Cardinal invariants associated with ideals between measure zero and strong measure zero

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Dedicado a
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Abstract

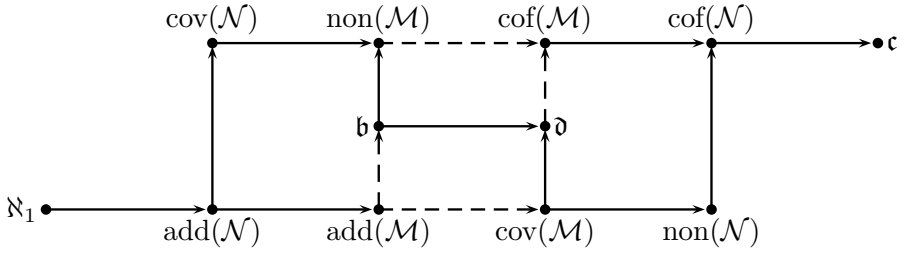
In 2002, Yorioka [Y02] introduced the σ-ideal $\mathcal{I}_f$ for increasing functions f from ω to ω to analyze the cofinality of the strong measure zero ideal. We use and generalize some techiques of itearations with finite support to construct a model of ZFC, by a matrix iteration, satisfying $\mathrm{add}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ for every fast increasing f.

One technical tool we use is the preservation theory of Judah-Shelah [JS90] and Brendle [Br91]. We generalize this theory so that we can cover the example of preservation presented in [KO14] which is fundamental in this thesis. We also provide original proofs of two Claims from [O08], about the additivity and cofinality of Yorioka ideals, whose proofs have not been published anywhere.

Introduction

The cardinal invariants of the continuum characterize combinatorial properties of the structure of the real line. They are usually defined in terms of ideals on the reals. In almost all cases, these cardinals lie between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$. Some examples, relevant to this work, are defined in Section 1.2.

ZFC proves some inequalities involving cardinal invariants whose represent relations between the combinatorial concepts involved. The following well know *Cichoń's diagram* is an important and very well studied example related to category, measure and compactness of sets of irrationals. The arrows represent $\leq$ and the diagram illustrates precisely the only inequalities that can be proven in ZFC between the cardinal invariants associated with $\mathcal{N}$, *the ideal of measure zero subsets of* $\mathbb{R}$, $\mathcal{M}$, *the ideal of meagre subsets of* $\mathbb{R}$, and the ideal $\mathcal{K}$ of compact sets of irrational numbers (here, $\mathfrak{b} = \text{add}(\mathcal{K}) = \text{non}(\mathcal{K})$ and $\mathfrak{d} = \text{cov}(\mathcal{K}) = \text{non}(\mathcal{K})$). Besides, $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$ and $\text{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \text{non}(\mathcal{M})\}$. See more details in Section 1.2.



Another important example of ideals different from $\mathcal{N}$ and $\mathcal{M}$ is the ideal $\mathcal{SN}$ of strongly measure zero of subsets of $[0, 1]$. The ideal $\mathcal{SN}$ is very interesting in the sense that it cannot be proven that there is an uncountable set of strongly measure zero as Laver [L76] built a model by forcing where all the sets in $\mathcal{SN}$ are countable (i.e, Borel conjecture is consistent with ZFC).

The cardinal invariants associated with $\mathcal{SN}$ have also been studied extensively (see e.g [BJ]). Yorioka [Y02] showed the consistency of $\text{cof}(\mathcal{SN}) > \mathfrak{c}$ which was very unexpected because most cardinal invariants of the continuum are below $\mathfrak{c}$. To prove this, he introduced ideals of the form $\mathcal{I}_f$ for each increasing function f on ω, which we call *Yorioka ideals*. These ideals are subideals of the null ideal $\mathcal{N}$ and they include $\mathcal{SN}$.

Afterwards, the study of Yorioka ideals started to have an impact in set theory, specially in Japan. For example, Osuga [O06] proved the consistency of $\text{cov}(\mathcal{I}_f) < \mathfrak{c}$ and $\text{non}(\mathcal{I}_f) < \mathfrak{c}$. In 2008, Kamo and Osuga [KO08] studied the additivity and the cofinality

of $\mathcal{I}_f$ and they showed that $\mathrm{add}(\mathcal{I}_f) \leq \mathfrak{b}$ and $\mathfrak{d} \leq \mathrm{cof}(\mathcal{I}_f)$ and that no other inequality as in Figure 1.2 for $\mathcal{I}_f$ (page 8) can be proved in ZFC. Later, Kamo-Osuga [KO14] used covering numbers of the form $\mathfrak{c}^{\exists}_{b,h}$, defined by Kellner and Shelah [KeS09, KeS12], to prove the consistency of the existence of infinitely many pairwise different cardinals of the form $\mathrm{cov}(\mathcal{I}_f)$.

The purpose of this thesis is to attack the following problem:

Main Problem. *For which functions f is it consistent that* $\mathrm{add}(\mathcal{I}_f)$, $\mathrm{cov}(\mathcal{I}_f)$, $\mathrm{non}(\mathcal{I}_f)$, $\mathrm{cof}(\mathcal{I}_f)$ *are pairwise different?*

In this thesis, we solve the main problem in the positive for a large class of functions. Concretely, we construct a model of ZFC where there is a fast increasing function such that any function above it satisfies the statement. On the other hand, there are no known counterexamples, e.g., the problem is still open when f is the identity function.

We attack the Main Problem with the forcing method of matrix iterations. This method was introduced by Blass and Shelah [BlS84] and it has been refined and evolved in works such as [BrF11, M13]. The Main Problem for the ideal $\mathcal{N}$ has been solved by Mejía [M13] with a matrix iteration. The problem for $\mathcal{M}$ is still an open question, but there are partial results with three different values obtained in [M13] and [GMS16].

One of the technical tools we use in our proof is the preservation theory of Judah-Shelah [JS90] and Brendle [Br91]. We generalize this theory so that we can cover the example of preservation presented in [KO14]. These properties are important to preserve some combinatorial properties to determine the values of the respective cardinal invariants in generic extensions of posets produced finite support iterations of ccc forcing and by matrix iterations.

This work is structured as follows. The first chapter is a review of set theory of the reals, forcing, cardinal invariants, descriptive set theory and the Tukey-Galois order, which are necessary tools to understand the main results of this thesis. In the second chapter we define Yorioka ideals and show the relationship of its associated cardinal invariants with those in Cichoń's diagram. Additionally, we provide original proofs of two Claims from [O08], about the additivity and cofinality of the Yorioka ideals, which are presented without proof and whose proofs have not been published anywhere. In Chapter 3 we present a general version of the preservation theory mentioned in the previous paragraph. Chapter 4 is dedicated to prove the main result of this work. In the last chapter we present some related open questions about the main problem of the thesis and offer some forcing techniques that might be useful the solution of those questions.

Chapter 1

Preliminaries

The purpose of the Chapter to present the basic notions we need to develop the main results of this thesis. The notation used in this work is standard, see for example [Je], [Ke] and [Ku].

Given a set X and a cardinal number κ, $[X]^\kappa$ denotes the set of all subset of X of size κ. For sets X and Y, Y^X denotes the set of functions from X into Y. For an ordinal δ, X^δ denotes the set of all sequences of elements of X of length δ. Besides, $[X]^{<\kappa} = \bigcup_{\mu<\kappa}[X]^{<\mu}$ and $X^{<\delta} = \bigcup_{\alpha<\delta} X^\alpha$. For a function $f \in \omega^\omega$ define the functions f^+ and $\log f$ by

- $f^+(n) = \sum_{j\leq n} f(j)$
- $\log f(n) = \lceil \log_2 f(n) \rceil$ where $\lceil \log_2 m \rceil$ is the least $k < \omega$ such that $m \leq 2^k$.

For any set A, $\mathrm{id}_A = \{(x, x) :\in A\}$ is the *identity function on* A. In particular, id_ω is the identity function on ω.

Let $h \in \omega^\omega$. For $n < \omega$ define

$$\mathcal{S}_n(\omega, h) = \{s : n \to [\omega]^{<\omega} : \forall i < n(|s(i)| \leq h(i))\},$$

$\mathcal{S}_{<\omega}(\omega, h) = \bigcup_{n<\omega} \mathcal{S}_n(\omega, h)$ and

$$\mathcal{S}(\omega, h) = \{s : \omega \to [\omega]^{<\omega} : \forall i < \omega(|s(i)| \leq h(i))\}.$$

In general, for $h \in \omega^\omega$, $b : \omega \to \omega + 1$ and $n < \omega$, define

$$\mathcal{S}_n(b, h) = \{s : n \to [b(n)]^{<\omega} : \forall i < n(|s(i)| \leq h(i))\},$$

$\mathcal{S}_{<\omega}(b, h) = \bigcup_{n<\omega} \mathcal{S}_n(b, h)$ and

$$\mathcal{S}(b, h) = \big\{s \in \prod_{n<\omega} \mathcal{P}(b(n)) : |s(n)| \leq h(n)\big\}.$$

Then we have that $\mathcal{S}(b, h) = \mathcal{S}(\omega, h)$ and $\mathcal{S}_{<\omega}(b, h) = \mathcal{S}_{<\omega}(\omega, h)$ when b is a constant fuction with value ω.

For $f, g \in \omega^\omega$ define:

(i) the relation $f \leq^* g$ iff $\exists m \in \omega \forall n \geq m(f(n) \leq g(n))$ which we read g *(eventually) dominates* f.

(ii) $f \leq g$ iff $\forall n < \omega(f(n) \leq g(n))$.

(iii) The function $f+g \in \omega^\omega$ such that $(f+g)(n) = f(n)+g(n)$ for all $n < \omega$.

(iv) The function $f^g \in \omega^\omega$ such that $(f^g)(n) = f(n)^{g(n)}$ for all $n < \omega$.

(v) The function $f \cdot g \in \omega^\omega$ such that $(f \cdot g)(n) = f(n) \cdot g(n)$ for al $n < \omega$.

Given a formula $\varphi(x)$ in the languaje of ZFC, $\forall^\infty n < \omega\, \varphi(n)$ means that $\varphi(n)$ holds for all but finitely many $n < \omega$, and $\exists^\infty n < \omega\, \varphi(n)$ means that infinitely many $n < \omega$ satisfy $\varphi(n)$.

For $x \in \omega^\omega$ and $\psi \in ([\omega]^{<\omega})^\omega$, define the relation

$$x \in^* \psi \iff \forall^\infty i < \omega (x(i) \in \psi(i)),$$

which we read ψ *localizes* x. We also define the relation

$$x \in^\infty \psi \iff \exists^\infty n < \omega (x(n) \in \psi(n)).$$

For $b : \omega \to (\omega+1) \setminus \{0\}$ define $\mathbb{R}_b = \prod_{n<\omega} b(n)$.

Let X be a set. A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal* on X if it satisfies the following properties:

(i) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,

(ii) if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$,

(iii) $[X]^{<\omega} \subseteq \mathcal{I}$, and

(iv) $X \notin \mathcal{I}$

Let κ be an uncountable regular cardinal. An ideal $\mathcal{I}$ on a set X is said to be a κ-*ideal* if $\bigcup \mathcal{A} \in \mathcal{I}$ for any set $\mathcal{A} \subseteq \mathcal{I},\ |\mathcal{A}| < \kappa$. An $\aleph_1$-ideal is simply called σ-*ideal*.

For any collection $\mathcal{E}$ of subsets of a set X, put

$$\mathcal{E}_\sigma = \Big\{ \bigcup_n A_n : A_n \in \mathcal{E}, n \in \mathbb{N} \Big\} \text{ and}$$

$$\mathcal{E}_\delta = \Big\{ \bigcap_n A_n : A_n \in \mathcal{E}, n \in \mathbb{N} \Big\}$$

The *universal class*, or *universe*, is the class of all sets $\mathbf{V} = \{x : x = x\}$.

Now, we discuss the notions of relativization and absoluteness. A *standard model* in the language of ZF is a structure $\mathbf{M} = \langle \mathbf{M}, \in \rangle$. For any formula φ we define $\varphi^{\mathbf{M}}$, the *relativization of* φ *to* $\mathbf{M}$, by induction on φ:

(i) $(x = y)^{\mathbf{M}}$ is $x = y$.

(ii) $(x \in y)^{\mathbf{M}}$ is $x \in y$.

(iii) Given two formulas φ, ψ,

  (a) $(\neg\varphi)^{\mathbf{M}} \equiv \neg(\varphi)^{\mathbf{M}}$

  (b) $(\varphi \Rightarrow \psi)^{\mathbf{M}} \equiv (\varphi)^{\mathbf{M}} \Rightarrow (\psi)^{\mathbf{M}}$

  (c) $(\forall x \varphi^{\mathbf{M}}) \equiv \forall x \in \mathbf{M} \varphi^{\mathbf{M}}$

Also $\varphi^{\mathbf{M}}$ reads as φ *is true in* $\mathbf{M}$ or $\mathbf{M}$ *satisfies* φ, also denoted by $\mathbf{M} \models \varphi$. For a set S of sentences, $\mathbf{M}$ *is a model for* S means that each sentence in S is true in $\mathbf{M}$. In other words, a standard model interprets the language of ZF and gives a special meaning to all its formulas from its own point of view, i.e, it determines which formulas are true in $\mathbf{M}$.

Now let $\mathbf{M}, \mathbf{M_1}$ be two standard models such that $\mathbf{M} \subseteq \mathbf{M}_1$, and let φ be a formula of ZF.

(1) φ *is absolute for* $\mathbf{M}, \mathbf{M}_1$ iff

$$\forall x_1, \ldots, x_n \in \mathbf{M}(\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \Leftrightarrow \varphi^{\mathbf{M_1}}(x_1, \ldots, x_n))$$

(2) φ is absolute for $\mathbf{M}$ iff is absolute for $\mathbf{M}, \mathbf{V}$, i.e,

$$\forall x_1, \ldots, x_n \in \mathbf{M}(\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \Leftrightarrow \varphi(x_1, \ldots, x_n))$$

Absoluteness is a useful tool in the sense that it establishes when two models interpret formulas in the same way. Even more, a formula which is absolute in a model does not change its meaning when interpreted in that model. There are several statements that are absolute in all transitive standard models of ZF. For example, the notions of *function* and *real number*. However, being countable as well the notion of a cardinal are not absolute in general.

If S is a set of axioms such that $S \vdash \forall x_1, \ldots, x_n \exists! y \varphi(x_1, \ldots, x_n, y)$ we may define $F(x_1, \ldots, x_n)$ to be the unique y such that $\varphi(x_1, \ldots, x_n, y)$. Formally, expressions using F are abbreviations for expressions not using F. If $\mathbf{M}$ is a standar model of S then $\forall x_1, \ldots, x_n \in \mathbf{M} \exists! y \in \mathbf{M} \varphi^{\mathbf{M}}(x_1, \ldots, x_n, y)$. Given $x_1, \ldots, x_n \in \mathbf{M}$, define $F^{\mathbf{M}}(x_1, \ldots, x_n)$ to be the unique term $y \in \mathbf{M}$ such that $\varphi^{\mathbf{M}}(x_1, \ldots, x_n, y)$. We shall only discuss absoluteness for $\mathbf{M}, \mathbf{M}_1$ if the statement $\forall x_1, \ldots, x_n \exists! y \varphi(x_1, \ldots, x_n, y)$ is true in both models. Assuming this, F *is absolute for* $\mathbf{M}, \mathbf{M}_1$ iff, for all $x_1, \ldots, x_n \in \mathbf{M}$, $F^{\mathbf{M}}(x_1, \ldots, x_n) = F^{\mathbf{M}_1}(x_1, \ldots, x_n)$. Some example of absolute terms for transitive models of ZFC are $X \cup Y$, $\bigcap X$ and $\{X, Y\}$.

1.1 Category and measure

In this section we introduce several basic notions of descriptive set theory, Baire category and measure in Polish spaces. We also discuss interpretation of Polish spaces in transitive models of ZFC. For details of the contents of this section, see [Ke], [Ku84] and [Je].

Recall that a topological space X is a *Polish space* if X is completely metrizable and separable. Examples of Polish spaces are the real line $\mathbb{R}$ with the usual topology and the space $\mathbb{R}_b$ for $b : \omega \to (\omega + 1) \setminus \{0\}$ with the product topology when $b(n)$ is discrete. The same topology is obteined by the metric defined as $d(x, y) = 2^{-\bar{m}(x,y)}$ where $\bar{m}(x, y) = \min\{n < \omega : x(n) \neq y(n)\}$ (if $x = y$ then $\bar{m}(x, y) = \infty$, hence $d(x, y) = 2^{-\infty} = 0$). Note that $\mathbb{Q}_b = \{x \in \mathbb{R}_b : \exists N < \omega \forall n \geq N(x(n) = 0)\}$ is a dense countable subset of $\mathbb{R}_b$. Even more, if $\mathrm{Seq}_{<\omega}(b) = \bigcup_{n<\omega} \mathrm{Seq}_n(b)$ where $\mathrm{Seq}_n(b) = \prod_{k<n} b(k)$, we get that $\{[s] : s \in \mathrm{Seq}_{<\omega}(b)\}$ is a countable base of $\mathbb{R}_b$ where each $[s]$ is a clopen set, that is, open and closed. In practice, an uncountable Polish space is thought as of a space of reals. For example, 2^ω is homeomorphic to the Cantor ternary set, and the Baire space ω^ω is homeomorphic to the subspace of the irrational numbers in $(0, 1)$. Hereafter, we refer as *reals* to the members of any uncountable Polish space.

Closed sets in $\mathbb{R}_b$ are represented by *trees*: $T \subseteq \mathrm{Seq}_{<\omega}(b)$ is a tree if it is downwards closed under $\subseteq$, i.e. for all $s, t \in \mathrm{Seq}_{<\omega}(b)$, if $t \in T$ and $s \subseteq t$, then $s \in T$. For a tree T the set of its branches $[T] = \{f \in \mathbb{R}_b : \forall n < \omega (f \restriction n \in T)\}$ is closed. Moreover, if $C \subseteq \mathbb{R}_b$ is closed, there is a tree $T = T_C$ such that $C = [T]$.

Let X be a Topological space . A family $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra* if

(1) $X \in \mathcal{A}$,

(2) if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$, and

(3) if $A_n \in \mathcal{A}$ for all $n \in \omega$, then $\bigcup_{n\in\omega} A_n \in \mathcal{A}$.

We define *the Borel σ-algebra* $\mathcal{B}(X)$ as the smallest σ-algebra containing all open subsets of X.

Say that $A \subseteq X$ is *nowhere dense* (nwd) if $\mathrm{int}(\bar{A}) = 0$. A is *meager* if it is of the form $A = \bigcup_{n\in\mathbb{N}} F_n$ where each F_n is a nwd set. Let $\mathcal{M}(X)$ be the σ-ideal of meager subsets of X. When the space is clear from the context, we just write $\mathcal{M}$ to denote the ideal.

For a σ-algebra $\mathcal{A}$, a function $\mu : \mathcal{A} \to [0, +\infty]$ is a *measure* if it satisfies the following properties:

(1) $\mu(\varnothing) = 0$

(2) $\mu(\bigcup_{n<\omega} A_n) = \sum_{n<\omega} \mu(A_n)$ for any pairwise disjoint family $\{A_n : n < \omega\} \subseteq \mathcal{A}$

Denote by $\mathcal{N}(X, \mathcal{A}, \mu)$ the σ-ideal of μ-null subsets of X. We say that $N \subseteq X$ is *μ-null* if there is a $B \in \mathcal{A}$ such that $N \subseteq B$ and $\mu(B) = 0$. When the space and the measure are clear from the context, we just write $\mathcal{N}$. We say that μ is a *probability measurable* if it is a measura such that $\mu(X) = 1$. A measure μ is *σ-finite* if $X = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n has finite measure. A measure μ is *continuous* if $\mu(\{x\}) = 0$ for all $x \in X$.

Given $0 < \eta \leq \omega$. Define the measure $\mu_\eta : \mathcal{P}(\eta) \to [0, 1]$ as follows:

(i) if $\eta < \omega$, μ_η is the measure such that, for $k < \eta$, $\mu_\eta(\{k\}) = \frac{1}{\eta}$, and

(ii) if $\eta = \omega$, μ_ω is the measure such that, for $k < \eta$, $\mu_\omega(\{k\}) = \frac{1}{2^{k+1}}$.

The *Borel measure in* $\mathbb{R}_b$ for $b : \omega \to (\omega + 1) \setminus \{0\}$ is the product measure $\mu_b = \prod_{n<\omega} \mu_{b(n)} : \mathcal{B}(\mathbb{R}_b) \to [0, 1]$. Concretely, it is the unique measure on $\mathcal{B}(\mathbb{R}_b)$ such that, for any $s \in \mathrm{Seq}_{<\omega}(b)$, $\mu([b]) = \prod_{i<|s|} \mu_{b(i)}(\{s(i)\})$.

For ordinals $1 \leq \alpha < \omega_1$, we define the *Borel hierarchy* of Borel subsets of a topological space X by transfinite induction:

$$\Sigma^0_1(X) = \{U \subseteq X : U \text{ open}\},$$

$$\Pi^0_1(X) = \{F \subseteq X : F \text{ closed}\};$$

for $1 < \alpha \leq \omega_1$,

$$\Sigma^0_\alpha(X) = \Big(\bigcup_{\xi<\alpha} \Pi^0_\xi(X)\Big)_\sigma$$

and

$$\Pi^0_\alpha(X) = \Big(\bigcup_{\xi<\alpha} \Sigma^0_\xi(X)\Big)_\delta$$

For every $1 \leq \alpha < \omega_1$, put

$$\Delta^0_\alpha(X) = \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X).$$

Note that $\Delta^0_1(X)$ is the family of all *clopen subsets of* X, Σ^0_2 is the set of all F_σ *subsets of* X, and Π^0_2 is set of all G_δ *sets in* X. Sets in Σ^0_3 are also called $G_{\delta\sigma}$ *sets* and those in Π^0_3 are called $F_{\delta\sigma}$ *sets*, etc. Even more, for $1 \leq \alpha < \beta < \omega_1$, $\Sigma^0_\alpha(X) \cup \Pi^0_\alpha(X) \subseteq \Delta^0_\beta(X)$. Note that $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha(X)$.

Let X be a Polish space and $\pi_0 : X \times \omega^\omega \to X$ the projection onto the first coordinate, that is, $\pi_0(x, y) = x$. We say that a set A is *analytic* (*in the space* X) if $A = \pi[F]$ for some $F \subseteq X \times \omega^\omega$ closed. Equivalently, A is analytic in X iff either $A = \emptyset$ or there exists a continuous fuction $f : \omega^\omega \to A$ such that $A = f[\omega^\omega]$. The class of analytic subsets of X is denoted by $\Sigma^1_1(X)$. We define $\Pi^1_1(X) = \{X \setminus A : A \in \Sigma^1_1(X)\}$ whose members are called *co-analytic in* X. Note that, $\Delta^1_1(X) = \Sigma^1_1(X) \cap \Pi^1_1(X) = \mathcal{B}(X)$.

We often abuse the language and say that, for Polish spaces $X_1, \ldots, X_n$, a statement $\varphi(x_1, \cdots, x_n)$ is Σ^1_1 (of Π^1_1, or Borel, etc.) in $X_1 \times \ldots \times X_n$ when the set $\{(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n : \varphi(x_1, \ldots, x_n)\}$ is in $\Sigma^1_1(X_1 \times \ldots \times X_n)$ (respectively Π^1_1, Borel, etc.).

Every Polish space is determined by the completion of a countable metric space, a fact that allows us to interpret Polish spaces in models of ZFC. Fix $M \subseteq N$ transitive models of ZFC. If $\langle X, d\rangle$ is a Polish space and $D \subseteq X$ is dense and countable, given that X is the completion of $\langle D, d_0 = d \restriction (D \times D)\rangle$, if $\langle D, d_0\rangle \in M$ then the *interpretation of* X *in* M, denoted by X^M, is the object in M such that $M \models$ "X^M is the completion of $\langle D, d_0\rangle$" (we recall that the completion of a metric space is a inique modulo isometry). Furthermore, $N \models$ "X^N is the completion of X^M". For example, $\mathbb{R}$ is the completion of $\mathbb{Q}$ with the usual metric and $\mathbb{R}^M = \mathbb{R} \cap M$; given $b : \omega \to (\omega + 1) \setminus \{0\}$, $\mathbb{R}_b$ is the completion of $\mathbb{Q}_b$ with metric $d_b \restriction (\mathbb{Q}_b \times \mathbb{Q}_b)$ and $\mathbb{R}^M_b = \mathbb{R}_b \cap M$. In general, if we see X defined as the completion of $\langle D, d_0\rangle$ by Cauchy sequences, we can think of $X^M = X \cap M$.

Continuous functions between Polish spaces can also be interpreted in models of ZFC. Recall that, if $f_0 : \langle D, d_0\rangle \to \langle D', d_0'\rangle$ is a *Cauchy-continuous* function (i.e. preserves Cauchy sequences) then there exists a unique continuous function f from the completion of $\langle D, d_0\rangle$ into the completion of $\langle D', d_0'\rangle$ extending f_0. Conversely, given $f : X \to Y$ continuous between two Polish spaces and D, D' countable dense subsets of X, Y, respectively, then $f_0 = f \restriction D : D \to D^*$ is Cauchy-continuous where $D^* := D' \cup f[D]$ is dense in Y and countable. The function f_0 between countable metric spaces is usefull to interpret f. If $f_0, D^* \in M$, the *interpretation of* f *in* M, denoted by f^M, is the object of M such that $M \models$ "$f : X^M \to Y^M$ is the unique continuous function extending f_0". For example. if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \pi x$, it is the unique continuous extension of $f_0 = f \restriction \mathbb{Q} : \mathbb{Q} \to D^*$ where $D^* = \mathbb{Q} \cup \{\pi q : q \in \mathbb{Q}\}$ and $f^M : \mathbb{R} \cap M \to \mathbb{R} \cap M$ is just the restriction of f to $\mathbb{R} \cap M$.

Through the interpretation of continuous functions in models of ZFC we may also interpret analytic subsets of a Polish space. If $A \subseteq X$ is non-empty and analytic, then there is a continuous function $f : \omega^\omega \to X$ such that $f[\omega^\omega] = A$, so A^M, *the interpretation of* A *in* M, is $A^M = f^M[\omega^\omega \cap M]$ (note that $(\omega^\omega)^M = \omega^\omega \cap M$). If $A = \emptyset$, we define $A^M = \emptyset^M = \emptyset$

1.1 Theorem (Mostowski absoluteness theorem [Je, Thm. 25.4])**.** *Let $f_0 : \mathbb{Q}_{b^*} \to D^*$ be a Cauchy-continuous function between countable metric spaces where $b^* : \omega \to \omega+1$ is the constant function ω and X is the completion of D^*. If M is a transitive model of ZFC and $f_0, D^* \in M$, then $f^M[\omega^\omega \cup M] = f[\omega^\omega] \cup M$ where $f : \omega^\omega \to X$ is the unique continuous function that extends f_0. In other words, if $A \subseteq X$ is analytic, then $A^M = A \cap M$.*

In other words, if $\varphi(x)$ is an analytic statement in a Polish space X, then the theorem above implies that "$x \in X$ and $\varphi(x)$" is absolute for transitive models. Consequently, the co-analytic sets are also absolute. If $C \subseteq X$ is co-analytic with complement A, $C^M := X^M \setminus A^M$. Therefore, $C^M = C \cap M$. Likewise, if $\varphi(x)$ is a co-analytic statement in X, "$x \in X$ and $\varphi(x)$" is absolute for transitive models of ZFC. For example, it can be shown that the statement $\mathrm{Pd}(d, d') \equiv$ "$\langle \omega, d\rangle$ and $\langle \omega, d'\rangle$ are metric spaces with isometrically isomorphic completions" is analytic in $(\mathbb{R}^{\omega\times\omega})^2$, therefore absolute, which implies that the interpretation of a Polish space in a model does not depend on the countable dense subset that is chosen. The statement $\mathrm{Cc}(f_0, d, d') \equiv$ "$f_0 : \langle \omega, d\rangle \to \langle \omega, d'\rangle$ is Cauchy-continuous between metric spaces" is co-anlytic in $\omega^\omega \times (\mathbb{R}^{\omega\times\omega})^2$ and therefore absolute, and the statement "$\mathrm{Pd}(d_0, d_1), \mathrm{Pd}(d_0', d_1'), \mathrm{Cc}(f_0, d_0, d_0'), \mathrm{Cc}(f_0, d_1, d_1')$ and f_0, f_1 extend to the same continuous function between their completions" is a conjunction between a co-analytic and an analytic statement in $(\omega^\omega)^2 \times (\mathbb{R}^{\omega\times\omega})^4$ so the interpretation of a continuous function between Polish spaces does not depend on the function between the denses that are chosen to interpret.

1.2 Remark. When we work in a transitive model M of ZFC, we are going to omit the superindexes M for the interpretation of Polish spaces, countinuous functions and analytic and co-analytic sets in M. For example, if $\sqsubset$ is analytic in the Polish space $X \times Z$ and it can be defined in M, when we are working in M we write "$x, z \in X$ and $x \sqsubset y$" instead of "$x, y \in X^M$ and $x \sqsubset^M y$" because Theorem 1.1 guarantees that there is no place for ambiguity between both formulas.

Now we present a brief description of the *Borel coding*, which is a way to code the construction of Borel sets by real numbers. It can be defined a Π^1_1 set $\mathrm{BC} \subseteq 2^\omega$ such that each $c \in \mathrm{BC}$ indicates how a Borel set is constructed in a Polish space. Given a Polish space X, it has an onto function $\mathrm{Val}_X : \mathrm{BC} \to \mathcal{B}(X)$. Therefore, we say that $c \in \mathrm{BC}$ is a *Borel code of* $\mathrm{Val}_X(c)$. The function Val_X is absolute in the sense that, if M is a transitive model of ZFC and $c \in \mathrm{BC} \cap M$, then $[\mathrm{Val}_X(c)]^M = \mathrm{Val}_X(c) \cap M$.

For a Polish space X, there are subsets S_X and P_X of $X \times 2^\omega$ such that $S_X \in \Sigma^1_1$, $P_X \in \Pi^1_1$ and for all $x \in X$ and $c \in \mathrm{BC}$, $(x, c) \in S_X$ iff $(x, c) \in P_X$ iff $x \in \mathrm{Val}_X(c)$. Therefore, the statement "$c \in \mathrm{BC}$ and $x \in \mathrm{Val}_X(c)$" is absolute because it is equivalent to a Π^1_1 statement. Similarly, statements such as "$d, c \in \mathrm{BC}$ and $\mathrm{Val}_X(c) \subseteq \mathrm{Val}_X(d)$" and "$c \in \mathrm{BC}$, $\{\mathrm{Val}_X(c_n) : n < \omega\} \subseteq \mathrm{BC}$ and $\mathrm{Val}_X(c) = \bigcup_{n<\omega} c_n$" are Π^1_1. Hence, they are absolute for transitive models of ZFC. Even more, it can be proved that "$c \in \mathrm{BC}$ and $\mathrm{Val}_X(c) \in \mathcal{M}$" and "$c \in \mathrm{BC}$, $z \in [0, 1]$ and $\mu_b(\mathrm{Val}_{\mathbb{R}_b}(c)) = z$" are absolute statements for transitive models of ZFC.

1.2 Cardinal invariants

In this section, we present some classical cardinal invariants of the continuum that are relevant in this thesis. For the proofs of all the results of this section, see [BJ] and [B10].

For any family $\mathcal{I}$ of subsets of X, define the *cardinal invariants associated with* $\mathcal{I}$ as:

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) =& \{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \,\&\, \bigcup \mathcal{J} \notin \mathcal{I}\} \;\; \textit{the additivity of } \mathcal{I}.\\
\mathrm{cov}(\mathcal{I}) =& \{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \,\&\, \bigcup \mathcal{J} = X\} \;\; \textit{the covering of } \mathcal{I}.\\
\mathrm{non}(\mathcal{I}) =& \{|A| : A \subseteq \omega^\omega \,\&\, A \notin \mathcal{I}\} \;\; \textit{the uniformity of } \mathcal{I}.\\
\mathrm{cof}(\mathcal{I}) =& \{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \,\&\, \forall A \in \mathcal{I}\, \exists B \in \mathcal{J}(A \subseteq B)\} \;\; \textit{the cofinality of } \mathcal{I}.
\end{aligned}$$

If $\mathcal{I}$ is a ideal, $\mathrm{add}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$ and $\mathrm{add}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$ as in Figure 1.1. Also notice that $\mathrm{add}(\mathcal{I})$ is a regular cardinal.

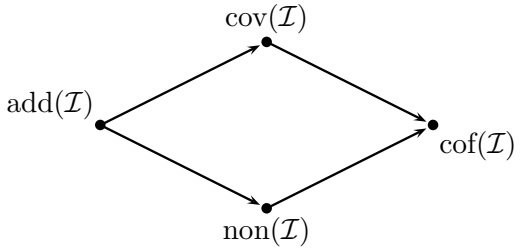


Figure 1.1: Diagram of provable inequalities in ZFC between the cardinal invariants associated with $\mathcal{I}$.

Bounding and dominating numbers

A family of functions $\mathcal{F} \subseteq \omega^\omega$ is called *bounded* if it is eventually dominated by a single funtion, i.e there is a $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in \mathcal{F}$. Otherwise $\mathcal{F}$ is *unbounded*. On the other hand, $\mathcal{F}$ is *dominating* if every $g \in \omega^\omega$ is eventually dominated by a member of $\mathcal{F}$.

Define the cardinal numbers $\mathfrak{b}$ and $\mathfrak{d}$ as follows:

$$\mathfrak{b} = \min\{|F| \; F \subseteq \omega^\omega \;\&\; \forall g \in \omega^\omega \exists f \in F(f \not\leq^* g)\} \;\; \textit{the (un)bounding number}$$

and

$$\mathfrak{d} = \min\{|F| \; F \subseteq \omega^\omega \;\&\; \forall g \in \omega^\omega \exists f \in F(g \leq^* f)\} \;\; \textit{the dominating number.}$$

In other words, $\mathfrak{b}$ is the size smallest a $\leq^*$-unbounded family in ω^ω and $\mathfrak{d}$ is the size smallest a dominating family. Moreover,

$$\mathfrak{b} = \mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K}) \text{ and } \mathfrak{d} = \mathrm{cof}(\mathcal{K}) = \mathrm{cov}(\mathcal{K})$$

where $\mathcal{K} = \{X \subseteq \omega^\omega : \exists K \subseteq \omega^\omega \sigma\text{-compact}(X \subseteq K)\}$.

The relationship between the notions of category, measure and compactness in the irrationals are illustrated in the so called *Cichoń's diagram* in Figure 1.2 where the arrows indicate $\leq$ provable in ZFC. Also, the dotted lines represent $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$. The inequalities between cardinals exhibited in this diagram were proved by Bartozynski, Raisonnier, Miller, Truss and Stern.

The following theorem shows that cardinals associated with $\mathcal{M}$ and $\mathcal{N}$ do not depend on the representing space.

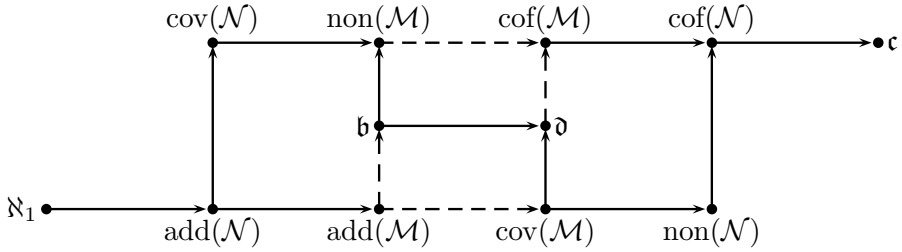


Figure 1.2: Cichoń's diagram

1.3 Theorem ([Ke]). *Let X, Y be uncountable Polish spaces.*

(a) $\mathrm{add}(\mathcal{M}(X)) = \mathrm{add}(\mathcal{M}(Y))$ *and likewise for* $\mathrm{cov}, \mathrm{non}$ *and* cof.

(b) If μ and ν are σ-finite continuous non-zero measures on $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively, then $\mathrm{add}(\mathcal{N}(X,\mu)) = \mathrm{add}(\mathcal{N}(Y,\nu))$ *and likewise for* $\mathrm{cov}, \mathrm{non}$ *and* cof.

The cardinals $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ can be characterized as follows.

1.4 Theorem ([BJ, Thm. 2.3.9]). *Let $h \in \omega^\omega$ monotone increasing that goes to infinity. Then,*

(a) $\mathrm{add}(\mathcal{N}) = \min\{|F| : F \subseteq \omega^\omega \ \&\ \forall \varphi \in S(\omega,h) \exists f \in F \exists^\infty n < \omega(f(n) \notin \varphi(n))\}$,

(b) $\mathrm{cof}(\mathcal{N}) = \min\{|G| : G \subseteq S(\omega,h) \ \&\ \forall f \in \omega^\omega \exists \varphi \in G(f \notin^* \varphi)\}$.

The following is a well known characterization of covering and uniformity of category.

1.5 Theorem ([BJ, Thm. 2.4.1 and 2.4.7]). *(a)* $\mathrm{non}(\mathcal{M}) = \min\{|J| : J \subseteq \omega^\omega \ \&\ \forall x \in \omega^\omega \exists y \in J \exists^\infty n < \omega(x(n) = y(n))\}$.

(b) $\mathrm{cov}(\mathcal{M}) = \min\{|I| : I \subseteq \omega^\omega \ \&\ \forall x \in \omega^\omega \exists y \in I \forall^\infty n < \omega(y(n) \neq x(n))\}$.

1.3 Forcing

Forcing is a technique to extend a transitive model V of ZFC to other transitive model $V[G]$ of ZFC through a *generic object* G. This generic object is, in practice, a new subset of some partially ordered set $\mathbb{P}$ in V, known as a *forcing notion*. This procedure has the following desirable properties:

(I) $V[G]$ is also a transitive model of ZFC and it is the minimal transitive model of ZFC which contains $V \cup \{G\}$. Besides, V and $V[G]$ contain the same ordinals.

(II) The truth of mathematical statements in $V[G]$ can be determined by a formalism within V, known as the *forcing relation*, which is completely specified by $\mathbb{P}$.

1.6 Definition (Forcing Notion). A *forcing notion* or *poset* is a set $\mathbb{P}$ equipped with a reflexive and transitive relation $\leq_{\mathbb{P}}$ which contains a greatest element $\mathbb{1}_{\mathbb{P}}$. When $\mathbb{P}$ is clear from the context, the subscripts are usually suppressed.

Elements of a forcing notion $\mathbb{P}$ are often referred to as *conditions* and are regarded as being approximations of the generic object. For two conditions $p, q \in \mathbb{P}$ we say that p *stronger that* q if $p \leq q$, which means, intuitively, that p carries more information that q, that is, it provides a better approximation of the generic object. We say that p, q are *compatible* iff there is an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$, which is denoted by $p||q$. Otherwise, they are *incompatible* and denoted by $p \perp q$.

By Gödel's Second Incompleteness Theorem, it is not posible to construct a model of ZFC within ZFC, but it is possible to build transitive countable models of finite fragments of ZFC as large as necessary to carry out the arguments in question, justify absoluteness and to prove consistency results. For this reason, we abuse the language from now on and say "model of ZFC" to refer to a transitive model of of a large enough finite fragment of ZFC.

1.7 Definition. Suppose that $D \subseteq \mathbb{P}$.

(i) D is *dense* if
$$\forall p \in \mathbb{P} \exists q \in D(q \leq p).$$

(ii) D is an *antichain* if is consists of incompatible elements. D is a *maximal antichain* if it is an antichain and each element of $\mathbb{P}$ is compatible with some element of D.

(iii) $G \subseteq \mathbb{P}$ is a *filter* if:

- $\mathbb{1}_{\mathbb{P}} \in G$,
- for each $p, q \in G$ there is an $r \in G$ such that $r \leq p$ and $r \leq q$, and
- for each $p \in G$ and $q \in \mathbb{P}$, if $p \leq q$ then $q \in G$.

(iv) Let V be a transitive model of ZFC. A filter G is said to be $\mathbb{P}$-*generic over* V iff, for every dense subset $D \in V$ of $\mathbb{P}$, $D \cap G \neq 0$.

The existence of a generic filter is not true in general, but it can be guaranteed for countable transitive models of ZFC.

1.8 Theorem. *Let V be a transitive countable model of* ZFC *and let $\mathbb{P} \in V$ be a forcing notion. Then, there exists a $\mathbb{P}$-generic filter over V.*

If V is a transitive model of ZFC and $\mathbb{P} \in V$ is *atomless*, i.e. below each $p \in \mathbb{P}$ there are two incompatible elements, then any $\mathbb{P}$-generic filter G over V is outside V. V can be extended to $V[G]$ through the generic filter G. In this context, V is called the *ground model*, and $V[G]$ a *generic forcing extension of* V.

Consider the *forcing trivial* $\mathbb{1} = (\{\emptyset\}, \subseteq)$. A forcing notion $\mathbb{P}$ is *non-trivial* if $\exists p, q \in \mathbb{P}(p \perp q)$, otherwise we say it is trivial. Note that a $\mathbb{P}$ is trivial iff $\mathbb{P} \simeq \mathbb{1}$(see Definition 1.22). Even more, if $\mathbb{P}$ is trivial then any $\mathbb{P}$-generic G is in V and $V[G] = V$.

We briefly present the construction of $V[G]$. People living in V are not able, in general, to know what an element in $V[G]$ is since it depends on the choice of the generic G. However, what people in V have are *names (or codes)* for all the elements of $V[G]$. Formally, a $\mathbb{P}$-*name* can be defined recursively as follows:

τ is a $\mathbb{P}$-name iff it is a relation and, for every $(\sigma, p) \in \tau, \sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

The class of $\mathbb{P}$-names in V is denoted by $V^{\mathbb{P}}$. The idea is that $\mathbb{P}$-names codify the objects of $V[G]$ and G is required for decoding these codes.

1.9 Definition. The *evaluation of τ via G* is denoted by τ_G and it is defined by recursion as:

$$\tau_G = \{\sigma_G : \exists p \in G((\sigma, p) \in \tau), \}$$

Define

$$V[G] = \{\tau_G : \tau \in V^{\mathbb{P}}\}.$$

The following theorem shows that $V[G]$ is a model for set theory.

1.10 Theorem (The Generic Model Theorem)**.** *Let V be a transitive model of ZFC. Let $\mathbb{P} \in V$ be a poset. If $G \subseteq \mathbb{P}$ is a generic filter over V then*

(a) $V[G]$ is a transitive model of ZFC,

(b) $V \subseteq V[G]$ and $G \in V[G]$,

(c) V and $V[G]$ have the same ordinals, and

(d) if N is a transitive model of ZFC such that $V \subseteq N$ and $G \in N$, then $V[G] \subseteq N$.

The model V has also access to the semantics of $V[G]$ through the forcing relation $\Vdash$.

1.11 Definition. Let $p \in \mathbb{P}$, $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$ and $\varphi(x_1, \ldots, x_n)$ be a formula in the language of ZFC. Say that p *forces* $\varphi(\tau_1, \ldots, \tau_n)$, denoted by $p \Vdash^V_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$, when

$$V[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G)$$

for every $\mathbb{P}$-generic filter G over V such that $p \in G$.

The forcing relation $p \Vdash^V_{\mathbb{P}} \varphi(\tau_1, \cdots, \tau_n)$ is definable in V for each formula $\varphi(x_1, \ldots, x_n)$ in the language of ZFC when V is a countable transitive model of ZFC. Also, $\Vdash^V_{\mathbb{P}} \varphi(\tau \cdots, \tau_n)$ means that $\mathbb{1}_{\mathbb{P}} \Vdash^V_{\mathbb{P}} \varphi(\tau \cdots, \tau_n)$. Usually, in $\Vdash^V_{\mathbb{P}}$ both V and $\mathbb{P}$ are omitted when there is not confusion.

1.12 Lemma (The truth lemma)**.** *Let V be a transitive model of ZFC and $\mathbb{P} \in V$ a forcing notion. Let G be $\mathbb{P}$-generic over V. Then, for any formula $\varphi(x_1, \ldots, \varphi_n)$ of the language of ZFC and for any $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$, $V[G] \models \varphi((\tau_1)_G, \ldots, (\tau_n)_G)$ iff there is some $p \in G$ such that $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$.*

1.13 Theorem (Maximal Principle)**.** *Let $\mathbb{P} \in V$ be a forcing notion, $p \in \mathbb{P}$ and $\varphi(x)$ a formula in the languaje of ZFC. Then,*

(a) $p \Vdash \exists x \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for some $\tau \in V^{\mathbb{P}}$.

(b) If $\tau \in V^{\mathbb{P}}$ and $p \Vdash \varphi(\tau)$, then there exists a $\sigma \in V^{\mathbb{P}}$ such that

(i) $p \Vdash \tau = \sigma$ and

(ii) $\Vdash$ "$\varphi(\sigma) \Leftrightarrow \exists x \varphi(x)$".

1.14 Definition. For a forcing poset $\mathbb{P}$ and any $x \in V$, define the $\mathbb{P}$-name $\check{x} = \{(\check{y}, \mathbb{1}) : y \in x\}$. "$\check{x}$" is read "*$x$ check*" and it is know as a *the canonical name for x*. Note that, for any filter G, $(\check{x})_G = x$. This is reason why $V \subseteq V[G]$. Usually, the check is omitted when there is no confusion.

Define $\dot{G} = \{(\check{p}, p) : p \in \mathbb{P}\}$ *the canonical name* of the $\mathbb{P}$-generic set. Besides, $(\dot{G})_G = G$ so $G \in V[G]$.

For $\tau, \sigma \in V^{\mathbb{P}}$, define

(i) $\mathrm{up}(\sigma,\tau)=\{(\sigma,\mathbb{1}),(\tau,\mathbb{1})\}$ and

(ii) $\mathrm{op}(\sigma,\tau)=\mathrm{up}(\mathrm{up}(\sigma,\sigma),\mathrm{up}(\sigma,\tau))$.

Even more, $\Vdash \mathrm{up}(\sigma,\tau)=\{\sigma,\tau\}$ and $\Vdash \mathrm{op}(\sigma,\tau)=(\sigma,\tau)$.

1.15 Definition. Let $\mathbb{P}$ be a poset, B and D sets. Let A be an antichain in $\mathbb{P}$ and $h: A\to V^{\mathbb{P}}$. Define $\tau\langle A,h\rangle=\bigcup_{q\in A}\{(\pi,r)\in \mathrm{dom}(h(q))\times q{\downarrow}\colon r\Vdash \pi\in h(q)\}$ where $q{\downarrow}:=\{r: r\leq q\}$.

(a) We say that a $\mathbb{P}$-name $\dot{b}$ is a *nice name for a member of* B if $\dot{b}$ has the form $\tau\langle A,h\rangle$, where A is a maximal antichain and $h: A\to \mathrm{dom}(\check{B})$. We often abuse the language and say that $h: A\to B$.

(b) If $\dot{f}=\{(\mathrm{op}(d,\tau\langle A_d,h_h\rangle),\mathbb{1}): d\in D\}$ where $\tau\langle A_d,h_h\rangle$ is a nice name for a member of B for all $d\in D$, then we say $\dot{f}$ is a *nice name for a function from* D *into* B.

In the notation of the previous definition, it is known that $p\Vdash \tau\langle A,h\rangle=h(p)$ for every $p\in\mathbb{P}$.

1.16 Lemma. *Let $\mathbb{P}$ be a poset, B and D sets.*

(i) If $\dot{b}$ is a nice name for a member of B then $\Vdash \dot{b}\in B$.

(ii) If $p\in\mathbb{P}$ and τ is a $\mathbb{P}$-name such that $p\Vdash \tau\in B$, then there is a nice name $\dot{b}$ for an object in B such that $p\Vdash \tau=\dot{b}$.

(i) If $\dot{f}$ is a nice name for a function B^D, then $\Vdash \dot{f}: D\to B$.

(iii) If $p\in\mathbb{P}$ and σ is a $\mathbb{P}$-name such that $p\Vdash \sigma: D\to B$, then there is a nice name $\dot{f}$ for a function from D into B such that $p\Vdash \sigma=\dot{f}$.

When extending from V to $V[G]$ it is possible that a cardinal in V is no longer a cardinal in $V[G]$. The following definition provides sufficient conditions to avoid the previous situation.

1.17 Definition. Let $\mathbb{P}$ be a forcing notion and κ an infinite cardinal. We say that:

(1) For $n<\omega$, $B\subseteq\mathbb{P}$ is *n-linked* if, for every $F\subseteq B$ of size $\leq n$, $\exists p\in\mathbb{P}\forall q\in F(p\leq q)$.

(2) $C\subseteq\mathbb{P}$ is *centered* if it is n-linked for every $n<\omega$.

(3) $\mathbb{P}$ is *κ-linked* if $\mathbb{P}=\bigcup_{\alpha<\kappa}P_\alpha$ where each P_α is 2-linked. When $\kappa=\omega$, we say that $\mathbb{P}$ is *σ-linked*.

(4) $\mathbb{P}$ is *κ-centered* if $\mathbb{P}=\bigcup_{\alpha<\kappa}P_\alpha$ where each P_α is centered. When $\kappa=\omega$, we say that $\mathbb{P}$ is *σ-centered*.

(5) $\mathbb{P}$ has *κ-cc (the κ-chain condition)* if every antichain in $\mathbb{P}$ has size $<\kappa$, $\mathbb{P}$ has ccc (*the countable chain contidion*) if it has $\aleph_1$-cc.

Any κ-centered poset is κ-linked. Even more, any κ-linked poset has κ^+-cc. It is well known that, if κ is an uncountable regular cardinal and a poset $\mathbb{P}$ has κ-cc, then cofinalities and cardinals $\geq\kappa$ are preserved from V to any $\mathbb{P}$-generic extension. Therefore, ccc posets preserve all the cardinals and cofinalities from the ground model.

1.18 Lemma ([KO14, Lemma 7]). *Let $n < \omega$ and $B \subseteq \mathbb{P}$ be n-linked. If $F \in V$ has size $\leq n$ and $\dot{a}$ is a $\mathbb{P}$-name for a member of F, then there exists a $c \in F$ such that no $p \in B$ forces $\dot{a} \neq c$.*

The following generalization of σ-linked is fundamental in this work.

1.19 Definition ([KO14]). Let $h, \pi \in \omega^\omega$. A forcing notion $\mathbb{P}$ is (h, π)-*linked* if there exists a sequence $\langle Q_{n,j} : n < \omega \text{ and } j < h(n)\rangle$ of subsets of $\mathbb{P}$ such that

(1) $Q_{n,j}$ is $\pi(n)$-linked for all $n < \omega$ and $j < h(n)$, and

(2) $\forall p \in \mathbb{P} \forall^\infty n \exists j < h(n) \exists p' \in Q_{n,j}(p' \leq p)$

1.20 Lemma. *If $\mathbb{P}$ is σ-centered, then $\mathbb{P}$ is (h, π)-linked when h is a monotone increasing function that goes to $+\infty$.*

Proof. Suppose that $\mathbb{P} = \bigcup_{n<\omega} P_n$ where each P_n is centered. For each $n \in \omega$, define $Q_{n,j} = P_j$ for $j < h(n)$. Since h is a monotone increasing function, $\langle Q_{n,j} : n < \omega \text{ and } j < h(n)\rangle$ satisfies (1) and (2) of Definition 1.19. □

1.21 Lemma ([KO14, Lemma 6]). *If $\mathbb{P}$ is (h, π)-linked and $\pi \geq^* 2$, then $\mathbb{P}$ is σ-linked.*

We say that $i : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if is satisfies,

(i) $\forall p, p' \in \mathbb{P}(p \leq p' \Rightarrow i(p) \leq i(p'))$,

(ii) $\forall p, p' \in \mathbb{P}(p \perp p' \Rightarrow i(p) \perp i(p'))$, and

(iii) $i[\mathbb{P}]$ is dense in $\mathbb{Q}$.

Say that a complete Boolean algebra $\mathbb{B}$ is a *completion* of a poset $\mathbb{P}$ if there is a dense embedding from $\mathbb{P}$ into $\mathbb{B}$. A completation always exists and it is unique modulo isomorphism.

1.22 Definition. Given two forcing notions $\mathbb{P}, \mathbb{Q}$, we say that:

(i) $\mathbb{P} \lessdot \mathbb{Q}$ if $\mathbb{A}$ is completely embedded into $\mathbb{B}$, when $\mathbb{A}$ and $\mathbb{B}$ are the completations of $\mathbb{P}$ and $\mathbb{Q}$ respectively.

(ii) $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent*, denoted by $\mathbb{P} \simeq \mathbb{Q}$, if the completations of $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic.

If $\mathbb{P} \lessdot \mathbb{Q}$ then there is a transformation i from the family of $\mathbb{Q}$-generic sets over V into the family of $\mathbb{P}$-generic sets over V such that $V[i(G)] \subseteq V[G]$; $\mathbb{P} \simeq \mathbb{Q}$ implies that there is a bijection i between both families of generic sets such that $V[G] = V[i(G)]$. Even more, $\mathbb{P} \lessdot \mathbb{Q}$, implies that there exists a function $i_* : V^{\mathbb{P}} \to V^{\mathbb{Q}}$ such that, if $\varphi(x_1, \ldots, x_n)$ is a formula in the language of ZFC that is absolute for transitive models of ZFC, then $\Vdash_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$ iff $\Vdash_{\mathbb{Q}} \varphi(i_*(\tau_1), \ldots, i_*(\tau_n))$ for all $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$. Also $i_*(\check{x}) = \check{x}$ for all $x \in V$ (the check with respect to the corresponding poset). In the case $\mathbb{P} \simeq \mathbb{Q}$, i_* is onto and the previous equivalence holds for any formula $\varphi(x_1, \ldots, x_n)$. For this reason, equivalent posets force the same sentences.

Say that a poset $\mathbb{P}$ is *homogeneous* if $\mathbb{P} \simeq q{\downarrow}$ for all $q \in \mathbb{P}$.

1.23 Lemma. *If $\mathbb{P}$ and $\mathbb{Q}$ are posets and $\mathbb{P}$ is homogeneous, then $\mathbb{P} \lessdot \mathbb{Q}$ iff $\mathbb{Q}$ adds a $\mathbb{P}$-generic over the ground model.*

Throughout this thesis, the meaning of $\mathbb{P} \lessdot \mathbb{Q}$ is restricted to the particular case when $\mathbb{P}$ is a suborder of $\mathbb{Q}$ such that any maximal antichain in $\mathbb{P}$ is also a maximal antichain in $\mathbb{Q}$.

1.3.1 Iterated forcing

Throughout this thesis we will use iterated forcing. In particular, finite support iterations of ccc forcings is the main technique used in our applications. It was first introduced by Solovay and Tennenbaum in their proof of the independence of Martin Axiom. The idea of iterated forcing is to construct a $\lessdot$-increasing sequence $\langle \mathbb{P}_\alpha \rangle_{\alpha \leq \lambda}$ of forcing notions so that, for every α, $\mathbb{P}_{\alpha+1}$ is the form $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ and, at limit stages, $\mathbb{P}_\alpha$ is some limit of $\{\mathbb{P}_\beta\}_{\beta<\alpha}$. In this section, we describe finite support iterations of forcing notions, starting with two-step iterations.

Two-step iterations

Let $\mathbb{P}$ be a forcing notion, and $\dot{\mathbb{Q}} = \langle \dot{\mathbb{Q}}, \leq_{\dot{\mathbb{Q}}}, \mathbb{1}_{\dot{\mathbb{Q}}} \rangle$ a $\mathbb{P}$-name for a forcing notion. We can find the least μ and a set of $\mathbb{P}$-names $\{\dot{q}_\alpha : \alpha < \mu\}$ such that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} = \{\dot{q}_\alpha : \alpha < \mu\}$. For any $\mathbb{P}$-name $\dot{q}$ for a member of $\dot{\mathbb{Q}}$, $\{p \in \mathbb{P} : \exists \alpha < \mu (p \Vdash \dot{q} = \dot{q}_\alpha)\}$ is dense in $\mathbb{P}$, so there is a maximal antichain A contained in this dense set. This maximal antichain decides which $\dot{q}_\alpha$ corresponds to $\dot{q}$, i.e, there is a function $h : A \to \mu$ such that $p \Vdash \dot{q} = \dot{q}_{h(p)}$ for all $p \in A$.

Let $\dot{q}_{A,h}$ be the nice name associted with a A and h^*(as in the Definition 1.15) where h^* is defined as $h^*(p) = \dot{q}_{h(p)}$. Define

$$[\dot{\mathbb{Q}}] = \{\dot{q}_{A,h} : A \subseteq \mathbb{P} \text{ maximal antichain and } h : A \to \mu\}$$

Clearly,

(i) if $\dot{q} \in [\dot{\mathbb{Q}}]$ then $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$, and

(ii) if $p \Vdash_{\mathbb{P}} \dot{q}' \in \dot{\mathbb{Q}}$ then there is a $\dot{q} \in [\dot{\mathbb{Q}}]$ such that $p \Vdash \dot{q}' = \dot{q}$.

1.24 Definition. Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name of a poset. Define *the two-step iteration* $\mathbb{P} * \dot{\mathbb{Q}} = \mathbb{P} \times [\dot{\mathbb{Q}}]$ ordered by

$$(p, \dot{q}) \leq_{\mathbb{P}*\dot{\mathbb{Q}}} (p', \dot{q}') \iff p \leq p' \ \ \& \ \ p \Vdash_{\mathbb{P}} \dot{q} \leq \dot{q}'$$

1.25 Lemma. *Assume that $\mathbb{P}, \dot{\mathbb{Q}}$ are as above. Then,*

(a) If G is $\mathbb{P}$-generic over V and H is a $\dot{\mathbb{Q}}_G$ generic filter over $V[G]$ then

$$G * H = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G, \dot{q}_G \in H\}$$

is $\mathbb{P} * \dot{\mathbb{Q}}$-generic for over V, and $V[G * H] = V[G][H]$.

(b) Let K be a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter over V. Then, G is $\mathbb{P}$-generic for over V, H is $\dot{\mathbb{Q}}$-generic over $V[G]$ and $K = G * H$ where

$$G = \{p \in \mathbb{P} : \exists \dot{q} \in [\dot{\mathbb{Q}}]((p, \dot{q}) \in K)\} \text{ and } H = \{\dot{q}_G : \exists p \in G((p, \dot{q}) \in K)\}$$

Lemma 1.25 means that any generic extension of $\mathbb{P} * \dot{\mathbb{Q}}$ is also obtained by extending generically with $\mathbb{P}$ and then with $\dot{\mathbb{Q}}$. This lemma also allows us to identify canonically $\mathbb{P} * \dot{\mathbb{Q}}$-names with $\mathbb{P}$ names for $\dot{\mathbb{Q}}$-names. Also note that $\mathbb{P} \lessdot \mathbb{P} * \dot{\mathbb{Q}}$.

1.26 Lemma. *Let $\kappa > \aleph_0$ be a regular cardinal. Suppose that $\mathbb{P}$ is κ-cc and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is κ-cc. Then, $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc. In particular, this holds for ccc.*

Finite support iteration

1.27 Definition. Let δ be an ordinal. Recursively, define the *finite support iteration* $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ *of length* δ where all $\mathbb{P}_\alpha$, for $\alpha \leq \delta$, are forcing notions consisting of partial functions on α defined as follows:

(i) (Basic stage) $\mathbb{P}_0 = \mathbb{1}$

(ii) (Succesor stage) $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset and

$$\mathbb{P}_{\alpha+1} = \{p : p \restriction \alpha \in \mathbb{P}_\alpha \text{ and } \alpha \in \mathrm{dom}(p) \Rightarrow p(\alpha) \in [\dot{\mathbb{Q}}_\alpha]\}$$

ordered by

$$q \leq_{\alpha+1} p \iff q \restriction \alpha \leq_\alpha p \restriction \alpha \text{ and } q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} q(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha)$$

(ignore the latter part when either $q(\alpha)$ or $p(\alpha)$ are not defined). Note that $\mathbb{P}_{\alpha+1} \simeq \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

(iii) (Limit stage) For limit $\beta \leq \delta$, $\mathbb{P}_\beta$ is the *direct limit* of $\langle \mathbb{P}_\alpha : \alpha < \beta \rangle$, i.e., $\mathbb{P}_\beta = \bigcup_{\alpha<\beta} \mathbb{P}_\alpha = \{p : \exists \alpha < \beta (\mathrm{dom}(p) \subseteq \alpha) \text{ and } \forall \xi < \beta (p \restriction \xi \in \mathbb{P}_\xi)\}$. The order $\leq_\beta$ in $\mathbb{P}_\beta$ is defined as

$$q \leq_\beta p \iff \forall \alpha < \beta (q \restriction \alpha \leq_\alpha p \restriction \alpha).$$

Here $\mathrm{dom}(p)$ is also called the *support of* p. Note that, for all $\alpha \leq \delta$ and for all $p \in \mathbb{P}_\alpha$, $\mathrm{supp}(p)$ is finite. In the limit case, $p \in \mathbb{P}_\beta$ iff $\mathrm{dom}(p)$ is finite and $\forall \alpha < \beta (p \restriction \alpha \in \mathbb{P}_\alpha)$.

Fix $\alpha \leq \delta$. Define $\mathbb{P}_\delta / \mathbb{P}_\alpha$ as the poset in the $\mathbb{P}_\alpha$ extention that represents the remaining portion of the iteration to get to $\mathbb{P}_\delta$, i.e, $\mathbb{P}_\delta / \mathbb{P}_\alpha$ is a $\mathbb{P}_\alpha$-name and $\mathbb{P}_\delta \simeq \mathbb{P}_\alpha * (\mathbb{P}_\delta / \mathbb{P}_\alpha)$. Formally, $\mathbb{P}_\delta / \mathbb{P}_\alpha$ is the $\mathbb{P}_\alpha$-name of the finite support iteration $\langle \mathbb{P}_{\alpha+\xi} / \mathbb{P}_\alpha, \dot{\mathbb{Q}}_{\alpha+\xi} : \xi < \delta - \alpha \rangle$.

For a finite support iteration $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\lambda}$, if G is $\mathbb{P}_\alpha$-generic over V(the ground model) we denote $V_\alpha = V[G \cap \mathbb{P}_\alpha]$ for $\alpha \leq \lambda$.

The following lemma shows that finite support iterations of ccc forcing notions are ccc, which is very important to preserve cardinals and cofinalities.

1.28 Lemma. *Let* $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ *be a finite suport iteration and* κ *an uncountable regular cardinal such that* $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ *is* κ*-cc for all* $\alpha < \delta$*. Then,* $\mathbb{P}_\delta$ *is* κ*-cc.*

The following theorem shows that, in finite support iterations of ccc forcing notions, no new reals are added in limit steps of uncountable cofinality.

1.29 Theorem. *Let* κ *be an uncountable regular cardinal and* $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ *a finite suport iteration where* $\mathrm{cf}(\delta) \geq \kappa$ *and* $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ κ*-cc for any* $\alpha < \delta$*. If* $A, B \in V$*,* $|A| < \kappa$ *and* $\dot{f}$ *is a* $\mathbb{P}_\delta$*-name such that* $\Vdash_{\mathbb{P}_\delta} \dot{f} : A \to B$*, then there exists an* $\alpha < \delta$ *and a nice* $\mathbb{P}_\alpha$*-name* $\dot{f}^*$ *of a function from* A *to* B *such that* $\Vdash_{\mathbb{P}_\delta} \dot{f} = \dot{f}^*$*.*

In particular, if G *is* $\mathbb{P}_\delta$*-generic over* V*, then* $B^A \cap V[G] = \bigcup_{\alpha<\delta} (B^A \cap V[G \cap \mathbb{P}_\alpha])$*.*

From this point on, in this thesis we use $V^{\mathbb{P}}$ to denote a $\mathbb{P}$-generic extension when the filter is not specified.

1.4 Suslin ccc posets

1.30 Definition (Suslin ccc poset)**.** A poset $\mathbb{S}$ is called a *Suslin ccc forcing notion* if it is ccc and there is a Polish space Z such that

(i) $\mathbb{S} \subseteq Z$,

(ii) $\leq_{\mathbb{S}} \subseteq Z \times Z$ is Σ^1_1 and

(iii) $\perp_{\mathbb{S}} \subseteq Z \times Z$ is Σ^1_1.

Assume that $M \models \text{ZFC}$. If the parameters in the definition of $\mathbb{S}, \leq_{\mathbb{S}}$ and$\perp_{\mathbb{S}}$ are in M, we can interpret it in M. Then we say that $\mathbb{S}$ *is coded in* M and denote its interpretation by $\mathbb{S}^M$, as usual for analytic sets.

Note that $\mathbb{S}$ itself is a Σ^1_1-set because $x \in \mathbb{S}$ iff $x \leq x$. The main feature of this type of poset is that, thanks to its definability, many of its features, when relativized to models of ZFC, become absolute. For example, by Theorem 1.1 the statements "$p \in \mathbb{S}$", "$p \leq_{\mathbb{S}} q$" and "$p \perp_{\mathbb{S}} q$" are absolute for transitive models of ZFC.

1.31 Lemma ([BJ, Lemma 3.6.4])**.** *Let M be a transitive countable model of* ZFC*. Suppose that $\mathbb{S} \in M$ is a Suslin ccc forcing, and $A := \langle s_n : n < \omega \rangle \in Z^\omega$ where Z is the Polish space where $\mathbb{S}$ is defined. Then, the statement "A is a maximal antichain in $\mathbb{S}$" is a conjunction of a Σ^1_1 statement with a Π^1_1 statement in Z^ω. Moreover, if $\mathbb{S}$ is a Borel set, then the previous statement is Π^1_1 in Z^ω.*

Now, we present briefly how to define nice names for reals in a Polish space. If D is countable dense in a Polish space X and d is a metric in D for which X is its completion, then any object in X can be represented by some sequence $\langle q_n : n < \omega \rangle \in D^\omega$ such that $d(q_n, q_{n+1}) < 2^{-(n+1)}$ for all $n < \omega$. Denote the set of these sequences by X^*, which is a closed subset of D^ω(with the product topology where D has the discrete topology). Therefore, a nice $\mathbb{S}$-name of a member of X is represented by a nice $\mathbb{S}$-name $\dot{z}$ of a member of D^ω such that $\Vdash \dot{z} \in X^*$. Note that, as each antichain in $\mathbb{S}$ is member of Z^ω, then any nice name of a member of D^ω can seen as a member of $(D \times Z)^{\omega \times \omega}$.

1.32 Lemma. *Let X be a Polish space, $\langle D, d \rangle$ a countable metric space for which X is its completion, and $\mathbb{S}$ a Suslin ccc poset defined in a Polish space Z.*

(a) The statement "$\dot{x}$ is a nice $\mathbb{S}$-name of a member of X" is the conjuction of a Σ^1_1 statement with a Π^1_1 statement in $(D \times Z)^{\omega \times \omega}$.

(b) If $E \subseteq X$ is a G_δ set, the statement "$p \in \mathbb{S}$, $\dot{x}$ is a nice name $\mathbb{S}$-name of a member of X and $p \Vdash \dot{x} \in E$" is the conjuction of a Σ^1_1 statement with a Π^1_1 statement in $Z \times (D \times Z)^{\omega \times \omega}$(even more, the statement that corresponds to $p \Vdash \dot{x} \in E$ is Borel).

Additionally, if $\mathbb{S}$ is Borel in Z, then the previous statement are Π^1_1.

1.4.1 Examples of forcing notions

Cohen forcing

For any I, J, $\text{Fn}(I, J)$ denotes the collection of finite partial functions from I to J. It is ordered by reverse inclusion, i.e., $p \leq q$ iff $p \supseteq q$ iff p extends q as a function. A *Cohen*

poset is the of form $\mathbb{C}_I = \mathrm{Fn}(I, 2)$, in particular, *Cohen forcing* is $\mathbb{C} = \mathbb{C}_\omega$. Any Cohen poset is ccc, moreover, it is σ-centered when $|I| \leq \mathfrak{c}$. If G is $\mathbb{C}_I$-generic over V, let $c_I = \bigcup\{s \in \mathbb{C}_I : s \in G\} \in 2^\omega$. Say that $c \in 2^I$ is a *Cohen real over* V iff there exist a $\mathbb{C}$-generic G over V such that $c = \bigcup G$. For $\mathbb{C}_I$, it is important to know that the generic filter can be reconstructed from c_I, that is,

1.33 Lemma. *$V[G] = V[c_I]$ when G is $\mathbb{C}_I$-generic over V.*

1.34 Lemma. *Any pair of countable atomless posets are forcing equivalent.*

Given a perfect Polish space X and a countable dense set $Q \subseteq X$, let $\mathbb{C}[Q] = \{q \in Q^{<\omega} : \forall i < |q| - 1(d(q_i, q_{i+1}) < \frac{1}{2^{i+1}}))\}$ ordered by $\supseteq$ where d is the metric on D (or X). By Lemma 1.34, $\mathbb{C}[Q] \simeq \mathbb{C}$. If G is $\mathbb{C}[G]$-generic over V, $\bigcup G$ is a Cauchy sequence which converges to a real $c \in X^{V[G]}$. In this context, c is a *Cohen real* in X. In general, a Cohen real over X is characterized as follows:

1.35 Theorem (Solovay's characterization of genericity)**.** *Assume that V is a model of* ZFC *and X is a perfect Polish space. A real $c \in X$ is Cohen over V iff c evades all the Borel meager sets of X coded in V, that is $\forall e \in \mathrm{BC} \cap V(\mathrm{Val}_X(e) \in \mathcal{M}(X) \Rightarrow c \notin \mathrm{Val}_X(e))$.*

Now, consider $\mathbb{C} = 2^{<\omega}$ ordered by $\supseteq$. Note that $\mathbb{C}$ is a Suslin ccc poset. Indeed, define $g : \mathbb{C} \to 2^\omega$ such that $g(s) = \langle |s| \rangle^\frown s^\frown \langle 0 \cdots 0 \rangle$. Let $\mathbb{S} = g[\mathbb{C}]$ where $x \leq_{\mathbb{S}} y$ iff $x \restriction [1, x(0)) \supseteq y \restriction [1, y(0))$. Clearly, $\mathbb{S} \simeq \mathbb{C}$, $\mathbb{S}$ is a closed set in 2^ω and $\leq_{\mathbb{S}}$, $\perp_{\mathbb{S}}$ are closed in $2^\omega \times 2^\omega$.

Hereafter, we will use $\mathbb{C}$ to indicate any poset as in Lemma 1.34.

1.36 Lemma. *Let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\delta}$ be a fsi where each $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be non-trivial. Then, $Fn(\delta, 2) \lessdot \mathbb{P}_\delta$.*

Proof. Thanks to Lemma 1.23, it suffices to prove that $\mathbb{P}_\delta$ adds a $\mathrm{Fn}(\delta, 2)$-generic over the ground model. We show the case $\delta = \omega$, as the general case is similar. For each n, let $\dot{p}_n$, $\dot{q}_n$ be $\mathbb{P}_n$-names for a condition in $\dot{\mathbb{Q}}_n$ such that $\Vdash_{\mathbb{P}_n} \dot{p}_n \perp \dot{q}_n$. In the extension, define $c \in 2^\omega$ by $c(n) = 1$ iff the interpretation of $\dot{q}_n$ belongs to the generic filter. Now, we prove that c is Cohen over V.

Let D be a dense subset of $\mathbb{C}$ in V and let $p \in \mathbb{P}_\omega$. We need to find $r \leq p$ and $t \in D$ such that r forces that $t \subseteq \dot{c}$. Fix k such that $p \in \mathbb{P}_k$. Strengthening p if necessary, we may assume that, for all $n < k$, either $p \restriction n \Vdash_{\mathbb{P}_n} p(n) \leq \dot{q}_n$ or $p \restriction n \Vdash_{\mathbb{P}_n} p(n)$ and $\dot{q}_n$ are incompatible. This defines $s \in 2^k$ given by $s(n) = 1$ iff the first alternative holds. Clearly p forces that $s \subseteq \dot{c}$. Find $t \in D$ with $s \subseteq t$. Let $l = |t|$ and extend p to $r \in \mathbb{P}_l$ such that $r \restriction k = p$ and, for $k \leq n < l$, $r(n) = \dot{q}_n$ if $t(n) = 1$, and $r(n) = \dot{q}_n$ if $t(n) = 0$. Then r forces that $t \subseteq \dot{c}$. □

1.37 Corollary. *If $\mathbb{P}_\delta$ is a fsi(finite support iteration) as in Lemma 1.36, $\alpha < \delta$ and δ is limit, then $\mathbb{P}_\delta$ forces that there is a Cohen real over $V[\dot{G} \cap \mathbb{P}_\alpha]$.*

Random forcing

For $A, B \in \mathcal{B}(2^\omega)$ we define the equivalence relation $A \equiv B$ by $A \vartriangle B \in \mathcal{N}(2^\omega)$. Let

$$\mathbb{B} = \{[A]_{\mathcal{N}} : A \in \mathcal{B}(2^\omega)\} = \mathcal{B}(2^\omega)/\mathcal{N}$$

For $[A]_{\mathcal{N}}, [B]_{\mathcal{N}} \in \mathbb{B}$ define

$$[A]_{\mathcal{N}} \leq [B]_{\mathcal{N}} \iff A \setminus B \in \mathcal{N}$$

where $[A]_{\mathcal{N}}$ denotes the equivalence class of the set A. $\mathbb{B}$ is a Complete Boolean algebra and $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$ is called *random forcing*. In the context of posets, we write $\mathbb{B}$ instead of $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$.

We say that $r \in 2^\omega$ is a *random real over* V iff there exist a $\mathbb{B}$-generic G over V such that $r = \bigcup\{p \in \mathrm{Fn}(\omega, 2) : [p] \in G\}$. It is known that $\mathbb{C}$ does not add random reals, random forcing $\mathbb{B}$ does not add Cohen reals, $\mathbb{B}$ is ω^ω-bounding and $\mathbb{B}$ is σ-linked. It is important to know that the generic filter can be reconstructed from the random real:

1.38 Lemma. *$V[G] = V[r]$ when G is $\mathbb{B}$-generic over V.*

A dual version of Theorem 1.35 for random forcing is:

1.39 Theorem (Solovay's characterization of genericity)**.** *Assume that V is a model of* ZFC*. A real $r \in 2^\omega$ is random over V iff r evades all the Borel null sets of 2^ω coded in V, that is, $\forall e \in \mathrm{BC} \cap V(\mathrm{val}_{2^\omega}(e) \in \mathcal{N} \Rightarrow c \notin \mathrm{val}_{2^\omega}(e))$.*

Consider the poset $\mathbb{S} = \{T \subseteq 2^{<\omega} : \mu_2([T]) > 0 \,\text{and}\, \forall t \in T(\mu_2([T] \cap [t]) > 0)\}$ ordered by $\subseteq$. As $T \mapsto [T]$ is a dense embedding from $\mathbb{S}$ into $\mathbb{B}$, then $\mathbb{S} \simeq \mathbb{B}$. On the other hand, its order and incompatibility relations are Borel in $2^{2^{<\omega}}$(since any $T, T_1 \in \mathbb{S}$ are incompatible iff $\mu([T] \cap [T_1]) = 0$, which is a Borel statement, therefore analytic), so we can say random forcing is a Suslin ccc poset.

Hechler forcing

Let $\mathbb{D} = \{(s, f) : s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subseteq f\}$.

For $(t, g), (s, f) \in \mathbb{D}$ define

$$(t, g) \leq (s, f) \iff s \subseteq t \text{ and } f \leq g.$$

$\mathbb{D}$ is called *Hechler forcing*. Consider G a $\mathbb{D}$-generic over V and put $d = \bigcup\{s : \exists f \in \omega^\omega \cap V\,((s, f) \in G)\} \in \omega^\omega$. It follows that $\forall f \in V \cap \omega^\omega (f \leq^* d)$. In other words, d *is dominating over* V. The real d added by $\mathbb{D}$ is called *Hechler real*. Also, $V[G] = V[d]$. Moreover, $\mathbb{D}$ adds Cohen reals, i.e., $\mathbb{C} \lessdot \mathbb{D}$, and $\mathbb{D}$ is σ-centered. As $\mathbb{D} = \bigcup_{s \in \omega^{<\omega}} \{s\} \times [s]$, it is clopen in $\omega^{<\omega} \times \omega^\omega$ ($\omega^{<\omega}$ with the discrete topology) and therefore Borel. Even more, the relations $\leq_{\mathbb{D}}$, and $\perp_{\mathbb{D}}$ are closed. Then, $\mathbb{D}$ is a Suslin ccc poset.

Localization forcing

For a monotone increasing function $h \in \omega^\omega$ that goes to $+\infty$, let

$$\mathbb{LOC}_h = \{\varphi \in \mathcal{S}(\omega, h) : \exists k < \omega \forall^\infty i < \omega(|\varphi(i)| \leq k)\}.$$

For $\varphi, \psi \in \mathbb{LOC}_h$, define

$$\psi \leq \varphi \iff \forall i < \omega(\varphi(i) \subseteq \psi(i)).$$

$\mathbb{LOC}_h$ is called *localization forcing for* h.

If G is $\mathbb{LOC}_h$-generic over V, put $\psi \in \mathcal{S}(\omega, h)$ such that $\psi(i) = \bigcup\{\varphi(i) : \varphi \in G\}$. Then, $\forall f \in V \cap \omega^\omega (f \in^* \psi)$. In other words, ψ localizes all the reals in $\omega^\omega \cap V$. Also, $V[G] = V[\psi]$. The localization forcing $\mathbb{LOC}_h$ is σ-linked.

$\mathbb{LOC}_h$ is G_δ in $\mathcal{S}(\omega, h) = \prod_{i<\omega} [\omega]^{\leq h(i)}$ and therefore Borel. Moreover, the relations $\leq_{\mathbb{LOC}_h}$ and $\perp_{\mathbb{LOC}_h}$ are Borel, so $\mathbb{LOC}_h$ is a Suslin ccc poset.

Eventually different reals bounded by b, h forcing

Let $h, b \in \omega^\omega$ such that $b \geq 1$ and $\frac{b(n)}{h(n)} \to +\infty$.
Define $\mathbb{E}^h_b$ the *eventually different real bounded by b,h forcing*, as the poset whose conditions are of the form (s, F) and satisfy:

(i) $s \in \mathrm{Seq}_{<\omega}(b)$,

(ii) $F \subseteq \mathcal{S}(b, h)$ finite, and

(iii) $|F|h(n) < b(n)$ for each $n \geq |s|$.

For $(t, F'), (s, F) \in \mathbb{E}^h_b$, define the order

$$(t, F') \leq (s, F) \iff s \subseteq t, F \subseteq F' \text{ and } \forall i \in |t| \setminus |s| \forall \varphi \in F(t(i) \notin \varphi(i))$$

If $S \subseteq \mathcal{S}(b, h)$, define $\mathbb{E}^h_b(S) = \{(s, F) \in \mathbb{E}^h_b : F \subseteq S\}$ with the same order of $\mathbb{E}^h_b$.
If G is $\mathbb{E}^h_b(S)$-generic over V and $r = \bigcup\{s : \exists F((s, F) \in G)\} \in \mathbb{R}_b$, then

$$\forall \varphi \in V \cap S \forall^\infty n(r(n) \notin \varphi(n)).$$

Also, $V[G] = V[r]$. $\mathbb{E}^h_b(S)$ is σ-linked because, for each $s \in \mathrm{Seq}_{<\omega}b$, $L_s = \{(t, F') \in \mathbb{E}^h_b(S) : \exists F((s, F) \in \mathbb{E}^h_b(S), (s, F) \leq (t, F') \text{ and } 2|F|h(n) < b(n) \text{ for all } n \geq |s|)\}$ is 2-linked and $\mathbb{E}^h_b(S) = \bigcup_{s \in Seq_{<\omega}(b)} L_s$.
By coding $(s, F) \in \mathbb{E}^h_b$ by

$$(|F|, s, \varphi_0(0), \cdots, \varphi_{k-1}(0), \varphi_0(1), \cdots, \varphi_{k-1}(1) \cdots, \varphi_0(n), \cdots, \varphi_{k-1}(n), \cdots)$$

in $\omega \times \omega^{<\omega} \times ([\omega]^{<\omega})^\omega$ when $F = \{\varphi_k : k < n\}$, it is clear that $\mathbb{E}^h_b$, $\leq_{\mathbb{E}^h_b}$ and $\perp_{\mathbb{E}^h_b}$ are Borel, therefore analytic, so $\mathbb{E}^h_b$ is a Suslin ccc poset.

1.40 Lemma ([KO14])**.** *Let $c, b, h \in \omega^\omega$ such that c and h are monotone increasing, $c, b, h \geq 1$, $b >^* c \cdot h \cdot \mathrm{id}_\omega$ and $S \subseteq \mathcal{S}(b, h)$. Then the forcing notion $\mathbb{E}^h_b(S)$ is $((h \cdot \mathrm{id}_\omega)^{\mathrm{id}_\omega}, c)$-linked. In particular, if h is the constant function 1, then it is $((\mathrm{id}_\omega)^{\mathrm{id}_\omega}, c)$-linked.*

1.5 Tukey-Galois connections

Many of the classical cardinal invariants are defined by relational systems (see Definitions 1.41 and 1.42) and inequalities between these cardinal invariants are induced by an order between the corresponding relational systems which is known as *the Tukey-Galois order* (Definition 1.46). Votjas [Voj] introduced a framework that captures this additional information. Moreover, the construction of the cardinal invariants by means of relational systems clarifies the duality of cardinals and their inequalities.

1.41 Definition. A *relational system* is a triple $\mathbf{A} = \langle A_-, A_+, \sqsubset \rangle$ consisting of two non-empty sets A_+, A_- and a binary relation $\sqsubset \subseteq A_- \times A_+$. If $\mathbf{A} = \langle A_-, A_+, \sqsubset \rangle$ is a relational system, define *the dual of* $\boldsymbol{A}$ as the relation system $\mathbf{A}^\perp = \langle A_+, A_-, \not\sqsupset \rangle$

For $x \in A_-$ and $y \in A_+$, $x \sqsubset y$ is often read y $\sqsubset$-*dominates* x. A family $X \subseteq A_-$ is $\mathbf{A}$-*unbounded* if there is no member in A_+ that $\sqsubset$-dominates every member of X. Dually, $Y \subseteq A_+$ is $\mathbf{A}$-*dominating* family if every member of A_- is $\sqsubset$-dominated by some member of Y.

1.42 Definition. Let $\mathbf{A} = \langle A_-, A_+, \sqsubset \rangle$ be a relational system. Define $\mathfrak{d}(\mathbf{A})$ as the smallest size of an $\mathbf{A}$-dominating family, and $\mathfrak{b}(\mathbf{A})$ as the smallest size of an $\mathbf{A}$-unbounded family.

Note that $\mathfrak{d}(\mathbf{A}) = \mathfrak{b}(\mathbf{A}^\perp)$ and $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$. Many classical cardinal invariants can be expressed within this framework.

1.43 Example. For a family $\mathcal{J}$ of subsets of X:

(1) $\mathrm{cof}(\mathcal{J}) = \mathfrak{d}(\mathcal{J}, \mathcal{J}, \subseteq)$.

(2) $\mathrm{add}(\mathcal{J}) = \mathfrak{b}(\mathcal{J}, \mathcal{J}, \subseteq)$.

(3) $\mathrm{cov}(\mathcal{J}) = \mathfrak{d}(X, \mathcal{J}, \in)$.

(4) $\mathrm{non}(\mathcal{J}) = \mathfrak{b}(X, \mathcal{J}, \in)$.

Also,

(5) $\mathfrak{d} = \mathfrak{d}(\omega^\omega, \omega^\omega, \leq^*)$, and

(6) $\mathfrak{b} = \mathfrak{b}(\omega^\omega, \omega^\omega, \leq^*)$.

Kellner and Shelah [KeS09, KeS12] have studied cardinal coefficients of the form $\mathfrak{c}^\exists_{b,h}$ defined as below. Here, we also study its dual cardinal $\mathfrak{n}^\exists_{b,h}$.

1.44 Definition. Let $b : \omega \to (\omega + 1) \setminus \{0\}$ and $h \in \omega^\omega$. Define

$$\mathfrak{c}^\exists_{b,h} = \min\{|S| : S \subseteq \mathcal{S}(b,h) \,\text{and}\, \forall x \in \mathbb{R}_b \exists \varphi \in S(x \in^\infty \varphi)\}$$

and

$$\mathfrak{n}^\exists_{b,h} = \min\{|F| : F \subseteq \mathbb{R}_b \,\text{and}\, \forall \varphi \in \mathcal{S}(b,h) \exists x \in F(x \notin^\infty \varphi)\}$$

1.45 Remark. $\mathfrak{c}^\exists_{b,h} = \mathfrak{d}(\mathbb{R}_b, \mathcal{S}(b,h), \in^\infty)$ and $\mathfrak{n}^\exists_{b,h} = \mathfrak{b}(\mathbb{R}_b, \mathcal{S}(b,h), \in^\infty)$ by the previous definition.

1.46 Definition ([B10, Def. 4.8])**.** Let $\mathbf{A} = \langle A_-, A_+, \sqsubset \rangle$ and $\mathbf{B} = \langle B_-, B_+, \sqsubset' \rangle$ be two relational systems. Define the *Tukey-Galois order* $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{B}$ when there exist $\varphi_- : A_- \to B_-$ and $\varphi_+ : B_+ \to A_+$ such that, for any $x \in A_-$ and $b \in B_+$,

$$\varphi_-(x) \sqsubset' b \Rightarrow x \sqsubset \varphi_+(b).$$

Say that $\mathbf{A}$ and $\mathbf{B}$ are *Tukey equivalent*, denoted by $\mathbf{A} \simeq_{\mathbf{T}} \mathbf{B}$, iff $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{B}$ and $\mathbf{B} \preceq_{\mathrm{T}} \mathbf{A}$.

1.47 Theorem ([B10, Thm. 4.9])**.** *Let $\varphi_- : A_- \to B_-$ and $\varphi_+ : B_+ \to A_+$ be as in Definition 1.46.*

(a) If $D \subseteq B_+$ is $\boldsymbol{B}$-dominating, then $\varphi_+[D]$ is $\boldsymbol{A}$-dominating.

(b) If $C \subseteq A_-$ is $\boldsymbol{A}$-unbounded, then $\varphi_-[D]$ is $\boldsymbol{B}$-unbounded.

In particular $\mathfrak{d}(\boldsymbol{A}) \leq \mathfrak{d}(\boldsymbol{B})$ and $\mathfrak{b}(\boldsymbol{B}) \leq \mathfrak{b}(\boldsymbol{A})$.

1.48 Example. If $\mathcal{I} \subseteq \mathcal{J}$ are two family of subsets of X, then $\langle X, \mathcal{J}, \in\rangle \preceq_{\mathbf{T}} \langle X, \mathcal{I}, \in\rangle$.

$$\begin{aligned}\varphi_- &: X \to X\\ &\quad x \mapsto x,\\ \varphi_+ &: \mathcal{I} \to \mathcal{J}\\ &\quad y \mapsto y.\end{aligned}$$

Clearly, for any $x \in X$ and $y \in \mathcal{I}$,

$$\varphi_-(x) \in y \Rightarrow x \in \varphi_+(y).$$

In particular, $\mathrm{cov}(\mathcal{J}) \leq \mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I}) \leq \mathrm{non}(\mathcal{J})$.

1.49 Example. Fix $b : \omega \to (\omega + 1) \setminus \{0\}$ and $h \in \omega^\omega$ such that $h \geq^* 1$. Define

$$\begin{aligned}\varphi_- &: \mathbb{R}_b \to \mathbb{R}_b\\ &\quad x \mapsto x,\\ \varphi_+ &: \mathbb{R}_b \to \mathcal{S}(b,h)\\ &\quad y \mapsto s_y = \langle\{y(n)\}\rangle_{n<\omega}.\end{aligned}$$

Note that, for any $x \in \mathbb{R}_b$ and $y \in \mathbb{R}_b$,

$$x =^\infty y \Rightarrow x \in^\infty s_y.$$

Therefore, $\langle \mathbb{R}_b, \mathcal{S}(b,h), \in^\infty\rangle \preccurlyeq_{\mathbf{T}} \langle \mathbb{R}_b, \mathbb{R}_b, =^\infty\rangle$. On the other hand, $\langle \mathbb{R}_b, \mathcal{S}(b,1), \in^\infty\rangle \simeq_{\mathbf{T}} \langle \mathbb{R}_b, \mathbb{R}_b, =^\infty\rangle$. Indeed,

$$\begin{aligned}\varphi'_- &: \mathbb{R}_b \to \mathbb{R}_b\\ &\quad x \mapsto x,\\ \varphi'_+ &: \mathcal{S}(b,1) \to \mathbb{R}_b\\ &\quad s \mapsto f_s, \text{ where } f_s(n) = 0 \text{ if } s(n) \text{ is empty, else } f_s(n) = a \text{ where } s(n) = \{a\}.\end{aligned}$$

Note that, for any $x \in \mathbb{R}_b$ and $s \in \mathcal{S}(b,1)$,

$$x \in^\infty s \Rightarrow x =^\infty f_s$$

1.50 Example. Let $b, b' : \omega \to (\omega + 1) \setminus \{0\}$ and $h, h' \in \omega^\omega$.

(1) If $h \leq^* h'$, then $\langle \mathbb{R}_b, \mathcal{S}(b,h'), \in^\infty\rangle \preceq_{\mathrm{T}} \langle \mathbb{R}_b, \mathcal{S}(b,h), \in^\infty\rangle$, so $\mathfrak{c}^\exists_{b,h'} \leq \mathfrak{c}^\exists_{b,h}$ and $\mathfrak{n}^\exists_{b,h} \leq \mathfrak{n}^\exists_{b,h'}$. Indeed, define

$$\begin{aligned}\varphi_- &: \mathbb{R}_b \to \mathbb{R}_b\\ &\quad x \mapsto x,\\ \varphi_+ &: \mathcal{S}(b,h) \to \mathcal{S}(b,h')\\ &\quad s \mapsto s \,(\text{with finitely many modifications}).\end{aligned}$$

Clearly, for any $x \in \mathbb{R}_b$ and $s \in \mathcal{S}(b,h)$,

$$\varphi_-(x) \in^\infty s \Rightarrow x \in^\infty \varphi_+(s).$$

(2) If $b \leq^* b'$, then $\langle \mathbb{R}_b, \mathcal{S}(b,h), \in^\infty \rangle \preceq_{\mathrm{T}} \langle \mathbb{R}_{b'}, \mathcal{S}(b',h), \in^\infty \rangle$, so $\mathfrak{c}^\exists_{b,h} \leq \mathfrak{c}^\exists_{b',h}$ and $\mathfrak{n}^\exists_{b',h} \leq \mathfrak{n}^\exists_{b,h}$. Indeed, define

$$\begin{aligned}
\varphi_- &: \mathbb{R}_b \to \mathbb{R}_{b'} \\
&\quad y \mapsto y' \quad \text{where } y'(n) = y(n) \text{ if } b(n) \leq b'(n), \text{ otherwise } y'(n) = 0, \\
\varphi_+ &: \mathcal{S}(b',h) \to \mathcal{S}(b,h) \\
&\quad s \mapsto s' \text{ where } s'(n) = s(n) \cap b(n)
\end{aligned}$$

Clearly, for any $y \in \mathbb{R}_b$ and $s \in \mathcal{S}(b',h)$,

$$\varphi_-(y) \in^\infty s \Rightarrow y \in^\infty \varphi_+(s).$$

1.51 Definition ([B10, Def. 4.10])**.** Let $\mathbf{A} = \langle A_-, A_+, \sqsubset \rangle$ and $\mathbf{B} = \langle B_-, B_+, \sqsubset' \rangle$ be relational systems. We define:

(1) The *categorical product* $\mathbf{A} \times \mathbf{B} = \langle A_- \sqcup B_-, A_+ \times B_+, \sqsubset_\times \rangle$, where $\sqcup$ is the disjoint union and the binary relation $\sqsubset_\times$ is defined as $x \sqsubset_\times (a,b)$ iff either $x \in A_-$ and $x \sqsubset a$, or $x \in B_-$ and $x \sqsubset' b$.

(2) The *sequential composition* $(\mathbf{A};\mathbf{B}) = \langle A_- \times B_-^{A_+}, A_+ \times B_+, \sqsubset_; \rangle$, where the binary relation $(x,f) \sqsubset_; (a,b)$ means $x \sqsubset a$ and $f(a) \sqsubset' b$.

The following theorem describes the effect of these operations on the corresponding cardinal invariants.

1.52 Theorem ([B10, Thm. 4.11])**.** *Let* $\boldsymbol{A} = \langle A_-, A_+, \sqsubset \rangle$ *and* $\boldsymbol{B} = \langle B_-, B_+, \sqsubset' \rangle$ *two relational systems. Then*

(a) $\mathfrak{d}(\boldsymbol{A} \times \boldsymbol{B}) = \max\{\mathfrak{d}(\boldsymbol{A}), \mathfrak{d}(\boldsymbol{B})\}$.

(b) $\mathfrak{d}(\boldsymbol{A};\boldsymbol{B}) = \mathfrak{d}(\boldsymbol{A}) \cdot \mathfrak{d}(\boldsymbol{B})$.

(c) $\mathfrak{b}(\boldsymbol{A} \times \boldsymbol{B}) = \min\{\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{B})\}$.

(d) $\mathfrak{b}(\boldsymbol{A};\boldsymbol{B}) = \min\{\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{B})\}$.

Proof. (a) and (c). Define,

$$\begin{aligned}
\varphi_- &: A_- \to A_- \sqcup B_- \\
&\quad x \mapsto x, \\
\varphi_+ &: A_+ \times B_+ \to A_+ \\
&\quad (y,z) \mapsto y.
\end{aligned}$$

Clearly, for any $x \in A_-$ and $(y,z) \in A_+ \times B_+$, $x \sqsubset_\times (y,z)$ iff $x \sqsubset y$. Therefore, $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{A} \times \mathbf{B}$. Analogously, $\mathbf{B} \preceq_{\mathrm{T}} \mathbf{A} \times \mathbf{B}$. So $\max\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\} \leq \mathfrak{d}(\mathbf{A} \times \mathbf{B})$ and $\mathfrak{b}(\mathbf{A} \times \mathbf{B}) \leq \min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\}$.

Now, we prove $\max\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\} \geq \mathfrak{d}(\mathbf{A} \times \mathbf{B})$. Without loss of generality, assume that $\mathfrak{d}(\mathbf{A}) \geq \mathfrak{d}(\mathbf{B})$, so there exists an onto function $f : D \to D'$ from some $\mathbf{A}$-dominating $D \subseteq A_+$ of size $\mathfrak{d}(\mathbf{A})$ onto some $\mathbf{B}$-dominating $D' \subseteq B_+$ of size $\mathfrak{d}(\mathbf{B})$. Define

$$Y' = \{(a, f(a)) : a \in D\}.$$

It is enough to prove that Y' is $\mathbf{A}\times\mathbf{B}$-dominating. Let $x \in A_- \sqcup B_-$, that is, either $x \in A_-$ or $x \in B_-$. If $x \in A_-$ then there exists an $a \in D$ such that $x \sqsubset a$. So $x \sqsubset_\times (a, f(a))$. Otherwise, if $x \in B_-$ then there exists a $b \in D'$ such that $x \sqsubset' b$. As f is onto there exists a $a \in D$ such that $x \sqsubset' f(a) = b$. Therefore $x \sqsubset_\times (a, f(a))$.

Finally, we show $\mathfrak{b}(\mathbf{A}\times\mathbf{B}) \geq \min\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\}$. Let $X \subseteq A_+ \sqcup B_+$ of size $< \min\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\}$. We want to prove that there is a $(a, b) \in A_- \cap B_-$ such that $\forall x \in X(x \sqsubset_\times (a, b))$. Define $X_1 = X \cap A_-$ and $X_2 = X \setminus B_-$. As both have size $< \min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\}$, there exist $a \in A_-$ and $b \in B_-$ such that $\forall x \in X_1(x \sqsubset a)$ and $\forall x \in X_2(x \sqsubset' b)$, that is, $\forall x \in X(x \sqsubset_\times (a, b))$. Therefore, $\mathfrak{b}(\mathbf{A}\times\mathbf{B}) \geq \min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\}$.

(b) To prove $\geq$, suppose $Y \subseteq A_+ \times B_+$ such that $|Y| < \mathfrak{d}(\mathbf{A}) \cdot \mathfrak{d}(\mathbf{B})$. We want to show that Y is not $(\mathbf{A};\mathbf{B})$-dominating. For each $a \in \mathrm{dom}(Y)$ define $Y_a = \{b : (a, b) \in Y\}$. Clearly, $|Y| = \sum_{a\in\mathrm{dom}(Y)} |Y_a|$. Partition $\mathrm{dom}(Y) = C \cup D$ where $C = \{a \in \mathrm{dom}(Y) : |Y_a| < \mathfrak{d}(\mathbf{B})\}$ and D is its complement. Note that $|D| < \mathfrak{d}(\mathbf{A})$ (if not, $|Y| \geq \sum_{a\in D} |Y_a| \geq |D| \cdot \mathfrak{d}(\mathbf{B}) \geq \mathfrak{d}(\mathbf{A}) \cdot \mathfrak{d}(\mathbf{B})$). Then, there exists an $x \in A_-$ such that $x \not\sqsubset a$ for all $a \in D$. Define $f : A_+ \to B_-$ such that $f(a) \not\sqsubset' b$ for all $b \in Y_a$ if $a \in C$, otherwise $f(a)$ can be anything. Clearly, for all $(a, b) \in Y$, if $x \sqsubset a$ then $f(a) \not\sqsubset' b$, so Y is not $(\mathbf{A};\mathbf{B})$-dominating. Hence, $\mathfrak{d}(\mathbf{A}) \cdot \mathfrak{d}(\mathbf{B}) \leq \mathfrak{d}(\mathbf{A};\mathbf{B})$.

To prove that $\mathfrak{d}(\mathbf{A};\mathbf{B}) \leq \mathfrak{d}(\mathbf{A}) \cdot \mathfrak{d}(\mathbf{B})$, suppose that $X \subseteq A_+$ and $Y \subseteq B_+$ are dominating with $|X| = \mathfrak{d}(\mathbf{A})$ and $|Y| = \mathfrak{d}(\mathbf{B})$. We want to show that $X \times Y$ is $(\mathbf{A};\mathbf{B})$-dominating. For $(x, f) \in A_- \times B_-^{A_+}$, there are $a \in A_+$ and $b \in B_+$ such that $x \sqsubset a$ and $f(a) \sqsubset' b$. Therefore, $(x, f) \sqsubset_; (a, b)$.

(d) Chose f_0 some function from A_+ into B_-. Define,

$$\begin{aligned}
\varphi_- &: A_- \to A_- \times B_-^{A_+} \\
&\quad x \mapsto (x, f_0), \\
\varphi_+ &: A_+ \times B_+ \to A_+ \\
&\quad (a, b) \mapsto a.
\end{aligned}$$

Clearly, for any $x \in A_-$ and $(a, b) \in A_+ \times B_+$, $\varphi_-(x) \sqsubset_; (a, b)$ implies $x \sqsubset \varphi_+(a, b)$. So, $\mathbf{A} \preceq_{\mathrm{T}} (\mathbf{A};\mathbf{B})$. On the other hand, define choose $x_0 \in B_-$ and

$$\begin{aligned}
\varphi'_- &: B_- \to A_- \times B_-^{A_+} \\
&\quad y \mapsto (x_0, f_y), \text{ where } f_y \text{ is the constant function } y, \\
\varphi'_+ &: A_+ \times B_+ \to B_+ \\
&\quad (a, b) \mapsto b.
\end{aligned}$$

Clearly, for any $y \in B_-$ and $(a, b) \in A_+ \times B_+$, $\varphi'_-(y) \sqsubset_; (a, b)$ implies $y \sqsubset' \varphi'_+(a, b)$. So, $\mathbf{B} \preceq_{\mathrm{T}} (\mathbf{A};\mathbf{B})$. Therefore, $\mathfrak{b}(\mathbf{A};\mathbf{B}) \leq \min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\}$ and $\max\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\} \leq \mathfrak{d}(\mathbf{A};\mathbf{B})$. To finish, we prove that $\min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{A})\} \leq \mathfrak{b}(\mathbf{A};\mathbf{B})$. Let $X \subseteq A_- \times B_-^{A_+}$ with $|X| < \min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\}$. Define $X_1 = \{x : \exists f((x, f) \in X)\}$. Clearly, $|X_1| < \min\{\mathfrak{b}(\mathbf{A})), \mathfrak{b}(\mathbf{B}))\}$ and $X_1 \subseteq A_-$, so there is an $a \in A_+$ such that $\forall x \in X_1(x \sqsubset a)$. On the other hand, put $X_2 = \{f(a) : \exists x((x, f) \in X)\}$. Note that $|X_2| < \min\{\mathfrak{b}(\mathbf{A})), \mathfrak{b}(\mathbf{B}))\}$ and $X_2 \subseteq B_-$, so there is a $b \in A_+$ such that $\forall x \in X_2(f(a) \sqsubset' b)$. Therefore, $\forall (x, f) \in X((x, f) \sqsubset_; (a, b))$, so $\min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\} \leq \mathfrak{b}(\mathbf{A};\mathbf{B})$.

□

Chapter 2

Cardinal invariants associated with Yorioka ideals

In 2002, Yorioka [Y02] introduced some σ-ideals, which we call *Yorioka ideals*, to prove that $\mathrm{cof}(\mathcal{SN}) > \mathfrak{c}$ is consistent with ZFC where $\mathcal{SN}$ is the ideal of strongly measure zero subsets of the real line. These ideals are contained in the null ideal $\mathcal{N}$. The main results of this chapter concern the relationship between the cardinal coefficients of Yorioka ideals with the cardinals of Cichoń's digram as illustrated in Figure 2.1. In Section 2.1 we define Yorioka ideals and its associated cardinal invariants. In Section 2.2 we prove the inequalities between the additivity and cofinaliy of Yorioka ideals with the cardinals in Cichoń's diagram. We also give two original proofs of Lemmas 2.5 and 2.7, which were stated without proof in [O08], due to the author and D. Mejía. In Section 2.3 we prove inequalities between the covering and uniformity of Yorioka ideals and the cardinals in Cichoń's diagram.

2.1 Yorioka ideals

2.1 Definition. (1) For $\sigma \in (2^{<\omega})^{\omega}$, define

$$[\sigma] = \{x \in 2^{\omega} : \exists^{\infty} n \in \omega(\sigma(n) \subseteq x)\} = \bigcap_{n<\omega} \bigcup_{m \geqslant n} [\sigma(m)]$$

where $[t] = \{x \in 2^{\omega} : t \subseteq x\}$ for each $t \in 2^{<\omega}$.

(2) For each $\sigma \in (2^{<\omega})^{\omega}$ define $h_{\sigma} \in \omega^{\omega}$ such that $\forall n \in \omega(h_{\sigma}(n) = |\sigma(n)|)$.

(3) For each $g \in \omega^{\omega}$, we define the family $\mathcal{J}_g$ by

$$\mathcal{J}_g = \{X \subseteq 2^{\omega} : \exists \sigma \in (2^{<\omega})^{\omega}(X \subseteq [\sigma] \wedge h_{\sigma} = g)\}.$$

Note that $g \leq^* g'$ implies $\mathcal{J}_g \supseteq \mathcal{J}_{g'}$.

Kamo-Osuga [KO08] proved that $\mathcal{J}_g$ is not an ideal for any function $g \in \omega^{\omega}$ that satifies $\forall^{\infty} n(g(n+1) \geq g(n) + 3)$.

2.2 Definition ([Y02])**.** Let $f \in \omega^\omega$. Define the relation $\ll$ on ω^ω by

$$f \ll g \text{ iff } \forall k < \omega \forall^\infty n < \omega (f(n^k) \leq g(n)).$$

For $f \in \omega^\omega$, define

$$\mathcal{I}_f = \bigcup_{g \gg f} \mathcal{J}_g.$$

Any ideal of the form $\mathcal{I}_f$ where f is increasing is called a *Yorioka ideal*.

Note that $f \leq^* f'$ implies $\mathcal{I}_f \supseteq \mathcal{I}_{f'}$. Thus, $f \leq^* f'$ implies $\mathrm{cov}(\mathcal{I}_f) \leq \mathrm{cov}(\mathcal{I}_{f'})$ and $\mathrm{non}(\mathcal{I}_f) \geq \mathrm{non}(\mathcal{I}_{f'})$. Also $\mathcal{J}_f \supseteq \mathcal{I}_f$ holds for any $f \in \omega^\omega$. Moreover, $f \ll g$ implies $\mathcal{J}_g \subseteq \mathcal{I}_f$, so $\mathrm{cov}(\mathcal{J}_f) \leq \mathrm{cov}(\mathcal{I}_f) \leq \mathrm{cov}(\mathcal{J}_g)$ and $\mathrm{non}(\mathcal{J}_g) \leq \mathrm{non}(\mathcal{I}_f) \leq \mathrm{non}(\mathcal{J}_f)$.

2.3 Theorem (Yorioka [Y02])**.** *Let $f \in \omega^\omega$ be a strictly increasing function. Then $\mathcal{SN} \subseteq \mathcal{I}_f \subseteq \mathcal{N}$ and $\mathcal{I}_f$ is a σ-ideal. Moreover,* $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{I}_f) \leq \mathrm{cov}(\mathcal{SN})$ *and* $\mathrm{non}(\mathcal{SN}) \leq \mathrm{non}(\mathcal{I}_f) \leq \mathrm{non}(\mathcal{N})$.

Fix a strictly increasing function $f \in \omega^\omega$. For $d \in \omega^{\uparrow\omega}$, define $g^f_d \in \omega^\omega$ by

$$g^f_d(n) = f(n^{k+10}) \quad \text{for} \quad n \in [d(k), d(k+1)).$$

denote $\omega^{\uparrow\omega} := \{d \in \omega^\omega : d(0) = 0 \text{ and is monote increasing}\}$. Note that $d \leq^* e$ iff $g_e \leq^* g_d$, and $d \leq e$ iff $g_e \leq g_d$.

2.4 Lemma (Osuga [O06])**.** *For each $d \in \omega^{\uparrow\omega}$, g_d is strictly increasing and $g^f_d \gg f$ for each $d \in \omega^{\uparrow\omega}$. Also, for each $g \gg f$ there exists a $d \in \omega^{\uparrow\omega}$ such that $g^f_d \leq^* g$. Even more, if $D \subseteq \omega^{\uparrow\omega}$ is dominating, then $\mathcal{I}_f = \bigcup_{d \in D} \mathcal{J}_{g^f_d}$.*

We are also interested in the following cardinal invariants related to the ideals of the form $\mathcal{I}_f$:

$$\mathrm{minadd} = \min\{\mathrm{add}(\mathcal{I}_f) : f \in \omega^\omega \text{ and } f \text{ is strictly increasing}\}$$

$$\mathrm{supcov} = \sup\{\mathrm{cov}(\mathcal{I}_f) : f \in \omega^\omega \text{ and } f \text{ is strictly increasing}\}$$

$$\mathrm{minnon} = \min\{\mathrm{non}(\mathcal{I}_f) : f \in \omega^\omega \text{ and } f \text{ is strictly increasing}\}$$

$$\mathrm{supcof} = \sup\{\mathrm{cof}(\mathcal{I}_f) : f \in \omega^\omega \text{ and } f \text{ is strictly increasing}\}$$

It is not necessary to talk about $\mathrm{mincov}, \mathrm{supnon}, \mathrm{supadd}$ and mincof because $\mathrm{mincov} = \mathrm{cov}(\mathrm{id}_\omega)$, $\mathrm{supnon} = \mathrm{non}(\mathrm{id}_\omega)$, $\mathrm{supadd} = \mathrm{add}(\mathrm{id}_\omega)$ and $\mathrm{mincof} = \mathrm{cof}(\mathrm{id}_\omega)$, which follows from the fact that $f \leq^* f'$ implies $\mathrm{cov}(\mathcal{I}_f) \leq \mathrm{cov}(\mathcal{I}_{f'})$ and $\mathrm{non}(\mathcal{I}_f) \geq \mathrm{non}(\mathcal{I}_{f'})$, and that $\mathrm{add}(\mathcal{I}_f) \leq \mathrm{add}(\mathcal{I}_{\mathrm{id}_\omega})$ and $\mathrm{cof}(\mathcal{I}_f) \geq \mathrm{cof}(\mathcal{I}_{\mathrm{id}_\omega})$ for any increasing $f \in \omega^\omega$ by Lemma 2.7.

2.2 Additivity and cofinality

This section presents results about the additivity and cofinality of Yorioka ideals in relation with Cichoń's diagram.

2.5 Lemma. $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{I}_f)$ *and* $\mathrm{cof}(\mathcal{I}_f) \leq \mathrm{cof}(\mathcal{N})$. *In consequence* $\mathrm{add}(\mathcal{N}) \leq \mathrm{minadd}$ *and* $\mathrm{supcof} \leq \mathrm{cof}(\mathcal{N})$.

Proof. Assume $\kappa < \mathrm{add}(\mathcal{N})$. Let $\langle \sigma_\alpha : \alpha < \kappa \rangle \subseteq (2^{<\omega})^\omega$ with $h_{\sigma_\alpha} = g^f_{d_\alpha}$ for some $d_\alpha \in \omega^{\uparrow\omega}$, which means that $[\sigma_\alpha] \in \mathcal{J}_{g^f_{d_\alpha}} \subseteq \mathcal{I}_f$. We want to find a $\tau \in (2^{<\omega})^\omega$ such that $[\tau] \in \mathcal{I}_f$ and $\forall \alpha < \kappa([\sigma_\alpha] \subseteq [\tau])$.
Since $\kappa < \mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$, we can find an $e \in \omega^{\uparrow\omega}$ such that $\forall \alpha < \kappa(d_\alpha \leq^* e)$. In particular, $g^f_e \leq^* g^f_{d_\alpha}$.

Define $\sigma'_\alpha(i) := \sigma_\alpha(i) \restriction g_e(i)$ if $g^f_e(i) \leq g^f_{d_\alpha}(i)$ or else, $\sigma'_\alpha(i)$ is some member of $2^{g^{f(i)}_e}$. It is clear that $[\sigma_\alpha] \subset [\sigma'_\alpha] \in \mathcal{J}_{g^f_e}$.

As $\mathrm{add}(\mathcal{N}) = \min\{|F| : F \subset (2^{<\omega})^\omega \text{ and } \forall s \in \mathcal{S}(2^{<\omega}, \mathrm{id}_\omega) \exists x \in F(x \in^* s)\}$ by Theorem 1.4, there exists an $s^* \in \mathcal{S}(2^{<\omega}, \mathrm{id}_\omega)$ such that $\forall \alpha < \kappa(\sigma'_\alpha \in^* s)$.
Without loss of generality, suppose that $\forall i \forall \tau \in s^*(i)(|\tau| = g^f_e(i))$(Here, for $s^*(i)$, we are only interested in sequences of length $g^f_e(i)$, like $\sigma'_\alpha(i)$ for all $\alpha < \kappa$). Let $\langle I_n \rangle_{n<\omega}$ be the interval partition of ω such that $|I_n| = |s^*(n)|$.
Enumerate

$$s^*(n) = \{\tau_{n,i} : i \in I_n\}.$$

Define $\tau \in (2^{<\omega})^\omega$ by $\tau(i) = \tau_{n,i}$ when $i \in I_n$. We want to show

(a) $[\tau] \in \mathcal{I}_f$ and

(b) $\forall \alpha < \kappa([\sigma'_\alpha] \subset [\tau])$.

For (a) it is enough to prove that $h_\tau \gg f$. Let us show that $\forall k < \omega \exists N' < \omega \forall i \geq N'(f(i^k) \leq h_\tau(i))$. Fix $k < \omega$. As $g^f_e \gg f$ there exists $N > 2$ such that $\forall n \geq N(f(n^{3k}) \leq g^f_e(n))$. For $i \in I_n$ with $n \geq N$, $i < \sum_{k \leq n} s^*(k) \leq \sum_{k \leq n} k = \sum_{k \leq n} k^2 \leq n^3$, thus, $h_\tau(i) = |\tau(i)| = g^f_e(n) \geq f(n^{3k}) \geq f(i^k)$.

For (b) assume that $x \in [\sigma'_\alpha]$. Choose $N < \omega$ such that $\forall n \geq N(\sigma'(n) \in s^*(n))$. By definition of $s^*(n)$, $\exists i_n \in I_n(\sigma'_\alpha(n) = \tau_{n,i_n})$. Therefore, $\exists^\infty i(x \in [\tau])$.

The previous argument allows us to describe a Tukey connection between the corresponding structures. For $X \in \mathcal{I}_f$ there exists a $\sigma_X \in (2^{<\omega})^\omega$, with $h_{\sigma_X} = g^f_{d_X}$ for some $d_X \in \omega^{\omega\uparrow}$, such that $X \subseteq [\sigma_X]$. Define

$$\begin{aligned} F_X : \omega^{\uparrow\omega} &\to (2^{<\omega})^\omega \\ e &\mapsto \sigma_X \end{aligned}$$

where $\sigma'_{X,e}(i) := \sigma_X(i) \restriction g^f_e(i)$ if $g^f_e(i) \leq g^f_{d_X}(i)$ or else, $\sigma'_{X,e}(i)$ is some fixed member of $2^{g^{f(i)}_e}$, so $[\sigma'_{X,e}] \in \mathcal{J}_{g^f_e}$.

Define φ_- as

$$\begin{aligned} \varphi_- : \mathcal{I}_f &\to \omega^{\uparrow\omega} \times ((2^{<\omega})^\omega)^{\omega^{\uparrow\omega}} \\ X &\mapsto (d_X, F_X) \end{aligned} \tag{2.1}$$

For $e \in \omega^{\uparrow\omega}$ and $s \in \mathcal{S}(2^{<\omega}, \mathrm{id}_\omega)$, define $s^*(n) = \{t \in s(n) : |t| = g^f_e(i)\}$. We can find an interval partition $\langle I_n \rangle_{n<\omega}$ of ω such that $|I_n| = |s^*(n)|$ when $\exists^\infty n < \omega(s^*(n) \neq \emptyset)$, otherwise, $\varphi_+(e, s)$ below is some fixed member of $\mathcal{I}_f$.
In the firts case, enumerate

$$s^*(n) = \{\tau_{n,i} : i \in I_n\}.$$

Now define $\tau_{e,s} \in (2^{<\omega})^\omega$ by $\tau_{e,s}(i) = \tau_{n,i}$ when $i \in I_n$, so $[\tau_{e,s}] \in \mathcal{I}_f$ by the some argument as in (a).

Define φ_+ as

$$\begin{aligned}\varphi_+ : \omega^{\uparrow\omega} \times \mathcal{S}(2^{<\omega}, \mathrm{id}_\omega) &\to \mathcal{I}_f \\ (e, s) &\to [\tau_{e,s}]\end{aligned}$$

From an argument as in (b), if $X \in \mathcal{I}_f$, $e \in \omega^{\uparrow\omega}$, $s \in \mathcal{S}(2^{<\omega}, \mathrm{id}_\omega)$ such that $d_X \leq^* e$ and $\sigma'_{e,X} \in^* s^*$, then $X \subseteq [\tau_{e,s}]$.

Put $\mathbf{A} = \langle \omega^{\uparrow\omega}, \omega^{\uparrow\omega}, \leq^* \rangle$ and $\mathbf{B} = \langle (2^{<\omega})^\omega, \mathcal{S}(2^{<\omega}, \mathrm{id}_\omega), \in^* \rangle$(Note that $\langle \omega^{\uparrow\omega}, \omega^{\uparrow\omega}, \leq^* \rangle) \cong_{\mathrm{T}} \langle \omega^\omega, \omega^\omega, \leq^* \rangle$), so φ_+, φ_- witness that $(\mathcal{I}_f, \mathcal{I}_f, \subseteq) \preceq_{\mathbf{T}} (\mathbf{A}; \mathbf{B})$. Therefore $\mathfrak{d}(\mathcal{I}_f, \mathcal{I}_f, \subseteq) \leq \mathfrak{d}(\mathbf{A}; \mathbf{B})$ but, by Theorem 1.52, $\mathfrak{d}(\mathbf{A}; \mathbf{B}) = \max\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\} = \max\{\mathfrak{d}, \mathrm{cof}(\mathcal{N})\} = \mathrm{cof}(\mathcal{N})$. As $\mathfrak{d}(\mathcal{I}_f, \mathcal{I}_f, \subseteq) = \mathrm{cof}(\mathcal{I}_f)$, hence $\mathrm{cof}(\mathcal{I}_f) \leq \mathrm{cof}(\mathcal{N})$. Also, this Tukey-Galois order allows us to simplify the previous proof because $\mathfrak{b}(\mathbf{A}; \mathbf{B}) \leq \mathfrak{b}(\mathcal{I}_f, \mathcal{I}_f, \subseteq)$ and, by Theorem 1.52, $\mathfrak{b}(\mathbf{A}; \mathbf{B}) = \min\{\mathfrak{b}(\mathbf{A}), \mathfrak{b}(\mathbf{B})\} = \min\{\mathfrak{b}, \mathrm{add}(\mathcal{N})\} = \mathrm{add}(\mathcal{N})$ and $\mathfrak{b}(\mathcal{I}_f, \mathcal{I}_f, \subseteq) = \mathrm{add}(\mathcal{I}_f)$, so $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{I}_f)$. □

The following theorem illustrates the relationship between the additivity and cofinality of Yorioka ideals with $\mathfrak{b}$ and $\mathfrak{d}$.

2.6 Theorem (Kamo and Osuga [KO08])**.** $\mathrm{add}(\mathcal{I}_f) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \mathrm{cof}(\mathcal{I}_f)$. *Even more* $\langle \omega^\omega, \omega^\omega, \leq^* \rangle \preceq_{\mathrm{T}} \langle \mathcal{I}_f, \mathcal{I}_f, \subseteq \rangle$.

The following lemma shows the monotonicity of aditivity and cofinality of Yorioka ideals.

2.7 Lemma. *Let* $f, f' \in \omega^\omega$ *be strictly increasing. If* $\forall^\infty n < \omega(f(n+1) - f(n) \leq f'(n+1) - f'(n))$, *then* $\mathrm{add}(\mathcal{I}_f) \geq \mathrm{add}(\mathcal{I}_{f'})$ *and* $\mathrm{cof}(\mathcal{I}_f) \leq \mathrm{cof}(\mathcal{I}_{f'})$. *In particular* $\mathrm{add}(\mathcal{I}_f) \leq \mathrm{add}(\mathcal{I}_{\mathrm{id}_\omega})$ *and* $\mathrm{cof}(\mathcal{I}_{\mathrm{id}_\omega}) \leq \mathrm{cof}(\mathcal{I})$ *for any increasing* $f \in \omega^\omega$.

Proof. If $c \in \omega$ then $\mathcal{I}_{f'} = \mathcal{I}_{f'+c}$ because

$$f' \ll g \Leftrightarrow f' + c \ll g$$

for all $g \in \omega^\omega$. Indeed, suppose that $f' \ll g$. Let $k < \omega$ and $m > c + 2$ such that $\forall n \geq m(f'(n^{k+1}) \leq g(n))$. For $n \geq m$, $f'(n^k) + c \leq f'(n^k + c) \leq f'(n^k + n) \leq f'(n^{k+1})$ because f' is increasing, so

$$f'(n^k) + c \leq f'(n^{k+1}) \leq g(n).$$

Therefore, $f' + c \ll g$.

By the previous fact, we can assume wlog that $f(0) \leq f'(0)$ and $\forall n < \omega(f(n+1) - f(n) \leq f'(n+1) - f'(n))$. Note that $m < n \Rightarrow f(n) - f(m) \leq f'(n) - f'(m)$. By induction on n,

$$\begin{aligned}f(n+1) - f(m) &\leq f(n) - f(m) + f'(n+1) - f'(n) \\ &\leq f'(n) - f'(m) + f'(n+1) - f'(n) \\ &\leq f'(n+1) - f'(m).\end{aligned}$$

In particular, $f \leq f'$. As a consequence, for any $d \in \omega^{\uparrow\omega}$,

(a) $g_d^f(0) \leq g_d^{f'}(0)$ and

(b) $g_d^f(i+1) - g_d^f(i) \leq g_d^{f'}(i+1) - g_d^{f'}(i)$ for all $i < \omega$.

Assume that $\kappa < \mathrm{add}(\mathcal{I}_{f'})$. Let $\langle \sigma_\alpha : \alpha < \kappa \rangle \subseteq (2^{<\omega})^\omega$ where $h_{\sigma_\alpha} = g^f_{d_\alpha}$ for some $d_\alpha \in \omega^{\uparrow\omega}$, which means that $[\sigma_\alpha] \in \mathcal{J}_{g^f_{d_\alpha}} \subseteq \mathcal{I}_f$. As $\kappa < \mathrm{add}(\mathcal{I}_{f'}) \leq \mathfrak{b}$, find $e \in \omega^{\uparrow\omega}$ dominating $\{d_\alpha : \alpha < \kappa\}$ and σ'_α such that $h_{\sigma'} = g^f_e$ and $[\sigma_\alpha] \subseteq [\sigma'_\alpha] \in \mathcal{J}_{g^f_e}$ as in the proof of Lemma 2.5. We want to find a $\rho \in (2^{<\omega})^\omega$ such that $[\rho] \in \mathcal{I}_f$ and $[\sigma_\alpha] \subseteq [\rho]$ for all $\alpha < \kappa$.

Define $\tau_\alpha \in (2^{<\omega})^\omega$ by

$$\tau_\alpha(n) = \sigma'_\alpha(n) \restriction g^f_e(0)^\frown \underbrace{\langle 0 \cdots 0 \rangle}_{\text{lenght } g^{f'}_e(0) - g^f_e(0)} {}^\frown \cdots {}^\frown \sigma'_\alpha(n) \restriction (g^f_e(n) \setminus g^f_e(n-1))^\frown \underbrace{\langle 0 \cdots 0 \rangle}_{\text{lenght } g^{f'}_e(n) - g^{f'}_e(n-1) - (g^f_e(n) - g^f_e(n-1))}$$

whenever $d_\alpha(n) \leq e(n)$, otherwise, τ_α is a fixed member of $2^{g^{f'}_e(n)}$.

Note that $h_{\tau_\alpha} = g^{f'}_e$, so $[\tau_\alpha] \in \mathcal{I}_{f'}$. Since $\kappa < \mathrm{add}(\mathcal{I}_{f'})$, there exists a $\tau \in (2^{<\omega})^\omega$ such that $\forall \alpha < \kappa([\tau_\alpha] \subseteq [\tau])$ and $h_\tau = g^{f'}_{e^*}$ for some $e^* \in \omega^{\uparrow\omega}$ with $e \leqslant e^*$, so $g^{f'}_{e^*} \leqslant g^{f'}_e$.

Define ρ by

$$\rho(n) = \tau(n) \restriction g^f_e(0)^\frown \cdots {}^\frown \tau(n) \restriction [g^{f'}_e(n-1), g^{f'}_e(n-1) + g^f_e(n) - g^f_e(n-1))$$

Note that

$$|\rho(n)| = \begin{cases} g^f_e(m) & \text{if } g^{f'}_{e^*}(n) \geq g^{f'}_e(m-1) + g^f_e(m) - \\ & -g^f_e(m-1) \\ g^f_e(m-1) + g^{f'}_{e^*}(n) - g^{f'}_e(m-1) & \text{otherwise,} \end{cases}$$

where m is the minimum natural number such that $g^{f'}_{e^*}(n) \leq g^{f'}_e(m)$ (put $g^f_e(-1) = g^{f'}_e(-1) = 0$). Clearly $m \leqslant n$.

2.8 Claim. $h_\rho \gg f$

Proof. We prove that $\forall c < \omega \exists N < \omega \forall n \geq N(f(n^c) \leq h_\rho(n))$. Let $k_0 = 0$ and $k_{m+1} = \min\{k < \omega : g^{f'}_{e^*}(n) > g^{f'}_e(m) \text{ for all } n > k\}$. Note that $\{k_m\}_{m<\omega}$ is a monotone increasing sequence that goes to $+\infty$. Put $I_m = [k_m, k_{m+1})$ for $m < \omega$. Note that, for $n \in I_m$, $m = \min\{m' : g^{f'}_{e^*}(n) \geq g^{f'}_e(m')\}$ so $|\rho(n)| \geq g^f_e(m-1)$. Fix $c > 0$. Find $N > e^*(3c)$ such that, for all $m \geq N^{3c}$

(i) $k_m > e^*(3c)$ and

(ii) $(m-1)^2 \geq m$.

2.9 Subclaim. *If $m \geq N^{3c}$ and $m \in [e(k), e(k+1))$ then $k_{m+1} \leq m^{\frac{k+10}{3c}}$*

Proof. If $n > m^{\frac{k+10}{3c}}$ then $n^{3c} > m^{k+10}$, so $g^{f'}_{e^*}(n) \geq f'(n^{3c}) > f'(m^{k+10}) = g^{f'}_e(m)$, because $e^*(3c) < N \leq N^{k+10} \leq m^{(\frac{k+10}{3c})} < n$. □

Fix $n > k_{N^{3c}}$, so $n \in I_m$ for some $m \geq N^{3c}$. Therefore, $n < k_{m+1} \geq m^{\frac{k+10}{3c}}$ where $m \in [e(k), e(k+1))$. Then, $h_\rho(n) = |\rho(n)| \geq g^f_e(m-1) \geq f((m-1)^{k+9}) \geq f(n^c)$ since $(m-1)^{k+9} \geq m^{\frac{k+9}{2}}$ and $\frac{k+9}{2} \geq \frac{k+10}{3}$, so $(m-1)^{k+9} \geq m^{\frac{k+10}{3}} \geq m^{(\frac{k+10}{3c})c} \geq n^c$. □

Therefore $[\rho] \in \mathcal{I}_f$. We now prove that $\forall \alpha < \kappa([\sigma'_\alpha] \subseteq [\rho])$. Let $x \in [\sigma'_\alpha]$, i.e. $\exists^\infty n(\sigma'_\alpha(n) \subseteq x)$. Put

$$x' = x \restriction g^f_e(0)^\frown \underbrace{\langle 0 \cdots 0 \rangle}_{\text{length } g^{f'}_e(0) - g^f_e(0)} {}^\frown \cdots {}^\frown x \restriction (g^f_e(n) \setminus g^f_e(n-1))^\frown \underbrace{\langle 0 \cdots 0 \rangle}_{\text{length } g^{f'}_e(n) - g^{f'}_e(n-1) - (g^f_e(n) - g^f_e(n-1))} {}^\frown \cdots$$

Clearly, $x' \in [\tau_\alpha] \subseteq [\tau]$. As $\tau(n) \subseteq x'$ implies $\rho(n) \subseteq x$, we conclude that $x \in [\rho]$.

The proof above actually gives an argument for $(\mathcal{I}_f, \mathcal{I}_f, \subseteq) \preceq_{\mathrm{T}} (\mathbf{A}; \mathbf{B})$ which we prove below. Here, $\mathbf{A} = \langle \omega^{\uparrow\omega}, \omega^{\uparrow\omega}, \leq^* \rangle$ and $\mathbf{B} = \langle \mathcal{I}_{f'}, \mathcal{I}_{f'}, \subseteq \rangle$.

For $X \in \mathcal{I}_f$ there are $d_X \in \omega^{\uparrow\omega}$ and $\sigma_X \in (2^{<\omega})^\omega$ such that $X \subseteq [\sigma_X]$ and $h_{\sigma_X} = g^f_{d_X}$. For $e \in \omega^{\uparrow\omega}$, define $F_X(e)$ by

$F_X(e) = [\tau_{X,e}]$ where

$$\tau_{X,e}(n) = \sigma_e(n) \restriction g^f_e(0)^\frown \underbrace{\langle 0 \cdots 0 \rangle}_{\text{lenght } g^{f'}_e(0) - g^f_e(0)} {}^\frown \cdots {}^\frown \sigma_e(n) \restriction (g^f_e(n) \setminus g^f_e(n-1))^\frown \underbrace{\langle 0 \cdots 0 \rangle}_{\text{lenght } g^{f'}_e(n) - g^{f'}_e(n-1) - (g^f_e(n) - g^f_e(n-1))}$$

if $d_X(n) \leq e(n)$, otherwise, it is just a fixed member of $2^{g^{f'}_e(n)}$. Note that $h_{\tau_{X,e}} = 2^{g^{f'}_e}$, so $[\tau_{X,e}] \in \mathcal{J}_{g^{f'}_e} \subseteq \mathcal{I}_{f'}$.

Define

$$\begin{gathered} \varphi_- : \mathcal{I}_f \to \omega^{\uparrow\omega} \times (\mathcal{I}_{f'})^{\omega^{\uparrow\omega}} \\ X \to (d_X, F_X) \end{gathered}$$

Now, for $e \in \omega^{\uparrow\omega}$ and $Y \in \mathcal{I}_{f'}$, there are $e^* \geq e$ in $\omega^{\uparrow\omega}$ and $\tau_{e,Y} \in (2^{<\omega})^\omega$ with $h_{\tau_{e,Y}} = g^{f'}_{e^*}$ such that $Y \subseteq [\tau_{e,Y}]$. Put

$$\rho_{Y,e} = \tau_{e,Y}(n) \restriction g^f_e(0)^\frown \cdots {}^\frown \tau_{e,Y}(n) \restriction [g^{f'}_e(n-1), g^{f'}_e(n-1) + g^f_e(n) - g^f_e(n-1))$$

By the previous argument for ρ (Claim 2.8), $[\rho_{Y,e}] \in \mathcal{I}_f$.

Define

$$\begin{gathered} \varphi_+ : \omega^{\uparrow\omega} \times \mathcal{I}_{f'} \to \mathcal{I}_f \\ (e, Y) \to [\rho_{Y,e}] \end{gathered}$$

To finish the proof we have to check, that for $X \in \mathcal{I}_f$, $Y \in \mathcal{I}_{f'}$ and $e \in \omega^{\uparrow\omega}$, if $d_X \leq^* e$ and $F_X(e) \subseteq Y$, then $X \subseteq [\rho_{Y,e}]$. But this holds by the same argument after Claim 2.8.

Therefore, φ_-, φ_+ witness $(\mathcal{I}_f, \mathcal{I}_f, \subseteq) \preceq_{\mathrm{T}} (\mathbf{A}; \mathbf{B})$, so $\mathfrak{d}(\mathcal{I}_f, \mathcal{I}_f, \subseteq) \leq \mathfrak{d}(\mathbf{A}; \mathbf{B})$. But $\mathfrak{d}(\mathcal{I}_f, \mathcal{I}_f, \subseteq) = \mathrm{cof}(\mathcal{I}_f)$ and, $\mathfrak{d}(\mathbf{A}; \mathbf{B}) = \max\{\mathfrak{d}(\mathbf{A}), \mathfrak{d}(\mathbf{B})\} = \max\{\mathfrak{d}, \mathrm{cof}(\mathcal{I}_{f'})\} = \mathrm{cof}(\mathcal{I}_{f'})$ by Theorems 1.52 and 2.6, so $\mathrm{cof}(\mathcal{I}_f) \leq \mathrm{cof}(\mathcal{I}_{f'})$. On the other hand, $\mathrm{add}(\mathcal{I}_{f'}) = \min\{\mathrm{add}(\mathcal{I}_{f'}), \mathfrak{b}\} = \mathfrak{b}(\mathbf{A}.\mathbf{B}) \leq \mathfrak{b}(\mathcal{I}_f, \mathcal{I}_f, \subseteq) = \mathrm{add}(\mathcal{I}_f)$. □

2.3 Covering and uniformity

The following lemmas show the relationships between two kinds of covering numbers for slaloms and $\mathcal{J}_g$.

2.10 Lemma (Kamo and Osuga [KO14]). *Let $b \in \omega^\omega$ satisfying $b \geq 2$. If $g \in \omega^\omega$ satisfies $g \geq (\log b)^+$, then $\mathfrak{c}^\exists_{b,1} \leq \mathrm{cov}(\mathcal{J}_g)$ and $\mathrm{non}(\mathcal{J}_g) \leq \mathfrak{n}^\exists_{b,1}$. Moreover, $\langle \mathbb{R}_b, \mathbb{R}_b, =^\infty \rangle \preceq_{\mathrm{T}} \langle 2^\omega, \mathcal{J}_g, \in \rangle$.*

2.11 Lemma (Kamo and Osuga [KO14])**.** *Let $h, b \in \omega^\omega$ and $g \in \omega^\omega$ monotone increasing. If $1 \leq h \leq b$ and $b \geq 2^{g \circ (h^+ - 1)}$, then $\langle 2^\omega, \mathcal{J}_g, \in\rangle \preceq_{\mathrm{T}} \langle \mathbb{R}_b, \mathcal{S}(b,h), \in^\infty\rangle$. So, $\mathrm{cov}(\mathcal{J}_g) \leq \mathfrak{c}^\exists_{b,h}$ and $\mathfrak{n}^\exists_{b,h} \leq \mathrm{non}(\mathcal{J}_g)$.*

We can characterize minnon and supcov as follows:

2.12 Proposition. *If $h \in \omega^\omega$ with $h \geq 1$, then*

$$\mathrm{supcov} = \sup\{\mathfrak{c}^\exists_{b,h} : b \in \omega^\omega\} \text{ and } \mathrm{minnon} = \min\{\mathfrak{n}^\exists_{b,h} : b \in \omega^\omega\}.$$

Proof. We first prove that, for all $b \in \omega^\omega$, $\mathfrak{c}^\exists_{b,h} \leqslant \mathrm{supcov}$ and $\mathrm{minnon} \leq \mathfrak{n}^\exists_{b,h}$. Let $f = (\log b)^+$. As $\mathcal{J}_f \supseteq \mathcal{I}_f$ then $\mathrm{cov}(\mathcal{J}_f) \leqslant \mathrm{cov}(\mathcal{I}_f)$ and $\mathrm{non}(\mathcal{I}_f) \leqslant \mathrm{non}(\mathcal{J}_f)$. By Lemma 2.10, $\mathfrak{c}^\exists_{b,1} \leq \mathrm{cov}(\mathcal{J}_f)$ and $\mathrm{non}(\mathcal{J}_f) \leq \mathfrak{n}^\exists_{b,1}$ and, from Example, 1.50 $\mathfrak{c}^\exists_{b,h} \leq \mathfrak{c}^\exists_{b,1}$ and $\mathfrak{n}^\exists_{b,1} \leq \mathfrak{n}^\exists_{b,h}$ (since $h \geq 1$). Therefore, so $\mathfrak{c}^\exists_{b,h} \leq \mathrm{cov}(\mathcal{I}_f) \leq \mathrm{supcov}$ and $\mathrm{minnon} \leq \mathrm{non}(\mathcal{I}_f) \leq \mathfrak{n}^\exists_{b,h}$.

To prove the converse inequalities, assume that $f \in \omega^{\uparrow\omega}$. Choose some $f \gg g$. Then $\mathcal{J}_g \subseteq \mathcal{I}_f$. Hence $\mathrm{cov}(\mathcal{I}_f) \leqslant \mathrm{cov}(\mathcal{J}_g)$ and $\mathrm{non}(\mathcal{J}_g) \leq \mathrm{non}(\mathcal{I}_f)$. Put $b = 2^{g \circ h^+}$. By Lemma 2.11, $\mathrm{cov}(\mathcal{J}_g) \leq \mathfrak{c}^\exists_{b,h}$ and $\mathfrak{n}^\exists_{b,h} \leq \mathrm{non}(\mathcal{J}_g)$, so $\mathrm{cov}(\mathcal{I}_f) \leq \mathfrak{c}^\exists_{b,h} \leq \sup\{\mathfrak{c}^\exists_{b,h} : b \in \omega^\omega\}$ and $\min\{\mathfrak{n}^\exists_{b,h} : b \in \omega^\omega\} \leq \mathfrak{n}^\exists_{b,h} \leq \mathrm{non}(\mathcal{I}_f)$. □

We know that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$, but we can refine this.

We will first give the following proposition that shows the relationship between the cardinal invariants $\mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M})$ with $\mathfrak{c}^\exists_{b,h}$ and $\mathfrak{n}^\exists_{b,h}$.

2.13 Proposition. *For any $b, h \in \omega^\omega$, if $h \geq^* 1$ then*

$$\mathrm{cov}(\mathcal{M}) \leq \mathfrak{n}^\exists_{b,h} \text{ and } \mathfrak{c}^\exists_{b,h} \leq \mathrm{non}(\mathcal{M}).$$

Proof. From Example, 1.49 $\langle \mathbb{R}_b, \mathcal{S}(b,h), \in^\infty\rangle \preccurlyeq_{\mathbf{T}} \langle \mathbb{R}_b, \mathbb{R}_b, =^\infty\rangle$. It is clear that $\langle \mathbb{R}_b, \mathbb{R}_b, =^\infty\rangle \preceq_{\mathbf{T}} \langle \omega^\omega, \omega^\omega, =^\infty\rangle$, so $\langle \mathbb{R}_b, \mathcal{S}(b,h), \in^\infty\rangle \preceq_{\mathbf{T}} \langle \omega^\omega, \omega^\omega, =^\infty\rangle$. By Theorem 1.47, $\mathfrak{b}(\omega^\omega, \omega^\omega, =^\infty) \leq \mathfrak{b}(\mathbb{R}_b, \mathcal{S}(b,h), \in^\infty)$ and $\mathfrak{d}(\mathbb{R}_b, \mathcal{S}(b,h), \in^\infty) \leq \mathfrak{d}(\omega^\omega, \omega^\omega, =^\infty)$. On the other hand, $\mathfrak{b}(\mathbb{R}_b, \mathcal{S}(b,h), \in^\infty) = \mathfrak{n}^\exists_{b,h}$, $\mathfrak{d}(\mathbb{R}_b, \mathcal{S}(b,h), \in^\infty) = \mathfrak{c}^\exists_{b,h}$ and, by Theorem 1.5, $\mathfrak{b}(\omega^\omega, \omega^\omega, =^\infty) = \mathrm{cov}(\mathcal{M})$ and $\mathfrak{d}(\omega^\omega, \omega^\omega, =^\infty) = \mathrm{non}(\mathcal{M})$, so $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{n}^\exists_{b,h}$ and $\mathfrak{c}^\exists_{b,h} \leq \mathrm{non}(\mathcal{M})$. □

2.14 Corollary. $\mathrm{supcov} \leq \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathrm{minnon}$. *In particular,* $\mathrm{cov}(\mathcal{I}_f) \leq \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{I}_f)$ *for any $f \in \omega^\omega$ increasing.*

Proof. It follows from Propositions 2.12 and 2.13. □

2.15 Lemma. $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{minnon}\}$.

Proof. The inequality $\leq$ follows from Corollary 2.14.

Let us prove $\geq$. Assume $\kappa < \min\{\mathfrak{b}, \mathrm{minnon}\}$, that is, $\kappa < \mathfrak{b}$ and $\kappa < \mathrm{minnon}$. Suppose that $F \subseteq \omega^\omega$ has size κ. As $\kappa < \mathfrak{b}$ there exists a $b \in \omega^\omega$ such that $\forall f \in F(f \leq^* b)$. On the other hand, $\kappa < \mathrm{minnon} = \min\{\mathfrak{n}^\exists_{b,1} : b \in \omega^\omega\}$, so $\kappa < \mathfrak{n}^\exists_{b,1}$. Since $\forall f \in F(f \leq^* b)$, without loss of generality we assume $F \subseteq \mathbb{R}_b$. As $\kappa < \mathfrak{n}^\exists_{b,1}$, there exists a $y \in \mathbb{R}_b \subseteq \omega^\omega$ such that $\forall x \in F(x =^\infty y)$. Therefore, $\kappa < \mathfrak{b}\langle \omega^\omega, \omega^\omega, =^\infty\rangle$ for any $\kappa < \min\{\mathfrak{b}, \mathrm{minnon}\}$, so $\min\{\mathfrak{b}, \mathrm{minnon}\} \leq \mathfrak{b}(\omega^\omega, \omega^\omega, =^\infty) = \mathrm{cov}(\mathcal{M})$ by Theorem 1.5. □

As a consequence

2.16 Corollary. $\mathrm{minadd} \leq \mathrm{add}(\mathcal{M})$.

Proof. It follows from Theorem 2.6 and Lemma 2.15. □

We can also characterize $\mathrm{cof}(\mathcal{M})$.

2.17 Lemma. $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{supcov}\}$

Proof. The inequality $\geq$ follows from Corollary 2.14.

Let us prove $\leq$. Let $D \subseteq \omega^{\uparrow\omega}$ be dominating, $|D| = \mathfrak{d}$. By Proposition 2.12, $\mathrm{supcov} = \sup\{\mathfrak{c}^{\exists}_{b,1} : b \in D\}$. For each $d \in D$ let $E_d \subseteq \mathbb{R}_d$ be a witness of $\mathfrak{c}^{\exists}_{d,1}$. Then $E = \bigcup_{d\in D} E_d$ satisfies $\forall x \in \omega^\omega \exists y \in E(x =^\infty y)$. Indeed, for $x \in \omega^\omega$ there exists a $d \in D$ such that $x \leq^* d$, wlog $x \in \mathbb{R}_d$. As $E_d \subseteq \mathbb{R}_d$ is a witness of $\mathfrak{c}^{\exists}_{d,1}$, there exists $y \in E_d$ such that $x =^\infty y$, hence $y \in E$. Therefore, $\mathrm{non}(\mathcal{M}) \leq |E| \leq \mathfrak{d} \cdot \mathrm{supcov}$ by Theorem 1.5, so, $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\} \leq \max\{\mathfrak{d}, \mathrm{supcov}\}$. □

2.18 Corollary. $\mathrm{cof}(\mathcal{M}) \leq \mathrm{supcof}$

Proof. It follows from Theorem 2.6 and Lemma 2.17. □

Figure 2.1 summarizes the relations between the cardinal invariants associated to Yorioka ideals with the cardinals in Cichoń's diagram that we have represented in this section.

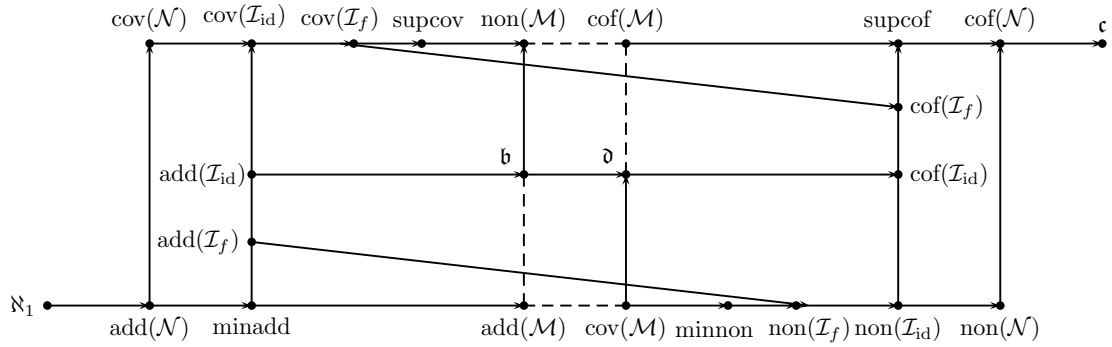


Figure 2.1: Extended Cichoń's diagram.

Chapter 3

Preservation properties

In this chapter, we present a general theory of preservation for finite support iterations and extend its use to matrix iterations. This type of preservation was introduced for the first time in [JS90] and [Br91] by Judah-Shelah and Brendle, respectively, to get models of cardinal invariants with large continuum from fsi of ccc forcing, later generalized and summarized in [BJ] and [M13]. In Section 3.1 we introduce this theory and generalize several known results. In Section 3.2 we present some typical examples of preservation properties. In the last Section 3.3, we generalize properties studied in [BlS84], [BrF11] and in [M13] to preserve unbounded reals along fsi, which are useful for matrix iterations in Chapter 4.

3.1 Preservation properties

3.1 Definition. Say that $\mathcal{R} = \langle X, Y, \sqsubset \rangle$ is a *Polish relational system* if

(I) X is an uncountable Polish space,

(II) $Y = \bigcup_{e\in\Omega} Y_e$ where Ω is a non-empty set and, for some Polish space Z, Y_e is non-empty and analytic in Z for all $e \in \Omega$, and

(III) $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ where $\langle \sqsubset_n : n < \omega \rangle$ is some increasing sequence of closed sets in $X \times Z$ such that, for any $n < \omega$ and for any $y \in Y$

(i) $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense.

For a set A and $x \in X$ say that x is *$\mathcal{R}$-unbounded over A* if $\forall y \in A \cap Y (x \not\sqsubset y)$.
Fix, for this section, a Polish relational system $\mathcal{R} = \langle X, Y, \sqsubset \rangle$.

3.2 Lemma. $\langle X, \mathcal{M}(X), \in \rangle \preceq_{\boldsymbol{T}} \langle X, Y, \sqsubset \rangle$. *In particular,* $\mathfrak{b}(\mathcal{R}) \leq \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathcal{R})$.

Proof. Define

$$\begin{aligned} \varphi_- &: X \to X \\ &\quad x \mapsto x \\ \varphi_+ &: Y \to \mathcal{M}(X) \\ &\quad y \mapsto \varphi_+(y) = \{x \in X : x \sqsubset y\} = \bigcup_{n<\omega} \{x \in X : x \sqsubset_n y\} \end{aligned}$$

Clearly, for any $x \in X$ and $y \in Y$, $x \sqsubset y$ implies $x \in \varphi_+(y)$ Therefore, (φ_-, φ_-) is a Galois-Tukey connection that witnesses $\langle X, \mathcal{M}(X), \in\rangle \preceq_{\mathbf{T}} \langle X, Y, \sqsubset\rangle$. □

Fix, for this section, an uncountable regular cardinal θ.

3.3 Definition. A family $F \subseteq X$ is *θ-$\mathcal{R}$-unbounded* if, for any $A \subseteq Y$ of size $< \theta$, there exists is an $x \in F$ which is $\mathcal{R}$-unbounded over A.

Clearly, any θ-$\mathcal{R}$-unbounded family is $\mathcal{R}$-unbounded, so

3.4 Lemma. *If $F \subseteq X$ is θ-$\mathcal{R}$-unbounded, then $\mathfrak{b}(\mathcal{R}) \leq |F|$ and $\theta \leq \mathfrak{d}(\mathcal{R})$.*

The following are useful properties to preserve θ-$\mathcal{R}$-unbounded families from the ground model in generic extensions. In this context, X, Z and $\sqsubset$ are interpreted in models of ZFC as explained in Section 1.1. In the case of Y, it is interpreted as $Y^M = \bigcup_{e\in\Omega} Y_e^M$ for a transitive model M of ZFC containing the information to define Y. We also omit the superindex M on the interpretation of Y.

3.5 Definition. Let $\mathbb{P}$ be a forcing notion.

(1) $\mathbb{P}$ is *θ-$\mathcal{R}$-good* if for any $\mathbb{P}$-name $\dot{h}$ for a member of Y, there exists a nonempty $H \subseteq Y$ (in the ground model) of size $< \theta$ such that, for any $x \in X$, if x is $\mathcal{R}$-unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.

(2) $\mathbb{P}$ is *θ-$\mathcal{R}$-nice* if, for all $e \in \Omega$ and for any $\mathbb{P}$-name $\dot{h}$ for a member of Y_e, there exists a nonempty $H \subseteq Y$ (in the ground model) of size $< \theta$ such that, for any $x \in X$, if x is $\mathcal{R}$-unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.

Note that $\theta < \theta'$ implies that any θ-$\mathcal{R}$-good poset is θ'-$\mathcal{R}$-good. Also, if $\mathbb{P} \lessdot \mathbb{Q}$ and $\mathbb{Q}$ is θ-$\mathcal{R}$-good, then $\mathbb{P}$ is θ-$\mathcal{R}$-good

Say that $\mathbb{P}$ is *$\mathcal{R}$-good* (*$\mathcal{R}$-nice*) if it is $\aleph_1$-$\mathcal{R}$-good ($\aleph_1$-$\mathcal{R}$-nice). It is clear that any θ-$\mathcal{R}$-good forcing notion is θ-$\mathcal{R}$-nice. The converse holds in some cases as below.

3.6 Lemma. *If either $\mathbb{P}$ is θ-cc or $|\Omega| < \theta$, then $\mathbb{P}$ is θ-$\mathcal{R}$-nice iff it is θ-$\mathcal{R}$-good.*

Proof. Assume that either $\mathbb{P}$ is θ-cc or $|\Omega| < \theta$. Let $\dot{h}$ an $\mathbb{P}$-name for a member of Y. Choose A a maximal antichain in $\mathbb{P}$ and $\{e_p : p \in A\} \subseteq \Omega$ such that $p \Vdash \dot{h} \in Y_{e_p}$ for all $p \in A$. Put $\Gamma := \{e_p : p \in A\}$. By the hypothesis, it follows that Γ has size $< \theta$. For each $\beta \in \Gamma$, define $A_\beta = \{p \in A : e_p = \beta\}$. As $p \Vdash \dot{h} \in Y_\beta$ for any $p \in A_\beta$, we can find a $\mathbb{P}$-name $\dot{y}_\beta$ of a member of Y_β such that $p \Vdash \dot{h} = \dot{y}_\beta$ for any $p \in A_\beta$.

As $\mathbb{P}$ is θ-$\sqsubset$-nice, for each $\beta \in \Gamma$ there exists a nonemty $H_\beta \subseteq Y$ of size $< \theta$ such that, for any $x \in \mathcal{R}$-unbounded over H_β, $\Vdash x \not\sqsubset \dot{y}_\beta$. Put $H = \bigcup_{\beta\in\Gamma} H_\beta$ which has size $< \theta$ because $|H_\beta| < \theta$ for all $\beta \in \Gamma$, $|\Gamma| < \theta$ and θ is regular.

Assume that $x \in X$ is $\mathcal{R}$-unbounded over H. Given $p \in A$, we have that $p \in \bigcup_{\beta\in\Gamma} A_\beta$, so there is a $\beta \in \Gamma$ such that $p \in A_\beta$. As $x \in X$ is $\mathcal{R}$-unbounded over H_β, then $p \Vdash x \not\sqsubset \dot{y}_\beta$, but on the other hand $p \Vdash \dot{h} = \dot{y}_\beta$, so $p \Vdash x \not\sqsubset \dot{h}$. As A is a maximal antichain and $p \Vdash x \not\sqsubset \dot{h}$ for all $p \in A$, then $\Vdash x \not\sqsubset \dot{h}$. □

3.7 Lemma. *Assume that $\mathbb{P}$ is θ-$\mathcal{R}$-good and $\lambda \geq \theta$ is a cardinal.*

(i) If $F \subseteq X$ is θ-$\mathcal{R}$-unbounded, then $\mathbb{P}$ forces that F is θ-$\mathcal{R}$-unbounded.

(ii) If $\mathfrak{d}(\mathcal{R}) \geq \lambda$, then $\mathbb{P}$ forces that $\mathfrak{d}(\mathcal{R}) \geq \lambda$.

Proof. (i) Let $\{\dot{h}_\alpha\}_{\alpha<\beta}$ be a sequence of $\mathbb{P}$-names for members of Y with $\beta<\theta$. For $\alpha<\theta$, let $H_\alpha\subseteq Y$ of size $<\theta$ that witnesses θ-$\mathcal{R}$-good for $\dot{h}_\alpha$. Put $H=\bigcup_{\alpha<\beta}H_\alpha$, which has size $<\theta$ because of the regularity of θ. As F is θ-$\mathcal{R}$-unbounded, there exist an $x\in F$ which is $\mathcal{R}$-unbounded over H. Therefore, $\Vdash x\not\sqsubset\dot{h}_\alpha$ for all $\alpha<\beta$.

(ii) Let $\{\dot{g}_\alpha\}_{\alpha<\eta}$ be a sequence of $\mathbb{P}$-names for members of Y with $\eta<\lambda$. For $\alpha<\eta$, let $C_\alpha\subseteq Y$ of size $<\theta$ that witnesses the goodness of $\mathbb{P}$ for $\dot{g}_\alpha$. Put $C=\bigcup_{\alpha<\eta}C_\alpha$. As $\mathfrak{d}(\mathcal{R})\geq\lambda$ and $|C|<\lambda$ then C is not $\mathcal{R}$-dominating, so there is some $x\in X$ $\mathcal{R}$-unbounded over C. Then, $\Vdash x\not\sqsubset\dot{g}_\alpha$ for any $\alpha<\eta$, that is, $\mathbb{P}$ forces that $\{\dot{g}_\alpha\}_{\alpha<\eta}$ is not $\mathcal{R}$-dominating.

□

We first aim to prove that the property θ-$\mathcal{R}$-good is satisfied in fsi of θ-cc posets (see Theorem 3.13).

3.8 Definition. Let $\mathbb{P}$ be a forcing notion, p a condition in $\mathbb{P}$, and let $\dot{z}$ be a $\mathbb{P}$-name for a real in ω^ω. A pair $(\{p_n\}_{n<\omega},g)$ is called an *interpretation of $\dot{z}$ in $\mathbb{P}$ below p* if $g\in\omega^\omega$ and, for all $n<\omega$,

(1) $p_n\in\mathbb{P}$, $p_0\leq p$ and $p_{n+1}\leq p_n$, and

(2) $p_n\Vdash\dot{z}\restriction n=g\restriction n$.

3.9 Lemma. *If $p\in\mathbb{P}$ and $\dot{z}$ is $\mathbb{P}$-name for a real in ω^ω then there is an interpretation of $\dot{z}$ below p.*

Proof. Assume that $p\in\mathbb{P}$ and $\dot{z}$ is $\mathbb{P}$-name for a real in ω^ω. By induction on $n<\omega$, we construct $\{p_n\}_{n<\omega}$ decreasing in $\mathbb{P}$ and $\{s_n\}_{n<\omega}$ $\subseteq$-increasing such that $s_n\in\omega^n$ and $p_n\Vdash\dot{z}\restriction n=s_n$. For $n=0$, put $p_0=p$, $s_0=\emptyset$ and clearly $p_0\Vdash\dot{z}\restriction 0=s_0$. Now assume $p_n\in\mathbb{P}$ and $s_n\in\omega^n$ such that $p_n\Vdash\dot{z}\restriction n=s_n$. There is a $p_{n+1}\leq p_n$ that decide $\dot{z}\restriction(n+1)$, i.e., there exists an $s_{n+1}\in\omega^{n+1}$ such that $p_{n+1}\Vdash\dot{z}\restriction(n+1)=s_{n+1}$. Moreover, as $p_{n+1}\Vdash\dot{z}\restriction n=s_n$ and $p_{n+1}\Vdash\dot{z}\restriction(n+1)=s_{n+1}$ then $p_{n+1}\Vdash s_n\subseteq s_{n+1}$, so $s_n\subseteq s_{n+1}$ in the ground model. Define $g=\bigcup_{n<\omega}s_n$. Clearly, $p_n\Vdash g\restriction n=s_n=\dot{z}\restriction n$. So $(\{p_n\}_{n<\omega},g)$ is an interpretation of $\dot{z}$ below p. □

3.10 Lemma. *Assume that $\mathbb{P}$ is a poset , $e\in\Omega$, $f:\omega^\omega\to Y_e$ is a continuous and surjective function, $\dot{z}$ a $\mathbb{P}$-name for a real in ω^ω and let $(\{p_n\}_{n<\omega},g)$ be an interpretation of $\dot{z}$ below $\mathbb{1}$ in $\mathbb{P}$. If $x\in X$ $n<\omega$ and $x\not\sqsubset_n f(g)$, then there is a $k<\omega$ such that $p_k\Vdash x\not\sqsubset_n f(\dot{z})$.*

Proof. Suppose that $x\in X$ such that $x\not\sqsubset_n f(g)$. As $\{y\in Y_e:x\sqsubset_n y\}$ is closed in Y_e (see Definition 3.1) and $f:\omega^\omega\to Y_e$ is continuous, then $f^{-1}[\{y\in Y_e:x\sqsubset_n y\}]$ is closed in ω^ω. Define $C_x:=\{w\in\omega^\omega:x\sqsubset_n f(w)\}$ and note that $f^{-1}[\{y\in Y_e:x\sqsubset_n y\}]=C_x$, so C_x is a closed set in ω^ω. As $x\not\sqsubset_n f(g)$, then there is a $k<\omega$ such that $[g\restriction k]\cap C_x=\emptyset$. On the other hand, by hypothesis $p_k\Vdash\dot{z}\restriction k=g\restriction k$, so $p_k\Vdash[\dot{z}\restriction k]\cap C_x=\emptyset$, which implies $p_k\Vdash\dot{z}\notin C_x$. Hence, $p_k\Vdash x\not\sqsubset_n f(\dot{z})$. □

3.11 Lemma. *Any poset of size $<\theta$ is θ-$\mathcal{R}$-good. In particular, $\mathbb{C}$ is $\mathcal{R}$-good.*

Proof. Put $\mathbb{P}=\{p_\alpha:\alpha<\mu\}$ where $\mu=|\mathbb{P}|$. Let $e\in\Omega$ and $\dot{h}$ be a $\mathbb{P}$-name for a member of Y_e. It is enough to prove that $\mathbb{P}$ is θ-$\mathcal{R}$-nice by Lemma 3.6. Choose a continuous and surjective function $f:\omega^\omega\to Y_e$. Let $\dot{z}$ be a $\mathbb{P}$-name for a real in ω^ω such that $\mathbb{P}$ forces

that $f(\dot{z}) = \dot{h}$. By Lemma 3.9, for each $\alpha < \mu$, choose an interpretation $(\{p_{\alpha,n}\}_{n<\omega}, z_\alpha)$ of $\dot{z}$ below p_α. We prove that, if $x \in X$ and $\forall \alpha < \mu (x \not\sqsubset f(z_\alpha))$ then $\Vdash x \not\sqsubset \dot{h}$, that is, $\forall p \in \mathbb{P} \forall m < \omega \exists q \leq p (q \Vdash x \not\sqsubset_m \dot{h})$. Fix $p \in \mathbb{P}$ and $m < \omega$, so there exists an $\alpha < \mu$ such that $p = p_\alpha$. From Lemma 3.10 there exists a $k < \omega$ such that $p_{\alpha,k} \Vdash x \not\sqsubset_m f(\dot{z})$. Therefore, $p_{\alpha,k} \Vdash x \not\sqsubset_m \dot{h}$ and $p_{\alpha,k} \leq p_\alpha = p$. □

3.12 Lemma. *Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset. If $\mathbb{P}$ is θ-cc, θ-$\mathcal{R}$-good and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is θ-$\mathcal{R}$-good, then $\mathbb{P} * \dot{\mathbb{Q}}$ is θ-$\mathcal{R}$-good*

Proof. Let $\dot{h}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a member of Y. Wlog $\mathbb{P}$ forces that $\dot{h}$ is a $\dot{\mathbb{Q}}$-name for a member in Y. As $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is θ-$\mathcal{R}$-good, then $\mathbb{P}$ forces that there is a nonempty $H \subseteq Y$ of size $< \theta$ such that, for any $x \in X$, if x $\mathcal{R}$-unbounded over H, then $\Vdash_{\dot{\mathbb{Q}}} x \not\sqsubset \dot{h}$. By the Maximal principle, there is a $\mathbb{P}$-name $\dot{H}$ for such an H. Choose a $\mathbb{P}$-name $\dot{\mu}$ for a cardinal number such that $\mathbb{P}$ forces that $|\dot{H}| = \dot{\mu} < \theta$ and choose a maximal antichain A in $\mathbb{P}$ and a function $f : A \to \theta \setminus \{0\}$ such that any $p \in A$ forces $\dot{\mu} = f(p)$. Then, put $\nu := \sup\{f(p) : p \in A\}$ which is $< \theta$ by regularity of θ. Clearly, $\mathbb{P}$ forces $|\dot{H}| \leq \dot{\mu} \leq \nu$, so there exists an onto function from ν into $\dot{H}$. By the Maximal principle, there exists a $\mathbb{P}$-name $\dot{g}$ for that onto fuction. Put $\dot{y}_\alpha := \dot{g}(\alpha)$ for all $\alpha < \nu$. Clearly, $\mathbb{P}$ forces $\dot{H} = \{\dot{y}_\alpha : \alpha < \nu\}$. Now, for each $\alpha < \nu$ let B_α be a witness of goodness of $\mathbb{P}$ for $\dot{y}_\alpha$, so put $B = \bigcup_{\alpha<\nu} B_\alpha$ which has size $< \theta$ and witnesses goodness of $\mathbb{P} * \dot{\mathbb{Q}}$ for $\dot{h}$. We prove that, if $x \in X$ is $\mathcal{R}$-unbounded over B, $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} x \not\sqsubset \dot{h}$. Assume that $x \in X$ is $\mathcal{R}$-unbounded over B, then $\mathbb{P}$ forces that $\Vdash x \not\sqsubset \dot{y}_\alpha$, therefore, $\mathbb{P}$ forces that $x \in X$ is $\mathcal{R}$-unbounded over $\dot{H}$, so $\mathbb{P}$ forces that $\Vdash_{\dot{\mathbb{Q}}} x \not\sqsubset \dot{h}$ by Hypothesis. Hence, we get that $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} x \not\sqsubset \dot{h}$. □

3.13 Theorem. (Preservation of goodness in fsi). *Let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\delta}$ be a fsi of θ-cc forcing notions. If for each $\alpha < \delta$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is θ-$\mathcal{R}$-good, then $\mathbb{P}_\delta$ is θ-$\mathcal{R}$-good.*

Proof. We prove by induction on $\alpha \leq \delta$ that $\mathbb{P}_\alpha$ is θ-$\mathcal{R}$-good. If $\alpha = 0$, $\mathbb{P}_0 = \mathbb{1}$ is θ-$\mathcal{R}$-good by Lemma 3.11; the succesor step $\mathbb{P}_{\alpha+1} \simeq \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ is clear from Lemma 3.12. For the limit step we consider two cases. First assumme that $\mathrm{cf}(\alpha) < \theta$. It is enough to prove that $\mathbb{P}$ is θ-$\mathcal{R}$-nice by Lemma 3.6. Let $e \in \Omega$ and let $\dot{h}$ be a $\mathbb{P}$-name for member in Y_e. Choose a continuous and surjective function $f : \omega^\omega \to Y_e$. Let $\dot{z}$ be a $\mathbb{P}_\alpha$-name for a real in ω^ω such that $\mathbb{P}$ forces that $f(\dot{z}) = \dot{h}$. Fix an increasing sequence $\{\alpha_\xi\}_{\xi<\mathrm{cf}(\alpha)}$ that converges to α. By Lemma 3.9 for each $\xi < \mathrm{cf}(\alpha)$, find a $\mathbb{P}_{\alpha_\xi}$-name for a real $\dot{z}_\xi$ in ω^ω and a sequence $\langle \dot{p}_{\xi,k} : k < \omega \rangle$ of $\mathbb{P}_{\alpha_\xi}$-names that represents a decreasing sequence of conditions in $\mathbb{P}_\alpha/\mathbb{P}_{\alpha_\xi}$ such that $\mathbb{P}_{\alpha_\xi}$ forces that $\dot{p}_{\xi,k} \Vdash_{\mathbb{P}/\mathbb{P}_{\alpha_\xi}} \dot{z} \restriction k = \dot{z}_\xi \restriction k$. Choose $H_\xi \subseteq Y$ of size $< \theta$ that witnesses goodness of $\mathbb{P}_{\alpha_\xi}$ for $f(\dot{z}_\xi)$. Put $H = \bigcup_{\xi<\mathrm{cf}(\alpha)} H_\xi$ which has size $< \theta$ since $\mathrm{cf}(\alpha) < \theta$ and θ is regular.

We prove that, if $x \in X$ is $\mathcal{R}$-unbounded over H, then $\Vdash_{\mathbb{P}_\alpha} x \not\sqsubset \dot{h}$. Assume towards a contradiction that there are $p \in \mathbb{P}_\alpha$ and $n < \omega$ such that $p \Vdash_{\mathbb{P}_\alpha} x \sqsubset_n \dot{h}$. Choose $\xi < \mathrm{cf}(\alpha)$ such that $p \in \mathbb{P}_{\alpha_\xi}$. Let G be $\mathbb{P}_{\alpha_\xi}$-generic over the ground model V with $p \in G$. Then, by the choice of H_ξ, $x \not\sqsubset f(z_\xi)$, in particular, $x \not\sqsubset_n f(z_\xi)$. From Lemma 3.10 there is a $k < \omega$ such that $p_{\xi,k} \Vdash_{\mathbb{P}/\mathbb{P}_{\alpha_\xi}} x \not\sqsubset_n f(\dot{z}) = \dot{h}$. On the other hand, by hypothesis, $\Vdash_{\mathbb{P}/\mathbb{P}_{\alpha_\xi}} x \sqsubset_n f(\dot{z}_\xi)$, a contradiction.

Now assume that $\mathrm{cf}(\alpha) \geq \theta$. Let $\dot{h}$ be a $\mathbb{P}_\alpha$ for a member in Y. By θ-cc and Theorem 1.29, there exists an $\zeta < \alpha$ such that $\dot{h}$ is a $\mathbb{P}_\zeta$-name. As $\mathbb{P}_\zeta$ is θ-$\mathcal{R}$-good, then there is an $H \subseteq Y$ of size $< \theta$ that witnesses goodness of $\mathbb{P}_\zeta$ for $\dot{h}$. It is clear that H also witnesses goodness of $\mathbb{P}_\alpha$ for $\dot{h}$.

□

The following results show how to add $\mathcal{R}$-unbounded families with Cohen reals, in order to get values for $\mathfrak{b}(\mathcal{R})$ and $\mathfrak{d}(\mathcal{R})$. These are essential to prove the main result of this thesis.

3.14 Lemma. *If μ is a regular uncountable cardinal and $\mathbb{P}_\mu = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\mu}$ is a fsi where each $\dot{\mathbb{Q}}_\alpha$ is forced (by $\mathbb{P}_\alpha$) to be ccc and non-trivial, then $\mathbb{P}_\mu$ adds a μ-$\mathcal{R}$-unbounded family (of Cohen reals) of size μ.*

Proof. Fix an increasing enumeration $\{\delta_\alpha : \alpha < \mu\}$ of all the limit ordinals less that μ. By Lemma 1.36 and Corollary 1.37, for each $\alpha < \mu$ there exists a $\mathbb{P}_{\delta_{\alpha+1}}$-name $\dot{c}_\alpha$ of a Cohen real over $V^{\mathbb{P}_{\delta_\alpha}}$. We prove that $\mathbb{P}_\mu$ forces that $\{\dot{c}_\alpha : \alpha < \mu\}$ is μ-$\mathcal{R}$-unbounded. Let $\{\dot{y}_\xi\}_{\xi<\nu}$ be a sequence of $\mathbb{P}_\mu$-names for members of Y with $\nu < \mu$, so there is a $\beta < \mu$ such that $\{\dot{y}_\xi\}_{\xi<\nu}$ is a sequence of $\mathbb{P}_\beta$-names. Wlog put $\beta = \delta_\alpha$ for some $\alpha < \mu$. By Definition 3.1, $\{x \in X : x \sqsubset \dot{y}_\xi\}$ is an F_σ meager set coded in $V^{\mathbb{P}_{\delta_\alpha}}$ for all $\xi < \nu$, so $\mathbb{P}_{\delta_{\alpha+1}}$ forces that $\dot{c}_\alpha \not\sqsubset \dot{y}_\xi$ for all $\xi < \nu$. □

3.15 Theorem. *Let $\delta \geq \theta$ be an ordinal and let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\delta}$ be a fsi such that, for each $\alpha < \delta$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name of a non-trivial ccc and θ-$\mathcal{R}$-good poset. Then, $\mathbb{P}_\delta$ forces that $\mathfrak{b}(\mathcal{R}) \leq \theta$ and $|\delta| \leq \mathfrak{d}(\mathcal{R})$.*

Proof. Let ν be a uncountable regular cardinal such that $\theta \leq \nu \leq |\delta|$. By Lemma 3.14, $\mathbb{P}_\nu$ adds a ν-$\mathcal{R}$-unbounded family of size ν. In V_ν, $\mathbb{P}_\delta/\mathbb{P}_\nu = \langle \mathbb{P}_{\nu+\xi}/\mathbb{P}_\nu, \dot{\mathbb{Q}}_{\nu+\xi} : \xi < \delta - \nu \rangle$ is θ-$\mathcal{R}$-good by Theorem 3.13 and thus ν-$\mathcal{R}$-good because $\theta \leq \nu$. Hence, by Lemma 3.7, $\mathbb{P}_\delta/\mathbb{P}_\nu$ preserves ν-$\mathcal{R}$-unbounded families. Therefore, $\mathbb{P}_\delta = \mathbb{P}_\nu * \mathbb{P}_\delta/\mathbb{P}_\nu$ forces that $\mathfrak{b}(\mathcal{R}) \leq \nu$ and $\nu \leq \mathfrak{d}(\mathcal{R})$. In particular, when $\nu = \theta$, $\mathbb{P}_\delta$ forces $\mathfrak{b}(\mathcal{R}) \leq \theta$. For $|\delta|$ we have two cases: if $|\delta|$ is regular, when $\nu = |\delta|$ $\mathbb{P}_\delta$ forces $|\delta| \leq \mathfrak{d}(\mathcal{R})$. If $|\delta|$ is singular, we can find a sequence $\{\kappa_\alpha\}_{\alpha<\mathrm{cf}(|\delta|)}$ of regular cardinals $< |\delta|$ such that $\sup_{\alpha<\mathrm{cf}(|\delta|)} \kappa_\alpha = |\delta|$. As above, for each $\alpha < \mathrm{cf}(|\delta|)$, $\mathbb{P}_\delta$ forces $\kappa_\alpha \leq \mathfrak{d}(\mathcal{R})$, so $\mathbb{P}_\delta$ forces $|\delta| \leq \mathfrak{d}(\mathcal{R})$. □

3.2 Examples of preservation properties

3.16 Example (Preserving non-meager sets)**.** $\mathbf{Ed} = (\omega^\omega, \omega^\omega, \neq^*)$ is a Polish relational system where $\Omega = 1, Z = \omega^\omega$ and $\neq^* = \bigcup_{n<\omega} \neq^*_n$ where $x \neq^*_n y$ iff $\forall m \geq n(x(m) \neq y(m))$ and $x \neq^* y$ iff $\exists n \forall m \geq n(x(m) \neq y(m))$. As a consequence of Bartoszyński's characterization (Theorem 1.5) $\mathfrak{b}(\mathbf{Ed}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{Ed}) = \mathrm{cov}(\mathcal{M})$.

Every forcing notion that adds an eventually different real is not $\mathrm{cov}(\mathcal{M})$-$\mathbf{Ed}(\omega^\omega)$-good, so this applies to $\mathbb{LOC}$, $\mathbb{D}$ and $\mathbb{B}$.

3.17 Example (Preserving unbounded families)**.** $\mathbf{Dom} = (\omega^\omega, \omega^\omega, \leq^*)$ is a Polish relational system where $\Omega = 1, Z = Y = Y_0 = \omega^\omega$ and $\leq^* = \bigcup_{n<\omega} \leq^*_n$ where $x \leq^*_n y$ iff $\forall m \geq n(x(m) \leq y(m))$ and $x \leq^* y$ iff $\exists n \forall m \geq n(x(m) \leq y(m))$. Clearly, $\mathfrak{b}(\mathbf{Dom}) = \mathfrak{b}$ and $\mathfrak{d}(\mathbf{Dom}) = \mathfrak{d}$. $\mathbb{B}$ is $\mathbf{Dom}$-good because $\mathbb{B}$ is ω^ω-bounding.

3.18 Example (Preserving union of null sets is not null)**.** For $k < \omega$ let $\mathrm{id}^k : \omega \to \omega$ such that $\mathrm{id}^k(i) = i^k$ for all $i < \omega$ and put $\mathcal{H} = \{\mathrm{id}^{k+1} : k < \omega\}$. Let $\mathbf{Lc} := \langle \omega, \mathcal{S}(\omega, \mathcal{H}), \in^* \rangle$ be the Polish relational system where $\Omega = \mathcal{H}$, $Z = ([\omega]^{<\omega})^\omega$, $Y_h = \mathcal{S}(\omega, h)$ for each $h \in \mathcal{H}$ and

$$\mathcal{S}(\omega, \mathcal{H}) := \{\varphi : \omega \to \omega : \exists h \in \mathcal{H} \forall i < \omega(|\varphi(i)| \leq h(i))\} = \bigcup_{h \in \mathcal{H}} Y_h.$$

As a consequence of Bartoszyński's characterization (Theorem 1.4), $\mathfrak{b}(\mathbf{Lc}) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc}) = \mathrm{cof}(\mathcal{N})$.

3.19 Lemma ([Ka89]). *Every boolean algebra with a strictly positive finitely additive measure is* ***Lc****-good. In particular, any subalgebra of random forcing is* ***Lc****-good.*

3.20 Lemma ([JS90]). *If $\mu < \theta$ is a infinite cardinal, then every μ-centered poset is θ-****Lc****-good.*

3.21 Example (Preserving covering families related to $\mathfrak{c}^{\exists}_{b,h^{\mathrm{id}_\omega}}$). Fix a family $\mathcal{E} \subseteq \omega^\omega$ of non-decreasing functions of size $\aleph_1$ which satisfies

(1) $\forall e \in \mathcal{E}(e \leq \mathrm{id}_\omega)$,

(2) $\forall e \in \mathcal{E}(\lim_{n\to\infty} e(n) = \infty$ and $\lim_{n\to\infty}(n - e(n)) = \infty)$,

(3) $\forall e \in \mathcal{E}\exists e' \in \mathcal{E}(e + 1 \leq^* e')$ and

(4) $\forall \mathcal{E}' \in [\mathcal{E}]^{\leq\omega}\exists e \in \mathcal{E}\forall e' \in \mathcal{E}'(e' \leq^* e)$.

The existence of the family $\mathcal{E}$ is a consequence of [BrM14, Lemma 6.5].

For $b, h \in \omega^\omega$ such that $b > 0$ and $h \geq^* 1$, we define $\hat{\mathcal{S}}(b,h)(= \hat{\mathcal{S}}(b,h)[\mathcal{E}])$ by

$$\hat{\mathcal{S}}(b,h) := \bigcup_{e\in\mathcal{E}} \mathcal{S}(b,h^e) = \Big\{\varphi \in \prod_{n<\omega} \mathcal{P}(b(n)) : \exists e \in \mathcal{E}\forall n < \omega(|\varphi(n)| \leq h(n)^{e(n)})\Big\}$$

Let $n < \omega$. For $\psi, \varphi : \omega \to [\omega]^{<\omega}$, define the relation $\psi \blacktriangleright_n \varphi$ iff $\forall k \geq n(\psi(k) \not\supseteq \varphi(k))$, and define $\psi \blacktriangleright \varphi$ iff $\forall^\infty k < \omega(\psi(k) \not\supseteq \varphi(k))$, i.e., $\blacktriangleright = \bigcup_{n<\omega} \blacktriangleright_n$. $\mathbf{Slm}(b,h) = \langle \mathcal{S}(b,h^{\mathrm{id}_\omega}), \hat{\mathcal{S}}(b,h), \blacktriangleright\rangle$ is a Polish relational system where $\Omega = \mathcal{E}$, $Z = \prod_{n<\omega} \mathcal{P}(b(n))$ with the product topology where each $\mathcal{P}(b(n))$ has the discrete topology and $Y_e = \mathcal{S}(b,h^e)$ for each $e \in \mathcal{E}$.

We show a relation between $\mathbf{Slm}(b,h)$ and the cardinals $\mathfrak{c}^{\exists}_{b,h^{\mathrm{id}_\omega}}$ and $\mathfrak{n}^{\exists}_{b,h^{\mathrm{id}_\omega}}$. Define,

$$\begin{aligned}
\varphi_- &: \mathcal{S}(b,h^{\mathrm{id}_\omega}) \to \mathcal{S}(b,h^{\mathrm{id}_\omega}) \\
&\quad \psi \mapsto \psi \\
\varphi_+ &: \mathbb{R}_b \to \hat{\mathcal{S}}(b,h) \\
&\quad y \mapsto \varphi_y = \langle y(n)\rangle_{n<}
\end{aligned}$$

Note that, for any $x \in X$ and $y \in \mathbb{R}$

$$\psi \not\blacktriangleright^* y \text{ iff } \psi \blacktriangleright \varphi_y$$

Therefore, $\langle \mathcal{S}(b,h^{\mathrm{id}_\omega}), \hat{\mathcal{S}}(b,h), \blacktriangleright\rangle \preceq_{\mathrm{T}} \langle \mathcal{S}(b,h^{\mathrm{id}_\omega}), \mathbb{R}_b, \not\blacktriangleright^*\rangle$, so

$$\mathfrak{c}^{\exists}_{b,h^{\mathrm{id}_\omega}} \leq \mathfrak{b}(\mathbf{Slm}(b,h)) \text{ and } \mathfrak{d}(\mathbf{Slm}(b,h)) \leq \mathfrak{n}^{\exists}_{b,h^{\mathrm{id}_\omega}}.$$

This is the reason why we can use the preservation property of $\mathbf{Slm}(b,h)$ to decide the values of $\mathfrak{c}^{\exists}_{b,h^{\mathrm{id}_\omega}}$ and $\mathfrak{n}^{\exists}_{b,h^{\mathrm{id}_\omega}}$ by forcing. A particular case is when $h = 1$ we where have $\mathfrak{c}^{\exists}_{b,1} = \mathfrak{b}(\mathbf{Slm}(b,1))$ and $\mathfrak{n}^{\exists}_{b,1} = \mathfrak{d}(\mathbf{Slm}(b,1))$ because $\mathcal{S}(b,1)$ is Galois-Tukey equivalent with the Polish relational system $\mathbf{Ed}_b := \langle \mathbb{R}_b, \mathbb{R}_b, \neq^*\rangle$ (with $\Omega = 1$ and $Z = \mathbb{R}_b$).

3.22 Lemma. *Let $h, b \in \omega^\omega$ such that $b \geq^* 2$, $h \geq^* 1$ and let $\mathbb{P}$ be a $(h, b^{h^{id_\omega}})$-linked forcing notion. Then, $\mathbb{P}$ is* ***Slm****(b,h)-good.*

Proof. By Lemma 3.6 it is enough to show that $\mathbb{P}$ is $\mathbf{Slm}^*(b,h)$-nice, even more, we prove that for $e \in \mathcal{E}$ and any $\mathbb{P}$-name $\dot\varphi$ for a member of $\mathcal{S}(b,h^e)$, there exists a $\psi \in \hat{\mathcal{S}}(b,h)$ such that, for any $\vartheta \in \mathcal{S}(b,h^{\mathrm{id}_\omega})$, if $\vartheta \blacktriangleright \psi$ then $\Vdash \vartheta \blacktriangleright \dot\varphi$.

3.23 Claim ([KO14, Lemma 8])**.** *Let $h, c, H \in \omega^\omega$. Assume that $\mathbb{P}$ is (h, c^H)-linked. Then for each $\mathbb{P}$-name $\dot\psi$ with $\Vdash_{\mathbb{P}} \dot\psi \in \mathcal{S}(c,H)$, there exists a $\varphi \in \mathcal{S}(c, h\cdot H)$ such that $\Vdash_{\mathbb{P}} \exists^\infty n \in X \dot\psi \subseteq \varphi$ for all infinite subsets X of ω.*

By Claim 3.23, there exists a $\psi \in \mathcal{S}(b, h\cdot h^e)$ such that $\Vdash \exists^\infty n \in X(\dot\varphi(n) \subseteq \psi(n))$ for all $X \in [\omega]^\omega$. As $h \cdot h^e = h^{e+1}$, there is an $e' \in \mathcal{E}$ such that $e' \geq^* e+1$. Wlog, we assume that $\psi \in \mathcal{S}(b, h^{e'})$, so $\psi \in \hat{S}_{b,h}$.

To finish the proof, let $\vartheta \in \mathcal{S}(b,h^{\mathrm{id}_\omega})$ such that $\vartheta \blacktriangleright \psi$, that is, $\exists^\infty n < \omega(\psi(n) \subseteq \vartheta(n))$. Put $X = \{n < \omega : \psi(n) \subseteq \vartheta(n)\} \in [\omega]^\omega$. Therefore, $\Vdash \exists^\infty n \in X(\dot\varphi(n) \subseteq \psi(n) \subseteq \vartheta(n))$, in particular, $\Vdash \vartheta \blacktriangleright \dot\varphi$. □

3.24 Lemma. *If θ is uncountable regular and $\mu < \theta$ is an infinite cardinal, then any μ-centered poset is θ-$\boldsymbol{Slm}(b,h)$-good.*

Proof. Let $\mathbb{P}$ be a poset such that $\mathbb{P} = \bigcup_{\alpha<\mu} P_\alpha$ where each P_α is centered. It is sufficient to prove that $\mathbb{P}$ is $\mathbf{Slm}(b,h)$-nice by Lemma 3.6. Let $\dot\varphi$ be a $\mathbb{P}$-name for a member of $\mathcal{S}(b,h^e)$. For $\alpha<\mu$, by Lemma 1.21, for each $m \in \omega$ find an $a_{\alpha,m} \in [b(m)]^{\leq (h(m)^{e(m)})}$ such that no $p \in P_\alpha$ forces $a_{\alpha,m} \neq \dot\varphi(m)$.

For each $\alpha < \mu$, define $\psi_\alpha \in \mathcal{S}(b,h^e)$ where $\psi_\alpha(m) = a_{\alpha,m}$ for all $m < \omega$. Put $H := \{\psi_\alpha : \alpha < \mu\}$. Assume that $\vartheta \in \mathcal{S}(b,h^{\mathrm{id}_\omega})$ and $\vartheta \blacktriangleright \psi_\alpha$ for all $\alpha<\mu$. Fix $p \in \mathbb{P}$ and $m<\omega$. Choose $\alpha<\mu$ such that $p \in P_\alpha$ and find a $k \geq m$ such that $\vartheta(k) \supseteq \psi_\alpha(k)$. As $p \nVdash \psi_\alpha(k) \neq \dot\varphi(k)$, there is a $q \leq p$ that forces $\vartheta(k) \supseteq \psi_\alpha(k) = \dot\varphi(k)$. □

In particular, we have that $\mathbb{D}$ is $\mathbf{Slm}(b,h)$-good.

As an example, we give an aplication of preservation properties in fsi to construct a model where many values in Cichoń's diagram are separated.

3.25 Theorem ([M13, Theorem 4])**.** *In V, assume that $\mu_1 \leq \mu_2 \leq \kappa$ regular uncountable cardinals, $\lambda \geq \kappa$ cardinal and $\lambda^{<\mu_2} = \lambda$. Then, there exists a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \mu_1, \mathfrak{b} = \mu_2, \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \kappa$ and $\mathfrak{d} = \mathfrak{c} = \lambda$.*

Proof. Perform a fsi $\mathbb{P}_{\lambda\kappa} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle_{\alpha<\lambda\kappa}$ where $\lambda\kappa$ is the product as ordinals. Note that $\lambda\kappa$ is an ordinal of cofinality κ, moreover $\lambda\kappa = \bigcup_{\xi<\kappa}[\lambda\xi, \lambda(\xi+1))$ and $[\lambda\xi,\lambda(\xi+1))$ has length λ. Now, by induction on $\xi \leq \kappa$, we construct the iteration up to $\lambda\xi$, i.e., $\mathbb{P}_{\lambda\xi}$. We see how to define $\mathbb{P}_{\lambda(\xi+1)}$ because the limit step is only the direct limit of what had already been constructed. We construct the iteration on the interval $[\lambda\xi,\lambda(\xi+1))$. Enumerate $\{\dot{\mathbb{LOC}}^\xi_\epsilon : \epsilon<\lambda\}$ and $\{\dot{\mathbb{D}}^\xi_\epsilon : \epsilon<\lambda\}$ the nice $\mathbb{P}_{\lambda\xi}$-names for all the σ-linked subposets of $\mathbb{LOC}_h$ of size $<\mu_1$ and all the σ-centered subposets of $\mathbb{D}$ of size $<\mu_2$, respectively. Fix $C_1 \cup C_2 = \lambda \setminus \{0\}$ a disjoint union such that each C_i has size λ and $C_i = \{\rho^i_\epsilon : \epsilon<\lambda\}$ is an one-one enumeration.

Now, for each $\alpha \in [\lambda\xi,\lambda(\xi+1))$ we define $\dot{\mathbb{Q}}_\alpha$. There is a unique $\rho<\lambda$ such that $\alpha = \lambda\xi+\rho$. We consider three cases for ρ:

(i) If $\rho = 0$ then $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{B}$,

(ii) If $\rho \in C_1$ and $\rho = \rho^1_\epsilon$ for some (unique) $\epsilon<\lambda$, $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{LOC}}^\xi_\epsilon$.

(iii) If $\rho \in C_2$ and $\rho = \rho^2_\epsilon$ for some (unique) $\epsilon<\lambda$, $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{D}}^\xi_\epsilon$.

Note that, in V, by Lemmas 3.11, 3.19 and 3.20, Example 3.17 and Theorem 3.13, $\mathbb{P}_{\lambda\kappa}$ is μ_1-**Lc**-good and μ_2-**Dom**-good. Therefore, in $V_{\lambda\kappa} = V[G]$ for a fixed $\mathbb{P}_{\lambda\kappa}$-generic G over V, $\mathrm{add}(\mathcal{N}) \leq \mu_1$ and $\mathfrak{b} \leq \mu_2$ by Theorem 3.15. To see that $\mathfrak{b} \geq \mu_2$ in $V_{\lambda\kappa}$, let $\mathcal{F} \subseteq \omega^\omega$ be a set of size less than μ_1. By Theorem 1.29, there is an $\alpha < \kappa$ such that $\mathcal{F} \subseteq V_{\lambda\alpha}$. We work in $V_{\lambda\alpha}$. Let N be a transitive model of a finite large fragment of ZFC of size $< \mu_1$ such that $\mathcal{F} \subseteq N$. Note that $\mathbb{D}^N$ is a σ-centered subposet of $\mathbb{D}$ of size $< \mu_1$. Then, there is a $\rho < \lambda$ such that $\mathbb{D}^N = (\dot{\mathbb{D}}^\rho_\alpha)_{G\cap\mathbb{P}_{\lambda\alpha}}$. Since the Hechler added real by $\dot{\mathbb{D}}^\rho_\alpha$ dominates all the reals in N, $\mathcal{F}$ is bounded in $V_{\lambda\alpha+\rho+1}$. Hence, $\mathfrak{b} \leq \mu_2$ in $V_{\lambda\kappa}$. A similar argument can be used to get $\mathrm{add}(\mathcal{N}) \geq \mu_1$.

On the other hand, as $\mathbb{P}_{\lambda\kappa}$ is obtained by a fsi of cofinality κ, $\mathbb{P}_{\lambda\kappa}$ adds a κ-**Ed**-unbounded family of Cohen reals of size κ by Theorem 3.14, so $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\mathbf{Ed}) \leq \kappa$ and $\kappa \leq \mathfrak{d}(\mathbf{Ed}) = \mathrm{cov}(\mathcal{M})$. To see that $\mathrm{cov}(\mathcal{N}) \geq \kappa$ in $V_{\lambda\kappa}$, let $\mathcal{J}$ be a family of Borel null sets of size less than κ. By Theorem 1.29 there is a $\alpha < \kappa$ such that all members of $\mathcal{J}$ are coded in $V_{\lambda\alpha}$. Since the random real r_α added in the $\lambda\alpha + 1$-th step by $\mathbb{B}$ evades all the Borel null sets coded in $V_{\kappa\alpha}$, $r_\alpha \notin \bigcup \mathcal{J}$. Hence, $\mathcal{J}$ is not covering, so $\mathrm{cov}(\mathcal{N}) \geq \kappa$. A similar argument can be used to get that the set $\{r_\alpha : \alpha < \kappa\}$ of random reals is not null, so $\mathrm{non}(\mathcal{N}) \leq \kappa$. As $\kappa \leq \mathrm{cov}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \kappa$ and $\kappa \leq \mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{N}) \leq \kappa$, we get $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{M}) = \kappa$.

Note that, by Theorem 3.15, $\mathfrak{d} = \mathfrak{d}(\mathbf{Dom}) \geq \lambda$ and $\mathfrak{d} = \mathfrak{d}(\mathbf{Dom}) \leq \mathfrak{c} \leq \lambda$. Hence, $\mathfrak{d} = \mathfrak{c} = \lambda$ in $V_{\lambda\kappa}$. □

As a consequence of Theorem 3.25, we have a model where some cardinal inavariants associated with Yorioka ideals are separated.

3.26 Corollary. *In V, assume that $\mu \leq \kappa$ are uncountable regular cardinals, $\lambda \geq \kappa$ and $\lambda^{<\mu} = \lambda$. Then there exists a ccc poset that forces* $\mathrm{add}(\mathcal{I}_f) = \mu$, $\mathrm{cov}(\mathcal{I}_f) = \mathrm{non}(\mathcal{I}_f) = \kappa$ *and* $\mathrm{cof}(\mathcal{I}_f) = \lambda$ *for all increasing $f \in \omega^\omega$.*

3.3 Preservation of $\mathcal{R}$-unbounded reals

In this section we present the notions and results that are fundamental for the construction of matrix iterations that yield, in the next Chapter, the model for our main result.

Fix M, N a transitive models of (a sufficiently large finite fragment of) ZFC such that $M \subseteq N$.

3.27 Definition. Given two posets $\mathbb{P} \in M$ and $\mathbb{Q}$ (not necessarily in M) say that $\mathbb{P}$ *is a complete suborder of* $\mathbb{Q}$ *with respect to* M, denoted by $\mathbb{P} \lessdot_M \mathbb{Q}$, if $\mathbb{P}$ is a suborder of $\mathbb{Q}$ and every maximal antichain in $\mathbb{P}$ that belongs to M is also a maximal antichain in $\mathbb{Q}$.

Recall that, if $\mathbb{S}$ is a Suslin ccc poset coded in M, then $\mathbb{S}^M \lessdot_M \mathbb{S}^N$. Also, if $\mathbb{P} \in M$ is a poset, then $\mathbb{P} \lessdot_M \mathbb{P}$. For the following results, consider a Polish relational system $\mathcal{R} = \langle X, Y, \sqsubset\rangle$ coded in M (in the sense that all its components are coded in M). Clearly, if $\mathbb{P} \lessdot_M \mathbb{Q}$ and G is $\mathbb{Q}$-generic over N, then $G\cap\mathbb{P}$ is $\mathbb{P}$-generic over M and $M[G\cap\mathbb{P}] \subseteq N[G]$. We are interested in preserving $\mathcal{R}$-unbounded reals between forcing extensions of M and N.

3.28 Lemma ([M13, Theorem 7])**.** *Let $\mathbb{S}$ be a Suslin ccc poset coded in M. If $M \models \mathbb{S} \sqsubset$-good, then, in N, $\mathbb{S}^N$ forces that every real in $X \cap N$ which is $\mathcal{R}$-unbounded over M is $\mathcal{R}$-unbounded over $M^{\mathbb{S}^M}$.*

Proof. In M, fix $\dot{h}$ a $\mathbb{S}$-name for a real in Y and $H \subseteq Y$ a witness of the goodness of $\mathbb{S}$ for $\dot{h}$. As H is a countable set, enumerate $Y = \{y_n : n < \omega\}$. By Lemma 1.32, we have

3.29 Claim ([M13, Claim 1])**.** *The statement "$\dot{h}$ is a $\mathbb{S}$-name for a member of Z and for all $x \in X$, if $x \not\sqsubset_n y_n$ for each $n < \omega$, then $\Vdash x \not\sqsubset \dot{h}$" is a conjunction of a Σ^1_1-statement with a Π^1_1-statement in $(D \times Z')^{\omega\times\omega} \times Z^\omega$ where Z' is the Polish space where $\mathbb{S}$ is defined . Even more, if $\mathbb{S}$ is Borel in Z', then the statement above is Π^1_1.*

Now, by Claim 3.29, as the statement "$\dot{h}$ is a $\mathbb{S}$-name for a member of Z and for all $x \in X$, if $x \not\sqsubset_n y_n$ for each $n < \omega$, then $\Vdash x \not\sqsubset \dot{h}$" is true in M, then it is also true in N. In N, as c is $\mathcal{R}$-unbounded over M, we get $\forall g \in H(c \not\sqsubset g)$, so $\Vdash c \not\sqsubset \dot{h}$. □

3.30 Lemma ([BrF11, Lemma 11])**.** *Assume that $\mathbb{P} \in M$ is a poset. Then, in N, $\mathbb{P}$ forces that every real in $X \cap N$ which is $\mathcal{R}$-unbounded over M is $\mathcal{R}$-unbounded over $M^{\mathbb{P}}$.*

Proof. Work within M. Let $e \in \Omega$ and $\dot{h}$ be a $\mathbb{P}$-name for a member in Y_e. Fix $p \in \mathbb{P}$ and $n < \omega$. Choose a continuous and surjective function $f : \omega^\omega \to Y_e$. Let $\dot{z}$ be a $\mathbb{P}$-name for a real in ω^ω such that $\mathbb{P}$ forces that $f(\dot{z}) = \dot{h}$. From Lemma 3.9 choose $\{p_k\}_{k<\omega}$ a decreasing sequence in $\mathbb{P}$ and $g \in \omega^\omega$ such that $p_0 \leq p$ and $p_k \Vdash \dot{z} \restriction k = g \restriction k$. In N, as c is $\mathcal{R}$-unbounded over M, then $c \not\sqsubset f(g)$, so $c \not\sqsubset_n f(g)$. By Lemma 3.12, there is a $k < \omega$ such that $p_k \Vdash^N c \not\sqsubset_n f(\dot{z})$. So $p_k \Vdash^N c \not\sqsubset_n \dot{h}$. □

3.31 Lemma ([BlS84],[BrF11, Lemma 10, 12 and 13])**.** *Let δ be an ordinal, $\langle \mathbb{P}^0_\alpha, \dot{\mathbb{Q}}^0_\alpha \rangle_{\alpha<\delta} \in M$ and $\langle \mathbb{P}^1_\alpha, \dot{\mathbb{Q}}^1_\alpha \rangle_{\alpha<\delta} \in N$ both fsi such that, for any $\alpha < \delta$, $\mathbb{P}^0_\alpha \lessdot_M \mathbb{P}^1_\alpha$ and, in N, $\mathbb{P}^1_\alpha$ forces that $\dot{\mathbb{Q}}^0_\alpha \lessdot_{M^{\mathbb{P}^0_\alpha}} \dot{\mathbb{Q}}^1_\alpha$. Then, $\mathbb{P}^0_\delta \lessdot_M \mathbb{P}^1_\delta$.*

Additionally, if $c \in X \cap N$ is $\mathcal{R}$-unbounded over M, δ is limit and, for each $\alpha < \delta$, $\mathbb{P}^1_\alpha$ forces (in N) that c is $\mathcal{R}$-unbounded over $M^{\mathbb{P}^0_\alpha}$, then $\mathbb{P}^1_\delta$ forces (in N) that c is $\mathcal{R}$-unbounded over $M^{\mathbb{P}^0_\delta}$.

Proof. Working within M. Let $e \in \Omega$ and $\dot{h}$ be a $\mathbb{P}$-name for a member in Y_e. Choose a continuous and surjective function $f : \omega^\omega \to Y_e$. Let $\dot{z}$ be a $\mathbb{P}$-name for a real in ω^ω such that $\mathbb{P}^0_\delta$ forces that $f(\dot{z}) = \dot{h}$. In N, assume, towards a contradiction, that there are $q \in \mathbb{P}^1_\delta$ and $n < \omega$ such that $q \Vdash_{\mathbb{P}^1_\delta} c \sqsubset_n \dot{h}$. Choose $\alpha < \delta$ such that $q \in \mathbb{P}^1_\alpha$.

Let G be a $\mathbb{P}^1_\alpha$-generic over N such that $q \in G$. By assumption, $\Vdash_{\mathbb{P}_\delta/\mathbb{P}_\alpha} c \sqsubset \dot{h}$. In $M[G \cap \mathbb{P}^0_\alpha]$, by Lemma 3.9, find $g \in \omega^\omega$ and decreasing chain $\{p_k\}_{k<\omega}$ in $\mathbb{P}^0_\delta/\mathbb{P}^0_\alpha$ such that $p_k \Vdash_{\mathbb{P}^0_\delta/\mathbb{P}^0_\alpha} g \restriction k = \dot{z} \restriction k$. In $N[G]$, as $c \not\sqsubset f(g)$, there is a $k < \omega$ such that $p_k \Vdash^{N[G]}_{\mathbb{P}^1_\delta/\mathbb{P}^1_\alpha} c \not\sqsubset_n f(\dot{z})$ by (the first part of this Lemma implies $\mathbb{P}^0_\delta/\mathbb{P}^0_\alpha \lessdot_{M[G\cap\mathbb{P}^0_\alpha]} \mathbb{P}^1_\delta/\mathbb{P}^1_\alpha$, so the previous interpretation of $\dot{z}$ in $\mathbb{P}^0_\delta/\mathbb{P}^0_\alpha$ is also an interpretation in $\mathbb{P}^1_\delta/\mathbb{P}^1_\alpha$). Thus, $p_k \Vdash_{\mathbb{P}^1_\delta/\mathbb{P}^1_\alpha} c \not\sqsubset_n \dot{h}$ which is a contradiction. □

Note that if $c \in X$ is a Cohen real over M then c is $\mathcal{R}$-unbounded over M by Definition 3.1. In fact, the unbounded reals used in our applications are actually Cohen.

Chapter 4

Matrix iterations

The first intance of a matrix iteration was constructed by Blass and Shelah [BlS84] to force $\mathfrak{u} < \mathfrak{d}$ with large continuum. Years later, Brendle and Fischer [BrF11] improved this construction by showing how to add dominating reals up to some intermediate extensions in the matrix. Also Mejía [M13] constructed a matrix iteration to construct models where the cardinals on Cichoń's diagram assume several different values, in particular $\mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$.

In Section 4.1 we define matrix iterations in general and fix a type of matrix that we use for our applications. In Section 4.2 we prove our main (consitency) result about cardinal invariants associated with Yorioka ideals through a model construted by a matrix iteration.

4.1 Matrix iterations of ccc posets

4.1 Definition ([BlS84, Blass and Shelah])**.** Let δ and γ be ordinals. A *matrix iteration of ccc posets* $\mathbb{P}_{\delta,\gamma} = \langle\langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}\rangle_{\alpha\leq\delta}$ satifies: for all $\alpha \leq \delta$ and $\xi < \gamma$

(i) $\mathbb{P}_{0,0}$ is trivial forcing, $\mathbb{P}_{\alpha,0}$ is ccc for all $\alpha \leq \delta$,

(ii) $\dot{\mathbb{Q}}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$-name of a ccc poset and $\mathbb{P}_{\alpha,\xi+1} = \mathbb{P}_{\alpha,\xi} * \dot{\mathbb{Q}}_{\alpha,\xi}$,

(iii) if $\alpha \leq \beta \leq \delta$ and $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ then $\mathbb{P}_{\beta,\xi}$ forces $\dot{\mathbb{Q}}_{\alpha,\xi} \lessdot_{V^{\mathbb{P}_{\alpha,\xi}}} \dot{\mathbb{Q}}_{\beta,\xi}$, and

(iv) if $\xi \leq \delta$ is limit, then $\mathbb{P}_{\alpha,\xi} = \mathrm{limdir}_{\eta<\xi}\mathbb{P}_{\alpha,\eta}$.

For fixed $\alpha \leq \delta$ note that $\mathbb{P}_{\alpha,\gamma} = \langle \mathbb{P}^*_{\alpha,\xi}, \dot{\mathbb{Q}}^*_{\alpha,\xi}\rangle_{\xi<1+\gamma}$ is a fsi where $\dot{\mathbb{Q}}^*_{\alpha,0} = \mathbb{P}_{\alpha,0}$ and $\dot{\mathbb{Q}}^*_{\alpha,1+\xi} = \dot{\mathbb{Q}}_{\alpha,\xi}$ for all $\xi < \gamma$, so $\mathbb{P}^*_{\alpha,1+\xi} = \mathbb{P}_{\alpha,\xi}$ for all $\xi \leq \gamma$. Therefore, $\mathbb{P}_{\alpha,\xi}$ is ccc and $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\alpha,\eta}$ whenever $\xi \leq \eta \leq \gamma$. On the other hand, by Lemma 3.31, $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi}$ for all $\alpha \leq \beta \leq \delta$ and $\xi \leq \eta$.

For a matrix iteration as in Definition 4.1, if G is a $\mathbb{P}_{\delta,\gamma}$-generic over V(the ground model), we denote $V_{\alpha,\xi} = V[G \cap \mathbb{P}_{\alpha,\xi}]$ for all $\alpha \leq \delta$ and $\xi \leq \gamma$. Note that $V_{0,0} = V$ and clearly $V_{\alpha,\xi} \subseteq V_{\beta,\eta}$ for all $\alpha \leq \beta \leq \delta$ and $\xi \leq \eta \leq \gamma$. The idea of the contruction is to obtain a matrix $\langle V_{\alpha,\beta} : \alpha \leq \gamma,\ \beta \leq \delta\rangle$ of generic extensions as illustated in Figure 4.1.

The construction of the matrix iteration for the model in Section 4.2 corresponds to the following particular case.

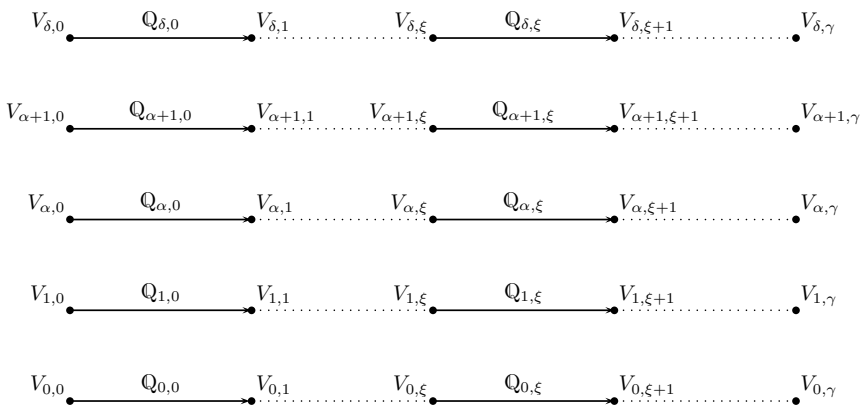


Figure 4.1: Matrix iteration

4.2 Definition. A *standard matrix iteration* $\mathbf{m}$ consists of

(I) ordinal numbers $\delta^{\mathbf{m}}$ and $\gamma^{\mathbf{m}}$,

(II) a partition $\gamma^{\mathbf{m}} = S^{\mathbf{m}} \cup T^{\mathbf{m}}$,

(III) a sequence $\langle \mathbb{S}^{\mathbf{m}}_\xi : \xi \in S^{\mathbf{m}} \rangle$ of Suslin ccc posets coded in the ground model V,

(IV) a function $\Delta^{\mathbf{m}} : T^{\mathbf{m}} \to \delta^{\mathbf{m}}$,

(V) a matrix iteration $\mathbb{P}^{\mathbf{m}}_{\delta^{\mathbf{m}},\gamma^{\mathbf{m}}} = \langle \langle \mathbb{P}^{\mathbf{m}}_{\alpha,\xi}, \dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\xi} \rangle_{\xi<\gamma^{\mathbf{m}}} \rangle_{\alpha\le\delta^{\mathbf{m}}}$ constructed by induction on $\xi < \gamma^{\mathbf{m}}$ where

(i) $\mathbb{P}^{\mathbf{m}}_{\alpha,0} = \mathrm{Fn}(\alpha \times \omega, 2)$ for all $\alpha \le \delta^{\mathbf{m}}$,

(ii) if $\xi \in S^{\mathbf{m}}$ then $\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\xi}$ is a $\mathbb{P}^{\mathbf{m}}_{\alpha,\xi}$-name for $(\mathbb{S}^{\mathbf{m}}_\xi)^{V_{\alpha,\xi}}$ for each $\alpha \le \delta^{\mathbf{m}}$, and

(iii) if $\xi \in T^{\mathbf{m}}$, a $\mathbb{P}^{\mathbf{m}}_{\Delta(\xi),\xi}$-name $\dot{\mathbb{T}}^{\mathbf{m}}_\xi$ of a poset is chosen such that $\mathbb{P}^{\mathbf{m}}_{\delta^{\mathbf{m}},\xi}$ forces $\dot{\mathbb{T}}^{\mathbf{m}}_\xi$ to be ccc. For each $\alpha \le \delta^{\mathbf{m}}$,

$$\dot{\mathbb{Q}}^{\mathbf{m}}_{\alpha,\xi} := \begin{cases} \mathbb{1} & \text{if } \alpha \le \Delta(\xi), \\ \dot{\mathbb{T}}_\xi & \text{if } \alpha > \Delta(\xi). \end{cases}$$

By Lemma 3.31, it is clear that (V) yields a matrix iteration. When there is no confusion we omit the super index $\mathbf{m}$.

Fix a standard matrix iteration $\mathbf{m}$.

4.3 Theorem (Brendle and Fischer[BrF11, Lemma 5])**.** *Assume that δ has uncountable cofinality and $\xi \le \gamma$. Then,*

(a) If $p \in \mathbb{P}_{\delta,\xi}$, then there exists an $\alpha < \delta$ such that $p \in \mathbb{P}_{\alpha,\xi}$, that is, $\mathbb{P}_{\delta,\xi} = \mathrm{limdir}_{\alpha<\delta}\mathbb{P}_{\alpha,\xi}$.

(b) If $\dot{h} \in \mathbb{P}_{\delta,\xi}$-name for a real, then there exists an $\alpha < \delta$ such that $\dot{h}$ is forced to be equal to a $\mathbb{P}_{\alpha,\xi}$-name for a real.

Fix a Polish relational system $\mathcal{R}$ coded in V. Given $\alpha < \delta$, $\mathbb{P}_{\alpha+1,0}$ adds a real $\dot{c}_\alpha \in X \cap V_{\alpha+1,0}$ which is Cohen over $V_{\alpha,0}$ and, therefore, $\mathcal{R}$-unbounded over $V_{\alpha,0}$. The following result ilustates conditions to preserve $\dot{c}_\alpha$ unbounded, i.e, to guarantee that $\dot{c}_\alpha$ is $\mathcal{R}$-unbounded over $V_{\alpha,\gamma}$.

4.4 Theorem ([M13, Theorem 10 and Corollary 1])**.** *Assume that, for any $\beta \in S$ and $\alpha < \delta$, $\mathbb{P}_{\alpha,\beta}$ forces that $\dot{\mathbb{Q}}_{\alpha,\beta} = \mathbb{S}_\beta^{V_{\alpha,\beta}}$ is $\mathcal{R}$-good. Then, for any $\alpha < \delta$ and $\xi \leq \gamma$, $\mathbb{P}_{\alpha+1,\xi}$ forces that $\dot{c}_\alpha$ is $\mathcal{R}$-unbounded over $V_{\alpha,\xi}$.*
Even more, if δ has uncountable cofinality then $\mathbb{P}_{\delta,\gamma}$ forces $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\delta) \leq \mathfrak{d}(\mathcal{R})$.

Proof. The fist statement is a direct consequence of Lemmas 3.28, 3.30 and 3.31. For the second statement, fix an increasing sequence $\{\alpha_\zeta : \zeta < \mathrm{cf}(\delta)\}$ cofinal in δ and we prove that $\mathbb{P}_{\delta,\gamma}$ forces that $\{\dot{c}_{\alpha_\zeta} : \zeta < \mathrm{cf}(\delta)\}$ is $\mathrm{cf}(\delta)$-$\mathcal{R}$-unbounded. Let G be a $\mathbb{P}_{\delta,\gamma}$-generic over V and let $\{y_\eta\}_{\eta<\nu}$ be a subset of Y in $V_{\delta,\gamma}$ with $\nu < \mathrm{cf}(\delta)$. From Theorem 4.3, there is an $\alpha < \delta$ such that $\{y_\eta\}_{\eta<\nu}$ is a subset of $V_{\alpha,\gamma}$. Wlog, assume that $\alpha = \alpha_\zeta$ for some $\zeta < \mathrm{cf}(\delta)$. As $c_{\alpha_\zeta} = \dot{c}_{\alpha_\zeta}[G]$ is $\mathcal{R}$-unbounded over $V_{\alpha_\zeta,\xi}$, then no real in $\{y_\eta : \eta < \nu\}$ can dominate c_{α_ζ}, so the family $\{c_{\alpha_\zeta} : \zeta < \mathrm{cf}(\delta)\}$ is $\mathrm{cf}(\delta)$-$\mathcal{R}$-unbounded of size $\mathrm{cf}(\delta)$. Thus, Lemma 3.15, $\mathfrak{b}(\mathcal{R}) \leq \mathrm{cf}(\delta) \leq \mathfrak{d}(\mathcal{R})$. □

4.2 Main Result

4.5 Theorem. *In V, let $\mu \leq \nu \leq \kappa$ be uncountable regular cardinals, $\lambda \geq \kappa$ a cardinal, let $b, g_0, c \in \omega^\omega$ and $h = \mathrm{id}_\omega^{\mathrm{id}_\omega}$ satisfying $H = h^{\mathrm{id}_\omega} \leq c$. Assume that $\lambda^{<\mu} = \lambda$, $g_0 \gg \mathrm{id}_\omega$ is non-decreasing, $2^{g_0 \circ (H^+ - 1)} \leq c$ and $b >^* c^{h^{\mathrm{id}_\omega}} * \mathrm{id}_\omega$. Then, there is a ccc poset forcing that $\mathrm{add}(\mathcal{I}_f) = \mu$, $\mathrm{cov}(\mathcal{I}_f) = \nu$, $\mathrm{non}(\mathcal{I}_f) = \kappa$ and $\mathrm{cof}(\mathcal{I}_f) = \lambda = \mathfrak{c}$ for all increansing $f \geq (\log b)^+$ in ω^ω.*

Proof. Fix a bijection $g : \lambda \to \kappa \times \lambda$, denote by $(\ )_0 : \kappa \times \lambda \to \kappa$ the projection onto the first coordinate and fix $t : \kappa\nu \to \kappa$ such that $t(\kappa\delta + \alpha) = \alpha$ for $\delta < \nu$ and $\alpha < \kappa$.

Accoding to Defintion 4.2, construct a standard matrix iteration $\mathbf{m}$ such that:

(i) $\delta^{\mathbf{m}} = \kappa$ and $\gamma^{\mathbf{m}} = \lambda\kappa\nu$ (as product of ordinal numbers).

(ii) $S^{\mathbf{m}} = S = \{\lambda\rho : \rho < \kappa\nu\}$.

(iii) $\mathbb{S}_\xi = \mathbb{D}$ for any $\xi \in S$.

(iv) If $\xi = \lambda\rho + 1$ for some $\rho < \kappa\nu$, put $\Delta(\xi) = t(\rho)$ and let $\dot{\mathbb{T}}_\xi$ be a $\mathbb{P}_{t(\rho),\xi}$-name for $(\mathbb{E}_b^1)^{V_{t(\rho),\xi}} = \mathbb{E}_b^1(\mathcal{S}(b,1) \cap V_{t(\rho),\xi})$.

For $\alpha < \kappa$ and $\rho < \kappa\nu$, fix a sequence of nice $\mathbb{P}_{\alpha,\lambda\rho}$-names $\langle \mathbb{L}\dot{\mathbb{O}}\mathbb{C}^\rho_{\alpha,\gamma}\rangle_{\gamma<\lambda}$ for all the σ-linked subposets of $\mathbb{L}\mathbb{O}\mathbb{C}$ of size $< \mu$.

(v) If $\xi = \lambda\rho + 1 + \epsilon$ for some $\rho < \kappa\nu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \mathbb{L}\dot{\mathbb{O}}\mathbb{C}^\rho_{g(\epsilon)}$.

Note that, in V, by Lemmas 1.40, 3.11, 3.13, 3.22 and 3.24. $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ is μ-$\mathbf{Slm}(c,h)$-good because it can be seen as a fsi of length $1 + \lambda\kappa\nu = \lambda\kappa\nu$ (see the dicussion after Definition 4.1). Therefore, in $V_{\kappa,\lambda\kappa\nu} = V[G]$ for a fixed $\mathbb{P}_{\kappa,\lambda\kappa\nu}$-generic G over V, $\mathfrak{c}^\exists_{c,H} \leq \mu$ and $\lambda = |\lambda\kappa\nu| \leq \mathfrak{n}^\exists_{c,H}$ by Theorem 3.15. By Lemma 2.11, $\mathrm{cov}(\mathcal{J}_{g_0}) \leq \mathfrak{c}^\exists_{c,H}$ and $\mathfrak{n}^\exists_{c,H} \leq \mathrm{non}(\mathcal{J}_{g_0})$. As $g_0 \gg \mathrm{id}_\omega$, then $\mathrm{cov}(\mathcal{I}_{\mathrm{id}_\omega}) \leq \mathrm{cov}(\mathcal{J}_{g_0})$ and $\mathrm{non}(\mathcal{J}_{g_0}) \leq \mathrm{non}(\mathcal{I}_{\mathrm{id}_\omega})$ so, in $V_{\kappa,\lambda\kappa\nu}$, $\mathrm{add}(\mathcal{I}_f) \leq \mu$

and $\lambda \leq \mathrm{cof}(\mathcal{I}_f)$ for all $f \in \omega^\omega$ increasing because $\mathrm{add}(\mathcal{I}_f) \leq \mathrm{add}(\mathcal{I}_{\mathrm{id}_\omega}) \leq \mathrm{cov}(\mathcal{I}_{\mathrm{id}_\omega})$ and $\mathrm{non}(\mathcal{I}_{\mathrm{id}_\omega}) \leq \mathrm{cof}(\mathcal{I}_{\mathrm{id}_\omega}) \leq \mathrm{cof}(\mathcal{I}_f)$ by Lemma 2.7.

Now, we prove that $\mathfrak{c} \leq \lambda$. As $|\mathbb{P}_{\kappa,\lambda\kappa\nu}| \leq \lambda$ and $\lambda^\omega = \lambda$ we can count nice names in the usual way to verify that $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ forces $\mathfrak{c} \leq \lambda$.

Now we prove that $\mathrm{add}(\mathcal{N}) \geq \mu$ in $V_{\kappa,\lambda\kappa\nu}$. Let $\{x_\zeta : \zeta < \theta\} \subseteq \omega^\omega$, $\theta < \mu$. By Theorem 4.3 and Theorem 1.29, there are $\alpha < \kappa$ and $\rho < \kappa\nu$ such that $\{x_\zeta : \zeta < \theta\} \in V_{\alpha,\lambda\rho}$. We work in $V_{\alpha,\lambda\rho}$. Let N be a transitive model of a finite large fragment of ZFC of size $\leq \theta$ such that $\{x_\zeta : \zeta < \theta\} \subseteq N$. Note that $\mathbb{LOC}^N$ is a σ-linked subposet of $\mathbb{LOC}$ of size $< \mu$. Then, there is a $\gamma < \lambda$ such that $\mathbb{LOC}^N = (\mathbb{L}\dot{\mathbb{O}}\mathbb{C}^\rho_{\alpha,\gamma})_{G\cap\mathbb{P}_{\alpha,\lambda\rho}}$, so the generic slalom added by $\mathbb{L}\dot{\mathbb{O}}\mathbb{C}^\rho_{g(\epsilon)}$, with $\epsilon < \lambda$ such that $g(\epsilon) = (\alpha, \gamma)$, localizes all the reals in N, in particular, x_ζ for $\zeta < \theta$.

From Lemma 2.5 $\mathrm{add}(\mathcal{I}_f) \geq \mathrm{add}(\mathcal{N})$ in $V_{\kappa,\lambda\kappa\nu}$ for all increasing $f \in \omega^\omega$, so $\mathrm{add}(\mathcal{I}_f) \geq \mu$ in $V_{\kappa,\lambda\kappa\nu}$.

For each $\rho < \kappa\nu$ let $r^\rho \in \mathbb{R}_b \cap V_{t(\rho)+1,\lambda\rho+2}$ be the generic real added by $\dot{\mathbb{Q}}_{t(\rho)+1,\lambda\rho+1} = \dot{\mathbb{Q}}_{\kappa,\lambda\rho+1} = (\mathbb{E}^1_b)^{V_{t(\rho),\lambda\rho+1}}$ over $V_{t(\rho),\lambda\rho+1}$ which is eventually different from all the reals of $\mathbb{R}_b \cap V_{t(\rho),\lambda\rho+1}$. Then $\nu \leq \mathfrak{c}^\exists_{b,1}$ is a consequence of the following.

4.6 Claim. *In $V_{\kappa,\lambda\kappa\nu}$, for any $F \subseteq \mathbb{R}_b$ of size $< \nu$, there is some r^ρ eventually different from all the members of F.*

Proof. By Theorem 4.3, there are $\alpha < \kappa$ and $\delta < \kappa\nu$ such that $F \subseteq V_{\alpha,\lambda\delta}$. By the definition of t find $\rho \in [\delta, \kappa\nu)$ such that $t(\rho) = \alpha$. Clearly, $F \subseteq V_{\alpha,\xi}$ where $\xi = \lambda\rho + 1$, so their members are all eventually different from r^ρ. □

To see that $\mathfrak{n}^\exists_{b,1} \leq \kappa$, note that $\{r^\rho : \rho < \kappa\nu\}$ is a family of reals of size $< \kappa$ which, by Claim 4.6, satisfies that any member of $\mathbb{R}_b \cap V_{\kappa,\lambda\kappa\nu}$ is eventually different from a member of $\{r^\rho : \rho < \kappa\nu\}$. So $\mathfrak{n}^\exists_{b,1} \leq \kappa$.

Fix $f \in \omega^\omega$ increasing such that $f \geq (\log b)^+$. By Lemma 2.10, $\mathfrak{c}^\exists_{b,1} \leq \mathrm{cov}(\mathcal{J}_f)$ and $\mathfrak{n}^\exists_{b,1} \geq \mathrm{non}(\mathcal{J}_f)$, so $\nu \leq \mathrm{cov}(\mathcal{I}_f)$ and $\kappa \geq \mathrm{non}(\mathcal{I}_f)$. On the other hand, as $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ is obtained by a fsi of cofinality ν, $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ adds a ν-$\mathrm{Ed}(\omega^\omega)$-unbounded family of Cohen reals of size ν by Theorem 3.14, so $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\mathrm{Ed}(\omega^\omega)) \leq \nu$ and $\nu \leq \mathfrak{d}(\mathrm{Ed}(\omega^\omega))) = \mathrm{cov}(\mathcal{M})$.

To see $\mathfrak{b} \geq \nu$, let $\mathcal{F} \subseteq \omega^\omega$ be of size less than ν in $V_{\kappa,\lambda\kappa\nu}$. By Theorem 1.29, there is a $\beta < \kappa\nu$ such that $\mathcal{F}$ is contained in $V_{\kappa,\lambda\kappa\beta}$. Since the Hechler real d_β added in the $\lambda\kappa\beta$-th step by $\mathbb{D}$ dominates $V_{\kappa,\lambda\kappa\beta}$, $\mathcal{F}$ is bounded. Hence $\mathfrak{b} \geq \nu$. On the other hand, using once again that a Hechler real is dominating, we have that $\{d_\beta : \beta < \nu\}$ is a dominating family of size ν. Hence, $\mathfrak{d} \leq \nu$ and $\mathfrak{b} = \mathfrak{d} = \nu$. Even more, $\mathfrak{b} = \mathrm{non}(\mathcal{M}) = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \nu$, so $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \nu$. Therefore, $\mathrm{cov}(\mathcal{I}_f) = \nu$ because $\mathrm{cov}(\mathcal{I}_f) \leq \mathrm{non}(\mathcal{M})$.

Finally, we prove that $\kappa \leq \mathrm{non}(\mathcal{I}_f)$. By Proposition 2.12 it is sufficient to show that $\kappa \leq \mathrm{minnon}$. Note that, for any $b' \in \omega^\omega$, by Theorem 4.4, Example 3.21 with $h = 1$ and Lemma 3.24, $\kappa \leq \mathfrak{n}^\exists_{b',1}$, so $\kappa \leq \mathrm{minnon}$. As $\mathrm{minnon} \leq \mathrm{non}(\mathcal{I}_f)$ for any increasing $f \in \omega^\omega$, then $\kappa \leq \mathrm{non}(\mathcal{I}_f)$. □

Open Questions

1 Question. *Is there a model of ZFC satisfying* $\mathrm{add}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ *for all increasing* $f : \omega \to \omega$*?.*

This problem was partially solved in Theorem 4.6. However, it is not known whether there is such model for $f = \mathrm{id}_\omega$. One way to attack Question 1 is using large products of creature forcing (see [KeS09, KeS12]). Quite recently, A. Fischer, Goldstern, Kellner and Shelah [FGKS] used this technique to prove that 5 cardinal invariants on the right of Cichoń's diagram are pairwise different.

Other way to attack Question 1 is by finding a suitable preservation property for $\mathrm{add}(\mathcal{I}_f)$ in the context of Section 3. Two related open problems concerning this preservation property are the following.

2 Question. *Is it consistent with ZFC that infinitely many cardinal invariants of the form* $\mathrm{add}(\mathcal{I}_f)$ *are pairwise different?*

3 Question. *Are the following statements true in ZFC?*

(a) $\mathrm{add}(\mathcal{I}_f) = \mathrm{add}(\mathcal{I}_{\mathrm{id}_\omega})$ *(and* $\mathrm{cof}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_{\mathrm{id}_\omega})$*) for all (or some) increasing* f*.*

(b) $\mathrm{add}(\mathcal{I}_f) = \mathrm{add}(\mathcal{N})$ *(and* $\mathrm{cof}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{N})$*) for all (or some) increasing* f*.*

A third way to attack Question 1 is to adapt the method of finite support iterations with ultrafilters of [GMS16] to matrix iterations. If such a method can be defined successfully, the following problem, which is stronger, can be solved.

4 Question. *Is there a model of ZFC satisfying* $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M}) < \mathrm{cof}(\mathcal{M})$?

The method of large products of creature forcing may also work to attack the following problem.

5 Question. *Is there a model of* $\mathrm{add}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ *for all increasing* $f : \omega \to \omega$*?.*

6 Question. *Is there a model of ZFC satisfying* $\mathrm{add}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$?

The alternative case to this question, that is, $\mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$, was solved by Mejía [M13] with a matrix iteration. The method of large products of creature forcing may work to attack the previous question. However, it is more difficult for Questions 1 and 5 because it is hard to control adding random reals with this method.

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