the iterated Ore extension $R[x;\sigma_x][y;\sigma_y][z;\sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. For the SPBW extension of endomorphism type $\sigma(R)\langle x, y, z\rangle$, $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r'_0 + r'_1x + r'_2y + r'_3z$, and $zy = d_{2,3}yz + r''_0 + r''_1x + r''_2y + r''_3z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r'_0, r''_0, r_1, r'_1, r''_1, r_2, r'_2, r''_2, r_3, r'_3, r''_3$ of R. As the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

(ii) From Definition 5.2 (iv), it is clear that SPBW extensions are more general than iterated skew polynomial rings. For example, universal enveloping algebras of finite dimensional Lie algebras and some 3-dimensional skew polynomial algebras in the sense of Bell and Smith [BS90] cannot be expressed as iterated skew polynomial rings but are SPBW extensions. Quasi-commutative SPBW extensions are isomorphic to iterated Ore extensions of endomorphism type [LR14, Theorem 2.3].

(iii) PBW extensions introduced by Bell and Goodearl [BG88] are particular examples of SPBW extensions. More exactly, the first objects satisfy the relation $x_i r = r x_i + \delta_i(r)$ for every $i = 1, \ldots, n$ and each $r \in R$, and the elements d_{ij} in Definition 5.2 (iv) are equal to the identity of R. As examples of PBW extensions, we mention the following: the enveloping algebra of a finite-dimensional Lie algebra; any differential operator ring $R[\theta_1, \ldots, \theta_1; \delta_1, \ldots, \delta_n]$ formed from commuting derivations $\delta_1, \ldots, \delta_n$; differential operators introduced by Rinehart; twisted or smash product differential operator rings, and others [BG88, p. 27].

(iv) *3-dimensional skew polynomial algebras* considered in Section 1.2.2.

(v) *Diffusion algebras* were introduced from the physicist point of view by Isaev et al. [IPR01] as quadratic algebras that appear as algebras of operators that model the stochastic flow of motion of particles in a one dimensional discrete lattice, while

Pyatov and Twarock [PT02] presented a construction formalism for these algebras and to use the latter to prove the results in [IPR01]: "Diffusion algebras play a key role in the understanding of one-dimensional stochastic processes. In the case of N species of particles with only nearest-neighbor interactions with exclusion on a one-dimensional lattice, diffusion algebras are useful tools in finding expressions for the probability distribution of the stationary state of these processes. Following the idea of matrix product states, the latter are given in terms of monomials built from the generators of a quadratic algebra" [PT02, p. 3268].

Following Pyatov and Twarock's notation and let α, β be two elements belonging to the set $I_N := \{1, \ldots, N\}$ with $\alpha < \beta$. Consider quadratic relations of the form

$$g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha = x_\beta D_\alpha - x_\alpha D_\beta, \tag{5.1.1}$$

with $g_{\alpha\beta} \in \mathbb{R} \setminus \{0\}$, $g_{\beta\alpha} \in \mathbb{R}$, and $x_\alpha, x_\beta \in \mathbb{C}$.

From [PT02, Definition 1.1], an algebra with set of generators given by $\{D_\alpha \mid \alpha \in I_N\}$ and relations of type (5.1.1) is called *diffusion algebra*, if it admits a linear PBW-basis of ordered monomials of the form

$$D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_N}^{k_N}, \quad \text{with} \quad k_j \in \mathbb{N} \quad \text{and} \quad \alpha_1 > \alpha_2 > \cdots > \alpha_N. \tag{5.1.2}$$

Due to physical reasons only relations with positive coefficients $g_{\alpha\beta} \in \mathbb{R}_{>0}$ and $g_{\beta\alpha} \in \mathbb{R}_{\geq 0}$ $(\alpha < \beta)$ are relevant because they are interpreted as hopping rates in stochastic models [PT02, p. 3268].

(vi) The *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1, \ldots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ ... It is clear from the definition that bi-quadratic algebras having PBW basis are particular examples of SPBW extensions.

The following proposition establishes an interesting relationship between SPBW extensions and SAS regular algebras.

Proposition 5.1 ([Lez21, Theorem 4.14]). *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension over R that satisfies the following conditions:*

(i) *R and A are $\Bbbk$-algebras.*

(ii) $\operatorname{lgld}(R) < \infty$.

(iii) *R is an FSG ring with semi-graduation $R = \bigoplus_{p \geq 0} R_p$.*

(iv) *R is connected, i.e. $R_0 = \Bbbk$.*

(v) *For $1 \leq i \leq n$, σ_i and δ_i in Definition 5.2 are homogeneous, and there exist i, j such that the parameter d_{ij} satisfies $d_{ij} \in \Bbbk \setminus \{0\}$.*

Then, A is SAS.

5.2 Differential smoothness of SPBW extensions

One of the topics of research on Ore extensions has been carried out by several authors concerning to the ring-theoretical notions of *Baer*, *quasi-Baer*, *p.p.*, *p.q.*, *Armendariz* and *reduced* rings (e.g. [BKP01, HMJ03, HM05, HKK00, HKK03, HLS02, Kap68, Kre96, LW03, Mat04] and references therein; for a detailed description of each one of these notions, see the excellent treatment developed by Birkenmeier et al. [BPR13]). All these properties are formulated to the ring of coefficients R of the Ore extension $R[x;\sigma,\delta]$ under certain adequate conditions on the maps σ and δ. The *extension* of these theoretical notions from R to $R[x;\sigma,\delta]$ was considered by Nasr-Isfahani and Moussavi [NIM08]. In their paper, for a ring R with an automorphism σ and σ-derivation δ where $\alpha\delta = \delta\alpha$, they considered the $\overline{\sigma}$-derivation $\overline{\delta}$ on the Ore extension $R[x;\sigma,\delta]$ which, precisely, *extends* δ. More exactly, for an element $f(x) = r_0 + r_1x + \cdots + r_nx^n \in R[x;\sigma,\delta]$ they defined the automorphism $\overline{\sigma}$ and the $\overline{\sigma}$-derivation $\overline{\delta}$ on $R[x;\sigma,\delta]$ as

$$\overline{\sigma}(f(x)) = \sigma(r_0) + \sigma(r_1)x + \cdots + \sigma(r_n)x^n, \quad \text{and}$$
$$\overline{\delta}(f(x)) = \delta(r_0) + \delta(r_1)x + \cdots + \delta(r_n)x^n,$$

respectively. Such an automorphism $\overline{\sigma}$ and derivation $\overline{\delta}$ of $R[x;\sigma,\delta]$ are called *extended automorphism* and *derivation of* δ.

Related to the study of transfer of ring-theoretical properties from coefficient rings to noncommutative polynomial extensions over these rings, Kwak et al. [KYL14] extended the reflexive property to the skewed reflexive property by ring endomorphisms. An endomorphism σ of a ring R is called *right* (resp., *left*) *skew reflexive* if for $a, b \in R$, $aRb = 0$ implies $bR\sigma(a) = 0$ (resp., $\sigma(b)Ra = 0$), and R is called *right* (resp., *left*) σ-*skew reflexive* if there exists a right (resp., left) skew reflexive endomorphism σ of R. R is said to be σ-*skew reflexive* if it is both right and left σ-skew reflexive. It is clear that σ-rigid rings are right σ-skew reflexive. More precisely, a ring R is reduced and right σ-skew reflexive for a monomorphism σ of R if and only if R is σ-rigid [KYL14, Theorem 2.6]. Bhattacharjee [Bha20] extend the notion of RNP rings (Kheradmand et al. [KKKL17], where *RNP* means *reflexive-nilpotents-property*) to ring endomorphisms σ and introduced the notion of σ-skew RNP rings as a generalization of σ-skew reflexive rings. An endomorphism σ of a ring R is called *right* (resp., *left*) *skew* RNP if for $a, b \in N(R)$ (the set of nilpotent elements of R), $aRb = 0$ implies $bR\sigma(a) = 0$ (resp., $\sigma(b)Ra = 0$). A ring R is called *right* (resp., *left*) σ-*skew* RNP if there exists a right (resp., left) skew RNP endomorphism σ of R. R is said to be σ-*skew* RNP if it is both right and left σ-skew RNP. From [Bha20, Remark 1.2], we know that reduced rings are σ-skew RNP for any endomorphism σ, and every right (resp., left) σ-skew reflexive ring is right (resp., left) σ-skew RNP. By [Bha20, Example 1.3], we have that the notion of σ-skew RNP ring is not left-right symmetric. However, if R is an RNP ring with an endomorphism σ, then R is right σ-skew RNP if and only if R is left σ-skew RNP.

Due to the relation between Ore extensions and SPBW extensions in terms of endomorphisms and derivations (Definition 5.2), Reyes and Suárez [RS17b] carried out a similar work to the presented by by Nasr-Isfahani and Moussavi [NIM08] but now in the more general setting of SPBW extensions (see Section 5.1 for all details), while Suárez et al. [SHR24] studied the reflexive-nilpotents-property for skew PBW extensions. There,

they introduced the Σ-*skew CN* and Σ-*skew reflexive* (*RNP*) *rings* and under conditions of compatibility, they investigated the transfer of the reflexive-nilpotents-property from a ring of coefficients to a skew PBW extension over it. Their results extend those corresponding presented by Bhattacharjee [Bha20] for Ore extensions.

With all above facts in mind, our purpose in this paper is to investigate the differential smoothness of skew PBW extensions considering extended automorphisms and derivations, from the ring of coefficients, to the skew PBW over this ring. In this way, we contribute to the the study of algebraic and geometric properties of the Ore extensions considered by Nasr-Isfahani and Moussavi [NIM08] and Bhattacharjee [Bha20], and hence continue the research on the differential smoothness of noncommutative algebras related to SPBW extensions.

5.2.1 Extension of automorphisms and derivations

With the aim of setting up notation and render this paper self-contained, next we present the details of extended derivations over SPBW extensions on one (this is the case of the classical Ore extensions), two and n generators (Propositions 5.3, 5.7 and 5.7, respectively).

We start with the following well-known result (e.g. Lam et al. [LLM97, p. 2468]).

Lemma 5.2. *Let $A = \sigma(R)\langle x\rangle$ be a SPBW extension over R, that is, $A = R[x;\sigma,\delta]$. For any $n \in \mathbb{N}$, we have that*

$$x^n r = \sum_{k=0}^{n} \sum_{f_k \in C_{n-k,k}} f_k(r) x^{n-k},$$

where $C_{n-k,k}$ is the set of all possible compositions between $n-k$ σ's and k δ's. In addition, if $\sigma \circ \delta = \delta \circ \sigma$, we get that

$$x^n r = \sum_{k=0}^{n} \binom{n}{k} \sigma^{n-k} \circ \delta^k(r) x^{n-k}.$$

Proof. We proceed by induction on n. For $n = 1$, we know that $C_0 = \{\sigma\}$ and $C_1 = \{\delta\}$, whence

$$\begin{aligned} xr &= \sigma(r)x + \delta(r) \\ &= \sum_{k=0}^{1} \sum_{f_k \in C_{n-k,k}} f_k(r) x^{1-k}. \end{aligned}$$

Suppose the relation holds for n. Then

$$\begin{aligned} x^{n+1}r &= x\left(\sum_{k=0}^{n}\sum_{f_k\in C_{n-k,k}} f_k(r)x^{n-k}\right) \\ &= \sum_{k=0}^{n}\sum_{f_k\in C_{n-k,k}} xf_k(r)x^{n-k} = \sum_{k=0}^{n}\sum_{f_k\in C_{n-k,k}} (\sigma(f_k(r))x+\delta(f_k(r)))x^{n-k} \\ &= \sum_{k=0}^{n}\sum_{f_k\in C_{n-k,k}} \left[\sigma(f_k(r))x^{n+1-k}+\delta(f_k(r))x^{n-k}\right] \\ &= \sum_{k=0}^{n+1}\sum_{g_k\in C_{n+1-k,k}} g_k(r)x^{n+1-k}. \end{aligned}$$

If σ and δ commute, it follows that $C_k = \{\sigma^{n-k}\circ\delta^k\}$ and the number of times that the element is repeated is $\binom{n}{k}$, which implies that

$$x^n r = \sum_{k=0}^{n}\sum_{f_k\in C_{n-k,k}} f_k(r)x^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\sigma^{n-k}\circ\delta^k(r)x^{n-k}.$$

□

The importance of extending endomorphisms and derivations of the ring R to the extension A over R can be appreciated in works such as [NIM08, RS17b] for the study of ring-theoretical properties. In particular, the following result appears without proof in [NIM08, p. 514].

Proposition 5.3. *Let $A = \sigma(R)\langle x\rangle$ be a SPBW extension over R of automorphism type. Consider the automorphism $\widetilde{\sigma}: A\to A$ given by $\widetilde{\sigma}(x) = x$ and $\widetilde{\sigma}(r) = \sigma(r)$ for each $r\in R$. If $\sigma\circ\delta = \delta\circ\sigma$, then the map $\widetilde{\delta}: A\to A$ with $\widetilde{\delta}(f(x)) = \widetilde{\delta}(a_0+a_1x+\cdots+a_nx^n) = \delta(a_0)+\delta(a_1)x+\cdots+\delta(a_n)x^n$ for all $a_i\in R$, $0\leq i\leq n$ is a $\widetilde{\sigma}$-derivation of A.*

Proof. Let $p(x) = \sum_{i=0}^{n} r_ix^i \in A$. Then

$$\widetilde{\sigma}(p(x)) = \widetilde{\sigma}\left(\sum_{i=0}^{n} r_ix^i\right) = \sum_{i=0}^{n}\widetilde{\sigma}(r_ix^i) = \sum_{i=0}^{n}\widetilde{\sigma}(r_i)x^i = \sum_{i=0}^{n}\sigma(r_i)x^i.$$

It is clear that the bijectivity of $\widetilde{\sigma}$ is inherited by σ.

Now, for $p(x) = rx^k$, we have that

$$\widetilde{\delta}(p(x)) = \widetilde{\delta}(rx^k) = \widetilde{\sigma}(r)\widetilde{\delta}(x^k)+x\widetilde{\delta}(r)x^k = \sigma(r)\widetilde{\delta}(x^k)+\delta(r)x^k.$$

Let $r(x) = rx^i$ and $s(x) = sx^j$ for some $i,j\in\mathbb{N}$. The idea is to satisfy the relation given by

$$\widetilde{\delta}(r(x)s(x)) = \widetilde{\sigma}(r(x))\widetilde{\delta}(s(x))+\widetilde{\delta}(r(x))s(x).$$

With this aim, note that

$$\begin{aligned}
\widetilde{\delta}(r(x)s(x)) &= \widetilde{\delta}(rx^i sx^j)\\
&= \widetilde{\delta}\left(r\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} f_k(s)x^{i-k}x^j\right)\\
&= \widetilde{\delta}\left(\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} rf_k(s)x^{i-k+j}\right)\\
&= \sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} \widetilde{\delta}\left(rf_k(s)x^{i-k+j}\right)\\
&= \sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} \left(\sigma(rf_k(s))\widetilde{\delta}(x^{i-k+j}) + \delta(rf_k(s))x^{i-k+j}\right)\\
&= \sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} \left(\sigma(r)\sigma(f_k(s))\widetilde{\delta}(x^{i-k+j}) + (\sigma(r)\delta(f_k(s)) + \delta(r)f_k(s))\, x^{i-k+j}\right)\\
&= \sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} \Big(\sigma(r)\sigma(f_k(s))\left(x^{i-k}\widetilde{\delta}(x^j) + \widetilde{\delta}(x^{i-k})x^j\right)\\
&\quad + (\sigma(r)\delta(f_k(s)) + \delta(r)f_k(s))\, x^{i-k+j}\Big). \qquad (5.2.1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\widetilde{\sigma}(r(x))\widetilde{\delta}(s(x)) + \widetilde{\delta}(r(x))s(x) &= \widetilde{\sigma}(rx^i)\widetilde{\delta}(sx^j) + \widetilde{\delta}(rx^i)sx^j\\
&= \sigma(r)x^i\left(\sigma(s)\widetilde{\delta}(x^j) + \delta(s)x^j\right) \qquad (5.2.2)\\
&\quad + \left(\sigma(r)\widetilde{\delta}(x^i) + \delta(r)x^i\right)sx^j\\
&= \sigma(r)x^i\sigma(s)\widetilde{\delta}(x^j) + \sigma(r)x^i\delta(s)x^j \qquad (5.2.3)\\
&\quad + \sigma(r)\widetilde{\delta}(x^i)sx^j + \delta(r)x^i sx^j\\
&= \sigma(r)\left(\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} f_k(\sigma(s))x^{i-k}\right)\widetilde{\delta}(x^j) \qquad (5.2.4)\\
&\quad + \sigma(r)\left(\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} f_k(\delta(s))x^{i-k}\right)x^j\\
&\quad + \sigma(r)\widetilde{\delta}(x^i)sx^j + \delta(r)\left(\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}} f_k(s)x^{i-k}\right)x^j,
\end{aligned}$$

that is,

$$\widetilde{\sigma}(r(x))\widetilde{\delta}(s(x))+\widetilde{\delta}(r(x))s(x)=\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}}\Big(\sigma(r)f_k(\sigma(s))x^{i-k}\widetilde{\delta}(x^j)+\sigma(r)f_k(\delta(s))x^{i-k+j} \tag{5.2.5}$$

$$+\,\delta(r)f_k(s)x^{i-k+j}\Big)+\sigma(r)\widetilde{\delta}(x^i)sx^j. \tag{5.2.6}$$

If we compare (5.2.1) and (5.2.6), we get that

$$\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}}\Big((\sigma(f_k(s))-f_k(\sigma(s)))x^{i-k}\widetilde{\delta}(x^j)+(\delta(f_k(s))-f_k(\delta(s)))x^{i-k+j}$$
$$+\,\sigma(f_k(s))\widetilde{\delta}(x^{i-k})x^j\Big)-\widetilde{\delta}(x^i)sx^j=0.$$

Let $\widetilde{\delta}(x^i)=\sum_{l=0}^{n}a_lx^l$, $\widetilde{\delta}(x^j)=\sum_{l=0}^{n}b_lx^l$ and $\widetilde{\delta}(x^{i-k})=\sum_{l=0}^{n}c_lx^l$. Then,

$$\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}}\Bigg((\sigma(f_k(s))-f_k(\sigma(s)))x^{i-k}\sum_{l=0}^{n}b_lx^l+(\delta(f_k(s))-f_k(\delta(s)))x^{i-k+j}$$
$$+\,\sigma(f_k(s))\sum_{l=0}^{n}c_lx^lx^j\Bigg)-\sum_{l=0}^{n}a_lx^lsx^j=0.$$

$$\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}}\Bigg((\sigma(f_k(s))-f_k(\sigma(s)))\sum_{l=0}^{n}\Bigg(\sum_{p=0}^{i-k}\sum_{h_p\in C_{i-k-p,p}}h_p(b_l)x^{i-k-p}\Bigg)x^l$$
$$+\,(\delta(f_k(s))-f_k(\delta(s)))x^{i-k+j}+\sigma(f_k(s))\sum_{l=0}^{n}c_lx^lx^j\Bigg)$$
$$-\sum_{l=0}^{n}a_l\Bigg(\sum_{k=0}^{l}\sum_{g_k\in C_{l-k,k}}g_k(s)x^{l-k}\Bigg)x^j=0.$$

$$\sum_{k=0}^{i}\sum_{f_k\in C_{i-k,k}}\Bigg((\sigma(f_k(s))-f_k(\sigma(s)))\sum_{l=0}^{n}\sum_{p=0}^{i-k}\sum_{h_p\in C_{i-k-p,p}}h_p(b_l)x^{i-k-p+l} \tag{5.2.7}$$
$$+\,(\delta(f_k(s))-f_k(\delta(s)))x^{i-k+j}+\sigma(f_k(s))\sum_{l=0}^{n}c_lx^{l+j}\Bigg)$$
$$-\sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{g_k\in C_{l-k,k}}a_lg_k(s)x^{l-k+j}=0.$$

When $k = 0$, we put attention on the monomial $\sum_{f_0 \in C_{i,0}} (\delta(f_0(s)) - f_0(\delta(s)))x^{i+j}$. This is the monomial of the highest degree and since it is equal to zero, it follows that $\sum_{f_0 \in C_{i,0}} (\delta(f_0(s)) - f_0(\delta(s))) = 0$. Since the power i is arbitrary, it can be seen that $\delta(\sigma(s)) = \sigma(\delta(s))$. In the same way, the value of s is arbitrary, which means that $\delta\sigma = \sigma\delta$.

Now, expression (5.2.7) can be written as

$$\sum_{k=0}^{i} \sum_{f_k \in C_{i-k,k}} \left(\sigma(f_k(s)) \sum_{l=0}^{n} c_l t^{l+j} \right) - \sum_{l=0}^{n} \sum_{k=0}^{l} \sum_{g_k \in C_{l-k,k}} a_l g_k(s) x^{l-k+j} = 0,$$

or equivalently,

$$\sum_{l=0}^{n} \left(\sum_{k=0}^{i} \sum_{f_k \in C_{i-k,k}} \sigma(f_k(s)) c_l x^l - \sum_{k=0}^{l} \sum_{g_k \in C_{l-k,k}} a_l g_k(s) x^{l-k} \right) = 0.$$

Since this equality holds for all $s \in R$, any automorphism σ and every σ-derivation δ of R, necessarily $a_l = c_l = 0$ for all $0 \leq l \leq n$. By changing the values of i with j we obtain that $b_l = 0$. Then $\widetilde{\delta}(x) = 0$. This means that $\widetilde{\delta}(f(x)) = \widetilde{\delta}(a_0 + a_1 x + \cdots + a_n x^n) = \delta(a_0) + \delta(a_1)x + \cdots + \delta(a_n)x^n$. □

Lemma 5.4. *Let $A = \sigma(R)\langle x_1, x_2 \rangle$ be a SPBW extension over R such that σ_i commutes with δ_i for $i = 1, 2$. Then for any $m \in \mathbb{N}$ we have that*

$$x_i^m r = \sum_{k=0}^{m} \binom{m}{k} \sigma_i^{m-k} \circ \delta_i^k(r) x_i^{m-k}, \quad \text{for } i = 1, 2.$$

Proof. This result can be proven in a completely analogous way to the Lemma 5.2; it is enough to apply induction on x_1 and then on x_2. It must also be taken into account that σ_i commutes with δ_i for $i = 1, 2$. □

Proposition 5.5. *Let $A = \sigma(R)\langle x_1, x_2 \rangle$ be a skew PBW extension over R of automorphism type. Consider the endomorphisms $\widetilde{\sigma_i} : A \to A$ and $\widetilde{\delta_i} : A \to A$ defined as*

$$\widetilde{\sigma_i}\left(\sum r_k x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}} \right) = \sum \sigma_i(r_k) x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}},$$
$$\widetilde{\delta_i}\left(\sum r_k x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}} \right) = \sum \delta_i(r_k) x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}},$$

for $i = 1, 2$, and such that σ_i and δ_i commute, $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, $\delta_i \circ \sigma_j = \sigma_j \circ \delta_i$ and $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$ for $i, j, k, l = 1, 2$. Then $\widetilde{\sigma}$ is an automorphism of A and $\widetilde{\delta}$ is a $\widetilde{\sigma_i}$-derivation of A.

Proof. Let $f = \sum_{k=0}^{m} r_k x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}}$, where $r_k \in R$ and $\alpha_{1,k}, \alpha_{2,k} \in \mathbb{N}$, for $1 \leq k \leq m$. Then

$$\widetilde{\sigma_i}(f) = \widetilde{\sigma_i}\left(\sum_{k=0}^{m} r_k x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}} \right) = \sum_{k=0}^{m} \widetilde{\sigma_i}\left(r_k x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}} \right) = \sum_{k=0}^{m} \widetilde{\sigma_i}\left(r_k \right) x_1^{\alpha_{1,k}} x_2^{\alpha_{2,k}}.$$

The bijectivity of $\widetilde{\sigma_i}$ is inherited by σ_i for $i = 1, 2$.

Now, for $p = rx_1^{\alpha_1}x_2^{\alpha_2}$, we have that

$$\begin{aligned}\widetilde{\delta_i}(p) &= \widetilde{\delta_i}(rx_1^{\alpha_1}x_2^{\alpha_2}) = \widetilde{\sigma_i}(r)\widetilde{\delta_i}(x_1^{\alpha_1}x_2^{\alpha_2}) + \widetilde{\delta_i}(r)x_1^{\alpha_1}x_2^{\alpha_2}\\ &= \sigma_i(r)\left(\widetilde{\delta_i}(x_1^{\alpha_1})x_2^{\alpha_2} + x_1^{\alpha_1}\widetilde{\delta_i}(x_2^{\alpha_2})\right) + \delta_i(r)x_1^{\alpha_1}x_2^{\alpha_2}\\ &= \delta_i(r)x_1^{\alpha_1}x_2^{\alpha_2}.\end{aligned}$$

Consider $p = rx_1^{\alpha_1}x_2^{\alpha_2}$ and $s = sx_1^{\beta_1}x_2^{\beta_2}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$. Then,

$$\begin{aligned}\widetilde{\delta_i}(ps) &= \widetilde{\delta}(rx_1^{\alpha_1}x_2^{\alpha_2}sx_1^{\beta_1}x_2^{\beta_2})\\ &= \widetilde{\delta_i}\left(rx_1^{\alpha_1}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)x_2^{\alpha_2-k_2}x_1^{\beta_1}x_2^{\beta_2}\right)\\ &= \widetilde{\delta_i}\left(\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}rx_1^{\alpha_1}\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)x_2^{\alpha_2-k_2}x_1^{\beta_1}x_2^{\beta_2}\right)\\ &= \widetilde{\delta_i}\left(\sum_{k_2=0}^{\alpha_2}\binom{\alpha_2}{k_2}r\sum_{k_1=0}^{\alpha_1}\binom{\alpha_1}{k_1}\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)x_1^{\alpha_1-k_1}x_2^{\alpha_2-k_2}x_1^{\beta_1}x_2^{\beta_2}\right)\\ &= \sum_{k_1=0}^{\alpha_1}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\delta_i\left(r\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)\right)x_1^{\alpha_1-k_1}x_2^{\alpha_2-k_2}x_1^{\beta_1}x_2^{\beta_2}. \qquad (5.2.8)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\widetilde{\sigma_i}(p)\widetilde{\delta_i}(s) + \widetilde{\delta_i}(p)s = \widetilde{\sigma_i}(rx_1^{\alpha_1}x_2^{\alpha_2})\widetilde{\delta_i}(sx_1^{\beta_1}x_2^{\beta_2}) + \widetilde{\delta_i}(rx_1^{\alpha_1}x_2^{\alpha_2})sx_1^{\beta_1}x_2^{\beta_2}\\ &= \sigma_i(r)x_1^{\alpha_1}x_2^{\alpha_2}\delta_i(s)x_1^{\beta_1}x_2^{\beta_2} + \delta_i(r)x_1^{\alpha_1}x_2^{\alpha_2}sx_1^{\beta_1}x_2^{\beta_2}\\ &= \sum_{k_1=0}^{\alpha_1}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\sigma_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(\delta_i(s))\right)x_1^{\alpha_1-k_1}x_2^{\alpha_2-k_2}x_1^{\beta_1}x_2^{\beta_2}\\ &\quad + \sum_{k_1=0}^{\alpha_1}\sum_{k_2=0}^{\alpha_2}\binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\delta_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)x_1^{\alpha_1-k_1}x_2^{\alpha_2-k_2}x_1^{\beta_1}x_2^{\beta_2}. \qquad (5.2.9)\end{aligned}$$

We focus on the monomial $x_2^{\alpha_2-k_2}x_1^{\beta_1}$. With the aim of comparing the terms, necessarily the commutation rule of the ring must be applied: there, the elements $d_{1,2}$, $r_k^{(1,2)}$, $0 \leq k \leq 2$, appear. As expected, these must be eliminated except $r_0^{(1,2)}$. For this reason, we assume that $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$ for $i, j, k, l = 1, 2$. If we compare (5.2.8) with (5.2.9), then we get that

$$\begin{aligned}&\sigma_i(r)\delta_i\left(\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)\right) + \delta_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)\\ &= \sigma_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(\delta_i(s))\right) + \delta_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)\end{aligned}$$

$$\begin{aligned}&\sigma_i(r)\left(\delta_i\left(\sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(s)\right)\right) - \sigma_1^{\alpha_1-k_1}\circ\delta_1^{k_1}\left(\sigma_2^{\alpha_2-k_2}\circ\delta_2^{k_2}(\delta_i(s))\right)\right)\\ &= 0.\end{aligned}$$

Since δ_i commute with σ_j and δ_j, for $j = 1, 2$, this last equality holds, whence $\widetilde{\delta}_i$ is a $\widetilde{\sigma}_i$-derivation. □

Lemma 5.6. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over R such that σ_i commutes with δ_i, $1 \leq i \leq n$ for all i. Then, for any $m \in \mathbb{N}$ we have that*

$$x_i^m r = \sum_{k=0}^{m} \binom{m}{k} \sigma_i^{m-k} \circ \delta_i^k(r) x_i^{m-k}, \quad \text{for all } 1 \leq i \leq n.$$

Proof. The proof of this lemma, similar to the previous Lemmas 5.2 and 5.4, is done by induction on each x_i, $1 \leq i \leq n$. □

Proposition 5.7. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over R of automorphism type. Consider the endomorphisms $\widetilde{\sigma}_i : A \to A$ and $\widetilde{\delta}_i : A \to A$ defined as*

$$\begin{aligned}
\widetilde{\sigma}_i\left(\sum r_k x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}}\right) &= \sum \sigma_i(r_k) x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}} \\
\widetilde{\delta}_i\left(\sum r_k x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}}\right) &= \sum \delta_i(r_k) x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}},
\end{aligned}$$

for $1 \leq i \leq n$, and such that σ_i and δ_i commute, $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, $\delta_i \circ \sigma_j = \sigma_j \circ \delta_i$ and $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$, for $1 \leq i, j, k, l \leq n$. Then $\widetilde{\sigma}$ is an automorphism of A and $\widetilde{\delta}$ is a $\widetilde{\sigma}_i$-derivation of A.

Proof. Let $f = \sum_{k=0}^{m} c_k x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}}$, where $\alpha_{1,k}, \ldots, \alpha_{n,k} \in \mathbb{N}$, for $1 \leq k \leq m$. Then

$$\begin{aligned}
\widetilde{\sigma}_i(a) &= \widetilde{\sigma}_i\left(\sum_{k=0}^{m} c_k x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}}\right) \\
&= \sum_{k=0}^{m} \widetilde{\sigma}_i\left(c_k x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}}\right) \\
&= \sum_{k=0}^{m} \widetilde{\sigma}_i\left(c_k\right) x_1^{\alpha_{1,k}} \cdots x_n^{\alpha_{n,k}}.
\end{aligned}$$

With the above, the bijectivity of $\widetilde{\sigma}_i$ is inherited by σ_i, for $1 \leq i \leq n$.

For $p = r x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, we have that

$$\begin{aligned}
\widetilde{\delta}_i(p) &= \widetilde{\delta}_i(r x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \\
&= \widetilde{\sigma}_i(r)\widetilde{\delta}_i(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) + \widetilde{\delta}_i(r) x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\
&= \sigma_i(r)\left(\widetilde{\delta}_i(x_1^{\alpha_1}) x_2^{\alpha_2} \cdots x_n^{\alpha_n} + \cdots + x_1^{\alpha_1} \cdots \widetilde{\delta}_i(x_n^{\alpha_n})\right) + \delta_i(r) x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\
&= \delta_i(r) x_1^{\alpha_1} \cdots x_n^{\alpha_n}.
\end{aligned}$$

Let $p = rx_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $s = sx_1^{\beta_1}\cdots x_n^{\beta_n}$ for some $\alpha_1,\ldots,\alpha_n \in \mathbb{N}$ and elements $\beta_1,\ldots,\beta_n$ belonging to $\mathbb{N}$. Then

$$\begin{aligned}
\widetilde{\delta}_i(ps) &= \widetilde{\delta}(rx_1^{\alpha_1}\cdots x_n^{\alpha_n}sx_1^{\beta_1}\cdots x_n^{\beta_n}) \\
&= \widetilde{\delta}_i\left(\sum_{k_1=0}^{\alpha_1}\cdots\sum_{k_n=0}^{\alpha_n}\left(\prod_{j=1}^{n}\binom{l_j}{k_j}\right) r\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right)x_1^{\alpha_1-k_1}\right. \\
&\quad \left.\cdots\, x_{n-1}^{l_{n-1}-k_{n-1}}x_n^{\alpha_n-k_n}x_1^{\beta_1}\cdots x_n^{\beta_n}\right) \\
&= \sum_{k_1=0}^{\alpha_1}\cdots\sum_{k_n=0}^{\alpha_n}\left(\prod_{j=1}^{n}\binom{l_j}{k_j}\right)\delta_i\left(r\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right)\right)x_1^{\alpha_1-k_1} \\
&\quad \cdots x_{n-1}^{l_{n-1}-k_{n-1}}x_n^{\alpha_n-k_n}x_1^{\beta_1}\cdots x_n^{\beta_n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\widetilde{\sigma}_i(p)\widetilde{\delta}_i(s) + \widetilde{\delta}_i(p)s &= \widetilde{\sigma}_i(rx_1^{\alpha_1}\cdots x_n^{\alpha_n})\widetilde{\delta}_i(sx_1^{\beta_1}\cdots x_n^{\beta_n}) + \widetilde{\delta}_i(rx_1^{\alpha_1}\cdots x_n^{\alpha_n})sx_1^{\beta_1}\cdots x_n^{\beta_n} \\
&= \sigma_i(r)x_1^{\alpha_1}\cdots x_n^{\alpha_n}\delta_i(s)x_1^{\beta_1}\cdots x_n^{\beta_n} + \delta_i(r)x_1^{\alpha_1}\cdots x_n^{\alpha_n}sx_1^{\beta_1}\cdots x_n^{\beta_n} \\
&= \sum_{k_1=0}^{\alpha_1}\cdots\sum_{k_n=0}^{\alpha_n}\left(\prod_{j=1}^{n}\binom{l_j}{k_j}\right)\sigma_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(\delta_i(s))\cdots\right) \\
&\quad x_1^{\alpha_1-k_1}\cdots x_{n-1}^{l_{n-1}-k_{n-1}}x_n^{\alpha_n-k_n}x_1^{\beta_1}\cdots x_n^{\beta_n} \\
&\quad + \sum_{k_1=0}^{\alpha_1}\cdots\sum_{k_n=0}^{\alpha_n}\left(\prod_{j=1}^{n}\binom{l_j}{k_j}\right)\delta_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right) \\
&\quad x_1^{\alpha_1-k_1}\cdots x_{n-1}^{l_{n-1}-k_{n-1}}x_n^{\alpha_n-k_n}x_1^{\beta_1}\cdots x_n^{\beta_n}
\end{aligned}$$

For the same reason as in Proposition 5.5, when the elements $d_{i,j}$, $r_k^{(i,j)}$, $1 \leq i < j \leq n$, $0 \leq k \leq n$, appear, these must cancel out when the terms are equalized, except $r_0^{(i,j)}$, $1 \leq i < j \leq n$. For this reason, we assume that $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$ for $0 \leq i,j,k,l \leq n$. In this way,

$$\begin{aligned}
&\sigma_i(r)\delta_i\left(\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right)\right) \\
&\quad + \delta_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right) \\
&= \sigma_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(\delta_i(s))\cdots\right) \\
&\quad + \delta_i(r)\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right) \\
&= \sigma_i(r)\left(\delta_i\left(\sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(s)\cdots\right)\right)\right. \\
&\quad \left. - \sigma_1^{\alpha_1-k_1}\circ\delta_1^{\alpha_1-k_1}\left(\cdots\sigma_n^{\alpha_n-k_n}\circ\delta_n^{k_n}(\delta_i(s))\cdots\right)\right) \\
&= 0.
\end{aligned}$$

By using that δ_i commute with σ_j and δ_j, for $1 \leq j \leq n$, the previous equation holds. Therefore, $\widetilde{\delta}_i$ is a $\widetilde{\sigma}_i$-derivation. □

Remark 17. The commutativity of the maps σ's and δ's, that is, of the system of endomorphisms Σ and Σ-derivations Δ of R, as in Proposition 5.7, was considered by Lezama et al. [LAR15] to characterize prime ideals of SPBW extensions.

Since we are considering SPBW extensions over any associative and unital ring R, we need some natural conditions to compute the Gelfand-Kirillov dimension of a skew PBW extension A over R. Having in mind that Lezama and Venegas [LV20] generalized the classical notion of Gelfand-Kirillov dimension [GK66a, GK66b] considered by Brzeziński in his definition of differential smoothness (Section 1.4.2), throughout this section we follow this more general setting: we assume that R is an S-algebra (S a commutative domain) with a generator frame V and the *rank* of R is understood as $\operatorname{rank}(R) = \dim_Q(Q \otimes R) < \infty$ where Q is the field of fractions of S (see [LV20, Theorem 2.1] for more details).

The following theorem is the most important result of the paper.

Theorem 5.8. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension such that σ_i and δ_i are S-linear, $\sigma_i(V) \subseteq V$ for all $1 \leq i \leq n$, and $d_{i,j} = 1$, $r_k^{(i,j)} = 0$ for $1 \leq k \leq n$, $1 \leq i, j, \leq n$ (Definition 5.2 (iv)). Consider $\widetilde{\sigma}_i$ and $\widetilde{\delta}_i$ as in Proposition 5.7 for $1 \leq i \leq n$. If $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ for all $1 \leq i < j \leq n$, then A is differentially smooth.*

Proof. From [LV20, Theorem 2.1], we get that $\operatorname{GKdim}(\sigma(R)\langle x_1, \dots, x_n\rangle) = n$. Hence, we proceeed to construct an n-dimensional integrable calculus.

Consider $\Omega^1(\sigma(R)\langle x_1, \dots, x_n\rangle)$ a free right $\sigma(R)\langle x_1, \dots, x_n\rangle$-module of rank n with generators $dx_1, \dots, dx_n$. Define a left $\sigma(R)\langle x_1, \dots, x_n\rangle$-module structure by

$$f dx_i = dx_i \widetilde{\sigma}_i(f), \quad \text{for all } 1 \leq i \leq n,\ f \in \sigma(R)\langle x_1, \dots, x_n\rangle. \tag{5.2.10}$$

Notice that the relations in $\Omega^1(\sigma(R)\langle x_1, \dots, x_n\rangle)$ are given by

$$x_i dx_j = dx_j x_i, \quad \text{for all } 1 \leq i < j \leq n. \tag{5.2.11}$$

We want to extend the assignments $x_i \mapsto dx_i$ and $1 \leq i \leq n$ to a map

$$d : \sigma(R)\langle x_1, \dots, x_n\rangle \to \Omega^1(\sigma(R)\langle x_1, \dots, x_n\rangle)$$

satisfying the Leibniz's rule. As expected, this is possible if we guarantee its compatibility with the non-trivial relations (5.2), i.e. if

$$dx_j x_i + x_j dx_i = d_{i,j} dx_i x_j + d_{i,j} x_i dx_j + \sum_{k=1}^{n} r_k^{(i,j)} dx_k, \quad \text{for } i < j.$$

Define S-linear maps

$$\partial_{x_i} : \sigma(R)\langle x_1, \dots, x_n\rangle \to \sigma(R)\langle x_1, \dots, x_n\rangle$$

such that

$$d(f) = \sum_{i=1}^{n} dx_i \partial_{x_i}(f), \quad \text{for all } f \in \sigma(R)\langle x_1, \dots, x_n\rangle.$$

These maps are well-defined since the elements dx_i for $1 \leq i \leq n$ are free generators of the right $\sigma(R)\langle x_1, \ldots, x_n\rangle$-module $\Omega^1(\sigma(R)\langle x_1, \ldots, x_n\rangle)$. In this way, $d(f) = 0$ if and only if $\partial_{x_i}(f) = 0$ for $1 \leq i \leq n$. Using relations (5.3.78) and definitions of the maps $\widetilde{\sigma}_i$ for $1 \leq i \leq n$, we find that

$$\partial_{x_i}(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = l_i x_1^{\alpha_1} \cdots x_i^{l_i - 1} x_{i+1}^{l_{i+1}} \cdots x_n^{\alpha_n}.$$

Then $d(f) = 0$ if and only if f is a scalar multiple of the identity. This shows that $(\Omega(\sigma(R)\langle x_1, \ldots, x_n\rangle), d)$ is connected, where

$$\Omega(\sigma(R)\langle x_1, \ldots, x_n\rangle) = \bigoplus_{i=0}^{n} \Omega^i\left(\sigma(R)\langle x_1, \ldots, x_n\rangle\right).$$

The universal extension of d to higher forms compatible with (5.2.11) gives the following rules for $\Omega^l(\sigma(R)\langle x_1, \ldots, x_n\rangle)$ for $2 \leq l \leq n-1$,

$$\bigwedge_{k=1}^{l} dx_{q(k)} = (-1)^{\sharp} \bigwedge_{k=1}^{l} dx_{p(k)}, \tag{5.2.12}$$

where $q : \{1, \ldots, l\} \to \{1, \ldots, n\}$ is an injective map, $p : \{1, \ldots, l\} \to \mathrm{Im}(q)$ is an increasing injective map and $\sharp$ is the number of 2-permutation needed to transform q into p.

By assumption the automorphisms $\widetilde{\sigma}_i$'s commute with each other, which implies that there are no additional relations to the previous ones. Then

$$\begin{aligned}\Omega^{n-1}(\sigma(R)\langle x_1, \ldots, x_n\rangle) = \; & [dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1} \oplus dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n \\ & \oplus \cdots \oplus \; dx_2 \wedge \cdots \wedge dx_n]\, \sigma(R)\langle x_1, \ldots, x_n\rangle.\end{aligned}$$

Since

$$\Omega^n(\sigma(R)\langle x_1, \ldots, x_n\rangle) = \omega\sigma(R)\langle x_1, \ldots, x_n\rangle \cong \sigma(R)\langle x_1, \ldots, x_n\rangle$$

as a right and left $\sigma(R)\langle x_1, \ldots, x_n\rangle$-module, with $\omega = dx_1 \wedge \cdots \wedge dx_n$, where $\widetilde{\sigma}_\omega = \widetilde{\sigma}_1 \circ \cdots \circ \widetilde{\sigma}_n$, it follows that ω is a volume form of the SPBW extension $\sigma(R)\langle x_1, \ldots, x_n\rangle$. From Proposition 1.7 (2), we get that ω is an integral form by setting

$$\omega_i^j = \bigwedge_{k=1}^{j} dx_{p_{i,j}(k)}, \quad \text{for } 1 \leq i \leq \binom{n}{j}, \quad \text{and}$$

$$\bar{\omega}_i^{n-j} = (-1)^{\sharp_i} \bigwedge_{k=j+1}^{n} dx_{\bar{p}_{i,j}(k)}, \quad \text{for } 1 \leq i \leq \binom{n}{j},$$

where $p_{i,j} : \{1, \ldots, j\} \to \{1, \ldots, n\}$ is an increasing injective map, $\bar{p}_{i,j} : \{j+1, \ldots, n\} \to (\mathrm{Im}(p_{i,j}))^c$ is also an increasing injective map and $\sharp_{i,j}$ is the number of 2-permutation needed to transform $\{\bar{p}_{i,j}(j+1), \ldots, \bar{p}_{i,j}(n), p_{i,j}(1), \ldots, p_{i,j}(j)\}$ into $\{1, \ldots, n\}$.

Let $\omega' \in \Omega^j(\sigma(R)\langle x_1, \dots, x_n\rangle)$. Then:

$$\omega' = \sum_{i=1}^{\binom{n}{j}} \bigwedge_{k=1}^{j} dx_{p_{i,j}(k)} a_i, \quad \text{with } a_i \in R.$$

This implies that we have the equalities given by

$$\begin{aligned}
\sum_{i=1}^{\binom{n}{j}} \omega_i^j \pi_\omega(\bar{\omega}_i^{n-j} \wedge \omega') &= \sum_{i=1}^{\binom{n}{j}} \bigwedge_{k=1}^{j} dx_{p_{i,j}(k)} \pi_\omega \left(a_i (-1)^{\sharp_{i,j}} \bigwedge_{k=j+1}^{n} dx_{\bar{p}_{i,j}(k)} \wedge \bigwedge_{k=1}^{j} dx_{p_{i,j}(k)} \right) \\
&= \sum_{i=1}^{\binom{n}{j}} \bigwedge_{k=1}^{j} dx_{p_{i,j}(k)} a_i \\
&= \omega',
\end{aligned}$$

and finally, by Proposition 1.7 (2), we conclude that $\sigma(R)\langle x_1, \dots, x_n\rangle$ is differentially smooth. □

Example 5.3. Consider the skew polynomial ring $R[x; \sigma, \delta]$ studied by Nasr-Isfahani and Moussavi [NIM08]. As we saw in the Introduction, σ is an automorphism of R, δ is a σ-derivation such that $\alpha\delta = \delta\alpha$, and the extended automorphism $\overline{\sigma}$ and the $\overline{\sigma}$-derivation $\overline{\delta}$ on $R[x; \sigma, \delta]$ are given by

$$\begin{aligned}
\overline{\sigma}(f(x)) &= \sigma(r_0) + \sigma(r_1)x + \cdots + \sigma(r_n)x^n, \quad \text{and} \\
\overline{\delta}(f(x)) &= \delta(r_0) + \delta(r_1)x + \cdots + \delta(r_n)x^n,
\end{aligned}$$

respectively. As it is clear, these assumptions satisfy those corresponding in Proposition 5.7, and hence Theorem 5.8 shows that $R[x; \sigma, \delta]$ is differentially smooth when $\mathrm{GKdim}(R) = 0$.

By using a similar reasoning, and under the natural assumptions, iterated Ore extensions $R[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \dots R[x_1; \sigma_n, \delta_n]$ are also differentially smooth.

Example 5.4. Artamonov et al. [VAF16] studied extended modules, Vaserstein's, Quillen's patching, Horrock's, and Quillen-Suslin's theorems for a special class of Ore extensions. They assumed that for a ring R, A denotes the Ore extension $A := R[x_1, \dots, x_n; \sigma]$ for which σ is an automorphism of R, $x_i x_j = x_j x_i$ and $x_i r = \sigma_i(r)x_i$, for every $1 \leq i, k \leq n$. As it is clear, this kind of Ore extensions satisfies the assumptions in Theorem 5.8.

The results appearing in Example 1.6 motivate us to formulate the following question for the near future:

Question 2. Let A be a skew PBW extension over a differential smooth algebra R. Under which conditions does the differential smooth property pass from R to A?

5.3 Differential smoothness of SPBW extensions over $\Bbbk[t]$

In this section, we investigate the differential smoothness of bijective SPBW extensions over the commutative polynomial ring $\Bbbk[t]$.

5.3.1 SPBW extensions in two indeterminates

Consider a SPBW extension of the form $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$. From Definition 5.2, we get the relations

$$x_1 r(t) = \sigma_1(r(t))x_1 + \delta_1(r(t)), \quad x_2 r(t) = \sigma_2(r(t))x_2 + \delta_2(r(t)), \quad \text{and}$$
$$x_2 x_1 = c_{1,2}(t)x_1 x_2 + q_{1,2}^{(0)}(t) + q_{1,2}^{(1)}(t)x_1 + q_{1,2}^{(2)}(t)x_2,$$

where $r(t), c_{1,2}(t), q_{1,2}^{(0)}(t), q_{1,2}^{(1)}(t), q_{1,2}^{(2)}(t)$ belong to $\Bbbk[t]$ with $c_{1,2}(t)$ non-zero.

Let $\sigma_1(t) = a_1 t + b_1$ and $\sigma_2(t) = a_2 t + b_2$ be automorphisms of $\Bbbk[t]$ (this is precisely the form of the elements of $\mathrm{Aut}(\Bbbk[x])$ [SU03, DE00]) with the corresponding σ_i-derivations $(i = 1, 2)$ expressed as in (2.2.1), that is,

$$\delta_1(f(t)) = \frac{f(\sigma_1(t)) - f(t)}{\sigma_1(t) - t} p_1(t), \quad \text{and} \quad \delta_2(f(t)) = \frac{f(\sigma_2(t)) - f(t)}{\sigma_2(t) - t} p_2(t),$$

where $p_1(t), p_2(t)$ are fixed elements of $\Bbbk[t]$. Thus, the relations between the indeterminates t, x_1 and x_2 can be expressed as

$$x_1 t = a_1 t x_1 + b_1 x_1 + p_1(t), \quad x_2 t = a_2 t x_2 + b_2 x_2 + p_2(t), \quad \text{and} \tag{5.3.1}$$
$$x_2 x_1 = c_{1,2}(t)x_1 x_2 + q_{1,2}^{(0)}(t) + q_{1,2}^{(1)}(t)x_1 + q_{1,2}^{(2)}(t)x_2. \tag{5.3.2}$$

Proposition 5.9. *From Equation (5.3.1), we obtain the commutation relations*

$$x_1 t^n = (a_1 t + b_1)^n x_1 + p_1(t) \sum_{l=0}^{n-1} (a_1 t + b_1)^l t^{n-1-l}, \quad \text{and}$$
$$x_2 t^n = (a_2 t + b_2)^n x_2 + p_2(t) \sum_{l=0}^{n-1} (a_2 t + b_2)^l t^{n-1-l}.$$

Proof. We proceed by induction on n. For $n = 1$, the assertion is clear. Suppose that the relation holds for $n = k$. Since

$$\begin{aligned}
x_1 t^{k+1} &= (x_1 t^k)t = \left((a_1 t + b_1)^k x_1 + p_1(t) \sum_{l=0}^{k-1} (a_1 t + b_1)^l t^{k-1-l} \right) t \\
&= (a_1 t + b_1)^k x_1 t + p_1(t) \sum_{l=0}^{k-1} (a_1 t + b_1)^l t^{(k+1)-1-l} \\
&= (a_1 t + b_1)^k ((a_1 t + b_1)x_1 + p_1(t)) + p_1(t) \sum_{l=0}^{k-1} (a_1 t + b_1)^l t^{(k+1)-1-l} \\
&= (a_1 t + b_1)^{k+1} x_1 + \left((a_1 t + b_1)^k p_1(t) + p_1(t) \sum_{l=0}^{k-1} (a_1 t + b_1)^l t^{(k+1)-1-l} \right) \\
&= (a_1 t + b_1)^{k+1} x_1 + \sum_{l=0}^{k} (a_1 t + b_1)^l t^{(k+1)-1-l},
\end{aligned}$$

the assertion follows. The proof of the second relation is similar. □

Proposition 5.10. *Let*

$$\nu_t(t) = t, \qquad \nu_t(x_1) = a_1x_1 + p_1'(t), \qquad \nu_t(x_2) = a_2x_2 + p_2'(t), \tag{5.3.3}$$

$$\nu_{x_1}(t) = \sigma_1^{-1}(t), \qquad \nu_{x_1}(x_1) = x_1, \qquad \nu_{x_1}(x_2) = c_{1,2}x_2 + q_{1,2}^{(1)}, \tag{5.3.4}$$

$$\nu_{x_2}(t) = \sigma_2^{-1}(t), \qquad \nu_{x_2}(x_1) = c_{1,2}^{-1}x_1 - c_{1,2}^{-1}q_{1,2}^{(2)}, \qquad \nu_{x_2}(x_2) = x_2, \tag{5.3.5}$$

where $p_1'(t)$ *and* $p_2'(t)$ *are the* t*-derivatives of* $p_1(t)$ *and* $p_2(t)$*, respectively, and* $c_{1,2}, q_{1,2}^{(0)}, q_{1,2}^{(1)}, q_{1,2}^{(2)} \in \Bbbk$*, with* $c_{1,2}$ *non-zero. Then:*

(1) *Leibniz's rule holds in the cases listed in Table 5.1. The maps defined by (5.3.3), (5.3.4) and (5.3.5) simultaneously extend to algebra automorphisms* $\nu_t, \nu_{x_1}, \nu_{x_2}$ *of* $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$ *only in cases* (a), (b), (c), (d), (e), (g) *and* (i).

(2) *In cases* (a), (b), (c), (d), (e), (g) *and* (i), *we get that*

$$\nu_t \circ \nu_{x_1} = \nu_{x_1} \circ \nu_t, \quad \nu_t \circ \nu_{x_2} = \nu_{x_2} \circ \nu_t, \quad \nu_{x_2} \circ \nu_{x_1} = \nu_{x_1} \circ \nu_{x_2}. \tag{5.3.6}$$

Table 5.1: Leibniz's rule

Case	Possibilities for a_1, b_1, a_2, b_2	*Polynomials* $p_1(t)$ *and* $p_2(t)$	*Restrictions*
(a)	$a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$	$p_1(t) = p_2(t) = 0$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} \in \Bbbk^*$
		$p_1(t), p_2(t) \in \Bbbk[t]$	$q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} = 1$, $q_{1,2}^{(0)} \in \Bbbk$
(b)	$a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 \neq 0$	$p_1(t) = p_1$, $p_2(t) = p_2$, $p_1, p_2 \in \Bbbk$	$c_{1,2} = 1$, $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $q_{1,2}^{(0)} \in \Bbbk$
		$p_1(t) = p_2(t) = 0$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} \in \Bbbk^*$
(c)	$a_1 = 1$, $b_1 \neq 0$, $a_2 = 1$, $b_2 = 0$	$p_1(t) = p_1$, $p_2(t) = p_2$, $p_1, p_2 \in \Bbbk$	$c_{1,2} = 1$, $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $q_{1,2}^{(0)} \in \Bbbk$
		$p_1(t) = p_2(t) = 0$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} \in \Bbbk^*$
(d)	$a_1 = 1$, $b_1 \neq 0$, $a_2 = 1$, $b_2 \neq 0$	$p_1(t) = p_1$, $p_2(t) = p_2$, $p_1, p_2 \in \Bbbk$	$c_{1,2} = 1$, $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $q_{1,2}^{(0)} \in \Bbbk$
		$p_1(t) = p_2(t) = 0$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} \in \Bbbk^*$
(e)	$a_1 = 1$, $b_1 = 0$, $a_2 \neq 1$, $b_2 \in \Bbbk$	$p_1(t) = 0$, $p_2(t) = p_2\left(t + \frac{b_2}{a_2 - 1}\right)$, $p_2 \in \Bbbk$	$c_{1,2} = 1$, $q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$
		$p_1(t) = p_1$ $p_2(t) = 0$,$p_1 \in \Bbbk$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} = a_2^{-1}$
		$p_1(t) = p_2(t) = 0$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} \notin \{1, a_2^{-1}\}$
(f)	$a_1 = 1$, $b_1 \neq 0$, $a_2 \neq 1$	$p_1(t) = p_1$, $p_2(t) = p_2\left(t + \frac{b_2}{a_2 - 1}\right)$, $p_1, p_2 \in \Bbbk$	There is not solution for all relations
(g)	$a_1 \neq 1$, $a_2 = 1$, $b_2 = 0$	$p_1(t) = p_1\left(t + \frac{b_1}{a_1 - 1}\right)$, $p_2(t) = 0$, $p_1 \in \Bbbk$	$c_{1,2} = 1$, $q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$
		$p_1(t) = 0$ $p_2(t) = p_2$,$p_2 \in \Bbbk$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} = a_1^{-1}$
		$p_1(t) = p_2(t) = 0$	$q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, $c_{1,2} \notin \{1, a_1^{-1}\}$
(h)	$a_1 \neq 1$, $b_1 \in \Bbbk$, $a_2 = 1$, $b_2 \neq 0$	$p_1(t) = p_1\left(t + \frac{b_1}{a_1 - 1}\right)$, $p_2(t) = p_2$, $p_1, p_2 \in \Bbbk$	There is not solution for all relations
(i)	$a_1 \neq 1$, $a_2 \neq 1$, $b_1 = 0$ $b_2 = 0$	$p_1(t) = p_1 t$, $p_2(t) = p_2 t$, $p_1, p_2 \in \Bbbk$	$c_{1,2} = 1$, $q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$

Proof. For the first assertion, the map ν_t can be extended to an algebra homomorphism if and only if the definitions of $\nu_t(t)$, $\nu_t(x_1)$ and $\nu_t(x_2)$ respect relations (5.3.1), and (5.3.2), i.e.

$$\begin{aligned}
&\nu_t(x_1)\nu_t(t) - \nu_t(a_1t + b_1)\nu_t(x_1) = \nu_t(p_1(t)),\\
&\nu_t(x_2)\nu_t(t) - \nu_t(a_2t + b_2)\nu_t(x_2) = \nu_t(p_2(t)), \quad \text{and}\\
&\nu_t(x_2)\nu_t(x_1) - c_{1,2}\nu_t(x_1)\nu_t(x_2) = q_{1,2}^{(0)} + q_{1,2}^{(1)}\nu_t(x_1) + q_{1,2}^{(2)}\nu_t(x_2).
\end{aligned}$$

In this way, we obtain the equations

$$\begin{aligned}((a_1-1)t+b_1)p_1'(t) &= (a_1-1)p_1(t),\\ ((a_2-1)t+b_2)p_2'(t) &= (a_2-1)p_2(t),\end{aligned} \tag{5.3.7}$$

and

$$\begin{aligned}&(a_1a_2-1)q_{1,2}^{(0)}+(a_2-1)a_1q_{1,2}^{(1)}x_1+a_1(p_2'(t)x_1-c_{1,2}x_1p_2'(t))+(a_1-1)a_2q_{1,2}^{(2)}x_2\\ &+a_2(x_2p_1'(t)-c_{1,2}p_1'(t)x_2)+(1-c_{1,2})p_1'(t)p_2'(t)-q_{1,2}^{(1)}p_1'(t)-q_{1,2}^{(2)}p_2'(t)=0.\end{aligned} \tag{5.3.8}$$

Note that the map ν_{x_1} can be extended to an algebra automorphism if and only if the definitions of $\nu_{x_1}(t)$, $\nu_{x_1}(x_1)$ and $\nu_{x_1}(x_2)$ respect relations (5.3.1), and (5.3.2), that is,

$$\begin{aligned}\nu_{x_1}(x_1)\nu_{x_1}(t)-\nu_{x_1}(a_1t+b_1)\nu_{x_1}(x_1)&=\nu_{x_1}(p_1(t)),\\ \nu_{x_1}(x_2)\nu_{x_1}(t)-\nu_{x_1}(a_2t+b_2)\nu_{x_1}(x_2)&=\nu_{x_1}(p_2(t)),\quad\text{and}\\ \nu_{x_1}(x_2)\nu_{x_1}(x_1)-c_{1,2}\nu_{x_1}(x_1)\nu_{x_1}(x_2)&=q_{1,2}^{(0)}+q_{1,2}^{(1)}\nu_{x_1}(x_1)+q_{1,2}^{(2)}\nu_{x_1}(x_2).\end{aligned}$$

Therefore,

$$a_1^{-1}p_1(t)=p_1(a_1^{-1}(t-b_1)), \tag{5.3.9}$$

$$\begin{aligned}c_{1,2}(a_1^{-1}(a_2b_1+b_2-b_1)-b_2)x_2+a_1^{-1}(c_{1,2}p_2(t)-(1+a_2)q_{1,2}^{(1)}t)\\ +q_{1,2}^{(1)}(a_1^{-1}b_1(a_2-1)-b_2)=p_2(a_1^{-1}(t-b_1)),\quad\text{and}\end{aligned} \tag{5.3.10}$$

$$(c_{1,2}-1)q_{1,2}^{(0)}-q_{1,2}^{(1)}q_{1,2}^{(2)}=0. \tag{5.3.11}$$

The map ν_{x_2} can be extended to an algebra automorphism if and only if the definitions of $\nu_{x_2}(t)$, $\nu_{x_2}(x_1)$ and $\nu_{x_2}(x_2)$ respect relations (5.3.1), and (5.3.2), i.e.

$$\begin{aligned}\nu_{x_2}(x_1)\nu_{x_2}(t)-\nu_{x_2}(a_1t+b_1)\nu_{x_2}(x_1)&=\nu_{x_2}(p_1(t)),\\ \nu_{x_2}(x_2)\nu_{x_2}(t)-\nu_{x_2}(a_2t+b_2)\nu_{x_2}(x_2)&=\nu_{x_2}(p_2(t)),\quad\text{and}\\ \nu_{x_2}(x_2)\nu_{x_2}(x_1)-c_{1,2}\nu_{x_2}(x_1)\nu_{x_2}(x_2)&=q_{1,2}^{(0)}+q_{1,2}^{(1)}\nu_{x_2}(x_1)+q_{1,2}^{(2)}\nu_{x_2}(x_2).\end{aligned}$$

In other words,

$$\begin{aligned}c_{1,2}^{-1}(a_2^{-1}(b_1+a_1b_2-b_2)-b_1)x_1+c_{1,2}^{-1}a_2^{-1}(p_1(t)-(1+a_1)q_{1,2}^{(2)}t)\\ +q_{1,2}^{(2)}c_{1,2}^{-1}(a_2^{-1}b_2(1+a_1)+b_1)=p_1(a_2^{-1}(t-b_2)),\end{aligned} \tag{5.3.12}$$

$$a_2^{-1}p_2(t)=p_2(a_2^{-1}(t-b_2)),\quad\text{and} \tag{5.3.13}$$

$$(c_{1,2}^{-1}-1)q_{1,2}^{(0)}+c_{1,2}^{-1}q_{1,2}^{(1)}q_{1,2}^{(2)}=0. \tag{5.3.14}$$

Notice that expressions (5.3.7) are the same as in [Brz15, Lemma 3.1], and that these equations are independent of each other, so we have nine possible combinations for the values of a_1, b_1, a_2 and b_2. For each of these combinations, equations (5.3.11) and (5.3.14) will be used to determine the possible values for $c_{1,2}$, $p_1(t)$, $p_2(t)$ $q_{1,2}^{(i)}$, $i=0,1,2$. Let us see.

Consider $p_1(t):=\sum_{j=0}^{n}m_jt^j$ and $p_2(t):=\sum_{j=0}^{n}k_jt^j$.

(a) Equation (5.3.8) leads to the equalities

$$\sum_{j=1}^{n}[jk_jt^{j-1}x_1 - c_{1,2}x_1jk_jt^{j-1} + x_2jm_jt^{j-1} - c_{1,2}jm_jt^{j-1}x_2 + \sum_{s=1}^{n}(1-c_{1,2})jk_jsm_st^{s+j-2} - q_{1,2}^{(1)}jm_jt^{j-1} - q_{1,2}^{(2)}jk_jt^{j-1}] = 0,$$

$$\sum_{j=1}^{n}[jk_jt^{j-1}x_1 - c_{1,2}jk_j\left(t^{j-1}x_1 + (j-1)p_1(t)t^{n-1}\right) + jm_j\left(t^{j-1}x_2 + (j-1)p_2(t)t^{n-1}\right) - c_{1,2}jm_jt^{j-1}x_2 + \sum_{s=1}^{n}(1-c_{1,2})jk_jsm_st^{s+j-2} - q_{1,2}^{(1)}jm_jt^{j-1} - q_{1,2}^{(2)}jk_jt^{j-1}] = 0,$$

and

$$\sum_{j=1}^{n}t^{j-1}[jk_j(1-c_{1,2})x_1 + jm_j(1-c_{1,2})x_2 + j(j-1)\sum_{i=0}^{n}(m_jk_i - c_{1,2}k_jm_i)\,t^{i+n-j} + \sum_{s=1}^{n}(1-c_{1,2})jk_jsm_st^{s-1} - j(q_{1,2}^{(1)}m_j + q_{1,2}^{(2)}k_j)] = 0.$$

If we focus on the coefficients of x_1 and x_2, these must be zero, that is, $k_j(1-c_{1,2}) = 0$ and $m_j(1-c_{1,2}) = 0$. This implies that $m_j = k_j = 0$, for $1 \leq j \leq n$ and so the polynomials $p_1(t)$ and $p_2(t)$ are constants or $c_{1,2} = 1$. From relations (5.3.10), (5.3.11), (5.3.12) and (5.3.14), we get that

$$\begin{aligned}(c_{1,2}-1)p_2(t) - 2q_{1,2}^{(1)}t &= 0,\\ (c_{1,2}^{-1}-1)p_1(t) - 2c_{1,2}^{-1}q_{1,2}^{(2)}t &= 0,\\ (c_{1,2}-1)q_{1,2}^{(0)} &= q_{1,2}^{(1)}q_{1,2}^{(2)}, \quad \text{and}\\ (c_{1,2}^{-1}-1)q_{1,2}^{(0)} &= -c_{1,2}^{-1}q_{1,2}^{(1)}q_{1,2}^{(2)}.\end{aligned}$$

If $p_1(t) = p_1, p_2(t) = p_2 \in \Bbbk$, then $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$ and we obtain the following options:

- $c_{1,2} = 1$, $q_{1,2}^{(0)}$ has no restrictions.
- $p_2 = 0$, $q_{1,2}^{(0)} = 0$ and $p_1 = 0$, with no restriction over $c_{1,2}$.

If $c_{1,2} = 1$ it is necessary that $q_{1,2}^{(1)}m_j + q_{1,2}^{(2)}k_j = 0$ for all $1 \leq i \leq n$. One possibility is precisely when $m_j = k_j = 0$, which means that $p_1(t)$ and $p_2(t)$ are constants (as in the previous case). The other option is that $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$, with no restrictions on the polynomials $p_1(t)$ and $p_2(t)$. We have considered all possible options.

(b) Equation (5.3.8) leads to the following way of relating the coefficients

$$\sum_{j=1}^{n}\left[jm_j\left((t+b_2)^{j-1}x_2 - c_{1,2}t^{j-1}x_2 + \sum_{l=0}^{j-2}p_2(t+b_2)^l t^{j-2-l}\right) - q_{1,2}^{(1)}jm_j t^{j-1}\right] = 0.$$

The coefficient of x_2 must be zero, that is, $jm_j((t+b_2)^{j-1} - c_{1,2}t^{j-1}) = 0$. This implies that $m_j = 0$ for $1 \leq i \leq n$ whence the polynomial $p_1(t)$ is constant. From relations (5.3.10), (5.3.11), (5.3.12) and (5.3.14), it follows that

$$\begin{aligned}(c_{1,2}-1)p_2 - b_2q_{1,2}^{(1)} - 2q_{1,2}^{(1)}t &= 0,\\ (c_{1,2}^{-1}-1)p_1 + 2b_2c_{1,2}^{-1}q_{1,2}^{(2)} - 2c_{1,2}^{-1}q_{1,2}^{(2)}t &= 0,\\ (c_{1,2}-1)q_{1,2}^{(0)} &= q_{1,2}^{(1)}q_{1,2}^{(2)}, \quad \text{and}\\ (c_{1,2}^{-1}-1)q_{1,2}^{(0)} &= -c_{1,2}^{-1}q_{1,2}^{(1)}q_{1,2}^{(2)},\end{aligned}$$

and thus $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$. If $c_{1,2} = 1$, then there are no restrictions over $q_{1,2}^{(0)}$. If $c_{1,2} \neq 1$, then $p_1 = p_2 = q_{1,2}^{(0)} = 0$. Again, all possible options are covered.

(c) Note that in this case the conditions are the same as in (b) by considering x_2 instead of x_1.

(d) It is clear that (5.3.8) holds. By using the relations (5.3.10), (5.3.11), (5.3.12) and (5.3.14) we obtain that

$$\begin{aligned}(c_{1,2}-1)p_2 - b_2q_{1,2}^{(1)} - 2q_{1,2}^{(1)}t = (c_{1,2}^{-1}-1)p_1 + (2b_2+b_1)c_{1,2}^{-1}q_{1,2}^{(2)} - 2c_{1,2}^{-1}q_{1,2}^{(2)}t &= 0,\\ (c_{1,2}-1)q_{1,2}^{(0)} &= q_{1,2}^{(1)}q_{1,2}^{(2)}, \quad \text{and}\\ (c_{1,2}^{-1}-1)q_{1,2}^{(0)} &= -c_{1,2}^{-1}q_{1,2}^{(1)}q_{1,2}^{(2)}.\end{aligned}$$

These equalities are satisfied when $q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$. If $c_{1,2} = 1$ then there are no restrictions on $q_{1,2}^{(0)}$; in other case, then $p_1 = p_2 = q_{1,2}^{(0)} = 0$.

(e) From expression (5.3.8) we have that

$$\begin{aligned}&((a_2-1)q_{1,2}^{(1)} + (1-c_{1,2})p_2)x_1 + (a_2-1)q_{1,2}^{(0)} - q_{1,2}^{(2)}p_2\\ &+ \sum_{j=1}^{n}[a_2jm_j\left((a_2t+b_2)^{j-1}x_2 - c_{1,2}t^{j-1}x_2 + p_2(t)\sum_{l=0}^{j-2}(a_2t+b_2)^l t^{j-2-l}\right)\\ &- q_{1,2}^{(1)}jm_jt^{j-1}] = 0.\end{aligned}$$

Again, necessarily the coefficient of x_2 is zero, that is, $jm_j((a_2t+b_2)^{j-1} - c_{1,2}t^{j-1}) = 0$, and hence necessarily $m_j = 0$, for $1 \leq i \leq n$, which shows that the polynomial $p_1(t)$ is constant.

With respect to the coefficient of x_1 and the constant term, both must be zero, and so

$$(a_2-1)q_{1,2}^{(1)} + (1-c_{1,2})p_2 = 0 \quad \text{and} \quad (a_2-1)q_{1,2}^{(0)} - p_2q_{1,2}^{(2)} = 0,$$

or equivalently,

$$q_{1,2}^{(0)} = \frac{q_{1,2}^{(2)}}{a_2 - 1} p_2 \quad \text{and} \quad q_{1,2}^{(1)} = \frac{c_{1,2} - 1}{a_2 - 1} p_2.$$

Expressions (5.3.10), (5.3.11), (5.3.12) and (5.3.14) imply that

$$\begin{aligned} ((c_{1,2} - 1)p_2 - (a_2 + 1)q_{1,2}^{(1)})t + (c_{1,2} - 1)\frac{b_2}{a_2 - 1} - b_2 q_{1,2}^{(1)} &= 0, \\ -2c_{1,2}^{-1}a_2^{-1}q_{1,2}^{(2)}t + (c_{1,2}^{-1}a_2^{-1} - 1)p_1 + 2c_{1,2}^{-1}a_2^{-1}b_2 q_{1,2}^{(2)} &= 0, \\ (c_{1,2} - 1)q_{1,2}^{(0)} &= q_{1,2}^{(1)}q_{1,2}^{(2)}, \quad \text{and} \\ (c_{1,2}^{-1} - 1)q_{1,2}^{(0)} &= -c_{1,2}^{-1}q_{1,2}^{(1)}q_{1,2}^{(2)}. \end{aligned}$$

In this way,

$$\begin{aligned} q_{1,2}^{(1)} &= \frac{c_{1,2} - 1}{a_2 - 1} p_2, \\ (c_{1,2} - 1)p_2 &= 0, \\ (c_{1,2}^{-1}a_2^{-1} - 1)p_1 &= 0, \quad \text{and} \\ q_{1,2}^{(2)} &= 0, \end{aligned}$$

so we get the restrictions $q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$. Note that if $c_{1,2} = 1$, then $p_2 \in \Bbbk$ and $p_1 = 0$; or $c_{1,2} = a_2^{-1}$ with $p_1 \in \Bbbk$ and $p_2 = 0$; or in other value of $c_{1,2}$, $p_1 = p_2 = 0$.

(f) Equation (5.3.8) becomes

$$\begin{aligned} (a_2 - 1)q_{1,2}^{(0)} + (a_2 - 1)q_{1,2}^{(1)}x_1 + p_2x_1 - c_{1,2}p_2x_1 - q_{1,2}^{(2)}p_2 = 0, \quad \text{and} \\ ((a_2 - 1)q_{1,2}^{(1)} + p_2 - c_{1,2}p_2)x_1 + (a_2 - 1)q_{1,2}^{(0)} - q_{1,2}^{(2)}p_2 = 0, \end{aligned}$$

whence

$$q_{1,2}^{(1)} = \frac{c_{1,2} - 1}{a_2 - 1} p_2 \quad \text{and} \quad q_{1,2}^{(0)} = \frac{q_{1,2}^{(2)}}{a_2 - 1} p_2.$$

From expression (5.3.10) we have that $c_{1,2}b_1(a_2 - 1)x_2 = 0$, where the only options are $c_{1,2} = 0$, $b_1 = 0$ or $a_2 = 1$. However, as it is clear none of these are possible.

(g) The conditions corresponding to this case are the same as (e) since the hypotheses are completely analogous but replacing the indeterminate x_1 with x_2.

(h) This case is the same as (f) by replacing the indeterminate x_1 with x_2.

(i) Equation (5.3.8) leads to the following way of relating the coefficients:

$$\begin{aligned} (a_1a_2 - 1)q_{1,2}^{(0)} + (a_2 - 1)a_1q_{1,2}^{(1)}x_1 + a_1(p_2x_1 - c_{1,2}x_1p_2) + (a_1 - 1)a_2q_{1,2}^{(2)}x_2 \\ + a_2(x_2p_1 - c_{1,2}p_1x_2) + (1 - c_{1,2})p_1p_2 - q_{1,2}^{(1)}p_1 - q_{1,2}^{(2)}p_2 = 0. \end{aligned}$$

After some computations, we get that $c_{1,2} = 1$, $q_{1,2}^{(0)} = q_{1,2}^{(1)} = q_{1,2}^{(2)} = 0$. Thus, expression (5.3.10) becomes

$$(a_1^{-1}(a_2-1)b_1 + (a_1^{-1}-1)b_2)x_1 + a_1^{-1}((1-a_1^{-1})p_2 - (1+a_2)q_{1,2}^{(1)})t + q_{1,2}^{(1)}(a_1^{-1}b_1a_2 - a_1^{-1}b_1 - b_2) + a_1^{-1}b_1p_2 - \frac{b_2}{a_2-1}p_2 = 0.$$

By replacing the values found previously, we obtain that $b_1 = b_2 = 0$. Note that relations (5.3.11), (5.3.12) and (5.3.14) are trivially satisfied.

For the second assertion, it is enough to prove it for the generators t, x_1 and x_2:

$$(\nu_t \circ \nu_{x_1})(t) = \nu_t(\sigma_1^{-1}(t)) = a_1^{-1}(t-b_1), \tag{5.3.15}$$
$$(\nu_{x_1} \circ \nu_t)(t) = \nu_{x_1}(t) = a_1^{-1}(t-b_1), \tag{5.3.16}$$
$$(\nu_t \circ \nu_{x_1})(x_1) = \nu_t(x_1) = a_1x_1 + p_1'(t), \tag{5.3.17}$$
$$(\nu_{x_1} \circ \nu_t)(x_1) = a_1x_1 + p_1'(a_1^{-1}(t-b_1)), \tag{5.3.18}$$
$$(\nu_t \circ \nu_{x_1})(x_2) = c_{1,2}a_2x_2 + c_{1,2}p_2'(t) + q_{1,2}^{(1)}, \quad \text{and} \tag{5.3.19}$$
$$(\nu_{x_1} \circ \nu_t)(x_2) = a_2c_{1,2}x_2 + a_2q_{1,2}^{(1)} + p_2'(a_1^{-1}(t-b_1)). \tag{5.3.20}$$

In any case, the two compositions shown in (5.3.15) and (5.3.16) are the same. Relation (5.3.18) was used to find the conditions of the polynomial $p_1(t)$ to be equal to the expression (5.3.17). Thus, all of them are satisfied in every possible case. As it is clear, relation (5.3.20) holds in all cases to be equal to (5.3.19). So, $\nu_t \circ \nu_{x_1} = \nu_{x_1} \circ \nu_t$.

Next,

$$\nu_t \circ \nu_{x_2}(t) = \nu_t(\sigma_2^{-1}(t)) = a_2^{-1}(t-b_2),$$
$$\nu_{x_2} \circ \nu_t(t) = \nu_{x_2}(t) = a_2^{-1}(t-b_2), \tag{5.3.21}$$
$$\nu_t \circ \nu_{x_2}(x_1) = c_{1,2}^{-1}a_1x_1 + c_{1,2}^{-1}p_1'(t) - c_{1,2}^{-1}q_{1,2}^{(2)},$$
$$\nu_{x_2} \circ \nu_t(x_1) = a_1c_{1,2}^{-1}x_1 - a_1c_{1,2}^{-1}q_{1,2}^{(2)} + p_1'(a_2^{-1}(t-b_2)), \tag{5.3.22}$$
$$\nu_t \circ \nu_{x_2}(x_2) = a_2x_2 + p_2'(t), \quad \text{and}$$
$$\nu_{x_2} \circ \nu_t(x_2) = a_2x_2 + p_2'(a_2^{-1}(t-b_2)). \tag{5.3.23}$$

In any case, the two compositions shown in (5.3.21) are the same. Relation (5.3.23) was similarly used to find the conditions of the polynomial $p_1(t)$, whence they are satisfied in all cases. Note that relation (5.3.22) works in all cases but case (g) only works when $c_{1,2} = 1$. In this way, $\nu_t \circ \nu_{x_2} = \nu_{x_2} \circ \nu_t$.

Note that

$$\nu_{x_1} \circ \nu_{x_2}(t) = a_2^{-1}(a_1^{-1}(t - b_1)) - a_2^{-1}b_2, \tag{5.3.24}$$
$$\nu_{x_2} \circ \nu_{x_1}(t) = a_1^{-1}(a_2^{-1}(t - b_2)) - a_1^{-1}b_1, \tag{5.3.25}$$
$$\nu_{x_1} \circ \nu_{x_2}(x_1) = c_{1,2}^{-1}x_1 - c_{1,2}^{-1}q_{1,2}^{(2)}, \tag{5.3.26}$$
$$\nu_{x_2} \circ \nu_{x_1}(x_1) = c_{1,2}^{-1}x_1 - c_{1,2}^{-1}q_{1,2}^{(2)}, \tag{5.3.27}$$
$$\nu_{x_1} \circ \nu_{x_2}(x_2) = c_{1,2}x_2 + q_{1,2}^{(1)}, \quad \text{and} \tag{5.3.28}$$
$$\nu_{x_2} \circ \nu_{x_1}(x_2) = c_{1,2}x_2 + q_{1,2}^{(1)}. \tag{5.3.29}$$

In any case, relations (5.3.26), (5.3.27), (5.3.28) and (5.3.29) hold. Expressions (5.3.24) and (5.3.25) coincide when $b_2 = \frac{a_2-1}{a_1-1}b_1$. □

Next, we formulate the first important result of the chapter.

Theorem 5.11. *If a SPBW extension $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$ satisfies one of the conditions* (a)-(i)*, except* (f) *and* (h)*, in Proposition 5.10, then it is differentially smooth.*

Proof. We know that SPBW extensions of the form $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$ have Gelfand-Kirillov dimension three [Rey13, Theorems 14 and 18], so we are able to formulate a three-dimensional integrable calculus. With this aim, consider $\Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2\rangle)$ a free right $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$-module of rank three with generators dt, dx_1 and dx_2. Define a left $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$-module structure by

$$f dt = dt\nu_t(f), \quad f dx_1 = dx_1\nu_{x_1}(f) \quad \text{and} \quad f dx_2 = dx_2\nu_{x_2}(f), \tag{5.3.30}$$

for all $f \in \sigma(\Bbbk[t])\langle x_1, x_2\rangle$, where ν_t, ν_{x_1} and ν_{x_2} are the algebra automorphisms established in Proposition 5.10. Notice that the relations in $\Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2\rangle)$ are given by

$$t dt = dt t, \qquad t dx_1 = a_1^{-1}dx_1 t - a_1^{-1}b_1 dx_1, \qquad t dx_2 = a_2^{-1}dx_2 t - a_2^{-1}b_2 dx_2, \tag{5.3.31}$$
$$x_1 dt = a_1 dt x_1 + dt p_1'(t), \quad x_1 dx_1 = dx_1 x_1, \qquad x_1 dx_2 = dx_2 c_{1,2}^{-1}x_1 - dx_2 c_{1,2}^{-1}q_{1,2}^{(2)}, \tag{5.3.32}$$
$$x_2 dt = a_2 dt x_2 + dt p_2'(t), \quad x_2 dx_1 = dx_1 c_{1,2}x_2 + dx_1 q_{1,2}^{(1)}, \quad x_2 dx_2 = dx_2 x_2. \tag{5.3.33}$$

We want to extend $t \mapsto dt$, $x_1 \mapsto dx_1$ and $x_2 \mapsto dx_2$ to a map $d : \sigma(\Bbbk[t])\langle x_1, x_2\rangle \to \Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2\rangle)$ satisfying Leibniz's rule. As expected, this is possible if Leibniz's rule is compatible with the non-trivial relations (5.3.1) and (5.3.2), i.e. if the equalities

$$\begin{aligned} dx_1 t + x_1 dt &= a_1 dt x_1 + a_1 t dx_1 + b_1 dx_1 + dp_1(t), \\ dx_2 t + x_2 dt &= a_2 dt x_2 + a_2 t dx_2 + b_2 dx_2 + dp_2(t), \quad \text{and} \\ dx_2 x_1 + x_2 dx_1 &= c_{1,2} dx_1 x_2 + c_{1,2} x_1 dx_2 + q_{1,2}^{(1)} dx_1 + q_{1,2}^{(2)} dx_2, \end{aligned}$$

hold. In view of $t dt = dt t$ which defines the usual commutative calculus on the polynomial ring $\Bbbk[t]$, it follows that $dp_1(t) = dt p_1'(t)$ and $dp_2(t) = dt p_2'(t)$.

Now, we define $\Bbbk$-linear maps

$$\partial_t, \partial_{x_1}, \partial_{x_2} : \sigma(\Bbbk[t])\langle x_1, x_2\rangle \to \sigma(\Bbbk[t])\langle x_1, x_2\rangle$$

such that

$$d(f) = dt\partial_t(f) + dx_1\partial_{x_1}(f) + dx_2\partial_{x_2}(f), \quad \text{for all } f \in \sigma(\Bbbk[t])\langle x_1, x_2\rangle.$$

Since dt, dx_1 and dx_2 are free generators of the right $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$-module $\Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2\rangle)$, these maps are well-defined. Then $d(a) = 0$ if and only if $\partial_t(a) = \partial_{x_1}(a) = \partial_{x_2}(a) = 0$. Using relations (5.3.30) and the definitions of the maps ν_t, ν_{x_1} and ν_{x_2}, we get that

$$\begin{aligned}
\partial_t(t^k x_1^l x_2^s) &= k t^{k-1} x_1^l x_2^s,\\
\partial_{x_1}(t^k x_1^l x_2^s) &= l a_1^{-k}(t - b_1)^k x_1^{l-1} x_2^s, \quad \text{and}\\
\partial_{x_2}(t^k x_1^l x_2^s) &= a_2^{-k} c_{1,2}^{-l} s (t - b_2)^k (x_1 - q_{1,2}^{(2)})^l x_2^{s-1}.
\end{aligned}$$

Thus $d(f) = 0$ if and only if f is a scalar multiple of the identity. This shows that $(\Omega(\sigma(\Bbbk[t])\langle x_1, x_2\rangle), d)$ is connected with $\Omega(\sigma(\Bbbk[t])\langle x_1, x_2\rangle) = \bigoplus_{i=0}^{3} \Omega^i(\sigma(\Bbbk[t])\langle x_1, x_2\rangle)$.

The universal extension of d to higher forms compatible with (5.3.31), (5.3.32) and (5.3.33) gives the following rules for $\Omega^2(\sigma(\Bbbk[t])\langle x_1, x_2\rangle)$:

$$\begin{aligned}
dx_1 \wedge dt &= -a_1 dt \wedge dx_1,\\
dx_2 \wedge dt &= -a_2 dt \wedge dx_2, \quad \text{and}\\
dx_2 \wedge dx_1 &= -c_{1,2} dx_1 \wedge dx_2.
\end{aligned} \tag{5.3.34}$$

Since the automorphisms ν_t, ν_{x_1} and ν_{x_2} commute with each other, there are no additional relationships to the previous ones, so we can write

$$\begin{aligned}
\Omega^2(\sigma(\Bbbk[t])\langle x_1, x_2\rangle) = {} & dt \wedge dx_1 \sigma(\Bbbk[t])\langle x_1, x_2\rangle\\
& \oplus dt \wedge dx_2 \sigma(\Bbbk[t])\langle x_1, x_2\rangle \oplus dx_1 \wedge dx_2 \sigma(\Bbbk[t])\langle x_1, x_2\rangle.
\end{aligned}$$

Note that

$$\Omega^3(\sigma(\Bbbk[t])\langle x_1, x_2\rangle) = \omega\sigma(\Bbbk[t])\langle x_1, x_2\rangle \cong \sigma(\Bbbk[t])\langle x_1, x_2\rangle$$

as a right and left $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$-module, with $\omega = dt \wedge dx_1 \wedge dx_2$, where $\nu_\omega = \nu_t \circ \nu_{x_1} \circ \nu_{x_2}$. This means that ω is a volume form of $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$. From Proposition 1.7 (2), ω is an integral form by setting

$$\begin{aligned}
&\omega_1^1 = dt, && \omega_2^1 = dx_1, && \omega_3^1 = dx_2,\\
&\omega_1^2 = dx_1 \wedge dx_2, && \omega_2^2 = dt \wedge dx_2, && \omega_3^2 = dt \wedge dx_1,\\
&\bar{\omega}_1^1 = dt, && \bar{\omega}_2^1 = -a_1^{-1} dx_1, && \bar{\omega}_3^1 = a_2^{-1} c_{1,2}^{-1} dx_2,\\
&\bar{\omega}_1^2 = a_1^{-1} a_2^{-1} dx_1 \wedge dx_2, && \bar{\omega}_2^2 = -c_{1,2}^{-1} dt \wedge dx_2, && \bar{\omega}_3^2 = dt \wedge dx_1.
\end{aligned}$$

Indeed, let $\omega' = dta + dx_1b + dx_2c$ with $a, b, c \in \Bbbk$. Then

$$\begin{aligned}\sum_{i=1}^{3} \omega_i^1 \pi_\omega(\bar{\omega}_i^2 \wedge \omega') &= dt\pi_\omega(a_1^{-1}a_2^{-1}a dx_1 \wedge dx_2 \wedge dt) \\ &\quad + dx_1\pi_\omega(-c_{1,2}^{-1}b dt \wedge dx_2 \wedge dx_1) + dx_2\pi_\omega(c dt \wedge dx_1 \wedge dx_2) \\ &= dta + dx_1b + dx_2c = \omega',\end{aligned}$$

and let $\omega'' = dt \wedge dx_1a + dt \wedge dx_2b + dx_1 \wedge dx_2c$, with $a, b, c \in \Bbbk$. We obtain that

$$\begin{aligned}\sum_{i=1}^{3} \omega_i^2 \pi_\omega(\bar{\omega}_i^1 \wedge \omega'') &= dx_1 \wedge dx_2\pi_\omega(c dt \wedge dx_1 \wedge dx_2) \\ &\quad + dt \wedge dx_2\pi_\omega(-a_1^{-1}b dx_1 \wedge dt \wedge dx_2) \\ &\quad + dt \wedge dx_1\pi_\omega(a_2^{-1}c_{1,2}^{-1}a dx_2 \wedge dt \wedge dx_1) \\ &= dt \wedge dx_1a + dt \wedge dx_2b + dx_1 \wedge dx_2 = \omega''.\end{aligned}$$

Therefore, we have proved that $\sigma(\Bbbk[t])\langle x_1, x_2\rangle$ is differentially smooth. □

5.3.2 SPBW extensions in three indeterminates

In this section we develop a similar treatment to the presented in Section 5.3.1 but now we consider a SPBW extension of the form $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$ satisfying the defining relations

$$\begin{aligned}x_1r(t) &= \sigma_1(r(t))x_1 + \delta_1(r(t)),\\ x_2r(t) &= \sigma_2(r(t))x_2 + \delta_2(r(t)),\\ x_3r(t) &= \sigma_3(r(t))x_2 + \delta_3(r(t)),\\ x_2x_1 &= c_{1,2}(t)x_1x_2 + q_{1,2}^{(0)}(t) + q_{1,2}^{(1)}(t)x_1 + q_{1,2}^{(2)}(t)x_2 + q_{1,2}^{(3)}(t)x_3,\\ x_3x_1 &= c_{1,3}(t)x_1x_3 + q_{1,3}^{(0)}(t) + q_{1,3}^{(1)}(t)x_1 + q_{1,3}^{(2)}(t)x_2 + q_{1,3}^{(3)}(t)x_3, \quad \text{and}\\ x_3x_2 &= c_{2,3}(t)x_2x_3 + q_{2,3}^{(0)}(t) + q_{2,3}^{(1)}(t)x_1 + q_{2,3}^{(2)}(t)x_2 + q_{2,3}^{(3)}(t)x_3,\end{aligned}$$

where the elements $r(t)$, $c(t)$'s and $q(t)$'s belong to $\Bbbk[t]$ with $c_{1,2}(t), c_{1,3}(t)$ and $c_{2,3}(t)$ non-zero. Consider the automorphisms of $\Bbbk[t]$ given by $\sigma_i(t) = a_it + b_i$, for $a_i, b_i, \in \Bbbk$, with $a_i \neq 0$, $i = 1, 2, 3$, with the corresponding σ_i-derivations expressed as in (2.2.1), that is,

$$\delta_i(f) = \frac{f(\sigma_i(t)) - f(t)}{\sigma_i(t) - t}p_i(t), \text{ for } i = 1, 2, 3, \tag{5.3.35}$$

where $p_i(t)$ is a fixed element of $\Bbbk[t]$ for each i. The relations between t, x_1, x_2, x_3 can be expressed as

$$x_it = a_itx_i + b_ix_i + p_i(t), \quad \text{for every } i, \quad \text{and} \tag{5.3.36}$$

$$x_jx_i = c_{i,j}(t)x_ix_j + q_{i,j}^{(0)} + q_{i,j}^{(1)}x_1 + q_{i,j}^{(2)}x_2 + q_{i,j}^{(3)}x_3, \quad \text{for } i < j. \tag{5.3.37}$$

The following result is the natural extension of Proposition 5.10.

Proposition 5.12. *Let*

$$\nu_t(t) = t, \qquad \nu_t(x_i) = a_i x_i + p_i'(t), \quad i = 1,2,3, \tag{5.3.38}$$

$$\nu_{x_i}(t) = \sigma_i^{-1}(t), \qquad \nu_{x_i}(x_i) = x_i, \quad i = 1,2,3, \tag{5.3.39}$$

$$\nu_{x_i}(x_j) = c_{i,j}x_j + q_{i,j}^{(i)}, \qquad \nu_{x_j}(x_i) = c_{j,i}^{-1}x_i - c_{j,i}^{-1}q_{j,i}^{(j)}, \quad i < j, \tag{5.3.40}$$

where $p_i'(t)$ *are the* t*-derivatives of* $p_i(t)$ *for* $i = 1,2,3$*, and* $c_{i,j}, q_{i,j}^{(k)} \in \Bbbk$*,* $c_{i,j} \neq 0$*, for all* $1 \leq i,j \leq 3$ *and* $0 \leq k \leq 3$*.*

(1) *Leibniz's rule holds in the cases listed in Table 5.2. The maps defined by (5.3.38), (5.3.39) and (5.3.40) simultaneously extend to algebra automorphisms* ν_t, ν_{x_i}*,* $i = 1,2,3$*, of* $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$ *in cases* (a) - (d)*.*

(2) *Precisely, in cases* (a) - (d)*, we have that*

$$\nu_t \circ \nu_{x_i} = \nu_{x_i} \circ \nu_t \quad \text{and} \quad \nu_{x_i} \circ \nu_{x_j} = \nu_{x_j} \circ \nu_{x_i}, \quad \text{for } i = 1,2,3. \tag{5.3.41}$$

Table 5.2: Leibniz's rule

Case	Possibilities for a_i, b_i, $i = 1,2,3$	*Polynomials* $p_i(t)$, $i = 1,2,3$	*Restrictions*
(a)	$a_i = 1$, $b_i = 0$, for all $i = 1,2,3$	$p_i(t) = 0$ for all $i = 1,2,3$	$q_{i,j}^{(k)} = 0$, $c_{i,j} \in \Bbbk^*$ for all $i,j = 1,2,3$, $k \geq 0$
		$p_i(t) \in \Bbbk[t]$, for all $i = 1,2,3$	$q_{i,j}^{(k)} = 0$, $c_{i,j} = 1$, $q_{i,j}^{(0)} \in \Bbbk$, for all $i,j = 1,2,3$, $k > 0$
(b)	$a_i = 1$, for all $i = 1,2,3$, $b_i \neq 0$, for some $i = 1,2,3$	$p_i(t) = p_i$, $p_i \in \Bbbk$, for all $i = 1,2,3$	$c_{i,j} = 1$, $q_{i,j}^{(k)} = 0$, $q_{i,j}^{(0)} \in \Bbbk$, for all $i,j = 1,2,3$, $k > 0$
		$p_i(t) = 0$, for all $i = 1,2,3$	$q_{i,j}^{(k)} = 0$, $c_{i,j} \in \Bbbk^*$, for all $i,j = 1,2,3$, $k \geq 0$
(c)	$a_r \neq 1$, $a_s = 1$, $b_s = 0$, for $r \in S \subsetneq \{1,2,3\}$ and $s \in S^c$	$p_s(t) = 0$, $p_r(t) = p_r\left(t + \frac{b_r}{a_r - 1}\right)$, for $r \in S \subsetneq \{1,2,3\}$ and $s \in S^c$, $p_r \in \Bbbk$	$q_{i,j}^{(k)} = 0$, $c_{i,j} = 1$, for all $i,j = 1,2,3$, $k \geq 0$
(d)	$a_i \neq 1$, $b_i = 0$, for all $i = 1,2,3$	$p_i(t) = p_i t$, $p_i \in \Bbbk$, for all $i = 1,2,3$	$q_{i,j}^{(k)} = 0$, $c_{i,j} = 1$, for all $i,j = 1,2,3$, $k \geq 0$

Proof. For the first assertion, the map ν_t can be extended to an algebra homomorphism if and only if the definitions of $\nu_t(t)$ and $\nu_t(x_i)$, $i = 1,2,3$ respect relations (5.3.36), and (5.3.37), i.e.

$$\nu_t(x_i)\nu_t(t) - \nu_t(a_i t + b_i)\nu_t(x_i) = \nu_t(p_i(t)),$$
$$\nu_t(x_j)\nu_t(x_i) - c_{i,j}\nu_t(x_i)\nu_t(x_j) = q_{i,j}^{(0)} + q_{i,j}^{(1)}\nu_t(x_1) + q_{i,j}^{(2)}\nu_t(x_2) + q_{i,j}^{(3)}\nu_t(x_3),$$

for $i < j$. This yields the equalities

$$((a_i - 1)t + b_i)p_i'(t) = (a_i - 1)p_i(t), \quad i = 1,2,3, \tag{5.3.42}$$

and

$$(a_i a_j - 1)q_{i,j}^{(0)} + a_i(p_j'(t)x_i - c_{i,j}x_i p_j'(t)) + a_j(x_j p_i'(t) - c_{i,j}p_i'(t)x_j) + (a_j a_i - 1)q_{i,j}^{(0)}$$
$$+ \sum_{r=1}^{3} q_{i,j}^{(r)}(a_i a_j - a_r)x_r + (1 - c_{i,j})p_i'(t)p_j'(t) - q_{i,j}^{(1)}p_1'(t) - q_{i,j}^{(2)}p_2'(t) - q_{i,j}^{(3)}p_3'(t) = 0. \tag{5.3.43}$$

The map ν_{x_i} can be extended to an algebra homomorphism if and only if the definitions of $\nu_{x_i}(t)$ and $\nu_{x_i}(x_j)$, for each i, respect relations (5.3.36), and (5.3.37), i.e.

$$\begin{aligned}
&\nu_{x_i}(x_i)\nu_{x_i}(t) - \nu_{x_i}(a_i t + b_i)\nu_{x_i}(x_i) = \nu_{x_i}(p_i(t)),\\
&\nu_{x_i}(x_j)\nu_{x_i}(t) - \nu_{x_i}(a_j t + b_j)\nu_{x_i}(x_j) = \nu_{x_i}(p_j(t)), \quad \text{and}\\
&\nu_{x_i}(x_j)\nu_{x_i}(x_i) - c_{i,j}\nu_{x_i}(x_i)\nu_{x_i}(x_j) = q_{i,j}^{(0)} + q_{i,j}^{(1)}\nu_{x_i}(x_1) + q_{i,j}^{(2)}\nu_{x_i}(x_2) + q_{i,j}^{(3)}\nu_{x_i}(x_3),
\end{aligned}$$

for $i < j$. In this way,

$$a_i^{-1}p_i(t) = p_i(a_i^{-1}(t - b_i)), \tag{5.3.44}$$

$$\begin{aligned}
&c_{i,j}(a_i^{-1}(a_j b_i + b_j - b_i) - b_j)x_j + a_i^{-1}(c_{i,j}p_j(t) - (1 + a_j)q_{i,j}^{(i)}t),\\
&\quad + q_{i,j}^{(i)}(a_i^{-1}b_i(a_j - 1) - b_j) = p_j(a_i^{-1}(t - b_i)), \quad \text{and}
\end{aligned} \tag{5.3.45}$$

$$\begin{aligned}
&(c_{1,2} - 1)q_{1,2}^{(0)} + q_{1,2}^{(3)}(c_{1,2} - c_{1,3})x_3 - q_{1,2}^{(2)}q_{1,2}^{(1)} - q_{1,2}^{(3)}q_{1,3}^{(1)} = 0,\\
&(c_{1,3} - 1)q_{1,3}^{(0)} + q_{1,3}^{(2)}(c_{1,3} - c_{1,2})x_2 - q_{1,3}^{(3)}q_{1,3}^{(1)} - q_{1,3}^{(2)}q_{1,2}^{(1)} = 0,\\
&(c_{2,3} - 1)q_{2,3}^{(0)} + q_{2,3}^{(1)}(c_{2,3} - c_{1,2}^{-1})x_1 - q_{2,3}^{(3)}q_{2,3}^{(2)} + c_{1,2}^{-1}q_{1,2}^{(2)}q_{2,3}^{(1)} = 0.
\end{aligned} \tag{5.3.46}$$

Then, the map ν_{x_j} can be extended to an algebra homomorphism if and only if the definitions of $\nu_{x_j}(t)$ and $\nu_{x_j}(x_i)$, $i = 1, 2, 3$ respect relations (5.3.36), and (5.3.37), and so

$$\begin{aligned}
&\nu_{x_j}(x_i)\nu_{x_j}(t) - \nu_{x_j}(a_i t + b_i)\nu_{x_j}(x_i) = \nu_{x_j}(p_i(t)),\\
&\nu_{x_j}(x_j)\nu_{x_j}(t) - \nu_{x_j}(a_j t + b_j)\nu_{x_j}(x_j) = \nu_{x_j}(p_j(t)), \quad \text{and}\\
&\nu_{x_j}(x_j)\nu_{x_j}(x_i) - c_{i,j}\nu_{x_j}(x_i)\nu_{x_j}(x_j) = q_{i,j}^{(0)} + q_{i,j}^{(1)}\nu_{x_j}(x_1) + q_{i,j}^{(2)}\nu_{x_j}(x_2) + q_{i,j}^{(3)}\nu_{x_j}(x_3),
\end{aligned}$$

for $i < j$. We obtain the expressions given by

$$\begin{aligned}
&c_{i,j}^{-1}(a_j^{-1}(b_i + a_i b_j - b_j) - b_i)x_i + c_{i,j}^{-1}a_j^{-1}(p_i(t) - (1 + a_i)q_{i,j}^{(j)}t)\\
&\quad + q_{i,j}^{(j)}c_{i,j}^{-1}(a_j^{-1}b_j(1 + a_i) + b_i) = p_i(a_j^{-1}(t - b_j))
\end{aligned} \tag{5.3.47}$$

$$a_j^{-1}p_j(t) = p_j(a_j^{-1}(t - b_j)) \tag{5.3.48}$$

$$\begin{aligned}
&(c_{1,2}^{-1} - 1)q_{1,2}^{(0)} + q_{1,2}^{(3)}(c_{1,2}^{-1} - c_{2,3}^{-1})x_3 + c_{1,2}^{-1}q_{1,2}^{(2)}q_{1,2}^{(1)} - q_{1,2}^{(3)}q_{2,3}^{(2)} = 0,\\
&(c_{1,3}^{-1} - 1)q_{1,3}^{(0)} + q_{1,3}^{(2)}(c_{1,3}^{-1} - c_{2,3}^{-1})x_2 + c_{1,3}^{-1}q_{1,3}^{(3)}q_{1,3}^{(1)} + c_{2,3}^{-1}q_{1,3}^{(2)}q_{2,3}^{(3)} = 0,\\
&(c_{2,3}^{-1} - 1)q_{2,3}^{(0)} + q_{2,3}^{(1)}(c_{2,3}^{-1} - c_{1,3}^{-1})x_1 + c_{2,3}^{-1}q_{2,3}^{(3)}q_{2,3}^{(2)} + c_{1,3}^{-1}q_{1,3}^{(3)}q_{2,3}^{(1)} = 0.
\end{aligned} \tag{5.3.49}$$

Expressions (5.3.42) are the same as in [Brz15, Lemma 3.1]. It should be noted that these equations are independent of each other, which means that there are different combinations considering the values of a_i, b_i for $i = 1, 2, 3$. Let us see.

We consider the expression $p_i(t) = \sum_{j=0}^{n} m_{i,j}t^j$, for every $i = 1, 2, 3$.

(a) Equation (5.3.43) leads to the coefficients that accompany x_i, so these must be zero, $m_{i,j}(1 - c_{i,k}) = 0$. This implies that $m_{i,j} = 0$, for $1 \leq j \leq n$, and so the polynomials $p_i(t)$ are constants or $c_{i,k} = 1$, for $i = 1, 2, 3$ and $i < k$. From relations (5.3.45), (5.3.46), (5.3.47) and (5.3.49), we get that if $p_i(t) = p_i \in \Bbbk$ then $q_{i,j}^{(k)} = 0$

with $k > 0$, whence $c_{i,j} = 1$ and $q_{i,j}^{(0)}$ has no restrictions. Also, it is necessary that $q_{i,k}^{(i)} m_{i,j} + q_{i,k}^{(k)} m_{k,j} = 0$ for all $1 \leq i \leq 3$, which shows that $q_{i,j}^{(k)} = 0$ with $k > 0$ and there is not restrictions over polynomials $p_i(t)$.

In this way, we have considered all possibilities.

(b) Equation (5.3.43) leads that the coefficient that accompany x_l must be zero, that is

$$j m_{i,j}((t + b_l)^{j-1} - c_{i,l} t^{j-1}) = 0.$$

This implies that all the coefficients $m_{i,j}$ are zero, whence the polynomial $p_i(t)$ is constant.

From relations (5.3.45), (5.3.46), (5.3.47) and (5.3.49), we obtain that $q_{i,j}^{(k)} = 0$ with $k > 0$, $c_{i,j} = 1$ and there are no restrictions on $q_{i,j}^{(0)}$.

Again, all options are covered.

(c) Equation (5.3.8) implies that the coefficient of x_r is zero,

$$j m_{i,j}((a_r t + b_r)^{j-1} - c_{i,r} t^{j-1}) = 0.$$

Thus, $m_{i,j} = 0$ for $1 \leq i \leq 3$, whence the polynomial $p_i(t)$ is constant. Also, if we focus on the coefficient that accompany x_s and the constant element, both must be zero,

$$q_{i,r}^{(0)} = \frac{q_{i,r}^{(r)}}{a_r - 1} p_r \quad \text{and} \quad q_{i,r}^{(s)} = \frac{c_{i,r} - 1}{a_r - 1} p_r.$$

By using expressions (5.3.45), (5.3.46), (5.3.47) and (5.3.49) we obtain the restrictions $q_{i,j}^{(k)} = 0$ for $1 \leq i, j \leq 3$ and $k \geq 0$. Also, note that $c_{s,j} = 1$ and $p_s = 0$, for $s \in S$.

(d) Equation (5.3.43) shows that $c_{i,j} = 1$, $q_{i,j}^{(k)} = 0$ for $1 \leq i, j \leq 3$ and $k \geq 0$. If we consider the expression (5.3.45) then we get the condition $b_i = 0$ for each i. It is clear that relations (5.3.46), (5.3.47) and (5.3.49) hold.

For the second assertion, it is enough to prove it for the generators t, x_1 and x_2. Note that

$$\nu_t \circ \nu_{x_i}(t) = \nu_t(\sigma_i^{-1}(t)) = a_i^{-1}(t - b_i), \tag{5.3.50}$$

$$\nu_{x_i} \circ \nu_t(t) = \nu_{x_i}(t) = a_i^{-1}(t - b_i), \tag{5.3.51}$$

$$\nu_t \circ \nu_{x_i}(x_i) = \nu_t(x_i) = a_i x_i + p_i'(t), \tag{5.3.52}$$

$$\nu_{x_i} \circ \nu_t(x_i) = a_i x_i + p_i'(a_i^{-1}(t - b_i)), \tag{5.3.53}$$

$$\nu_t \circ \nu_{x_i}(x_j) = c_{i,j} a_j x_j + c_{i,j} p_j'(t) + q_{i,j}^{(i)}, \tag{5.3.54}$$

$$\nu_{x_i} \circ \nu_t(x_j) = a_j c_{i,j} x_j + a_j q_{i,j}^{(i)} + p_j'(a_i^{-1}(t - b_i)), \quad i < j, \tag{5.3.55}$$

$$\nu_t \circ \nu_{x_i}(x_k) = c_{k,i}^{-1} a_k x_k + c_{k,i}^{-1} p_k'(t) - c_{k,i}^{-1} q_{k,i}^{(i)}, \quad \text{and} \tag{5.3.56}$$

$$\nu_{x_i} \circ \nu_t(x_k) = a_k c_{k,i}^{-1} x_k - a_k c_{k,i}^{-1} q_{k,i}^{(i)} + p_k'(a_i^{-1}(t - b_i)), \quad i > k. \tag{5.3.57}$$

In any case, the two compositions shown in (5.3.51) are the same. Relation (5.3.53) was similarly used to find the conditions of the polynomial $p_i(t)$. Thus, they hold in all cases, and relations (5.3.55) and (5.3.57) are correct. Then, $\nu_t \circ \nu_{x_i} = \nu_{x_i} \circ \nu_t$.

We have that

$$\nu_{x_i} \circ \nu_{x_j}(t) = a_j^{-1}(a_i^{-1}(t - b_i)) - a_j^{-1}b_j, \tag{5.3.58}$$

$$\nu_{x_j} \circ \nu_{x_i}(t) = a_i^{-1}(a_j^{-1}(t - b_j)) - a_i^{-1}b_i, \tag{5.3.59}$$

$$\nu_{x_i} \circ \nu_{x_j}(x_i) = c_{i,j}^{-1}x_i - c_{i,j}^{-1}q_{i,j}^{(j)}, \tag{5.3.60}$$

$$\nu_{x_j} \circ \nu_{x_i}(x_i) = c_{i,j}^{-1}x_i - c_{i,j}^{-1}q_{i,j}^{(j)}, \tag{5.3.61}$$

$$\nu_{x_i} \circ \nu_{x_j}(x_j) = c_{i,j}x_j + q_{i,j}^{(i)}, \quad \text{and} \tag{5.3.62}$$

$$\nu_{x_j} \circ \nu_{x_i}(x_j) = c_{i,j}x_j + q_{i,j}^{(i)}. \tag{5.3.63}$$

In any case, relations (5.3.61) and (5.3.63) hold. Relation (5.3.59) works in all cases. Then, $\nu_{x_i} \circ \nu_{x_j} = \nu_{x_j} \circ \nu_{x_i}$. □

Theorem 5.13. *If a SPBW extension $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$ satisfies one of the conditions* (a)-(d) *in Proposition 5.12, then it is differentially smooth.*

Proof. Since the SPBW extension $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$ has Gelfand-Kirillov dimension 4, a 4-dimensional integrable calculus can be constructed. We know that we have to consider $\Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle)$, a free right $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$-module of rank 4 with generators dt, dx_1, dx_2, dx_3. Define a left $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$-module structure by

$$adt = dt\nu_t(a), \quad adx_i = dx_i\nu_{x_i}(a), \quad \text{for all } 1 \leq i \leq 3, a \in \sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle, \tag{5.3.64}$$

where ν_t, ν_{x_i}, $1 \leq i \leq 3$ are the algebra automorphisms established in Proposition 5.12. Notice that the relations in $\Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle)$ are given by

$$tdt = dtt \qquad tdx_i = dx_i a_i^{-1}(t - b_i), \quad \text{for all } 1 \leq i \leq 3, \tag{5.3.65}$$

$$x_i dx_i = dx_i x_i, \qquad x_i dt = dt(a_i x_i + p_i'(t)), \quad \text{for all } 1 \leq i \leq 3, \tag{5.3.66}$$

and

$$x_i dx_j = dx_j(c_{i,j}^{-1}x_i - c_{i,j}^{-1}q_{i,j}^{(j)}), \text{ for } i < j, \tag{5.3.67}$$

$$x_i dx_j = dx_j(c_{j,i}x_i + q_{j,i}^{(j)}), \text{for } i > j. \tag{5.3.68}$$

We want to extend $t \mapsto dt$, $x_i \mapsto dx_i$, $1 \leq i \leq 3$ to a map $d : \sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle \to \Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle)$ satisfying the Leibniz's rule. This is possible if the Leibniz's rule is compatible with the non-trivial relations (5.3.36) and (5.3.37), i.e.

$$dx_i t + x_i dt = a_i dtx_i + a_i tdx_i + b_i dx_i + dp_i(t), \quad \text{for } 1 \leq i \leq 3$$

$$dx_j x_i + x_j dx_i = c_{i,j}dx_i x_j + c_{i,j}x_i dx_j + \sum_{k=1}^{3} q_{i,j}^{(k)} dx_k, \text{ for } i < j.$$

Due to that $tdt = dtt$, which defines the usual commutative calculus on the polynomial ring $\Bbbk[t]$, $dp_i(t) = dtp_i'(t)$, $1 \leq i \leq 3$.

Define $\Bbbk$-linear maps

$$\partial_t, \partial_{x_i} : \sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle \to \sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$$

such that

$$d(a) = dt\partial_t(a) + \sum_{i=1}^{3} dx_i\partial_{x_i}(a), \text{ for all } a \in \sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle.$$

These maps are well-defined since dt, dx_i, $1 \leq i \leq 3$ are free generators of the right $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$-module $\Omega^1(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle)$. With that, $d(a) = 0$ if and only if $\partial_t(a) = \partial_{x_i}(a) = 0$, $1 \leq i \leq 3$. Using relations (5.3.64) and definitions of the maps ν_t, ν_{x_i}, $1 \leq i \leq 3$, we obtain that

$$\begin{aligned}
\partial_t(t^k x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= k t^{k-1} x_1^{l_1} x_2^{l_2} x_3^{l_3}, \\
\partial_{x_1}(t^k x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= l_1 a_1^{-k}(t - b_1)^k x_1^{l_1-1} x_2^{l_2} x_3^{l_3}, \\
\partial_{x_2}(t^k x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= l_2 a_2^{-k}(t - b_2)^k c_{1,2}^{-l_1}(x_1 - q_{1,2}^{(2)})^{l_1} x_2^{l_2-1} x_3^{l_3}, \quad \text{and} \\
\partial_{x_3}(t^k x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= l_3 a_3^{-k}(t - b_3)^k c_{1,3}^{-l_1}(x_1 - q_{1,3}^{(3)})^{l_1} c_{2,3}^{-l_2}(x_2 - q_{2,3}^{(3)})^{l_2} x_3^{l_3-1}.
\end{aligned} \tag{5.3.69}$$

Then, $d(a) = 0$ if and only if a is a scalar multiple of the identity. This shows that $\Omega(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle, d)$ is connected, where

$$\Omega(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle) = \bigoplus_{i=0}^{4} \Omega^i(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle).$$

The universal extension of d to higher forms compatible with (5.3.65), (5.3.66) and (5.3.68) gives the following rules for $\Omega^l(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle)$ $(l = 2, 3)$:

$$\begin{aligned}
dx_i \wedge dt &= -a_i dt \wedge dx_i, \quad \text{for } 1 \leq i \leq 3, \\
dx_j \wedge dx_i &= -c_{i,j} dx_i \wedge dx_j, \quad \text{for } 1 \leq i < j \leq 3, \\
dx_2 \wedge dx_1 \wedge dt &= -c_{1,2} a_1 a_2 dt \wedge dx_1 \wedge dx_2, \\
dx_3 \wedge dx_2 \wedge dt &= -c_{2,3} a_2 a_3 dt \wedge dx_2 \wedge dx_3, \\
dx_3 \wedge dx_2 \wedge dx_1 &= -c_{1,2} c_{1,3} c_{2,3} dx_1 \wedge dx_2 \wedge dx_3, \quad \text{and} \\
dx_3 \wedge dx_1 \wedge dt &= -a_1 a_3 c_{1,3} dt \wedge dx_1 \wedge dx_3.
\end{aligned} \tag{5.3.70}$$

Since the automorphisms ν_t, ν_{x_i}, $1 \leq i \leq 3$ commute with each other, there are no additional relationships to the previous ones, so

$$\begin{aligned}
\Omega^3(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle) = [&dt \wedge dx_1 \wedge dx_2 \oplus dt \wedge dx_2 \wedge dx_3 \oplus dt \wedge dx_1 \wedge dx_3 \\
&\oplus dx_1 \wedge dx_2 \wedge dx_3]\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle.
\end{aligned}$$

Now,

$$\Omega^4(\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle) = \omega\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle \cong \sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$$

as a right and left $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$-module, with $\omega = dt \wedge dx_1 \wedge dx_2 \wedge dx_3$, where $\nu_\omega = \nu_t \circ \nu_{x_1} \circ \nu_{x_2} \circ \nu_{x_3}$, this means that ω is a volume form of $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$. From Proposition 1.7 (2), it follows that ω is an integral form by setting

$$
\begin{aligned}
&\omega_1^1 = dt, \quad \omega_j^1 = dx_{j-1}, \text{ for } 2 \leq j \leq 4,\\
&\omega_1^2 = dt \wedge dx_1, \quad \omega_2^2 = dt \wedge dx_2, \quad \omega_3^2 = dt \wedge dx_3, \quad \omega_4^2 = dx_1 \wedge dx_2,\\
&\omega_5^2 = dx_1 \wedge dx_3, \quad \omega_6^2 = dx_2 \wedge dx_3,\\
&\omega_1^3 = dt \wedge dx_1 \wedge dx_2, \quad \omega_2^3 = dt \wedge dx_2 \wedge dx_3, \quad \omega_3^3 = dx_1 \wedge dx_2 \wedge dx_3,\\
&\omega_4^3 = dt \wedge dx_1 \wedge dx_3,\\
&\bar{\omega}_1^1 = -a_3^{-1}c_{1,3}^{-1}c_{2,3}^{-1}dx_3, \quad \bar{\omega}_2^1 = -a_1^{-1}dx_1, \quad \bar{\omega}_3^1 = dt, \quad \bar{\omega}_4^1 = c_{1,2}^{-1}a_2^{-1}dx_2\\
&\bar{\omega}_1^2 = a_2^{-1}a_3^{-1}c_{1,3}^{-1}c_{1,2}^{-1}dx_2 \wedge dx_3,\\
&\bar{\omega}_2^2 = -a_1^{-1}a_3^{-1}c_{2,3}^{-1}dx_1 \wedge dx_3, \quad \bar{\omega}_3^2 = a_1^{-1}a_2^{-1}dx_1 \wedge dx_2, \quad \bar{\omega}_4^2 = c_{1,3}^{-1}c_{2,3}^{-1}dt \wedge dx_3,\\
&\bar{\omega}_5^2 = -c_{1,2}^{-1}dt \wedge dx_2, \quad \bar{\omega}_6^2 = dt \wedge dx_1,\\
&\bar{\omega}_1^3 = -a_1^{-1}a_2^{-1}a_3^{-1}dx_1 \wedge dx_2 \wedge dx_3, \quad \bar{\omega}_2^3 = c_{1,3}^{-1}c_{1,2}^{-1}dt \wedge dx_2 \wedge dx_3,\\
&\bar{\omega}_3^3 = -c_{2,3}^{-1}dt \wedge dx_1 \wedge dx_3, \quad \bar{\omega}_4^3 = dt \wedge dx_1 \wedge dx_2.
\end{aligned}
$$

Let $\omega' = dta + dx_1b + dx_2c + dx_3d$, $a, b, c, d \in \Bbbk$. Then

$$
\begin{aligned}
\sum_{i=1}^{4} \omega_i^1 \pi_\omega(\bar{\omega}_i^3 \wedge \omega') &= dt\pi_\omega(-aa_1^{-1}a_2^{-1}a_3^{-1}dx_1 \wedge dx_2 \wedge dx_3 \wedge dt)\\
&\quad + dx_1\pi_\omega(bc_{1,3}^{-1}c_{1,2}^{-1}dt \wedge dx_2 \wedge dx_3 \wedge dx_1)\\
&\quad + dx_2\pi_\omega(-cc_{2,3}^{-1}dt \wedge dx_1 \wedge dx_3 \wedge dx_2)\\
&\quad + dx_3\pi_\omega(ddt \wedge dx_1 \wedge dx_2 \wedge dx_3)\\
&= dta + dx_1b + dx_2c + dx_3d\\
&= \omega'.
\end{aligned}
$$

On the other hand, if

$$\omega'' = dt \wedge dx_1a + dt \wedge dx_2b + dt \wedge dx_3c + dx_1 \wedge dx_2d + dx_1 \wedge dx_3e + dx_2 \wedge dx_3f$$

with $a, b, c, d, e, f \in \Bbbk$, it yields that

$$
\begin{aligned}
\sum_{i=1}^{6} \omega_i^2 \pi_\omega(\bar{\omega}_i^2 \wedge \omega'') &= dt \wedge dx_1\pi_\omega(aa_2^{-1}a_3^{-1}c_{1,3}^{-1}c_{1,2}^{-1}dx_2 \wedge dx_3 \wedge dt \wedge dx_1)\\
&\quad + dt \wedge dx_2\pi_\omega(-ba_1^{-1}a_3^{-1}c_{2,3}^{-1}dx_1 \wedge dx_3 \wedge dt \wedge dx_2)\\
&\quad + dt \wedge dx_3\pi_\omega(ca_1^{-1}a_2^{-1}dx_1 \wedge dx_2 \wedge dt \wedge dx_3)\\
&\quad + dx_1 \wedge dx_2\pi_\omega(dc_{1,3}^{-1}c_{2,3}^{-1}dt \wedge dx_3 \wedge dx_1 \wedge dx_2)\\
&\quad + dx_1 \wedge dx_3\pi_\omega(-ec_{1,2}^{-1}dt \wedge dx_2 \wedge dx_1 \wedge dx_3)\\
&\quad + dx_2 \wedge dx_3\pi_\omega(fdt \wedge dx_1 \wedge dx_2 \wedge dx_3)\\
&= dt \wedge dx_1a + dt \wedge dx_2b + dt \wedge dx_3c\\
&\quad + dx_1 \wedge dx_2d + dx_1 \wedge dx_3e + dx_2 \wedge dx_3f\\
&= \omega''.
\end{aligned}
$$

Let

$$\omega''' = dt \wedge dx_1 \wedge dx_2 a + dt \wedge dx_2 \wedge dx_3 b + dx_1 \wedge dx_2 \wedge dx_3 c + dt \wedge dx_1 \wedge dx_3 d,$$

with $a, b, c, d \in \Bbbk$. Since that

$$\begin{aligned}\sum_{i=1}^{3} \omega_i^3 \pi_\omega(\bar{\omega}_i^1 \wedge \omega''') &= dt \wedge dx_1 \wedge dx_2 \pi_\omega(-a a_3^{-1} c_{1,3}^{-1} c_{2,3}^{-1} dx_3 \wedge dt \wedge dx_1 \wedge dx_2) \\ &\quad + dt \wedge dx_2 \wedge dx_3 \pi_\omega(-b a_1^{-1} dx_1 \wedge dt \wedge dx_2 \wedge dx_3) \\ &\quad + dx_1 \wedge dx_2 \wedge dx_3 \pi_\omega(c dt \wedge dx_1 \wedge dx_2 \wedge dx_3) \\ &\quad + dt \wedge dx_1 \wedge dx_3 \pi_\omega(d c_{1,2}^{-1} a_2^{-1} dx_2 \wedge dt \wedge dx_1 \wedge dx_2) \\ &= dt \wedge dx_1 \wedge dx_2 a + dt \wedge dx_2 \wedge dx_3 b + dx_1 \wedge dx_2 \wedge dx_3 c \\ &\quad + dt \wedge dx_1 \wedge dx_3 d = \omega''',\end{aligned}$$

we conclude that $\sigma(\Bbbk[t])\langle x_1, x_2, x_3\rangle$ is differentially smooth. □

5.3.3 SPBW extensions in n indeterminates

The noncommutative differential geometry of SPBW extensions of the form $\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle$ satisfying the defining relations

$$\begin{aligned} x_i r(t) &= \sigma_i(r(t)) x_i + \delta_i(r(t)), \quad \text{and} \\ x_j x_i &= c_{i,j}(t) x_i x_j + q_{i,j}^{(0)}(t) + \sum_{k=1}^{n} q_{i,j}^{(k)}(t) x_k, \end{aligned}$$

In this case, we consider $\sigma_i(t) = a_i t + b_i$, for $a_i, b_i, \in \Bbbk$, and $a_i \neq 0$, $1 \leq i \leq n$, and the derivations δ_i are motivated by (2.2.1), that is,

$$\delta_i(f) = \frac{f(\sigma_i(t)) - f(t)}{\sigma_i(t) - t} p_i(t), \text{ for } 1 \leq i \leq n, \tag{5.3.71}$$

where $p_i(t) \in \Bbbk[t]$, $1 \leq i \leq n$. The relations between $t, x_1, \ldots, x_n$ can be expressed as

$$x_i t = a_i t x_i + b_i x_i + p_i(t), \quad \text{and} \tag{5.3.72}$$

$$x_j x_i = c_{i,j} x_i x_j + q_{i,j}^{(0)} + \sum_{k=1}^{n} q_{i,j}^{(k)} x_k, \quad \text{for } 1 \leq i, j \leq n. \tag{5.3.73}$$

Lemma 5.14. *Let*

$$\nu_t(t) = t, \quad \nu_t(x_i) = a_i x_i + p_i'(t), \quad \text{for } 1 \leq i \leq n \tag{5.3.74}$$

$$\nu_{x_i}(t) = \sigma_i^{-1}(t), \quad \nu_{x_i}(x_i) = x_i, \quad \text{for } 1 \leq i \leq n \tag{5.3.75}$$

$$\nu_{x_i}(x_j) = c_{i,j} x_j + q_{i,j}^{(i)}, \quad \text{for } i < j \quad \text{and} \quad \nu_{x_i}(x_j) = c_{j,i}^{-1} x_j - c_{j,i}^{-1} q_{j,i}^{(i)}, \quad \text{for } i > j, \tag{5.3.76}$$

where $p_i'(t)$ *are the* t*-derivatives of* $p_i(t)$ *for* $1 \leq i \leq n$*, and* $c_{i,j}, q_{i,j}^{(k)} \in \Bbbk$*,* $c_{i,j} \neq 0$*, for all* $1 \leq i, j, k \leq n$*.*

Table 5.3: Leibniz's rule

Case	Possibilities for a_i, b_i, $i = 1,2,3$	*Polynomials* $p_i(t)$, $i = 1,2,3$	*Restrictions*
(a)	$a_i = 1$, $b_i = 0$, for all $1 \leq i \leq n$	$p_i(t) = 0$ for all $1 \leq i \leq n$	$q_{i,j}^{(k)} = 0$, $c_{i,j} \in \Bbbk^*$ for all $1 \leq i,j \leq n$, $k \geq 0$
		$p_i(t) \in \Bbbk[t]$, for all $1 \leq i \leq n$	$q_{i,j}^{(k)} = 0$, $c_{i,j} = 1$, $q_{i,j}^{(0)} \in \Bbbk$, for all $1 \leq i,j \leq n$, $k > 0$
(b)	$a_i = 1$, for all $1 \leq i \leq n$, $b_i \neq 0$, for some $1 \leq l \leq n$	$p_i(t) = p_i$, $p_i \in \Bbbk$, for all $1 \leq i \leq n$	$c_{i,j} = 1$, $q_{i,j}^{(k)} = 0$, $q_{i,j}^{(0)} \in \Bbbk$, for all $1 \leq i,j \leq n$, $k > 0$
		$p_i(t) = 0$, for all $1 \leq i \leq n$	$q_{i,j}^{(k)} = 0$, $c_{i,j} \in \Bbbk^*$, for all $1 \leq i,j \leq n$, $k \geq 0$
(c)	$a_r \neq 1$, $a_s = 1$, $b_s = 0$, for $r \in S \subsetneq \{1,\ldots,n\}$ and $s \in S^c$	$p_s(t) = 0$, $p_r(t) = p_r\left(t + \frac{b_r}{a_r - 1}\right)$, for $r \in S \subsetneq \{1,\ldots,n\}$ and $s \in S^c$, $p_r \in \Bbbk$	$q_{i,j}^{(k)} = 0$, $c_{i,j} = 1$, for all $1 \leq i,j \leq n$, $k \geq 0$
(d)	$a_i \neq 1$, $b_i = 0$, for all $1 \leq i \leq n$	$p_i(t) = p_i t$, $p_i \in \Bbbk$, for all $1 \leq i \leq n$	$q_{i,j}^{(k)} = 0$, $c_{i,j} = 1$, for all $1 \leq i,j \leq n$, $k \geq 0$

(1) *Leibniz's rule holds in the cases listed in Table 5.3.*

The symbols defined by (5.3.74), (5.3.75) and (5.3.76) simultaneously extend to algebra automorphisms ν_t, ν_{x_i}, $1 \leq i \leq n$ *of* $\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle$ *in cases* (a) - (d).

(2) *In cases* (a) - (d), *we have that*

$$\nu_t \circ \nu_{x_i} = \nu_{x_i} \circ \nu_t \quad \text{and} \quad \nu_{x_i} \circ \nu_{x_j} = \nu_{x_j} \circ \nu_{x_i}, \quad \textit{for } 1 \leq i,j \leq n. \tag{5.3.77}$$

Theorem 5.15. *If a SPBW extension* $\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle$ *satisfies one of the conditions* (a)-(d) *in Lemma 5.14, then it is differentially smooth.*

Proof. Since $\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle$ has Gelfand-Kirillov dimension $n+1$, we can construct an $n+1$-dimensional integrable. Consider $\Omega^1(\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle)$ a free right $\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle$-module of rank $n+1$ with generators dt, $dx_1,\ldots,dx_n$. Define a left $\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle$-module structure by

$$adt = dt\nu_t(a) \quad \text{and} \quad adx_i = dx_i\nu_{x_i}(a), \text{ for all } 1 \leq i \leq n, \ a \in \sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle, \tag{5.3.78}$$

where ν_t and ν_{x_i} with $i = 1,\ldots,n$ are the algebra automorphisms established in Lemma 5.14. The relations in $\Omega^1(\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle)$ are given by

$$tdt = dtt, \qquad tdx_i = dx_i a_i^{-1}(t - b_i), \quad \text{for all } 1 \leq i \leq n, \tag{5.3.79}$$

$$x_i dx_i = dx_i x_i, \qquad x_i dt = dt(a_i x_i + p_i'(t)), \quad \text{for all } 1 \leq i \leq n, \tag{5.3.80}$$

and

$$x_i dx_j = dx_j(c_{i,j}^{-1}x_i - c_{i,j}^{-1}q_{i,j}^{(j)}), \quad \text{for } i < j, \quad \text{and} \tag{5.3.81}$$

$$x_i dx_j = dx_j(c_{j,i}x_i + q_{j,i}^{(j)}), \quad \text{for } i > j. \tag{5.3.82}$$

We want to extend the assignments $t \mapsto dt$, $x_i \mapsto dx_i$, $1 \leq i \leq n$ to a map

$$d : \sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle \to \Omega^1(\sigma(\Bbbk[t])\langle x_1,\ldots,x_n\rangle)$$

satisfying the Leibniz's rule, so we need to impose the compatibility between this rule and the non-trivial relations (5.3.72) and (5.3.73). In this way, we have that

$$dx_i t + x_i dt = a_i dt x_i + a_i t dx_i + b_i dx_i + dp_i(t), \quad \text{for } 1 \leq i \leq n, \quad \text{and}$$

$$dx_j x_i + x_j dx_i = c_{i,j} dx_i x_j + c_{i,j} x_i dx_j + \sum_{k=1}^{n} q_{i,j}^{(k)} dx_k, \quad \text{for } i < j.$$

Note that in view of the equality $tdt = dtt$, which defines the usual commutative calculus on the polynomial ring $\Bbbk[t]$, we get that $dp_i(t) = dtp_i'(t)$ for $1 \leq i \leq n$.

Define $\Bbbk$-linear maps

$$\partial_t, \partial_{x_i} : \sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle \to \sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle, \quad i = 1, \ldots, n,$$

such that

$$d(f) = dt\partial_t(f) + \sum_{i=1}^{n} dx_i\partial_{x_i}(f), \quad \text{for all } f \in \sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle.$$

These maps are well-defined since dt and dx_i $(1 \leq i \leq n)$ are free generators of the right $\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle$-module $\Omega^1(\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle)$. Thus, $d(a) = 0$ if and only if $\partial_t(a) = \partial_{x_i}(a) = 0$ for $1 \leq i \leq n$. Using relations appearing in (5.3.78) and the definitions of the maps ν_t and ν_{x_i} $(1 \leq i \leq n)$, we obtain that

$$\partial_t(t^k x_1^{l_1} \cdots x_n^{l_n}) = kt^{k-1}x_1^{l_1} \cdots x_n^{l_n}, \quad \text{and} \tag{5.3.83}$$

$$\partial_{x_i}(t^k x_1^{l_1} \cdots x_n^{l_n}) = l_i a_i^{-k}(t - b_i)^k \prod_{s=1}^{i-1} c_{s,i}^{-l_s}(x_s - q_{s,i}^{(i)})^{l_s} x_i^{l_i - 1} x_{i+1}^{l_{i+1}} \cdots x_n^{l_n}.$$

Hence, $d(a) = 0$ if and only if a is a scalar multiple of the identity. This fact shows that $(\Omega(\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle, d))$ is connected, where

$$\Omega(\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle) = \bigoplus_{i=0}^{n+1} \Omega^i(\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle).$$

The universal extension of d to higher forms compatible with (5.3.79), (5.3.80) and (5.3.82) gives the following rules for $\Omega^l(\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle)$ $(2 \leq l \leq n)$:

$$dx_{q(1)} \wedge \cdots \wedge dx_{q(s)} \wedge dt \wedge dx_{q(s+1)} \wedge \cdots \wedge dx_{q(l)} = (-1)^s \prod_{r=1}^{s} a_{q(r)}^{-1} dt \wedge \bigwedge_{\substack{k=1,\\ k\neq s_1}}^{l} dx_{q(k)}, \tag{5.3.84}$$

$$\bigwedge_{k=1}^{l} dx_{q(k)} = (-1)^{\sharp} \prod_{r,s\in P} c_{r,s}^{-1} \bigwedge_{k=1}^{l} dx_{p(k)}, \tag{5.3.85}$$

where where $s_1 \in \{1, \ldots, l\}$ do not appear in Relation (5.3.84), $q : \{1, \ldots, l\} \to \{1, \ldots, n\}$ is an injective map, $p : \{1, \ldots, l\} \to \mathrm{Im}(q)$ is an increasing injective map and $\sharp$ is the number of 2-permutations needed to transform q into p, and $P := \{(s, t) \in \{1, \ldots, l\} \times \{1, \ldots, l\} \mid q(s) > q(t)\}$.

Since the automorphisms ν_t, ν_{x_i}, $1 \leq i \leq n$ commute with each other, there are no additional relations to the previous ones, so we get that

$$\begin{aligned}\Omega^n(\sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle) = \Bigg[&\bigoplus_{r=2}^{n-1} dt \wedge dx_1 \wedge \cdots dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_n \\ &\oplus dt \wedge dx_2 \wedge \cdots \wedge dx_n \oplus dt \wedge dx_1 \wedge \cdots \wedge dx_{n-1} \\ &\oplus dx_1 \wedge \cdots \wedge dx_n\Bigg]\, \sigma(\Bbbk[t])\langle x_1, \ldots, x_n\rangle.\end{aligned}$$

Now, since

$$\Omega^{n+1}(\sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle) = \omega\sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle \cong \sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle$$

as a right and left $\sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle$-module, with

$$\omega = dt \wedge dx_1 \wedge \cdots \wedge dx_n \quad \text{and} \quad \nu_\omega = \nu_t \circ \nu_{x_1} \circ \cdots \circ \nu_{x_n},$$

it follows that ω is a volume form of $\sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle$. In order to make the calculations easier, we consider the following notation $t = x_0$, $c_{0,i} = a_i$ for $1 \leq i \leq n$.

From Proposition 1.7 (2) we get that ω is an integral form by setting

$$\omega_i^j = \bigwedge_{k=0}^{j-1} dx_{p_{i,j}(k)}, \text{ for } 1 \leq i \leq \binom{n+1}{j},$$

$$\bar{\omega}_i^{n+1-j} = (-1)^{\sharp_{i,j}} \prod_{r,s\in P_{i,j}} c_{r,s}^{-1} \bigwedge_{k=j}^{n} dx_{\bar{p}_{i,j}(k)}, \text{ for } 1 \leq i \leq \binom{n+1}{j},$$

for $1 \leq j \leq n+1$ and where

$$p_{i,j} : \{0,\dots,j-1\} \to \{0,\dots,n\}, \quad \text{and}$$
$$\bar{p}_{i,j} : \{j,\dots,n\} \to (\mathrm{Im}(p_{i,j}))^c$$

(the symbol $\square^c$ denotes the complement of the set $\square$), are increasing injective maps, and $\sharp_{i,j}$ is the number of 2-permutation needed to transform

$$\{\bar{p}_{i,j}(j),\dots,\bar{p}_{i,j}(n),p_{i,j}(0),\dots,p_{i,j}(j-1)\} \quad \text{into the set} \quad \{0,\dots,n\},$$

and

$$P_{i,j} := \{(s,t) \in \{0,\dots,j-1\} \times \{j,\dots,n\} \mid p_{i,j}(s) < \bar{p}_{i,j}(t)\}.$$

Consider $\omega' \in \Omega^j(\sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle)$, that is,

$$\omega' = \sum_{i=1}^{\binom{n+1}{j}} \bigwedge_{k=0}^{j-1} dx_{p_{i,j}(k)} b_i, \quad \text{with } b_i \in \Bbbk.$$

Then

$$\begin{aligned}\sum_{i=1}^{\binom{n+1}{j}} \omega_i^j \pi_\omega(\bar{\omega}_i^{n+1-j} \wedge \omega') &= \sum_{i=1}^{\binom{n+1}{j}} \left[\bigwedge_{k=0}^{j-1} dx_{p_i(k)}\right] \cdot \pi_\omega\left[(-1)^{\sharp_{i,j}} \square^* \wedge \omega'\right] \\ &= \sum_{i=1}^{\binom{n+1}{j}} \bigwedge_{k=0}^{j-1} dx_{p_{i,j}(k)} b_i = \omega',\end{aligned}$$

where

$$\square^* := \prod_{r,s\in P_{i,j}} c_{r,s}^{-1} \bigwedge_{k=j}^{n} dx_{\bar{p}_{i,j}(k)}.$$

By Proposition 1.7 (2), it follows that $\sigma(\Bbbk[t])\langle x_1,\dots,x_n\rangle$ is differentially smooth. $\square$

Remark 18. Note that there is no unique way to define ω_i^j and $\bar{\omega}_i^{n-j}$. Our way of defining them is because it is the simplest.

5.4 Differential smoothness of SPBW extensions over $\Bbbk[t_1, t_2]$

Finally, we investigate the differential smoothness of bijective SPBW extensions over the commutative polynomial ring $\Bbbk[t_1, t_2]$.

$\mathrm{Aut}(\Bbbk[t_1, t_2])$ are compositions of automorphisms of two types (see McKay and Wang [MW88], Shestakov and Umirbaev [SU03] or Van den Essen [DE00] for more details):

- First type:

$$t_1 \longmapsto a_{11}t_1 + a_{12}t_2 + a_{13} \quad \text{and} \quad t_2 \longmapsto a_{21}t_1 + a_{22}t_2 + a_{23}, \tag{5.4.1}$$

  where $a_{i,j} \in \Bbbk$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

- Second type:

$$t_1 \longmapsto t_1, \quad t_2 \longmapsto t_2 + h(t_1), \tag{5.4.2}$$

  where $h(t_1) \in \Bbbk[t_1]$.

With these facts in our hands, we proceed to study the differential smoothness of SPBW extension on two generators.

5.4.1 SPBW extensions in two indeterminates

Let $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$. From Definition 5.2 we know that

$$\begin{aligned}
x_1 r(t_1) &= \sigma_1(r(t_1))x_1 + \delta_1(r(t_1)), \quad x_2 r(t_1) = \sigma_2(r(t_1))x_2 + \delta_2(r(t_1)),\\
x_1 r(t_2) &= \sigma_1(r(t_2))x_1 + \delta_1(r(t_2)), \quad x_2 r(t_2) = \sigma_2(r(t_2))x_2 + \delta_2(r(t_2)), \quad \text{and}\\
x_2x_1 &= c_{1,2}(t_1, t_2)x_1x_2 + q_{1,2}^{(0)}(t_1, t_2) + q_{1,2}^{(1)}(t_1, t_2)x_1 + q_{1,2}^{(2)}(t_1, t_2)x_2,
\end{aligned}$$

where the polynomials $r(t_1, t_2), c_{1,2}(t_1, t_2), q_{1,2}^{(0)}(t_1, t_2), q_{1,2}^{(1)}(t_1, t_2), q_{1,2}^{(2)}(t_1, t_2)$ belong to $\Bbbk[t_1, t_2]$, and $c_{1,2}(t_1, t_2)$ is a non-zero element.

Considering the notation above, we write $\sigma_1, \sigma_2 \in \mathrm{Aut}(\Bbbk[t_1, t_2])$ as follows:

$$\begin{aligned}
\sigma_1(t_1) &= a_{111}t_1 + a_{112}t_2 + b_{11},\\
\sigma_1(t_2) &= a_{121}t_1 + a_{122}t_2 + b_{12},\\
\sigma_2(t_1) &= a_{211}t_1 + a_{212}t_2 + b_{21}, \quad \text{and}\\
\sigma_2(t_2) &= a_{221}t_1 + a_{222}t_2 + b_{22}.
\end{aligned}$$

As in Section 5.3.1, the polynomials $p_1(t_1, t_2), p_2(t_1, t_2) \in \Bbbk[t_1, t_2]$ are considered in such a way that the following identities

$$\begin{aligned} x_1t_1 &= a_{111}t_1x_1 + a_{112}t_2x_1 + b_{11}x_1 + p_1(t_1, t_2), \\ x_2t_1 &= a_{211}t_1x_2 + a_{212}t_2x_2 + b_{21}x_2 + p_2(t_1, t_2), \\ x_1t_2 &= a_{121}t_1x_1 + a_{122}t_2x_1 + b_{12}x_1 + p_1(t_1, t_2), \\ x_2t_2 &= a_{221}t_1x_2 + a_{222}t_2x_2 + b_{22}x_2 + p_2(t_1, t_2), \quad \text{and} \\ x_2x_1 &= c_{1,2}(t_1, t_2)x_1x_2 + q^{(0)}_{1,2}(t_1, t_2) + q^{(1)}_{1,2}(t_1, t_2)x_1 + q^{(2)}_{1,2}(t_1, t_2)x_2, \end{aligned} \tag{5.4.3}$$

hold. Since the map $d : \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle \to \Omega^1(\sigma(\Bbbk[t_1, t_2]))\langle x_1, x_2\rangle$ must satisfy Leibniz's rule, it is straightforward to see that we need to guarantee the conditions

- $c_{1,2}, q^{(0)}_{1,2}, q^{(1)}_{1,2}, q^{(2)}_{1,2} \in \Bbbk$, with $c_{1,2}$ non-zero.
- $p_1(t_1, t_2) = p_1$ and $p_2(t_1, t_2) = p_2$, where $p_1, p_2 \in \Bbbk$.
- $a_{221} = a_{212} = a_{112} = a_{121} = 0$.

Indeed, the first four relations in (5.4.3) can be written as

$$x_it_j = a_{ij1}t_1x_i + a_{ij2}t_2x_i + b_{ij}x_i + p_i(t_1, t_2), \quad \text{for } i, j \in \{1, 2\}. \tag{5.4.4}$$

By applying d to (5.4.4) we get that

$$0 = -d(x_it_j) + d(a_{ij1}t_1x_i + a_{ij2}t_2x_i + b_{ij}x_i + p_i(t_1, t_2)).$$

Since d is $\Bbbk$-linear, the Leibniz's rule implies that

$$\begin{aligned} 0 = &- dx_it_j - x_idt_j + a_{ij1}dt_1x_i + a_{ij1}t_1dx_i + a_{ij2}dt_2x_i + a_{ij2}t_2dx_i \\ &+ b_{ij}dx_i + d(p_i(t_1, t_2)). \end{aligned}$$

By (1.4.2), the action of the module is written using the automorphisms ν_{t_1}, ν_{t_2}, ν_{x_1} and ν_{x_2}, that is,

$$\begin{aligned} 0 = &- dx_it_j - dt_j\nu_{t_j}(x_i) + a_{ij1}dt_1x_i + a_{ij1}dx_i\nu_{x_i}(t_1) + a_{ij2}dt_2x_i + a_{ij2}dx_i\nu_{x_i}(t_2) \\ &+ b_{ij}dx_i + dt_1\frac{\partial p_i}{\partial t_1} + dt_2\frac{\partial p_i}{\partial t_2}. \end{aligned}$$

If we put together the terms that multiply the different differentials, then

$$\begin{aligned} 0 = dx_i(-t_j + a_{ij1}\nu_{x_i}(t_1) + a_{ij2}\nu_{x_i}(t_2) + b_{ij}) + dt_1\left(a_{ij1}x_i + \frac{\partial p_i}{\partial t_1}\right) \\ dt_2\left(a_{ij2}x_i + \frac{\partial p_i}{\partial t_2}\right) - dt_j\nu_{t_j}(x_i). \end{aligned}$$

For $j = 1$, we obtain the term

$$a_{i12}x_i + \frac{\partial p_i}{\partial t_2} = 0,$$

whence $a_{i12} = 0$ and $\frac{\partial p_i}{\partial t_2} = 0$ for $i \in \{1, 2\}$.

Next, when $j = 2$,

$$a_{i21}x_i + \frac{\partial p_i}{\partial t_1} = 0.$$

Once more again, it follows that $a_{i21} = 0$ and $\frac{\partial p_i}{\partial t_1} = 0$ for $i \in \{1, 2\}$.

Since the partial derivatives of p_i are zero, we conclude that p_i is a constant element for $i \in \{1, 2\}$.

By applying d to the last equation in (5.4.3) we get that

$$\begin{aligned}
d(x_2x_1) &= d(c_{1,2}(t_1, t_2)x_1x_2 + q^{(0)}_{1,2}(t_1, t_2) + q^{(1)}_{1,2}(t_1, t_2)x_1 + q^{(2)}_{1,2}(t_1, t_2)x_2) \\
&= d(c_{1,2}(t_1, t_2))x_1x_2 + c_{1,2}(t_1, t_2)d(x_1x_2) + d(q^{(0)}_{1,2}(t_1, t_2)) \\
&\quad + d(q^{(1)}_{1,2}(t_1, t_2))x_1 + q^{(1)}_{1,2}(t_1, t_2)dx_1 + d(q^{(2)}_{1,2}(t_1, t_2))x_2 + q^{(2)}_{1,2}(t_1, t_2)dx_2 \\
&= dt_1\frac{\partial c_{1,2}}{\partial t_1}x_1x_2 + dt_2\frac{\partial c_{1,2}}{\partial t_2}x_1x_2 + c_{1,2}(t_1, t_2)dx_1x_2 + c_{1,2}(t_1, t_2)x_1dx_2 \\
&\quad + dt_1\frac{\partial q^{(0)}_{1,2}}{\partial t_1} + dt_2\frac{\partial q^{(0)}_{1,2}}{\partial t_2} + dt_1\frac{\partial q^{(1)}_{1,2}}{\partial t_1}x_1 + dt_2\frac{\partial q^{(1)}_{1,2}}{\partial t_2}x_1 + q^{(1)}_{1,2}(t_1, t_2)dx_1 \\
&\quad + dt_1\frac{\partial q^{(2)}_{1,2}}{\partial t_1}x_2 + dt_2\frac{\partial q^{(2)}_{1,2}}{\partial t_2}x_2 + q^{(2)}_{1,2}(t_1, t_2)dx_2.
\end{aligned}$$

The expression (1.4.2) implies that the action of the module is written using the automorphisms ν_{t_1}, ν_{t_2}, ν_{x_1} and ν_{x_2} as follows:

$$\begin{aligned}
0 &= -dx_2x_1 - dx_1\nu_{x_1}(x_2) + dt_1\frac{\partial c_{1,2}}{\partial t_1}x_1x_2 + dt_2\frac{\partial c_{1,2}}{\partial t_2}x_1x_2 + dx_1\nu_{x_1}(c_{1,2}(t_1, t_2))x_2 \\
&\quad + dx_2\nu_{x_2}(c_{1,2}(t_1, t_2))\nu_{x_2}(x_1) + dt_1\frac{\partial q^{(0)}_{1,2}}{\partial t_1} + dt_2\frac{\partial q^{(0)}_{1,2}}{\partial t_2} + dt_1\frac{\partial q^{(1)}_{1,2}}{\partial t_1}x_1 + dt_2\frac{\partial q^{(1)}_{1,2}}{\partial t_2}x_1 \\
&\quad + dx_1\nu_{x_1}(q^{(1)}_{1,2}(t_1, t_2)) + dt_1\frac{\partial q^{(2)}_{1,2}}{\partial t_1}x_2 + dt_2\frac{\partial q^{(2)}_{1,2}}{\partial t_2}x_2 + dx_2\nu_{x_2}(q^{(2)}_{1,2}(t_1, t_2)).
\end{aligned}$$

In this way,

$$\begin{aligned}
0 &= dt_1\left(\frac{\partial c_{1,2}}{\partial t_1}x_1x_2 + \frac{\partial q^{(0)}_{1,2}}{\partial t_1} + \frac{\partial q^{(1)}_{1,2}}{\partial t_1}x_1 + \frac{\partial q^{(2)}_{1,2}}{\partial t_1}x_2\right) \\
&\quad + dt_2\left(\frac{\partial c_{1,2}}{\partial t_2}x_1x_2 + \frac{\partial q^{(0)}_{1,2}}{\partial t_2} + \frac{\partial q^{(1)}_{1,2}}{\partial t_2}x_1 + \frac{\partial q^{(2)}_{1,2}}{\partial t_2}x_2\right).
\end{aligned}$$

From the reasoning above, it can be seen that all partial derivatives must be equal to zero. Equivalently, $c_{1,2}, q^{(0)}_{1,2}, q^{(1)}_{1,2}, q^{(2)}_{1,2} \in \Bbbk$, with $c_{1,2}$ a non-zero element of the field $\Bbbk$.

The five relations in (5.4.3) are reduced to

$$x_1t_1 = a_{111}t_1x_1 + b_{11}x_1 + p_1, \tag{5.4.5}$$
$$x_2t_1 = a_{211}t_1x_2 + b_{21}x_2 + p_2, \tag{5.4.6}$$
$$x_1t_2 = a_{122}t_2x_1 + b_{12}x_1 + p_1, \tag{5.4.7}$$
$$x_2t_2 = a_{222}t_2x_2 + b_{22}x_2 + p_2, \quad \text{and} \tag{5.4.8}$$
$$x_2x_1 = c_{1,2}x_1x_2 + q_{1,2}^{(0)} + q_{1,2}^{(1)}x_1 + q_{1,2}^{(2)}x_2. \tag{5.4.9}$$

All these facts allow us to formulate the following proposition.

Proposition 5.16. *Let*

$$\nu_{t_1}(t_1) = t_1, \quad \nu_{t_1}(t_2) = t_2, \quad \nu_{t_1}(x_1) = a_{111}x_1, \tag{5.4.10}$$
$$\nu_{t_1}(x_2) = a_{211}x_2, \quad \nu_{t_2}(t_1) = t_1, \quad \nu_{t_2}(t_2) = t_2, \tag{5.4.11}$$
$$\nu_{t_2}(x_1) = a_{122}x_1, \quad \nu_{t_2}(x_2) = a_{222}x_2, \quad \nu_{x_1}(t_1) = a_{111}^{-1}(t_1 - b_{11}), \tag{5.4.12}$$
$$\nu_{x_1}(t_2) = a_{122}^{-1}(t_2 - b_{12}), \quad \nu_{x_1}(x_1) = x_1, \quad \nu_{x_1}(x_2) = c_{1,2}x_2 + q_{1,2}^{(1)}, \tag{5.4.13}$$
$$\nu_{x_2}(t_1) = a_{211}^{-1}(t_1 - b_{21}), \quad \nu_{x_2}(t_2) = a_{222}^{-1}(t_2 - b_{22}), \quad \nu_{x_2}(x_1) = c_{1,2}^{-1}x_1 - c_{1,2}^{-1}q_{1,2}^{(2)}, \tag{5.4.14}$$
$$\nu_{x_2}(x_2) = x_2. \tag{5.4.15}$$

Then:

(1) *Leibniz's rule holds in the cases listed in Table 5.4. The maps defined by (5.4.10), (5.4.11), (5.4.12), (5.4.13), (5.4.14) and (5.4.15) simultaneously extend to algebra automorphisms* $\nu_{t_1}, \nu_{t_2}, \nu_{x_1}, \nu_{x_2}$ *of* $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$ *in cases* (a) - (p).

(2) *In cases* (a) - (p), *we have that*

$$\nu_{t_i} \circ \nu_{t_j} = \nu_{t_j} \circ \nu_{t_i}, \quad \nu_{t_i} \circ \nu_{x_j} = \nu_{x_j} \circ \nu_{t_i}, \quad \nu_{x_i} \circ \nu_{x_j} = \nu_{x_j} \circ \nu_{x_i}, \quad \text{for } i, j = 1, 2. \tag{5.4.16}$$

Proof. For the first assertion, the map ν_{t_1} can be extended to an algebra homomorphism if and only if the definitions of $\nu_{t_1}(t_1)$, $\nu_{t_1}(t_2)$, $\nu_{t_1}(x_1)$ and $\nu_{t_1}(x_2)$ respect relations (5.4.5), (5.4.6), (5.4.7), (5.4.8) and (5.4.9), i.e.

$$\begin{aligned}
&\nu_{t_1}(x_1)\nu_{t_1}(t_1) - \nu_{t_1}(a_{111}t_1 + b_{11})\nu_{t_1}(x_1) = \nu_{t_1}(p_1),\\
&\nu_{t_1}(x_2)\nu_{t_1}(t_1) - \nu_{t_1}(a_{211}t_1 + b_{21})\nu_{t_1}(x_2) = \nu_{t_1}(p_2),\\
&\nu_{t_1}(x_1)\nu_{t_1}(t_2) - \nu_{t_1}(a_{122}t_2 + b_{12})\nu_{t_1}(x_1) = \nu_{t_1}(p_1),\\
&\nu_{t_1}(x_2)\nu_{t_1}(t_2) - \nu_{t_1}(a_{222}t_2 + b_{22})\nu_{t_1}(x_2) = \nu_{t_1}(p_2), \quad \text{and}\\
&\qquad \nu_{t_1}(x_2)\nu_{t_1}(x_1) - c_{1,2}\nu_{t_1}(x_1)\nu_{t_1}(x_2) = q_{1,2}^{(0)} + q_{1,2}^{(1)}\nu_{t_1}(x_1) + q_{1,2}^{(2)}\nu_{t_1}(x_2).
\end{aligned}$$

Table 5.4: Leibniz's rule

Case	Possibilities for a_{111}, a_{122}, a_{211}, a_{222}	*Polynomials* p_1 *and* p_2	*Restrictions*
(a)	$a_{111}=1$, $a_{122}=1$, $a_{211}=1$, $a_{222}=1$	$p_1, p_2 \in \Bbbk$	$c_{1,2}=1$, $q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{11}, b_{12}, b_{21}, b_{22}, q_{1,2}^{(0)} \in \Bbbk$
			$c_{1,2}=1$, $q_{1,2}^{(1)}=0, q_{1,2}^{(2)}\neq 0$, $b_{11}=b_{12}=0$, $b_{21}, b_{22}, q_{1,2}^{(0)} \in \Bbbk$
			$c_{1,2}=1$, $q_{1,2}^{(1)}\neq 0, q_{1,2}^{(2)}=0$, $b_{21}=b_{22}=0$, $b_{11}, b_{12}, q_{1,2}^{(0)} \in \Bbbk$
		$p_1=p_2=0$	$c_{1,2}\neq 1$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{11}, b_{12}, b_{21}, b_{22} \in \Bbbk$
		$p_1=\frac{b_{11}q_{1,2}^{(2)}}{c_{1,2}-1}$, $p_2=0$	$c_{1,2}\neq 1$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$ $q_{1,2}^{(2)}\neq 0$, $b_{11}=b_{12}$, $b_{11}, b_{21}, b_{22} \in \Bbbk$
		$p_1=0, p_2=\frac{b_{22}q_{1,2}^{(1)}}{c_{1,2}-1}$	$c_{1,2}\neq 1$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$ $q_{1,2}^{(1)}\neq 0$, $b_{22}=b_{21}$, $b_{11}, b_{12}, b_{22} \in \Bbbk$
		$p_1=\frac{b_{11}q_{1,2}^{(2)}}{c_{1,2}-1}, p_2=\frac{b_{22}q_{1,2}^{(1)}}{c_{1,2}-1}$	$c_{1,2}\neq 1$, $q_{1,2}^{(0)}=\frac{q_{1,2}^{(1)}q_{1,2}^{(2)}}{c_{1,2}-1}$ $q_{1,2}^{(1)}\neq 0$ $q_{1,2}^{(2)}\neq 0$, $b_{11}=b_{12}$, $b_{22}=b_{21}$, $b_{11}, b_{22} \in \Bbbk$
(b)	$a_{111}\neq 1$, $a_{122}=1$, $a_{211}=1$, $a_{222}=1$	$p_1=p_2=0$	$b_{21}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$, $c_{1,2}, b_{11}, b_{12} \in \Bbbk$ with $b_{22}=0$ and $q_{1,2}^{(1)} \in \Bbbk$ or $q_{1,2}^{(1)}=0$ and $b_{22} \in \Bbbk$
		$p_1=0$, $p_2=\frac{b_{22}q_{1,2}^{(1)}}{c_{1,2}-1}$	$b_{21}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$, $c_{1,2}=a_{111}$, $b_{11}, b_{12}, b_{22}, q_{1,2}^{(1)} \in \Bbbk$
(c)	$a_{111}=1$, $a_{122}\neq 1$, $a_{211}=1$, $a_{222}=1$	$p_1=p_2=0$	$b_{22}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$, $c_{1,2}, b_{11}, b_{12} \in \Bbbk$ with $b_{21}=0$ and $q_{1,2}^{(1)} \in \Bbbk$ or $q_{1,2}^{(1)}=0$ and $b_{21} \in \Bbbk$
		$p_1=0$, $p_2=\frac{b_{21}q_{1,2}^{(1)}}{c_{1,2}-1}$	$b_{22}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$, $c_{1,2}=a_{122}$, $b_{11}, b_{12}, b_{21}, q_{1,2}^{(1)} \in \Bbbk$
(d)	$a_{111}=1$, $a_{122}=1$, $a_{211}\neq 1$, $a_{222}=1$	$p_1=p_2=0$	$b_{11}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$, $c_{1,2}, b_{22}, b_{21} \in \Bbbk$ with $b_{12}=0$ and $q_{1,2}^{(2)} \in \Bbbk$ or $q_{1,2}^{(2)}=0$ and $b_{12} \in \Bbbk$
		$p_1=\frac{b_{12}q_{1,2}^{(2)}}{c_{1,2}-1}$, $p_2=0$	$b_{11}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$, $c_{1,2}=a_{211}$, $b_{12}, b_{21}, b_{22}, q_{1,2}^{(2)} \in \Bbbk$
(e)	$a_{111}=1$, $a_{122}=1$, $a_{211}=1$, $a_{222}\neq 1$	$p_1=p_2=0$	$b_{12}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$, $c_{1,2}, b_{22}, b_{21} \in \Bbbk$ with $b_{11}=0$ and $q_{1,2}^{(2)} \in \Bbbk$ or $q_{1,2}^{(2)}=0$ and $b_{11} \in \Bbbk$
		$p_1=\frac{b_{11}q_{1,2}^{(2)}}{c_{1,2}-1}$, $p_2=0$	$b_{12}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$, $c_{1,2}=a_{211}$, $b_{11}, b_{21}, b_{22}, q_{1,2}^{(2)} \in \Bbbk$
(f)	$a_{111}\neq 1$, $a_{122}\neq 1$, $a_{211}=1$, $a_{222}=1$	$p_1=p_2=0$	$b_{21}=b_{22}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$, $c_{1,2}, q_{1,2}^{(1)}, b_{11}, b_{12} \in \Bbbk$
		$p_1=0$, $p_2 \in \Bbbk$	$b_{21}=b_{22}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(2)}=0$, $a_{111}=a_{122}\, c_{1,2}=a_{111}$, $q_{1,2}^{(1)}, b_{11}, b_{12} \in \Bbbk$
(g)	$a_{111}\neq 1$, $a_{122}=1$, $a_{211}\neq 1$, $a_{222}=1$	$p_1=p_2=0$	$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{21}=\frac{b_{11}(a_{211}-1)}{a_{111}-1}\, c_{1,2}, b_{11}, b_{12}, b_{22} \in \Bbbk$
			$a_{111}=a_{211}^{-1}\, q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{21}=-a_{211}^{-1}b_{11}, b_{11}, b_{12}, b_{22} \in \Bbbk$ with $q_{1,2}^{(0)}=0$ and $c_{1,2} \in \Bbbk$ or $q_{1,2}^{(0)} \in \Bbbk$ and $c_{1,2}=0$
(h)	$a_{111}\neq 1$, $a_{122}=1$, $a_{211}=1$, $a_{222}\neq 1$	$p_1=p_2=0$	$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{21}=b_{12}=0$, $c_{1,2}, b_{11}, b_{22} \in \Bbbk$
(i)	$a_{111}=1$, $a_{122}\neq 1$, $a_{211}\neq 1$, $a_{222}=1$	$p_1=p_2=0$	$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{11}=b_{22}=0$, $c_{1,2}, b_{12}, b_{21} \in \Bbbk$
(j)	$a_{111}=1$, $a_{122}\neq 1$, $a_{211}=1$, $a_{222}\neq 1$	$p_1=p_2=0$	$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{12}=\frac{b_{22}(a_{122}-1)}{a_{222}-1}\, c_{1,2}, b_{11}, b_{21}, b_{22} \in \Bbbk$
			$a_{222}=a_{122}^{-1}\, q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{12}=-a_{122}^{-1}b_{22}, b_{11}, b_{21}, b_{22} \in \Bbbk$ with $q_{1,2}^{(0)}=0$ and $c_{1,2} \in \Bbbk$ or $q_{1,2}^{(0)} \in \Bbbk$ and $c_{1,2}=0$
(k)	$a_{111}=1$, $a_{122}=1$, $a_{211}\neq 1$, $a_{222}\neq 1$	$p_1=p_2=0$	$b_{11}=b_{12}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$, $c_{1,2}, q_{1,2}^{(2)}, b_{21}, b_{22} \in \Bbbk$
		$p_1 \in \Bbbk$, $p_2=0$	$b_{11}=b_{12}=0$, $q_{1,2}^{(0)}=q_{1,2}^{(1)}=0$, $a_{211}=a_{222}\, c_{1,2}=a_{222}$, $q_{1,2}^{(2)}, b_{21}, b_{22} \in \Bbbk$
(l)	$a_{111}=1$, $a_{122}\neq 1$, $a_{211}\neq 1$, $a_{222}\neq 1$	$p_1=p_2=0$	$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{11}=0$, $b_{12}=\frac{b_{22}(a_{122}-1)}{a_{222}-1}$, $c_{1,2}, b_{21}, b_{22} \in \Bbbk$
(m)	$a_{111}\neq 1$, $a_{122}=1$, $a_{211}\neq 1$, $a_{222}\neq 1$		$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{12}=0$, $b_{21}=\frac{b_{11}(a_{211}-1)}{a_{111}-1}$, $c_{1,2}, b_{11}, b_{22} \in \Bbbk$
(n)	$a_{111}\neq 1$, $a_{122}\neq 1$, $a_{211}=1$, $a_{222}\neq 1$		$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{21}=0$, $b_{12}=\frac{b_{22}(a_{122}-1)}{a_{222}-1}$, $c_{1,2}, b_{11}, b_{22} \in \Bbbk$
(o)	$a_{111}\neq 1$, $a_{122}\neq 1$, $a_{211}\neq 1$, $a_{222}=1$		$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{22}=0$, $b_{21}=\frac{b_{11}(a_{211}-1)}{a_{111}-1}$, $c_{1,2}, b_{12}, b_{11} \in \Bbbk$
(p)	$a_{111}\neq 1$, $a_{122}\neq 1$, $a_{211}\neq 1$, $a_{222}\neq 1$	$p_1=p_2=0$	$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $b_{12}=\frac{b_{22}(a_{122}-1)}{a_{222}-1}$, $b_{21}=\frac{b_{11}(a_{211}-1)}{a_{111}-1}$, $c_{1,2}, b_{11}, b_{22} \in \Bbbk$
			$q_{1,2}^{(0)}=q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $a_{211}=a_{111}$, $b_{12}=\frac{b_{22}(a_{122}-1)}{a_{222}-1}$, $b_{21}=-a_{111}^{-1}b_{11}$, $c_{1,2}, b_{11}, b_{22} \in \Bbbk$
			$q_{1,2}^{(1)}=q_{1,2}^{(2)}=0$, $a_{211}=a_{111}^{-1}$, $a_{122}=a_{222}^{-1}$, $b_{21}=-a_{111}^{-1}b_{11}$, $b_{12}=-a_{222}^{-1}b_{22}$, $b_{11}, b_{22} \in \Bbbk$ with $c_{1,2}=1$ and $q_{1,2}^{(0)} \in \Bbbk$ or $q_{1,2}^{(0)}=0$ and $c_{1,2} \in \Bbbk$

We obtain the equations given by

$$\begin{aligned} p_1(a_{111}-1)&=0,\\ p_2(a_{211}-1)&=0,\\ q_{1,2}^{(0)}(a_{211}a_{111}-1)&=0, \qquad (5.4.17)\\ q_{1,2}^{(1)}(a_{211}-1)&=0, \quad \text{and} \qquad (5.4.18)\\ q_{1,2}^{(2)}(a_{111}-1)&=0. \end{aligned}$$

Again, the map ν_{t_2} can be extended to an algebra homomorphism if and only if the definitions of $\nu_{t_2}(t_1)$, $\nu_{t_2}(t_2)$, $\nu_{t_2}(x_1)$ and $\nu_{t_2}(x_2)$ respect relations (5.4.5), (5.4.6), (5.4.7), (5.4.8) and (5.4.9), that is,

$$\begin{aligned} \nu_{t_2}(x_1)\nu_{t_2}(t_1)-\nu_{t_2}(a_{111}t_1+b_{11})\nu_{t_2}(x_1)&=\nu_{t_2}(p_1),\\ \nu_{t_2}(x_2)\nu_{t_2}(t_1)-\nu_{t_2}(a_{211}t_1+b_{21})\nu_{t_2}(x_2)&=\nu_{t_2}(p_2),\\ \nu_{t_2}(x_1)\nu_{t_2}(t_2)-\nu_{t_2}(a_{122}t_2+b_{12})\nu_{t_2}(x_1)&=\nu_{t_2}(p_1),\\ \nu_{t_2}(x_2)\nu_{t_2}(t_2)-\nu_{t_2}(a_{222}t_2+b_{22})\nu_{t_2}(x_2)&=\nu_{t_2}(p_2), \quad \text{and}\\ \nu_{t_2}(x_2)\nu_{t_2}(x_1)-c_{1,2}\nu_{t_2}(x_1)\nu_{t_2}(x_2)&=q_{1,2}^{(0)}+q_{1,2}^{(1)}\nu_{t_2}(x_1)+q_{1,2}^{(2)}\nu_{t_2}(x_2). \end{aligned}$$

Then

$$\begin{aligned} p_1(a_{122}-1) &= 0,\\ p_2(a_{222}-1) &= 0,\\ q_{1,2}^{(0)}(a_{222}a_{122}-1) &= 0,\\ q_{1,2}^{(1)}(a_{222}-1) &= 0, \quad \text{and}\\ q_{1,2}^{(2)}(a_{122}-1) &= 0. \end{aligned} \tag{5.4.19}$$

The map ν_{x_1} can be extended to an algebra homomorphism if and only if the definitions of $\nu_{x_1}(t_1)$, $\nu_{x_1}(t_2)$, $\nu_{x_1}(x_1)$ and $\nu_{x_1}(x_2)$ respect relations (5.4.5), (5.4.6), (5.4.7), (5.4.8) and (5.4.9). This yields that

$$\begin{aligned} &\nu_{x_1}(x_1)\nu_{x_1}(t_1) - \nu_{x_1}(a_{111}t_1 + b_{11})\nu_{x_1}(x_1) = \nu_{x_1}(p_1),\\ &\nu_{x_1}(x_2)\nu_{x_1}(t_1) - \nu_{x_1}(a_{211}t_1 + b_{21})\nu_{x_1}(x_2) = \nu_{x_1}(p_2),\\ &\nu_{x_1}(x_1)\nu_{x_1}(t_2) - \nu_{x_1}(a_{122}t_2 + b_{12})\nu_{x_1}(x_1) = \nu_{x_1}(p_1),\\ &\nu_{x_1}(x_2)\nu_{x_1}(t_2) - \nu_{x_1}(a_{222}t_2 + b_{22})\nu_{x_1}(x_2) = \nu_{x_1}(p_2), \quad \text{and}\\ &\qquad \nu_{x_1}(x_2)\nu_{x_1}(x_1) - c_{1,2}\nu_{x_1}(x_1)\nu_{x_1}(x_2) = q_{1,2}^{(0)} + q_{1,2}^{(1)}\nu_{x_1}(x_1) + q_{1,2}^{(2)}\nu_{x_1}(x_2). \end{aligned}$$

Thus, we get that

$$\begin{aligned} p_1(a_{111}^{-1}-1) &= 0,\\ p_1(a_{122}^{-1}-1) &= 0,\\ a_{111}^{-1}(b_{21} - b_{11} + a_{211}b_{11}) - b_{21} &= 0,\\ p_2(a_{111}^{-1}c_{1,2}-1) &= a_{111}^{-1}b_{21}q_{1,2}^{(1)},\\ a_{122}^{-1}(b_{22} - b_{12} + a_{222}b_{12}) - b_{22} &= 0,\\ p_2(a_{122}^{-1}c_{1,2}-1) &= a_{122}^{-1}b_{22}q_{1,2}^{(1)}, \quad \text{and}\\ q_{1,2}^{(0)}(c_{1,2}-1) &= q_{1,2}^{(1)}q_{1,2}^{(2)}. \end{aligned} \tag{5.4.20}$$

As above, the map ν_{x_2} can be extended to an algebra homomorphism if and only if the definitions of $\nu_{x_2}(t_1)$, $\nu_{x_2}(t_2)$, $\nu_{x_2}(x_1)$ and $\nu_{x_2}(x_2)$ respect relations (5.4.5), (5.4.6), (5.4.7), (5.4.8) and (5.4.9). Then:

$$\begin{aligned} &\nu_{x_2}(x_1)\nu_{x_2}(t_1) - \nu_{x_2}(a_{111}t_1 + b_{11})\nu_{x_2}(x_1) = \nu_{x_2}(p_1),\\ &\nu_{x_2}(x_2)\nu_{x_2}(t_1) - \nu_{x_2}(a_{211}t_1 + b_{21})\nu_{x_2}(x_2) = \nu_{x_2}(p_2),\\ &\nu_{x_2}(x_1)\nu_{x_2}(t_2) - \nu_{x_2}(a_{122}t_2 + b_{12})\nu_{x_2}(x_1) = \nu_{x_2}(p_1),\\ &\nu_{x_2}(x_2)\nu_{x_2}(t_2) - \nu_{x_2}(a_{222}t_2 + b_{22})\nu_{x_2}(x_2) = \nu_{x_2}(p_2), \quad \text{and}\\ &\qquad \nu_{x_2}(x_2)\nu_{x_2}(x_1) - c_{1,2}\nu_{x_2}(x_1)\nu_{x_2}(x_2) = q_{1,2}^{(0)} + q_{1,2}^{(1)}\nu_{x_2}(x_1) + q_{1,2}^{(2)}\nu_{x_2}(x_2). \end{aligned}$$

Equivalently,

$$\begin{aligned}
p_2(a_{211}^{-1} - 1) &= 0,\\
p_2(a_{222}^{-1} - 1) &= 0,\\
a_{211}^{-1}(b_{11} - b_{21} + a_{111}b_{21}) - b_{11} &= 0,\\
p_1(a_{211}^{-1}c_{1,2}^{-1} - 1) &= -c_{1,2}^{-1}a_{211}^{-1}b_{11}q_{1,2}^{(2)},\\
a_{222}^{-1}(b_{12} - b_{22} + a_{122}b_{22}) - b_{12} &= 0,\\
p_1(a_{122}^{-1}c_{1,2}^{-1} - 1) &= -c_{1,2}^{-1}a_{222}^{-1}b_{12}q_{1,2}^{(2)}, \quad \text{and}\\
q_{1,2}^{(0)}(c_{1,2}^{-1} - 1) &= -c_{1,2}^{-1}q_{1,2}^{(1)}q_{1,2}^{(2)}.
\end{aligned} \tag{5.4.21}$$

These equations are satisfied by the conditions formulated in the Table 5.4.

For the second assertion, it is enough to prove it for the generators t_1, t_2, x_1 and x_2:

$$\begin{aligned}
(\nu_{t_1} \circ \nu_{t_2})(t_1) &= \nu_{t_1}(t_1) = t_1,\\
(\nu_{t_2} \circ \nu_{t_1})(t_1) &= \nu_{t_2}(t_1) = t_1,\\
(\nu_{t_1} \circ \nu_{t_2})(t_2) &= \nu_{t_1}(t_2) = t_2,\\
(\nu_{t_2} \circ \nu_{t_1})(t_2) &= \nu_{t_2}(t_2) = t_2,\\
(\nu_{t_1} \circ \nu_{t_2})(x_1) &= \nu_{t_1}(a_{122}x_1) = a_{111}a_{122}x_1,\\
(\nu_{t_2} \circ \nu_{t_1})(x_1) &= \nu_{t_2}(a_{111}x_1) = a_{111}a_{122}x_1,\\
(\nu_{t_1} \circ \nu_{t_2})(x_2) &= \nu_{t_1}(a_{222}x_2) = a_{222}a_{211}x_2, \quad \text{and}\\
(\nu_{t_2} \circ \nu_{t_1})(x_2) &= \nu_{t_2}(a_{211}x_2) = a_{222}a_{211}x_2.
\end{aligned} \tag{5.4.22}$$

In each case, the conditions shown in (5.4.22) hold, and so $\nu_{t_1} \circ \nu_{t_2} = \nu_{t_2} \circ \nu_{t_1}$.

Next,

$$\begin{aligned}
(\nu_{t_1} \circ \nu_{x_1})(t_1) &= \nu_{t_1}(a_{111}^{-1}(t_1 - b_{11})) = a_{111}^{-1}(t_1 - b_{11}),\\
(\nu_{x_1} \circ \nu_{t_1})(t_1) &= \nu_{x_1}(t_1) = a_{111}^{-1}(t_1 - b_{11}),\\
(\nu_{t_1} \circ \nu_{x_1})(t_2) &= \nu_{t_1}(a_{122}^{-1}(t_2 - b_{12})) = a_{122}^{-1}(t_2 - b_{12}),\\
(\nu_{x_1} \circ \nu_{t_1})(t_2) &= \nu_{x_1}(t_2) = a_{122}^{-1}(t_2 - b_{12}),\\
(\nu_{t_1} \circ \nu_{x_1})(x_1) &= \nu_{t_1}(x_1) = a_{111}x_1,\\
(\nu_{x_1} \circ \nu_{t_1})(x_1) &= \nu_{x_1}(a_{111}x_1) = a_{111}x_1,\\
(\nu_{t_1} \circ \nu_{x_1})(x_2) &= \nu_{t_1}(c_{1,2}x_2 + q_{1,2}^{(1)}) = a_{211}c_{1,2}x_2 + q_{1,2}^{(1)}, \quad \text{and}\\
(\nu_{x_1} \circ \nu_{t_1})(x_2) &= \nu_{x_1}(a_{211}x_2) = a_{211}(c_{1,2}x_2 + q_{1,2}^{(1)}).
\end{aligned} \tag{5.4.23}$$

Note that the only conditions that appear to be different in (5.4.23) are $(\nu_{t_1} \circ \nu_{x_1})(x_2)$ and $(\nu_{x_1} \circ \nu_{t_1})(x_2)$; these are satisfied when $q_{1,2}^{(1)} = a_{211}q_{1,2}^{(1)}$. As we can see, every case in Table 5.4 satisfies these conditions.

Now,

$$\begin{aligned}
&(\nu_{t_1} \circ \nu_{x_2})(t_1) = \nu_{t_1}(a_{211}^{-1}(t_1 - b_{21})) = a_{211}^{-1}(t_1 - b_{21}),\\
&(\nu_{x_2} \circ \nu_{t_1})(t_1) = \nu_{x_2}(t_1) = a_{211}^{-1}(t_1 - b_{21}),\\
&(\nu_{t_1} \circ \nu_{x_2})(t_2) = \nu_{t_1}(a_{222}^{-1}(t_2 - b_{22})) = a_{222}^{-1}(t_2 - b_{22}),\\
&(\nu_{x_2} \circ \nu_{t_1})(t_2) = \nu_{x_2}(t_2) = a_{222}^{-1}(t_2 - b_{22}),\\
&(\nu_{t_1} \circ \nu_{x_2})(x_1) = \nu_{t_1}(c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)})) = c_{1,2}^{-1}(a_{111}x_1 - q_{1,2}^{(2)}),\\
&(\nu_{x_2} \circ \nu_{t_1})(x_1) = \nu_{x_2}(a_{111}x_1) = a_{111}c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)}),\\
&(\nu_{t_1} \circ \nu_{x_2})(x_2) = \nu_{t_1}(x_2) = a_{211}x_2, \quad \text{and}\\
&(\nu_{x_2} \circ \nu_{t_1})(x_2) = \nu_{x_2}(a_{211}x_2) = a_{211}x_2.
\end{aligned} \tag{5.4.24}$$

Once more again, note that the only conditions that appear to be different are $(\nu_{t_1} \circ \nu_{x_2})(x_1)$ and $(\nu_{x_2} \circ \nu_{t_1})(x_1)$; these are satisfied if $q_{1,2}^{(2)} = a_{111}q_{1,2}^{(2)}$, and all cases in Table 5.4 satisfy both conditions.

Consider

$$\begin{aligned}
&(\nu_{t_2} \circ \nu_{x_1})(t_1) = \nu_{t_2}(a_{111}^{-1}(t_1 - b_{11})) = a_{111}^{-1}(t_1 - b_{11}),\\
&(\nu_{x_1} \circ \nu_{t_2})(t_1) = \nu_{x_1}(t_1) = a_{111}^{-1}(t_1 - b_{11}),\\
&(\nu_{t_2} \circ \nu_{x_1})(t_2) = \nu_{t_2}(a_{122}^{-1}(t_2 - b_{12})) = a_{122}^{-1}(t_2 - b_{12}),\\
&(\nu_{x_1} \circ \nu_{t_2})(t_2) = \nu_{x_1}(t_2) = a_{122}^{-1}(t_2 - b_{12}),\\
&(\nu_{t_2} \circ \nu_{x_1})(x_1) = \nu_{t_2}(x_1) = a_{122}x_1,\\
&(\nu_{x_1} \circ \nu_{t_2})(x_1) = \nu_{x_1}(a_{122}x_1) = a_{122}x_1,\\
&(\nu_{t_2} \circ \nu_{x_1})(x_2) = \nu_{t_2}(c_{1,2}x_2 + q_{1,2}^{(1)}) = c_{1,2}a_{222}x_2 + q_{1,2}^{(1)}, \quad \text{and}\\
&(\nu_{x_1} \circ \nu_{t_2})(x_2) = \nu_{x_1}(a_{222}x_2) = a_{222}(c_{1,2}x_2 + q_{1,2}^{(1)}).
\end{aligned} \tag{5.4.25}$$

By using a similar reasoning, it can be seen that all cases in Table 5.4 satisfy these conditions.

We continue with the following compositions:

$$\begin{aligned}
&(\nu_{t_2} \circ \nu_{x_2})(t_1) = \nu_{t_2}(a_{211}^{-1}(t_1 - b_{21})) = a_{211}^{-1}(t_1 - b_{21}),\\
&(\nu_{x_2} \circ \nu_{t_2})(t_1) = \nu_{x_2}(t_1) = a_{211}^{-1}(t_1 - b_{21}),\\
&(\nu_{t_2} \circ \nu_{x_2})(t_2) = \nu_{t_2}(a_{222}^{-1}(t_2 - b_{22})) = a_{222}^{-1}(t_2 - b_{22}),\\
&(\nu_{x_2} \circ \nu_{t_2})(t_2) = \nu_{x_2}(t_2) = a_{222}^{-1}(t_2 - b_{22}),\\
&(\nu_{t_2} \circ \nu_{x_2})(x_1) = \nu_{t_2}(c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)}) = c_{1,2}^{-1}(a_{122}x_1 - q_{1,2}^{(2)}),\\
&(\nu_{x_2} \circ \nu_{t_2})(x_1) = \nu_{x_2}(a_{122}x_1) = a_{122}c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)}),\\
&(\nu_{t_2} \circ \nu_{x_2})(x_2) = \nu_{t_2}(x_2) = a_{222}x_2, \quad \text{and}\\
&(\nu_{x_2} \circ \nu_{t_2})(x_2) = \nu_{x_2}(a_{222}x_2) = a_{222}x_2.
\end{aligned} \tag{5.4.26}$$

All cases in Table 5.4 satisfy conditions in (5.4.26).

Finally,

$$\begin{aligned}
(\nu_{x_1} \circ \nu_{x_2})(t_1) &= \nu_{x_1}(a_{211}^{-1}(t_1 - b_{21})) = a_{211}^{-1}(a_{111}^{-1}(t_1 - b_{11}) - b_{21}),\\
(\nu_{x_2} \circ \nu_{x_1})(t_1) &= \nu_{x_2}(a_{111}^{-1}(t_1 - b_{11})) = a_{111}^{-1}(a_{211}^{-1}(t_1 - b_{21}) - b_{11}),\\
(\nu_{x_1} \circ \nu_{x_2})(t_2) &= \nu_{x_1}(a_{222}^{-1}(t_2 - b_{22})) = a_{222}^{-1}(a_{122}^{-1}(t_2 - b_{12}) - b_{22}),\\
(\nu_{x_2} \circ \nu_{x_1})(t_2) &= \nu_{x_2}(a_{122}^{-1}(t_2 - b_{12})) = a_{122}^{-1}(a_{222}^{-1}(t_2 - b_{22}) - b_{12}),\\
(\nu_{x_1} \circ \nu_{x_2})(x_1) &= \nu_{x_1}(c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)}) = c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)}),\\
(\nu_{x_2} \circ \nu_{x_1})(x_1) &= \nu_{x_2}(x_1) = c_{1,2}^{-1}(x_1 - q_{1,2}^{(2)}),\\
(\nu_{x_1} \circ \nu_{x_2})(x_2) &= \nu_{x_1}(x_2) = c_{1,2}x_2 + q_{1,2}^{(1)}, \quad \text{and}\\
(\nu_{x_2} \circ \nu_{x_1})(x_2) &= \nu_{x_2}(c_{1,2}x_2 + q_{1,2}^{(1)}) = c_{1,2}x_2 + q_{1,2}^{(1)}.
\end{aligned} \tag{5.4.27}$$

At first glance, it seems that compositions $(\nu_{x_1} \circ \nu_{x_2})(t_1)$ and $(\nu_{x_2} \circ \nu_{x_1})(t_1)$ are different. However, due to the expression (5.4.21), they coincide. Similarly, it occurs with the compositions $(\nu_{x_1} \circ \nu_{x_2})(t_2)$ and $(\nu_{x_2} \circ \nu_{x_1})(t_2)$. All are satisfied. □

We arrive to the important result of this section.

Theorem 5.17. *If a SPBW extension $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$ satisfies one of the conditions* (a)-(p) *in Proposition 5.16, then it is differentially smooth.*

Proof. As $\operatorname{GKdim}(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle) = 4$, we proceed to construct a 4-dimensional integrable calculus. Consider $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle)$, a free right $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$-module of rank 4 with generators dt_1, dt_2, dx_1, dx_2. Define a left $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$-module structure by

$$a\,dt_i = dt_i\nu_{t_i}(a), \quad a\,dx_i = dx_i\nu_{x_i}(a), \quad \text{for all } i \in \{1, 2\}, a \in \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle, \tag{5.4.28}$$

where ν_{t_i}, ν_{x_i}, $i \in \{1, 2\}$ are the algebra automorphisms established in Proposition 5.16. Notice that the relations in $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle)$ are given by

$$t_i dt_j = dt_j t_i \qquad t_i dx_j = dx_j a_{jii}^{-1}(t_i - b_{ij}), \quad \text{for all } i, j \in \{1, 2\}, \tag{5.4.29}$$

$$x_i dx_i = dx_i x_i, \qquad x_i dt_j = dt_j a_{ijj} x_i, \quad \text{for all } i, j \in \{1, 2\}, \tag{5.4.30}$$

and

$$x_1 dx_2 = dx_2(c_{1,2}^{-1}x_1 - c_{1,2}^{-1}q_{1,2}^{(2)}), \tag{5.4.31}$$

$$x_2 dx_1 = dx_1(c_{1,2}x_2 + q_{1,2}^{(1)}). \tag{5.4.32}$$

We extend the maps $t_i \mapsto dt_i$, $x_i \mapsto dx_i$ for $i \in \{1, 2\}$ to a map

$$d : \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle \to \Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle)$$

satisfying the Leibniz's rule. From relations (5.4.5), (5.4.6), (5.4.7), (5.4.8) and (5.4.9) we get that

$$\begin{aligned} dx_i t_j + x_i dt_j &= a_{ijj} dt_j x_i + a_{ijj} t_j dx_i + b_{ij} dx_i, \quad \text{for } i, j \in \{1,2\}, \quad \text{and} \\ dx_2 x_1 + x_2 dx_1 &= c_{1,2} dx_1 x_2 + c_{1,2} x_1 dx_2 + q_{1,2}^{(1)} dx_1 + q_{1,2}^{(2)} dx_2. \end{aligned}$$

Define $\Bbbk$-linear maps

$$\partial_{t_i}, \partial_{x_i} : \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle \to \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$$

such that

$$d(a) = dt_1 \partial_{t_1}(a) + dt_2 \partial_{t_2}(a) + \sum_{i=1}^{2} dx_i \partial_{x_i}(a), \text{ for all } a \in \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle.$$

These maps are well-defined since dt_i and dx_i for $i \in \{1,2\}$ are free generators of the right $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$-module $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle)$. Hence, $d(a) = 0$ if and only if $\partial_{t_i}(a) = \partial_{x_i}(a) = 0$ for $i \in \{1,2\}$. Using relations (5.4.28) and the definitions of the maps ν_{t_i}, ν_{x_i}, $i \in \{1,2\}$, we obtain that

$$\begin{aligned} \partial_{t_1}(t_1^k t_2^s x_1^{l_1} x_2^{l_2}) &= k t_1^{k-1} t_2^s x_1^{l_1} x_2^{l_2}, \\ \partial_{t_2}(t_1^k t_2^s x_1^{l_1} x_2^{l_2}) &= s t_1^k t_2^{s-1} x_1^{l_1} x_2^{l_2}, \\ \partial_{x_1}(t_1^k t_2^s x_1^{l_1} x_2^{l_2}) &= l_1 a_{111}^{-k} a_{122}^{-s} (t_1 - b_{11})^k (t_2 - b_{12})^s x_1^{l_1-1} x_2^{l_2}, \quad \text{and} \\ \partial_{x_2}(t_1^k t_2^s x_1^{l_1} x_2^{l_2}) &= l_2 c_{1,2}^{-l_1} a_{211}^{-k} a_{222}^{-s} (t_1 - b_{21})^k (t_2 - b_{22})^s (x_1 - q_{1,2}^{(2)})^{l_1} x_2^{l_2-1}. \end{aligned} \tag{5.4.33}$$

Then, $d(a) = 0$ if and only if a is a scalar multiple of the identity. This shows that $\Omega(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle, d)$ is connected, where

$$\Omega(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle) = \bigoplus_{i=0}^{4} \Omega^i(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle).$$

The universal extension of d to higher forms compatible with (5.4.29), (5.4.30) and (5.4.32) gives the following rules for $\Omega^l(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle)$ with $l = 2, 3$:

$$dt_2 \wedge dt_1 = -dt_1 \wedge dt_2, \tag{5.4.34}$$

$$\begin{aligned} dx_i \wedge dt_j &= -a_{ijj} dt_j \wedge dx_i, \quad \text{for } i, j \in \{1,2\}, \\ dx_2 \wedge dx_1 &= -c_{i,j} dx_1 \wedge dx_2, \end{aligned}$$

$$\begin{aligned} dx_1 \wedge dt_2 \wedge dt_1 &= -a_{111} a_{122} dt_1 \wedge dt_2 \wedge dx_1, \\ dx_2 \wedge dt_2 \wedge dt_1 &= -a_{211} a_{222} dt_1 \wedge dt_2 \wedge dx_2, \\ dx_2 \wedge dx_1 \wedge dt_1 &= -c_{1,2} a_{111} a_{211} dt_1 \wedge dx_1 \wedge dx_2, \quad \text{and} \\ dx_2 \wedge dx_1 \wedge dt_2 &= -c_{1,2} a_{122} a_{222} dt_2 \wedge dx_1 \wedge dx_2. \end{aligned} \tag{5.4.35}$$

Since the automorphisms ν_{t_i} and ν_{x_i} for $i \in \{1,2\}$ commute with each other, there are no additional relationships to the previous ones, so we write

$$\begin{aligned} \Omega^3(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle) = [&dt_1 \wedge dx_1 \wedge dx_2 \oplus dt_2 \wedge dx_1 \wedge dx_2 \oplus dt_1 \wedge dt_2 \wedge dx_2 \\ &\oplus dt_1 \wedge dt_2 \wedge dx_1] \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle. \end{aligned}$$

Now, due to that

$$\Omega^4(\sigma(\Bbbk[t_1,t_2])\langle x_1,x_2\rangle) = \omega\sigma(\Bbbk[t_1,t_2])\langle x_1,x_2\rangle \cong \sigma(\Bbbk[t_1,t_2])\langle x_1,x_2\rangle,$$

as a right and left $\sigma(\Bbbk[t_1,t_2])\langle x_1,x_2\rangle$-module, with $\omega = dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2$, where $\nu_\omega = \nu_{t_1} \circ \nu_{t_2} \circ \nu_{x_1} \circ \nu_{x_2}$, then ω is a volume form of $\sigma(\Bbbk[t_1,t_2])\langle x_1,x_2\rangle$. From Proposition 1.7 (2), it follows that ω is an integral form by setting

$$\begin{aligned}
&\omega_1^1 = dt_1, \quad \omega_2^1 = dt_2, \quad \omega_3^1 = dx_1, \quad \omega_4^1 = dx_2,\\
&\omega_1^2 = dt_1 \wedge dt_2, \quad \omega_2^2 = dt_1 \wedge dx_1, \quad \omega_3^2 = dt_1 \wedge dx_2, \quad \omega_4^2 = dt_2 \wedge dx_1,\\
&\omega_5^2 = dt_2 \wedge dx_2, \quad \omega_6^2 = dx_1 \wedge dx_2,\\
&\omega_1^3 = dt_1 \wedge dt_2 \wedge dx_1, \quad \omega_2^3 = dt_1 \wedge dt_2 \wedge dx_2, \quad \omega_3^3 = dt_1 \wedge dx_1 \wedge dx_2,\\
&\omega_4^3 = dt_2 \wedge dx_1 \wedge dx_2,\\
&\bar{\omega}_1^1 = -a_{211}^{-1}a_{222}^{-1}c_{1,2}^{-1}dx_2, \quad \bar{\omega}_2^1 = a_{122}^{-1}a_{111}^{-1}dx_1, \quad \bar{\omega}_3^1 = -dt_2, \quad \bar{\omega}_4^1 = dt_1,\\
&\bar{\omega}_1^2 = a_{211}^{-1}a_{111}^{-1}a_{122}^{-1}a_{222}^{-1}dx_1 \wedge dx_2,\\
&\bar{\omega}_2^2 = -a_{211}^{-1}c_{1,2}^{-1}dt_2 \wedge dx_2, \quad \bar{\omega}_3^2 = a_{111}^{-1}dt_2 \wedge dx_1, \quad \bar{\omega}_4^2 = a_{222}^{-1}c_{1,2}^{-1}dt_1 \wedge dx_2,\\
&\bar{\omega}_5^2 = -a_{122}^{-1}dt_1 \wedge dx_1, \quad \bar{\omega}_6^2 = dt_1 \wedge dt_2,\\
&\bar{\omega}_1^3 = -a_{111}^{-1}a_{211}^{-1}dt_2 \wedge dx_1 \wedge dx_2, \quad \bar{\omega}_2^3 = a_{122}^{-1}a_{222}^{-1}dt_1 \wedge dx_1 \wedge dx_2,\\
&\bar{\omega}_3^3 = -c_{1,2}^{-1}dt_1 \wedge dt_2 \wedge dx_2, \quad \bar{\omega}_4^3 = dt_1 \wedge dt_2 \wedge dx_1
\end{aligned}$$

Let $\omega' = dt_1a + dt_2b + dx_1c + dx_2d$, $a, b, c, d \in \Bbbk$. Then

$$\begin{aligned}
\sum_{i=1}^{4} \omega_i^1 \pi_\omega(\bar{\omega}_i^3 \wedge \omega') &= dt_1\pi_\omega(-aa_{111}^{-1}a_{211}^{-1}dt_2 \wedge dx_1 \wedge dx_2 \wedge dt_1)\\
&\quad + dt_2\pi_\omega(ba_{122}^{-1}a_{222}^{-1}dt_1 \wedge dx_1 \wedge dx_2 \wedge dt_2)\\
&\quad + dx_1\pi_\omega(-cc_{1,2}^{-1}dt_1 \wedge dt_2 \wedge dx_2 \wedge dx_1)\\
&\quad + dx_2\pi_\omega(ddt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2)\\
&= dt_1a + dt_2b + dx_1c + dx_2d\\
&= \omega'.
\end{aligned}$$

On the other hand, if

$$\omega'' = dt_1 \wedge dt_2a + dt_1 \wedge dx_1b + dt_1 \wedge dx_2c + dt_2 \wedge dx_1d + dt_2 \wedge dx_2e + dx_1 \wedge dx_2f$$

with $a, b, c, d, e, f \in \Bbbk$, it yields that

$$\begin{aligned}\sum_{i=1}^{6} \omega_i^2 \pi_\omega(\bar{\omega}_i^2 \wedge \omega'') &= dt_1 \wedge dt_2 \pi_\omega(a a_{211}^{-1} a_{111}^{-1} a_{122}^{-1} a_{222}^{-1} dx_1 \wedge dx_2 \wedge dt_1 \wedge dt_2) \\ &\quad + dt_1 \wedge dx_1 \pi_\omega(-b a_{211}^{-1} c_{1,2}^{-1} dt_2 \wedge dx_2 \wedge dt_1 \wedge dx_1) \\ &\quad + dt_1 \wedge dx_2 \pi_\omega(c a_{111}^{-1} dt_2 \wedge dx_1 \wedge dt_1 \wedge dx_2) \\ &\quad + dt_2 \wedge dx_1 \pi_\omega(d a_{222}^{-1} c_{1,2}^{-1} dt_1 \wedge dx_2 \wedge dt_2 \wedge dx_1) \\ &\quad + dt_2 \wedge dx_2 \pi_\omega(-e a_{122}^{-1} dt_1 \wedge dx_1 \wedge dt_2 \wedge dx_2) \\ &\quad + dx_1 \wedge dx_2 \pi_\omega(f dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2) \\ &= dt_1 \wedge dt_2 a + dt_1 \wedge dx_1 b + dt_1 \wedge dx_2 c \\ &\quad + dt_2 \wedge dx_1 d + dt_2 \wedge dx_2 e + dx_1 \wedge dx_2 f \\ &= \omega''.\end{aligned}$$

Let

$$\begin{aligned}\omega''' = dt_1 \wedge dt_2 \wedge dx_1 a + dt_1 \wedge dt_2 \wedge dx_2 b dt_1 \wedge dx_1 \\ \wedge dx_2 c + dt_2 \wedge dx_1 \wedge dx_2 d, \quad \text{with } a, b, c, d \in \Bbbk.\end{aligned}$$

Since

$$\begin{aligned}\sum_{i=1}^{3} \omega_i^3 \pi_\omega(\bar{\omega}_i^1 \wedge \omega''') &= dt_1 \wedge dt_2 \wedge dx_1 \pi_\omega(-a a_{211}^{-1} a_{222}^{-1} c_{1,2}^{-1} dx_2 \wedge dt_1 \wedge dt_2 \wedge dx_1) \\ &\quad + dt_1 \wedge dt_2 \wedge dx_2 \pi_\omega(b a_{122}^{-1} a_{111}^{-1} dx_1 \wedge dt_1 \wedge dt_2 \wedge dx_2) \\ &\quad + dt_1 \wedge dx_1 \wedge dx_2 \pi_\omega(-c dt_2 \wedge dt_1 \wedge dx_1 \wedge dx_2) \\ &\quad + dt_2 \wedge dx_1 \wedge dx_2 \pi_\omega(d dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2) \\ &= dt_1 \wedge dt_2 \wedge dx_1 a + dt_1 \wedge dt_2 \wedge dx_2 b dt_1 \wedge dx_1 \wedge dx_2 c \\ &\quad + dt_2 \wedge dx_1 \wedge dx_2 d = \omega''',\end{aligned}$$

we conclude that $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2\rangle$ is differentially smooth. □

Remark 19. Only automorphisms of the form $t_1 \longmapsto a_{11}t_1 + a_{12}t_2 + a_{13}$, $t_2 \longmapsto a_{21}t_1 + a_{22}t_2 + a_{23}$, where $a_{i,j} \in \Bbbk$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$ are taken. This holds for the following reason. Without loss of generality, suppose that we have a relation with an automorphism of the form $t_1 \longmapsto t_1$, $t_2 \longmapsto t_2 + h(t_1)$, where $h(t_1) \in \Bbbk[t_1]$ as follows

$$\begin{aligned}x_1 t_1 &= t_1 x_1 + p(t_1, t_2), \\ x_1 t_2 &= t_2 x_1 + h(t_1) x_1 + p_1(t_1, t_2).\end{aligned}$$

When we want to take the automorphisms that can work to build the differential calculus, we obtain the following:

$$\begin{aligned}d(x_1 t_2) &= d(t_2 x_1) + d(h(t_1) x_1) + d(p_1(t_1, t_2)), \\ dx_1 t_2 + x_1 dt_2 &= dt_2 x_1 + t_2 dx_1 + d(h(t_1)) x_1 + h(t_1) dx_1 + d(p_1(t_1, t_2)).\end{aligned}$$

Consider $h(t_1) = \sum_s a_s t_1^s$, $dt_1 h'(t_1)$ is the usual derivative with respect to t_1 and $dt_1 \frac{\partial p_1}{\partial t_1}$ is the usual partial derivative with respect to t_1. Then

$$dx_1 t_2 + dt_2 \nu_{t_2}(x_1) = dt_2 x_1 + dx_1 \nu_{x_1}(t_2) + dt_1 h'(t_1) x_1 + dx_1 \sum_s a_s \left[\nu_{x_1}(t_1)\right]^s + dt_1 \frac{\partial p_1}{\partial t_1},$$

and

$$dx_1 \left(t_2 - \nu_{x_1}(t_2) - \sum_s a_s [\nu_{x_1}(t_1)]^s \right) + dt_2(\nu_{t_2}(x_1) - x_1) + dt_1 \left(h'(t_1) x_1 + \frac{\partial p_1}{\partial t_1} \right) = 0.$$

Since that dt_1, dt_2, dx_1 and dx_2 are generators for $\Omega^1(\Bbbk[t_1, t_2])$, the elements that multiply them must be zero. In particular,

$$h'(t_1) x_1 + \frac{\partial p_1}{\partial t_1} = 0. \tag{5.4.36}$$

Now,

$$\begin{aligned} h'(t_1) x_1 &= \sum_s s a_s t_1^{s-1} x_1 = \sum_s s a_s \left(x_1 t_1^{s-1} - (s-1) t_1^{s-2} p_1(t_1, t_2) \right) \\ &= x_1 h'(t_1) - \sum_s s a_s (s-1) t_1^{s-2} p_1(t_1, t_2). \end{aligned}$$

Therefore, expression (5.4.36) can be written as

$$x_1 h'(t_1) - \sum_s s a_s (s-1) t_1^{s-2} p_1(t_1, t_2) + \frac{\partial p_1}{\partial t_1} = 0.$$

In this way, it is necessary that $h'(t_1) = 0$, i.e. $h(t_1) = h \in \Bbbk$, whence the automorphism has the form $t_1 \longmapsto t_1$, $t_2 \longmapsto t_2 + h$, where $h \in \Bbbk$. Note that this automorphism coincides with the first type above when $a_{11} = 1$, $a_{12} = 0$, $a_{13} = 0$, $a_{21} = 0$, $a_{22} = 1$ and $a_{23} = h$.

Sections 5.4.2 and 5.4.3 contain sufficient conditions to guarantee the differential smoothness of SPBW extensions over $\sigma(\Bbbk[t_1, t_2])$ on three and n generators. However, as we saw in the previous sections, the number of cases in which the Leibniz's rule holds increases considerably as we consider more indeterminates on the SPBW extension. Due to this reason, in the next two sections we will not include any table but will simply formulate the adequate conditions to the extension of this rule.

5.4.2 SPBW extensions in three indeterminates

Let $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$. Consider the family of automorphisms as in the previous section. Since the conditions in (5.4.3) also hold in this case, it yields that

$$x_i t_j = a_{ijj} t_j x_i + b_{ij} x_i + p_i, \quad \text{for } i = 1,2,3 \text{ and } j = 1,2, \tag{5.4.37}$$

$$x_j x_i = c_{i,j} x_i x_j + q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)} x_k, \quad \text{for } i,j,k \in \{1,2,3\}, \text{ and } i < j, \tag{5.4.38}$$

where $c_{i,j} \in \Bbbk^*$, $q_{i,j}^{(0)}, q_{i,j}^{(k)} \in \Bbbk$, for $i,j,k \in \{1,2,3\}$ with $i < j$, and $a_{ijj} \in \Bbbk^*$, $b_{ij}, p_i \in \Bbbk$ for $i \in \{1,2,3\}$ and $j \in \{1,2\}$.

Proposition 5.18. *Let*

$$\nu_{t_i}(t_j) = t_j, \qquad \nu_{t_i}(x_k) = a_{kii} x_k, \tag{5.4.39}$$

$$\nu_{x_k}(t_i) = a_{kii}^{-1}(t_i - b_{ki}), \qquad \nu_{x_k}(x_k) = x_k, \tag{5.4.40}$$

$$\nu_{x_i}(x_j) = c_{i,j} x_j + q_{i,j}^{(i)}, \quad \text{for } i < j, \qquad \nu_{x_i}(x_j) = c_{j,i}^{-1} x_j - c_{j,i}^{-1} q_{j,i}^{(i)}, \quad \text{for } i > j, \tag{5.4.41}$$

Then:

(1) *Leibniz's rule holds if the following relationships hold:*

$$b_{sl}(a_{ill} - 1) = b_{il}(a_{sll} - 1), \tag{5.4.42}$$

$$q_{s,i}^{(s)}(a_{ill} - 1) = 0, \tag{5.4.43}$$

$$p_i(c_{s,i} - a_{sll}) = b_{il} q_{s,i}^{(s)}, \tag{5.4.44}$$

$$q_{i,j}^{(s)}(c_{s,j} c_{s,i} - 1) = 0, \tag{5.4.45}$$

$$q_{s,j}^{(s)}(c_{i,j} - 1) = q_{i,j}^{(i)}(c_{s,j} - 1), \tag{5.4.46}$$

$$q_{s,i}^{(s)}(c_{i,j} - 1) = q_{i,j}^{(j)}(c_{s,i} - 1), \tag{5.4.47}$$

$$q_{i,j}^{(k)}(c_{s,j} c_{s,i} - c_{k,s}^{-1}) = 0, \quad 1 \leq k \leq 3, \quad \text{and} \tag{5.4.48}$$

$$\sum_{k=1}^{s-1} c_{k,s}^{-1} q_{i,j}^{(k)} q_{k,s}^{(s)} - \sum_{k=s+1}^{n} q_{i,j}^{(k)} q_{k,s}^{(s)} + q_{s,i}^{(s)} q_{s,j}^{(s)}(1 - c_{i,j}) + (c_{s,j} c_{s,i} - 1) q_{i,j}^{(0)} = 0. \tag{5.4.49}$$

for $1 \leq i,j,s \leq n$, $i < j$, $l \in \{1,2\}$, *where* $c_{r,t} := c_{t,r}^{-1}$, $q_{r,t}^{(p)} := -c_{t,r}^{-1} q_{t,r}^{(p)}$, *for all* $1 \leq r,t,p \leq n$, $t < r$, *and* $c_{r,r} = 1$ *and* $q_{r,r}^{(p)} = 0$ *for all* $1 \leq r,p \leq n$.

The maps defined by (5.4.39), (5.4.40) and (5.4.41) simultaneously extend to algebra automorphisms $\nu_{t_1}, \nu_{t_2}, \nu_{x_1}, \nu_{x_2}, \nu_{x_3}$ *of* $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$ *when the previous relations are satisfied.*

(2) *If the mentioned conditions in (1) hold, then we have that*

$$\nu_{t_i} \circ \nu_{t_j} = \nu_{t_j} \circ \nu_{t_i}, \quad \nu_{t_j} \circ \nu_{x_k} = \nu_{x_k} \circ \nu_{t_j}, \quad \nu_{x_k} \circ \nu_{x_l} = \nu_{x_l} \circ \nu_{x_k}, \tag{5.4.50}$$

for $i,j = 1,2$ *and* $1 \leq k,l \leq 3$.

Proof. For the first assertion, the map ν_{t_l}, $l = 1, 2$ can be extended to an algebra homomorphism if and only if the definitions of $\nu_{t_l}(t_j)$, and $\nu_{t_l}(x_i)$ respect relations (5.4.38):

$$\nu_{t_l}(x_i)\nu_{t_l}(t_j) - \nu_{t_l}(a_{ijj}t_j + b_{ij})\nu_{t_l}(x_i) = \nu_{t_l}(p_i), \quad \text{for } i \in \{1,2,3\},\ j, l \in \{1,2\},$$

and

$$\nu_{t_l}(x_j)\nu_{t_l}(x_i) - c_{i,j}\nu_{t_l}(x_i)\nu_{t_l}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)}\nu_{t_l}(x_k),$$

for $i, j \in \{1,2,3\},\ l \in \{1,2\}$ with $i < j$. Then

$$\begin{aligned} p_i(a_{ill} - 1) &= 0, \quad \text{for } i \in \{1,2,3\},\ l \in \{1,2\}, \\ q_{i,j}^{(0)}(a_{jll}a_{ill} - 1) &= 0, \quad \text{and} \\ q_{i,j}^{(l)}(a_{jll}a_{ill} - a_{kll}) &= 0, \quad \text{for } i, j, k \in \{1,2,3\},\ l \in \{1,2\}. \end{aligned} \tag{5.4.51}$$

It is necessary that ν_{x_s} for $s = 1, 2, 3$ can be extended to $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$, that is,

$$\nu_{x_s}(x_i)\nu_{x_s}(t_j) - \nu_{x_s}(a_{ijj}t_j + b_{ij})\nu_{x_s}(x_i) = \nu_{x_s}(p_i), \quad \text{for } i \in \{1,2,3\},\ j, l \in \{1,2\}, \tag{5.4.52}$$

and

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)}\nu_{x_s}(x_k). \tag{5.4.53}$$

From Equation (5.4.52) we get the following three cases to be considered:

- $s < i$:

$$\begin{aligned} \nu_{x_s}(x_i)\nu_{x_s}(t_j) &- \nu_{x_s}(a_{ijj}t_j + b_{ij})\nu_{x_s}(x_i) = \nu_{x_s}(p_i) \\ c_{s,i}(x_i + q_{s,i}^{(s)})a_{sjj}^{-1}(t_j - b_{sj}) &- a_{ijj}a_{sjj}^{-1}(t_j - b_{sj})c_{s,i}(x_i + q_{s,i}^{(s)}) \\ &- b_{ij}c_{s,i}(x_i + q_{s,i}^{(s)} = p_i. \end{aligned}$$

  In this way,

$$\begin{aligned} b_{sj}(a_{ijj} - 1) &= b_{ij}(a_{sjj} - 1), \\ q_{s,i}^{(s)}(a_{ijj} - 1) &= 0, \quad \text{and} \\ p_i(c_{s,i} - a_{sjj}) &= b_{ij}q_{s,i}^{(s)}. \end{aligned}$$

- $s = i$: It is straightforward that no new conditions are obtained.
- $s > i$:

$$\begin{aligned} \nu_{x_s}(x_i)\nu_{x_s}(t_j) &- \nu_{x_s}(a_{ijj}t_j + b_{ij})\nu_{x_s}(x_i) = \nu_{x_s}(p_i) \\ c_{i,s}^{-1}(x_i - q_{i,s}^{(s)})a_{sjj}^{-1}(t_j - b_{sj}) &- a_{ijj}a_{sjj}^{-1}(t_j - b_{sj})c_{i,s}^{-1}(x_i - q_{i,s}^{(s)}) \\ &- b_{ij}c_{i,s}^{-1}(x_i - q_{i,s}^{(s)}) = p_i. \end{aligned}$$

  Then,

$$q_{i,s}^{(s)}(a_{ijj} - 1) = 0 \quad \text{and} \quad p_i(c_{i,s}a_{sjj} - 1) = a_{sjj}b_{ij}q_{i,s}^{(s)}.$$

Due to expression (5.4.53) we have to consider the following cases:

- $s < i$:

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)}\nu_{x_s}(x_k)$$

From this equation, we obtain the following new conditions

$$\begin{aligned}
q_{i,j}^{(s)}(c_{s,j}c_{s,i} - 1) &= 0,\\
q_{s,j}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{i}(c_{s,j} - 1),\\
q_{s,i}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{j}(c_{s,i} - 1),\\
q_{i,j}^{(k)}(c_{s,j}c_{s,i} - c_{k,s}^{-1}) &= 0, \quad 1 \leq k \leq s-1,\\
q_{i,j}^{(k)}(c_{s,j}c_{s,i} - c_{s,k}) &= 0, \quad s+1 \leq k \leq 3, k \neq i, j, \quad \text{and}\\
\sum_{k=1}^{s-1} c_{k,s}^{-1}q_{i,j}^{(k)}q_{k,s}^{(s)} - \sum_{k=s+1}^{3} q_{i,j}^{(k)}q_{k,s}^{(s)} &+ q_{s,i}^{(s)}q_{s,j}^{(s)}(1 - c_{i,j}) + (c_{s,j}c_{s,i} - 1)q_{i,j}^{(0)} = 0.
\end{aligned}$$

- $i < s < j$:

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)}\nu_{x_s}(x_k).$$

Hence,

$$\begin{aligned}
q_{i,j}^{(s)}(c_{s,j} - c_{i,s}) &= 0,\\
q_{s,j}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(i)}(c_{s,j} - 1),\\
q_{i,s}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(j)}(c_{i,s} - 1),\\
q_{i,j}^{(k)}(c_{s,j}c_{i,s}^{-1} - c_{k,s}^{-1}) &= 0, \quad 1 \leq k \leq s-1, k \neq i,\\
q_{i,j}^{(k)}(c_{s,j}c_{i,s}^{-1} - c_{s,k}) &= 0, \quad s+1 \leq k \leq n, k \neq j, \quad \text{and}\\
\sum_{k=1}^{s-1} c_{k,s}^{-1}q_{i,j}^{(k)}q_{k,s}^{(s)} - \sum_{k=s+1}^{3} q_{i,j}^{(k)}q_{s,k}^{(s)} &+ c_{i,s}^{-1}q_{s,j}^{(s)}q_{i,s}^{(s)}(c_{i,j} - 1) + (c_{s,j}c_{i,s}^{-1} - 1)q_{i,j}^{(0)} = 0.
\end{aligned}$$

- $s \geq j$:

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)}\nu_{x_s}(x_k).$$

Then,

$$\begin{aligned} q_{i,j}^{(s)}(c_{j,s}c_{i,s}-1) &= 0,\\ q_{j,s}^{(s)}(c_{i,j}-1) &= q_{i,j}^{(i)}(c_{j,s}-1),\\ q_{i,j}^{(k)}(c_{j,s}^{-1}c_{i,s}^{-1}-c_{k,s}^{-1}) &= 0, \quad 1\leq k\leq s-1, k\neq i,j\\ q_{i,j}^{(k)}(c_{j,s}^{-1}c_{i,s}^{-1}-c_{s,k}) &= 0, \quad s+1\leq k\leq n, \quad \text{and}\\ \sum_{k=1}^{s-1} c_{k,s}^{-1}q_{i,j}^{(k)}q_{k,s}^{(s)} - \sum_{k=s+1}^{3} q_{i,j}^{(k)}q_{s,k}^{(s)} &+ c_{j,s}^{-1}c_{i,s}^{-1}q_{j,s}^{(s)}q_{i,s}^{(s)}(1-c_{i,j}) + (c_{j,s}^{-1}c_{i,s}^{-1}-1)q_{i,j}^{(0)} = 0. \end{aligned}$$

If we put together all the conditions for $s\in\{1,2,3\}$, then we obtain the restrictions for the extension of the automorphisms.

For the second assertion, it is enough to prove it for the generators t_j, x_i, $1\leq i\leq 3$ and $j\in\{1,2\}$. Since

$$\nu_{t_k}\circ\nu_{t_m}(t_j) = \nu_{t_k}(t_j) = t_j, \tag{5.4.54}$$
$$\nu_{t_m}\circ\nu_{t_k}(t_j) = \nu_{t_m}(t_j) = t_j, \tag{5.4.55}$$
$$\nu_{t_k}\circ\nu_{t_m}(x_i) = \nu_{t_k}(a_{imm}x_i) = a_{imm}a_{ikk}x_i, \quad \text{and} \tag{5.4.56}$$
$$\nu_{t_m}\circ\nu_{t_k}(x_i) = \nu_{t_m}(a_{ikk}x_i) = a_{imm}a_{ikk}x_i. \tag{5.4.57}$$

for all $1\leq i\leq 3$ and $k,m,j\in\{1,2\}$, then all relations are satisfied, and hence $\nu_{t_k}\circ\nu_{t_m} = \nu_{t_m}\circ\nu_{t_k}$ for $k,m\in\{1,2\}$.

Now,

$$\nu_{t_k}\circ\nu_{x_m}(t_j) = \nu_{t_k}(a_{mjj}^{-1}(t_j-b_{mj})) = a_{mjj}^{-1}(t_j-b_{mj}), \tag{5.4.58}$$
$$\nu_{x_m}\circ\nu_{t_k}(t_j) = \nu_{x_m}(t_j) = a_{mjj}^{-1}(t_j-b_{mj}), \tag{5.4.59}$$
$$\nu_{t_k}\circ\nu_{x_m}(x_i) = \nu_{t_k}(c_{m,i}x_i+q_{m,i}^{(m)}) = c_{m,i}a_{ikk}x_i+q_{m,i}^{(m)}, \quad \text{and} \tag{5.4.60}$$
$$\nu_{x_m}\circ\nu_{t_k}(x_i) = \nu_{x_m}(a_{ikk}x_i) = a_{ikk}(c_{m,i}x_i+q_{m,i}^{(m)}), \tag{5.4.61}$$

for all $1\leq i,m\leq 3$ and $k,j\in\{1,2\}$. As it is clear, expressions (5.4.58) and (5.4.59) hold, while relations (5.4.60) and (5.4.61) are also satisfied due to expression (5.4.45). In this way, $\nu_{t_k}\circ\nu_{x_m} = \nu_{x_m}\circ\nu_{t_k}$, for $1\leq m\leq 3$ and $k\in\{1,2\}$.

Finally,

$$\nu_{x_k}\circ\nu_{x_m}(t_j) = \nu_{x_k}(a_{mjj}^{-1}(t_j-b_{mj})) = a_{mjj}^{-1}a_{kjj}^{-1}(t_j-b_{kj}) - a_{mjj}^{-1}b_{jm}, \tag{5.4.62}$$
$$\nu_{x_m}\circ\nu_{x_k}(t_j) = \nu_{x_m}(a_{kjj}^{-1}(t_j-b_{kj})) = a_{kjj}^{-1}a_{mjj}^{-1}(t_j-b_{mj}) - a_{kjj}^{-1}b_{kj}, \tag{5.4.63}$$
$$\nu_{x_k}\circ\nu_{x_m}(x_i) = \nu_{x_k}(c_{m,i}x_i+q_{m,i}^{(m)}) = c_{m,i}(c_{k,i}x_i+q_{k,i}^{(k)})+q_{m,i}^{(m)}, \quad \text{and} \tag{5.4.64}$$
$$\nu_{x_m}\circ\nu_{x_k}(x_i) = \nu_{x_m}(c_{k,i}x_i+q_{k,i}^{(k)}) = c_{k,i}(c_{m,i}x_i+q_{m,i}^{(m)})+q_{k,i}^{(k)}, \tag{5.4.65}$$

for all $1\leq i,m,k\leq 3$ and $j\in\{1,2\}$. Then relations (5.4.62) and (5.4.63) hold because relation (5.4.42) is satisfied. With respect to relations (5.4.62) and (5.4.63), both are satisfied due to the expressions (5.4.46) and (5.4.47). This yields that $\nu_{x_k}\circ\nu_{x_m} = \nu_{x_m}\circ\nu_{x_k}$, for $1\leq k,m\leq 3$. □

We formulate the important result of this section.

Theorem 5.19. *If a SPBW extension $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$ satisfies the conditions in Proposition 5.18, then it is differentially smooth.*

Proof. We know that $\operatorname{GKdim}(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle) = 5$ and that we have to consider $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle)$, a free right $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$-module of rank 5 with generators dt_1, dt_2, dx_1, dx_2, dx_3.

Define a left $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$-module structure by

$$f dt_i = dt_i \nu_{t_i}(f), \quad f dx_j = dx_j \nu_{x_j}(f), \tag{5.4.66}$$

for all $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$, $f \in \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$, where ν_{t_i}, ν_{x_j} are the algebra automorphisms for $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$ established in Proposition 5.18.

The relations in $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle)$ are given by

$$t_i dt_j = dt_j t_i \qquad t_i dx_j = dx_j a_{jii}^{-1}(t_i - b_{ij}), \quad \text{for all } i \in \{1, 2\},\ j \in \{1, 2, 3\}, \tag{5.4.67}$$

$$x_i dx_i = dx_i x_i, \qquad x_i dt_j = dt_j a_{ijj} x_i, \quad \text{for all } i \in \{1, 2\},\ j \in \{1, 2, 3\}, \tag{5.4.68}$$

and

$$x_i dx_j = dx_j (c_{i,j}^{-1} x_i - c_{i,j}^{-1} q_{i,j}^{(j)}), \quad \text{for } i < j, \quad \text{and} \tag{5.4.69}$$

$$x_i dx_j = dx_j (c_{j,i} x_i + q_{j,i}^{(j)}), \quad \text{for } i > j. \tag{5.4.70}$$

We extend $t_i \mapsto dt_i$ and $x_j \mapsto dx_j$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ to a map

$$d : \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle \to \Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle)$$

satisfying the Leibniz's rule. This must satisfy the relations given by

$$dx_i t_j + x_i dt_j = a_{ijj} dt_j x_i + a_{ijj} t_j dx_i + b_{ij} dx_i, \quad \text{for } i \in \{1, 2\},\ j \in \{1, 2, 3\}$$

$$dx_j x_i + x_j dx_1 = c_{i,j} dx_i x_j + c_{i,j} x_j dx_1 + \sum_{k=1}^{3} q_{i,j}^{(k)} dx_k, \quad \text{for } i, j, k \in \{1, 2, 3\},\ i < j.$$

Define $\Bbbk$-linear maps

$$\partial_{t_i}, \partial_{x_i} : \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle \to \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$$

such that

$$d(a) = dt_1 \partial_{t_1}(a) + dt_2 \partial_{t_2}(a) + \sum_{i=1}^{3} dx_i \partial_{x_i}(a), \text{ for all } a \in \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle.$$

These maps are well-defined since dt_i, dx_j, $i \in \{1, 2\}, j \in \{1, 2, 3\}$ are free generators of the right $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$-module $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle)$. With that, $d(a) = 0$ if

and only if $\partial_{t_i}(a) = \partial_{x_j}(a) = 0$, $i \in \{1,2\}$ and $j \in \{1,2,3\}$. Using relations (5.4.66) and definitions of the maps ν_{t_i}, ν_{x_j}, $i \in \{1,2\}$ and $j \in \{1,2,3\}$, we obtain that

$$
\begin{aligned}
\partial_{t_1}(t_1^k t_2^s x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= k t_1^{k-1} t_2^s x_1^{l_1} x_2^{l_2} x_3^{l_3}, \\
\partial_{t_2}(t_1^k t_2^s x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= s t_1^k t_2^{s-1} x_1^{l_1} x_2^{l_2} x_3^{l_3}, \\
\partial_{x_1}(t_1^k t_2^s x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= l_1 a_{111}^{-k} a_{122}^{-s} (t_1 - b_{11})^k (t_2 - b_{12})^s x_1^{l_1-1} x_2^{l_2} x_3^{l_3}, \\
\partial_{x_2}(t_1^k t_2^s x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= l_2 c_{1,2}^{-l_1} a_{211}^{-k} a_{222}^{-s} (t_1 - b_{21})^k (t_2 - b_{22})^s (x_1 - q_{1,2}^{(2)})^{l_1} x_2^{l_2-1} x_3^{l_3}, \quad \text{and} \\
\partial_{x_3}(t_1^k t_2^s x_1^{l_1} x_2^{l_2} x_3^{l_3}) &= l_3 c_{2,3}^{-l_2} c_{1,3}^{-l_1} a_{311}^{-k} a_{322}^{-s} (t_1 - b_{31})^k (t_2 - b_{32})^s (x_1 - q_{1,3}^{(3)})^{l_1} \\
&\quad (x_2 - q_{2,3}^{(3)})^{l_2} x_3^{l_3-1}.
\end{aligned} \tag{5.4.71}
$$

Then, $d(a) = 0$ if and only if a is a scalar multiple of the identity. This shows that $\Omega(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle, d)$ is connected, where

$$
\Omega(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle) = \bigoplus_{i=0}^{4} \Omega^i(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle).
$$

The universal extension of d to higher forms compatible with (5.4.67), (5.4.68) and (5.4.70) gives the following rules for $\Omega^l(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle)$ with $l = 2, 3, 4$:

$$
\begin{aligned}
dt_2 \wedge dt_1 &= -dt_1 \wedge dt_2, \\
dx_i \wedge dt_j &= -a_{ijj} dt_j \wedge dx_i, \quad \text{for } i \in \{1,2\}, j \in \{1,2,3\}, \\
dx_k \wedge dx_j &= -c_{j,k} dx_j \wedge dx_k, \quad \text{for } j,k \in \{1,2,3\}, j < k \\
dx_j \wedge dt_2 \wedge dt_1 &= -a_{j11} a_{j22} dt_1 \wedge dt_2 \wedge dx_j, \text{ for } j \in \{1,2,3\}, \\
dx_k \wedge dx_j \wedge dt_i &= -c_{j,k} a_{jii} a_{kii} dt_i \wedge dx_j \wedge dx_k, \text{ for } i \in \{1,2\}, j,k \in \{1,2,3\}, j < k, \\
dx_3 \wedge dx_2 \wedge dx_1 &= -c_{2,3} c_{1,2} c_{1,3} dx_1 \wedge dx_2 \wedge dx_3, \quad \text{and} \\
dx_k \wedge dx_j \wedge dt_2 \wedge dt_1 &= c_{j,k} a_{1jj} a_{1kk} a_{2jj} a_{2kk} dt_1 \wedge dt_2 \wedge dx_j \wedge dx_k, \text{ for } j,k \in \{1,2,3\}, j < k, \\
dx_3 \wedge dx_2 \wedge dx_1 \wedge dt_i &= c_{1,2} c_{1,3} c_{2,3} a_{1ii} a_{2ii} a_{3ii} dt_i \wedge dx_1 \wedge dx_2 \wedge dx_3, \text{ for } i \in \{1,2\},
\end{aligned} \tag{5.4.72}
$$

Since the automorphisms ν_{t_i}, ν_{x_j}, $i \in \{1,2\}, j \in \{1,2,3\}$ commute with each other, there are no additional relationships to the previous ones, so we write

$$
\begin{aligned}
\Omega^4(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle) = [&dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2 \oplus dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_3 \\
&\oplus dt_1 \wedge dt_2 \wedge dx_2 \wedge dx_3 \oplus dt_1 \wedge dx_1 \wedge dx_2 \wedge dx_3 \\
&\oplus dt_2 \wedge dx_1 \wedge dx_2 \wedge dx_3] \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle.
\end{aligned}
$$

Now,

$$
\Omega^5(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle) = \omega\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle \cong \sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle
$$

as a right and left $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$-module, with

$$
\omega = dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2 \wedge x_3, \quad \text{where } \nu_\omega = \nu_{t_1} \circ \nu_{t_2} \circ \nu_{x_1} \circ \nu_{x_2} \circ \nu_{x_3}.
$$

Then ω is a volume form of $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$. From Proposition 1.7 (2), it follows that ω is an integral form by setting

$$\omega_1^1 = dt_1, \quad \omega_2^1 = dt_2, \quad \omega_3^1 = dx_1, \quad \omega_4^1 = dx_2, \quad \omega_5^1 = dx_3$$
$$\omega_1^2 = dt_1 \wedge dt_2, \quad \omega_2^2 = dt_1 \wedge dx_1, \quad \omega_3^2 = dt_1 \wedge dx_2, \quad \omega_4^2 = dt_1 \wedge dx_3,$$
$$\omega_5^2 = dt_2 \wedge dx_1, \quad \omega_6^2 = dt_2 \wedge dx_2, \quad \omega_7^2 = dt_2 \wedge dx_3, \quad \omega_8^2 = dx_1 \wedge dx_2,$$
$$\omega_9^2 = dx_1 \wedge dx_3, \quad \omega_{10}^2 = dx_2 \wedge dx_3,$$
$$\omega_1^3 = dt_1 \wedge dt_2 \wedge dx_1, \quad \omega_2^3 = dt_1 \wedge dt_2 \wedge dx_2, \quad \omega_3^3 = dt_1 \wedge dt_2 \wedge dx_3,$$
$$\omega_4^3 = dt_1 \wedge dx_1 \wedge dx_2, \quad \omega_5^3 = dt_1 \wedge dx_1 \wedge dx_3, \quad \omega_6^3 = dt_1 \wedge dx_2 \wedge dx_3,$$
$$\omega_7^3 = dt_2 \wedge dx_1 \wedge dx_2, \quad \omega_8^3 = dt_2 \wedge dx_1 \wedge dx_3, \quad \omega_9^3 = dt_2 \wedge dx_2 \wedge dx_3,$$
$$\omega_{10}^3 = dx_1 \wedge dx_2 \wedge dx_3,$$
$$\omega_1^4 = dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2, \quad \omega_2^4 = dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_3,$$
$$\omega_3^4 = dt_1 \wedge dt_2 \wedge dx_2 \wedge dx_3, \quad \omega_4^4 = dt_1 \wedge dx_1 \wedge dx_2 \wedge dx_3,$$
$$\omega_5^4 = dt_2 \wedge dx_1 \wedge dx_2 \wedge dx_3,$$
$$\bar{\omega}_1^1 = a_{311}^{-1} a_{322}^{-1} c_{1,3}^{-1} c_{2,3}^{-1} dx_3, \quad \bar{\omega}_2^1 = -c_{1,2}^{-1} a_{222}^{-1} a_{211}^{-1} dx_2, \quad \bar{\omega}_3^1 = a_{122}^{-1} a_{111}^{-1} dx_1,$$
$$\bar{\omega}_4^1 = -dt_2, \quad \bar{\omega}_5^1 = dt_1$$
$$\bar{\omega}_1^2 = c_{1,2}^{-1} c_{1,3}^{-1} a_{211}^{-1} a_{311}^{-1} a_{222}^{-1} a_{322}^{-1} dx_2 \wedge dx_3, \quad \bar{\omega}_2^2 = -c_{2,3}^{-1} a_{111}^{-1} a_{311}^{-1} a_{122}^{-1} a_{322}^{-1} dx_1 \wedge dx_3,$$
$$\bar{\omega}_3^2 = a_{111}^{-1} a_{211}^{-1} a_{122}^{-1} a_{222}^{-1} a_{111}^{-1} dx_1 \wedge dx_2, \quad \bar{\omega}_4^2 = c_{1,3}^{-1} c_{2,3}^{-1} a_{311}^{-1} dt_2 \wedge dx_3,$$
$$\bar{\omega}_5^2 = -c_{1,2}^{-1} a_{211}^{-1} dt_2 \wedge dx_2, \quad \bar{\omega}_6^2 = -a_{111}^{-1} dt_2 \wedge dx_1, \quad \bar{\omega}_7^2 = -c_{1,3}^{-1} c_{2,3}^{-1} a_{322}^{-1} dt_1 \wedge dx_3,$$
$$\bar{\omega}_8^2 = c_{1,2}^{-1} a_{222}^{-1} dt_1 \wedge dx_2, \quad \bar{\omega}_9^2 = -a_{122}^{-1} dt_1 \wedge dx_1, \quad \bar{\omega}_{10}^2 = dt_1 \wedge dt_2,$$
$$\bar{\omega}_1^3 = -a_{111}^{-1} a_{211}^{-1} a_{311}^{-1} a_{122}^{-1} a_{222}^{-1} a_{322}^{-1} dx_1 \wedge dx_2 \wedge dx_3,$$
$$\bar{\omega}_2^3 = -c_{1,2}^{-1} c_{1,3}^{-1} a_{211}^{-1} a_{311}^{-1} dt_2 \wedge dx_2 \wedge dx_3, \quad \bar{\omega}_3^3 = c_{2,3}^{-1} a_{311}^{-1} a_{111}^{-1} dt_2 \wedge dx_1 \wedge dx_3,$$
$$\bar{\omega}_4^3 = -a_{211}^{-1} a_{111}^{-1} dt_2 \wedge dx_1 \wedge dx_2, \quad \bar{\omega}_5^3 = c_{1,2}^{-1} c_{1,3}^{-1} a_{222}^{-1} a_{322}^{-1} dt_1 \wedge dx_2 \wedge dx_3,$$
$$\bar{\omega}_6^3 = -c_{2,3}^{-1} a_{122}^{-1} a_{322}^{-1} dt_1 \wedge dx_1 \wedge dx_3, \quad \bar{\omega}_7^3 = a_{122}^{-1} a_{222}^{-1} dt_1 \wedge dx_1 \wedge dx_2,$$
$$\bar{\omega}_8^3 = c_{2,3}^{-1} c_{1,3}^{-1} dt_1 \wedge dt_2 \wedge dx_3, \quad \bar{\omega}_9^3 = -c_{1,2}^{-1} dt_1 \wedge dt_2 \wedge dx_2, \quad \bar{\omega}_{10}^3 = dt_1 \wedge dt_2 \wedge dx_1,$$
$$\bar{\omega}_1^4 = a_{311}^{-1} a_{211}^{-1} a_{111}^{-1} dt_2 \wedge dx_1 \wedge dx_2 \wedge dx_3,$$
$$\bar{\omega}_2^4 = -a_{322}^{-1} a_{222}^{-1} a_{122}^{-1} dt_1 \wedge dx_1 \wedge dx_2 \wedge dx_3,$$
$$\bar{\omega}_3^4 = c_{1,2}^{-1} c_{1,3}^{-1} dt_1 \wedge dt_2 \wedge dx_2 \wedge dx_3, \quad \bar{\omega}_4^4 = -c_{2,3}^{-1} dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_3,$$
$$\bar{\omega}_5^4 = dt_1 \wedge dt_2 \wedge dx_1 \wedge dx_2.$$

It can be seen that any element $\omega' \in \Omega^l(\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle)$, $l \in \{1, 2, 3, 4\}$ can be generated by ω_i^l, $\bar{\omega}_i^{5-l}$, $1 \leq i \leq \binom{5}{l}$. Hence, $\sigma(\Bbbk[t_1, t_2])\langle x_1, x_2, x_3\rangle$ is differentially smooth. □

5.4.3 SPBW extensions in n indeterminates

Let $\sigma(\Bbbk[t_1, t_2])\langle x_1, \ldots, x_n\rangle$. Consider once more again the family of automorphisms of the previous section. We get that

$$\begin{aligned} x_i t_j &= a_{ijj} t_j x_i + b_{ij} x_i + p_i, \quad \text{for } 1 \leq i \leq n \text{ and } j = 1, 2, \\ x_j x_i &= c_{i,j} x_i x_j + q_{i,j}^{(0)} + \sum_{k=1}^{n} q_{i,j}^{(k)} x_k, \quad \text{for } 1 \leq i, j, k \leq n,\ i < j, \end{aligned} \tag{5.4.73}$$

where $c_{i,j} \in \Bbbk^*$, $q_{i,j}^{(0)}, q_{i,j}^{(k)} \in \Bbbk$, for $1 \leq i, j, k \leq n$, $i < j$ and $a_{ijj} \in \Bbbk^*$, $b_{ij}, p_i \in \Bbbk$ where $1 \leq i \leq n$ and $j \in \{1, 2\}$.

Proposition 5.20. *Let*

$$\nu_{t_i}(t_j) = t_j, \qquad \nu_{t_i}(x_k) = a_{kii} x_k, \tag{5.4.74}$$

$$\nu_{x_k}(t_i) = a_{kii}^{-1}(t_i - b_{ki}), \qquad \nu_{x_k}(x_k) = x_k, \tag{5.4.75}$$

$$\nu_{x_i}(x_j) = c_{i,j} x_j + q_{i,j}^{(i)}, \quad \text{for } i < j, \qquad \nu_{x_i}(x_j) = c_{j,i}^{-1} x_j - c_{j,i}^{-1} q_{j,i}^{(i)}, \quad \text{for } i > j, \tag{5.4.76}$$

Then:

(1) *Leibniz's rule holds if the following relationships are satisfied:*

$$b_{sl}(a_{ill} - 1) = b_{il}(a_{sll} - 1), \tag{5.4.77}$$

$$q_{s,i}^{(s)}(a_{ill} - 1) = 0, \tag{5.4.78}$$

$$p_i(c_{s,i} - a_{sll}) = b_{il} q_{s,i}^{(s)}, \tag{5.4.79}$$

$$q_{i,j}^{(s)}(c_{s,j} c_{s,i} - 1) = 0, \tag{5.4.80}$$

$$q_{s,j}^{(s)}(c_{i,j} - 1) = q_{i,j}^{(i)}(c_{s,j} - 1), \tag{5.4.81}$$

$$q_{s,i}^{(s)}(c_{i,j} - 1) = q_{i,j}^{(j)}(c_{s,i} - 1), \tag{5.4.82}$$

$$q_{i,j}^{(k)}(c_{s,j} c_{s,i} - c_{k,s}^{-1}) = 0, \quad 1 \leq k \leq n, \tag{5.4.83}$$

$$\sum_{k=1}^{s-1} c_{k,s}^{-1} q_{i,j}^{(k)} q_{k,s}^{(s)} - \sum_{k=s+1}^{n} q_{i,j}^{(k)} q_{k,s}^{(s)} + q_{s,i}^{(s)} q_{s,j}^{(s)}(1 - c_{i,j}) + (c_{s,j} c_{s,i} - 1) q_{i,j}^{(0)} = 0. \tag{5.4.84}$$

for $1 \leq i, j, s \leq n$, $i < j$, $l \in \{1, 2\}$, *having the convention of* $c_{r,t} := c_{t,r}^{-1}$, $q_{r,t}^{(p)} := -c_{t,r}^{-1} q_{t,r}^{(p)}$, *for all* $1 \leq r, t, p \leq n$, $t < r$ *and* $c_{r,r} = 1$ *and* $q_{r,r}^{(p)} = 0$ *for all* $1 \leq r, p \leq n$. *The maps defined by (5.4.74), (5.4.75) and (5.4.76) simultaneously extend to algebra automorphisms* $\nu_{t_1}, \nu_{t_2}, \nu_{x_i}$, $1 \leq i \leq n$, *of* $\sigma(\Bbbk[t_1, t_2])\langle x_1, \ldots, x_n\rangle$ *when the previous relations are satisfied.*

(2) *If the relations in* (1) *hold, then*

$$\nu_{t_i} \circ \nu_{t_j} = \nu_{t_j} \circ \nu_{t_i}, \quad \nu_{t_j} \circ \nu_{x_k} = \nu_{x_k} \circ \nu_{t_j}, \quad \nu_{x_k} \circ \nu_{x_l} = \nu_{x_l} \circ \nu_{x_k}, \tag{5.4.85}$$

for $i, j = 1, 2$ *and* $1 \leq k, l \leq n$.

Proof. For the first assertion, the map ν_{t_l}, $l = 1, 2$ can be extended to an algebra homomorphism if and only if the definitions of $\nu_{t_l}(t_j)$, and $\nu_{t_l}(x_i)$ respect relations (5.4.73), i.e.

$$\nu_{t_l}(x_i)\nu_{t_l}(t_j) - \nu_{t_l}(a_{ijj}t_j + b_{ij})\nu_{t_l}(x_i) = \nu_{t_l}(p_i), \quad \text{for } 1 \leq i \leq n,\ j, l \in \{1, 2\},$$

and

$$\nu_{t_l}(x_j)\nu_{t_l}(x_i) - c_{i,j}\nu_{t_l}(x_i)\nu_{t_l}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{3} q_{i,j}^{(k)}\nu_{t_l}(x_k),$$

for $1 \leq i, j \leq n$, $l \in \{1, 2\}$, and $i < j$. Then

$$\begin{aligned} p_i(a_{ill} - 1) &= 0, \quad \text{for } 1 \leq i \leq n,\ l \in \{1, 2\} \\ q_{i,j}^{(0)}(a_{jll}a_{ill} - 1) &= 0, \quad \text{and} \\ q_{i,j}^{(k)}(a_{jll}a_{ill} - a_{kll}) &= 0, \quad \text{for } 1 \leq i, j, k \leq n,\ l \in \{1, 2\}. \end{aligned} \tag{5.4.86}$$

Now, it is necessary that ν_{x_s}, $1 \leq s \leq n$, can also be extended to the entire SPBW extension $\sigma(\Bbbk[t_1, t_2])\langle x_1, \ldots, x_n\rangle$. Thus,

$$\nu_{x_s}(x_i)\nu_{x_s}(t_j) - \nu_{x_s}(a_{ijj}t_j + b_{ij})\nu_{x_s}(x_i) = \nu_{x_s}(p_i), \quad \text{for } i \in \{1, \ldots, n\},\ j, l \in \{1, 2\}, \tag{5.4.87}$$

and

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{n} q_{i,j}^{(k)}\nu_{x_s}(x_k), \tag{5.4.88}$$

For Equation (5.4.87), the following three cases are considered:

- $s < i$. We obtain

$$\nu_{x_s}(x_i)\nu_{x_s}(t_j) - \nu_{x_s}(a_{ijj}t_j + b_{ij})\nu_{x_s}(x_i) = \nu_{x_s}(p_i)$$

$$c_{s,i}(x_i + q_{s,i}^{(s)})a_{sjj}^{-1}(t_j - b_{sj}) - a_{ijj}a_{sjj}^{-1}(t_j - b_{sj})c_{s,i}(x_i + q_{s,i}^{(s)}) - b_{ij}c_{s,i}(x_i + q_{s,i}^{(s)}) = p_i$$

  From this equation, we obtain the following conditions

$$\begin{aligned} b_{sj}(a_{ijj} - 1) &= b_{ij}(a_{sjj} - 1), \\ q_{s,i}^{(s)}(a_{ijj} - 1) &= 0, \\ p_i(c_{s,i} - a_{sjj}) &= b_{ij}q_{s,i}^{(s)}. \end{aligned}$$

- $s = i$. In this case, when the calculations are made, no new conditions are obtained.
- $s > i$. We obtain

$$\nu_{x_s}(x_i)\nu_{x_s}(t_j) - \nu_{x_s}(a_{ijj}t_j + b_{ij})\nu_{x_s}(x_i) = \nu_{x_s}(p_i)$$

$$c_{i,s}^{-1}(x_i - q_{i,s}^{(s)})a_{sjj}^{-1}(t_j - b_{sj}) - a_{ijj}a_{sjj}^{-1}(t_j - b_{sj})c_{i,s}^{-1}(x_i - q_{i,s}^{(s)}) - b_{ij}c_{i,s}^{-1}(x_i - q_{i,s}^{(s)}) = p_i$$

  From this equation, we obtain the following new conditions

$$\begin{aligned} q_{i,s}^{(s)}(a_{ijj} - 1) &= 0, \\ p_i(c_{i,s}a_{sjj} - 1) &= b_{ij}q_{i,s}^{(s)}. \end{aligned}$$

By Equation (5.4.88), we have the following three cases:

- $s \leq i$. We obtain

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{n} q_{i,j}^{(k)}\nu_{x_s}(x_k)$$

  From this equation, we obtain the following new conditions

$$\begin{aligned}
q_{i,j}^{(s)}(c_{s,j}c_{s,i} - 1) &= 0,\\
q_{s,j}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(i)}(c_{s,j} - 1),\\
q_{s,i}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(j)}(c_{s,i} - 1),\\
q_{i,j}^{(k)}(c_{s,j}c_{s,i} - c_{k,s}^{-1}) &= 0, \quad 1 \leq k \leq s-1,\\
q_{i,j}^{(k)}(c_{s,j}c_{s,i} - c_{s,k}) &= 0, \quad s+1 \leq k \leq n, k \neq i, j,\\
\sum_{k=1}^{s-1} c_{k,s}^{-1} q_{i,j}^{(k)} q_{k,s}^{(s)} - \sum_{k=s+1}^{n} q_{i,j}^{(k)} q_{k,s}^{(s)} &+ q_{s,i}^{(s)} q_{s,j}^{(s)}(1 - c_{i,j}) + (c_{s,j}c_{s,i} - 1)q_{i,j}^{(0)} = 0.
\end{aligned}$$

- $i < s < j$. We obtain

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{n} q_{i,j}^{(k)}\nu_{x_s}(x_k)$$

  From this equation, we obtain the following new conditions

$$\begin{aligned}
q_{i,j}^{(s)}(c_{s,j} - c_{i,s}) &= 0,\\
q_{s,j}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(i)}(c_{s,j} - 1),\\
q_{i,s}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(j)}(c_{i,s} - 1),\\
q_{i,j}^{(k)}(c_{s,j}c_{i,s}^{-1} - c_{k,s}^{-1}) &= 0, \quad 1 \leq k \leq s-1, k \neq i\\
q_{i,j}^{(k)}(c_{s,j}c_{i,s}^{-1} - c_{s,k}) &= 0, \quad s+1 \leq k \leq n, k \neq j,\\
\sum_{k=1}^{s-1} c_{k,s}^{-1} q_{i,j}^{(k)} q_{k,s}^{(s)} - \sum_{k=s+1}^{n} q_{i,j}^{(k)} q_{s,k}^{(s)} &+ c_{i,s}^{-1} q_{s,j}^{(s)} q_{i,s}^{(s)}(c_{i,j} - 1) + (c_{s,j}c_{i,s}^{-1} - 1)q_{i,j}^{(0)} = 0.
\end{aligned}$$

- $s \geq j$. We obtain

$$\nu_{x_s}(x_j)\nu_{x_s}(x_i) - c_{i,j}\nu_{x_s}(x_i)\nu_{x_s}(x_j) = q_{i,j}^{(0)} + \sum_{k=1}^{n} q_{i,j}^{(k)}\nu_{x_s}(x_k)$$

From this equation, we obtain the following new conditions

$$\begin{aligned} q_{i,j}^{(s)}(c_{j,s}c_{i,s} - 1) &= 0, \\ q_{j,s}^{(s)}(c_{i,j} - 1) &= q_{i,j}^{(i)}(c_{j,s} - 1), \\ q_{i,j}^{(k)}(c_{j,s}^{-1}c_{i,s}^{-1} - c_{k,s}^{-1}) &= 0, \quad 1 \leq k \leq s-1, k \neq i, j \\ q_{i,j}^{(k)}(c_{j,s}^{-1}c_{i,s}^{-1} - c_{s,k}) &= 0, \quad s+1 \leq k \leq n, \\ \sum_{k=1}^{s-1} c_{k,s}^{-1} q_{i,j}^{(k)} q_{k,s}^{(s)} - \sum_{k=s+1}^{n} q_{i,j}^{(k)} q_{s,k}^{(s)} &+ c_{j,s}^{-1}c_{i,s}^{-1} q_{j,s}^{(s)} q_{i,s}^{(s)}(1 - c_{i,j}) + (c_{j,s}^{-1}c_{i,s}^{-1} - 1) q_{i,j}^{(0)} = 0. \end{aligned}$$

Putting together all the conditions for s, we obtain the restrictions for the extension of the automorphisms.

For the second assertion, it is enough to prove it for the generators t_j, x_i, $1 \leq i \leq n$ and $j \in \{1, 2\}$. Note that

$$\nu_{t_k} \circ \nu_{t_m}(t_j) = \nu_{t_k}(t_j) = t_j, \tag{5.4.89}$$
$$\nu_{t_m} \circ \nu_{t_k}(t_j) = \nu_{t_m}(t_j) = t_j, \tag{5.4.90}$$
$$\nu_{t_k} \circ \nu_{t_m}(x_i) = \nu_{t_k}(a_{imm}x_i) = a_{imm}a_{ikk}x_i, \tag{5.4.91}$$
$$\nu_{t_m} \circ \nu_{t_k}(x_i) = \nu_{t_m}(a_{ikk}x_i) = a_{imm}a_{ikk}x_i. \tag{5.4.92}$$

for all $1 \leq i \leq n$ and $k, m, j \in \{1, 2\}$, so all relations are satisfied. It yields that $\nu_{t_k} \circ \nu_{t_m} = \nu_{t_m} \circ \nu_{t_k}$ for $k, m \in \{1, 2\}$.

Now,

$$\nu_{t_k} \circ \nu_{x_m}(t_j) = \nu_{t_k}(a_{mjj}^{-1}(t_j - b_{mj})) = a_{mjj}^{-1}(t_j - b_{mj}), \tag{5.4.93}$$
$$\nu_{x_m} \circ \nu_{t_k}(t_j) = \nu_{x_m}(t_j) = a_{mjj}^{-1}(t_j - b_{mj}), \tag{5.4.94}$$
$$\nu_{t_k} \circ \nu_{x_m}(x_i) = \nu_{t_k}(c_{m,i}x_i + q_{m,i}^{(m)}) = c_{m,i}a_{ikk}x_i + q_{m,i}^{(m)}, \tag{5.4.95}$$
$$\nu_{x_m} \circ \nu_{t_k}(x_i) = \nu_{x_m}(a_{ikk}x_i) = a_{ikk}(c_{m,i}x_i + q_{m,i}^{(m)}), \tag{5.4.96}$$

for all $1 \leq i, m \leq n$ and $k, j \in \{1, 2\}$. In this way, expressions (5.4.93) and (5.4.94) hold. With respect to the relations (5.4.95) and (5.4.96), both also hold since (5.4.80) is satisfied. So $\nu_{t_k} \circ \nu_{x_m} = \nu_{x_m} \circ \nu_{t_k}$, for $1 \leq m \leq n$ and $k \in \{1, 2\}$.

Finally,

$$\nu_{x_k} \circ \nu_{x_m}(t_j) = \nu_{x_k}(a_{mjj}^{-1}(t_j - b_{mj})) = a_{mjj}^{-1}a_{kjj}^{-1}(t_j - b_{kj}) - a_{mjj}^{-1}b_{jm}, \tag{5.4.97}$$
$$\nu_{x_m} \circ \nu_{x_k}(t_j) = \nu_{x_m}(a_{kjj}^{-1}(t_j - b_{kj})) = a_{kjj}^{-1}a_{mjj}^{-1}(t_j - b_{mj}) - a_{kjj}^{-1}b_{kj}, \tag{5.4.98}$$
$$\nu_{x_k} \circ \nu_{x_m}(x_i) = \nu_{x_k}(c_{m,i}x_i + q_{m,i}^{(m)}) = c_{m,i}(c_{k,i}x_i + q_{k,i}^{(k)}) + q_{m,i}^{(m)}, \tag{5.4.99}$$
$$\nu_{x_m} \circ \nu_{x_k}(x_i) = \nu_{x_m}(c_{k,i}x_i + q_{k,i}^{(k)}) = c_{k,i}(c_{m,i}x_i + q_{m,i}^{(m)}) + q_{k,i}^{(k)}, \tag{5.4.100}$$

for all $1 \leq i, m, k \leq n$ and $j \in \{1, 2\}$. Relations (5.4.97) and (5.4.98) hold due to expression (5.4.77). Relations (5.4.97) and (5.4.98) work, since we have the expressions (5.4.81) and (5.4.82). So $\nu_{x_k} \circ \nu_{x_m} = \nu_{x_m} \circ \nu_{x_k}$, for $1 \leq k, m \leq n$. □

Theorem 5.21. *If a SPBW extension $\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle$ satisfies the conditions in Proposition 5.20, then it is differentially smooth.*

Proof. It is clear that $\operatorname{GKdim}(\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle) = n + 2$. We know that we have to consider $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle)$, a free right $\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle$-module of rank $n + 2$ with generators dt_1, dt_2, dx_j, $1 \leq j \leq n$. Define a left $\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle$-module structure by

$$a dt_i = dt_i \nu_{t_i}(a), \quad a dx_j = dx_j \nu_{x_j}(a), \tag{5.4.101}$$

for all $i \in \{1, 2\}$, $j \in \{1, \dots, n\}$, $a \in \sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle$, where ν_{t_i}, ν_{x_j}, $i \in \{1, 2\}, j \in \{1, \dots, n\}$ are the algebra automorphisms established in Proposition 5.20. Notice that the relations in $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle)$ are given by

$$t_i dt_j = dt_j t_i \qquad t_i dx_j = dx_j a_{jii}^{-1}(t_i - b_{ij}), \quad \text{for all } i \in \{1, 2\},\ j \in \{1, \dots, n\}, \tag{5.4.102}$$

$$x_i dx_i = dx_i x_i, \quad x_i dt_j = dt_j a_{ijj} x_i, \quad \text{for all } i \in \{1, 2\},\ j \in \{1, \dots, n\}, \tag{5.4.103}$$

and

$$x_i dx_j = dx_j (c_{i,j}^{-1} x_i - c_{i,j}^{-1} q_{i,j}^{(j)}), \quad \text{for } 1 \leq i < j \leq n, \quad \text{and} \tag{5.4.104}$$

$$x_i dx_j = dx_j (c_{j,i} x_i + q_{j,i}^{(j)}), \quad \text{for } 1 \leq i < j \leq n. \tag{5.4.105}$$

We extend $t_i \mapsto dt_i$, $x_j \mapsto dx_j$, $i \in \{1, 2\}$, $j \in \{1, \dots, n\}$ to a map

$$d : \sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle \to \Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle)$$

satisfying the Leibniz's rule. This is possible if we guarantee its compatibility with the non-trivial relations (5.4.73), i.e.

$$dx_i t_j + x_i dt_j = a_{ijj} dt_j x_i + a_{ijj} t_j dx_i + b_{ij} dx_i, \quad \text{for } i \in \{1, 2\},\ j \in \{1, \dots, n\}$$

$$dx_j x_i + x_j dx_1 = c_{i,j} dx_i x_j + c_{i,j} x_j dx_1 + \sum_{k=1}^{n} q_{i,j}^{(k)} dx_k, \ \text{for } i, j, k \in \{1, \dots, n\},\ i < j.$$

Define $\Bbbk$-linear maps

$$\partial_{t_i}, \partial_{x_j} : \sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle \to \sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle$$

such that

$$d(a) = dt_1 \partial_{t_1}(a) + dt_2 \partial_{t_2}(a) + \sum_{i=1}^{n} dx_i \partial_{x_i}(a), \ \text{for all } a \in \sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle.$$

These maps are well-defined since dt_i, dx_j, $i \in \{1, 2\}, j \in \{1, \dots, n\}$ are free generators of the right $\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle$-module $\Omega^1(\sigma(\Bbbk[t_1, t_2])\langle x_1, \dots, x_n\rangle)$. Then $d(a) = 0$ if

and only if $\partial_{t_i}(a) = \partial_{x_j}(a) = 0$ for $i \in \{1,2\}, j \in \{1,\ldots,n\}$. Using relations (5.4.101) and definitions of the maps ν_{t_i}, ν_{x_j}, $i \in \{1,2\}, j \in \{1,\ldots,n\}$, we obtain that

$$\partial_{t_1}(t_1^k t_2^s x_1^{l_1}\cdots x_n^{l_n}) = k t_1^{k-1} t_2^s x_1^{l_1}\cdots x_n^{l_n}, \tag{5.4.106}$$

$$\partial_{t_2}(t_1^k t_2^s x_1^{l_1}\cdots x_n^{l_n}) = s t_1^k t_2^{s-1} x_1^{l_1}\cdots x_n^{l_n},$$

$$\partial_{x_j}(t_1^k t_2^s x_1^{l_1}\cdots x_n^{l_n}) = l_j a_{j11}^{-k} a_{j22}^{-s}(t_1 - b_{j1})^k (t_2 - b_{j2})^s \tag{5.4.107}$$

$$\prod_{s=1}^{j-1} c_{s,j}^{-l_s}(x_s - q_{s,j}^{(j)})^{l_s} x_j^{l_j - 1} x_{j+1}^{l_{j+1}}\cdots x_n^{l_n}, \quad 1 \leq j \leq n.$$

Since $d(a) = 0$ if and only if a is a scalar multiple of the identity, it follows that $\Omega(\sigma(\Bbbk[t_1,t_2])\langle x_1,\ldots,x_n\rangle, d)$ is connected, where

$$\Omega(\sigma(\Bbbk[t_1,t_2])\langle x_1,\ldots,x_n\rangle) = \bigoplus_{i=0}^{n+1} \Omega^i(\sigma(\Bbbk[t_1,t_2])\langle x_1,\ldots,x_n\rangle).$$

The universal extension of d to higher forms compatible with (5.4.102), (5.4.103) and (5.4.105) gives the following rules for $\Omega^l(\sigma(\Bbbk[t_1,t_2])\langle x_1,\ldots,x_n\rangle)$ $(l = 2,\ldots,n+1)$:

$$dx_{q(1)} \wedge \cdots \wedge dx_{q(s)} \wedge dt_2 \wedge dt_1 \wedge dx_{q(s+1)} \wedge \cdots \wedge dx_{q(l)} \tag{5.4.108}$$

$$= (-1)^{s+1} \prod_{\substack{r=1,\\ r\neq s_1,s_2}}^{s} a_{q(r)11}^{-1} a_{q(r)22}^{-1} dt_1 \wedge dt_2 \wedge \bigwedge_{\substack{k=1,\\ k\neq s_1,s_2}}^{l} dx_{q(k)}, \tag{5.4.109}$$

$$dx_{q(1)} \wedge \cdots \wedge dx_{q(s)} \wedge dt_1 \wedge dt_2 \wedge dx_{q(s+1)} \wedge \cdots \wedge dx_{q(l)} \tag{5.4.110}$$

$$= (-1)^{s} \prod_{\substack{r=1,\\ r\neq s_1,s_2}}^{s} a_{q(r)11}^{-1} a_{q(r)22}^{-1} dt_1 \wedge dt_2 \wedge \bigwedge_{\substack{k=1,\\ k\neq s_1,s_2}}^{l} dx_{q(k)}, \tag{5.4.111}$$

$$dx_{q(1)} \wedge \cdots \wedge dx_{q(s)} \wedge dt_i \wedge dx_{q(s+1)} \wedge \cdots \wedge dx_{q(l)} \tag{5.4.112}$$

$$= (-1)^{s} \prod_{\substack{r=1,\\ r\neq s_1}}^{s} a_{q(r)ii}^{-1} dt_i \wedge \bigwedge_{\substack{k=1,\\ k\neq s_1}}^{l} dx_{q(k)}, \tag{5.4.113}$$

$$\bigwedge_{k=1}^{l} dx_{q(k)} = (-1)^{\sharp} \prod_{r,s\in P} c_{r,s}^{-1} \bigwedge_{k=1}^{l} dx_{p(k)}, \tag{5.4.114}$$

where $s_1, s_2 \in \{1,\ldots,l\}$ do not appear in expression (5.4.109), (5.4.111) or (5.4.113). Besides,

$$q : \{1,\ldots,l\} \to \{1,\ldots,n\}$$

is an injective map, and

$$p : \{1,\ldots,l\} \to \mathrm{Im}(q)$$

is an increasing injective map, $\sharp$ is the number of 2-permutations needed to transform q into p, and $P := \{(s,t) \in \{1,\ldots,l\} \times \{1,\ldots,l\} \mid q(s) > q(t)\}$.

Since the automorphisms ν_{t_i}, ν_{x_j}, $i \in \{1,2\}, j \in \{1,\dots,n\}$ commute with each other, there are no additional relationships to the previous ones, so

$$\begin{aligned}\Omega^{n+1}(\sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle) = \Bigg[&\bigoplus_{r=2}^{n-1} dt_1 \wedge dt_2 \wedge dx_1 \wedge \cdots dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_n \\ &\oplus dt_1 \wedge dx_1 \wedge \cdots \wedge dx_n \\ &\oplus dt_2 \wedge dx_1 \wedge \cdots \wedge dx_n\Bigg]\, \sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle.\end{aligned}$$

Now,

$$\Omega^{n+2}(\sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle) = \omega\sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle \cong \sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle$$

as a right and left $\sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle$-module, with $\omega = dt_1 \wedge dt_2 \wedge dx_1 \wedge \cdots \wedge x_n$, where $\nu_\omega = \nu_{t_1} \circ \nu_{t_2} \circ \nu_{x_1} \circ \cdots \circ \nu_{x_n}$, this means that ω is a volume form of $\sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle$. In order to make the calculations easier, we consider the following notation:

$$t_1 = x_{-1},\ t_2 = x_0,\ c_{-1,0} = -1,\ c_{-1,i} = a_{i11},\ c_{0,i} = a_{i22}, \quad \text{for } 1 \le i \le n.$$

From Proposition 1.7 (2) we get that ω is an integral form by setting

$$\omega_i^j = \bigwedge_{k=-1}^{j-2} dx_{p_{i,j}(k)},\ \text{for } 1 \le i \le \binom{n+2}{j}, \quad \text{and}$$

$$\bar{\omega}_i^{n+2-j} = (-1)^{\sharp_{i,j}} \prod_{r,s \in P_{i,j}} c_{r,s}^{-1} \bigwedge_{k=j-1}^{n} dx_{\bar{p}_{i,j}(k)},\ \text{for } 1 \le i \le \binom{n+2}{j},$$

for $1 \le j \le n+2$, where

$$\begin{aligned} p_{i,j} &: \{-1,\dots,j-2\} \to \{-1,\dots,n\}, \quad \text{and} \\ \bar{p}_{i,j} &: \{j-1,\dots,n\} \to (\mathrm{Im}(p_{i,j}))^c \end{aligned}$$

(the symbol $\square^c$ denotes the complement of the set $\square$), are increasing injective maps, and $\sharp_{i,j}$ is the number of 2-permutation needed to transform

$$\{\bar{p}_{i,j}(j-1),\dots,\bar{p}_{i,j}(n), p_{i,j}(-1),\dots,p_{i,j}(j-2)\} \text{ into the set } \{-1,\dots,n\},$$

and

$$P_{i,j} := \{(s,t) \in \{-1,\dots,j-2\} \times \{j-1,\dots,n\} \mid p_{i,j}(s) < \bar{p}_{i,j}(t)\}.$$

Consider $\omega' \in \Omega^j(\sigma(\Bbbk[t_1,t_2])\langle x_1,\dots,x_n\rangle)$, that is,

$$\omega' = \sum_{i=1}^{\binom{n+2}{j}} \bigwedge_{k=-1}^{j-2} dx_{p_{i,j}(k)} \alpha_i, \quad \text{with } \alpha_i \in \Bbbk.$$

Then

$$\sum_{i=1}^{\binom{n+2}{j}} \omega_i^j \pi_\omega(\bar{\omega}_i^{n+2-j} \wedge \omega') = \sum_{i=1}^{\binom{n+2}{j}} \left[\bigwedge_{k=-1}^{j-2} dx_{p_i(k)} \right] \cdot \pi_\omega \left[(-1)^{\sharp_{i,j}} \square^* \wedge \omega' \right]$$
$$= \sum_{i=1}^{\binom{n+2}{j}} \bigwedge_{k=-1}^{j-2} dx_{p_{i,j}(k)} \alpha_i = \omega',$$

where

$$\square^* := \prod_{r,s \in P_{i,j}} c_{r,s}^{-1} \bigwedge_{k=j}^{n} dx_{\bar{p}_{i,j}(k)}.$$

Therefore, $\sigma(\Bbbk[t_1, t_2])\langle x_1, \ldots, x_n \rangle$ is differentially smooth. □

5.5 Future work

As expected, a natural task is to investigate the differential smoothness of SPBW extensions over commutative polynomial rings on three and more indeterminates. With this aim, we recall briefly some interesting facts on automorphisms of these polynomial rings. We follow Shestakov and Umirbaev's presentation [SU03, p. 197].

For $\Bbbk[X] = \Bbbk[x_1, \ldots, x_n]$, an automorphism $\tau \in \mathrm{Aut}(\Bbbk[X])$ is called *elementary* if it has a form

$$\tau(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, x_n) \mapsto (x_1, \ldots, x_{i-1}, ax_i + f, x_{i+1}, \ldots, x_n),$$

where $0 \neq a \in \Bbbk, f \in \Bbbk[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n]$. The subgroup of $\mathrm{Aut}(\Bbbk[X])$ generated by all the elementary automorphisms is called *tame subgroup*, and the elements from this subgroup are called *tame automorphisms* of $\Bbbk[X]$. Non-tame automorphisms of $\Bbbk[X]$ are called *wild*.

In the literature it has been shown that the automorphisms of polynomial rings and free associative algebras in two indeterminates are tame (e.g. [MW88, DE00]). Nevertheless, in the case of three or more indeterminates the similar question was open and known as "The generation gap problem" [DE00] or "Tame generators problem" [DE00]. The general belief was that the answer is negative, and the best known counterexample is the following automorphism $\sigma \in \mathrm{Aut}(\Bbbk[x, y, z])$, constructed by Nagata [Nag72]:

$$\begin{aligned} \sigma(x) &= x + (x^2 - yz)z, \\ \sigma(y) &= y + 2(x^2 - yz)x + (x^2 - yz)^2 z, \quad \text{and} \\ \sigma(z) &= z. \end{aligned}$$

Note that Nagata automorphism is *stably tame*; that is, it becomes tame after adding new variables. Shestakov and Umirbaev [SU03] gave a negative answer to the above question; in particular, the Nagata automorphism σ is wild.

In a sequel paper we study of the differential smoothness of SPBW extensions over $\Bbbk[X]$ by considering automorphisms following the ideas above.

On the other hand, since Artamonov [Art15] and Venegas [Ven15] presented some results concerning automorphisms and derivations of SPBW extensions, it is natural to study relationships between this kind of morphisms and those adequate to characterize the smoothness of these extensions. This will also be our topic of interest in the immediate future.

Of course, the Question 2 formulated at the end of Section 5.2.1 is of great importance for us.

CHAPTER 6

Serre duality for noncommutative projective schemes

Our purpose in this chapter is to present some bridging ideas with the aim of extending the Serre duality theorem for noncommutative projective schemes on $\mathbb{N}$-graded algebras to the setting of semi-graded algebras.

With this aim, in Section 6.1 we present a sketch of the key facts considered in the formulation of the Serre duality theorem for noncommutative projective schemes on $\mathbb{N}$-graded algebras following Yekutieli and Zhang's ideas [YZ97]. Next, Section 6.2 contains some facts of the research about noncommutative projective schemes on semi-graded rings: we recall briefly key results on the Serre-Artin-Zhang-Verevkin theorem [LL17, Lez21, Cha22, CR24a] and mention schematic semi-graded rings [Cha22, CR24b, CR24a]. Finally, we formulate some possible ideas that will allow us to have a version of the Serre duality theorem for semi-graded algebras.

6.1 Serre duality theorem for $\mathbb{N}$-graded algebras

Throughout this section, we follow the presentation (and the notation) on the topic carried out by Artin and Zhang [AZ94] and Yekutieli and Zhang [YZ97].

As is well-known, Serre [Ser55] proved a theorem that describes the coherent sheaves on a projective scheme in terms of graded modules. Briefly, a commutative graded $\Bbbk$-algebra is associated to a projective scheme $\mathsf{Proj}\ A$, and the geometry of this scheme can be described in terms of the quotient category $\mathsf{qgr}\ A = \mathsf{gr}\ A/\mathsf{tors}\ A$, where $\mathsf{gr}\ A$ denotes the category of graded modules and $\mathsf{tors}\ A$ denotes its subcategory of torsion modules. For A a finitely generated commutative graded $\Bbbk$-algebra and X its associated projective scheme, if $\mathsf{coh}\ X$ denotes the category of coherent sheaves on X and $\mathcal{O}_X(n)$ is the nth power of the twisting sheaf on X [Har77, p. 117], then we have a functor $\Gamma_* : \mathsf{coh}\ X \to \mathsf{qgr}\ A$ given by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d=-\infty}^{\infty} \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{O}_X(d)).$$

Serre's theorem [Ser55, Section 59, Proposition. 7.8, p. 252], [GD61, 3.3.5] and [Har77, Proposition. II. 5.15], asserts that if A is generated over $\Bbbk$ by elements of degree one, then Γ_* defines an equivalence of categories $\mathsf{coh}\ X \to \mathsf{qgr}\ A$.

Artin and Zhang [AZ94] extended Serre's theorem to the noncommutative setting as follows. Let A be an $\mathbb{N}$-graded algebra over a commutative Noetherian ring. Briefly, they defined the associated projective scheme to be the pair $\mathsf{Proj}\ A = (\mathsf{qgr}\ A, \mathcal{A})$, where $\mathsf{qgr}\ A$ is the quotient category above, $\mathcal{A}$ is the image of A in $\mathsf{qgr}\ A$ and plays the role of the structure sheaf of $\mathsf{Proj}\ A$, and s, called the *polarization* defined by the projective embedding (this definition is the same as is given by Verevkin [Ver92a, Ver92b]), is given by the shifting of the degrees in $\mathsf{gr}\ A$. Since Serre's theorem does not hold for all commutative graded algebras, i.e., the functor defined by Γ_* need not be an equivalence, Artin and Zhang's definition of $\mathsf{Proj}\ A$ is compatible with the classical definition for commutative graded rings only under some additional hypotheses, such as that A is generated in degree one and the following condition χ_1:

If A is left Noetherian, for $i \geq 0$ it is said that A satisfied the χ_i *condition* if for every finitely generated graded A-module M, we have that $\dim_{\Bbbk}\mathrm{Ext}^j_A(\Bbbk, M) < \infty$ for any $j \leq i$, where $\underline{\mathrm{Ext}}$ is the graded Ext. The algebra A satisfied the χ *condition* if it satisfies the χ_i condition for all $i \geq 0$. Note that since $\Bbbk$ is finitely generated as A-module, then

$$\dim_{\Bbbk}(\underline{\mathrm{Ext}}^j_A(\Bbbk, M)) = \dim_{\Bbbk}(\mathrm{Ext}^j_A(\Bbbk, M)).$$

In the literature, the noncommutative version of Serre's theorem is known as *Serre-Artin-Zhang-Verevkin theorem* [AZ94, Ver92a, Ver92b].

Three years later, Yekutieli and Zhang [YZ97] proved the Serre duality theorem for the noncommutative projective scheme $\mathsf{proj}\ A$ when A is a graded Noetherian PI ring or a graded Noetherian AS-Gorenstein ring. Let us see the details.

Let $\Bbbk$ be a field and let $A = \bigoplus_{i\geq 0} A_i$ be an $\mathbb{N}$-graded right Noetherian $\Bbbk$-algebra. A is called *locally finite* if each A_i is finite dimensional over $\Bbbk$, and if $A_0 = \Bbbk$ then A is called *connected graded*. Let $\mathsf{Gr}\ A$ be the *category of graded right* A*-modules* and $\mathsf{gr}\ A$ be the *subcategory consisting of Noetherian right* A*-modules*. The *augmentation ideal* $\mathfrak{m}$ of A is $A_{\geq 1} := \bigoplus_{i\geq 1} A_i$. Let $M = \bigoplus_i M_i$ be a graded right A-module and x a homogenous element of M. We say that x is $\mathfrak{m}$*-torsion* if $x\mathfrak{m}^n = 0$ for some n. All $\mathfrak{m}$-torsion elements form a submodule of M, which is denoted by $\tau(M)$. M is said to be $\mathfrak{m}$*-torsion* (resp. $\mathfrak{m}$*-torsion-free*) if $\tau(M) = M$ (resp. $\tau(M) = \{0\}$).

Let $\mathsf{Tor}\ A$ denote the subcategory of $\mathsf{Gr}\ A$ of all $\mathfrak{m}$-torsion modules and $\mathsf{tor}\ A$ denote the intersection of $\mathsf{Tor}\ A$ and $\mathsf{gr}\ A$. The (*degree*) *shift of* M, denoted by $s(M)$, is defined by $s(M)_i = M_{i+1}$, and we use $M(n)$ for the n-th power of a shift $s^n(M)$. Given an $\mathbb{N}$-graded right Noetherian ring A, the *noncommutative projective scheme of* A is defined as

$$\mathrm{Proj}\ A := (\mathsf{QGr}\ A, \mathcal{A}, s),$$

where $\mathsf{QGr}\ A$ is the quotient category $\mathsf{Gr}\ A/\mathsf{Tor}\ A$, $\mathcal{A}$ is the image of A_A in $\mathsf{QGr}\ A$ and s is the degree shift. If we work on Noetherian objects, the triple

$$\mathrm{proj}\ A := (\mathsf{qgr}\ A, \mathcal{A}, s)$$

is also called the *projective scheme of* A, where $\mathsf{qgr}\ A = \mathsf{gr}\ A/\mathsf{tor}\ A$ (for more details on the subject, see [AZ94]). The canonical functor from $\mathsf{Gr}\ A$ to $\mathsf{QGr}\ A$ (and from $\mathsf{gr}\ A$ to $\mathsf{qgr}\ A$) is denoted by π. If $M \in \mathsf{Gr}\ A$, the symbol $\mathfrak{M}$ denotes $\pi(M)$.

Let $X = \operatorname{proj} A$. The global section is $H^0(X, \mathcal{N}) = \operatorname{Hom}_{\mathsf{QGr}\ A}(\mathcal{A}, \mathcal{N})$ and the cohomology is $H^i(X, \mathcal{N}) = \operatorname{Ext}^i_{\mathsf{QGr}\ A}(\mathcal{A}, \mathcal{N})$ for all $i > 0$. Recall that in the commutative case, the Serre duality theorem [Har77, III.7.6] says there is an object ω^0 in $\mathsf{qgr}\ A$ such that

$$\theta^0 = \operatorname{Hom}_{\mathsf{qgr}\ A}(\mathcal{M}, \omega^0) \to H^d(X, \mathcal{M})^* \tag{6.1.1}$$

is a natural isomorphism for all $\mathcal{M} \in \mathsf{qgr}\ A$. The symbol $*$ means the vector space dual and d is the cohomological dimension of X defined by

$$\mathsf{cd}(X) = \max\left\{ i \mid H^i(X, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \mathsf{QGr}\ A \right\}.$$

It can be seen that such ω^0 is unique, and it is called a *dualizing sheaf* on $X = \operatorname{proj} A$.

Yekutieli and Zhang [YZ97] studied $H^d(X, -)^*$ as a functor from $\mathsf{qgr}\ A$ to $\mathsf{Mod}\ \Bbbk$. In general, $\mathsf{Mod}\ A$ is the category of (ungraded) right A-modules. Using a graded version of Watt's theorem they showed (6.1.1) for noncommutative rings under some hypothesis. In fact, these hypotheses can be checked for a class of rings including Noetherian PI rings. Second, they used the balanced dualizing complex which was introduced and studied by Yekutieli [Yek92]. Note that the existence of balanced dualizing complex implies (6.1.1) holds for some ω^0 constructed from the dualizing complex (we should not expect a dualizing sheaf ω^0 to exist in $\mathsf{qgr}\ A$, the question if (6.1.1) holds for some ω^0 in $\mathsf{QGr}\ A$ was answered affirmatively by Jørgensen), and since Noetherian AS-Gorenstein rings admits balanced dualizing complexes, (6.1.1) holds for this kind of rings.

6.2 Noncommutative projective schemes on semi-graded rings

Lezama and Latorre [LL17, Theorem 6.12] investigated the Serre-Artin-Zhang-Verevkin theorem following the ideas presented in the graded case. They assumed that the semi-graded left Noetherian ring is a domain. Nevertheless, as is well-known, the theorem for finitely graded algebras does not include this restriction. Precisely, this assumption was eliminated by Lezama [Lez21, Section 1.4] (see also [FGL+20, Section 18.4, Theorem 18.5.13]).

More exactly, he proved the theorem for an SG ring $R = \bigoplus_{n \geq 0} R_n$ satisfying the following conditions:

(C1) R is left Noetherian;

(C2) R_0 is left Noetherian;

(C3) for every n, R_n is a finitely generated left R_0-module;

(C4) $R_0 \subset Z(R)$;

(C5) Condition χ_1 above.

Notice that condition (C4) implies that R_0 is a commutative Noetherian ring.

Universal enveloping algebras of finite-dimensional Lie algebras, some quantum algebras with three generators, and some examples of 3-dimensional skew polynomial algebras [BS90, Red99, RS22] illustrate the Serre–Artin–Zhang–Verevkin theorem [Lez21, Example 1.26] and [FGL+20, Example 18.5.15].

On the other hand, having in mind that Manin [Man91] commented the failure of attempts to obtain a noncommutative scheme theory à la Grothendieck for quantized algebras, Van Oystaeyen and Willaert [vOW95] studied this Proj by developing a kind of scheme theory similar to the commutative theory. They noticed that this theory is possible only if the connected and $\mathbb{N}$-graded algebra considered contains "enough" Ore sets. Algebras satisfying this condition are called *schematic*. They constructed a *generalized Grothendieck topology* for the free monoid on all Ore sets of a schematic algebra R, and defined a *noncommutative site* (c.f. [vOW96b]) as a category with coverings on which sheaves can be defined, and formulated the Serre's theorem. As a consequence of their treatment, an equivalence between the category of all coherent sheaves and the category Proj R was obtained in the sense of Artin [Art92]. Some years later, Van Oystaeyen and Willaert [vOW96b, vOW96a, vOW97] presented a sequel of [vOW95] in which they studied the cohomology of these algebras and proved a lifting property for Ore sets. This allowed to present many examples of schematic algebras like homogenizations of almost commutative algebras, Rees rings of universal enveloping algebras of Lie algebras, and three-dimensional Sklyanin algebras. A detailed treatment about schematic algebras can be found in Van Oystaeyen's book [vO00].

Recently, Chacón and Reyes [Cha22, CR24a] presented a noncommutative scheme theory for the semi-graded rings generated in degree one defined by Lezama and Latorre [LL17] following the ideas about schematicness introduced by Van Oystaeyen and Willaert [vOW95] for $\mathbb{N}$-graded algebras. With this theory, they proved the Serre-Artin-Zhang-Verevkin theorem for several families of non-$\mathbb{N}$-graded algebras and finitely non-$\mathbb{N}$-graded algebras which are not necessarily connected. Their treatment contributed to the research on this theorem presented by Lezama [Lez21, LL17] from a different point of view.

Let us some technical details. For a positively SG ring R, if $R_+ := \bigoplus_{k\geq 1} R_k$, then we say that a left Ore set S is *non-trivial* if $S \cap R_+ \neq \emptyset$.

Definition 6.1 ([CR24b, Definition 4.1])**.** Let R be a positively SG left Noetherian ring. R is called (*left*) *schematic* if there is a finite set I of non-trivial good left Ore sets of R such that for each $(x_S)_{S\in I} \in \prod_{S\in I} S$, there exist $t, m \in \mathbb{N}$ such that $(R_{\geq t})^m \subseteq \sum_{S\in I} Rx_s$.

If R is schematic by considering the *good* left Ore sets S_i, and defining for each i,

$$\kappa_{S_i}(M) = \{m \in M \mid \exists s \in S_i, sm = 0\},$$

then $\bigcap_{i=1}^{n} \kappa_{S_i}(M) = T(M)$ for every SG R-module M.

Some authors have investigated geometrical properties of schematic semi-graded rings [CRR24, CR24b, Ni3, NRR24, NR24, Ram23].

The following examples illustrate Serre-Artin-Zhang-Verevkin theorem in the case of non-$\mathbb{N}$-graded rings where [vOW95, Theorem 3] cannot be applied.

Example 6.1 ([CR24a, Example 5.25]). (i) Consider the first Weyl algebra $A_1(\Bbbk) = \Bbbk\{x,y\}/\langle yx - xy - 1\rangle$ over a field $\Bbbk$ of $\operatorname{char}(\Bbbk) = p > 0$. It is well-known that $A_1(\Bbbk)$ is a non-$\mathbb{N}$-graded ring, the set $\{x^n y^m \mid n,m \in \mathbb{N}\}$ is $\Bbbk$-basis of $A_1(\Bbbk)$, and that $A_1(\Bbbk)$ is a Noetherian ring. Since $x^p, y^p \in Z(A_1(\Bbbk))$ it is clear that $\{x^{pk} \mid k \in \mathbb{N}\}$ and $\{y^{pk} \mid k \in \mathbb{N}\}$ are good left Ore sets. Besides, if $k_1, k_2 \in \mathbb{N}$ then $A_1(\Bbbk)x^{pk_1} + A_1(\Bbbk)y^{pk_2}$ is a two-sided ideal of $A_1(\Bbbk)$ which is a left SG submodule, whence $A_1(\Bbbk)_{\geq pk_1+pk_2} \subseteq A_1(\Bbbk)x^{pk_1} + A_1(\Bbbk)y^{pk_2}$. Therefore $A_1(\Bbbk)$ is a schematic algebra and satisfies the theorem.

(ii) In a similar way, it can be shown that the n-th Weyl algebra $A_n(\Bbbk)$ is schematicness when $\operatorname{char}(\Bbbk) = p > 0$.

(iii) The well-known Jordan plane $\Bbbk\{x,y\}/\langle yx - xy - y^2\rangle$ is schematic when $\operatorname{char}(\Bbbk) = p > 0$ since the sets $\{x^{pk} \mid k \in \mathbb{N}\}$ and $\{y^{pk} \mid k \in \mathbb{N}\}$ are good left Ore sets.

Examples 6.2 and 6.3 show that the theory presented by Lezama [Lez21, LL17] about Serre-Artin-Zhang-Verevkin theorem and the one developed in this chapter are independent.

Example 6.2 ([CR24a, Example 5.27]). If R is a left Noetherian noncommutative ring, then $A = R[x]$ is schematic, whence A satisfies the Serre-Artin-Zhang-Verevkin theorem. Notice that this result cannot be obtained from the the theory developed by Lezama [Lez21, LL17] because does not satisfy Lezama's assumption (C4) that says that $A_0 = R$ is a commutative ring. Notice that in the particular case of the $\Bbbk$-algebra $R = M_n(\Bbbk)$, since R is not connected it does not satisfy the definition of schematicness given by Van Oystaeyen and Willaert, and it is not a *finitely semi-graded algebra* in the sense of Lezama 1.5. However, from Chacón and Reyes's point point of view, the algebra is schematic and satisfies the Serre-Artin-Zhang-Verevkin theorem.

Example 6.3. Consider A as the 3-dimensional skew polynomial algebra subject to the relations

$$yz = zy, \quad xz = zx, \quad \text{and} \quad yx = xy - z.$$

This algebra satisfies the Serre-Artin-Zhang-Verevkin theorem [FGL+20, Example 18.5.15 (v)] following the ideas presented by Lezama [Lez21, LL17]. However, A is not schematic.

Finally, we want to say that Willaert [Wil97] presented new results on cohomological properties of Auslander-Gorenstein algebras that might be useful in the research on schematic algebras. He considered necessary and sufficient conditions to relate the property of being schematic with the Gorenstein property: "At this moment, we do not know the strongness of the hypothesis that a certain algebra is schematic. For instance, is a schematic algebra with finite global dimension automatically Gorenstein? [...] It would also be interesting to know that the cohomological dimension of a schematic algebra" [Wil97, p. 155].

At the end of his paper, he mentioned Yekutieli and Zhang's paper [YZ97]: "if the connected Noetherian $\Bbbk$-algebra A is Gorenstein, then Serre-duality holds for A" [Wil97, p. 155 and Theorem 2].

As we can see from the definitions of Artin-Schelter regular algebra and semi-graded Artin-Schelter regular algebra, the Gorenstein condition is fundamental, so we think that a natural task is to investigate possible relationships between this condition and the schematic property in the setting of semi-graded rings following the approach carried out by Willaert [Wil97].

Related with this, we also consider important to investigate if the *category* $\mathsf{SGR}-\mathsf{R}$ *of semi-graded modules over* a ring R whose morphisms are the homogeneous R–homomorphisms, or an appropriate subcategory of this, has modules of quotients. Thinking about it, a possible way to solve this problem is due to Gabriel [Gab62], which consists to prove that $\mathsf{SGR}-\mathsf{R}$ has enough injectives (in fact, this would allow to formulate of notion of *cohomology* in the semi-graded setting following Villaert's ideas). Precisely, it is important to note that until now we do not know if $\mathsf{SGR}-\mathsf{R}$ is Grothendieck; we only know that $\mathsf{SGR}-\mathsf{R}$ is $\mathtt{Ab5}$ [Cha22, Section 1.5]. Of course, if we found a positive answer then automatically we guarantee the existence of enough injectives in $\mathsf{SGR}-\mathsf{R}$. This will be our line of research in the near future (c.f. [CRR24]).

`P.S.` At the time when this thesis was written, the following preprint appeared in `arXiv`:

> Serre duality for dg-algebras (`https://arxiv.org/abs/2410.07204`).

There, Brown and Sridhar generalized Yekutieli-Zhang's noncommutative Serre duality theorem to the setting of noncommutative spaces associated to dg-algebras. This fact motivates us considerably to advance our research in the setting of semi-graded rings.

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