Theory of associated and attached prime ideals over quantum algebras

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Title in English

Theory of associated and attached prime ideals over quantum algebras

Título en español

Teoría de ideales primos asociados y adjuntos sobre álgebras cuánticas

Abstract: In this thesis we study the reflexive-nilpotents-property (RNP for short) over quantum algebras such as skew polynomial rings and skew PBW extensions. We investigate the transfer of this property between ring of coefficients and noncommutative rings of polynomial type over these rings. We consider this property for localizations by regular elements of skew PBW extensions. Related to RNP property, we study the notions of weak annihilator and nilpotent associated prime ideal, and formulate several results that extend or contribute those corresponding in the literature. Additionally, we investigate the associated prime ideals of induced modules over quantum algebras and characterize them. Our work is a contribution to an extensive research presented by different authors on ring-theoretical properties of noncommutative rings and its description of associated prime ideals. On the other hand, motivated by the research about the attached prime ideals of the inverse polynomial module over skew polynomial rings of automorphism type, and considering a very important family of quantum algebras known as skew Ore polynomials of higher order, we characterize the attached prime ideals of those modules over skew Ore polynomials. We also formulate results about the uniform dimension and the associated primes of induced modules over some families of quantum algebras. Finally, we study the couniform dimension for inverse polynomial modules over skew Ore polynomials.

Resumen: En esta tesis estudiamos la propiedad nilpotente-reflexiva (RNP para abreviar) sobre álgebras cuánticas como anillos de polinomios torcidos y extensiones PBW torcidas. Investigamos la transferencia de esta propiedad entre anillos de coeficientes y anillos no conmutativos de tipo polinomial sobre estos anillos. Consideramos esta propiedad para localizaciones mediante elementos regulares de extensiones PBW torcidas. En relación con la propiedad RNP, estudiamos las nociones de anulador débil e ideal primo asociado nilpotente, y formulamos varios resultados que extienden o contribuyen a los correspondientes en la literatura. Adicionalmente, investigamos los ideales primos asociados de módulos inducidos sobre álgebras cuánticas y los caracterizamos. Nuestro trabajo es una contribución a una extensa investigación presentada por diferentes autores sobre las propiedades teóricas de los anillos no conmutativos y su descripción de los ideales primos asociados. Por otro lado, motivados por la investigación sobre los ideales primos adjuntos del módulo polinomial inverso sobre anillos de polinomios torcidos de tipo automorfismo, y considerando una familia muy importante de álgebras cuánticas conocidas como polinomios de Ore torcidos de orden superior, caracterizamos los ideales primos adjuntos de esos módulos sobre polinomios de Ore torcidos. También formulamos resultados sobre la dimensión uniforme y los primos asociados de módulos inducidos sobre algunas familias de álgebras cuánticas. Finalmente, estudiamos la dimensión couniforme para módulos polinomiales inversos sobre polinomios de Ore torcidos.

Keywords: Skew polynomial ring, skew Ore polynomials, skew PBW extension, semi-graded ring, RNP ring, reflexive ring, associated prime, weak annihilator, nilpotent associated prime, nilpotent good polynomial, induced module, attached prime, inverse polynomial module, Bass module, uniform dimension, couniform dimension, perfect ring.

Palabras clave: Anillo de polinomios torcido, polinomios de Ore torcidos, extensión PBW torcida, anillo semi-graduado, anillo RNP, anillo reflexivo, primo asociado, anulador débil, primo asociado nilpotente, buen polinomio nilpotente, módulo inducido, primo adjunto, módulo polinomial inverso, módulo Bass, dimensión uniforme, dimensión couniforme, anillo perfecto.

Dedicated to

My mother Martha Rincón

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Contents

Introduction

An important concept of commutative algebra and number theory is *unique factorization*. In commutative algebra, it is a well-known that the unique factorization of ideals as the intersection of other ideals is not a condition that always holds. This fact motivated the development of what we know as the *theory of primary decomposition*. Lasker [Las05] initially formulated a theory of primary decomposition for affine rings and power series rings. Noether [Noe21] presented a new formulation of the theory of primary decomposition under conditions of stability of ascending chains of ideals. The theory identifies a finite set of prime ideals and decomposes any ideal I as an intersection of these ideals which are called *associated prime ideals*.

A very important problem today is the characterization of these ideals on polynomial rings. The first result was presented by Brewer and Heinzer [BH74], where they described the associated prime ideals of the commutative polynomial ring $R[x]$. Faith [Fai00] presented a proof of the same result using different algebraic techniques. In the noncommutative setting, Annin introduced the *prime modules* and *associated prime ideals*. Additionally, he defined the notion of *compatibility* with the aim to characterizing the associated prime ideals of modules on noncommutative rings. The study of right annihilators of a module is related to the identification of elements such as idempotents, zero divisors, nilpotents and units. In the literature, we can find that an important work has been to characterize these elements on polynomial rings. In the commutative setting, Atiyah and Macdonald [AM18] showed a complete characterization for polynomial rings. Several authors have studied these elements on different noncommutative rings (c.f. [HHR20, CHR24] and references therein).

The description of these elements and the study of algebraic properties of noncommutative rings of polynomial type is related to conditions on the ring of coefficients of these rings. For instance, the *reflexive-nilpotents-property condition* (*RNP* for short) defined by Bhattacharjee [Bha20] considers the characterization of nilpotent elements of skew polynomial rings defined by Ore [Ore31, Ore33]. The RNP condition extends different notions investigated by several authors (c.f. [KL11] and [KLY14]). In this way, the condition presented by Bhattacharjee can be seen as an important contribution to the study of properties on noncommutative rings.

In addition to the study of algebraic properties of noncommutative rings, the characterization of elements is also related to research about annihilator ideals. For instance, Kaplansky [Kap65] defined *Baer rings* as those rings for which the right annihilator of every non-empty subset of R is generated by an idempotent element, while Birkenmeier [BKP01] defined the *right principally quasi-Baer rings* (*right p.q.-Baer* for short) as those rings where the right annihilator of each non-zero principal right ideal of R is generated by an idempotent. In the commutative and noncommutative settings, these ring-theoretical properties have been widely studied (see [BKP01, BPR13, HMHSJ03, HM05, HKK00, HKK03, HLS02, RS16, LW03, RS18a], and references therein). Ouyang and Birkenmeier [OB12] introduced the *weak annihilator ideals* as a generalization of the classical annihilator and investigated its properties over skew polynomial rings. They studied properties analogous to the conditions considered by Kaplansky and Birkenmeier, but considering these annihilators and nilpotent elements. Furthermore, they extended the concept of associated primes and introduced the *nilpotent associated prime ideals*, which they characterized for skew polynomial rings.

The theory of associated prime ideals on noncommutative polynomial modules has also been widely studied by several authors. Annin [Ann04] introduced the concept of *annihilator-compliant polynomials* to study the associated primes of induced modules over skew polynomial rings. Leroy and Matczuk [LM04] defined the concepts of *good polynomial* and *good module* to characterize the associated primes of induced modules over skew polynomial rings. Following Annin's ideas, Niño et al. [NRR20] extended the definition of annihilator-compliant polynomials to investigate the associated primes of induced modules over quantum algebras. Under certain compatibility conditions, they described the associated prime ideals of induced modules over different families of noncommutative rings.

Thinking about the theory of associated primes and the primary decomposition presented by Lasker [Las05], Noether [Noe21], Gordon [Gor74], and Lam [Lam98], the idea of a dual theory is of great interest and a natural task. Macdonald [Mac73] introduced the notions of *representable module* and *secondary representation*, and then considered an optimal dualization of the theory of associated prime ideals over representable modules. He studied these representations about commutative polynomial modules. The prime ideals of the theory of secondary representations are called *attached prime ideals*. Annin [Ann08] considered these ideals on arbitrary modules (not necessarily representable). He proved some basic properties of attached prime ideals and showed the duality of these results with those corresponding for the associated prime ideals. He also extended Macdonald's theory of secondary representation to the noncommutative setting.

On the other hand, Shock [Sho72] investigated the *uniform dimension* of the commutative polynomial ring $R[x]$. In the setting of quantum algebras, Grzeszczuk [Grz88] and Quinn [Qui88] studied the uniform dimension of skew polynomial rings of derivation type and characterized its uniform dimension. Leroy and Matczuk [LM05] established results on the same dimension for the skew polynomial ring $R[x;\sigma,\delta]$ and induced modules over these rings. Annin [Ann02a] characterized the uniform dimension of polynomial modules over $R[x;\sigma,\delta]$, while Reyes [Rey14] presented several results about the uniform dimension of different families of quantum algebras.

Varadarajan [Var79] defined the concept of *couniform dimension* as a dual theory of uniform dimension. Sarath and Varadarajan [SV79] formulated results concerning the finiteness of the couniform dimension of a module. In the noncommutative setting, Annin [Ann05] investigated the this dimension for inverse polynomial modules over skew polynomial rings of automorphism type $R[x;\sigma]$ and characterized its couniform dimension when R is a right perfect ring.

Having in mind all facts above, we describe the structure of the thesis. This is not a classical monograph, but rather it is built upon a collection of papers.

In Chapter 1, Section 1.1 contains a little about the history and origin of the term *quantum algebra* and its importance. Section 1.2 introduces different families of noncommutative rings that will be considered in the next chapters.

In Chapter 2, Section 2.1 contains some remarks about theory of associated prime ideals over commutative and noncommutative rings. We recall definitions, remarks and results that give an overview of the work presented to date, and motivate us to formulate new questions not investigated. In Section 2.2, we study the notions of *weak annihilator* and *nilpotent associated prime ideal* introduced by Ouyang and Birkenmeier [OB12]. We recall some definitions and results about weak annihilators and ideals. In Section 2.3, we present a summary of the theory of attached prime ideals over commutative and noncommutative rings.

In Chapter 3, Section 3.1 presents the Σ-*skew reflexive* and *skew RNP rings* as a generalization of the σ-skew reflexive and RNP rings, respectively. We study the Σ-skew RNP property for Ore localization by regular elements. In Section 3.2, we formulate the original results of the chapter concerning weak annihilator ideals of skew PBW extensions. In Section 3.3, we characterize the nilpotent associated prime ideals of these rings. Section 3.4 presents the definition of good polynomial, original results that characterize these polynomials and contains results related to the characterization of associated prime ideals of induced modules. Finally, we say some words about a future research.

In Chapter 4, Section 4.1 establishes some preliminaries and key results concerning a family of quantum algebras known as skew Ore polynomials and introduces the notion of *completely* (σ,δ)-*compatible module*. We formulate original results about these modules and characterize the attached prime ideals of the right module $M[x^{-1}]_A$ where A is a skew Ore polynomial ring, thus extending the results presented by Annin [Ann08, Ann11].

In Chapter 5, Section 5.1 presents several results on essential and uniform modules, within which we characterize the uniform dimension of induced modules over families of quantum algebras. We investigate the essential and uniform modules of $M[x^{-1}]_A$, and we show that if M_R is completely (σ,δ)-compatible, then M_R and $M[x^{-1}]_A$ have the same uniform dimension. In Section 5.2 we study the hollow modules of $M[x^{-1}]_A$ and prove that the couniform dimensions of M_R and $M[x^{-1}]_A$ are equal.

Notation and some terminology

Symbol	Meaning
$\mathbb{N}$	The set of natural numbers including zero
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
R	Associative ring (not necessarily commutative) with identity
K	Commutative ring with identity
D	Division ring
$\mathbb{Z}_n$	The ring of integers modulo n
$\Bbbk$	Field
$\mathrm{Idem}(R)$	The set of idempotent elements of R
$N(R)$	The set of nilpotent elements of R
$U(R)$	The set of units of R
$J(R)$	The Jacobson radical of R, i.e., the intersection of all maximal left ideals of R
$N^*(R)$	The upper radical of R, i.e. the sum of all nil ideals of R
$N_*(R)$	The prime radical of R, i.e., the intersection of all prime ideals of R
$N_*(I)$	The prime radical of an ideal I of R, i.e., the intersection of all prime ideals of R containing I
$Q(R)$	The right Goldie quotient ring of R

Throughout the thesis, the term ring means an associative (not necessarily commutative) ring with identity.

Statement of contributions

The chapters three, four and five in this thesis correspond to the following publications and preprints containing original results.

- **Chapter 3. Higuera, S.** and Reyes, A. On weak annihilators and nilpotent associated primes of skew PBW extensions. *Communications in Algebra* 51(11) (2023) 4839–4861.

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  https://www.tandfonline.com/doi/abs/10.1080/00927872.2023.2222393

  https://arxiv.org/abs/2203.09912

- **Chapter 3.** Suárez, H., **Higuera, S.**, Reyes, A. On Σ-skew reflexive-nilpotents-property for rings. *Algebra and Discrete Mathematics* 37(1) (2024) 134–159.

  Available online at

  https://admjournal.luguniv.edu.ua/index.php/adm/article/view/1922

  https://arxiv.org/abs/2110.14061

- **Chapters 3 and 5. Higuera, S.**, Ramírez, M. C., Reyes, A. On the uniform dimension and the associated primes of skew PBW extensions. *Bulletin of the Brazilian Mathematical Society, New Series* 55(45) (2024) 1–24.

  Available online at

  https://link.springer.com/article/10.1007/s00574-024-00419-2

  https://arxiv.org/abs/2404.18698

- **Chapter 4**. **Higuera, S.** and Reyes, A. Attached prime ideals over skew Ore polynomials. *Communications in Algebra* 53(3) (2024) 1076–1087.

  Available online at

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  https://arxiv.org/abs/2406.19935

- **Chapter 5. Higuera, S.** and Reyes, A. Uniform and couniform dimensions of inverse polynomial modules over skew Ore polynomials. *International Electronic Journal of Algebra* 38(38) (2025).

  Available online at

  https://dergipark.org.tr/en/pub/ieja/issue/68167/1629334

  https://arxiv.org/abs/2408.07073

Conferences

The topics of this thesis have been socialized in the following events and seminars.

- *Weak annihilators and nilpotent associated primes of noncommutative rings.* July 2024. XXIV Latin American algebra colloquium. Pontificia Universidad Católica de Chile, Santiago de Chile, Chile.

- *Reflexive-nilpotent property over skew PBW extensions.* September 2023. Primer Encuentro Nacional en Anillos de Grupo & Tópicos Relacionados. Universidad Industrial de Santander, Bucaramanga, Colombia. [Virtual]

- *On weak annihilators and nilpotent associated primes of noncommutative rings.* June 2023. XXII Congreso Colombiano de Matemáticas. Universidad Pedagógica y Tecnológica de Colombia, Tunja, Colombia.

- *Some remarks about associated and attached prime ideals.* September 2022. Seminario Álgebra Constructiva SAC^2. Universidad Nacional de Colombia, Bogotá, Colombia.

- *On attached prime ideals of inverse polynomial modules.* July 2022. Seminario de Matemática de Postgrado. Universidad de Santiago de Chile, Santiago de Chile, Chile.

- Poster: *On attached prime ideals of inverse polynomial modules.* July 2022. Encuentro Nacional de Estudiantes de Matemática - ENEMAT. Pontificia Universidad Católica de Chile, Santiago de Chile, Chile.

- *On the nilpotent associated primes over skew PBW extensions.* October 2020. Seminario Álgebra Constructiva SAC^2. Universidad Nacional de Colombia, Bogotá, Colombia.

CHAPTER 1

Quantum algebras

In this chapter, we study the algebraic structures of interest for this thesis: the *quantum algebras*.

Section 1.1 contains a little of history of the origin of term *Quantum algebra*. We recall Dirac's motivation to the study of these algebras.

Section 1.2 presents some families of quantum algebras that have been studied by different authors. Our aim is to show explicitly the generality of these rings in the setting of ring theory and noncommutative geometry. In the following chapters, we will exemplify different results formulated with some of these families.

1.1 Origin of the term *Quantum algebra*

According to Dirac [Dir26], atomic physicists introduced the idea of quantities that do not satisfy the commutative law of multiplication, but satisfy all the other laws of algebra. In the literature, these quantities are called *q-numbers*, and the numbers of classical mathematics c-numbers, while the word *number* alone is used to denote either a q-number or a c-number. Of course, a c-number may be regarded as a special case of the more general q-number. As is well-known, when two numbers x and y satisfy $xy = yx$, we say that x *commutes with* y. This means that a c-number is assumed to commute with every number.

Until 1920's, Dirac [Dir26] asserted that the development of the algebra of q-numbers has up to the present been greatly hampered by the fact that there has been no general definition of a function of a q-number variable. He proposed a general definition of a function which appears to enable one to establish theorems involving arbitrary functions with rigour, and hence, the differential coefficient of a function of a single variable can also be conveniently defined, in a such way that its most important properties being contained in, or easily deducible from, its definition.

It is well-known that the theories of quantum mechanics introduced by Heisenberg [Hei25] and Dirac [Dir25] are different in their conception and formulation but both made use of a non-commutative algebra. Heisenberg considered the algebra of infinite matrices, while Dirac's paper contains of abstract q-numbers. Note that Schrodinger's theory, although mathematically equivalent to Heisenberg's, does not make explicit use of some kind of algebra. Nevertheless, the operators which Schrödinger used satisfy the same commutation formulas as Heisenberg's matrices.

McCoy [McC29a] studied the algebra of the quantum mechanics from the matrix view point and showed that his results did not depend of the form of the variables. Following the ideas of quantum mechanics, he used the expression "conjugate quantum variables" in analogy with the concept of conjugate variables of the classical theory and these enter in pairs. For a single pair of variables the properties of the algebra are determined by the commutation rule given by

$$pq - qp = cI, \tag{1.1.1}$$

where q and p are matrices representing the coordinate and momentum, respectively, c is a real or complex number (in the quantum mechanics $c = \frac{h}{2\pi i}$) and I is the unit matrix. Note that the algebra does not depend upon the particular value assigned to c, and in general, the symbol I will be omitted but we understand wherever a real or complex number. If we consider more than a single pair of conjugate quantum variables, the relation (1.1.1) is replaced by:

$$p_r q_s - q_s p_r = c\delta_{rs}, \quad p_r p_s - p_s p_r = 0, \quad \text{and} \quad q_r q_s - q_s q_r = 0.$$

If f and g are two polynomials in the conjugate quantum variables p and q, subject to the condition (1.1.1), then $fg \neq gf$ but this relation makes it possible to compute $fg - gf$ when f and g are given polynomials. To simplify the calculations, McCoy considered the formulas

$$ph - hp = c\frac{\partial h}{\partial q} \quad \text{and} \quad hq - qh = c\frac{\partial h}{\partial p}, \tag{1.1.2}$$

where h is a polynomial in the conjugate quantum variables p and q. The relation (1.1.2) has been extended to functions of any number of quantum variables and applied to some special problems (see [McC29b]). McCoy considered these equations as defining differentiation and asserts that the usual formulas for differentiating polynomials hold. Dirac and McCoy's papers motivated the research on algebraic properties of several families of noncommutative (also called) quantum algebras (e.g. [McC29a, McC29b, McC31, McC32a, McC32b, McC34]).

1.2 Some families of quantum algebras

1.2.1 Skew polynomial rings and ambiskew polynomial rings

Skew polynomial rings (or *Ore extensions*) were introduced by Ore [Ore31, Ore33] (Noether and Schmeidler [NS20] were interested in some kind of differential operator rings) as a generalization of the commutative polynomial rings. If σ is an endomorphism of R, then an additive map δ of R is a *σ-derivation* of R if $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$ (strictly speaking, this is the definition of *left σ-derivation*, but we will not need the concept of *right σ-derivation*, which is any additive map δ of R satisfying the rule $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$ for all $r, s \in R$) [GJ04, p. 26].

Definition 1.2.1 ([GJ04, p. 34])**.** Let σ be an endomorphism of R and δ a σ-derivation of R. We write $R[x;\sigma,\delta]$ provided that,

(i) $R[x;\sigma,\delta]$ is a ring, containing R as a subring,

(ii) x is an element of $R[x;\sigma,\delta]$,

(iii) $R[x;\sigma,\delta]$ is a free left R-module with basis $\{1, x, x^2, \dots\}$,

(iv) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

A ring $R[x;\sigma,\delta]$ is a *skew polynomial ring* or an *Ore extension* of R. The expression $R[x;\sigma]$ denotes the *skew polynomial ring of endomorphism type* when $\delta := 0$. If σ is an automorphism of R, then $R[x;\sigma]$ is called a *skew polynomial ring of automorphism type.* We write $R[x;\delta]$ to denote the *skew polynomial ring of derivation type.*

We can iterate the construction of the skew polynomial rings to get the *iterated skew polynomial ring* $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$, where σ_i, δ_i are defined on $R[x_1;\sigma_1,\delta_1]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}]$.

Definition 1.2.2 ([GJ04, p. 36])**.** Any non-zero element $p(x) \in R[x;\alpha,\delta]$ is uniquely expressed as $p(x) = r_n x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0$, for some $n \in \mathbb{N}$ and $r_i \in R$ with $r_n \neq 0$. The natural number n is called the *degree* of $p(x)$, for short $\deg(p(x))$, and the element r_n is the *leading coefficient* of $p(x)$. The zero element of $R[x;\sigma,\delta]$ is defined to have degree $-\infty$.

We review some examples of skew polynomial rings (e.g. [BG02, GL94, GJ04, FGL+20, MR01, Sei10]).

Example 1.2.3. (i) Any commutative polynomial ring $R[x_1,\dots,x_n]$ over R. In this case, σ is the identity homomorphism of R and the σ-derivation δ is defined as $\delta := 0$.

(ii) An *Ore algebra* is a skew polynomial ring of $K[t_1,\dots,t_m]$, where the homomorphisms σ_i and the σ_i-derivations δ_i are K-linear. Notice that $\sigma_i(r) = r$ and $\delta_i(r) = 0$, for all $r \in R$. In this way, an Ore algebra is a ring of the form $K[t_1,\dots,t_m][x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$.

(iii) If σ_h is the endomorphism of $\Bbbk[t]$ defined by $\sigma_h(p(t)) = p(t-h)$ for all $p(t) \in \Bbbk[t]$, then the skew polynomial ring $S_h = \Bbbk[t][x_h;\sigma_h]$ over $\Bbbk[t]$ is called *the algebra of shift operators.*

(iv) For any $h \in \Bbbk$, we define $D_h := \Bbbk[t]\left[x; \frac{\partial}{\partial t}\right][x_h;\sigma_h]$, where σ_h is the endomorphism of $\Bbbk[t]$ given by $\sigma_h(p(t)) = p(t-h)$. Notice that $D_h = A_1(\Bbbk)\,[x_h;\sigma_h]$ and thus it is an Ore algebra which is called *the mixed algebra* (or *the algebra of delay differential operator*).

(v) The *algebra for multidimensional discrete linear systems* is defined as

$$D := \Bbbk[t_1,\dots,t_n][x_1;\sigma_1]\cdots[x_n;\sigma_n],$$

where

$$\sigma_i(p(t_1,\dots,t_n)) := p(t_1,\dots,t_{i-1},t_i+1,t_{i+1},\dots,t_n), \text{ for all } 1 \le i \le n.$$

If $\mathfrak{g}$ is a finite dimensional Lie $\Bbbk$-algebra with basis $\{x_1,\dots,x_n\}$, then the universal enveloping algebra $U(\mathfrak{g})$ is the algebra generated by $x_1,\dots,x_n$ subject to the relations $x_i r - r x_i = 0 \in \Bbbk$, for all $r \in \Bbbk$, and $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g}$, where $[x_i, x_j] \subseteq \Bbbk + \Bbbk x_1 + \cdots + \Bbbk x_n$, for every $1 \le i, j \le n$. In general, these algebras are not skew polynomial rings even including non-zero trivial derivations. However, some universal enveloping algebras can be expressed as skew polynomial rings.

Example 1.2.4. (i) The *universal enveloping algebra* $U(\mathfrak{sl}_2(\Bbbk))$ is the $\Bbbk$-algebra generated over $\Bbbk$ by three indeterminate e, f, g subject to the relations

$$ef = fe + h, \ \ he = eh + 2e, \ \text{ and } \ hf = fh - 2f.$$

If R is the subalgebra of $U(\mathfrak{sl}_2(\Bbbk))$ generated by e and h, then $R = \Bbbk[e][h;\delta_1] = \Bbbk[h][e;\sigma_1]$, where $\sigma_1(h) = h-2$ is a $\Bbbk$-algebra automorphism of $\Bbbk[h]$, and $\delta_1(e) = 2e$. Thus

$$U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk[e][h;\delta_1][f;\sigma_2,\delta_2] = \Bbbk[h][e;\sigma_1][f;\sigma_2,\delta_2],$$

where $\sigma_2(e) = e$, $\sigma_2(h) = h+2$, $\delta_2(e) = -h$ and $\delta_2(h) = 0$.

(ii) The *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ generated over $\Bbbk$ by the indeterminate E, F, H, H^{-1} subject to the relations

$$HH^{-1} = H^{-1}H = 1, \ EF - FE = \frac{H - H^{-1}}{q - q^{-1}}, \ HE = q^2EH, \text{ and } HF = q^{-2}FH,$$

for $q \in \Bbbk^*$ with $q \neq \pm 1$ [GJ04, p. 41]. Notice that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][H^{\pm 1};\sigma_1][F;\sigma_2,\delta_2]$ [GJ04, Exercise 2T].

Jordan [Jor93] introduced the *ambiskew polynomial rings* with the aim of studying a family of skew polynomial rings in two indeterminates. Some examples of these rings are the enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of $\mathfrak{sl}_2(\mathbb{C})$, the quantum enveloping algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$, the algebras similar to $U(\mathfrak{sl}_2(\mathbb{C}))$ considered by Smith [Smi90], and their quantum analogues. According to each research presented, different definitions of these rings are given by Jordan and Wells (see [Jor93, JW96, Jor00, JW13]). However, the differences between each definition are very subtle. We follow the definition and some observations presented in [Jor00].

Definition 1.2.5 ([Jor00, p. 314])**.** Let R be a $\Bbbk$-algebra, σ a $\Bbbk$-automorphism of R and $c \in R$. If σ is extended to $R[x;\sigma^{-1}]$ by $\sigma(x) = px$ for some $p \in \Bbbk^*$, then there exists a σ-derivation δ of $R[x;\sigma^{-1}]$ such that $\delta(R) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $A(R,\sigma,c,p)$ is the skew polynomial ring $R[x;\sigma^{-1}][y;\sigma,\delta]$, whence the following relations hold:

$$yx - pxy = c,$$

and

$$xr = \sigma^{-1}(r)x \quad \text{and} \quad yr = \sigma(r)y, \ \text{ for all } r \in R.$$

Notice that $A(R,\sigma,c,p) \cong R[y;\sigma][x;\sigma^{-1},\delta']$ where $\sigma(y) = p^{-1}y$, $\delta'(R) = 0$, and $\delta'(y) = -p^{-1}c$. If we consider the relation $xa = \sigma^{-1}(a)x$ as $ax = x\sigma(a)$, then we can see that the definition involves twists from both sides using σ. This is the reason for the name of this ring.

In the literature, we can find a lot of papers concerning ring-theoretical, module properties and applications of skew polynomial rings (see Brown and Goodearl [BG02], Fajardo et al. [FLP+24], Goodearl and Warfield [GJ04], McConnell and Robson [MR01] and references therein).

1.2.2 Generalized Weyl algebras and down-up algebras

According to Bavula [Bav92], if σ is an automorphism of R and a is a central element of R, then the *generalized Weyl algebra* GWA $R(\sigma, a)$ is the ring generated over R by the indeterminates x^- and x^+ subject to the relations $x^-x^+ = a$, $x^+x^- = \sigma(a)$ and $x^+b = \sigma(b)x^+$, $x^-\sigma(b) = bx^-$, for all $b \in R$. This family of algebras includes the classical Weyl algebras and primitive quotients of $U(\mathfrak{sl}_2)$. Generalized Weyl algebras have been extensively studied in the literature (e.g. [Bav92, Bav96, Jor00] and references therein).

If $c = \sigma(a) - pa$, for some $a \in R$, then $A(R, \sigma, c, p)$ is called *conformal*. In this case, consider $c_0 = 0$ and $c_m = \sum_{l=0}^{m-1} p^l \sigma^l(c)$ for $m \in \mathbb{N}$. It is clear that $c_1 = c$ and each c_m is central. Additionally, it can be seen that $xy^m - p^m y^m x = c_m y^{m-1}$ and,

$$x^m y - p^m yx^m = x^{m-1}c_m = \sigma^{1-m}(c_m)x^{m-1} \ \text{ for all } m \geq 0.$$

Ambiskew polynomial rings are related to the generalized Weyl algebras GWA. Proposition 1.2.6 shows that every ambiskew polynomial ring is isomorphic to a generalized Weyl algebra.

Proposition 1.2.6 ([Jor00, Proposition 2.1])**.** *If R is a $\Bbbk$-algebra, σ is a $\Bbbk$-automorphism of R, $c \in R$ and $p \in \Bbbk^*$, then $A(R, \sigma, c, p)$ is isomorphic to $R[w](\sigma, w)$, where σ is extended to $R[w]$ by setting $\sigma(w) = pw + \sigma(c)$. In the conformal case, $A(R, \sigma, c, p) \simeq R[z](\sigma, z + \sigma(a))$ with $\sigma(z) = pz$.*

If $A(R, a, c, p)$ is conformal, then $z = yx - \sigma(r) = p(yx - r)$ is called the *Casimir element* of $A(R, a, c, p)$ [Jor00, p. 314]. In contrast with the Proposition 1.2.6, every generalized Weyl algebra is isomorphic to a factor of an ambiskew polynomial ring. It is not difficult to see that the factor ring R/zR is the ring extension of $A(R, a, c, p)$ generated by x^- and x^+, the images of x and y, respectively. Thus, R/zR is the generalized Weyl algebra $A(\sigma, a)$.

If $\mathbb{C}P$ is a vector space over $\mathbb{C}$ for certain partially ordered set $(P, \prec)$ and the sets $\{x \in P \mid x \succ p\}$ and $\{x \in P \mid x \prec p\}$ are finite for any $p \in P$, then the $\mathbb{C}$-linear operators $u, d : \mathbb{C}P \to \mathbb{C}P$ are defined as $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$, respectively. Benkart and Roby [Ben99, BR98] defined the *down-up algebras* $A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$, as a generalization of algebras generated by the "down" and "up" operators.

Definition 1.2.7 ([BR98, p. 308])**.** The *down-up algebra* is the $\mathbb{C}$-algebra generated by d and u subject to the relations

$$d^2u = \alpha dud + \beta ud^2 + \gamma d \quad \text{and} \quad du^2 = \alpha udu + \beta u^2 d + \gamma u,$$

for any $\alpha, \beta, \gamma \in \mathbb{C}$.

According to [BR98], $A(\alpha, \beta, \gamma) \simeq A(\alpha, \beta, \lambda\gamma)$, for any $0 \neq \lambda \in \mathbb{C}$. This means that when $\gamma \neq 0$, we may assume that $\gamma = 1$.

Remarkable examples of down-up algebras include the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ and some of its deformations introduced by Woronowicz [Wor87]. Related to the theoretical properties, Kirkman et al. [KMP99] proved that a down-up algebra $A(\alpha,\beta,\gamma)$ is Noetherian if and only if β is non-zero, and they showed that $A(\alpha,\beta,\gamma)$ is a generalized Weyl algebra.

Witten [Wit90, Wit91] defined a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2)$. By definition, Witten's deformation is a unital associative algebra over a field $\Bbbk$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\underline{\xi} = (\xi_1,\dots,\xi_7)$ of elements of $\Bbbk$. This algebra, denoted by $W(\underline{\xi})$, is generated by the indeterminates x, y, z subject to the defining relations $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4$, and $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$.

Proposition 1.2.8 ([Ben99, Theorem 2.6])**.** *A Witten deformation algebra $W(\xi)$ with*

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4, \tag{1.2.1}$$

is isomorphic to the down-up algebra $A(\alpha,\beta,\gamma)$ with α,β,γ given by

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4. \tag{1.2.2}$$

Conversely, any down-up algebra $A(\alpha,\beta,\gamma)$ with not both α and β equal to 0 *is isomorphic to a Witten deformation algebra $W(\underline{\xi})$ whose parameters satisfy (1.2.1).*

Le Bruyn [Bru94b, Bru94a] investigated the algebras $W(\underline{\xi})$ whose associated graded algebras are Auslander regular and determined a 3-parameter family of deformation algebras which are called *conformal* $\mathfrak{sl}_2$ *algebras* that are generated by x, y, z over $\Bbbk$ and subject to the relations

$$zx - axz = x, \quad zy - ayz = y, \quad \text{and} \quad yx - cxy = bz^2 + z.$$

In the case $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with these three relations is isomorphic to the down-up algebra $A(\alpha,\beta,\gamma)$, where

$$\alpha = c^{-1}(1 + ac), \quad \beta = -ac^{-1} \quad \text{and} \quad \gamma = -c^{-1}.$$

Kulkarni [Kul99] proved that if $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $W(\underline{\xi})$ is isomorphic to a conformal $\mathfrak{sl}_2$ algebra or to an iterated skew polynomial ring [Kul99, Theorem 3.0.3].

1.2.3 Bi-quadratic algebras on 3 generators with PBW bases

Recently, Bavula [Bav23] defined the *skew bi-quadratic algebras* with the aim of giving an explicit description of bi-quadratic algebras on 3 generators with PBW basis.

If $n \geq 2$ then $M = (m_{ij})_{i>j}$ with $m_{ij} \in R$ $(1 \leq j < i \leq n)$ is called a *lower triangular half-matrix* with coefficients in R. The set of all such matrices is denoted by $L_n(R)$. If $\sigma := (\sigma_1,\dots,\sigma_n)$ is an n-tuple of commuting endomorphisms of R, $\delta := (\delta_1,\dots,\delta_n)$ is a n-tuple of σ-endomorphisms of R, $Q = (q_{ij}) \in L_n(Z(R))$, $\mathbb{A} := (a_{ij,k})$ where $a_{ij,k} \in R$, and $\mathbb{B} := (b_{ij}) \in L_n(R)$, the *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1,\dots,x_n;\sigma,\delta,Q,\mathbb{A},\mathbb{B}]$ is a ring generated by R and $x_1,\dots,x_n$ subject to the relations

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \quad \text{for all } 1 \le i \le n \text{ and } r \in R, \tag{1.2.3}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij}, \quad \text{for all } j < i. \tag{1.2.4}$$

In particular, if $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for all $1 \le i \le n$, then A is called the *bi-quadratic algebra* (*BQA*) and it is denoted by $A = R[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$. If $A = \bigoplus_{\alpha \in \mathbb{N}^n} Rx^\alpha$ with $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then A has *PBW basis* [Bav23, p. 696].

If $q_{ii} = 1$ for all $1 \le i \le n$, then A is called a *bi-quadratic algebra of Lie type* and it is denoted by $A := R[x_1, \dots, x_n; \mathbb{A}, \mathbb{B}]$ [Bav23, p. 698]. If $A = R[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$ is a bi-quadratic algebra such that R is a $\Bbbk$-algebra and all the elements $a_{i,j,k}$ and b_{ij} belong to $\Bbbk$, then the algebra A is a bi-quadratic algebra of Lie type if and only if $A \cong R \otimes_{\Bbbk} (U(\mathfrak{g})/\langle z-1 \rangle)$, where $\mathfrak{g}$ is a Lie algebra of dimension $n+1$ with z a non-zero central element of $\mathfrak{g}$ [Bav23, Theorem 1.2].

If $A = \Bbbk[x_1, x_2, x_3; Q, \mathbb{A}, \mathbb{B}]$ is a bi-quadratic algebra where $Q = (q_1, q_2, q_3) \in \Bbbk^{*3}$,

$$\mathbb{A} = \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \lambda & \mu & \nu \end{bmatrix},$$

and $\mathbb{B} = (b_1, b_2, b_3)$, then the algebra A is generated over $\Bbbk$ by x_1, x_2 and x_3 subject to the relations

$$x_2 x_1 - q_1 x_1 x_2 = a x_1 + b x_2 + c x_3 + b_1, \tag{1.2.5}$$

$$x_3 x_1 - q_2 x_1 x_3 = \alpha x_1 + \beta x_2 + \gamma x_3 + b_2, \tag{1.2.6}$$

$$x_3 x_2 - q_3 x_2 x_3 = \lambda x_1 + \mu x_2 + \nu x_3 + b_3. \tag{1.2.7}$$

We present some examples of bi-quadratic algebras on 3 generators.

Example 1.2.9 ([Bav23, p. 699])**.** (i) The universal enveloping algebra of any 3-dimensional Lie algebra.

(ii) The 3-*dimensional quantum space* $\mathbb{A}^3_{q_1,q_2,q_3} := \Bbbk[x_1, x_2, x_3; Q, \mathbb{A} = 0, \mathbb{B} = 0]$.

(iii) Havlíček et al. [HKP00] defined the algebra $U'_q(\mathfrak{so}_3)$ as the $\Bbbk$-algebra generated over $\Bbbk$ by I_1, I_2, and I_3 subject to the relations

$$I_2 I_1 - q I_1 I_2 = -q^{\frac{1}{2}} I_3, \quad I_3 I_1 - q^{-1} I_1 I_3 = q^{-\frac{1}{2}} I_2, \quad I_3 I_2 - q I_2 I_3 = -q^{\frac{1}{2}} I_1,$$

where $q \in \Bbbk \setminus \{0, \pm 1\}$ [HKP00, p. 79].

(iv) Zhedanov [Zhe91] introduced the *Askey-Wilson algebra AW*(3) as the $\mathbb{R}$-algebra generated over $\mathbb{R}$ by K_0, K_1, and K_2 subject to the relations

$$[K_0, K_1]_\omega = K_2, [K_2, K_0]_\omega = BK_0 + C_1 K_1 + D_1, \text{ and } [K_1, K_2]_\omega = BK_1 + C_0 K_0 + D_0,$$

where $B, C_0, C_1, D_0, D_1 \in \mathbb{R}$ and $[\square, \triangle]_\omega := e^\omega \square \triangle - e^{-\omega} \triangle \square$, with $\omega \in \mathbb{R}$ [Zhe91, Section 1]. Notice that in the limit $\omega \to 0$, the algebra AW(3) becomes an ordinary Lie algebra with three generators (D_0 and D_1 are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as

$$\begin{aligned} e^{\omega}K_0K_1 - e^{-\omega}K_1K_0 &= K_2, \\ e^{\omega}K_2K_0 - e^{-\omega}K_0K_2 &= BK_0 + C_1K_1 + D_1, \\ e^{\omega}K_1K_2 - e^{-\omega}K_2K_1 &= BK_1 + C_0K_0 + D_0. \end{aligned}$$

Proposition 1.2.10 classifies the bi-quadratic algebras on three generators of Lie type.

Proposition 1.2.10 ([Bav23, Theorem 1.4])**.** *If A is an algebra of Lie type over an algebraically closed field $\Bbbk$ of characteristic zero, then A is isomorphic to one of the following (pairwise non-isomorphic) algebras:*

(1) $P_3 = \Bbbk[x_1, x_2, x_3]$ *is the classical polynomial ring.*

(2) *The universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$.*

(3) *The universal enveloping algebra $U(\mathfrak{H}_3)$ of the Heisenberg Lie algebra $\mathfrak{H}_3$.*

(4) $U(\mathcal{N})/\langle c-1\rangle \cong \Bbbk\{x,y,z\}/\langle [x,y]=z, [x,z]=0, [y,z]=1\rangle \cong A_1 \otimes \Bbbk[x']$*, where $A_1(\Bbbk)$ is the Weyl algebra $\Bbbk\{y,z\}/\big\langle [y,z]=1\big\rangle$ and $\Bbbk[x']$ is the commutative polynomial ring with $x' = x + \frac{1}{2}z^2$.*

(5) $U(\mathfrak{n}_2 \times \Bbbk z) \cong \Bbbk\{x,y,z\}/\langle [x,y]=y\rangle$ *and z is a central element.*

(6) $U(\mathcal{M})/\langle c-1\rangle \cong \Bbbk\{x,y,z\}/\langle [x,y]=y, [x,z]=1, [y,z]=0\rangle \cong A_1(\Bbbk)[y;\sigma]$*, where $A_1(\Bbbk)$ is the Weyl algebra $\Bbbk\{x,z\}/\langle [x,z]=1\rangle$ and σ is defined by $\sigma(x) = x+1$ and $\sigma(z) = z$.*

The following proposition classifies the bi-quadratic algebras on two generators.

Proposition 1.2.11 ([Bav23, Theorem 2.1])**.** *Up to isomorphism, there are only five bi-quadratic algebras on two generators:*

(1) *The polynomial algebra $\Bbbk[x_1, x_2]$.*

(2) *The Weyl algebra $A_1(\Bbbk) = \Bbbk\{x,y\}/\langle [x,y]=1\rangle$.*

(3) *The universal enveloping algebra $U(\mathfrak{n}_2)$ of the Lie algebra $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$.*

(4) *The quantum plane $\mathcal{O}_q(\Bbbk) = \Bbbk\{x_1,x_2\}/\langle x_2x_1 - qx_1x_2\rangle$, where $q \in \Bbbk \setminus \{0,1\}$.*

(5) *The quantum Weyl algebra $A_1(q) = \Bbbk\{x_1,x_2\}/\langle x_2x_1 - qx_1x_2 = 1\rangle$, where $q \in \Bbbk \setminus \{0,1\}$.*

1.2.4 Skew Ore polynomials of higher order

Cohn [Coh61] defined the *skew Ore polynomials of higher order* as a generalization of the skew polynomial rings considering the relation $xr := \Psi_1(r)x + \Psi_2(r)x^2 + \cdots$ for all $r \in R$, where the

Ψ's are endomorphisms of R. Following Cohn's ideas, Smits [Smi68] introduced the ring of skew Ore polynomials of higher order over a division ring D and commutation rule defined by

$$xr := r_1 x + \cdots + r_k x^k, \text{ for all } r \in R \text{ and } k \geq 1. \tag{1.2.8}$$

The expression (1.2.8) induces a finite set of endomorphisms $\delta_1, \ldots, \delta_k$ of the group $(D, +)$ with $\delta_i(r) := r_i$ for all $1 \leq i \leq k$ [Smi68, p. 211]. Smits proved that if $\{\delta_2, \ldots, \delta_k\}$ is a set of left D-independient endomorphisms (i.e., if $c_2\delta_2(r) + \cdots + c_k\delta_k(r) = 0$ for all $r \in D$ then $c_i = 0$ for all $2 \leq i \leq k$ [Smi68, p. 212]), then δ_1 is injective [Smi68, p. 213]. If δ_1 is an automorphism of D and $\{\delta_2, \ldots, \delta_k\}$ is a set of left D-independent endomorphism, then $\delta := \delta_1^{-1}\delta_2$ is a δ_1-derivation of D, $\delta_{i+1}(r) = \delta_1\delta^i(r)$, and $\delta^k(r) = 0$ for all $r \in D$ [Smi68, p. 214].

Dumas [Dum91] studied the field of fractions of $D[x; \sigma, \delta]$ where σ is an automorphism of D and stated that one technique for this purpose is to consider it as a subfield of a certain field of series [Dum91, p. 193]. According to Dumas, if Q is the field of fractions of $D[x; \sigma, \delta]$ then Q is a subfield of the *field of series of Laurent* $D((x^{-1}; \sigma^{-1}, -\delta\sigma^{-1}))$ whose elements are of the form $r_{-k}x^{-k} + \cdots + r_{-1}x^{-1} + r_0 + r_1 x + \cdots$ for some $k \in \mathbb{N}$, and satisfies the commutation rule

$$\begin{aligned} xr &:= \sigma(r)x + \sigma\delta(r)x^2 + \cdots = \sigma(r)x + x\delta(r)x, \text{ and} \\ x^{-1}r &:= \sigma'(r)x^{-1} + \delta'(r), \text{ for all } r \in R. \end{aligned}$$

There are algebras such as Clifford algebras, Weyl-Heisenberg algebras, Sklyanin algebras among others, in which this commutation rule is not sufficient to define the noncommutative structure of the algebras since a free non-zero term Ψ_0 is required.

Maksimov [Mak00] considered the skew Ore polynomials of higher order with free non-zero term $\Psi_0(r)$ where Ψ_0 satisfies the relation

$$\Psi_0(rs) = \Psi_0(r)s + \Psi_1(r)\Psi_0(s) + \Psi_2(r)\Psi_0^2(s) + \cdots, \text{ for all } r, s \in R.$$

Golovashkin and Maksimov [GM05] defined the algebras $Q(a, b, c)$ over a field $\Bbbk$ of characteristic zero with two generators x and y subject to the quadratic relations $yx = ax^2 + bxy + cy^2$, where $a, b, c \in \Bbbk$. If $\{x^m y^n\}$ forms a basis for $Q(a, b, c)$, then the ring generated by the quadratic relation is an algebra of skew Ore polynomials and can be defined by a system of linear mappings $\delta_0, \ldots, \delta_k$ of $\Bbbk[x]$ into itself such that for any $p(x) \in \Bbbk[x]$, $yp(x) = \delta_0(p(x)) + \delta_1(p(x))y + \cdots + \delta_k(p(x))y^k$, for some $k \in \mathbb{N}$. They found conditions for such an algebra $Q(a, b, c)$ to be expressed as a skew polynomial ring over the polynomial ring $\Bbbk[x]$ (c.f. [GM98]), and proved that these conditions are equivalent to the existence of a PBW basis, i.e. basis of the form $\{x^m y^n\}$.

1.2.5 Poincaré-Birkhoff-Witt extensions

Bell and Goodearl [BG88] defined the *Poincaré-Birkhoff-Witt extensions* with the aim of cover several families of generalized operator rings as the enveloping algebra of a finite-dimensional Lie algebra, Weyl algebras, differential operators over Lie algebras, the twisted or smash product differential operator rings and universal enveloping rings [BG88, Section 5].

Definition 1.2.12 ([BG88, p. 27])**.** A ring A is a *Poincaré-Birkhoff-Witt* (*PBW*) *extension of* R if the following conditions hold:

(i) A contains R as a proper subring.

(ii) There exist finitely many elements $x_1, \ldots, x_n \in A$ such that A is a free left R-module with basis $\mathrm{Mon}(A) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$.

(iii) For each $r \in R$ and every $1 \le i \le n$, we have that $x_i r - r x_i \in R$.

(iv) For every $1 \le i, j \le n$, we have that $x_i x_j - x_j x_i \in R + R x_1 + \cdots + R x_n$.

We write $A = R\langle x_1, \ldots, x_n\rangle$, and R is called *the ring of coefficients* of the extension A.

We present some examples of PBW extensions. Some of these cannot be expressed as iterated skew polynomial rings. These examples are adapted from [BG88, McC74, Pas87, Rin63].

Example 1.2.13. (i) If R is a $\Bbbk$-algebra containing $\Bbbk$, then $R \otimes_{\Bbbk} U(\mathfrak{g})$ and the *crossed product* $R * U(\mathfrak{g})$ of R by $U(\mathfrak{g})$ are PBW extensions of R.

(ii) If L is a Lie algebra and also a free R-module equipped with a Lie algebra map to derivations on R, then the *differential operator ring* $V(R, L)$ satisfies the property PBW [Rin63, Theorem 3.1], and hence $V(R, L)$ is a PBW extension of R.

(iii) If $\mathfrak{g}$ acts on R by derivations and σ is Lie 2-cocycle, then the *twisted product differential operator ring* $R\#_{\sigma} U(\mathfrak{g})$ satisfies the property PBW by [McC74, Theorem 2.8], and thus $R\#_{\sigma} U(\mathfrak{g})$ is a PBW extension of R.

(iv) If R is a $\Bbbk$-algebra and V is a $\Bbbk$-vector space which is also a Lie ring containing R and $\Bbbk$ as Lie ideals, then the *universal enveloping ring* $\mathcal{U}(V, R, \Bbbk)$ satisfies the property PBW [Pas87, Theorem 1.3], and hence it is a PBW extension of R.

1.2.6 Skew Poincaré-Birkhoff-Witt extensions

Gallego and Lezama [GL10] introduced the *skew PBW extensions* with the aim of generalizing Poincaré-Birkhoff-Witt extensions (Section 1.2.5) and skew polynomial rings of injective type (Section 1.2.1). Several authors have shown that skew PBW extensions also generalize families of noncommutative rings such as those mentioned above, and have also studied ring-theoretical and geometrical properties of skew PBW extensions [AT24, Art15, Faj18, Gal15, HHR20, HR22, LG19, Rey13, SR17, SRS23]. Relations between these rings and other noncommutative algebras having PBW bases can be consulted in [FGL+20, GT14, Sei10], and references therein.

Definition 1.2.14 ([GL10, Definition 1])**.** A ring A is a *skew PBW extension* of R, which is denoted by $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist finitely many elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\mathrm{Mon}(A) := \{x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For every $1 \leq i \leq n$ and any non-zero element $r \in R$, there exists a non-zero element $c_{i,r} \in R$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i, j \leq n$, there exists a non-zero elements $d_{i,j} \in R$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n,$$

i.e. there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)} \in R$ with

$$x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{k=1}^{n} r_k^{(i,j)} x_k.$$

Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ are unique. In this way, every non-zero element $f \in A$ can be uniquely expressed as $f = \sum_{i=1}^{k} a_i x^{\alpha_i}$, and $x^{\alpha_i} \in \mathrm{Mon}(A)$, for $0 \leq i \leq k$ [GL10, Remark 2].

Proposition 1.2.15 ([GL10, Proposition 3])**.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, then there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $1 \leq i \leq n$ and $r \in R$.*

Definition 1.2.16. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R.

(i) [GL10, Definition 4] A is called *quasi-commutative* if the conditions (iii) and (iv) presented above are replaced by the following:

(iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For every $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = d_{i,j} x_i x_j.$$

(ii) [GL10, Definition 4] A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $d_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(iii) [LAR15, Definition 2.3] If σ_i is the identity homomorphism of R for all $1 \leq i \leq n$, (we write $\sigma_i = \mathrm{id}_R$), we say that A is a skew PBW extension of *derivation type*. Similarly, if δ_i is zero for all $1 \leq i \leq n$, then A is called a skew PBW extension of *endomorphism type*.

We present some relationships between skew polynomial rings and skew PBW extensions.

Remark 1.2.17. (i) If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a quasi-commutative skew PBW extension of R, then A is isomorphic to an iterated skew polynomial ring of endomorphism type of R [LR14, Theorem 2.3].

(ii) Skew polynomial rings of injective type are strictly contained in skew PBW extensions [LR14, Example 5(3)]. The quantum plane $\mathcal{O}_2(q)$ is a skew polynomial ring of injective type but cannot be expressed as a PBW extension. This shows that skew polynomial rings are not contained in PBW extensions.

(iii) Skew PBW extensions of endomorphism type of a ring R are more general than iterated skew polynomial rings of endomorphism type on the same ring [SCR22, Remark 2.4 (ii)].

Definition 1.2.18 ([GL10, Section 3])**.** If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, then:

(i) We will write $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$ and $\delta^\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, where $\circ$ denotes the usual composition of functions. Additionally, if $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) Let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. If f is a non-zero element of A, then we use expressions as $f = a_1 x^{\alpha_1} + \cdots + a_k x^{\alpha_k}$, with $a_i \in R$, and $x^{\alpha_k} \succ \cdots \succ x^{\alpha_1}$. With this notation, we define $\mathrm{lm}(f) := x^{\alpha_k}$, the *leading monomial* of f; $\mathrm{lc}(f) := a_k$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_k x^{\alpha_k}$, the *leading term* of f. Note that $\deg(f) := \max\{\deg(x^{\alpha_i})\}_{i=1}^k$. If $f = 0$, then we consider $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$.

The next proposition is useful when one need to make some computations with elements of skew PBW extensions.

Proposition 1.2.19 ([GL10, Theorem 7])**.** *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(1) *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(2) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

Proposition 1.2.20 ([RR21, Proposition 2.7 and Remark 2.8])**.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $r \in R$, then*

$$
\begin{aligned}
x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\
&+ x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n-1}^{j-1} \right) x_n^{\alpha_n} \\
&+ x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \cdots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2 - j} \delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))))) x_2^{j-1} \right) x_3^{\alpha_3} x_4^{\alpha_4} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \mathrm{id}_R \ \text{ for } 1 \leq j \leq n.
\end{aligned}
$$

If $a_i, b_j \in R$ and $x^{\alpha_i} := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $x^{\beta_j} := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, when we compute every summand of $a_i x^{\alpha_i} b_j x^{\beta_j}$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's *depending on the coordinates of* α_i*. This assertion follows from the expression:*

$$
\begin{aligned}
a_i x^{\alpha_i} b_j x^{\beta_j} = a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} &+ a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}(b_j)} x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b_j} x^{\beta_j},
\end{aligned}
$$

where the polynomials p's are given by Proposition 1.2.19.

In the localization of noncommutative rings the *Ore condition* plays an important role. A multiplicative subset X of R satisfies the *left Ore condition* if $Xr \cap Rx \neq \emptyset$ for all $r \in R$ and $x \in X$. If X satisfies the left Ore condition then X is called a *left Ore set*. If $r \in R$ is a left and right divisor, then r is called *regular*. If I is an ideal of R, then $\mathscr{C}_I(R)$ denotes the set of all elements $a \in R$ such that $\overline{a}$ regular in R/I [FGL+20, Definition 5.4.1]. The set of regular elements of R is denoted by $\mathscr{C}_0(R)$. Lezama et al. [LAC+13] studied the Ore and Goldie's theorems for skew PBW extensions. An interesting application of the article is the quantum version of the Gelfand-Kirillov conjecture for the skew quantum polynomials [LAC+13, Corollary 5.1]. Proposition 1.2.21 characterized the Ore localization of a skew PBW extension A built on R.

Proposition 1.2.21 ([LAC+13, Lemma 2.6])**.** *Let* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *be a skew PBW extension of* R *and* $S \subseteq \mathscr{C}_0(R)$ *such that* $\sigma_i(S) = S$*, for all* $1 \le i \le n$*, where* σ_i *is defined by Proposition 1.2.15.*

(1) *If* $S^{-1}R$ *exists, then* $S^{-1}A$ *exists and it is a bijective skew PBW extension of* $S^{-1}R$ *with* $S^{-1}A = \sigma(S^{-1}R)\langle x_1', \dots, x_n'\rangle$*, where* $x_i' := \frac{x_i}{1}$ *and the system of constants of* $S^{-1}R$ *is given by* $d_{i,j}' := \frac{d_{i,j}}{1}$, $c_{i,\frac{r}{s}}' := \frac{\sigma_i(r)}{\sigma_i(s)}$*, for all* $1 \le i, j \le n$.

(2) *If* RS^{-1} *exists, then* AS^{-1} *exists and it is a bijective skew PBW extension of* RS^{-1} *with* $AS^{-1} = \sigma(RS^{-1})\langle x_1', \dots, x_n'\rangle$*, where* $x_i'' := \frac{x_i}{1}$ *and the system of constants of* $S^{-1}R$ *is given by* $d_{i,j}'' := \frac{d_{i,j}}{1}$, $c_{i,\frac{r}{s}}'' := \frac{\sigma_i(r)}{\sigma_i(s)}$*, for all* $1 \le i, j \le n$.

Proposition 1.2.22 gave sufficient conditions to ensure that endomorphisms and derivations can be extended from the ring of coefficients to the skew PBW extension.

Proposition 1.2.22 ([RS17, Theorem 5.1])**.** *Let* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *be a skew PBW extension of* R*. Suppose that* $\sigma_i\delta_j = \delta_j\sigma_i$, $\delta_i\delta_j = \delta_j\delta_i$*, and* $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$*, for* $1 \le i, j, k, l \le n$*, where* $d_{i,j}$ *and* $r_l^{(i,j)}$ *are the elements of Definition 1.2.14. If* $\overline{\sigma_k} : A \to A$ *and* $\overline{\delta_k} : A \to A$ *are functions given by* $\overline{\sigma_k}(f) := \sigma_k(a_1)x^{\alpha_1} + \cdots + \sigma_k(a_m)x^{\alpha_m}$ *and* $\overline{\delta_k}(f) := \delta_k(a_1)x^{\alpha_1} + \cdots + \delta_k(a_m)x^{\alpha_m}$*, for all* $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m} \in A$ *and* $\overline{\sigma_k}(r) := \sigma_k(r)$*, for all* $1 \le k \le n$ *and* $r \in R$*, then* $\overline{\sigma_k}$ *is an injective endomorphism of* A *and* $\overline{\delta_k}$ *is a* $\overline{\sigma_k}$*-derivation of* A*, for each* k*.*

The following examples present several families of quantum algebras that are skew PBW extensions (c.f. [FGL+20, p. 31]).

Example 1.2.23. (i) The *additive analogue of the Weyl algebra* $A_n(q_1,\dots,q_n)$ is the $\Bbbk$-algebra generated by $x_1,\dots,x_n,t_1,\dots,t_n$ subject to the relations

$$x_jx_i = x_ix_j,\quad t_jt_i = t_it_j,\quad 1\le i,j\le n.$$
$$x_jt_i = t_ix_j,\quad i\neq j.$$
$$x_it_i = q_it_ix_i+1,\quad 1\le i\le n,$$

where $q_i\in\Bbbk^*$ for all $1\le i\le n$.

(ii) The *multiplicative analogue of the Weyl algebra* $\mathcal{O}_n(q_{ji})$ is the $\Bbbk$-algebra generated by the indeterminate $x_1,\dots,x_n$ subject to the relations

$$x_jx_i = q_{ji}x_ix_j,\ \text{for all}\ 1\le i<j\le n,$$

where $q_{ji}\in\Bbbk^*$.

(iii) The *quantum algebra* $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$ is generated by I_1,I_2,I_3 subject to the following relations

$$I_2I_1-qI_1I_2=-q^{1/2}I_3,\ \ I_3I_1-q^{-1}I_1I_3=q^{-1/2}I_2,\ \ I_3I_2-qI_2I_3=-q^{1/2}I_1,$$

where $q\in\Bbbk^*$. It is easy to see that $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))\cong\sigma(\Bbbk)\langle I_1,I_2,I_3\rangle$ [FGL+20, Example 1.3.3].

(iv) A *3-dimensional skew polynomial algebra* A is a $\Bbbk$-algebra generated by x,y,z subject to the relations $yz-\alpha zy=\lambda$, $zx-\beta xz=\mu$, $xy-\gamma yx=\nu$, such that $\lambda,\mu,\nu\in\Bbbk+\Bbbk x+\Bbbk y+\Bbbk z$, $\alpha,\beta,\gamma\in\Bbbk^*$ and the standard monomials $\{x^iy^jz^l\mid 0\le i,j,l\}$ are a $\Bbbk$-basis of A. Bell and Smith proved that there exists 15 different families of 3-dimensional skew polynomial algebras [Ros95, Theorem C.4.3.1]. Following Fajardo et al. [FGL+20], every 3-dimension skew polynomial algebra is a skew PBW extension $\sigma(\Bbbk)\langle x,y,z\rangle$ [FGL+20, p. 33].

(vi) The *Dispin algebra* $U(\mathfrak{osp}(1,2))$ is the algebra presented by the three generators x,y,z and the relations

$$yz-zy=z,\ zx+xz=y\ \text{and}\ xy-yx=x.$$

It is not difficult to see that $U(\mathfrak{osp}(1,2))\cong\sigma(\Bbbk)\langle x_1,x_2,x_3\rangle$ [FGL+20, p. 33]. However, this algebra cannot be interpreted as a skew polynomial ring of $\Bbbk$.

(vii) The *Woronowicz algebra* $\mathcal{W}_\nu(\mathfrak{sl}(2,\Bbbk))$ is the $\Bbbk$-algebra generated by the indeterminate x,y,z subject to the relations

$$xz-\nu^4zx=(1+\nu^2)x,\ xy-\nu^2yx=\nu z,\ zy-\nu^4yz=(1+\nu^2)y,$$

where $\nu\in\Bbbk^*$ is not a root of unity. The algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,\Bbbk))$ can be interpreted as the skew PBW extension $\sigma(\Bbbk)\langle x,y,z\rangle$ [FGL+20, p. 34].

(viii) The *q-Heisenberg algebra* $\mathrm{H}_n(q)$ is the $\Bbbk$-algebra generated by the set of variables $\{x_i,y_i,z_i\}$ with $1\le i\le n$ and subject to the relations

- $x_jx_i=x_ix_j,\ \ z_jz_i=z_iz_j,$ for all $1\le i,j\le n$.
- $z_jy_i=y_iz_j,\ \ z_jx_i=x_iz_j,\ \ y_jx_i=x_iy_j,\ \ i\neq j.$

- $z_i y_i = q y_i z_i,\ \ z_i x_i = q^{-1} x_i z_i + y_i,\ \ y_i x_i = q x_i y_i,$ for all $1 \le i \le n,\ q \in \Bbbk \setminus \{0\}$.

It is not difficult to show that $H_n(q) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n \rangle$.

Example 1.2.24. A great variety of noncommutative rings have been interpreted as skew PBW extensions. For instance, skew polynomial ring of injective type and ambiskew polynomial rings (Section 1.2.1), down-up algebras (Section 1.2.2), skew bi-quadratic algebras (Section 1.2.3), some skew Ore polynomials of higher order (Section 1.2.4), and enveloping algebras of finite dimensional Lie algebras and PBW extensions (Section 1.2.5). Solvable polynomial rings defined by Kandri-Rody and Weispfenning [KRW90], almost normalizing extensions considered by McConnell and Robson [MR01], among other, are also examples of skew PBW extensions.

1.2.7 Other families of quantum algebras

In this section, we recall the definitions of some examples of noncommutative rings.

In the setting of stochastic processes, the *diffusion algebras* introduced by Isaev et al. [IPR01] play a key role, as they are useful tools for finding expressions for the probability distribution of the stationary state of these processes. Just as Pyatov and Twarock [PT02] said, "Diffusion algebras play a key role in the understanding of one-dimensional stochastic processes. In the case of N species of particles with only nearest-neighbor interactions with exclusion on a one-dimensional lattice, diffusion algebras are useful tools in finding expressions for the probability distribution of the stationary state of these processes. Following the idea of matrix product states, the latter are given in terms of monomials built from the generators of a quadratic algebra" [PT02, p. 3268]. Hinchcliffe in his PhD Thesis [Hin05] and different researchers have investigated several ring, theoretical and homological properties of diffusion algebras [FLP+24, HHR20, Lev05, RR21, Twa02].

We adopt the terminology and notation presented by Pyatov and Twarock [PT02] who presented a construction formalism for these algebras from the mathematical point of view, and proved the results formulated in [IPR01].

Let α, β be two elements belonging to the set $I_N := \{1, \dots, n\}$ with $\alpha < \beta$. Consider quadratic relations of the form

$$g_{\alpha\beta} D_\alpha D_\beta - g_{\beta\alpha} D_\beta D_\alpha = x_\beta D_\alpha - x_\alpha D_\beta, \tag{1.2.9}$$

with $g_{\alpha\beta} \in \mathbb{R} \setminus \{0\}$, $g_{\beta\alpha} \in \mathbb{R}$, and $x_\alpha, x_\beta \in \mathbb{C}$.

Definition 1.2.25 ([PT02, Definition 1.1])**.** An algebra with set of generators given by $\{D_\alpha \mid \alpha \in I_N\}$ and relations of type (1.2.9) is called *diffusion algebra*, if it admits a linear PBW-basis of ordered monomials of the form

$$D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_n}^{k_n}, \quad \text{with} \quad k_j \in \mathbb{N} \quad \text{and} \quad \alpha_1 > \alpha_2 > \cdots > \alpha_n. \tag{1.2.10}$$

Due to physical reasons only relations with positive coefficients $g_{\alpha\beta} \in \mathbb{R}_{>0}$ and $g_{\beta\alpha} \in \mathbb{R}_{\ge 0}$ ($\alpha < \beta$) are relevant because they are interpreted as hopping rates in stochastic models [PT02, p. 3268].

Note that the requirement of having a PBW basis (1.2.10) implies conditions on the coefficients $g_{\alpha\beta}$ and x_α in (1.2.9) according the the *Diamond Lemma* in ring theory formulated by Bergman [Ber78]. This means that we have a criterion to verify under which conditions the relations in (1.2.9) are of PBW type: this is the case precisely if each subset of three generators $\{D_\alpha, D_\beta, D_\gamma\}$ with ordering $\alpha < \beta < \gamma$ is reduction unique with respect to the ordering, that is if the two ways of reducing the monomial $D_\alpha D_\beta D_\gamma$ to the monomial $D_\gamma D_\beta D_\alpha$ lead to the same result when expressed in the PBW basis (1.2.10) (see also Krebs and Sandow [KS97]).

Just as Pyatov and Twarock [PT02, p. 3269] asserted, the task of deriving all diffusion algebras with N generators reduces to the following two steps:

(1) Find all diffusion algebras with three generators.

(2) Find all algebras with N generators such that each subset of three generators coincides with one of the cases listed before.

As it can be seen, the step (1) is equivalent to find those coefficients $g_{\alpha\beta}$ and x_α in (1.2.9) for which a set $\{D_\gamma, D_\beta, D_\alpha\}$ of three generators is reduction unique in the above sense. The second step is a combinatorial problem: it requires one to combine in a consistent way the three generators algebras listed before to algebras with N generators for general $N > 3$.

With the aim of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [Gre53] and Greenberg and Messiah [GM65] defined by means of generators and relations the *parafermionic* and *parabosonic algebras*. For the completeness of the paper, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [KD09]. Let $[\square, \triangle] := \square\triangle - \triangle\square$ and $\{\square, \triangle\} := \square\triangle + \triangle\square$.

Consider the $\Bbbk$-vector space V_F freely generated by the elements f_i^+, f_j^-, with $1 \le i, j \le n$. If $T(V_F)$ is the tensor algebra of V_F and I_F is the two-sided ideal I_F generated by the elements $[[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon - \eta)^2 \delta_{jk} f_i^\xi + \frac{1}{2}(\varepsilon - \xi)^2 \delta_{ik} f_j^\eta$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $1 \le i, j, k \le n$, then the *parafermionic algebra* in $2n$ generators $P_F^{(n)}$ (n parafermions) is the quotient algebra of $T(V_F)$ with the ideal I_F, that is,

$$P_F^{(n)} = \frac{T(V_F)}{\langle [[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon - \eta)^2 \delta_{jk} f_i^\xi + \frac{1}{2}(\varepsilon - \xi)^2 \delta_{ik} f_j^\eta \mid \xi, \eta, \varepsilon = \pm 1, i, j, k = 1, \dots, n \rangle}.$$

It is well-known that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$ (according to the classification of the simple complex Lie algebras, e.g., Kac [Kac77]), i.e., $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$ [KD09, Section 18.2]. On the other hand, $P_F^{(n)}$ is a skew PBW extension of $\Bbbk$, that is, $P_F^{(n)} \cong \sigma(\Bbbk)\langle f_i^\xi, f_j^\eta \rangle$, for values $\xi, \eta = \pm 1$, and $1 \le i, j \le n$.

Similarly, if V_B denotes the $\Bbbk$-vector space generated by the elements b_i^+, b_j^-, with $1 \le i, j \le n$, $T(V_B)$ is the tensor algebra of V_B and I_B is the two-sided ideal of $T(V_B)$ generated by the elements $[\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon] - (\varepsilon - \eta)\delta_{jk} b_i^\xi - (\varepsilon - \xi)\delta_{ik} b_j^\eta$, for all $\xi, \eta, \varepsilon = \pm 1$, and $1 \le i, j \le n$, then the *parabosonic* *algebra* $P_B^{(n)}$ in $2n$ generators (n parabosons) is defined as the quotient algebra $P_B^{(n)}/I_B$, that is,

$$P_B^{(n)} = \frac{T(V_B)}{\langle [\{b_i^{\xi}, b_j^{\eta}\}, b_k^{\varepsilon}] - (\varepsilon-\eta)\delta_{jk} b_i^{\xi} - (\varepsilon-\xi)\delta_{ik} b_j^{\eta} \mid \xi,\eta,\varepsilon = \pm 1, i, j = 1,\dots,n\rangle}.$$

It is known that the parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0,n)$, i.e $P_B^{(n)} \cong U(B(0,n))$.

According to Burdík and Navrátil [BN09], the basis of the algebra $\mathfrak{g} = \mathfrak{so}(5,\mathbb{C})$ consists of the elements $\mathbf{J}_{\alpha\beta} = -\mathbf{J}_{\beta\alpha}$, $\alpha, \beta = 1,2,3,4,5$ satisfying the commutation relations

$$[\mathbf{J}_{\alpha\beta}, \mathbf{J}_{\mu\nu}] = \delta_{\beta\mu}\mathbf{J}_{\alpha\nu} + \delta_{\alpha\nu}\mathbf{J}_{\beta\mu} - \delta_{\beta\nu}\mathbf{J}_{\alpha\mu} - \delta_{\alpha\mu}\mathbf{J}_{\beta\nu}.$$

If we consider the elements

$$\begin{aligned}
\mathbf{H}_1 &= \mathbf{i}\mathbf{J}_{12}, & \mathbf{H}_2 &= \mathbf{i}\mathbf{J}_{34},\\
\mathbf{E}_1 &= \frac{1}{\sqrt{2}}(\mathbf{J}_{45} + \mathbf{i}\mathbf{J}_{35}), & \mathbf{E}_2 &= \frac{1}{2}(\mathbf{J}_{23} + \mathbf{i}\mathbf{J}_{13} - \mathbf{J}_{14} + \mathbf{i}\mathbf{J}_{24}),\\
\mathbf{E}_3 &= \frac{1}{\sqrt{2}}(\mathbf{J}_{15} - \mathbf{i}\mathbf{J}_{25}), & \mathbf{E}_4 &= \frac{1}{2}(\mathbf{J}_{23} + \mathbf{i}\mathbf{J}_{13} + \mathbf{J}_{14} - \mathbf{i}\mathbf{J}_{24}),\\
\mathbf{F}_1 &= \frac{1}{\sqrt{2}}(-\mathbf{J}_{45} + \mathbf{i}\mathbf{J}_{35}), & \mathbf{F}_2 &= \frac{1}{2}(-\mathbf{J}_{23} + \mathbf{i}\mathbf{J}_{13} + \mathbf{J}_{14} + \mathbf{i}\mathbf{J}_{24}),\\
\mathbf{F}_3 &= \frac{1}{\sqrt{2}}(-\mathbf{J}_{15} - \mathbf{i}\mathbf{J}_{25}), & \mathbf{F}_4 &= \frac{1}{2}(-\mathbf{J}_{23} + \mathbf{i}\mathbf{J}_{13} - \mathbf{J}_{14} - \mathbf{i}\mathbf{J}_{24}),
\end{aligned}$$

the commutation relations (1.2.7) can be expressed as

$$\begin{array}{llll}
[\mathbf{H}_1,\mathbf{E}_1] = 0, & [\mathbf{H}_1,\mathbf{E}_2] = \mathbf{E}_2, & [\mathbf{H}_1,\mathbf{E}_3] = \mathbf{E}_3, & [\mathbf{H}_1,\mathbf{E}_4] = \mathbf{E}_4,\\
[\mathbf{H}_2,\mathbf{E}_1] = \mathbf{E}_1, & [\mathbf{H}_2,\mathbf{E}_2] = -\mathbf{E}_2, & [\mathbf{H}_2,\mathbf{E}_3] = 0, & [\mathbf{H}_2,\mathbf{E}_4] = \mathbf{E}_4,\\
[\mathbf{H}_1,\mathbf{F}_1] = 0, & [\mathbf{H}_1,\mathbf{F}_2] = -\mathbf{F}_2, & [\mathbf{H}_1,\mathbf{F}_3] = -\mathbf{F}_3, & [\mathbf{H}_1,\mathbf{F}_4] = -\mathbf{F}_4,\\
[\mathbf{H}_2,\mathbf{F}_1] = -\mathbf{F}_1, & [\mathbf{H}_2,\mathbf{F}_2] = \mathbf{F}_2, & [\mathbf{H}_2,\mathbf{F}_3] = 0, & [\mathbf{H}_2,\mathbf{F}_4] = -\mathbf{F}_4,\\
[\mathbf{E}_1,\mathbf{E}_2] = \mathbf{E}_3, & [\mathbf{E}_1,\mathbf{E}_3] = \mathbf{E}_4, & [\mathbf{E}_1,\mathbf{E}_4] = 0, & \\
[\mathbf{F}_1,\mathbf{F}_2] = -\mathbf{F}_3 & [\mathbf{F}_1,\mathbf{F}_3] = -\mathbf{F}_4, & [\mathbf{F}_1,\mathbf{F}_4] = 0, & \\
[\mathbf{F}_2,\mathbf{F}_3] = 0, & [\mathbf{F}_2,\mathbf{F}_4] = 0, & [\mathbf{F}_3,\mathbf{F}_4] = 0, & \\
[\mathbf{E}_1,\mathbf{F}_1] = \mathbf{H}_2, & [\mathbf{E}_1,\mathbf{F}_2] = 0, & [\mathbf{E}_1,\mathbf{F}_3] = -\mathbf{F}_2, & [\mathbf{E}_1,\mathbf{F}_4] = -\mathbf{F}_3,\\
[\mathbf{E}_2,\mathbf{F}_1] = 0, & [\mathbf{E}_2,\mathbf{F}_2] = \mathbf{H}_1 - \mathbf{H}_2, & [\mathbf{E}_2,\mathbf{F}_3] = \mathbf{F}_1, & [\mathbf{E}_2,\mathbf{F}_4] = 0,\\
[\mathbf{E}_3,\mathbf{F}_1] = -\mathbf{E}_2 & [\mathbf{E}_3,\mathbf{F}_2] = \mathbf{E}_1, & [\mathbf{E}_3,\mathbf{F}_3] = \mathbf{H}_1, & [\mathbf{E}_3,\mathbf{F}_4] = \mathbf{F}_1,\\
[\mathbf{E}_4,\mathbf{F}_1] = -\mathbf{E}_3 & [\mathbf{E}_4,\mathbf{F}_2] = 0, & [\mathbf{E}_4,\mathbf{F}_3] = \mathbf{E}_1, & [\mathbf{E}_4,\mathbf{F}_4] = \mathbf{H}_1 + \mathbf{H}_2.
\end{array}$$

Having in mind the classical PBW theorem for the universal enveloping algebra $U(\mathfrak{so}(5,\mathbb{C}))$ of $\mathfrak{so}(5,\mathbb{C})$, and since $U(\mathfrak{so}(5,\mathbb{C}))$ is a PBW extension of $\mathbb{C}$ [BG88, Section 5], then $U(\mathfrak{so}(5,\mathbb{C}))$ is a skew PBW extension over $\mathbb{C}$, i.e., $U(\mathfrak{so}(5,\mathbb{C})) \cong \sigma(\mathbb{C})\langle \mathbf{J}_{\alpha\beta} \mid \mathbf{1} \leq \alpha \leq \beta \leq \mathbf{5}\rangle$.

Zhang and Zhang [ZZ08] introduced the *double extensions* and presented different families of Artin-Schelter regular algebras of global dimension four. It is possible to find some similarities between the double extensions and two-step iterated skew polynomial rings, nevertheless, there exist no inclusions between the classes of all double extensions and of all length two iterated skew polynomial rings (c.f. [CLM11]). Several authors have studied different relations of double extensions with Poisson, Hopf, Koszul and Calabi-Yau algebra (see [RR24] and references therein). We start by recalling the definition of a double extension in the sense of Zhang and Zhang, and since some typos occurred in their papers [ZZ08, p. 2674] and [ZZ09, p. 379] concerning the relations that the data of a double extension must satisfy, we follow the corrections presented by Carvalho et al. [CLM11].

Definition 1.2.26 ([ZZ08, Definition 1.3]; [CLM11, Definition 1.1])**.** If B is a $\Bbbk$-algebra and R is a subalgebra of B, then

(a) B is called a *right double extension* of R if the following conditions hold:

  (i) B is generated by R and two new variables y_1 and y_2.

  (ii) y_1 and y_2 satisfy the relation

$$y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0, \tag{1.2.11}$$

  where $p_{12}, p_{11} \in \Bbbk$ and $\tau_1, \tau_2, \tau_0 \in R$.

  (iii) B is a free left R-module with a basis $\left\{ y_1^i y_2^j \mid i, j \geq 0 \right\}$.

  (iv) $y_1 R + y_2 R + R \subseteq R y_1 + R y_2 + R$.

(b) A right double extension B of R is called a *double extension* if

  (i) $p_{12} \neq 0$.

  (iii) B is a free right R-module with a basis $\left\{ y_2^i y_i^j \mid i, j \geq 0 \right\}$.

  (iv) $y_1 R + y_2 R + R = R y_1 + R y_2 + R$.

Condition (a)(iv) from Definition 1.2.26 is equivalent to the existence of two maps

$$\sigma(r) := \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} \text{ and } \delta(r) := \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix} \text{ for all } r \in R,$$

such that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} r := \begin{pmatrix} y_1 r \\ y_2 r \end{pmatrix} = \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix}. \tag{1.2.12}$$

If B is a right double extension of R then $B := R_P[y_1, y_2; \sigma, \delta, \tau]$ where $P := \{p_{12}, p_{11}\} \subseteq \Bbbk$, $\tau := \{\tau_1, \tau_2, \tau_0\} \subseteq R$ and σ, δ are as above. The set P is called a *parameter* and τ a *tail*. If $\delta := 0$ and τ consists of zero elements then the double extension is denoted by $R_P[y_1, y_2; \sigma]$ and is called a *trimmed double extension* [ZZ08, Convention 1.6 (c)].

1.2.8 Semi-graded rings

Lezama and Latorre [LL17] introduced the *semi-graded rings* as a generalization of $\mathbb{N}$-graded rings and several families of noncommutative rings of polynomial type non-$\mathbb{N}$-graded. Properties of modules over families of semi-graded rings have been investigated by some authors [Art15, GL17, LG19, Lez21, LR20a, NRR20, NR23, Rey19, TRS20]. We recall some definitions and results about semi-graded rings which are key in the following chapters.

Definition 1.2.27 ([LL17, Definition 2.1])**.** R is called *semi-graded* (SG) if there exists a collection $\{R_n\}_{n\in\mathbb{Z}}$ of subgroups of the additive group R^+ such that the following conditions hold:

(i) $R = \bigoplus_{n\in\mathbb{Z}} R_n$.

(ii) For every $m, n \in \mathbb{Z}$, we have that $R_m R_n \subseteq \bigoplus_{k\le m+n} R_k$.

(iii) $1 \in R_0$.

The collection of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ is a *semi-graduation of* R and the elements of R_n are called *homogeneous of degree n*; R is *positively semi-graded* if $R_n = 0$ for all $n < 0$. If R, S are SG rings and $f : R \to S$ is a ring homomorphism, then f is *homogeneous* if $f(R_n) \subseteq S_n$ for all $n \in \mathbb{Z}$.

Definitions 1.2.28 and 1.2.29 present the notion of finitely semi-graded ring and finitely semi-graded algebra, respectively.

Definition 1.2.28 ([LL17, Definition 2.4])**.** R is called *finitely semi-graded* (FSG) if it satisfies the following conditions:

(i) R is SG.

(ii) There exist finitely many elements $x_1, \dots, x_n \in R$ such that the subring generated by R_0 and $x_1, \dots, x_n$ coincides with R.

(iii) For every $n \ge 0$, we have that R_n is a free R_0-module of finite dimension.

Definition 1.2.29 ([LG19, Definition 10])**.** A $\Bbbk$-algebra R is called *finitely semi-graded* (FSG) if the following conditions hold:

(i) R is an FSG ring with semi-graduation defined by $R = \bigoplus_{n\ge 0} R_n$.

(ii) For every $m, n \ge 1$, we have that $R_m R_n \subseteq R_1 \oplus \cdots \oplus R_{m+n}$.

(iii) R is connected, i.e. $R_0 = \Bbbk$.

(iv) R is generated in degree 1.

According to Definition 1.2.29, if R is a FSG $\Bbbk$-algebra, then $R_+ := \bigoplus_{n\ge 1} R_n$ is a maximal ideal of R. Finitely graded $\Bbbk$-algebras, PBW extensions [BG88], 3-dimensional skew polynomial rings [BS90], down-up algebras [Ben99, BR98], diffusion algebras [IPR01] and skew PBW extensions [GL10] are examples of FSG rings.

We present some results about modules in the setting of semi-graded rings.

Definition 1.2.30 ([LL17, Definition 2.1])**.** If R is an SG ring, then we say that M_R is a *semi-graded* (SG) module if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups M_n of the additive group M^+ such that the following conditions hold:

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) For every $n, m \in \mathbb{Z}$, we have that $M_n R_m \subseteq \bigoplus_{k\leq n+m} M_k$.

The collection of subgroups $\{M_n\}_{n\in\mathbb{Z}}$ is a *semi-graduation of* M_R and the elements of M_n are called *homogeneous of degree n*; M_R is *positively semi-graded* if $M_n = 0$ for all $n < 0$. If M_R, N_R are SG modules and $f : M_R \to N_R$ is a modules homomorphism, then f is called *homogeneous* if $f(M_n) \subseteq N_n$ for every $n \in \mathbb{Z}$. If M_R is an SG module and N_R is a submodule of M_R, then N_R is a *semi-graded* (SG) module if $N = \bigoplus_{p\in\mathbb{Z}} N_n$, where $N_n = M_n \cap N$ [LL17, Definition 2.3].

Proposition 1.2.31 ([LL17, Proposition 2.6])**.** *If M_R is an* SG *module and N_R is a submodule of M_R, then the following conditions are equivalent:*

(1) *N_R is an* SG *submodule of M_R.*

(2) *For every $z \in N_R$, the homogeneous components of z belong to N_R.*

(3) *$(M/N)_R$ is a right* SG *module with semi-graduation given by*

$$(M/N)_n = (M_n + N)/N, \quad \textit{for all } n \in \mathbb{Z}.$$

Remark 1.2.32. (i) If N_R is an SG submodule of M_R, the homomorphism $M_R \to (M/N)_R$ defined by $m \mapsto \overline{m}$ is homogeneous.

(ii) If $\{M_i\}_{i\in I}$ is a family of SG submodules of a SG module M_R, then the sets $\bigcap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are SG submodules of M_R.

If M_R is an SG module and $N \subseteq M$, then the SG *submodule generated by* N is defined as the intersection of all SG submodules of M_R containing N and is denoted by $\langle N\rangle^{\mathsf{SG}}$. Additionally, if $N := \{n_1, \dots, n_l\}$, then we write $\langle N\rangle^{\mathsf{SG}} = \langle n_1, \dots, n_l\rangle^{\mathsf{SG}}$. An SG module M_R is *finitely generated* if there exist finitely elements $m_1, \dots, m_t$ such that $M = \langle m_1, \dots, m_t\rangle^{\mathsf{SG}}$. If M is simultaneously a module over different kinds of rings and there is risk of confusion, we write $\langle -\rangle_R^{\mathsf{SG}}$ to indicate the ring R we are considering. If N_R is an SG submodule of M_R, the notion of *finitely generated* SG submodule is defined in the natural way.

Example 1.2.33. Semi-graded rings extend several kinds of noncommutative rings of polynomial type such as skew polynomial ring and ambiskew polynomial rings (Section 1.2.1), down-up algebras (Section 1.2.2), skew bi-quadratic algebras (Section 1.2.3), some skew Ore polynomials of higher order (Section 1.2.4), and enveloping algebras of finite dimensional Lie algebras and PBW extensions (Section 1.2.5), skew PBW extension (Section 1.2.6), and others.

CHAPTER 2

Associated and attached prime ideals

In this chapter, we recall some ring-theoretical notions of the associated and attached prime ideals that are necessary in the next chapters.

2.1 Associated prime ideals

2.1.1 Basic properties of associated prime ideals

In commutative algebra, primary ideals are important because every ideal of a Noetherian ring has a reduced primary decomposition, that is, every ideal can be expressed as an intersection of finitely many primary ideals. This result extended the fundamental theorem of arithmetic and the fundamental theorem of finitely generated Abelian groups. In this section, we present some concepts of the theory of primary decomposition on modules and associated prime ideals.

Definition 2.1.1 ([Bai09, Definition 1.1.1])**.** A submodule N_K of M_K is *primary* if $N_K \neq 0$ and the endomorphism ϕ_r of $(M/N)_K$ defined by $\phi_r(m) := \overline{m}r$ for all $m \in M_K$ is either injective or nilpotent (i.e., there exists $k \in \mathbb{N}$ such that $\phi_r^k = 0$), for each $r \in K$.

If N_K is primary then $\sqrt{\mathrm{ann}_K(M/N)}$ is a prime ideal, and thus N_K is called P*-primary* [Bai09, Remark 1.1.2]. If P is an ideal of K and $N_K = \bigcap_{i=0}^{n} N_i$ is an intersection of P-primary submodules of N_K then N_K is P-primary [Bai09, Lemma 2.1.3].

Definition 2.1.2 ([Bai09, Definition 2.1.1])**.** A *primary decomposition* of a submodule N_K of M_K is an expression of N_K as an intersection of primary submodules of M_K, that is $N_K = \bigcap_{i=0}^{n} N_i$. If N_i is P_i-primary, for all $1 \leq i \leq n$ with $P_i \neq P_j$ for $i \neq j$, and cannot be expressed as the intersection of a proper subcollection of the N_i, then the decomposition is *reduced*.

Remark 2.1.3 ([Bai09, Remark 2.1.2])**.** A reduced primary decomposition can be obtained from an unreduced decomposition by discarding those N_i that contain $\bigcap_{i\neq j} N_j$ and intersecting those N_i that are P-primary for the same prime ideal P of K.

Definition 2.1.4 ([Bai09, Definition 2.2.1])**.** An ideal P of K is called *associated prime* of M_K if P is prime and $P = \text{ann}_K(m)$ for some $m \in M_K$. The set of all associated primes of M_K is denoted by $\text{Ass}(M_K)$.

We present some examples of associated prime ideals.

Example 2.1.5. (i) [TF15, Example 12.2.3] If $M_{\mathbb{Z}} := \mathbb{Z}_n$ then

$$\text{Ass}(\mathbb{Z}_n) = \{\langle p\rangle \in \text{Spec}(\mathbb{Z}) \mid p \text{ is a prime factor of } n\}.$$

(ii) [TF15, Example 12.2.5] If $A = \Bbbk[x, y]$ and $M_A = \Bbbk[x, y]/\langle x^2, xy\rangle$, then $\text{Ass}(M) = \{\langle x, y\rangle, \langle x\rangle\}$.

Proposition 2.1.6. *The following assertions hold:*

(1) [Bai09, Proposition 2.2.3] *If* $N_K \subseteq M_K$ *then*

$$\text{Ass}(M_K) \subseteq \text{Ass}(N_K) \cup \text{Ass}((M/N)_K).$$

(2) [Bai09, Proposition 2.2.4] *If* $M_1, M_2, \dots, M_n$ *are right modules then*

$$\text{Ass}\left(\bigoplus_{i=1}^{n} M_i\right) = \bigcup_{i=1}^{n} \text{Ass}(M_i).$$

(3) [Bai09, Proposition 2.2.6] *If* P *is prime ideal then* $\text{Ass}(K/P) = \{P\}$.

(4) [Bai09, Corollary 2.2.7] *If* K *is Noetherian then* $\text{Ass}(M_K) \neq \emptyset$ *for any* $M_K \neq 0$.

If K is Noetherian then the set of associated primes of M_K is finite [Mat86, Theorem 6.5(i)]. In addition, we can see that the formation of the set $\text{Ass}(M_K)$ commutes with localization at an arbitrary multiplicatively closed set U [Eis95, Theorem 3.1(c)].

It is well-known that if M_K is Noetherian, then every proper submodule of M_K has a primary decomposition, and thus a reduced primary decomposition [Bai09, Theorem 2.1.6]. Proposition 2.1.7 establish the connection between associated prime ideals and primary decomposition, and shows that there are only finitely many associated primes.

Proposition 2.1.7 ([Bai09, Theorem 2.2.9])**.** *If* K *is Noetherian,* M_K *is a finitely generated module, then the zero module* $\{0\}$ *has a reduced primary decomposition* $\bigcap_{i=0}^{n} N_i$ *where* N_i *is* P_i*-primary, the* P_i *are uniquely determined by* $\{0\}$ *and* $\text{Ass}(M_K) = \{P_1, \dots, P_n\}$ *is a finite set.*

If $\text{ann}_R(N) = \text{ann}_R(N')$ for any submodule N'_R of N_R then N_R is called *prime* [Lam98, p. 85]. If N_R is prime, then $\text{ann}_R(N)$ is a prime ideal of R [Lam98, Lemma 3.54]. Definition 2.1.8 extends the concept of associated prime ideals to the noncommutative setting, by replacing elements of the module with submodules.

Definition 2.1.8 ([Ann04, Definition 1.2])**.** A right ideal P of R is *associated* of M_R if there exists a prime submodule N_R of M_R such that $P = \text{ann}_R(N)$. The set of all associated prime ideals of M_R is denoted by $\text{Ass}(M_R)$.

It follows from Definition 2.1.8 that if N_R is prime, then $\text{Ass}(N_R) = \{\text{ann}_R(N)\}$ [Lam98, p. 86]. Lam [Lam98] proved that P is a prime ideal of R if and only if $(R/P)_R$ is prime. In this case, we obtain that $\text{Ass}((R/P)_R) = \{P\}$ [Lam98, Example 3.55].

2.1.2 Commutative polynomial extensions

Brewer and Heinzer [BH74] presented the first result that characterized the associated prime ideals of $K[x]$. They considered the following definition of associated prime ideal.

Definition 2.1.9 ([Laz69, p. 92])**.** A prime ideal P of K is *associated* of M_K if there exists $m \in M_R$ such that P is a minimal prime containing $\operatorname{ann}_R(m)$. The set of all associated primes of M_K is denoted by $\operatorname{Ass}(M_K)$.

Proposition 2.1.10 characterizes the associated prime ideals of $K[x]$.

Proposition 2.1.10 ([BH74, Theorem 7])**.** *If P is an associated prime of $\langle 0\rangle$ in $K[x]$, then $P = Q[X]$, for some associated prime Q of $\langle 0\rangle$ in R.*

The following propositions shows that the associated primes of regular elements of $D[X]$ are closely tied to the associated prime ideals of regular elements of D.

Proposition 2.1.11 ([BH74, Corollary 8])**.** *Suppose that P is an associated prime of a regular element of $R[X]$ and $Q = P \cap R$. If Q contains a regular element, then $P = Q[X]$ and Q is an associated prime of a regular element. In particular, if D is a domain and P is an associated prime of a non-zero polynomial in $D[X]$, then $P \cap D = \langle 0\rangle$ or $P = (P \cap D)[X]$ and $P \cap D$ is an associated prime of a principal ideal.*

Proposition 2.1.12 presents relations between the finiteness of the set of associated primes of D and certain properties of polynomial rings over D.

Proposition 2.1.12.

(1) [BH74, Corollary 9] *If principal ideals of D have only finitely many associated prime ideals, the same is true of $D[X]$.*

(2) [BH74, Proposition 10] *Let D be a domain having the property that principal ideals in D have only finitely many associated primes. If S is locally a polynomial ring over D and it is contained in a finitely generated ring extension of D, then S is a finitely generated ring extension of D.*

In the study of uniform dimension and associated prime ideals, a very important tool is the concept of *good polynomials.* Shock [Sho72] was the first to use these polynomials to prove that the uniform dimensions of R and $R[x]$ are equal [Sho72, Theorem 2.6].

Proposition 2.1.13 ([Sho72, Proposition 2.2])**.** *For a non-zero polynomial $f(x) \in R[x]$, there exists $r \in R$ such that $f(x)r \neq 0$ and the coefficients of $f(x)r$ have equal right annihilators.*

As a consequence of Proposition 2.1.13, Faith presented the following result.

Proposition 2.1.14 ([Fai00, p. 3985])**.** *Every associated prime ideal P of the polynomial ring $K[x]$ is extended, and hence the restriction map*

$$\varphi : \operatorname{Ass}(K[x]) \to \operatorname{Ass}(K)$$

is bijective. In this way, if $P \in \operatorname{Ass}(K[x])$ then $P = Q[x]$, where $Q = P \cap K \in \operatorname{Ass}(K)$.

2.1.3 Skew polynomial rings

In this section, we present definitions, remarks, and results of the theory of associated prime ideals over noncommutative structures of polynomial type.

Throughout this section, we denote the skew polynomial ring $R[x;\sigma,\delta]$ by S. Since S is a free left R-module, we can define the right module $M[x]_S = \widehat{M}_S := M_R \otimes_R S$ whose can be interpreted as polynomials of the form $m(x) = m_0 + \cdots + m_k x^k$, where $n \in \mathbb{N}$ and $m_i \in M_R$. The addition of $M[x]_S$ is defined in a natural way and the product is induced by the product of S.

Thinking about the study of associated prime ideals of skew polynomial rings, Annin [Ann04] introduced the (σ,δ)-*compatible rings*. If σ is an endomorphism of R and δ is a σ-derivation of R, then R is called σ-*compatible* if $ab = 0$ if and only if $a\sigma(b) = 0$, for all $a, b \in R$; R is δ-*compatible* if $ab = 0$ implies that $a\delta(b) = 0$, for all $a, b \in R$. If R is both σ-compatible and δ-compatible, then R is (σ,δ)-*compatible*.

Definition 2.1.15 ([Ann04, Definition 2.1])**.** M_R is called σ-*compatible* if for each $m \in M_R$, $r \in R$, we have that $mr = 0$ if and only if $m\sigma(r) = 0$; M_R is δ-*compatible* if for each $m \in M_R$, $r \in R$, we get that $mr = 0$ implies that $m\delta(r) = 0$. If M_R is both σ-compatible and δ-compatible, then M_R is called (σ,δ)-*compatible*.

Following Shock's ideas, Annin introduced the *annihilator-compliant polynomials* with the aim to study of associated primes of $M[x]_S$ [Ann02b, Ann02a, Ann04].

Definition 2.1.16 ([Ann04, Definition 3.1])**.** A polynomial $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]_S$ is called *annihilator-complaint* if $\operatorname{ann}_R(m_k) \subseteq \operatorname{ann}_R(m_i)$ for all $i \leq k$.

Annin proved that given any polynomial $m(x) \in M[x]_S$, there exists $r \in R$ such that $m(x)r$ is annihilator-complaint [Ann04, Lemma 3.1]. Additionally, if $m(x)$ is annihilator-compliant then $m(x)r$ is annihilator-compliant, for all $r \in R$ [Ann04, Lemma 3.2]. The following proposition characterizes the annihilator of right modules generated by annihilator-compliant polynomials.

Proposition 2.1.17 ([Ann04, Lemma 3.3])**.** *If $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]_S$ is a non-zero annihilator-compliant polynomial with $m_k \neq 0$ and $I := \operatorname{ann}_R(m_k R)$, then $\operatorname{ann}_S(m(x)S) = I[x]$.*

The following result generalizes Proposition 2.1.10.

Proposition 2.1.18 ([Ann04, Theorem 2.1])**.** *If M_R is a (σ,δ)-compatible module, then*

$$\operatorname{Ass}(M[x]_S) = \{P[x] \mid P \in \operatorname{Ass}(M_R)\}.$$

In fact, every $Q \in \operatorname{Ass}(M[x]_S)$ is extended; that is $Q = P[x]$, where $P = Q \cap R \in \operatorname{Ass}(M_R)$.

Example 2.1.19. (i) [Ann04, Example 4.3] Let $R = D[s,t]$, δ the derivate operator with respect to the indeterminate t and s, and $M_R := D[s]$ with the action defined by $m \cdot t := 0$, for all $m \in M_R$. If $mf = 0$, then $f = tg$, for some $g \in R$ and each $m \in M_R$ and $f \in R$. Therefore, $m\delta(f) = m\delta(tg) = m(tg_s + t_s g) = 0$ which implies that M_R is δ-compatible. Notice that $\operatorname{Ass}(M_R) = \{\langle t\rangle\}$ and by Proposition 2.1.18 we have that $\operatorname{Ass}(M[x]_S) = \{\langle t\rangle [x]\}$.

(ii) [Ann02a, Example 64] Let K be a local ring with maximal ideal $\mathfrak{m}$, $M_K := K/\mathfrak{m}$ and σ an automorphism of K. If $0 \neq s \in M_K$, then $s \in K$ and $s \notin \mathfrak{m}$. In addition, we have that $sr = 0$ if and only if $sr \in \mathfrak{m}$, for all $r \in K$. We also get that $sr \in \mathfrak{m}$ if and only if $r \in \mathfrak{m}$. On the other hand, $r \in \mathfrak{m}$ if and only if $s\sigma(r) \in \mathfrak{m}$, and $s\sigma(r) \in \mathfrak{m}$ if and only if $s\sigma(r) = 0$ which implies that M_K is a σ-compatible module. Since M_K is simple then M_K prime, and so $\operatorname{Ass}(M_K) = \{\mathfrak{m}\}$. By Proposition 2.1.18, we have that $\operatorname{Ass}((K/\mathfrak{m})[x]_S) = \{\mathfrak{m}[x]\}$.

(iii) [Ann02a, Example 67] Let R_0 be a ring, $R := R_0[t]$ and σ an endomorphism of R defined by $\sigma(t) = 0$ and $\sigma(r) = r$, for all $r \in R_0$. Additionally, let $h(t)$ be an element of the center of R and $c \in R_0$ a root of the polynomial $h(t)$. Consider δ a σ-derivation of R given by $\delta(f(t)) := f'(t)h(t)$ and $M := R_0$ with the action of R on M given by $m \cdot f(t) := mf(c)$, for all $m \in M_R$ and $f(t) \in R$. It is easy to show that M_R is (σ, δ) compatible and so Proposition 2.1.18 applies.

Example 2.1.20 shows that the δ-compatibility condition is not superfluous.

Example 2.1.20 ([Ann04, Example 4.1])**.** Let D be a ring of characteristic zero, $R := R_0[t]$, $M := R_0$ with the action of t on M defined by $m \cdot t := 0$, for all $m \in M$ and δ the derivate operator with respect to the indeterminate t. We obtain that $mt = 0$ but $m\delta(t) = m$ for all $m \in M_R$, and thus the δ-compatibility condition fails. Some simple computations show that $\operatorname{Ass}(M_R) = \{\langle t \rangle\}$ but $\langle t \rangle[x] \notin \operatorname{Ass}(M[x]_S)$.

2.1.4 Skew PBW extensions

Motivated by the notion of compatibility for skew polynomial rings, Reyes and Suárez [RS18b] and Hashemi et al. [HKA17] introduced independently the (Σ, Δ)-*compatible rings* which are a natural generalization of (σ, δ)-compatible rings. Some examples, ring and module properties of these structures can be consulted in [HHR20, HKA19].

Definition 2.1.21 ([HKA17, Definition 3.1]; [RS18a, Definition 3.2])**.** R is Σ-*compatible* if for each $a, b \in R$, we have that $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, where $\alpha \in \mathbb{N}^n$; R is Δ-*compatible* if for all $a, b \in R$, it follows that $ab = 0$ implies $a\delta^{\beta}(b) = 0$, where $\beta \in \mathbb{N}^n$. If R is both Σ-compatible and Δ-compatible, then R is called (Σ, Δ)-*compatible*.

The following proposition is the natural generalization of [HM05, Lemma 2.1].

Proposition 2.1.22 ([RS18a, Proposition 3.8])**.** *If R is (Σ, Δ)-compatible, then for every elements a, b belonging to R we have the following assertions:*

(1) *If $ab = 0$ then $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$, where $\theta \in \mathbb{N}^n$.*

(2) *If $\sigma^{\beta}(a)b = 0$ for some $\beta \in \mathbb{N}^n$, then $ab = 0$.*

(3) *If $ab = 0$ then $\sigma^{\theta}(a)\delta^{\beta}(b) = \delta^{\beta}(a)\sigma^{\theta}(b) = 0$, where $\theta, \beta \in \mathbb{N}^n$.*

Example 2.1.23. Let $\mathbb{F}_4 = \{0, 1, a, a^2\}$ be the field, $\mathbb{F}_4[z]$ the commutative polynomial ring over $\mathbb{F}_4$ in the indeterminate z and $R = \mathbb{F}_4[z]/\langle z^2 \rangle$. For simplicity, we identify the elements of $\mathbb{F}_4[z]$ with their images in R. If $\Sigma = \{\sigma_{i,j}\}$ is a finite set of endomorphism of R defined by $\sigma_{i,j}(a) = a^i$ and $\sigma_{i,j}(z) = a^j z$, for $1 \le i \le 2$ and $0 \le j \le 2$, and $A = \sigma(R)\langle x_{i,j}\rangle$ is a skew PBW extension generated over R by the indeterminates $\{x_{i,j}\}$, with $1 \le i, j \le 2$ subject to the relations $x_{i,j}x_{i',j'} = x_{i',j'}x_{i,j}$, for all $1 \le i, i' \le 2$ and $0 \le j, j' \le 2$, and $a^r z \in R$, then

$$x_{i,j}a^r z = \sigma_{i,j}(a^r z)x_{i,j} = (a^r)^i a^j z x_{i,j} = a^{ri+j} z x_{i,j},$$

where $a^{ri+j} \in \mathbb{F}_4$, for some $1 \le r, i, j \le 2$. This example can be generalized to any finite field $\mathbb{F}_{p^n}$ with p a prime number. It is straightforward to show that R is Σ-compatible.

Example 2.1.24. We consider the identity endomorphism σ of $\Bbbk[t]$ and $\delta(t) := 1$. Since $\Bbbk[t]$ is reduced, we can show that $\Bbbk[t]$ is a (σ,δ)-compatible ring. Let R be the ring defined as

$$R = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in \Bbbk[t] \right\}.$$

The idea is to extend the usual derivation. The endomorphism σ of $\Bbbk[t]$ is extended to the endomorphism $\overline{\sigma}$ of R by defining $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ for every $1 \le i, j \le 2$, and the σ-derivation δ of $\Bbbk[t]$ is also extended to $\overline{\delta} : R \to R$ by considering $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$ for every $1 \le i, j \le 2$. Hence the skew polynomial ring $R[x;\overline{\sigma},\overline{\delta}]$ is a skew PBW extension of R which is $(\overline{\sigma},\overline{\delta})$-compatible.

Reyes and Suárez [RS20] introduced the *weak* (Σ,Δ)-*compatible rings* as a generalization of (Σ,Δ)-compatible and weak (σ,δ)-compatible rings defined by Ouyang and Liu [OL11].

Definition 2.1.25 ([RS20, Definition 4.1])**.** R is called *weak* Σ-*compatible* if for each $a, b \in R$, we have that $a\sigma^{\alpha}(b) \in N(R)$ if and only if $ab \in N(R)$, where $\alpha \in \mathbb{N}^n$; R is *weak* Δ-*compatible* if for each $a, b \in R$, we get that $ab \in N(R)$ implies $a\delta^{\beta}(b) \in N(R)$, where $\beta \in \mathbb{N}^n$. If R is both weak Σ-compatible and weak Δ-compatible, then R is called *weak* (Σ,Δ)-*compatible*.

The following result extends Proposition 2.1.22.

Proposition 2.1.26 ([RS20, Proposition 4.2])**.** *If R is weak (Σ,Δ)-compatible, then the following assertions hold:*

(1) *If $ab \in N(R)$ then $a\sigma^{\alpha}(b), \sigma^{\beta}(a)b \in N(R)$, for all $\alpha, \beta \in \mathbb{N}^n$.*

(2) *If $\sigma^{\alpha}(a)b \in N(R)$ for some element $\alpha \in \mathbb{N}^n$, then $ab \in N(R)$.*

(3) *If $a\sigma^{\beta}(b) \in N(R)$ for some element $\beta \in \mathbb{N}^n$, then $ab \in N(R)$.*

(4) *If $ab \in N(R)$ then $\sigma^{\alpha}(a)\delta^{\beta}(b), \delta^{\beta}(a)\sigma^{\alpha}(b) \in N(R)$, for $\alpha, \beta \in \mathbb{N}^n$.*

The following example shows a weak (Σ,Δ)-compatible ring which is not (Σ,Δ)-compatible.

Example 2.1.27. If ${}_RM_R$ is a left-right module, then the *trivial extension of R by M* is the ring $T(R,M) := R \oplus M$ with the usual addition of $R \oplus M$ and the multiplication is defined as follows: $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + m_1r_2)$, for $r_1, r_2 \in R$ and $m_1, m_2 \in {}_RM_R$. It is not hard to see that

$T(R, M)$ is isomorphic to the matrix ring (with the usual matrix operations) of the form $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in {}_R M_R$. We denote by $\mathrm{S}_2(\mathbb{Z})$ the ring of matrices isomorphic to $T(\mathbb{Z}, \mathbb{Z})$

$$\mathrm{S}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let σ_2 and σ_3 be the two endomorphisms of $\mathrm{S}_2(\mathbb{Z})$ defined by

$$\sigma_2\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \quad \text{and} \quad \sigma_3\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Notice that $N(\mathrm{S}_2(\mathbb{Z})) = \left\{\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \mid b \in \mathbb{Z}\right\}$. In addition, $\mathrm{S}_2(\mathbb{Z})$ is weak a Σ-compatible ring where $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ and $\sigma_1 = \mathrm{id}_{\mathrm{S}_2(\mathbb{Z})}$ is the identity map of $\mathrm{S}_2(\mathbb{Z})$, but $\mathrm{S}_2(\mathbb{Z})$ is not σ_3-compatible. Indeed, if $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), D = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{S}_2(\mathbb{Z})$ then $C\sigma_3(D) = 0$, but $CD = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \neq 0$, whence $\mathrm{S}_2(\mathbb{Z})$ is not Σ-compatible.

R is called an *NI* ring if $N^*(R) = N(R)$, and R is *2-primal* if $N_*(R) = N(R)$ [Mar01, p. 2114]. Ouyang and Liu [OL11] characterized the nilpotent elements of $R[x;\sigma,\delta]$, where R is weak (σ,δ)-compatible and NI [OL11, Lemma 2.13]. Reyes and Suárez [RS20] extended this result.

Proposition 2.1.28 ([RS20, Theorem 4.6])**.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where R is weak (Σ, Δ)-compatible and NI, then $f = a_1x^{\alpha_1} + \cdots + a_kx^{\alpha_k} \in N(A)$ if and only if $a_i \in N(R)$, for all $1 \le i \le k$.*

Proposition 2.1.28 also extends the results presented by Ouyang and Birkenmeier for skew polynomial rings [OB12, Lemma 2.6], and by Ouyang and Liu for differential polynomials rings [OL12, Lemma 2.12]. Since weak (Σ, Δ)-compatible rings contain strictly (Σ, Δ)-compatible rings, we have the following corollary.

Corollary 2.1.29. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where R is (Σ, Δ)-compatible and NI, then the following statements hold:*

(1) *$N(A)$ is an ideal of A and $N(A) = N(R)\langle x_1, \dots, x_n\rangle$.*

(2) *$f = a_1x^{\alpha_1} + \cdots + a_kx^{\alpha_k} \in N(A)$ if and only if $a_i \in N(R)$, for all $1 \le i \le k$.*

Corollary 2.1.29 generalizes the corresponding result for skew polynomial rings over 2-primal and (σ,δ)-compatible rings [OB12, Corollary 2.2], and for differential polynomials rings over δ-compatible reversible rings [OL12, Corollary 2.13].

The following examples shows that the NI condition is not superfluous in Corollary 2.1.29.

Example 2.1.30. Let $M_2(\mathbb{Z}_4)$ be the ring of matrices 2×2 over $\mathbb{Z}_4$ and $M_2(\mathbb{Z}_4)[x;\sigma]$ the skew polynomial ring of $M_2(\mathbb{Z}_4)$, where σ is the endomorphism of $M_2(\mathbb{Z}_4)$ given by $\sigma\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$. Notice that $M_2(\mathbb{Z}_4)$ is a σ-compatible module. Additionally, $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in N(M_2(\mathbb{Z}_4))$ but $A + B \notin N(M_2(\mathbb{Z}_4))$, which shows that $M_2(\mathbb{Z}_4)$ is not NI. Finally $f(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)x + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is an element of $N(M_2(\mathbb{Z}_4))[x;\sigma]$ but

$$f(x)^2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^2 x^2 + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)x + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)x + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)x \neq 0.$$

Thus $f(x) \notin N(M_2(\mathbb{Z}_4)[x;\sigma])$, and hence $N(M_2(\mathbb{Z}_4)[x;\sigma]) \neq N(M_2(\mathbb{Z}_4))[x;\sigma]$.

Example 2.1.31. Let $M_2(\mathbb{Z}_2)$ be the ring of matrices 2×2 over $\mathbb{Z}_2$ and $M_2(\mathbb{Z}_2)[x;\sigma]$ the skew polynomial ring of $M_2(\mathbb{Z}_2)$, where σ is the identity endomorphism of $M_2(\mathbb{Z}_2)$. It is not difficult to see that $M_2(\mathbb{Z}_2)$ is σ-compatible. In the same way as Example 2.1.30, $M_2(\mathbb{Z}_2)$ is not an NI ring, and also $f(x) = \left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)x + \left(\begin{smallmatrix} 0 & 0\\ 1 & 0\end{smallmatrix}\right) \in N(M_2(\mathbb{Z}_2))[x;\sigma]$ but $f(x) \notin N(M_2(\mathbb{Z}_2)[x;\sigma])$ which implies that $N(M_2(\mathbb{Z}_2)[x;\sigma]) \neq N(M_2(\mathbb{Z}_2))[x;\sigma]$.

Following Ouyang et al. [OLX13] , if σ is an endomorphism of R and δ is a σ-derivation of R, then R is *skew π-Armendariz* if for $f(x) = \sum_{i=0}^{l} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\sigma,\delta]$, we have that $f(x)g(x) \in N(R[x;\sigma,\delta])$ implies that $a_ib_j \in N(R)$, for each i, j. Ouyang and Liu [OL11] showed that if R is a weak (σ,δ)-compatible and NI, then R is skew π-Armendariz [OL11, Corollary 2.15]. Reyes [Rey18] formulated the analogue of skew π-Armendariz ring in the setting of skew PBW extensions. If A is a skew PBW extension of R, then R is called *skew Π-Armendariz* if for elements $f = a_1x^{\alpha_1} + \cdots + a_kx^{\alpha_k}$ and $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t}$ belong to A, we get that $fg \in N(A)$ implies that $a_ib_j \in N(R)$, for each $0 \le i \le k$ and $0 \le j \le t$. Reyes [Rey18] showed that if R is reversible (R is called *reversible* if $ab = 0$ implies $ba = 0$, where $a, b \in R$ [Coh99, p. 641]) and (Σ,Δ)-compatible, then R is skew Π-Armendariz [Rey18, Theorem 3.10]. The following proposition extends [OL12, Lemma 2.14] and [OB12, Corollary 2.3].

Proposition 2.1.32 ([RS20, Theorem 4.7])**.** *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension of R, where R is weak (Σ,Δ)-compatible and NI. If $f = a_1x^{\alpha_1} + \cdots + a_kx^{\alpha_k}$ and $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t}$ are elements of A, then $fg \in N(A)$ if and only if $a_ib_j \in N(R)$, for all i, j.*

According to Hashemi et al. [HM05], R satisfies the *(SQA1) condition* if $f(x)R[x;\sigma,\delta]g(x) = 0$ implies $a_iRb_j = 0$ for each i, j and $f(x) = a_0 + \cdots + a_mx^m, g(x) = b_0 + \cdots + b_lx^l \in R[x;\sigma,\delta]$. Reyes and Suárez [RS18a] defined this condition for skew PBW extensions. If A is a skew PBW extension over R, then R satisfies the *(SQA1) condition* if $fAg = 0$ implies $a_iRb_j = 0$, for every i, j and for every $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m}, g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t} \in A$. Relationships between the notions of compatibility, (SQA1) condition, and Armendariz rings in the context of skew PBW extensions have been studied by Reyes and Suárez [RS17].

According to Definition 1.2.14, if A is a skew PBW extension of R, then A is free left R-module, and we can consider the set $M\langle X\rangle_A$ where the elements are of the form $m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k}$, with $m_i \in M_R$ and $x^{\alpha_i} \in \mathrm{Mon}(A)$, for all $1 \le i \le k$. By Proposition 1.2.19 and the relations described by [Rey15, Proposition 2.9 and Remark 2.10 (iv)], $M\langle X\rangle_A$ has A-module structure defined as: if $mx^{\alpha_i} := mx_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}} \in M\langle X\rangle_A$ and $bx^{\beta_j} := bx_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}} \in A$, then we multiply these elements following the rule

$$\begin{aligned} mx_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}bx_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}} &= m\sigma^{\alpha_i}(b)x^{\alpha_i}x^{\beta_j} + mp_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\ &+ mx_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_{i3}^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\ &+ mx_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\ &+ \cdots + mx_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b)}x_n^{\alpha_{in}}x^{\beta_j}\\ &+ mx_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b}x^{\beta_j}. \end{aligned}$$

This commutation rule guarantees that $M\langle X\rangle_A$ is a right A-module and is called the *induced module* of M_R. If $\succeq$ is a total order defined on Mon(A) and $m \in M\langle X\rangle_A$ with $m \neq 0$, then we use expressions as $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k}$, where $m_i \in M_R$, and $x^{\alpha_k} \succ \cdots \succ x^{\alpha_1}$. With this notation, we define $\mathrm{lm}(m) := x^{\alpha_k}$, the *leading monomial* of m; $\mathrm{lc}(m) := m_k$, the *leading coefficient* of m; $\mathrm{lt}(m) := m_kx^{\alpha_k}$, the *leading term* of m. Note that $\deg(m) := \max\{\deg(x^{\alpha_i})\}_{i=1}^{k}$.

Reyes [Rey19] introduced the (Σ,Δ)*-compatible modules* and generalized some properties presented of induced modules over skew polynomial rings ([AM12] and references therein).

Definition 2.1.33 ([Rey19, Definition 3.3])**.** M_R is called Σ*-compatible* if for each $m \in M_R$, $r \in R$, we have that $mr = 0$ if and only if $m\sigma^{\alpha}(r) = 0$, for all $\alpha \in \mathbb{N}^n$; M_R is Δ*-compatible* if for each $m \in M_R$, $r \in R$, we obtain that $mr = 0$ implies that $m\delta^{\beta}(r) = 0$, for all $\beta \in \mathbb{N}^n$. If M_R is both Σ-compatible and Δ-compatible, then M_R is called (Σ,Δ)*-compatible*.

Niño et al. [NRR20] extended the notion of *annihilator-compliant polynomial* with the aim of characterizing the associated prime ideals of induced modules on skew PBW extensions. A polynomial $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in M\langle X\rangle_A$ of leading term $m_kx^{\alpha_k}$ is *annihilator-compliant* if for each $i \leq k$, we have that $\mathrm{ann}_R(m_k) \subseteq \mathrm{ann}_R(m_i)$ [NRR20, Definition 3.7]. They showed that if $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension of R and M_R is (Σ,Δ)-compatible, then there exists $r \in R$ such that mr is annihilator-compliant, for any $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in M\langle X\rangle_A$ [NRR20, Proposition 3.8]. They also characterized the right annihilators of ideals generated by annihilator-compliant polynomials [NRR20, Proposition 3.10], and used these results to describe the associated prime ideals of induced modules.

Proposition 2.1.34 ([NRR20, Theorem 3.12])**.** *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension of R. If M_R is (Σ,Δ)-compatible, then*

$$\mathrm{Ass}(M\langle X\rangle_A) = \{P\langle X\rangle \mid P \in \mathrm{Ass}(M_R)\}.$$

According to Krempa [Kre96], a right ideal I of R is σ*-rigid* if there exists an endomorphism σ of R such that $r\sigma(r) \in I$ implies that $r \in I$, for all $r \in R$. If $\{0\}$ is a σ-rigid ideal, then R is called a σ*-rigid* ring. Some properties of σ-rigid rings have been studied by several authors (see [HKK00] and references therein). Ouyang [Ouy08] introduced *weak* σ*-rigid rings* as a generalization of σ-rigid. If σ is an endomorphism of R, then R is called *weak* σ*-rigid* if for all $a \in R$, we have that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$. Reyes [Rey15] extended the notion of σ-rigid rings as follows: if $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ is a finite set of endomorphisms of R, then R is Σ*-rigid* if $r\sigma^{\alpha}(r) = 0$ implies that $r = 0$, for all $r \in R$ and $\alpha \in \mathbb{N}^n$ [Rey15, Definition 3.2].

Definition 2.1.35 ([Rey15, Definition 3.2])**.** If $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ is a finite set of endomorphisms of R, then R is called *weak* Σ*-rigid* if $r\sigma^{\alpha}(r) \in N(R)$ implies that $r \in N(R)$, for all $r \in R$ and $\alpha \in \mathbb{N}^n$.

Reyes and Suárez [RS18b] proved that if $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ is a finite set of endomorphisms of R, then R is Σ-rigid if and only if R is weak Σ-rigid and reduced (R is *reduced* if it has no non-zero nilpotent elements) [RS18b, Theorem 3.4].

The notion of *semiprime ideal* which arise as natural generalization of prime ideal. A right ideal I of R is called *semiprime* if I is an intersection of prime ideals of R. If $\{0\}$ is a semiprime ideal of R, then R is called *semiprime* [GJ04, p. 51]. Note that R is semiprime if $(aRa)^2 = 0$ implies $aRa = 0$, for all $a \in R$.

According to Lam [Lam98], M_R is uniform if the intersection of any two non-zero submodules of M_R is non-zero [Lam98, p. 84]. If R is right Noetherian and M_R is uniform, then there exists a unique prime ideal P de R such that $P = \text{ann}_R(N_R)$ some non-zero submodule N_R of M_R and P contains the annihilators of all non-zero submodules of M_R. Additionally, P is the unique associated prime of M_R [GJ04, Lemma 5.26]. This ideal of R is called the *assassinator* of M_R and it is denoted by $\text{assass}(M_R)$. The set of all the assassinator ideals of M_R is defined as $\mathbb{A}(M_R)$ [GJ04, p. 102].

Proposition 2.1.36. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R, where R is right Noetherian which is also an algebra over $\mathbb{Q}$.*

(1) [NRR20, Theorem 3.15]. *If R is semiprime and weak Σ-rigid, then $P \in \text{Ass}(A_A)$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $(P \cap R)A = P$ and $P \cap R = U$.*

(2) [NRR20, Theorem 3.16]. *Assume that $\sigma_i(\delta_j(a)) = \delta_j(\sigma_i(a))$, for all $a \in R$ and $1 \leq i, j \leq n$. If $\sigma_i(U) = U$ for all $U \in \mathbb{A}(R_R)$ and $1 \leq i \leq n$, then $P \in \text{Ass}(A_A)$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $P = UA$ and $P \cap R = U$.*

2.2 Nilpotent associated prime ideals

Ouyang and Birkenmeier [OB12] introduced the concept of *weak annihilator* as a generalization of the classical annihilator. We recall some important definitions and results.

Definition 2.2.1 ([OB12, Definition 2.1])**.** If Y is a subset of R, then the set of elements $a \in R$ that satisfy $Ya \subseteq N(R)$ is called *the weak annihilator* of Y in R. If Y is singleton, say $Y = \{r\}$, we use $N_R(r)$ in place of $N_R(\{r\})$.

The following example shows that the weak annihilator is a non-trivial generalization of the usual annihilator.

Example 2.2.2 ([OB12, Example 2.1])**.** Let $T_2(\mathbb{Z})$ be the 2×2 upper triangular matrix ring over $\mathbb{Z}$ and $Y = \left\{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\}$. It is straightforward to show that $r_{T_2(\mathbb{Z})}(Y) = 0$ and $N_{T_2(\mathbb{Z})}(Y) = \left\{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in \mathbb{Z}\right\}$, whence $r_{T_2(\mathbb{Z})}(Y) \neq N_{T_2(\mathbb{Z})}(Y)$.

Proposition 2.2.3 ([OB12, Proposition 2.1])**.** *If Y is a subset of R then:*

(1) *If $Y \subseteq Z$ then $N_R(Z) \subseteq N_R(Y)$, for any subset Z of R.*

(2) $Y \subseteq N_R(N_R(Y))$.

(3) $N_R(Y) = N_R(N_R(N_R(Y)))$.

Ouyang and Birkenmeier [OB12] defined a more general notion of associated prime ideal.

Definition 2.2.4. (i) [OB12, Definition 3.1] A right ideal I of R is called *right quasi-prime* if $I \nsubseteq N(R)$ and $N_R(I) = N_R(I')$, for every right ideal $I' \subseteq I$ and $I' \nsubseteq N(R)$.

(ii) [OB12, Definition 3.2] If R is NI, then a right ideal P of R is *nilpotent associated* of R if P is prime and there exists a right quasi-prime ideal I such that $P = N_R(I)$. The set of all nilpotent associated primes of R is denoted by $\mathrm{NAss}(R)$.

The following example shows that the definition of nilpotent associated prime ideal extends the notion of associated prime ideal.

Example 2.2.5 ([OB12, p. 346])**.** If $R = \left(\begin{smallmatrix} \Bbbk & 0 \\ \Bbbk & \Bbbk \end{smallmatrix}\right)$ is the ring of 2×2 lower triangular matrices over $\Bbbk$, the set of proper ideals is the following:

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ \Bbbk & \Bbbk \end{pmatrix},\ m_2 = \begin{pmatrix} \Bbbk & 0 \\ \Bbbk & 0 \end{pmatrix},\ m_3 = \begin{pmatrix} 0 & 0 \\ \Bbbk & 0 \end{pmatrix} \right\}.$$

Note that the zero ideal and m_3 are not prime ideals. In addition, the only associated prime ideal of R is m_1, that is, $\mathrm{Ass}(R) = \{m_1\}$. Finally, both m_1 and m_2 are nilpotent associated prime ideals of R, and thus $\mathrm{NAss}(R) = \{m_1, m_2\}$.

It is not difficult to see that if R is reduced, then P is a nilpotent associated prime ideal of R if and only if P is an associated prime ideal of R, and hence $\mathrm{NAss}(R) = \mathrm{Ass}(R)$ [OB12, p. 353].

Ouyang and Birkenmeier [OB12] defined the notion of nilpotent degree to study the nilpotent associated primes of skew polynomial rings. If $m(x) = m_0 + \cdots + m_k x^k + \cdots + m_n x^n \notin N(R)[x;\sigma,\delta]$, $m_k \notin N(R)$ and $m_i \in N(R)$ for all $i > k$, then the *nilpotent degree* of $m(x)$ is k, and it is denoted by $\mathrm{Ndeg}(m(x)) = k$. If $m(x) \in N(R)[x;\sigma,\delta]$ then $\mathrm{Ndeg}(m(x)) = -1$ [OB12, p. 354].

Definition 2.2.6 ([OB12, Definition 3.3])**.** Let $m(x) = m_0 + \cdots + m_k x^k + \cdots + m_n x^n \in N(R)[x;\sigma,\delta]$ with $\mathrm{Ndeg}(m(x)) = k$. If $N_R(m_k) \subseteq N_R(m_i)$ for all $i \leq k$, then $m(x)$ is a *nilpotent good polynomial*.

Proposition 2.2.7 characterized the nilpotent associated prime ideals of a skew polynomial rings. In some sense, the following result generalizes Proposition 2.1.18.

Proposition 2.2.7 ([OB12, Theorem 3.1])**.** *If R is (σ,δ)-compatible 2-primal, then*

$$\mathrm{NAss}(R[x;\sigma,\delta]) = \{P[x;\sigma,\delta] \mid P \in \mathrm{NAss}(R)\}.$$

2.3 Attached prime ideals

2.3.1 Basic properties of attached prime ideals

Macdonald [Mac73] considered a dual theory to the primary decomposition which is commonly known as *secondary representation* where the main ideals of this theory are called *attached*.

Definition 2.3.1 ([Bai09, Definition 3.1.1])**.** M_K is *secondary* if $M_K \neq 0$ and the endomorphism ϕ_r of M_K defined by $\phi_r(m) := mr$ for all $m \in M_K$ is either surjective or nilpotent (i.e., there exists $k \in \mathbb{N}$ such that $\phi_r^k = 0$), for each $r \in K$.

If M_R is secondary then $\sqrt{\text{ann}_R(M)}$ is a prime ideal [Bai09, Claim 3.1.2], and thus M_R is called P-*secondary* [Bai09, Definition 3.1.3]. Additionally, if P is an ideal of K and $M_R = M_1 + \cdots + M_n$ is a sum of P-secondary submodules of M_K then M_K is P-secondary [Bai09, Lemma 3.1.8]. This motivates the following definition.

Definition 2.3.2 ([Bai09, Definition 3.1.9]). A *secondary representation* of M_K is an expression of M_K as a sum of secondary submodules $M_K = \sum_{i=0}^{n} M_i$. If M_i is P_i-secondary, for all $1 \le i \le n$ with $P_i \neq P_j$ for $i \neq j$, and the sums $\sum_{i \neq k} M_i$ are proper submodules of M_K, then the representation is *minimal*. If M_K has a secondary representation, then M_K is called *representable*.

Proposition 2.3.3 shows that if M_K is representable, then the set of prime ideals $\{P_1, \ldots, P_n\}$ depends only of M_K and not of the minimal secondary representation.

Proposition 2.3.3 ([Bai09, Theorem 3.2.1]). *If M_K is representable and $\{P_1, \ldots, P_n\}$ is the set of prime ideals of the secondary representation of M_K, then the following conditions on a prime ideal P of K are equivalent:*

(1) *P is one of P_i.*

(2) *M_K has a P-secondary quotient module.*

(3) *M_K has a quotient Q_K such that $\sqrt{\text{ann}_K(Q)} = P$.*

(4) *M_K has a quotient Q_K such that P is minimal in the set of prime ideals containing* $\text{ann}_K(Q)$.

Definition 2.3.4 ([Bai09, Definition 3.2.2]). If M_K is representable, then the ideals $P_1, \ldots, P_n$ are called *attached primes* of M_K and the set of all attached primes of M_K is denoted by $\text{Att}^*(M_K)$.

Let ϕ_r be the endomorphism of M_K considered in Definition 2.3.1. The following proposition characterizes the elements that are outside the union or at the intersection of the attached prime ideals of a representable module.

Proposition 2.3.5 ([Bai09, Theorem 3.2.5]). *If M_K is representable and $r \in R$, then*

(1) *ϕ_r is surjective if and only if $r \notin \bigcup_{i=1}^{n} P_i$.*

(2) *ϕ_r is nilpotent if and only if $r \in \bigcap_{i=1}^{n} P_i$.*

Annin [Ann08] introduced the notion of *attached prime ideal* for arbitrary modules (not necessarily representables), and extended Macdonald's theory of secondary representation to the noncommutative setting. If $N_R \neq 0$ and $\text{ann}_R(N) = \text{ann}_R(Q)$ for every non-zero quotient module Q_R of N_R, then N_R is *coprime* [Ann08, Definition 2.1], and if N_R is coprime, then $P = \text{ann}(N_R)$ is a right prime ideal of R [Ann08, Lemma 2.2]. This fact motivates the following definition.

Definition 2.3.6 ([Ann08, Definition 2.3]). An ideal P of R is an *attached prime* of M_R if there exists a coprime quotient Q_R of M_R such that $P = \text{ann}_R(Q)$. The set of all the attached prime ideals of M_R is denoted by $\text{Att}(M_R)$.

Proposition 2.3.7. *The following assertions hold:*

(1) [Ann08, Proposition 2.5] *If* N_R *is a submodule of* M_R*, then*

$$\mathrm{Att}(M/N)_R \subseteq \mathrm{Att}(M_R) \subseteq \mathrm{Att}(N_R) \cup \mathrm{Att}(M/N)_R.$$

(2) [Ann08, Corollary 2.7] *If* $M_1, M_2, \ldots, M_n$ *are right* R*-modules, then*

$$\mathrm{Att}\left(\bigoplus_{i=1}^{n} M_i\right) = \bigcup_{i=1}^{n} \mathrm{Att}(M_i).$$

(3) [Ann08, Proposition 2.10] *If* $N_R \subseteq M_R$ *is a small submodule, then* $\mathrm{Att}(M/N)_R = \mathrm{Att}(M_R)$.

(4) [Ann08, Proposition 2.15] *If* M_R *is an Artinian module, then* $|\mathrm{Att}(M_R)| < \infty$.

If K is Noetherian and M_K is representable, then $\mathrm{Att}(M_K) = \mathrm{Att}^*(M_K)$ [Ann08, Proposition 2.4]. Following Macdonald, $\mathrm{Att}^*(M_R)$ is always a non-empty finite set. However, the set $\mathrm{Att}(M_R)$ may be empty (even for a non-zero module) or infinite [Ann08, Examples 2.12 and 2.14]. Annin proved that if R is right Noetherian, then $\mathrm{Att}(M_R) \neq \emptyset$, for all $M_R \neq 0$ [Ann08, Proposition 2.13].

Example 2.3.8. (i) [Ann08, Examples 2.6] $\mathbb{Q}_{\mathbb{Z}}$ is 0-secondary, and so is $(\mathbb{Q}/\mathbb{Z})_{\mathbb{Z}}$. If p is a prime integer, then the *Prüfer* p*-group* $\mathbb{Z}_{p^\infty} \cong E(\mathbb{Z}_p)$ (E denotes the injective hull) is 0-secondary, as it is a quotient of $\mathbb{Q}/\mathbb{Z}$. If $N_{\mathbb{Z}} := \mathbb{Z}_p \oplus \mathbb{Z}_q$, for $p \neq q$ primes, and $M_{\mathbb{Z}} := E(\mathbb{Z}_p) \oplus \mathbb{Z}_q$ then

$$\mathrm{Att}(N_{\mathbb{Z}}) \cup \mathrm{Att}((M/N)_{\mathbb{Z}}) = \{p\mathbb{Z}, q\mathbb{Z}\} \cup \{(0)\} = \{p\mathbb{Z}, q\mathbb{Z}, (0)\}, \quad \mathrm{Att}(M_{\mathbb{Z}}) = \left\{q\mathbb{Z}, (0)\right\}.$$

(ii) [Ann08, Examples 2.14] If $M_{\mathbb{Z}} := \mathbb{Z}_{\mathbb{Z}}$ then $\mathrm{Att}(\mathbb{Z}_{\mathbb{Z}}) = \{2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}, 11\mathbb{Z}, \ldots\}$.

Annin [Ann08] generalized the classical theory of secondary representation developed by Macdonald [Mac73]. The secondary modules of Macdonald's theory are those modules that have exactly one attached prime ideal. With this in mind, Annin presented the following definition.

Definition 2.3.9 ([Ann08, Definition 4.1])**.** M_R is called *secondary* if M_R has exactly one attached prime ideal and M_R is P*-secondary* provided that $\mathrm{Att}(M_R) = \{P\}$.

According to Annin [Ann08], if $\mathrm{Att}(M_R) \neq \emptyset$ and $M_R = M_1 + \cdots + M_n$ is a sum of P-secondary modules, for some prime ideal P of R, then M_R is P-secondary [Ann08, Proposition 4.5]. Notice that the Přufer p-group $M_{\mathbb{Z}} := \mathbb{Z}_{p^\infty} = \sum_{k=1}^{\infty} \mathbb{Z}/p^k\mathbb{Z}$ is an infinite sum of $\langle p \rangle$-secondary modules, but $\mathrm{Att}(M_{\mathbb{Z}}) = \{\langle 0 \rangle\}$ which shows that this result only applies to finite sums [Ann08, p. 517]. Annin also showed that the proposition cannot be generalized to a finite sum of modules $M_1, \ldots, M_n$ each with the same set of attached primes $\mathrm{Att}(M_i)$ and $|\mathrm{Att}(M_i)| > 1$ [Ann08, Example 4.6].

Definition 2.3.10 ([Ann08, Definition 4.7])**.** $M_R = M_1 + \cdots + M_n$ is a *secondary representation* of M_R if each M_i is a secondary submodule of M_R. If M_R has a secondary representation, then M_R is called *representable.*

R is right perfect if and only if $R/J(R)$ is a semisimple ring and every non-zero R-module has a maximal submodule [Lam91, Exercise 24.7]. The following proposition can be considered the noncommutative version of Proposition 2.3.5 (1).

Proposition 2.3.11 ([Ann08, Proposition 4.13])**.** *Assume that R is right perfect or that R satisfies the ascending chain condition on ideals. If $M_R = M_1 + \cdots + M_n$ is an irredundant secondary representation of M_R with M_i a P_i-secondary module, then for all $r \in R$, we have that $MrR = M$ if and only if $r \notin \bigcup_{i=1}^{n} P_i$.*

Proposition 2.3.5 characterizes the elements of K for which the endomorphism ϕ_r of M_K is nilpotent as those in the intersection of the attached prime ideals of a representable module. In the noncommutative setting, Annin showed that this does not even hold for secondary modules [Ann08, Remark 4.4]. With this in mind, he defined a stronger class of secondary modules by looking at the differences between the secondary modules in the sense of Macdonald with the secondary modules introduced by him.

Definition 2.3.12 ([Ann08, Definition 4.14])**.** M_R is *strongly P-secondary* if M_R is P-secondary and $M \cdot P^n = 0$ for some $n \in \mathbb{N}$. If $M_R = M_1 + \cdots + M_n$ is a secondary representation, where each M_i is strongly secondary, then this decomposition is called a *strong secondary representation*, and M_R is called *strongly representable.*

Proposition 2.3.13 can be seen as the noncommutative version of Proposition 2.3.5 (2).

Proposition 2.3.13 ([Ann08, Proposition 4.20])**.** *If $M_R = M_1 + \cdots + M_n$ is an irredundant strong secondary representation of M_R, with each M_i strongly P_i-secondary, then for all $r \in R$, we have that $M(rR)^k = 0$ for some $k \in \mathbb{N}$ if and only if $r \in \bigcap_{i=1}^{n} P_i$.*

2.3.2 Commutative polynomial extensions

Melkersson [Mel98] studied the attached prime ideals over commutative polynomial extensions. He investigated when multiplication by $f(x) \in K[x]$ defines a surjective endomorphism over the module $M[x^{-1}]_{K[x]}$ which consists of all the polynomials of the form $m(x) = m_0 + \cdots + m_k x^{-k}$ with $m_i \in M_K$ for all $0 \le i \le k$.

If g and h are endomorphisms of M_K such that $gh = hg$, g is surjective and h is nilpotent, then $f := g + h$ is surjective [Mel98, Lemma 2.1]. As a consequence of this result, Melkersson proved the following proposition.

Proposition 2.3.14 ([Mel98, Theorem 2.2])**.** *If M_K is Artinian or has a secondary representation and $f(x) \in K[x]$ such that $M = c(f)M$, where $c(f)$ is the ideal generated by the coefficients of $f(x)$, then the multiplication by $f(x)$ on the inverse polynomial module $M[x^{-1}]_{K[x]}$ is surjective. The converse also holds.*

Proposition 2.3.14 implies the following result that characterizes the attached prime ideals of inverse polynomial module $M[x^{-1}]_{K[x]}$.

Proposition 2.3.15 ([Mel98, Corollary 2.3])**.** (1) *If M_K is P-secondary, then so is $M[x^{-1}]_{K[x]}$.*

(2) *If M_K is representable, then $M[x^{-1}]_{K[x]}$ is representable. In fact if $M_K = M_1 + \cdots + M_n$ is a minimal secondary representation of M_K, then $M[x^{-1}]_{K[x]} = M_1[x^{-1}] + \cdots + M_n[x^{-1}]$ is a*

minimal secondary representation of $M[x^{-1}]_{K[x]}$ *and*

$$\mathrm{Att}(M[x^{-1}]_{K[x]}) = \{P[x] \mid P \in \mathrm{Att}(M_K)\}.$$

2.3.3 Noncommutative setting

Motivated by research on associated primes of $M[x]_S$ where $S = R[x;\sigma,\delta]$, Annin investigated the behavior of the attached primes of the inverse polynomial module over $R[x;\sigma]$, where σ is an automorphism of R. Following Annin [Ann11], if σ is an automorphism of R, then $M[x^{-1}]_{R[x;\sigma]}$ is a right module with the action defined as follows: $(mx^{-i})(rx^j) := m\sigma^{-i}(r)x^{-i+j}$ for all $m \in M_R$, $r \in R$, and $j \leq i$, and 0 otherwise [Ann11, p. 538]. Throughout this section S will denote the skew polynomial ring $R[x;\sigma]$ with σ an automorphism of R.

Definition 2.3.16 ([Ann08, Definition 1.4])**.** If σ is an endomorphism of R, then M_R is *completely* σ*-compatible* if $(M/N)_R$ is σ-compatible, for each submodule N_R of M_R.

It can be seen that every completely σ-compatible module is σ-compatible and every quotient of a completely σ-compatible module is completely σ-compatible. The following example shows that there exists a σ-compatible module that is not completely σ-compatible.

Example 2.3.17 ([Ann08, Example 2.2])**.** Let σ be the $\Bbbk$-automorphism of $\Bbbk[t]$ with $\sigma(t) = t+1$. Since $\Bbbk[t]$ is a domain and σ is an automorphism of $\Bbbk[t]$, we have that $\Bbbk[t]_{\Bbbk[t]}$ is σ-compatible. Nevertheless, for $(\Bbbk[t]/(t))_{\Bbbk[t]}$, we get that $\overline{1}\cdot t = \overline{0}$ while $\overline{1}\cdot(t+1) = \overline{1} \neq \overline{0}$, and so this quotient is not σ-compatible. Hence $\Bbbk[t]_{\Bbbk[t]}$ is σ-compatible, but is not completely σ-compatible.

According to Annin, there exists another description of the completely compatible condition. If σ is an endomorphism of R, then M_R is *completely* σ*-compatible* if for every submodule N_R of M_R, we have that $mr \in N_R$ if and only if $m\sigma(r) \in N_R$, for all $m \in M_R$ and $r \in R$. It is easy to see that if M_R is completely σ-compatible, then there exist $r', r'' \in R$ such that $m\sigma(r) = mrr'$ and $mr = m\sigma(r)r''$, for all $m \in M_R$ and $r \in R$. [Ann08, p. 539].

M_R is a *Bass module* if every proper submodule of M_R is contained in a maximal submodule of M_R [Ann02a, Definition 3.32]. It is straightforward to show that if R is right perfect, then M_R is Bass. Proposition 2.3.18 characterized the attached prime ideals of $M[x^{-1}]_S$.

Proposition 2.3.18 ([Ann11, Theorem 3.2])**.** *If* $M[x^{-1}]_R$ *is completely* σ*-compatible Bass, then*

$$\mathrm{Att}(M[x^{-1}]_S) = \{P[x] \mid P \in \mathrm{Att}(M_R)\}.$$

Example 2.3.19 ([Ann11, Example 3.3])**.** Let K be a local ring with maximal ideal $\mathfrak{m}$, $M_K := K/\mathfrak{m}$ and σ an automorphism of K. Annin proved that M_K is σ-compatible [Ann02b, Example 3.4], and since M_K is simple, then M_K is completely σ-compatible. He also proved that $M[x^{-1}]_K$ is completely σ-compatible [Ann11, Proposition 3.4]. Therefore, provided $M[x^{-1}]_K$ is Bass (e.g., K is perfect), Theorem 2.3.18 can be applied.

CHAPTER 3

Associated prime ideals of some quantum algebras

In this chapter, we investigate the reflexive-nilpotents-property of different noncommutative rings. We also study the weak annihilator ideals and the nilpotent associated prime ideals of skew PBW extensions. We formulate results about the associated primes of induced modules on these noncommutative rings.

In Section 3.1, we introduce the Σ-*skew reflexive rings* and present original results concerning different algebraic properties of these rings (Propositions 3.1.3, 3.1.4 and 3.1.5). We also define the Σ-skew RNP rings as a generalization of the σ-skew RNP rings considered by Bhattacharjee [Bha20]. We investigate properties of these rings and their relationships with different kind of noncommutative rings (Propositions 3.1.8, 3.1.9, 3.1.11, 3.1.12, 3.1.13). We formulate results concerning the Σ-skew RNP property for skew PBW extensions (Theorem 3.1.16 and Proposition 3.1.18) and study the Σ-skew RNP property for Ore localization by regular elements (Theorem 3.1.20). In particular, we present a result that characterizes this property for localizations of skew PBW extensions (Theorem 3.1.21).

Section 3.2 presents the notion of weak annihilator and investigates its properties over skew PBW extensions. This section also contains original results concerning weak annihilator ideals of principal ideals generated by nilpotent elements over these rings (Theorems 3.2.1, 3.2.2, 3.2.4, 3.2.5, 3.2.7, 3.2.8, and 3.2.10).

In Section 3.3, we characterize the nilpotent associated prime ideals of skew PBW extensions (Theorems 3.3.2 and 3.3.3). Our results extend those corresponding for skew polynomial extensions presented by Ouyang et al. [OB12, OL12].

Section 3.4 presents the concept of good polynomial and original results that characterize these polynomials (Lemmas 3.4.2, 3.4.3, 3.4.7, and Proposition 3.4.4). We also present several results related to the characterization of associated primes of induced modules over these rings (Lemma 3.4.12, and Theorems 3.4.15 and 3.4.17). Our results generalize those corresponding presented by Leroy and Matczuk [LM04]. It is worth mentioning that this section is a sequel of the study of ideals of skew PBW extensions that has been realized by different authors (e.g. [LAR15, RS18c, NRR20]). In this way, the results presented about associated prime ideals extend or contribute to those presented by Annin [Ann04], Brewer and Heinzer [BH74], Faith [Fai00], Leroy and Matczuk [LM04], Niño et al., [NRR20], and references therein.

Section 3.5 presents some ideas for a future work.

3.1 Reflexive-nilpotents-property for rings

According to Mason [Mas81], a right ideal I of R is called *reflexive* if $aRb \subseteq I$ implies that $bRa \subseteq I$ for all $a, b \in R$, and R is *reflexive* if the zero ideal of R is reflexive. Kwak et al. [KLY14] extended the reflexive property in the following way: an endomorphism σ of R is *right* (resp., *left*) *skew reflexive* if for $a, b \in R$, we have that $aRb = 0$ implies that $bR\sigma(a) = 0$ (resp., $\sigma(b)Ra = 0$). If there exists a right (resp., left) skew reflexive endomorphism σ of R, then R is *right* (resp., *left*) *σ-skew reflexive*. If R is both right and left σ-skew reflexive, then R is *σ-skew reflexive*. Following Kwak et al. [KLY14] R is reduced and right σ-skew reflexive if and only if R is σ-rigid, where σ is an injective endomorphism of R [KLY14, Theorem 2.6].

We introduce the following definition.

Definition 3.1.1. Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite set of endomorphisms of R. We say that Σ is *right* (resp., *left*) *skew reflexive* if for $a, b \in R$, we have that $aRb = 0$ implies that $bR\sigma^{\alpha}(a) = 0$ (resp., $\sigma^{\alpha}(b)Ra = 0$), for all $\alpha \in \mathbb{N}^n$ and R is *right* (resp., *left*) *Σ-skew reflexive* if there exists a right (resp., left) skew reflexive family of endomorphisms Σ of R. If R is both right and left Σ-skew reflexive, then R is *Σ-skew reflexive*.

Reflexive rings are Σ-skew reflexive. Reduced and reversible rings are reflexive, so both are Σ-skew reflexive. Right σ-skew reflexive rings are right Σ-skew reflexive.

We present an example of a right Σ-skew reflexive ring.

Example 3.1.2. If ${}_RM_R$ is a left-right module, then the *trivial extension of R by M* is the ring $T(R, M) := R \oplus M$ with the usual addition of $R \oplus M$ and the multiplication defined as follows: $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + m_1r_2)$, for $r_1, r_2 \in R$ and $m_1, m_2 \in {}_RM_R$. It is not hard to see that $T(R, M)$ is isomorphic to the matrix ring (with the usual matrix operations) of the form $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in {}_RM_R$. We denote by $\mathrm{S}_2(\mathbb{Z})$ the ring of matrices isomorphic to $T(\mathbb{Z}, \mathbb{Z})$

$$\mathrm{S}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let σ_2, σ_3 be two endomorphisms of $\mathrm{S}_2(\mathbb{Z})$ defined by:

$$\sigma_2\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \quad \text{and} \quad \sigma_3\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Consider $ARB = 0$, for all $R \in \mathrm{S}_2(\mathbb{Z})$, where

$$A = \begin{pmatrix} a & a' \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} b & b' \\ 0 & b \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} r & r' \\ 0 & r \end{pmatrix}.$$

We have that $arb = 0$ and $arb' + ar'b + bra' = 0$, for every $r \in \mathbb{Z}$, whence $a = 0$ or $b = 0$. If $a = 0$, then $bra' = 0$, for every $r \in \mathbb{Z}$, and so $b = 0$ or $a' = 0$. If $b = 0$, then $arb' = 0$, for each $r \in \mathbb{Z}$, which implies that $a = 0$ or $b' = 0$. First, we consider the case $BR\sigma_1(A) = BRA$:

$$BR\sigma_1(A) = \begin{pmatrix} bra & bra' + br'a + b'ra \\ 0 & bra \end{pmatrix}.$$

From the previous observation, it follows that $BR\sigma_1(A) = 0$.

Consider the case $BR\sigma_2(A)$. A calculation shows us that

$$BR\sigma_2(A) = \begin{pmatrix} bra & -bra' + br'a + b'ra \\ 0 & bra \end{pmatrix}.$$

Making use of the initial observation, we can see that $BR\sigma_2(A) = 0$, and so $S_2(\mathbb{Z})$ is right σ_2-skew reflexive [KLY14, Proposition 2.11].

Next, we present the case $BR\sigma_3(A)$:

$$BR\sigma_3(A) = \begin{pmatrix} bra & br'a + b'ra \\ 0 & bra \end{pmatrix}.$$

Since $BRA = 0$ we obtain that $bra = 0$. If $a = 0$, then $br'a + b'ra = 0$, for every $r, r' \in \mathbb{Z}$. If $b = 0$, it follows that $b'ra + br'a + bra' = br'a + b'ra = 0$, for each $r, r' \in \mathbb{Z}$. This implies that $BR\sigma_3(A) = 0$. Note that $\sigma_2 \circ \sigma_3 = \sigma_3 \circ \sigma_2 = \sigma_3$, whence $S_2(\mathbb{Z})$ is right Σ-skew reflexive with $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ where $\sigma_1 = \mathrm{id}_{S_2(\mathbb{Z})}$ is the identity endomorphism of $S_2(\mathbb{Z})$.

It is straightforward to see that $S_2(\mathbb{Z})$ is a reflexive ring. In addition, notice that $S_2(\mathbb{Z})$ is not σ_3-compatible. Consider the matrices $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Some computations show that $A\sigma_3(B) = 0$, but $AB = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \neq 0$, and thus $S_2(\mathbb{Z})$ is not a Σ-compatible ring.

The following proposition characterizes the right Σ-skew reflexive rings and shows that the composition of right skew reflexive endomorphisms is a right skew reflexive endomorphism.

Proposition 3.1.3. *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a family of endomorphisms of R. If R is Σ-compatible, then R is right Σ-skew reflexive if and only if σ_i is right skew reflexive for each $1 \leq i \leq n$.*

Proof. Assume that R is Σ-compatible and σ_i is a right skew reflexive endomorphism for all $1 \leq i \leq n$. It is enough to show that if σ_i and σ_j are right skew reflexive endomorphism of R for $1 \leq i, j \leq n$, then $\sigma_i \circ \sigma_j$ is also a right skew reflexive endomorphism of R. Let $a, b \in R$ such that $aRb = 0$, that is, $arb = 0$, for all $r \in R$. If σ_i and σ_j are right skew reflexive for all $1 \leq i, j \leq n$, then $arb = 0$ implies $br\sigma_i(a) = br\sigma_j(a) = 0$, whence $\sigma_i(br\sigma_j(a)) = \sigma_i(br)\sigma_i(\sigma_j(a)) = 0$, for all $1 \leq i, j \leq n$. If R is Σ-compatible then $\sigma_i(br)\sigma_i(\sigma_i(a)) = 0$ which implies that $br\sigma_i(\sigma_j(a)) = 0$. Hence $\sigma_i \circ \sigma_j$ is a right skew reflexive endomorphism of R, and so the composition of right skew reflexive endomorphisms is a right skew reflexive endomorphism, i.e. R is right Σ-skew reflexive.

Now, assume that R is right Σ-skew reflexive. If $aRb = 0$ for all $a, b \in R$, then $bR\sigma^{\alpha}(a) = 0$ for all $\alpha \in \mathbb{N}^n$ by the right Σ-skew reflexivity of R. In particular, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_i = 1$ and $\alpha_j = 0$ for $i \neq j$, we have that $bR\sigma^{\alpha}(a) = bR\sigma_i(a) = 0$, and hence σ_i is a right skew reflexive endomorphism of R for all $1 \leq i \leq n$. □

Proposition 3.1.4 guarantees that when a quasi-commutative skew PBW extension over a Σ-skew reflexive ring is a reflexive ring.

Proposition 3.1.4. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative skew PBW extension of R, where R is a right Σ-skew reflexive. If R satisfies the (SQA1) condition, then A is reflexive.*

Proof. Let $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m}$, $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t} \in A$. If $fAg = 0$, then $a_iRb_j = 0$ for each i, j by the (SQA1) condition of R. Since R is Σ-skew reflexive, $b_jR\sigma^{\alpha}(a_i) = \sigma^{\alpha}(b_j)Ra_i = 0$, for all $\alpha \in \mathbb{N}^n$, and each i, j. If A is quasi-commutative, then for every $x^{\alpha} \in \text{Mon}(A)$ and $0 \neq r \in R$, there exists a non-zero element $\sigma^{\alpha}(r) \in R$ such that $x^{\alpha}r = \sigma^{\alpha}(r)x^{\alpha}$, and for all $x^{\alpha}, x^{\beta} \in \text{Mon}(A)$, there exists an element $d_{\alpha,\beta} \in R$ such that $x^{\alpha}x^{\beta} = d_{\alpha,\beta}x^{\alpha+\beta}$, where $d_{\alpha,\beta}$ is left invertible by Proposition 1.2.19. Applying these commutation rules to the product ghf, where $h = c_1x^{\gamma_1} + \cdots + c_lx^{\gamma_l}$ is an arbitrary element of A, and taking into account that $b_jR\sigma^{\alpha}(a_i) = \sigma^{\alpha}(b_j)Ra_i = 0$, for all $\alpha \in \mathbb{N}^n$, and every i, j, then $gAf = 0$, that is, A is reflexive. □

Proposition 3.1.5 shows that every Σ-compatible semiprime ring is a Σ-skew reflexive ring.

Proposition 3.1.5. *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a family of endomorphims of R. If R is Σ-compatible semiprime, then R is Σ-skew reflexive.*

Proof. Let $a, b \in R$ such that $aRb = 0$. By the Σ-compatibility of R, $aR\sigma^{\alpha}(b) = 0$ for every $\alpha \in \mathbb{N}^n$, whence $\sigma^{\alpha}(b)RaR\sigma^{\alpha}(b)Ra = 0$. If R is a semiprime ring, then $\sigma^{\alpha}(b)Ra = 0$, and thus R is left Σ-skew reflexive. Since R is Σ-compatible, $aRb = 0$ implies $\sigma^{\alpha}(a)Rb = 0$, and hence $bR\sigma^{\alpha}(a)RbR\sigma^{\alpha}(a) = 0$. The semiprimeness of R implies $bR\sigma^{\alpha}(a) = 0$, whence R is right Σ-skew reflexive, and thus R is a Σ-skew reflexive ring. □

By Propositions 3.1.4 and 3.1.5, if A is a quasi-commutative skew PBW extension of R, where R is Σ-compatible semiprime and satisfies the (SQA1) condition, then A is reflexive.

Following Birkenmeier et al. [BKP01], a ring R is called *right principally quasi-Baer* (*right p.q.-Baer*) ring if the right annihilator of a principal right ideal of R is generated by an idempotent. Thinking about the reflexive property and the Baer properties studied in the setting of skew PBW extensions [RS17, RS18a], if A is a skew PBW extension and a right p.q.-Baer ring, then A is a semiprime ring if and only if A is a reflexive ring by [KL11, Proposition 3.15].

Bhattacharjee [Bha20] defined the *σ-skew RNP rings* as a generalization of σ-skew reflexive rings. An endomorphism σ of R is *right* (resp., *left*) *skew* RNP if for $a, b \in N(R)$, $aRb = 0$ implies that $bR\sigma(a) = 0$ (resp., $\sigma(b)Ra = 0$). If there exists a right (resp., left) skew RNP endomorphism σ of R, then R is *right* (resp., *left*) *σ-skew* RNP. If R is both right and left σ-skew RNP, then R is *σ-skew* RNP. Reduced rings are σ-skew RNP for any endomorphism σ, and every right (resp., left) σ-skew reflexive ring is right (resp., left) σ-skew RNP [Bha20, Remark 1.2]. The notion of σ-skew RNP ring is not left-right symmetric [Bha20, Example 1.3]. However, if R is RNP with an endomorphism σ, then R is right σ-skew RNP if and only if R is left σ-skew RNP.

We introduce the *Σ-skew RNP rings* and investigate some relationships of these rings with skew PBW extensions. Definition 3.1.6 generalizes the notions of skew RNP endomorphisms and the σ-skew RNP rings defined by Bhattacharjee [Bha20].

Definition 3.1.6. If $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ is a finite set of endomorphisms of R, then Σ is a *right* (resp., *left*) *skew RNP family* if for $a, b \in N(R)$, $aRb = 0$ implies $bR\sigma^{\alpha}(a) = 0$ (resp., $\sigma^{\alpha}(b)Ra = 0$), for all $\alpha \in \mathbb{N}^n$; R is *right* (resp., *left*) Σ-*skew RNP* if there exists a right (resp., left) skew RNP family of endomorphism Σ of R. If R is both right and left Σ-skew RNP, then R is called Σ-*skew RNP*.

It is straightforward to see that right Σ-skew reflexive rings are right Σ-skew RNP, and reduced and RNP rings are Σ-skew RNP.

Example 3.1.7. (i) Consider Example 2.1.23. Note that $\sigma_{1,0}$ is the identity homomorphism of R and Σ is closed under composition, that is, $\sigma_{i,j}^{\alpha} \in \Sigma$, for all $\alpha \in \mathbb{N}^6$. Additionally the set of nilpotent elements of R is the ideal generated by z, that is, $N(R) = \langle z \rangle$. Let $f, g \in N(R)$ such that $fRg = 0$. Since $\sigma_{i,j}^{\alpha}(f) = a^k z$, for all $f \in R$ and $\alpha \in \mathbb{N}^6$, and some $0 \le k \le 2$, it follows that $gR\sigma_{i,j}^{\alpha}(f) = 0$, and thus R is Σ-skew RNP.

(ii) Consider Example 3.1.2. It is not difficult to see that $\mathrm{S}_2(\mathbb{Z})$ is right Σ-skew RNP. Additionally, notice that $N(\mathrm{S}_2(\mathbb{Z})) = \left\{ \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \mid b \in \mathbb{Z} \right\}$.

Proposition 3.1.8 characterizes the right Σ-skew RNP rings and shows that the composition of right skew RNP endomorphisms is a right skew RNP endomorphism.

Proposition 3.1.8. *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite set of endomorphisms of R. If R is a Σ-compatible ring, then R is right Σ-skew RNP if and only if R is right σ_i-skew RNP for each $1 \le i \le n$.*

Proof. The proof is similar to that of Proposition 3.1.3 but using Proposition 2.1.26. □

Proposition 3.1.9 characterizes the Σ-skew RNP rings over Σ-compatible rings.

Proposition 3.1.9. *If R is a Σ-compatible then the following assertions are equivalent:*

(1) *R is RNP.*

(2) *R is right Σ-skew RNP.*

(3) *R is left Σ-skew RNP.*

Proof. (1) ⇒ (2) Suppose that R is RNP and let $a, b \in N(R)$ such that $aRb = 0$, that is, $arb = 0$ for all $r \in R$. By the Σ-compatibility of R, $\sigma^{\alpha}(a)rb = 0$, and thus $\sigma^{\alpha}(a)Rb = 0$. If $a \in N(R)$, then $\sigma^{\alpha}(a) \in N(R)$ by Proposition 2.1.26, and since R is RNP, $\sigma^{\alpha}(a)Rb = 0$ implies that $bR\sigma^{\alpha}(a) = 0$, whence R is right Σ-skew RNP.

(2) ⇒ (1) Assume that R is right Σ-skew RNP and let $a, b \in N(R)$ such that $aRb = 0$, i.e., $arb = 0$ for any $r \in R$. Since R is Σ-compatible, $ar\sigma^{\alpha}(b) = 0$, for every $\alpha \in \mathbb{N}^n$. If $b \in N(R)$, then $\sigma^{\alpha}(b) \in N(R)$ by Proposition 2.1.26, and if R is right Σ-skew RNP, then $\sigma^{\alpha}(b)R\sigma^{\alpha}(a) = 0$. So, for each $r \in R$, $\sigma^{\alpha}(bra) = \sigma^{\alpha}(b)\sigma^{\alpha}(r)\sigma^{\alpha}(a) = 0$. Finally, σ^{α} is injective for all $\alpha \in \mathbb{N}^n$ which implies that $bra = 0$ proving that R is RNP.

(1) ⇔ (3) The proof is similar to (1) ⇔ (2). □

Corollary 3.1.10 ([Bha20, Proposition 1.7])**.** *If R is a σ-compatible ring, then the following are equivalent:* (1) *R is RNP* (2) *R is right σ-skew RNP* (3) *R is left σ-skew RNP.*

Proposition 3.1.11 characterizes the right and left Σ-skew reflexive rings.

Proposition 3.1.11. *If R is Σ-compatible then the following assertions are equivalent:*

(1) *R is reflexive.*

(2) *R is right Σ-skew reflexive.*

(3) *R is left Σ-skew reflexive.*

Proof. Since Σ-skew reflexive rings are Σ-skew RNP, the result follows from Proposition 3.1.9. □

It is not difficult to see that nil-reversible rings are RNP. Proposition 3.1.12 shows that nil-reversible and Σ-compatible rings are Σ-skew RNP.

Proposition 3.1.12. *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a family of endomorphisms of R. If R is nil-reversible and Σ-compatible, then R is Σ-skew RNP.*

Proof. Since nil-reversible rings are RNP, if $aRb = 0$, for $a, b \in N(R)$, then $bRa = 0$, and thus $bR\sigma^{\alpha}(a) = 0$ for all $\alpha \in \mathbb{N}^n$, by Σ-compatibility of R. Hence, R is right Σ-skew RNP. By Proposition 2.1.22, $bRa = 0$ implies $\sigma^{\alpha}(b)Ra = 0$, and so R is left Σ-skew RNP. □

If A is a skew PBW extension of a domain R, then A is a domain [FGL+20, Proposition 3.2.1], and so $N(A) = 0$. If A also satisfies the conditions established in Proposition 1.2.22, then A is $\overline{\Sigma}$-skew RNP for $\overline{\Sigma} = \{\overline{\sigma_1}, \dots, \overline{\sigma_n}\}$, where $\overline{\sigma_k}$ is as in Proposition 1.2.22.

Proposition 3.1.13. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is skew PBW extension of R, where R is Σ-rigid, then the following assertions hold:*

(1) *Both R and A are reflexive.*

(2) *R is Σ-skew RNP.*

(3) *If the conditions established in Proposition 1.2.22 hold, then A is $\overline{\Sigma}$-skew RNP.*

Proof. (1) By [RS17, Theorem 4.4], we have that both R and A are reduced. Since reduced rings are reflexive, then R and A are reflexive.

(2) If R is Σ-rigid, then R is reduced by [RS17, Theorem 4.4], whence R is Σ-skew RNP for any finite family of endomorphisms Σ.

(3) Since A is reduced by [RS17, Theorem 4.4], then A is a Σ-skew RNP ring for any finite family of endomorphisms. In particular, A is $\overline{\Sigma}$-skew RNP for $\overline{\Sigma} = \{\overline{\sigma_1}, \dots, \overline{\sigma_n}\}$, where $\overline{\sigma_k}$ is as in Proposition 1.2.22. □

According to Kwak and Lee [KL11], R is called *CN* if $N(R[x]) \subseteq N(R)[x]$. Both NI rings and Armendariz rings are CN but not conversely. The classes of quasi-Armendariz and CN rings are independent of each other [KKKL17a, Examples 3.15(1) and (2)]. Bhattacharjee [Bha20] showed that if R is CN and quasi-Armendariz, there exists an equivalence of the RNP property between $R[x]$ and R. We introduce the following definition with the aim of studing the RNP property on skew PBW extensions.

Definition 3.1.14. If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension of R, then we say that R is Σ-*skew CN* if $N(A) \subseteq N(R)A$.

Hashemi et al. [HKA19] investigated the prime radicals and the upper nil radicals over skew PBW extensions. Some results establish conditions to ensure that $N^*(A) \subseteq N^*(R)A$, where A is bijective skew PBW extension over a (Σ,Δ)-compatible ring [HKA19, Theorem 3.15]. In this way, if A is NI, it follows that R is Σ-skew CN. Other important result states that if A is a PBW extension of R, where R is Σ-compatible and $N(R)$ is a Δ-ideal, then $N(A) \subseteq N(R)A$ [HKA19, Proposition 4.1], that is, R is a Σ-skew CN. If R is 2-primal Σ-compatible, then R is Σ-skew CN [HKA19, Corollary 4.12]. Reyes and Suárez [RS20] proved that if A is a skew PBW extension of R, where R is weak (Σ,Δ)-compatible and NI, then $f = a_1x^{\alpha_1} + \cdots + a_tx^{\alpha_t} \in N(A)$ if and only if $a_i \in N(R)$, for $0 \le i \le t$ [RS20, Theorem 4.6], and hence R is Σ-skew CN.

Example 3.1.15. We present some examples of Σ-skew CN rings.

(i) If R is σ-compatible and $N(R)$ is a right δ-ideal of R, then $N(R[x;\sigma,\delta]) \subseteq N(R)[x;\sigma,\delta]$ by [NI14, Proposition 2.2], and thus R is Σ-skew CN. If $R[x;\sigma,\delta]$ is NI and $N(R)$ is right σ-rigid, then $N(R[x;\sigma,\delta]) = N(R)[x;\sigma,\delta]$ by [NI15, Theorem 3.1], and so R is Σ-skew CN.

(ii) By [RS20, Theorem 4.6], it follows that the set of nilpotent elements of the enveloping algebra $U(\mathfrak{g})$ satisfies $N(U(\mathfrak{g})) \cong N(\Bbbk\langle x_1,\dots,x_n\rangle) \subseteq N(\Bbbk)A$, and thus $\Bbbk$ is Σ-skew CN.

(iii) Consider Example 2.1.23. Following the same ideas of the proof of [RS20, Theorem 4.6], it is not difficult to see that $N(A) = N(\sigma(R))\left\langle x_{i,j}\right\rangle$, and so R is Σ-skew CN.

Theorem 3.1.16 characterizes the skew PBW extensions over Σ-skew CN rings that are $\overline{\Sigma}$-skew RNP for $\overline{\Sigma} = \{\overline{\sigma_1},\dots,\overline{\sigma_n}\}$, where $\overline{\sigma_k}$ is as in Proposition 1.2.22.

Theorem 3.1.16. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension of R, where R is Σ-skew CN, (Σ,Δ)-compatible and satisfies the (SQA1) condition. If the conditions established in Proposition 1.2.22 hold, then R is right Σ-skew RNP if and only if A is right $\overline{\Sigma}$-skew RNP.*

Proof. Assume that R is right Σ-skew RNP and let $f = a_1x^{\alpha_1} + \cdots + a_mx^{\alpha_m}$, $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t}$ be two nilpotent elements of A such that $fAg = 0$. Since R satisfies the (SQA1) condition, then $a_iRb_j = 0$, for all i,j. On the other hand, R is Σ-skew CN, that is, $f,g \in N(R)A$, which implies that $a_i, b_j \in N(R)$ for all i,j. Since R is Σ-skew RNP, we have that $b_jR\sigma^{\alpha}(a_i) = 0$, for all i,j and $\alpha \in \mathbb{N}^n$. Consider an element $h \in A$. Note that each coefficient of $gh\overline{\sigma}^{\alpha}(f)$ is a product of the coefficients b_j with elements of R and several evaluations of a_i in σ's and δ's depending of the coordinates of α_i, β_j. From the previous statement and $b_jR\sigma^{\alpha}(a_i) = 0$, it follows that

$gh\overline{\sigma}^{\alpha}(f) = 0$ for all $\alpha \in \mathbb{N}^n$, by the (Σ, Δ)-compatibility of R. Since h is an arbitrary element of A, we have that $gA\overline{\sigma}^{\alpha}(f) = 0$ and thus A is right $\overline{\Sigma}$-skew RNP.

Conversely, suppose that A is right $\overline{\Sigma}$-skew RNP and let $a, b \in N(R)$ such that $aRb = 0$. If R is (Σ, Δ)-compatible, then $aAb = 0$. Thus, $bA\overline{\sigma}^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$ entailing $bR\sigma^{\alpha}(a) = 0$, by RNP property of A and $\overline{\sigma}^{\alpha}(a) \in N(R)$. Hence, R is a right Σ-skew RNP ring. □

Corollary 3.1.17. (1) [Bha20, Corollary 2.13] *If R is Armendariz and σ is an endomorphism of R, then R is right σ-skew RNP if and only if $R[x]$ is right $\overline{\sigma}$-skew RNP.*

(2) [Bha20, Proposition 2.12] *If R is quasi-Armendariz and CN and σ is an endomorphism of R, then R is right σ-skew RNP if and only if $R[x]$ is right $\overline{\sigma}$-skew RNP.*

Proposition 3.1.18 characterizes the skew PBW extensions over NI rings that are RNP rings.

Proposition 3.1.18. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where (Σ, Δ)-compatible and NI, then A is nil-reflexive if and only if A is RNP.*

Proof. If R is (Σ, Δ)-compatible then R is weak (Σ, Δ)-compatible. Thus, if R is NI then A is NI by [SCR22, Theorem 3.3], whence $N^*(A) = N(A)$. By [KKKL17b, Proposition 2.1], A is nil-reflexive if and only if $fAg = 0$ implies $gAf = 0$, for every $f, g \in N^*(A) = N(A)$, and hence A is nil-reflexive if and only if A is RNP. □

Corollary 3.1.19. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R where R is NI. If A is $(\overline{\Sigma}, \overline{\Delta})$-compatible and the conditions established in Proposition 1.2.22 hold, then A is nil-reflexive if and only if A is $\overline{\Sigma}$-skew RNP.*

Proof. The assertion follows from Propositions 3.1.9 and 3.1.18. □

If S is a multiplicatively closed subset of R consisting of central regular elements, and σ is an automorphism of R such that $\sigma(S) \subseteq S$, then σ induces an endomorphism $\overline{\sigma}$ of $S^{-1}R$ defined by $\overline{\sigma}(u^{-1}a) = \sigma(u)^{-1}\sigma(a)$ for $u \in S$ and $a \in R$. If Σ is a set of automorphisms of R, we denote Σ_S the set of automorphisms over $S^{-1}R$, induced by Σ.

Theorem 3.1.20. *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a finite set of automorphism of R and S a multiplicatively closed subset of R consisting of central regular elements such that $\sigma^{\alpha}(S) \subseteq S$, for every $\alpha \in \mathbb{N}^n$. Then R is right Σ-skew RNP if and only if $S^{-1}R$ is right Σ_S-skew RNP.*

Proof. Assume that R is right Σ-skew RNP. Note that if $s_1^{-1}a \in N(S^{-1}R) = S^{-1}N(R)$ then $a \in N(R)$. Let $s_1^{-1}a, s_2^{-1}b \in N(S^{-1}R)$ such that $(s_1^{-1}a)S^{-1}R(s_2^{-1}b) = 0$. Since $(s_1^{-1}a)s^{-1}r(s_2^{-1}b) = 0$ for all $s^{-1}r \in S^{-1}R$, we have that $(s_1ss_2)^{-1}(arb) = 0$. By definition of $S^{-1}R$, there exists $c \in S$ such that c is a central element of R and $0 = (arb)c = a(rc)b$. Since R is right Σ-skew RNP, it follows that $b(rc)\sigma^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$. Since $b(r)\sigma^{\alpha}(a)c = 0$, for all $\alpha \in \mathbb{N}^n$, we have that

$$\begin{aligned}(s_2^{-1}b)(s^{-1}r)\overline{\sigma}^{\alpha}(s_1^{-1}a) &= (s_2^{-1}b)(s^{-1}r)\sigma^{\alpha}(s_1)^{-1}\sigma^{\alpha}(a)\\ &= (s_2s\sigma^{\alpha}(s_1))^{-1}(br\sigma^{\alpha}(a))\\ &= 0,\end{aligned}$$

and hence $S^{-1}R$ is right Σ_S-skew RNP.

For the other implication, consider two non-zero elements $a, b \in N(R)$ such that $aRb = 0$. We have that $(1^{-1})(arb) = (1^{-1}a)(1^{-1}r)(1^{-1}b) = 0$. In addition, if $S^{-1}R$ is right Σ_S-skew RNP, then $(1^{-1}b)(1^{-1}r)\overline{\sigma}^{\alpha}(1^{-1}a) = (1^{-1}b)(1^{-1}r)\sigma^{\alpha}(1)^{-1}\sigma^{\alpha}(a) = (1^{-1})(br\sigma^{\alpha}(a))$ whence $br\sigma^{\alpha}(a) = 0$ since $(1^{-1}b)(1^{-1}r)\overline{\sigma}^{\alpha}(1^{-1}a) = 0$. Hence, R is right Σ-skew RNP. □

Theorem 3.1.21. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R, where R is (Σ, Δ)-compatible, S is a multiplicatively closed subset of R consisting of central regular elements such that $\overline{\sigma}^{\alpha}(S) \subseteq S$ and $S^{-1}R$ is a $\overline{\Sigma}_S$-skew CN which satisfies the (SQA1) condition. If the conditions established in Proposition 1.2.22 hold, then A is right $\overline{\Sigma}$-skew RNP if and only if $S^{-1}A$ is right $\overline{\Sigma}_S$-skew RNP.*

Proof. Suppose that A is right $\overline{\Sigma}$-skew RNP. If $S^{-1}R$ is Σ_S-skew CN, then R is also Σ-skew CN. Similarly, the condition (SQA1) also transfers from $S^{-1}R$ to R, whence R is right Σ-skew RNP by Theorem 3.1.16. If S is a multiplicatively closed subset of R consisting of central regular elements, then $S^{-1}R$ is right Σ_S-skew RNP by Theorem 3.1.20. In addition, $S^{-1}A$ is a bijective skew PBW extension over $S^{-1}R$ by Proposition 1.2.21. Since R is a (Σ, Δ)-compatible ring, then $S^{-1}R$ is $(\overline{\Sigma}, \overline{\Delta})$-compatible [RS18a, Theorem 4.20]. Notice that $S^{-1}A$ is a bijective skew PBW extension over $S^{-1}R$, where $S^{-1}R$ is $(\overline{\Sigma}, \overline{\Delta})$-compatible $\overline{\Sigma}_S$-skew CN, and satisfies the (SQA1) condition. Thus, if $S^{-1}R$ is right Σ-skew RNP then $S^{-1}A$ is right $\overline{\Sigma}_S$-skew RNP by Theorem 3.1.16.

On the other hand, if $S^{-1}A$ is right $\overline{\Sigma}_S$-skew RNP then $S^{-1}R$ is a right $\overline{\Sigma}_S$-skew RNP ring, because $S^{-1}R$ is $(\overline{\Sigma}, \overline{\Delta})$-compatible $\overline{\Sigma}_S$-skew CN and satisfies the condition (SQA1). The above statement follows from Theorem 3.1.16. Since $S^{-1}R$ is $\overline{\Sigma}_S$-skew RNP then R is Σ-skew RNP. Finally, R is a (Σ, Δ)-compatible ring and inherits the Σ-skew CN property and the condition (SQA1) from the ring $S^{-1}R$, which implies that A is right $\overline{\Sigma}$-skew RNP, by Theorem 3.1.16. □

3.2 Weak annihilator ideals

In this section, we introduce the weak annihilator ideals in the setting of skew PBW extensions.

If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, we define

$$\mathrm{NAnn}_R(R) := \{N_R(U) \mid U \subseteq R\} \text{ and } \mathrm{NAnn}_A(A) := \{N_A(V) \mid V \subseteq A\}.$$

If $f = a_1x^{\alpha_1} + \cdots + a_m x^{\alpha_l} \in A$, then we denote by C_f the set of the coefficients of f, and for a subset $V \subseteq A$, $C_V = \bigcup_{f\in V} C_f$.

Theorem 3.2.1 establishes a bijective correspondence between the weak annihilators of R and a skew PBW extension built on R, and extends [RS21, Theorem 3.21] which was formulated for classical annihilators.

Theorem 3.2.1. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where R is (Σ, Δ)-compatible and NI, then the correspondence $\varphi : \mathrm{NAnn}_R(R) \to \mathrm{NAnn}_A(A)$ defined by $\varphi(N_R(U)) = N_R(U)A$, for every $N_R(U) \in \mathrm{NAnn}_R(R)$, is bijective.*

Proof. Let us show that φ is well defined, that is, $N_A(U) = N_R(U)A$ for every nonempty subset U of R. Consider $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in N_R(U)A$ and $r \in U$. If $a_i \in N_R(U)$, for all $0 \le i \le l$ then $ra_i \in N(R)$, for each i. By Corollary 2.1.29, we have that $rf = ra_1x^{\alpha_1} + \cdots + ra_lx^{\alpha_l} \in N(R)A = N(A)$, and so $N_R(U)A \subseteq N_A(U)$.

If $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in N_A(U)$ then $rf = ra_1x^{\alpha_1} + \cdots + ra_lx^{\alpha_l} \in N(A)$ for all $r \in U$. Since $N(A) = N(R)A$ by Corollary 2.1.29, then $ra_i \in N(R)$ for all $1 \le i \le l$, and hence $a_i \in N_R(U)$. This shows that $f \in N_R(U)A$, and thus $N_A(U) = N_R(U)A$.

Let U, V be a two subsets of R such that $\varphi(N_R(U)) = \varphi(N_R(V))$. By definition of φ, we have that $N_A(U) = N_A(V)$. In particular, $N_R(U) = N_R(V)$, and thus φ is injective. Let us show that φ is surjective. Let $N_A(V) \in \text{NAnn}_A(A)$ and consider $g = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s} \in N_A(V)$, for a subset V of A. Then $fg \in N(A)$ for all $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in V$, and by Proposition 2.1.32, $a_ib_j \in N(R)$ for each i, j. Thus, $b_j \in N_R(C_V)$ for all $1 \le j \le s$, whence $g \in N_R(C_V)A$, and hence $N_A(V) \subseteq N_R(C_V)A$. Since $N_R(C_V)A \subseteq N_A(V)$, we get that $N_A(V) = N_R(C_V)A = \varphi(N_R(C_V))$, and so φ is surjective. □

The following theorem generalizes [OB12, Theorem 2.1].

Theorem 3.2.2. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R, where R is (Σ, Δ)-compatible and NI. If for each subset $X \nsubseteq N(R)$, $N_R(X)$ is generated as an ideal by a nilpotent element, then for each subset $U \nsubseteq N(A)$, $N_A(U)$ is generated as an ideal by a nilpotent element.*

Proof. Let U be a subset of A with $U \nsubseteq N(A)$. By Corollary 2.1.29, $C_U \nsubseteq N(R)$, and so there exists $c \in N(R)$ such that $N_R(C_U) = cR$. Let us show the equality $N_A(U) = cA$. If $f = a_1x_1 + \cdots + a_lx^{\alpha_l} \in U$ and $g = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s} \in A$, then

$$fcg = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_ix^{\alpha_i}cb_jx^{\beta_j}\right) = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(cb_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_i,cb_j}x^{\beta_j}\right).$$

Since $a_icb_j \in N(R)$, for all $1 \le i \le l$, $1 \le j \le s$, then $a_i\sigma^{\alpha_i}(cb_j) \in N(R)$, and $a_i\sigma^{\alpha}(\delta^{\beta}(cb_j))$ and $a_i\delta^{\beta}(\sigma^{\alpha}(cb_j))$ are elements of $N(R)$, for all $\alpha, \beta \in \mathbb{N}^n$ by [HKA19, Proposition 3.3]. By Proposition 1.2.20, the polynomial p_{α_i,cb_j} involves elements obtained evaluating σ's and δ's (depending on the coordinates of α_i) in the element cb_j. Thus, $a_ip_{\alpha_i,cb_j} \in N(R)$ and $a_icb_j \in N(R)$, for all i, j, whence $fcg \in N(R)A = N(A)$, and therefore $cg \in N_A(U)$.

If $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in N_A(U)$, then $fg \in N(A)$ for all element $g = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s} \in U$. By Corollary 2.1.29 we obtain that $fg \in N(R)A$, and so

$$fg = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_ix^{\alpha_i}b_jx^{\beta_j}\right) = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + p_{\alpha_i,b_j}x^{\beta_j}\right) \in N(R)A.$$

Since $a_i\sigma^{\alpha_i}(b_j) \in N(R)$, for all i, j, we have that $a_ib_j \in N(R)$ by [HKA19, Proposition 3.3]. In this way, $a_i \in N_R(C_U) = cR$, for each $1 \le i \le l$, and so there exist elements $r_i \in R$ such that $a_i = cr_i$ for every i, whence $f = c(r_1x^{\alpha_1} + \cdots + r_lx^{\alpha_l}) \in cA$, and hence $f \in cA$. □

Example 3.2.3. (i) Consider Example 2.1.23. If $A = \sigma\left(\mathbb{F}_4[z]/\langle z^2\rangle\right)\langle x_{i,j}\rangle$ and $Y \subseteq \mathbb{F}_4[z]/\langle z^2\rangle$ with $Y \nsubseteq N\left(\mathbb{F}_4[z]/\langle z^2\rangle\right)$ then $N_{\mathbb{F}_4[z]/\langle z^2\rangle}(Y) = z\mathbb{F}_4[z]/\langle z^2\rangle$, where $z \in N\left(\mathbb{F}_4[z]/\langle z^2\rangle\right)$. By Theorem 3.2.2, $N_AU = wA$, for any subset $U \subseteq A$ where $U \nsubseteq N(A)$ and $w \in N(A)$.

(ii) Consider Example 2.1.24. For any $Y \subseteq R$ with $Y \nsubseteq N(R)$, we have that $N_R(Y) = rR$, where $r \in N(R)$. By Theorem 3.2.2 it follows that $N_{R[x;\overline{\sigma},\overline{\delta}]}U = wR[x;\overline{\sigma},\overline{\delta}]$ for any $U \subseteq R[x;\overline{\sigma},\overline{\delta}]$, where $U \nsubseteq N(R[x;\overline{\sigma},\overline{\delta}])$ and $w \in N(R[x;\overline{\sigma},\overline{\delta}])$.

Theorem 3.2.4 is a consequence of Theorem 3.2.2 considering skew PBW extensions of the endomorphism type. This result generalizes [OB12, Theorem 2.2].

Theorem 3.2.4. *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension of endomorphism type of R, where R is Σ-compatible and NI, then the following statements are equivalent:*

(1) *For each subset $X \nsubseteq N(R)$, we have that $N_R(X)$ is generated by a nilpotent element.*

(2) *For each subset $U \nsubseteq N(A)$, we have that $N_A(U)$ is generated by a nilpotent element.*

Proof. By Theorem 3.2.2, it suffices to show (2) ⇒ (1). If X is a subset of R with $X \nsubseteq N(R)$, then $X \nsubseteq N(A)$, and so there exists $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in N(A)$ such that $N_A(X) = fA$. Notice that $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in N(A)$, whence $a_i \in N(R)$, for all $1 \le i \le l$, by Corollary 2.1.29. Suppose that $a_1 \neq 0$, and let us see that $N_R(X) = a_1R$. Since $a_1 \in N(R)$ and $N(R)$ is an ideal of R, we obtain $pa_1R \subseteq N(R)$, for each $p \in X$, whence $a_0R \subseteq N_R(X)$. Hence, if $m \in N_R(X) \subseteq N_A(X)$, there exists $g(x) = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s} \in A$ such that

$$m = fg = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_ix^{\alpha_i}b_jx^{\beta_j}\right) = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j}\right).$$

Thus, $m = a_1b_1 \in a_1R$ and so $N_R(X) \subseteq a_1R$, whence $N_R(X) = a_1R$ with $a_1 \in N(R)$. □

Now, we present a theorem for weak annihilators of principal right ideals that extends [OB12, Theorem 2.3].

Theorem 3.2.5. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension of R, where R is (Σ,Δ)-compatible and NI. If for each principal right ideal $pR \nsubseteq N(R)$, $N_R(pR)$ is generated as an ideal by a nilpotent element, then for each principal right ideal $fA \nsubseteq N(A)$, $N_A(fA)$ is generated as an ideal by a nilpotent element.*

Proof. Let $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in A$ with $fA \nsubseteq N(A)$. By Corollary 2.1.29, if $a_iR \subseteq N(R)$ for all $1 \le i \le l$, then $fA \subseteq N(A)$ a contradiction. Thus, there exists i such that $a_iR \nsubseteq N(R)$, and so there is $c \in N(R)$ such that $N_R(a_iR) = cR$. Let us show that $N_A(fA) = cA$. Let $g = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s}$ and $h = c_1x^{\gamma_1} + \cdots + c_tx^{\gamma_t}$ be two elements of A. By Theorem 3.2.2, $gch \in N(R)A = N(A)$ and by Corollary 2.1.29, for every $f \in A$, $fgch \in N(A) = N(R)A$, since $N(A)$ is an ideal of A. Therefore, $ch \in N_A(fA)$, and so $cA \subseteq N_A(fA)$. For the other inclusion, if $p = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s} \in N_A(fA)$, then $fRp \subseteq N(A) = N(R)A$ for every $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in A$, and thus

$$frp = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_ix^{\alpha_i}rb_jx^{\beta_j}\right) = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(rb_j)x^{\alpha_i}x^{\beta_i} + a_ip_{\alpha_i,rb_j}x^{\beta_j}\right) \in N(R)A.$$

Hence, $a_i\sigma^{\alpha_i}(rb_j) \in N(R)$ and so $a_irb_j \in N(R)$, for all i,j. Thus, $p_j \in N_R(a_iR) = cR$, and so there exists $r_j \in R$ such that $p_j = cr_j$ whence $p = c(r_1x^{\beta_1} + \cdots + r_sx^{\beta_s}) \in cA$. □

Example 3.2.6. (i) Let $A = \sigma\left(\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right)\langle x_{i,j}\rangle$ be as Example 2.1.23. If $p \in \mathbb{F}_4[z]/\left\langle z^2\right\rangle$ with $p\mathbb{F}_4[z]/\left\langle z^2\right\rangle \nsubseteq N\left(\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right)$, we obtain that $N_{\mathbb{F}_4[z]/\langle z^2\rangle}\left(p\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right) = z\mathbb{F}_4[z]/\left\langle z^2\right\rangle$, where $z \in N\left(\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right)$. By Theorem 3.2.5, the ideal $N_A(fA)$ is generated by a nilpotent element for any principal right ideal fA with $fA \nsubseteq N(A)$.

(ii) Consider Example 2.1.24. Let $p = \left(\begin{smallmatrix} p(t) & q(t) \\ 0 & p(t)\end{smallmatrix}\right) \in R$ for some $p(t), q(t) \in \Bbbk[t]$ and $p(t) \neq 0$ such that $pR \nsubseteq N(R)$. It is not hard to see that $N_R(pR) = \left\{\left(\begin{smallmatrix} 0 & q(t) \\ 0 & 0\end{smallmatrix}\right) \mid q(t) \in \Bbbk[t]\right\} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)R$, with $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right) \in N(R)$. By Theorem 3.2.5, $N_{R[x;\overline{\sigma},\overline{\delta}]}(fR[x;\overline{\sigma},\overline{\delta}])$ is generated by a nilpotent element of $R[x;\overline{\sigma},\overline{\delta}]$, for all principal right ideal $fR[x;\overline{\sigma},\overline{\delta}]$ where $fR[x;\overline{\sigma},\overline{\delta}] \nsubseteq N(R[x;\overline{\sigma},\overline{\delta}])$.

Theorem 3.2.7 generalizes [OB12, Theorem 2.4].

Theorem 3.2.7. *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension of endomorphism type of R, where is Σ-compatible and NI, then the following statements are equivalent:*

(1) *For each principal right ideal $pR \nsubseteq N(R)$, we have that $N_R(pR)$ is generated as an ideal by a nilpotent element.*

(2) *For each principal right ideal $fA \nsubseteq N(A)$, we have that $N_A(fA)$ is generated as an ideal by a nilpotent element.*

Proof. By Theorem 3.2.5, it suffices to show (2) ⇒ (1). The proof is very similar to the reasoning above and follows the same ideas. □

The next result extends [OB12, Theorem 2.5].

Theorem 3.2.8. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension of R, where R is (Σ,Δ)-compatible and NI. If for each $p \notin N(R)$, $N_R(p)$ is generated as an ideal by a nilpotent element, then for each $f \notin N(A)$, $N_A(f)$ is generated as an ideal by a nilpotent element.*

Proof. Let $f = a_1x^{\alpha_1} + \cdots + a_lx^{\alpha_l} \in A$ with $f \notin N(A)$. Let us see that $N_A(f)$ is generated as an ideal by a nilpotent element. By Corollary 2.1.29, if $a_iR \subseteq N(R)$, for all $1 \le i \le l$, then $f \in N(A)$ which is a contradiction. Hence, there exists $a_i \notin N(R)$, and so there exists $c \in N(R)$ such that $N_R(a_i) = cR$. The idea is to show that $N_A(f) = cA$. By Theorem 3.2.2, if $h = c_1x^{\gamma_1} + \cdots + c_tx^{\gamma_t} \in A$, it follows that $fch \in N(R)A = N(A)$ whence $ch \in N_A(f)$ which proves that $cA \subseteq N_A(f)$. For the other inclusion if $p = b_1x^{\beta_1} + \cdots + b_sx^{\beta_s} \in N_A(f)$, then $fp \in N(A) = N(R)A$, and thus

$$fp = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_ix^{\alpha_i}b_jx^{\beta_j}\right) = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_i,b_j}x^{\beta_j}\right) \in N(R)A.$$

Therefore, $a_i\sigma^{\alpha_i}(b_j) \in N(R)$ and so $a_ib_j \in N(R)$ for all i,j. Thus, $b_j \in N_R(a_i) = cR$ and so there exist $r_j \in R$ such that $b_j = cr_j$, whence $p = c(r_1x^{\beta_1} + \cdots + r_sx^{\beta_s}) \in cA$. □

Example 3.2.9. (i) Let $A = \sigma\left(\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right)\left\langle x_{i,j}\right\rangle$ be from Example 2.1.23. If $p \in \mathbb{F}_4[z]/\left\langle z^2\right\rangle$ with $p \notin N\left(\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right)$, then $N_{\mathbb{F}_4[z]/\langle z^2\rangle}(p) = z\mathbb{F}_4[z]/\left\langle z^2\right\rangle$, where $z \in N\left(\mathbb{F}_4[z]/\left\langle z^2\right\rangle\right)$. By Theorem 3.2.8, the ideal $N_A(f)$ is generated by a nilpotent element for any element f where $f \notin N(A)$.

(ii) Consider Example 2.1.24. Let $p = \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \in R$ for some $p(t), q(t) \in \Bbbk[t]$ and $p(t) \neq 0$ such that $p \notin N(R)$. It is not difficult to see that $N_R(p) = \left\{\begin{pmatrix} 0 & q(t) \\ 0 & 0 \end{pmatrix} \mid q(t) \in \Bbbk[t]\right\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R$, where $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$. By Theorem 3.2.8, the ideal $N_{R[x;\overline{\sigma},\overline{\delta}]}(f)$ is generated by a nilpotent element of $R[x;\overline{\sigma},\overline{\delta}]$, for any element $f \in R[x;\overline{\sigma},\overline{\delta}]$ where $f \notin N(R[x;\overline{\sigma},\overline{\delta}])$.

The following theorem characterizes the skew PBW extensions of endomorphism type for which every weak annihilator of an element of A is generated by a nilpotent element. This result generalizes [OB12, Theorem 2.6].

Theorem 3.2.10. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of endomorphism type of R, where R is Σ-compatible and NI, then the following statements are equivalent:*

(1) *For each $p \notin N(R)$, we have that $N_R(p)$ is generated as an ideal by a nilpotent element.*

(2) *For each $f \notin N(A)$, we have that $N_A(f)$ is generated as an ideal by a nilpotent element.*

Proof. By Theorem 3.2.8, it suffices to show (ii) ⇒ (i). If $p \in R$ with $p \notin N(R)$, then $p \notin N(A)$ and so there exists $f = a_1 x^{\alpha_1} + \cdots + a_l x^{\alpha_l} \in N(A)$ such that $N_A(p) = fA$. If $f \in N(A)$, then $a_i \in N(R)$, for all $1 \le i \le l$ by Corollary 2.1.29. We may assume that $a_1 \neq 0$. Now, we show that $N_R(p) = a_1 R$. Since $a_1 \in N(R)$ and $N(R)$ is a right ideal of R, we have that $pa_1 R \subseteq N(R)$, whence $a_1 R \subseteq N_R(p)$. If $m \in N_R(p)$, then $m \in N_A(p)$, and thus there exists $g = b_1 x^{\beta_1} + \cdots + b_s x^{\beta_s} \in A$ such that

$$m = fg = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i x^{\alpha_i} b_j x^{\beta_j}\right) = \sum_{k=2}^{s+l}\left(\sum_{i+j=k} a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j}\right).$$

Hence, $m = a_1 b_1 \in a_1 R$ and so $N_R(p) \subseteq a_1 R$, whence $N_R(p) = a_1 R$ with $a_1 \in N(R)$. □

We present some examples that illustrate Theorems 3.2.2, 3.2.5, 3.2.8, 3.2.4, 3.2.7 and 3.2.10.

(i) Let $U_q'(\mathfrak{so}_3)$ be the $\mathbb{C}$-algebra generated by the indeterminates I_1, I_2, and I_3. This algebra is a skew PBW extension over $\Bbbk$, i.e. $U_q'(\mathfrak{so}_3) \cong \sigma(\mathbb{C})\langle I_1, I_2, I_3\rangle$ [FGL+20, Example 1.3.3]. Notice that $\mathbb{C}$ is a (Σ,Δ)-compatible NI ring with the identity endomorphism of $\mathbb{C}$ and the zero derivation. Since that $\text{ann}_{\mathbb{C}}(X) = N_{\mathbb{C}}(X) = \{0\}$, for all $X \nsubseteq N(\mathbb{C})$, Theorem 3.2.2 ensures that $N_{U_q'(\mathfrak{so}_3)}(U)$ is an ideal generated by a nilpotent element of $U_q'(\mathfrak{so}_3)$, for each subset $U \nsubseteq N(U_q'(\mathfrak{so}_3))$. By Theorem 3.2.5, we get that $N_{U_q'(\mathfrak{so}_3)}(fU_q'(\mathfrak{so}_3))$ is an ideal generated by a nilpotent element, for each principal right ideal $fU_q'(\mathfrak{so}_3) \nsubseteq N(U_q'(\mathfrak{so}_3))$. Theorem 3.2.8 shows that $N_{U_q'(\mathfrak{so}_3)}(f)$ is a generated by a nilpotent, for all $f \notin N(U_q'(\mathfrak{so}_3))$.

(ii) Using techniques such as those presented in [FGL+20, Theorem 1.3.1], it can be shown that AW(3) is a skew PBW extension of endomorphism type, that is, $\text{AW}(3) \cong \sigma(\mathbb{R})\langle K_0, K_1, K_2\rangle$. Since that $\mathbb{R}$ is reduced, we have that $\text{ann}_{\mathbb{R}}(X) = N_{\mathbb{R}}(X) = \{0\}$, for all $X \nsubseteq N(\mathbb{R})$. Hence, the characterization of the weak annihilators of $N_{\text{AW}(3)}(U)$, $N_{\text{AW}(3)}(f\ \text{AW}(3))$, and $N_{\text{AW}(3)}(f)$, for certain subsets, principal right ideals, and elements of algebra AW(3), follows from Theorems 3.2.2, 3.2.5, and 3.2.8, respectively. Furthermore, Theorems 3.2.4, 3.2.7 and 3.2.10 are also valid for the description of the weak annihilators mentioned above.

(iii) One can check that if a, b, and c are not zero simultaneously, then $Q(a, b, c)$ is neither an skew polynomial ring of $\Bbbk$, $\Bbbk[x]$ nor of $\Bbbk[y]$. On the other hand, if $b \neq 0$ and $c = 0$, then it can be seen that $Q(a, b, c)$ is a skew PBW extension over $\Bbbk[x]$, i.e. $Q(a, b, c) \cong \sigma(\Bbbk[x])\langle y\rangle$. Note that $\Bbbk[x]$ is (Σ, Δ)-compatible and reduced with the identity endomorphism of $\Bbbk[x]$ and the zero derivation. Since $\Bbbk[x]$ is reduced, we get that $\text{ann}_{\Bbbk[x]}(X) = N_{\Bbbk[x]}(X) = \{0\}$, for all $X \nsubseteq N(\Bbbk[x])$. Hence, the characterization of the weak annihilators of $N_{Q(a,b,c)}(U)$, $N_{Q(a,b,c)}(f\, Q(a, b, c))$, and $N_{Q(a,b,c)}(f)$, for any subset, principal right ideal, and element of algebra $Q(a, b, c)$, follows from Theorems 3.2.2, 3.2.5, and 3.2.8, respectively. Theorems 3.2.4, 3.2.7 and 3.2.10 are also valid for the description of these weak annihilators.

(iv) Having in mind the classical PBW theorem for the universal enveloping algebra $U(\mathfrak{so}(5, \mathbb{C}))$, and since $U(\mathfrak{so}(5, \mathbb{C}))$ is a PBW extension of $\mathbb{C}$ [BG88, Section 5], then $U(\mathfrak{so}(5, \mathbb{C}))$ is a skew PBW extension of $\mathbb{C}$, that is $U(\mathfrak{so}(5, \mathbb{C})) \cong \sigma(\mathbb{C})\langle \mathbf{J}_{\alpha\beta} \mid 1 \leq \alpha \leq \beta \leq 5\rangle$. The weak annihilator of certain subsets, principal right ideals, and elements of $U(\mathfrak{so}(5, \mathbb{C}))$ are described by Theorems 3.2.2, 3.2.5, and 3.2.8, respectively. Furthermore, Theorems 3.2.4, 3.2.7 and 3.2.10 hold since $U(\mathfrak{so}(5, \mathbb{C}))$ is a skew PBW extension of endomorphism type.

(v) It is well-known that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$, i.e. $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$ (e.g., [KD09, Section 18.2]). Thus, $P_F^{(n)}$ is a skew PBW extension over $\Bbbk$, that is, $P_F^{(n)} \cong \sigma(\Bbbk)\langle f_i^{\xi}, f_j^{\eta}\rangle$, for values $\xi, \eta = \pm 1$, and $1 \leq i, j \leq n$. Since $\Bbbk$ is a field (hence reduced), then $\text{ann}_{\Bbbk}(X) = N_{\Bbbk}(X) = \{0\}$, for all $X \nsubseteq N(\Bbbk)$. This implies that the characterization of the weak annihilators of $N_{P_F^{(n)}}(U)$, $N_{P_F^{(n)}}(f\, P_F^{(n)})$, and $N_{P_F^{(n)}}(f)$, for some subsets, principal right ideals, and elements of $P_F^{(n)}$, follow from Theorems 3.2.2, 3.2.5, and 3.2.8, respectively. Furthermore, Theorems 3.2.4, 3.2.7 and 3.2.10 are also valid for the description of these weak annihilators. We recall that these subsets, principal right ideals and elements, are not contained in the set of nilpotent elements of the algebra $P_F^{(n)}$. Similar to the case of parafermionic algebra, the parabosonic algebra is a skew PBW extension over $\Bbbk$, that is, $P_B^{(n)} \cong \sigma(\Bbbk)\langle b_i^{\xi}, b_j^{\eta}\rangle$ for values $\xi, \eta = \pm 1$, and $1 \leq i, j \leq n$. In this case, the results mentioned for the parafermionic algebras hold for parabasonic algebras.

(vi) Following to definition of skew bi-quadratic algebra (Section 1.2.3), it is easy to see that all such algebras can be seen as skew PBW extensions of the form $\sigma(R)\langle x_1, \ldots, x_n\rangle$. In this way, if the ring of coefficients is a (Σ, Δ)-compatible NI ring, then the characterization of the weak annihilators for subsets, principal right ideals, and elements of the algebra, follow from Theorems 3.2.2, 3.2.5, and 3.2.8. If, in addition, the ring of coefficients is a field, Theorems 3.2.4, 3.2.7 and 3.2.10 are also valid for the description of all of them.

3.3 Nilpotent associated primes of skew PBW extensions

In this section, we characterize the nilpotent associated prime ideals of skew PBW extensions.

Ouyang and Birkenmeier [OB12] defined the notions of *nilpotent degree* and *nilpotent good polynomial* to study the nilpotent associated prime ideals of skew polynomial rings. In this way, we present the following definitions in the setting of skew PBW extensions.

Definition 3.3.1. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R and consider an element $f \in A$ such that $f = r_1 x^{\alpha_1} + \cdots + r_k x^{\alpha_k} + \cdots + r_l x^{\alpha_l} \notin N(R)A$, where $x^{\alpha_l} \succ \cdots \succ x^{\alpha_k} \succ \cdots \succ x^{\alpha_1}$ and $r_l \neq 0$ is the leading coefficient.

(i) If $r_k \notin N(R)$ and $r_i \in N(R)$, for all $i > k$, then the *nilpotent degree* of f is k, which is denoted as $\mathrm{Ndeg}(f)$. If $f \in N(R)A$, we define $\mathrm{Ndeg}(f) = -1$.

(ii) If $\mathrm{Ndeg}(f) = k$ and $N_R(r_k) \subseteq N_R(r_i)$ for all $i \leq k$, then f is a *nilpotent good polynomial.*

The following result generalizes [OB12, Lemma 3.1] and [OL12, Lemma 2.16].

Theorem 3.3.2. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where R is a (Σ, Δ)-compatible and NI, then for any $f = r_1 x^{\alpha_1} + \cdots + r_k x^{\alpha_k} + \cdots + r_l x^{\alpha_l} \notin N(R)A$, there exists $r \in R$ such that fr is a nilpotent good polynomial.*

Proof. Assume that the result is false and let $f = r_1 x^{\alpha_1} + \cdots + r_k x^{\alpha_k} + \cdots + r_l x^{\alpha_l} \notin N(A)$ be a counterexample of minimal nilpotent degree $\mathrm{Ndeg}(f) = k$, that is, fr is not a nilpotent good polynomial. In particular, if $r = 1$, we have that f is not a nilpotent good polynomial. Thus, there exists $i < k$ such that $N_R(r_k) \nsubseteq N_r(r_i)$. In this way, we can find $b \in R$ such that $b \in N_R(r_k)$ and $b \notin N_R(r_i)$, whence $r_i b \notin N(R)$ and $r_k b \in N(R)$.

The degree k coefficient of fb is $r_k \sigma^{\alpha_k}(b) + \sum_{i=k+1}^{l} r_i p_{\alpha_i, b}$, and since R is (Σ, Δ)-compatible, we get that $r_k \sigma^{\alpha_k}(b) \in N(R)$. On the other hand, we have $\mathrm{Ndeg}(f) = k$, whence $r_i \in N(R)$, for all $i > k$. This means that $r_i p_{\alpha_i, b} \in N(R)$, for all $i > k$ and so $\sum_{i=k+1}^{l} r_i p_{\alpha_i, b} \in N(R)$ since $N(R)$ is a right ideal of R. Therefore, fb has nilpotent degree at most $k-1$. If $r_i b \notin N(R)$ then $fb \notin N(R)A$, and by the minimality of k, there exists $c \in R$ with fbc a nilpotent good polynomial. However, this contradicts the fact that f is a counterexample since $bc \in R$ and $f(bc)$ is a nilpotent good polynomial. This concludes the proof. $\square$

Theorem 3.3.3 characterizes the nilpotent associated primes ideals of a skew PBW extension. This result extends different assertions about associated prime ideals (e.g., [Ann04, Theorem 2.1], [NRR20, Theorem 3.1.2], [OB12, Theorem 3.1], and [OL12, Theorem 3.1 and Corollary 3.2]).

Theorem 3.3.3. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, where R is (Σ, Δ)-compatible and NI, then*

$$\mathrm{NAss}(A) = \{PA \mid P \in \mathrm{NAss}(R)\}.$$

Proof. If $P \in \mathrm{NAss}(R)$, then there exists a right ideal $I \nsubseteq N(R)$ with I a right quasi-prime ideal of R and $P = N_R(I)$. Let us show that $PA = N_A(IA)$ and also that IA is a quasi-prime ideal. We show first that $PA = N_A(IA)$. If $i = a_0 + a_1 X_1 + \cdots + a_m X_m \in IA$ and $f = b_0 + b_1 Y_1 + \cdots + b_l Y_l \in PA$, then

$$if = \sum_{k=0}^{m+l} \left(\sum_{i+j=k} a_i X_i b_j Y_j \right) = \sum_{k=0}^{m+l} \left(\sum_{i+j=k} a_i \sigma^{\alpha_i}(b_j) X_i Y_j + a_i p_{\alpha_i, b_j} Y_j \right).$$

Since $a_i b_j \in N(R)$ for every i, j, from [HKA19, Proposition 3.3] it follows that $a_i \sigma^{\alpha_i}(b_j) \in N(R)$, Furthermore, the polynomial p_{α_i, b_j} involves elements obtained while evaluating σ's and δ's (depending on the coordinates of α_i) in the element b_j by Proposition 1.2.20. This means that $a_i p_{\alpha_i,b_j} \in N(R)$ for every i,j and thus, we obtain that $if \in N(R)A = N(A)$. Hence $PA \subseteq N_A(IA)$.

For the other inclusion, if $f = b_0 + b_1Y_1 + \cdots + b_mY_m \in N_A(IA)$ then $if \in N(A) = N(R)A$, for all $i = a_0 + a_1X_1 + \cdots + a_lX_l \in IA$, whence $a_i\sigma^{\alpha_i}(b_j) \in N(R)$, which implies that $a_ib_j \in N(R)$ [HKA19, Proposition 3.3]. Since $a_i \in I$, we have $b_j \in N_R(I) = P$. Therefore, we get that $f \in PA$ and thus $N_A(IA) \subseteq PA$. Hence, we conclude that $PA = N_A(IA)$.

Since I is a right quasi-prime ideal, we have that $I \nsubseteq N(R)$, which implies that $IA \nsubseteq N(A)$. Let us show that for any right ideal U of R if $U \nsubseteq N(A)$ and $U \subseteq IA$, then $N_A(U) = N_A(IA)$. Let us see first that $N_A(IA) \subseteq N_A(U)$. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in N_A(IA)$, this means that $if \in N(A)$, for all $i \in IA$. In particular, since $U \subseteq IA$ then $if \in N(A)$, for all $i \in U$, whence $f \in N_A(U)$. This shows that $N_A(IA) \subseteq N_A(U)$.

Conversely, let $C_U \subseteq R$ consisting of all coefficients of elements of U. Let us first consider P' the right ideal of R generated by C_U. Since $U \nsubseteq N(A) = N(R)A$, this means that $C_U \nsubseteq N(R)$, and hence $P' \subseteq I$ and $P' \nsubseteq N(R)$. As I is right quasi-prime, we have that $N_R(P') = N_R(I) = P$. Now, if $f = a_0 + a_1X_1 + \cdots + a_mX_m \in N_A(U)$ and $u = u_0 + u_1Y_1 + \cdots + u_tY_t \in U$, then $uf \in N(A)$, whence $u_i\sigma^{\beta_j}(a_j) \in N(R)$ and so $u_ia_j \in N(R)$, for all $0 \le i \le t$, $0 \le j \le m$. Since $N(R)$ is a right ideal of R, we get that $u_ia_j \in N(R)$ implies that $a_ju_i \in N(R)$. Thus, $(u_iRa_j)^2 \in N(R)$ which implies that $u_iRa_j \in N(R)$, and so $a_j \in N_R(P') = N_R(I) = P$, for all $0 \le j \le m$.

If $i = b_0 + b_1Z_1 + \cdots + b_rZ_r \in IA$, then $b_ma_j \in N(R)$, whence $b_m\sigma^{\alpha}(\delta^{\beta}(a_j))$ and $a_m\delta^{\beta}(\sigma^{\alpha}(a_j))$ are elements of $N(R)$, for all $\alpha, \beta \in \mathbb{N}^n$. Thus, $if \in N(R)A = N(A)$, which implies that $f \in N_A(IA)$. Hence $N_A(U) \subseteq N_A(IA)$ which proves that $PA = N_A(IA)$ and also that IA is a quasi-prime ideal.

Let $I \in \mathrm{NAss}(A)$. By Definition 2.2.4 (ii), there exists a right ideal $J \nsubseteq N(A)$ with J a right quasi-prime ideal of A and $I = N_A(J)$. Let $m = m_0 + m_1X_1 + \cdots + m_kX_k + \cdots + m_nX_n \notin N(A) = N(R)A$ and $m \in J$. Since $J \nsubseteq N(A)$, we may assume that m is nilpotent good and $\mathrm{Ndeg}(m) = k$, by Theorem 3.3.2. We consider $J_0 = mA$ the principal right ideal of A generated by m. Since $m \notin N(A) = N(R)A$ this implies that $J_0 = mA \nsubseteq N(R)A = N(A)$, whence $N_A(J) = N_A(J_0) = N_A(mA) = I$ because J is a quasi-prime ideal. Now, we consider the right ideal m_kR, and let us denote $U = N_R(m_kR)$.

Let us prove first that $I = UA$. Let $g = b_0 + b_1Y_1 + \cdots + b_lY_l \in UA$. Since $b_j \in U$, then $m_kRb_j \in N(R)$ for all $0 \le j \le l$. Furthermore, m is nilpotent good polynomial and $\mathrm{Ndeg}(m) = k$, hence $m_iRb_j \in N(R)$, for all $0 \le i \le k$, and $0 \le j \le l$. On the other hand, for all $i > k$, $m_i \in N(R)$. Thus, we get $m_iRb_j \in N(R)$, for all $0 \le i \le n$ and $0 \le j \le l$. Now, for any element $h \in A$ where $h = h_0 + h_1Z_1 + \cdots + h_pZ_p$, we have that $m_ih_db_j \in N(R)$, for all $0 \le i \le n$, $0 \le d \le p$ and $0 \le j \le l$. Therefore, by (Σ, Δ)-compatibility of R, we obtain $mhg \in N(A)$. This implies that $g \in N_A(mA) = I$, and so $UA \subseteq I$. Conversely, let $g = b_0 + b_1Y_1 + \cdots + b_lY_l \in I$. Since $mRg \in N(A)$, it follows that $m_iRb_j \in N(R)$, for all $0 \le i \le n$, and $0 \le j \le l$. Thus, we have that $b_j \in N_R(m_kR)$, for all $0 \le j \le l$, and so $g \in UA$. Hence, we conclude $I \subseteq UA$ which implies that $I = UA$.

Now, let m_kR be the principal right ideal of A generated by m_k. The idea is to show that m_kR is a quasi-prime ideal. Since $m_k \notin N(R)$, we have $m_kR \nsubseteq N(R)$. The idea is to show that for any right ideal $Q \subseteq m_kR$ with $Q \nsubseteq N(R)$, it follows that $N_R(Q) = N_R(m_kR)$. Assume that a right ideal $Q \subseteq m_kR$, and $Q \nsubseteq N(R)$. Then $N_R(m_kR) \subseteq N_R(Q)$ by Proposition 2.2.3. Now, we show that $N_R(Q) \subseteq N_R(m_kR)$. Let W be the following set $W = \{mr \mid r \in Q\}$, and let WA be the right ideal of A generated by W.

First, notice that $WA \subseteq mA$. Since $Q \nsubseteq N(R)$, there exists $a \in R$ such that $m_k a \in Q$ and $m_k a \notin N(R)$. If $m_k(m_k a) \in N(R)$, then we have $m_k a \in N(R)$ which contradicts to the fact that $m_k a \notin N(R)$. Thus $m_k(m_k a) \notin N(R)$, and hence $m(m_k a) \notin N(A)$, by Proposition 2.1.32. This implies that $WA \nsubseteq N(A)$. Since J is a quasi-prime ideal, we obtain $N_A(WA) = N_A(mA) = I$.

Suppose $q \in N_R(Q)$. Then $rq \in N(R)$, for each $r \in Q$. Now, for any $mrf \in WA$ where $f = a_0 + a_1 Y_1 + \cdots + a_l Y_l \in A$, the term of mrf is $m_i X_i r a_j Y_j$. The idea is to show that $m_i X_i r a_j Y_j \in N(R)$. Since $rq \in N(R)$ and $N(R)$ is an ideal, it follows that

$$rq \in N(R) \Rightarrow qr \in N(R) \Rightarrow ra_j(qr)a_j q \in N(R) \Rightarrow ra_j q \in N(R).$$

If $ra_j q \in N(R)$, then $m_i r a_j q \in N(R)$. Thus, due to the (Σ, Δ)-compatibility of R, we have that $m_i X_i r a_j Y_j q \in N(R)A$ which implies that $mrfq \in N(R)A = N(A)$. Hence, for any $\sum mr_i f_i \in WA$ it follows that $\sum(mr_i f_i)q \in N(A)$. Therefore, $q \in N_A(WA) = I = UA$, and so $q \in U = N_R(m_k R)$. Thus, $N_R(Q) \subseteq N_R(m_k R)$, and this implies that $N_R(Q) = N_R(m_k R)$. Hence $m_k R$ is quasi-prime ideal. □

Example 3.3.4. Consider Example 2.1.24. It is not difficult to see that R satisfies the conditions of Theorem 3.3.3, and so $\mathrm{NAss}(R[x;\overline{\sigma},\overline{\delta}]) = \mathrm{NAss}(R)[x;\overline{\sigma},\overline{\delta}]$. In addition, the right ideals of R are the following:

$$I_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} R, \qquad I_2 = \begin{pmatrix} 0 & 0 \\ 0 & q(t) \end{pmatrix} R, \qquad I_3 = \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} R,$$

$$I_4 = \begin{pmatrix} 0 & p(t) \\ 0 & q(t) \end{pmatrix} R, \quad I_5 = \begin{pmatrix} q(t) & p(t) \\ 0 & 0 \end{pmatrix} R, \quad I_6 = \begin{pmatrix} q(t) & p(t) \\ 0 & s(t) \end{pmatrix} R,$$

for $p(t), q(t) \in \Bbbk[t]$. We can observe that the only right quasi-prime ideals of R are I_2, I_4, I_5, and I_6. Furthermore, $N_R(I_2) = N_R(I_4) = N_R(I_5) = N_R(I_6) = I_3$, where $I_3 = N(R)$. Thus, it follows that $\mathrm{NAss}(R) = \{N(R)\} = \{I_3\}$, and therefore $\mathrm{NAss}(R[x;\overline{\sigma},\overline{\delta}]) = N(R)[x;\overline{\sigma},\overline{\delta}] = I_3[x;\overline{\sigma},\overline{\delta}]$.

We present more examples that illustrate Theorems 3.2.2 and 3.3.3.

Example 3.3.5. (i) Thinking about nilpotent good polynomials and nilpotent associated prime ideals, Theorem 3.3.2 allows us to conclude that the algebra $U'_q(\mathfrak{so}_3)$ has nilpotent good polynomials, while Theorem 3.3.3 describes the nilpotent associated primes ideals for these algebras.

(ii) Theorem 3.3.2 guarantees the existence of nilpotent good polynomials, and Theorem 3.3.3 describes the nilpotent associated prime ideals of the algebra AW(3). In particular, since that $\mathbb{R}$ is a field (hence reduced), the only nilpotent associated prime ideal is the zero ideal. All the theorems mentioned above are valid for AW(3) if we change $\mathbb{R}$ to any reduced ring R by considering e^{ω} as a non-zero element of R and $e^{-\omega}$ as its multiplicative inverse.

(iii) The existence of nilpotent good polynomials and the characterization of the nilpotent associated prime ideals of the algebra $Q(a,b,c)$, follows from Theorems 3.3.2 and 3.3.3, respectively. Notice that if $a=0$, then one can check that $Q(a,b,c)$ is a skew PBW extension over $\Bbbk[y]$, i.e., $Q(a,b,c) \cong \sigma(\Bbbk[y])\langle x\rangle$ and all the previous results are satisfied too. Similarly, if we change $\Bbbk$ to a reduced ring R, all results are still valid for $Q(a,b,c)$.

(iv) Since $U(\mathfrak{so}(5,\mathbb{C}))$ is a skew PBW extension of endomorphism type, Theorem 3.3.2 asserts the existence of nilpotent good polynomials and Theorem 3.3.3 describes the nilpotent associated prime ideals of $U(\mathfrak{so}(5,\mathbb{C}))$

(v) Since the algebra $P_F^{(n)}$ can be seen as a skew PBW extension of a (Σ,Δ)-compatible and NI ring, Theorem 3.3.2 ensures the existence of nilpotent good polynomials and Theorem 3.3.3 describes the nilpotent associated prime ideals of $P_F^{(n)}$. The results mentioned for the parafermionic algebras hold for parabasonic algebras.

(vi) About the theory of nilpotent associated primes, Theorem 3.3.2 establishes the existence of nilpotent good polynomials, and Theorem 3.3.3 describes the nilpotent associated primes of a skew bi-quadratic algebra with ring of coefficients (Σ,Δ)-compatible and NI.

The compatibility condition is not superfluous in Theorem 3.3.3. The next example show that if R is not a (Σ,Δ)-compatible ring, then this result 3.3.3 can be failed.

Example 3.3.6 ([HHK00, p. 3796])**.** Let $\mathbb{Z}_2[t]$ be the commutative polynomial ring over $\mathbb{Z}_2$, $\langle t^2\rangle$ the ideal of $\mathbb{Z}_2[t]$ generated by t^2, $R := \mathbb{Z}_2[t]/\langle t^2\rangle$, δ a derivation of R defined by $\delta(\overline{t}) = 1$ where $\overline{t} = t+\langle t^2\rangle$ and $R[x;\delta] = \mathbb{Z}_2[t]/\langle t^2\rangle[x;\delta]$ the corresponding skew polynomial ring. Since $t^2 = 0$ but $t\delta(t) \neq 0$, the δ-compatibility condition fails here. If we set $e_{11} = tx$, $e_{12} = t$, $e_{21} = tx^2+x$ and $e_{22} = 1+tx$ in $R[x;\delta]$, then they from a system of matrix units in $R[x;\delta]$. The centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So, $\langle\overline{t}\rangle$ is a nilpotent associated prime ideal of R, but $\langle\overline{t}\rangle[x;\delta]$ is not a nilpotent associated prime ideal of $R[x;\delta]$.

3.4 Associated prime ideals of induced modules

Leroy and Matczuk [LM04] used the good polynomials to study the uniform dimension and the associated primes of induced modules over skew polynomial rings. In this way, we introduce the following definition thinking about skew PBW extensions.

Definition 3.4.1. If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension of R, and $m = \sum_{i=1}^{k} m_i x^{\alpha_i}$ is a polynomial of $M\langle X\rangle_A$ with leading coefficient $m_k \neq 0$, then we say that m is *good* if for all $r\in R$, $\mathrm{lm}(mr) = \mathrm{lm}(m)$, as long as $mr \neq 0$.

If σ is an automorphism of R, then M_σ denotes the σ*-twisted module* defined on the same additive structure $M_\sigma = M$, where $m\cdot r := m\sigma(r)$, for all $r\in R$ [LM04, p. 2747]. With this in mind, we consider the following definition. If $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ is a finite set of automorphisms of R and $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, then M_{σ^α} denotes the σ^α*-twisted module* where the action of R over M_{σ^α} is defined by $m\cdot r := m\sigma^\alpha(r) = m\sigma_1^{\alpha_1}\circ\cdots\circ\sigma_n^{\alpha_n}(r)$, for all $r\in R$. If $m\in M_R$, we denote by $\langle m\rangle_{\sigma^\alpha}$

the σ^{α}*-twisted module generated* by m. If X is a subset of R and $\alpha \in \mathbb{N}^n$, then $\sigma^{-\alpha}(X)$ denotes the *inverse image* of X under σ^{α}, that is, if $r \in \sigma^{-\alpha}(X)$, then $\sigma^{\alpha}(r) \in X$.

Lemma 3.4.2. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R. If $m = \sum_{i=1}^{k} m_i x^{\alpha_i}$ is a polynomial of $M\langle X\rangle_A$ with leading coefficient $m_k \neq 0$, then the following statements are equivalent:*

(1) *m is a good polynomial.*

(2) *For all $f \in mR_R$, we have that* $\text{lm}(f) \succeq x^{\alpha_k}$.

(3) *For all $f \in mA_A$, we have that* $\text{lm}(f) \succeq x^{\alpha_k}$.

(4) *For any $r \in R$, we have that $m_k\sigma^{\alpha_k}(r) = 0$ if and only if $mr = 0$.*

(5) $\text{ann}_R(m) = \sigma^{-\alpha_k}(\text{ann}_R(m_k))$.

(6) $\text{ann}_A(m) = \sigma^{-\alpha_k}(\text{ann}_R(m_k))A$.

(7) *$mA \cong \langle m_k\rangle_{\sigma^{\alpha_k}} A$ as A-modules.*

Proof. (1) $\Rightarrow$ (2) If $f \in mR_R$, then $f = mr$, for some $r \in R$. If we assume that $\text{lm}(f) \prec x^{\alpha_k}$, then $\text{lm}(mr) \prec \text{lm}(m)$, which contradicts that m is a good polynomial. Hence $x^{\alpha_k} \preceq \text{lm}(f)$, for all $f \in mR_R$.

(2) $\Rightarrow$ (3) Suppose that $x^{\alpha_k} \preceq \text{lm}(f)$, for every $f \in mR_R$. It is clear that $\text{lm}(m) \preceq \text{lm}(mx^{\alpha})$, for all $\alpha \in \mathbb{N}^n$. Thus, if $f = mg$, for some $g \in A$, then $x^{\alpha_k} = \text{lm}(m) \prec \text{lm}(mg)$, whence $x^{\alpha_k} \preceq \text{lm}(f)$, for all $f \in mA_A$.

(3) $\Rightarrow$ (4) Suppose that $m_k\sigma^{\alpha_k}(r) = 0$. If $mr \neq 0$, then mr is a non-zero element of mA_A such that $\text{lm}(mr) \prec \text{lm}(m) = x^{\alpha_k}$, which is a contradiction. Thus, we have $mr = 0$. For the other implication, if $mr = 0$ then the leading coefficient of mr is zero, that is, $m_k\sigma^{\alpha_k}(r) = 0$ as desired.

(4) $\Rightarrow$ (5) If $r \in \text{ann}_R(m)$ then $mr = 0$. By statement (4) we have that $m_k\sigma^{\alpha_k}(r) = 0$, whence $\sigma^{\alpha_k}(r) \in \text{ann}(m_k)$. Hence, $r \in \sigma^{-\alpha_k}(\text{ann}(m_k))$, and so $\text{ann}_R(m) \subseteq \sigma^{-\alpha_k}(\text{ann}_R(m_k))$. For the other inclusion if $r \in \sigma^{-\alpha_k}(\text{ann}_R(m_k))$, then $\sigma^{\alpha_k}(r) \in \text{ann}(m_k)$, and thus $m_k\sigma^{\alpha_k}(r) = 0$. Therefore $mr = 0$ by hypothesis, and we get that $\sigma^{-\alpha_k}(\text{ann}_R(m_k)) \subseteq \text{ann}_R(m)$.

(5) $\Rightarrow$ (6) Assume that (5) holds, and let $g = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t} \in \text{ann}_A(m)$. If $mg = 0$, then $m_k\sigma^{\alpha_k}(b_t) = 0$, and so $b_t \in \text{ann}_R(m)$ by hypothesis. Thus, $m\left(b_1x^{\beta_1} + \cdots + b_{t-1}x^{\beta_{t-1}}\right) = 0$ whence $m_k\sigma^{\alpha_k}(b_{t-1}) = 0$, and so $b_{t-1} \in \text{ann}_R(m)$. Continuing this process, we have $b_i \in \sigma^{-\alpha_k}(\text{ann}_R(m_k))$, for every $1 \leq i \leq t$, and so $g \in \sigma^{-\alpha_k}(\text{ann}_R(m_k))A$. This proves the inclusion $\text{ann}_A(m) \subseteq \sigma^{-\alpha_k}(\text{ann}_R(m_k))A$. The other inclusion is clear.

(6) $\Rightarrow$ (7) Assume that (6) holds. If ϕ is the A-module homomorphism of A over mA defined by $\phi(f) = mf$ for all $f \in A$, then the kernel of ϕ is $\text{ann}_A(m)$, whence $mA \cong A/\text{ann}_A(m)$ by the first isomorphism theorem for A-modules. Moreover, if $\text{ann}_A(m) = \sigma^{-\alpha_k}(\text{ann}_R(m_k))A$ by assumption, then we get the isomorphism $mA \cong A/\sigma^{-\alpha_k}(\text{ann}_R(m_k))A$. Let φ be the

map of $R/\sigma^{-\alpha_k}(\text{ann}_R(m_k)) \times A$ over $A/\sigma^{-\alpha_k}(\text{ann}_R(m_k))A$ defined by $\varphi(\overline{r}, f) := \overline{rf}$. It is not difficult to see that φ is bilinear and due to the universal property of the tensorial product, there exists $\overline{\varphi}$ of $R/\sigma^{-\alpha_k}(\text{ann}_R(m_k)) \otimes_R A$ over $A/\sigma^{-\alpha_k}(\text{ann}_R(m_k))A$ give by $\overline{\varphi}(\overline{r} \otimes f) := \overline{rf}$ with inverse $\overline{\varphi}^{-1}$ of $A/\sigma^{-\alpha_k}(\text{ann}_R(m_k))A$ over $R/\sigma^{-\alpha_k}(\text{ann}_R(m_k)) \otimes_R A$ defined by $\overline{\varphi}^{-1}(\overline{f}) := \overline{1} \otimes f$. Thus, it follows that $A/\sigma^{-\alpha_k}(\text{ann}_R(m_k))A \cong R/\sigma^{-\alpha_k}(\text{ann}_R(m_k)) \otimes_R A$. Since $R/\sigma^{-\alpha_k}(\text{ann}_R(m_k))$ is isomorphic to the R-module $\langle m_k \rangle_{\sigma^{\alpha_k}}$, we have $mA \cong \langle m_k \rangle_{\sigma^{\alpha_k}} A$, where the isomorphism ϕ is defined as $\phi(mg) := m_k \sigma^{\alpha_k}(b_1)x^{\beta_1} + \cdots + m_k \sigma^{\alpha_k}(b_t)x^{\beta_t}$, for every $g = b_1 x^{\beta_1} + \cdots + b_t x^{\beta_t} \in A$.

(7) ⇒ (1) Let ϕ be the isomorphism defined above. If $\text{lm}(mr) \prec \text{lm}(m)$ for some $r \in R$, then $m_k \sigma^{\alpha_k}(r) = 0$. Thus, $\phi(mr) = m_k \sigma^{\alpha_k}(r) = 0$, and since ϕ is injective, we get that $mr = 0$. Hence, m is a good polynomial.

□

The following lemma characterizes the right annihilators of good polynomials over $M\langle X \rangle_A$, and extends [LM04, Corollary 3.5].

Lemma 3.4.3. *Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a bijective skew PBW extension of R. If $m = \sum_{i=1}^{k} m_i x^{\alpha_i}$ is a polynomial of $M\langle X \rangle_A$ with leading coefficient $m_k \neq 0$, then:*

(1) *There exists $r \in R$ such that mr is good polynomial.*

(2) *If m is good polynomial, then $\text{ann}_A(m) = \text{ann}_R(m)A$.*

Proof. (1) Assume that the result is false and let $m \in M\langle X \rangle_A$ be a counterexample of minimal leading monomial, that is, mr is not a good polynomial, for every $r \in R$. In particular, if $r = 1$, then m is not a good polynomial, and thus there is $r \in R$ such that $\text{lm}(mr) \prec \text{lm}(m)$. If $mr \neq 0$, then by the minimality of $\text{lm}(m)$, there exists $c \in R$ with mrc a good polynomial. However, this contradicts the fact that mr is not a good polynomial for all $r \in R$.

(2) If m is good polynomial, by using equivalences (5) and (6) of Lemma 3.4.2, we have that $\text{ann}_A(m) = \sigma^{-\alpha_k}(\text{ann}_R(m_k))A = \text{ann}_R(m)A$.

□

A submodule N_R of M_R is *essential* if $mR_R \cap N_R \neq 0$, for all $0 \neq m \in M_R$. The set of all elements $m \in M_R$ such that $\text{ann}_R(m)$ is an essential ideal of R is called the *singular submodule* of M_R and is denoted by $Z(M_R)$; M_R is *nonsingular* if $Z(M_R) = 0$ [Lam98, Definition 3.26]. Leroy and Matczuk [LM04] proved that there exist good polynomials of any degree in a submodule of a nonsingular module. The following proposition extends [LM04, Proposition 3.6].

Proposition 3.4.4. *If $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a bijective skew PBW extension and $m = \sum_{i=1}^{k} m_i x^{\alpha_i}$ is a good polynomial of $M\langle X \rangle_A$ with leading coefficient $m_k \neq 0$ and the submodule $m_k R$ of M_R is nonsingular, then:*

(1) *$m_k A$ contains a good polynomial f with leading monomial x_i for all $1 \leq i \leq n$.*

(2) *There exists a good polynomial $f \in mA$ with leading monomial x^{α_l}, for all $x^{\alpha_l} \succeq x^{\alpha_k}$.*

Proof. (1) Let $g \in A$ with leading term $m_k x_i$ for some $1 \leq i \leq n$. Since $m_k R$ is nonsingular, there exists $0 \neq b \in R$ such that $\sigma_i(b)R \bigcap \text{ann}_R(m_k) = 0$. Note that the leading coefficient of the polynomial gb is $m_k\sigma_i(b)$. Now, consider $r \in R$ such that $\sigma_i(r) \in \text{ann}_R(m_k\sigma_i(b))$. In this way, $\sigma_i(b)\sigma_i(r) \in \sigma_i(b)R \bigcap \text{ann}_R(m_k) = 0$, and so $\sigma_i(br) = 0$. By the injectivity of σ_i, if $br = 0$, then $gbr = 0$. Thus $f = gb$ is a good polynomial with leading monomial x_i.

(2) If m is a good polynomial, then there exists an isomorphism of A-modules ϕ of mA over $\langle m_k\rangle_{\sigma^{\alpha_k}} A$ such that $\phi(mg) := m_k\sigma^{\alpha_k}(b_1)x^{\beta_1} + \cdots + m_k\sigma^{\alpha_k}(b_j)x^{\beta_j}$, for every $g = b_1x^{\beta_1} + \cdots + b_jx^{\beta_j} \in A$, by Lemma 3.4.2 (7). Furthermore, we get that $\sigma^{-\alpha_k}(\text{ann}_R(m_k)) = \text{ann}_R(m)$, by Lemma 3.4.2 (5), which implies that the leading monomial of $\phi(mg)$ is x^{β_j} if and only if the leading monomial of mg is $x^{\alpha_k+\beta_j}$. Therefore, $mg \in mA$ is a good polynomial if and only if $m_k\sigma^{\alpha_k}(g) \in \langle m_k\rangle_{\sigma^{\alpha_k}} A$ is a good polynomial by Lemma 3.4.2 (2).

It is not difficult to see that, if $m_k R$ is nonsingular, then $\langle m_k\rangle_{\sigma^{\alpha_k}}$ also is. By part (1), there exists $g' = \phi(mg) \in \langle m_k\rangle_{\sigma^{\alpha_k}} A$ a good polynomial such that the leading monomial of g' is x_1. Thus, mg' is a good polynomial of mA with leading monomial x^β with $\beta \in \mathbb{N}^n$, where $\beta_1 = \alpha_{k1} + 1$ and $\beta_i = \alpha_{ki}$, for all $2 \leq i \leq n$. Since the leading coefficient of mg' belongs to $\langle m_k R\rangle_{\sigma^{\alpha_k}}$, its leading coefficient satisfies the hypothesis of the theorem. Thus, there exists $mg'' \in mA$ with leading monomial x^β where $\beta_1 = \alpha_{k1} + 2$ and $\beta_i = \alpha_{ki}$, for every $2 \leq i \leq n$. Following this argument, in at most $|\alpha_{l1} - \alpha_{k1}|$ steps, we find a good polynomial mg_1 with leading monomial x^β where $\beta_1 = \alpha_{l1}$ and $\beta_i = \alpha_{ki}$, for all $2 \leq i \leq n$. The idea is to continue with x_2. By part (1), there exists $g_1' \in \langle m_k\rangle_{\sigma^{\alpha_k}} A$ a good polynomial where the leading monomial of g_1' is x_2. Thus, mg_1' is a good polynomial of mA with leading monomial x^β with $\beta \in \mathbb{N}^n$, where $\beta_1 = \alpha_{l1}$, $\beta_2 = \alpha_{k2} + 1$ and $\beta_i = \alpha_{ki}$, for all $3 \leq i \leq n$. Repeating the process, in at most $|\alpha_{l2} - \alpha_{k2}|$ steps, we find a good polynomial mg_2 with leading monomial x^β where $\beta_i = \alpha_{li}$ for $i = 1, 2$ and $\beta_i = \alpha_{ki}$, for all $3 \leq i \leq n$. In this way, in at most $n \cdot \max\{|\alpha_{li} - \alpha_{ki}|\}$ steps, we have a good polynomial f with $\text{lm}(f) = x^{\alpha_l}$.

□

Let $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ be a finite set of endomorphisms of R and $\Delta := \{\delta_1, \ldots, \delta_n\}$ a finite set of Σ-derivations of R. If I is a two-sided ideal of R, then I is called Σ-*invariant* if $\sigma_i(I) \subseteq I$, for every $1 \leq i \leq n$; I is a Δ-*invariant* ideal if $\delta_i(I) \subseteq I$, for all $1 \leq i \leq n$; if I is both Σ and Δ-invariant, we say that I is (Σ, Δ)-*invariant* [LAR15, Definition 2.1]. These ideals have been widely studied in the literature [LAR15, LR20b, NR20, Rey14].

Leroy and Matczuk introduced some ideals associated with an ideal I of R, an endomorphism σ of R, and a σ-derivation δ of R (cf. [LM04, p. 2746]). Following their ideas, we consider some ideals with the aim of studying properties of induced modules over skew PBW extensions.

Definition 3.4.5. Let $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ be a finite set of endomorphisms of R and $\Delta := \{\delta_1, \ldots, \delta_n\}$ a finite set of Σ-derivations of R. If I is a two-sided ideal of R, then

$$I_\Sigma := \{a \in I \mid \sigma^\alpha(a) \in I, \text{ for all } \alpha \in \mathbb{N}^n\},$$

$$I_\Delta := \{a \in I \mid \delta^\beta(a) \in I, \text{ for all } \beta \in \mathbb{N}^n\},$$

$$I_{\Sigma,\Delta} := \{a \in I \mid \sigma^\alpha\delta^\beta(a) \in I, \text{ for all } \alpha, \beta \in \mathbb{N}^n\}.$$

Notice that I is Σ-invariant if and only if $I = I_\Sigma$; I is Δ-invariant if and only if $I = I_\Delta$, and I is a (Σ,Δ)-invariant ideal if and only if $I = I_{\Sigma,\Delta}$.

The following lemma extends [LM04, Lemma 3.1].

Lemma 3.4.6. *If $\Sigma := \{\sigma_1,\dots,\sigma_n\}$ is a finite set of endomorphisms of R and $\Delta := \{\delta_1,\dots,\delta_n\}$ is a finite set of Σ-derivations of R, then:*

(1) $I_{\Sigma,\Delta} \subseteq I_\Sigma \cap I_\Delta$.

(2) *If either I_Σ is Δ-invariant or I_Δ is Σ-invariant, then $I_{\Sigma,\Delta} = I_\Sigma$.*

(3) *If $I = \text{ann}_R(N)$ is an ideal Δ-invariant, then:*

  (a) $\text{ann}_R(N\langle X\rangle) = I_{\Sigma,\Delta}$.

  (b) *If $\text{ann}_A(N\langle X\rangle) = JA$ for some ideal J, then $J = I_{\Sigma,\Delta} = \text{ann}_R(N\langle X\rangle)$.*

Proof. (1) If $a \in I_{\Sigma,\Delta}$, then $\sigma^\alpha\delta^\beta(a) \in I$ for every $\alpha,\beta \in \mathbb{N}^n$. Thus, $\sigma^\alpha(a) \in I$ which implies that $a \in I_\Sigma$. Similarly, if $\sigma^\alpha\delta^\beta(a) \in I$ for all $\alpha,\beta \in \mathbb{N}^n$, then $\delta^\beta(a) \in I$ for all $\beta \in \mathbb{N}^n$, which implies that $a \in I_\Delta$, and so $a \in I_\Sigma \cap I_\Delta$.

(2) Suppose that I_Σ is a Δ-invariant ideal. If $a \in I_\Sigma$ and I_Σ is Δ-invariant, then $\delta^\beta(a) \in I_\Sigma$, and hence $\sigma^\alpha\delta^\beta(a) \in I$, for all $\alpha,\beta \in \mathbb{N}^n$. Thus, $I_\Sigma \subseteq I_{\Sigma,\Delta}$. If I_Δ is Σ-invariant, then the argument is similar.

(3) Let N_R be a right module with $I = \text{ann}_R(N)$ an ideal Δ-invariant.

  (a) Let $n = n_1x^{\alpha_1} + \cdots + n_kx^{\alpha_k} \in N\langle X\rangle_A$. The coefficients of nr are products of n_i with elements obtained evaluating σ's and δ's in the element r, for all $r \in R$ [Rey15, Remark 2.10]. If $r \in I_{\Sigma,\Delta}$, then $nr = 0$, and so $I_{\Sigma,\Delta} \subseteq \text{ann}_R(N\langle X\rangle)$. For the other inclusion, if $r \in \text{ann}_R(N\langle X\rangle)$, then $nx^\alpha r = 0$, for any $\alpha \in \mathbb{N}^n$ and every $n \in N$. Notice that the leading coefficient of $nx^\alpha r$ is zero, that is, $n\sigma^\alpha(r) = 0$, for all α, which implies that $r \in I_\Sigma$. Since I is a Δ-invariant ideal, we have $r \in I_\Sigma = I_{\Sigma,\Delta}$ and so $\text{ann}_R(N\langle X\rangle) \subseteq I_{\Sigma,\Delta}$.

  (b) By item (a), we obtain that $I_{\Sigma,\Delta} \subseteq J$. For the other inclusion, if we have the equality $\text{ann}_A(N\langle X\rangle) = JA$, for some ideal J, then $nx^\alpha rx^\beta = 0$, for every $\alpha,\beta \in \mathbb{N}^n$, $r \in J$, and $n \in N$. In particular, if $\beta = 0$, then $nx^\alpha r = 0$ and $n\sigma^\alpha(r) = 0$, for any α, which implies that $r \in I_\Sigma$. Since I is Δ-invariant, we get $r \in I_\Sigma = I_{\Sigma,\Delta}$ which shows that the equality $J = I_{\Sigma,\Delta} = \text{ann}_R(N\langle X\rangle)$ follows.

□

The following lemma characterizes annihilators of generated submodules by good polynomials and extends [LM04, Lemma 3.7].

Lemma 3.4.7. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a bijective skew PBW extension of R and $m = \sum_{i=1}^k m_ix^{\alpha_i}$ a good polynomial of $M\langle X\rangle_A$ with leading coefficient $m_k \neq 0$. If $I := \sigma^{-\alpha_k}(\text{ann}_R(m_kR))$, then:*

(1) $\text{ann}_R(mA) \subseteq \text{ann}_A(mA) \subseteq \text{ann}_A(mR)$.

(2) $\text{ann}_R(mR) = I$ *and* $\text{ann}_A(mR) = IA$.

(3) $\text{ann}_R(mA) = I_{\Sigma,\Delta}$.

(4) *If* I_Σ *is* Δ*-invariant, then* $\text{ann}_A(mA) = I_\Sigma A$.

Proof. (1) It is easy to see that if $R \subseteq A$, then $\text{ann}_R(mA) \subseteq \text{ann}_A(mA)$. On the other hand if $f \in \text{ann}_A(mA)$, then $gAf = 0$ whence $gRf = 0$. This proves that $\text{ann}_A(mA) \subseteq \text{ann}_A(mR)$.

(2) Let us show that $\text{ann}_R(mR) = I$. If $r \in I$, then $m_k R\sigma^{\alpha_k}(r) = 0$, and since m is a good polynomial, we have that $mRr = 0$ and thus $r \in \text{ann}_R(mR)$. Now, if $r \in \text{ann}_R(mR)$, then $m_k R\sigma^{\alpha_k}(r) = 0$ and so $r \in \sigma^{-\alpha_k}(\text{ann}_R(m_k R)) = I$. Let us prove that $\text{ann}_A(mR) = IA$. If $f = a_1 x^{\beta_1} + \cdots + a_t x^{\beta_t} \in \text{ann}_A(mR)$, then $mRf = 0$ whence $m_k R\sigma^{\alpha_k}(a_t) = 0$. Since m is good, we get that $a_t \in \text{ann}_R(mR) = I$. If $mRa_t = 0$, then $mR(a_1 x^{\beta_1} + \cdots + a_{t-1} x^{\beta_{t-1}}) = 0$ which implies that $m_k R\sigma^{\alpha_k}(a_{t-1}) = 0$. If m is good then $mRa_{t-1} = 0$, and so $a_{t-1} \in \text{ann}_R(mR) = I$. Continuing this argument, we have that $a_i \in \text{ann}_R(mR) = I$, for every $1 \le i \le t$, and thus $f \in \text{ann}_R(mR)A = IA$. Hence, $\text{ann}_A(mR) \subseteq IA$. The reverse inclusion is clear.

(3) We have the R-module isomorphism $(mR)\langle X\rangle = mR \otimes_R A \cong mA$. By item (2), we get that $\text{ann}_R(mR) = I$ and hence $\text{ann}_R((mR)\langle X\rangle) = \text{ann}_R(mA) = I_{\Sigma,\Delta}$ by Lemma 3.4.6 (3)(a).

(4) Suppose that I_Σ is Δ-invariant. By Lemma 3.4.6 (2), we get that $I_\Sigma = I_{\Sigma,\Delta}$. By items (1) and (3), we obtain that $I_\Sigma A = \text{ann}_R(mA)A \subseteq \text{ann}_A(mA)$. Now, if $f = a_1 x^{\beta_1} + \cdots + a_t x^{\beta_t}$ is an element of $\text{ann}_A(mA)$, then $mRx^\beta f = 0$ whence $\sigma^\beta(a_t) \in \sigma^{-\alpha_k}(\text{ann}_R(m_k R)) = I$, for every $\beta \in \mathbb{N}^n$, and so $a_t \in I_\Sigma$. Since m is good, we get that $mRx^\beta a_t = 0$ for all $\beta \in \mathbb{N}^n$ whence $mAa_t = 0$. Then $mA(f - a_t x^{\beta_t}) = 0$ and $\text{lm}(f - a_t x^{\beta_t}) \prec \text{lm}(f)$. An inductive argument proves that $\text{ann}_A(mA) \subseteq I_\Sigma A$ and so the equality $\text{ann}_A(mA) = I_\Sigma A$ holds.

□

If σ is an automorphism of R and δ is a σ-derivation of R, then a right ideal I of R is called $(\sigma,\sigma^{-1},\delta)$*-stable* if $\sigma(I) = I$ and $\delta(I) \subseteq I$ [GL94, p. 6]. Annin [Ann02b] investigated properties of induced modules over skew polynomial rings under the assumption that for any $m \in M_R$, $I = \text{ann}_R(m)$ is a $(\sigma,\sigma^{-1},\delta)$-stable ideal. Leroy and Matczuk [LM04] studied these same modules using the following condition: M_R satisfies the *weak* (σ,δ)*-compatibility condition* if every submodule N_R of M_R contains an element $m \neq 0$ such that $\text{ann}_R(m)$ is $(\sigma,\sigma^{-1},\delta)$-stable [LM04, Definition 4.2]. Thinking about skew PBW extensions, we consider the following definition: if $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ is a family of endomorphisms of R and $\Delta = \{\delta_1,\dots,\delta_n\}$ is a family of Σ-derivations of R, then a right ideal I of R is called $(\Sigma,\Sigma^{-1},\Delta)$*-stable* if $\sigma_i(I) = I$ and $\delta_i(I) \subseteq I$, for all $1 \le i \le n$. This definition allows us to extend the compatibility condition as follows.

Definition 3.4.8. Let Σ be a finite set of endomorphisms of R and Δ a finite set of Σ-derivations of R. We say that M_R satisfies the *weak* (Σ,Δ)*-compatibility condition*, if every submodule N_R of M_R contains an element $a \neq 0$ such that $I := \text{ann}_R(a)$ is $(\Sigma,\Sigma^{-1},\Delta)$-stable.

The following lemma is analogous to [LM04, Lemma 4.3].

Lemma 3.4.9. *Let $\Gamma = \{\gamma_i, \dots, \gamma_n\}$ be a finite set of automorphisms of R, and $\Lambda = \{\lambda_i, \dots, \lambda_n\}$ a finite set of Γ-derivations of R. If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension of R and M_R satisfies the weak (Γ, Λ)-compatibility condition, then:*

(1) *For any good polynomial $m \in M\langle X\rangle_A$, there exists $r \in R$ such that mr is a good polynomial and the annihilator of its leading coefficient is $(\Gamma, \Gamma^{-1}, \Lambda)$-stable.*

(2) *Suppose that for the family of endomorphisms $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ we have $\sigma_i\gamma_j = \gamma_j\sigma_i$ for $1 \le i, j \le n$. If there exist some central invertible elements $q_{ij} \in R$ such that $\sigma_i\lambda_j = q_{ij}\lambda_j\sigma_i$ and γ_i and λ_i can be extended to A, then $M\langle X\rangle_A$ satisfies the weak (Γ, Λ)-compatibility condition.*

Proof. (1) Let $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k}$ be a non-zero good polynomial with leading coefficient $m_k \neq 0$. If M_R satisfies the weak (Γ, Λ)-compatibility condition, then there exists $r \in R$ such that $m_kr \neq 0$ and $I = \text{ann}_R(m_kr)$ is $(\Gamma, \Gamma^{-1}, \Lambda)$-stable. By Lemma 3.4.2, $m\sigma^{-\alpha_k}(r)$ is good with leading coefficient m_kr and I is the annihilator of its leading coefficient.

(2) Let $N\langle X\rangle$ be a submodule of $M\langle X\rangle_A$ and $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in N\langle X\rangle$ be a polynomial of minimal leading monomial in mA, that is, $x^{\alpha_k} \prec \text{lm}(f)$ for every $f \in mA$. By Lemma 3.4.2 and item (1), m is a good polynomial, and so $m\sigma^{-\alpha_k}(r)$ is also a good polynomial for some $r \in R$ with $I = \text{ann}_R(m_kr)$ $(\Gamma, \Gamma^{-1}, \Lambda)$-stable. Additionally, $\text{ann}_A(m) = \sigma^{-\alpha_k}(I)A$ by Lemma 3.4.2, and since $\sigma_i\gamma_j = \gamma_j\sigma_i$, $\sigma_i\lambda_j = q_{ij}\lambda_j\sigma_i$ for every $1 \le i, j \le n$ and some central invertible element $q_{ij} \in R$, and also λ_i can be extended to A, it follows that $\text{ann}_A(m) = \sigma^{-\alpha_k}(I)A$ is $(\Gamma, \Gamma^{-1}\Lambda)$-stable.

□

Leroy and Matczuk showed that the induced modules over skew polynomial rings have good polynomials of any degree [LM04, Example 3.3]. This fact motivated the following definition: a submodule B_S of $\widehat{M}_S$ is called *good*, if for any good polynomial $g \in B_S$ and any $n \ge \deg(g)$, there exists a good polynomial of degree n in gS [LM04, Definition 4.4]. Following this idea and with the purpose of studying properties of induced modules over skew PBW extensions, we introduce the following definition.

Definition 3.4.10. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of R. A submodule $N\langle X\rangle_A$ of $M\langle X\rangle_A$ is called *good*, if for any good polynomial $m = \sum_{i=1}^{k} m_ix^{\alpha_i} \in N\langle X\rangle_A$ and any monomial x^β with $\beta \in \mathbb{N}^n$, there exists a good polynomial $f \in mA$ such that $\text{lm}(f) = x^\beta \succeq x^{\alpha_k}$.

The following lemma generalizes [LM04, Lemma 4.5].

Lemma 3.4.11. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R. If one of the following conditions is satisfied,*

(1) *M_R is nonsingular.*

(2) *$M_R = R_R$ and for any non-zero element $r \in R$, there is a good polynomial $m \in rA$ with leading monomial x_i for all $1 \le i \le n$.*

(3) *A is a skew PBW extension of endomorphism type.*

(4) *M_R satisfies the weak (Σ, Δ)-compatibility condition.*

then $M\langle X\rangle_A$ is a good module.

Proof. (1) This statement is a direct consequence of Proposition 3.4.4.

(2) If $f = a_1x^{\alpha_1} + \cdots + a_kx^{\alpha_k}$ is a good polynomial of A with leading monomial x^{α_k}, then there exists an A-module isomorphism ϕ of fA over $\langle a_k\rangle_{\sigma^{\alpha_k}}A$ such that $\phi(fg) := a_k\sigma^{\alpha_k}(b_1)x^{\beta_1} + \cdots + a_k\sigma^{\alpha_k}(b_j)x^{\beta_j}$, for every $g \in A$ with leading monomial x^{β_j} for any $\beta_j \in \mathbb{N}^n$, by Lemma 3.4.2 (7). Furthermore, we have $\sigma^{-\alpha_k}(\text{ann}_R(a_k)) = \text{ann}_R(f)$, by Lemma 3.4.2 (5), which implies that the leading monomial of $\phi(fg)$ is x^{β_j} if and only if the leading monomial of fg is $x^{\alpha_k+\beta_j}$. Therefore, $fg \in fA$ is a good polynomial if and only if $a_k\sigma^{\alpha_k}(g) \in \langle a_k\rangle_{\sigma^{\alpha_k}}A$ is a good polynomial by Lemma 3.4.2 (2).

If x^β is a monomial such that $x^{\alpha_k} \preceq x^\beta$, for some $\beta \in \mathbb{N}^n$, we must find a good polynomial with leading monomial x^β. By assumption, if $a_k \in R$, there exists $g' = \phi(fg) \in \langle a_k\rangle_{\sigma^{\alpha_k}}A$ a good polynomial such that the leading monomial of g' is x_i, for all i. Following the same argument of Proposition 3.4.4, we find a good polynomial $\overline{f}$ of fA with leading monomial x^β in at most $n\cdot\max\{|\beta_i - \alpha_{ki}|\}$ steps, proving that A_A is a good module.

(3) Let $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in M\langle X\rangle_A$ be a good polynomial with leading coefficient $m_k \neq 0$. If $\text{lm}(mx^\alpha r) \prec \text{lm}(mx^\alpha)$ for some $r \in R$, then $m_k\sigma^{\alpha_k}(\sigma^\alpha(r)) = 0$. Thus, $\text{lm}(m\sigma^\alpha(r)) \prec \text{lm}(m)$, which contradicts that m is good polynomial. Hence, mx^α is also good polynomial for any $\alpha \in \mathbb{N}^n$.

(4) Let $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in M\langle X\rangle_A$ be a good polynomial with leading coefficient $m_k \neq 0$. We claim that mx^β is good, for any $\beta \in \mathbb{N}^n$. If $x^{\alpha_k+\beta}$ is the leading monomial of mx^β and $\text{lm}(mx^\beta r) \prec \text{lm}(mx^\beta)$, for some $r \in R$, then $m_k\sigma^{\alpha_k+\beta}(r) = 0$. By Lemma 3.4.9 (1), $\text{ann}_R(m_k)$ is $(\Sigma, \Sigma^{-1}, \Delta)$-stable, and so $r \in \text{ann}_R(m_k)$. Furthermore, if $\text{ann}_R(m_k)$ is $(\Sigma, \Sigma^{-1}, \Delta)$-stable, then $mx^\beta r = 0$ which proves that mx^β is a good polynomial for any $\beta \in \mathbb{N}^n$.

□

According to Leroy and Matczuk [LM04], it may happen that $\text{Ass}(M_R) \neq \emptyset$ but $\text{Ass}(N_R) = \emptyset$, for some non-zero submodule N_R of M_R [LM04, p. 2756]. For this reason, they worked with modules where $\text{Ass}(N_R)$ is not empty for all non-zero submodule N_R of M_R and introduced the following definition: M_R has *enough prime submodules* if any non-zero submodule N_R of M_R contains a prime submodule [LM04, Definition 5.1].

Lemma 3.4.12 shows that if M_R has enough primes, then any non-zero submodule of $M\langle X\rangle_A$ contains a good polynomial, and generalizes [LM04, Lemma 5.4].

Lemma 3.4.12. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R. If M_R has enough prime submodules, then any non-zero submodule N_A of $M\langle X\rangle_A$ contains a good polynomial m, with leading coefficient $m_k \neq 0$, such that m_kR is a prime submodule of M_R.*

Proof. Let N_A be a submodule of $M\langle X\rangle_A$ and $n = n_1x^{\alpha_1}+\cdots+n_kx^{\alpha_k} \in N_A$ a non-zero polynomial with $\mathrm{lc}(n) = n_k \neq 0$ and minimal leading monomial in nA, that is, $x^{\alpha_k} \prec \mathrm{lm}(f)$, for every $f \in nA$. By Lemma 3.4.2, n is a good polynomial, and since M_R contains enough prime submodules, n_kR contains a non-zero prime submodule m_kR where $m_k = n_kr \neq 0$, for some $r \in R$. Thus, the polynomial $m = n\sigma^{-\alpha_k}(r) \in N_A$ is good with leading coefficient m_k such that m_kR is a prime submodule of M_R. □

Leroy and Matczuk characterized certain right annihilators of generated modules on $R[x;\sigma,\delta]$ where δ is a σ-derivation *q-quantized* of R [LM04, Lemma 5.6]. Goodearl and Letzter [GL94] introduced the notion of q-quantized derivation in the following way: a σ-derivation δ of R is *q-quantized* if $\delta\sigma = q\sigma\delta$ where q is a central, invertible element of R such that $\sigma(q) = q$ and $\delta(q) = 0$. The ring $R[x;\sigma,\delta]$ is called a *q-skew polynomial ring* if δ is q-quantized [GL94, p. 10].

Definition 3.4.13. Let Σ be a finite set of endomorphisms of R. A finite set of Σ-derivations Δ of R is called *quantized* if there exists $(q_1,\dots,q_n) \in R^n$ such that $\delta_i\sigma_i = q_i\sigma_i\delta_i$, $\sigma_i(q_j) = q_j$ and $\delta_i(q_j) = 0$, for all $0 \le i, j \le n$ where q_i is a central and invertible element of R for all $1 \le i \le n$.

A skew PBW extension $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is called *quantized* if the family of Σ-derivations Δ defined in Proposition 1.2.15 is quantized. Lemma 3.4.14 characterizes the annihilators of generated modules by good polynomials and generalizes [LM04, Lemma 5.6].

Lemma 3.4.14. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a bijective skew PBW extension of R and $m = \sum_{i=1}^{k} m_ix^{\alpha_i}$ a good polynomial of $M\langle X\rangle_A$ with leading coefficient $m_k \neq 0$. If $P := \mathrm{ann}_R(m_kR)$, then:*

(1) *If P is $(\Sigma,\Sigma^{-1},\Delta)$-stable, then* $\mathrm{ann}_A(mA) = P\langle X\rangle$.

(2) *If A is quantized and P is (Σ,Σ^{-1})-stable, then* $\mathrm{ann}_A(mA) = P_\Delta\langle X\rangle$.

(3) *If A is quantized, m_kR is prime, and mA contains good polynomial of any monomial greater than x^{α_k}, then* $\mathrm{ann}_A(mA) = \sigma^{-\alpha_k}(P_{\Sigma,\Delta})\langle X\rangle$.

Proof. (1) Since P is Δ-stable, we have $\mathrm{ann}_A(mA) = P_\Sigma\langle X\rangle$ by Lemma 3.4.7 (4). Since (Σ,Σ^{-1})-stable, we have $P_\Sigma = P$, and so $\mathrm{ann}_A(mA) = P\langle X\rangle$.

(2) If $r \in P_\Delta$, then $\delta^\beta(r) \in P$, and so $\sigma^\alpha(\delta^\beta(r)) \in P$ for every $\alpha,\beta \in \mathbb{N}^n$ by the (Σ,Σ^{-1})-stability of P. If A is quantized, $\delta^\beta(\sigma^\alpha(r)) = r_{\alpha,\beta}\sigma^\alpha(\delta^\beta(r)) \in P$ for some $r_{\alpha,\beta} \in R$ whence P_Δ is a Σ-invariant ideal of R. Hence, $P_\Delta\langle X\rangle$ is a two-sided ideal of A and so $P_\Delta\langle X\rangle \subseteq \mathrm{ann}_A(mA)$. Let $f = b_1x^{\beta_1}+\cdots+b_tx^{\beta_t}$ be an element of $\mathrm{ann}_A(mA)$. By induction on the monomials, we show that $\delta^\theta(b_i) \in P$, for any $\theta \in \mathbb{N}^n$ and $1 \le i \le t$. Since m is a good polynomial and P is a (Σ,Σ^{-1})-stable ideal of R, $\mathrm{ann}_A(mR) = \sigma^{-\alpha_k}(P)\langle X\rangle = P\langle X\rangle$ by Lemma 3.4.2, and thus $b_i \in P$ for every $1 \le i \le t$. Assume that for any leading monomial x^γ with $x^\gamma \prec x^\theta$, we have $\delta^\gamma(b_i) \in P$. If $f \in \mathrm{ann}_A(mA)$ and $\mathrm{ann}_A(mR) = P\langle X\rangle_A$, then $mrx^\theta f = 0$. Since A is quantized, it follows that $x^\theta b_i = r_1x^{\theta_1}+\cdots+r_sx^{\theta_s}$ where each r_j is a finite sum of several evaluations of σ^{θ_j}'s and $\delta^{\theta-\theta_j}$'s in the element b_i, for every $1 \le j \le s$. Thus, if P is a (Σ,Σ^{-1})-stable ideal and $\delta^\gamma(b_i) \in P$ for any $\gamma \in \mathbb{N}^n$ with $x^\gamma \prec x^\theta$, then $mRr_ix^{\theta_j} = 0$ for all $1 \le j \le s$ where $\theta_j \neq 0$. In this way, we have $mrx^\theta b_i = mr\delta^\theta(b_i)$, and hence $mrx^\theta f = mr(\delta^\theta(b_1)x^{\beta_1}+\cdots+\delta^\theta(b_t)x^{\beta_t})$ which shows that $\delta^\theta(b_i) \in P$ for all $1 \le i \le t$, and therefore $f \in P_\Delta\langle X\rangle$.

(3) Let $f = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t} \in \text{ann}_A(mA)$. We prove by induction on the leading monomials that $\delta^{\theta}(b_i) \in \sigma^{-\alpha_k}(P_\Sigma)$, for all $\theta \in \mathbb{N}^n$ and $1 \leq i \leq t$. Since mA contains good polynomials of any monomial greater than x^{α_k}, there exists $f_{\gamma_i} \in mA$ with leading monomial x^{γ_i}, for each $x^{\gamma_i} \succeq x^{\alpha_k}$. Let m_{γ_i} be the leading coefficient of f_{γ_i} where $m_{\gamma_i} \in m_kR$. If m_kR is prime, then $m_{\gamma_i}R$ is prime, and thus $\text{ann}(m_{\gamma_i}R) = P$ for all γ_i. Since the f_{γ_i} are good polynomials, $\text{ann}_A(f_{\gamma_i}R) = \sigma^{-\gamma_i}(P)\langle X\rangle$ by Lemma 3.4.2. If E denotes the submodule of $M\langle X\rangle_R$ defined by $E = \sum_{\gamma_i \geq \alpha_k} f_{\gamma_i}R$, then $E \subseteq mA$ and $\text{ann}_A(mA) \subseteq \text{ann}_A(E) = \bigcap_{\gamma_i \geq \alpha_k} \sigma^{-\gamma_i}(P)\langle X\rangle = \sigma^{-\alpha_k}(P_\Sigma)\langle X\rangle$. In this way, if $f \in \text{ann}_A(mA)$, then $f \in \sigma^{-\alpha_k}(P_\Sigma)\langle X\rangle$ and thus $b_i \in \sigma^{-\alpha_k}(P_\Sigma)$.

Now, assume that for any leading monomial x^γ with $x^\gamma \prec x^\theta$, we have $\delta^\gamma(b_i) \in \sigma^{-\alpha_k}(P_\Sigma)$. If $Ex^\theta \subseteq mA$ and $f = b_1x^{\beta_1} + \cdots + b_tx^{\beta_t} \in \text{ann}_A(mA)$, then $Ex^\theta f = 0$. In addition if A is quantized, then $x^\theta b_i = r_1x^{\theta_1} + \cdots + r_sx^{\theta_s}$ where each r_j is a finite sum of several evaluations of σ^{θ_j}'s and $\delta^{\theta-\theta_j}$'s in the element b_i, for every $1 \leq j \leq s$. If $\delta^\gamma(b_i) \in \sigma^{-\alpha_k}(P_\Sigma)$ for any $\gamma \in \mathbb{N}^n$ with $x^\gamma \prec x^\theta$, then $Er_ix^{\theta_j} = 0$ for all $1 \leq j \leq s$ where $\theta_j \neq 0$. Thus $Ex^\theta b_i = E\delta^\theta(b_i)$, and hence $Ex^\theta f = E(\delta^\theta(b_1)x^{\beta_1} + \cdots + \delta^\theta(b_t)x^{\beta_t})$ whence $\delta^\theta(b_i) \in \sigma^{-\alpha_k}(P_\Sigma)$ for all $1 \leq i \leq t$. So $\text{ann}_A(mA) \subseteq (\sigma^{-\alpha_k}(P_\Sigma))_\Delta A$, and since A is quantized, it follows that $(\sigma^{-\alpha_k}(P_\Sigma))_\Delta = \sigma^{-\alpha_k}(P_{\Sigma,\Delta})$, and hence $\text{ann}_A(mA) \subseteq \sigma^{-\alpha_k}(P_{\Sigma,\Delta})$.

To prove the other inclusion, if $r \in \sigma^{-\alpha_k}(P_{\Sigma,\Delta})$, then $\sigma^{\alpha_k}(r) \in P_{\Sigma,\Delta}$, and thus $\sigma^{\alpha_k}(\sigma^\alpha\delta^\beta(r))$ for all $\alpha, \beta \in \mathbb{N}$. Since m is a good polynomial of leading monomial x^{α_k} and leading coefficient $m_k \neq 0$, then $mR\sigma^\alpha\delta^\beta(r) = 0$. This implies that $mRx^\gamma r = 0$, for any $\gamma \in \mathbb{N}^n$, and so $\sigma^{-\alpha_k}(P_{\Sigma,\Delta}) \subseteq \text{ann}_A(mA)$.

□

The following theorem characterizes the associated prime ideals of $M\langle X\rangle_A$ where M_R is a right module that contains enough prime submodules.

Theorem 3.4.15. *If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a bijective skew PBW extension of R, $Q \in \text{Ass}(M\langle X\rangle_A)$, and M_R contains enough prime submodule, then:*

(1) *If P is $(\Sigma, \Sigma^{-1}, \Delta)$-stable for every $P \in \text{Ass}(M_R)$, then $Q = P\langle X\rangle$ for some $P \in \text{Ass}(M_R)$.*

(2) *If A is quantized and P is (Σ, Σ^{-1})-stable for every $P \in \text{Ass}(M_R)$, then $Q = P_\Delta\langle X\rangle$ for some $P \in \text{Ass}(M_R)$.*

(3) *If A is quantized and the module $M\langle X\rangle_A$ is good, then $Q = P_{\Sigma,\Delta}\langle X\rangle$ for some $P \in \text{Ass}(M_R)$ and $P_{\Sigma,\Delta}$ is (Σ, Σ^{-1})-stable.*

Proof. (1) If N_A is a prime submodule of $M\langle X\rangle_A$ with $\text{ann}_A(N) = Q$, then by Lemma 3.4.12, there is a good polynomial $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in N_A$ with leading coefficient $m_k \neq 0$ such that m_kR is prime submodule of M_R. Since N_A is prime, it follows that mA is also prime and $\text{ann}_A(mA) = \text{ann}_A(N) = Q$. By Lemma 3.4.14 (1), we have that $Q = P\langle X\rangle$ with $P = \text{ann}_R(m_kR) \in \text{Ass}(M_R)$.

(2) In the same way, if N_A is a prime submodule of $M\langle X\rangle_A$ with $\text{ann}_A(N) = Q$, then there exists a good polynomial $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in N_A$ with leading coefficient $m_k \neq 0$ such that m_kR is prime submodule of M_R by Lemma 3.4.12. Since N_A is prime, then

mA is also prime and $\operatorname{ann}_A(mA) = \operatorname{ann}_A(N) = Q$. By Lemma 3.4.14 (2), $Q = P_\Delta\langle X\rangle$ where $P = \operatorname{ann}_R(m_kR) \in \operatorname{Ass}(M_R)$.

(3) If N_A is a prime submodule of $M\langle X\rangle_A$ such that $\operatorname{ann}_A(N) = Q$, then there exists a good polynomial $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in N_A$ with leading coefficient $m_k \neq 0$ by Lemma 3.4.12. If $M\langle X\rangle_A$ is good, mA contains a good polynomial m' with leading monomial x^β such that $x^{\alpha_k} \preceq x^\beta$. Additionally, mA is prime module which implies that $\operatorname{ann}_A(mA) = Q$, and by Lemma 3.4.14 (3), $\operatorname{ann}_A(m'A) = \sigma^{-\beta}(P_{\Sigma,\Delta})\langle X\rangle$ and $\operatorname{ann}_A(mA) = \sigma^{-\alpha_k}(P_{\Sigma,\Delta})\langle X\rangle$. This implies that $P_{\Sigma,\Delta}$ is a (Σ,Σ^{-1})-stable ideal, and hence $Q = \operatorname{ann}_A(mA) = P_{\Sigma,\Delta}\langle X\rangle$.

□

Corollary 3.4.16 ([LM04, Theorem 5.7])**.** *Assume that M_R contains enough prime submodule and $Q \in \operatorname{Ass}(\widehat{M}_S)$.*

(1) *If for every $P \in \operatorname{Ass}(M_R)$, $\sigma(P) = P$ and $\delta(P) \subseteq P$, then $Q = PS$ for some $P \in \operatorname{Ass}(M_R)$.*

(2) *If δ is q-quantized and $\sigma(P) = P$ for all $P \in \operatorname{Ass}(M_R)$, then $Q = P_\delta S$ for some $P \in \operatorname{Ass}(M_R)$.*

(3) *If δ is q-quantized and $\widehat{M}_S$ is a good module, then $Q = P_{\sigma,\delta}S$ for some $P \in \operatorname{Ass}(M_R)$ and $\sigma(P_{\sigma,\delta}) = P_{\sigma,\delta}$.*

Leroy and Matczuk presented an example where Corollary 3.4.16 fails if M_R does not have enough prime submodules, which shows that this hypothesis is not superfluous. Assume that M is a $\Bbbk$-linear space with basis $\{v_i\}_{i\in\mathbb{Z}}$ and $R = \Bbbk\langle X\rangle$ is the free $\Bbbk$-algebra on the set $X = \{x_i\}_{i\in\mathbb{Z}}$, then M has a module structure over R defined by $v_ix_k = v_{i+1}$ if $i \leq k$ and 0 otherwise. Let σ be the automorphism of R given by $\sigma(x_k) := x_{k+1}$ for any $k \in \mathbb{Z}$. They showed that $\operatorname{Ass}(M_R) = \emptyset$ and that $\widehat{M}_{R[t,\sigma]}$ is prime with $\operatorname{Ass}(\widehat{M}_{R[t,\sigma]}) = \{0\}$ [LM04, Example 5.8].

Theorem 3.4.17 shows when $M\langle X\rangle_A$ is prime and characterizes its associated prime ideals in terms of the associated primes of M_R.

Theorem 3.4.17. *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a bijective skew PBW extension of R and M_R is a prime module with $P = \operatorname{ann}_R(M)$, then:*

(1) *If P is $(\Sigma,\Sigma^{-1},\Delta)$-stable, then the induced module $M\langle X\rangle_A$ is prime with the associated prime ideal equal to $Q = P\langle X\rangle$.*

(2) *If A is quantized and P is (Σ,Σ^{-1})-stable, then $M\langle X\rangle_A$ is prime with the associated prime ideal equal to $Q = P_\Delta\langle X\rangle$.*

(3) *If A is quantized and the module $M\langle X\rangle_A$ is good, then $M\langle X\rangle_A$ is a prime module if and only if $P_{\Sigma,\Delta}$ is (Σ,Σ^{-1})-stable. If $M\langle X\rangle_A$ is prime, then its associated prime ideal is equal to $Q = P_{\Sigma,\Delta}\langle X\rangle$.*

Proof. (1) Let N_A be a submodule of $M\langle X\rangle_A$ and $m = m_1x^{\alpha_1} + \cdots + m_kx^{\alpha_k} \in N_A$ with minimal leading monomial in mA_A, that is $x^{\alpha_k} \preceq \operatorname{lm}(f)$ for all $f \in mA_A$. By Lemma 3.4.2 m is good and since M_R is prime, then $m_k R$ is a submodule prime of M_R with $P = \text{ann}_R(m_k R)$ by Lemma 3.4.12. Additionally, by Lemma 3.4.14 (1) $P\langle X\rangle = \text{ann}_A(mA)$ and thus

$$P\langle X\rangle \subseteq \text{ann}_A(M\langle X\rangle) \subseteq \text{ann}_A(N) \subseteq \text{ann}_A(mA) = P\langle X\rangle.$$

In this way, we have that $\text{ann}_A(N) = P\langle X\rangle$ for any submodule N_A of $M\langle X\rangle_A$, whence $M\langle X\rangle_A$ is a prime module with associated prime ideal $Q = P\langle X\rangle$.

(2) Following the same argument in (1) and Lemma 3.4.14 (2), $M\langle X\rangle_A$ is a prime module with associated prime ideal $Q = P_\Delta\langle X\rangle$.

(3) In the same way, the argument of (1) and Lemma 3.4.14 (3) show that $M\langle X\rangle_A$ is a prime module with associated prime ideal $Q = P_{\Sigma,\Delta}\langle X\rangle$ and $P_{\Sigma,\Delta}$ is (Σ,Σ^{-1})-stable.

□

Corollary 3.4.18 ([LM04, Theorem 5.10])**.** *If M_R is prime with $P = \text{ann}_R(M)$, then:*

(1) *Suppose that $\sigma(P) = P$ and $\delta(P) \subseteq P$. Then the induced module $\widehat{M}_S$ is prime with the associated prime ideal equal to $PS = Q$.*

(2) *Suppose that δ is a q-quantized σ-derivation and $\sigma(P) = P$. Then $\widehat{M}_S$ is prime with the associated prime ideal equal to $P_\delta S = Q$.*

(3) *Suppose that δ is a q-quantized σ-derivation and the module $\widehat{M}_S$ is good. Then $\widehat{M}_S$ is a prime module if and only if $\sigma(P_{\sigma,\delta}) = P_{\sigma,\delta}$. Moreover, if $\widehat{M}_S$ is prime, then its associated prime ideal is equal to $P_{\sigma,\delta}S = Q$.*

The relevance of Theorems 3.4.15 and 3.4.17 is appreciated when we extend their application to to algebraic structures more general than those considered by Leroy and Matczuk [LM04].

Example 3.4.19. (i) If A is the diffusion algebra, M_R is a right module where $R = \Bbbk[x_1,\dots,x_n]$ and M_R contains enough prime submodules, then the characterization of the associated prime ideals of $M\langle D_1,\dots,D_n\rangle_A$ follows from Theorem 3.4.15, and if M_R is prime with $P = \text{ann}_R(M)$, then $M\langle D_1,\dots,D_n\rangle_A$ is prime with associated prime ideal $P\langle D_1,\dots,D_n\rangle$ by Theorem 3.4.17.

(ii) If A is the enveloping algebra $U(\mathfrak{so}(5,\mathbb{C}))$ and and $M_\mathbb{C}$ contains enough prime submodules, then Theorem 3.4.15 describes the associated prime ideals of $M\langle \mathbf{J}_{\alpha\beta}\rangle_A$.

(iii) If A is the $\mathbb{C}$-algebra $U'_q(\mathfrak{so}_3)$ and $M_\mathbb{C}$ contains enough prime submodules then Theorem 3.4.15 describes the associated primes of $M\langle I_1,I_2,I_3\rangle_A$, and if $M_\mathbb{C}$ is a prime module with $P = \text{ann}_\mathbb{C}(M)$ then $M\langle I_1,I_2,I_3\rangle_A$ is a prime module with associated prime ideal $P\langle I_1,I_2,I_3\rangle$ by Theorem 3.4.17.

(iv) If A is the Askey-Wilson algebra $AW(3)$ and $M_\mathbb{R}$ is a right module that contains enough prime submodules then Theorem 3.4.15 characterizes the associated prime ideals of $M\langle K_0,K_1,K_2\rangle_A$, and if $M_\mathbb{R}$ is prime with $P = \text{ann}_\mathbb{R}(M)$ then $M\langle K_0,K_1,K_2\rangle_A$ is prime with associated prime ideal $P\langle K_0,K_1,K_2\rangle$ by Theorem 3.4.17.

3.5 Future work

It is said that R satisfies *Shoda's condition* if every homomorphism between any two left ideals of R can be extended to a homomorphism of R [Ike52, p. 203]. Ikeda and Nakayama [IN54] studied Shoda's condition and the *quasi-Frobenius rings*, and showed that if every homomorphism from a left ideal of R into R can be represented by the right multiplication of an element of R, then

$$\mathrm{ann}_R(I) + \mathrm{ann}_R(J) = \mathrm{ann}_R(I \cap J), \tag{3.5.1}$$

for all left ideals I and J of R [IN54, Theorem 1]. Following Camillo et al. [CNY00], R is called a *left Ikeda-Nakayama ring* (left IN-ring for short) if it satisfies (3.5.1). All left self-injective rings, all left uniserial rings and all left uniform domains are left IN-ring. As a generalization of IN-rings, Birkenmeier et al. [BGT15] introduced the *right SA-rings*. If for all right ideals I and I' of R there exists a right ideal J of R such that $\mathrm{ann}_R(I) + \mathrm{ann}_R(I') = \mathrm{ann}_R(J)$, then R is called a *right SA-ring*. They showed that this class of rings is exactly the class of rings for which the lattice of annihilator ideals is a sublattice of the lattice of ideals.

The following proposition shows some relations between IN rings and other families of rings.

Proposition 3.5.1. (1) [PHA21, Theorem 2.1] *If $R[x;\sigma]$ is a left IN-ring, then R is a left IN-ring and Armendariz.*

(2) [PHA21, Theorem 2.8] *If R is a σ-compatible reduced left IN-ring and R has finitely many prime ideals, then $R[x;\sigma]$ is a left IN-ring.*

Proposition 3.5.2 characterized skew polynomial rings $R[x;\sigma]$ that are right SA-rings.

Proposition 3.5.2 ([PHA21, Theorem 3.12])**.** *If R is a σ-compatible, then $R[x;\sigma]$ is right SA if and only if R is right SA and quasi-Armendariz.*

Considering the study of the properties of annihilators and the existence of quantum algebras that cannot be expressed as skew polynomial rings of endomorphism type, a more general result about the IN and SA conditions for different families of noncommutative rings can be considered as a contribution to the theoretical properties of these rings. In this way, we think as future work to investigate the right IN and right SA rings on skew polynomial rings with non-zero derivations and skew PBW extensions, among others.

Marubayashi et al. [HMU16] introduced the *Ore-Rees rings* with the aim of studying maximal orders on these rings. If R is a Noetherian prime with quotient ring Q and σ is an automorphism of R, then σ is naturally extended to an automorphism of Q by $\overline{\sigma}(rc^{-1}) = \sigma(r)\sigma(c)^{-1}$, where c is a regular element of R. Similarly, if δ is a σ-derivation of R, then δ is extended to a $\overline{\sigma}$-derivation of Q by $\overline{\delta}(c^{-1}) = -\overline{\sigma}(c^{-1})\delta(c)c^{-1}$. In this way, if $R[t;\sigma,\delta]$ is a skew polynomial ring where R is Noetherian prime and σ is an automorphism of R, then we can consider the ring $T = Q[t;\overline{\sigma},\overline{\delta}]$ with the extended automorphism and derivation. A *fractional ideal* of R is a submodule I_R of Q such that $Ir \subseteq R$, for some non-zero $r \in R$. A fractional ideal is invertible if there exists another fractional ideal such that $IJ = R$. If I is invertible then $I^{-1} := J$.

Definition 3.5.3 ([HMU16, p. 407])**.** If R is right Noetherian prime, σ is an automorphism of R, δ is a σ-derivation of R and X is an invertible ideal of R, then the following subsets of T are defined:

$$S = R[Xt;\sigma,\delta] = R\bigoplus Xt\bigoplus X^2t^2\bigoplus\cdots\bigoplus X^nt^n\bigoplus\cdots$$

and

$$S_1 = R\bigoplus tX\bigoplus t^2X^2\bigoplus\cdots\bigoplus t^nX^n\bigoplus\cdots$$

If S is a ring, then it is called an *Ore-Rees ring* associated to X. In this case, S and $R[t;\sigma,\delta]$ have the same quotient ring $Q(S) = Q(R[t;\sigma,\delta])$ which is a simple Artinian ring.

The following proposition shows that when S is a ring.

Proposition 3.5.4 ([HMU16, Lemma 2.2])**.** *S is a ring if and only if $\sigma(X) = X$ if and only if $\sigma^{-1}(X) = X$ if and only if S_1 is a ring. In this case, we have that $S = S_1$ and is Noetherian.*

Marubayashi et al. [HMU16] showed that if $\sigma(X) = X$ and A is an ideal of S, then AT is an ideal of T [HMU16, Lemma 2.3]. In addition, he proved that if I is an ideal of R, then $I[Xt;\sigma,\delta]$ is an ideal of S if and only if $X\sigma(I) = IX$ and $X\delta(I) \subseteq I$ [HMU16, Lemma 2.5], and if $I[Xt;\sigma,\delta]$ is an ideal of S then $I[Xt;\sigma,\delta] = S_1I$ [HMU16, Lemma 2.7]. They also investigated some properties of prime ideals of S. We recall that $\mathrm{Spec}(R)$ denotes the set of prime ideals of any ring R.

Proposition 3.5.5.

(1) [HMU16, Proposition 2.12] *There is a bijective correspondence between* $\mathrm{Spec}(T)$ *and the set*

$$\mathrm{Spec}_0(S) := \left\{P \in \mathrm{Spec}(S) \mid P\cap R = \langle 0\rangle\right\},$$

via $P \mapsto PT$ and $P' \mapsto P'\cap S$.

(2) [HMU16, Proposition 2.14] *If P is a prime ideal of S such that $(P\cap R)[Xt;\sigma,\delta]$ is an ideal of S, then $(P\cap R)[Xt;\sigma,\delta]$ is a prime ideal of S.*

Following Annin's ideas [Ann02a] about associated prime ideals, and those of Ouyang and Birkenmeier [OB12] concerning nilpotent associated prime ideals over skew polynomial rings, we think that a natural task is to investigate these ideals over Ore-rees rings.

CHAPTER 4

Attached prime ideals of some quantum algebras

In this chapter, we present original results concerning the attached prime ideals of inverse polynomial modules over skew Ore polynomials.

Section 4.1 establishes some preliminaries and key results about skew Ore polynomials. We introduce the completely (σ,δ)-compatible modules and present original results (Propositions 4.1.9, 4.1.8 and 4.1.10). Under compatibility conditions, we also characterize the attached prime ideals of $M[x^{-1}]_A$ where A is a skew Ore polynomial ring (Theorems 4.1.13 and 4.1.17).

Section 4.2 presents some ideas for a future work.

4.1 Inverse polynomial modules over skew Ore polynomials

Motivated by Annin's research [Ann11] about the attached primes of $M[x^{-1}]_S$, where $S = R[x;\sigma]$ and σ and automorphism of R, we introduce a kind of noncommutative rings called *skew Ore polynomials* and we study the attached prime ideals of the inverse polynomial module over these rings. Our results extend those above corresponding to skew polynomial rings of automorphism type presented by Annin [Ann11].

We recall that a derivation δ of R is *locally nilpotent* if for all $r \in R$ there exists $n(r) \geq 1$ such that $\delta^{n(r)}(r) = 0$ [Fre06, p. 11]. Following the ideas of Cohn [Coh61] and Smits [Smi68], we define the following kind of skew Ore polynomials of higher order.

Definition 4.1.1. If σ is an automorphism of R and δ is a locally nilpotent σ-derivation of R, we define the *skew Ore polynomial ring* $A := R(x;\sigma,\delta)$ which consists of the uniquely representable elements $r_0 + r_1x + \cdots + r_kx^k$ where $r_i \in R$ and $k \in \mathbb{N}$, with the relation $xr := \sigma(r)x + x\delta(r)x$ for all $r \in R$.

According to Definition (4.1.1), if $r \in R$ and $\delta^{n(r)}(r) = 0$ for some $n(r) \geq 1$, then

$$xr = \sigma(r)x + \sigma\delta(r)x^2 + \cdots + \sigma\delta^{n(r)-1}(r)x^{n(r)}. \tag{4.1.1}$$

If we define the endomorphisms $\Psi_i := \sigma\delta^{i-1}$ for all $i \geq 1$ and $\Psi_0 := 0$, then A is a skew Ore polynomial of higher order in the sense of Cohn [Coh61].

Example 4.1.2. We present some examples of skew Ore polynomial rings.

(i) If $\delta = 0$ then $xr = \sigma(r)x$ and thus $R(x;\sigma) = R[x;\sigma]$ is the skew polynomial ring where σ is an automorphism of R.

(ii) The *quantum plane* $\Bbbk_q[x,y]$ is the free algebra generated by x, y over $\Bbbk$, and subject to the commutation rule $xy = qyx$ with $q \in \Bbbk^*$ and $q \neq 1$. We note that $\Bbbk_q[x,y] \cong \Bbbk[y](x;\sigma)$ where $\sigma(y) := qy$ is an automorphism of $\Bbbk[y]$.

(iii) The *Jordan plane* $\mathscr{J}(\Bbbk)$ defined by Jordan [Jor01] is the free algebra generated by the indeterminates x, y over $\Bbbk$ and the relation $yx = xy + y^2$. This algebra can be a written as the skew polynomial ring $\Bbbk[y][x;\delta]$ with $\delta(y) := -y^2$. On the other hand, notice that $\delta(x) = 1$ is a locally nilpotent derivation of $\Bbbk[x]$, and thus the Jordan plane also can be interpreted as $\Bbbk[x](y;\delta)$. Recall that the quantum plane and Jordan plane are not isomorphic [Shi05, Theorem 1.4].

(iv) Díaz and Pariguan [DP09] defined the *q-meromorphic Weyl algebra MW_q* as the algebra generated by x, y over $\mathbb{C}$, and defining relation $yx = qxy + x^2$, for $0 < q < 1$. Lopes [Lop23] showed that using the generator $Y = y + (q-1)^{-1}x$ instead of y, it follows that $Yx = qxY$ and thus the algebra MW_q can be written as a quantum plane $\mathbb{C}_q[x,y]$ [Lop23, Example 3.1]. Following example (2), we conclude that MW_q is a skew Ore polynomial ring.

(v) Let $Q(0,b,c)$ be the algebra introduced by Golovashkin and Maksimov [GM05] with $a = 0$. It is straightforward to see that $\sigma(x) = bx$ is an automorphism of $\Bbbk[x]$ with $b \neq 0$, $\delta(x) = c$ is a locally nilpotent σ-derivation of $\Bbbk[x]$ and so $Q(0,b,c)$ can be seen as $A = \Bbbk[x](y;\sigma,\delta)$.

(vi) If δ_1 is an automorphism of D and $\{\delta_2,\ldots,\delta_k\}$ is a set left D-independent, then $\delta := \delta_1^{-1}\delta_2$ is a δ_1-derivation of D, $\delta_{i+1}(r) = \delta_1\delta^i(r)$, and $\delta^k(r) = 0$ for all $r \in D$ [Smi68, p. 214], and thus (1.2.8) coincides with (4.1.1). In this way, the skew Ore polynomial rings of higher order defined by Smits can be seen as $D(x;\delta_1,\delta)$.

Proposition 4.1.3 shows that the set containing all powers of x satisfies the left Ore condition.

Proposition 4.1.3. *$X = \{x^k \mid k \geq 0\}$ is a left Ore set of the algebra A.*

Proof. It is clear that X is a multiplicative subset of A, so we have to show that X satisfies the left Ore condition. Let $a = r_0 + r_1x + \cdots + r_kx^k$ be an element of A with $r_k \neq 0$. Since δ is locally nilpotent, for each r_i in the expression of a there exists $m_i \geq 0$ such that

$$xr_i := \sum_{j=1}^{m_i} \sigma(\delta^{j-1}(r_i))x^j = a_ix,$$

where $a_i := \sigma(r_i) + \sigma(\delta(r_i))x + \cdots + \sigma(\delta^{m_i-1}(r_i))x^{m_i-1} \in A$. In this way, for each r_i there exists $a_i \in A$ such that $xr_i = a_ix$ for some $a_i \in A$, and so $xa = a'x$ for some $a' \in A$. By induction on p, assume that for any $a \in A$ and $x^p \in X$ there exists $\overline{a} \in A$ such that $x^pa = \overline{a}x$. Thus, $x^{p+1}a = x\overline{a}x$

and since $xa = \overline{a}'x$ for some $\overline{a}' \in A$, we get that $x^{p+1}a = a''x$ with $a'' = \overline{a}'x \in A$. Hence, X is a left Ore set of A. □

By Proposition 4.1.3, we can localize A by X, and so we denote this localization by $X^{-1}A$. It is straightforward to see that the indeterminate x^{-1} satisfies the relation $x^{-1}r := \sigma'(r)x^{-1} + \delta'(r)$, for all $r \in R$ with $\sigma'(r) := \sigma^{-1}(r)$ and $\delta'(r) := -\delta\sigma^{-1}(r)$. As a consequence of Definition 4.1.1 and Proposition 4.1.3, if σ is an automorphism of D and δ is a locally nilpotent σ-derivation of D then $X^{-1}A \subseteq D((x^{-1};\sigma^{-1},-\delta\sigma^{-1}))$ (see [AD95, DM23, Dum92] for more details).

Remark 4.1.4. According to Lam et al. [LLM97, p. 2468], if σ is an automorphism of R and δ is a σ-derivation of R, then we denote by f_j^i the endomorphism of R which is the sum of all possible words in σ', δ' built with i letters σ' and $j-i$ letters δ', for $i \leq j$. In particular, $f_0^0 = 1$, $f_j^j = \sigma'^j$, $f_j^0 = \delta'^j$, and $f_j^{j-1} = \sigma'^{j-1}\delta' + \sigma'^{j-2}\delta'\sigma' + \cdots + \delta'\sigma'^{j-1}$; if $\delta\sigma = \sigma\delta$, then $f_j^i = \binom{j}{i}\sigma'^i\delta'^{j-i}$. If $r \in R$ and $k \in \mathbb{N}$, then the following formula holds:

$$x^{-k}r = \sum_{i=0}^{k} f_k^i(r)x^{-i}. \tag{4.1.2}$$

In addition, if $r, s \in R$ and $k, k' \in \mathbb{N}$ then

$$(rx^{-k})(sx^{-k'}) = \sum_{i=0}^{k} r f_k^i(s)x^{-(k+k')}. \tag{4.1.3}$$

Taking into account the usual addition of polynomials and the product induced by (4.1.2) and (4.1.3), we define the ring of polynomials in the indeterminate x^{-1} with coefficients in R and denote it by $R[x^{-1}]$. The *inverse polynomial module* $M[x^{-1}]_R$ is defined as the set of all polynomials of the form $f(x) = m_0 + \cdots + m_k x^{-k}$ with $m_i \in M_R$ for all $0 \leq i \leq n$, the usual addition of polynomials and the action of R over any monomial mx^{-k} is given by (4.1.2) as follows:

$$mx^{-k}r := \sum_{i=0}^{k} m f_k^i(r)x^{-i}, \text{ for all } m \in M_R \text{ and } r \in R. \tag{4.1.4}$$

Remark 4.1.5. We use expressions as $m(x) = m_0 + m_1x^{-1} + \cdots + m_kx^{-k} \in M[x^{-1}]_R$. With this notation, we define the *leading monomial* of $m(x)$ as $\text{lm}(m(x)) := x^{-k}$, the *leading coefficient* of $m(x)$ by $\text{lc}(m(x)) := m_k$, and the *leading term* of $m(x)$ as $\text{lt}(m(x)) := m_kx^{-k}$. The *negative degree* of x^{-k} is defined by $\deg(x^{-k}) := -k$ for any $k \in \mathbb{N}$, and $\deg(m(x)) := \max\{\deg(x^{-i})\}_{i=0}^{k}$ for all $m(x) \in M[x^{-1}]_R$. For any $m(x) \in M[x^{-1}]_R$, we denote by C_m the set of all coefficients of $m(x)$.

We consider the *completely* (σ,δ)*-compatible modules* with the aim of characterizing the attached prime ideals of $M[x^{-1}]_A$.

Definition 4.1.6. If σ is an endomorphism of R and δ is a σ-derivation of R then M_R is called *completely* σ*-compatible* if for all submodule N_R of M_R, $(M/N)_R$ is a σ-compatible module; M_R is *completely* δ*-compatible* if for every submodule N_R of M_R, $(M/N)_R$ is δ-compatible. If it is both completely σ-compatible and δ-compatible, then M_R is called *completely* (σ,δ)*-compatible*.

Example 4.1.7. (i) If M_R is simple and (σ,δ)-compatible, then it is not difficult to see that M_R is completely (σ,δ)-compatible.

(ii) Let K be a local ring with maximal ideal $\mathfrak{m}$ and σ an automorphism of K. Annin [Ann02a] proved that $M_K := K/\mathfrak{m}$ is a σ-compatible module, and since M_K is simple it follows that M_K is completely σ-compatible [Ann02a, Example 3.35]. Additionally, if δ is a σ-derivation of K such that $\delta(r) \in \mathfrak{m}$ for every $r \in \mathfrak{m}$, then M_K is completely δ-compatible. Indeed, if $\overline{0} \neq \overline{s} \in M_K$ and $r \in K$ satisfy that $\overline{s}r = 0$ then $sr \in \mathfrak{m}$, and since $s \notin \mathfrak{m}$ we obtain that $r \in \mathfrak{m}$. If $\delta(r) \in \mathfrak{m}$ for every $r \in \mathfrak{m}$, then $s\delta(r) \in \mathfrak{m}$ and so $\overline{s}\delta(r) = 0$. Therefore, M_K is δ-compatible and thus M_K is completely δ-compatible.

Proposition 4.1.8 presents some properties of completely (σ,δ)-compatible modules.

Proposition 4.1.8. *If M_R is completely (σ,δ)-compatible and N_R is a submodule of M_R then the following assertions hold:*

(1) *If $ma \in N_R$ then $m\sigma^i(a), m\delta^j(a) \in N_R$ for each $i, j \in \mathbb{N}$.*

(2) *If $mab \in N_R$ then $m\sigma(\delta^j(a))\delta(b), m\sigma^i(\delta(a))\delta^j(b) \in N_R$ for all $i, j \in \mathbb{N}$.*

(3) *If $mab \in N_R$ then $ma\delta^j(b), m\delta^j(a)b \in N_R$ for all $j \in \mathbb{N}$.*

(4) *If $mab \in N_R$ or $m\sigma(a)b \in N_R$ then $m\delta(a)b \in N_R$.*

Proof. If M_R is completely (σ,δ)-compatible, then $(M/N)_R$ is (σ,δ)-compatible. Considering the elements $\overline{ma} = \overline{mab} = \overline{0} \in (M/N)_R$, the assertions follow from [AM12, Lemma 2.15]. □

Annin [Ann11] proved some properties of completely σ-compatible modules [Ann11, p. 539]. Proposition 4.1.9 extends these statements for completely (σ,δ)-compatible modules.

Proposition 4.1.9. *Let σ be an endomorphism of R and δ a σ-derivation of R.*

(1) *If M_R is completely (σ,δ)-compatible then M_R is (σ,δ)-compatible.*

(2) *If M_R is a completely (σ,δ)-compatible module then $(M/N)_R$ is completely (σ,δ)-compatible, for every submodule N_R of M_R.*

Proof. (1) If M_R is a completely (σ,δ)-compatible module then $(M/N)_R$ is a (σ,δ)-compatible module, for every submodule N_R of M_R. In particular, $(M/\{0\})_R \cong M_R$ is (σ,δ)-compatible for the submodule $\{0\}_R$ of M_R.

(2) Suppose that M_R is completely (σ,δ)-compatible and consider a submodule N'/N of $(M/N)_R$ where N'_R is a submodule of M_R with $N \subsetneq N'$. By the third isomorphism theorem for modules, $((M/N)/(N'/N))_R \cong (M/N')_R$, and since M_R is completely (σ,δ)-compatible, we have that $(M/N')_R$ is (σ,δ)-compatible and thus $((M/N)/(N'/N))_R$ is (σ,δ)-compatible, whence $(M/N)_R$ is a completely (σ,δ)-compatible module.

□

We present other important property of completely (σ,δ)-compatible modules.

Proposition 4.1.10. *If σ is bijective and M_R is a completely (σ,δ)-compatible module, then M_R is a completely (σ',δ')-compatible module.*

Proof. If M_R is completely (σ,δ)-compatible and N_R is a submodule of M_R, we get that $mr \in N_R$ if and only if $m\sigma(r) \in N_R$, for all $m \in M_R$ and $r \in R$. If σ is bijective, then $m\sigma^{-1}(r) \in N_R$ if and only if $mr \in N_R$, and thus $(M/N)_R$ is a σ'-compatible module proving that M_R is completely σ'-compatible. In addition, if M_R is completely σ'-compatible then $mr \in N_R$ which implies that $m\sigma^{-1}(r) \in N_R$. If M_R is completely δ-compatible and $m\sigma^{-1}(r) \in N_R$, then $m\delta\sigma^{-1}(r)$ and thus M_R is completely δ'-compatible. Therefore M_R is completely (σ',δ')-compatible. □

Now, we define an A-module structure to the inverse polynomial module $M[x^{-1}]_R$ and study the attached prime ideals of $M[x^{-1}]_A$. The action of A over $M[x^{-1}]_R$ is given by

$$mx^{-1}r := m\sigma'(r)x^{-1} + m\delta'(r) \text{ for all } r \in R \text{ and } m \in M_R, \text{ and} \tag{4.1.5}$$

$$x^{-i}x^{j} := x^{-i+j} \text{ if } j \leq i \text{ and 0 otherwise.} \tag{4.1.6}$$

Remark 4.1.11. By (4.1.5) and (4.1.6), if $\delta := 0$ then $mx^{-i}rx^{j} := m\sigma'^{i}(r)x^{-i+j}$ for all $r \in R$ and $i, j \in \mathbb{N}$ with $j \leq i$, which coincides with $M[x^{-1}]_S$ [Ann11, p. 538].

If $N[x^{-1}]_R$ (resp., $N[x^{-1}]_A$) is a right module, the right annihilator is denoted by $\text{ann}_R(N[x^{-1}])$ (resp., $\text{ann}_A(N[x^{-1}])$). The following lemma characterizes the ideals generated by right prime ideals of A that correspond to annihilators of quotient modules of $M[x^{-1}]_A$.

Lemma 4.1.12. *If M_R is completely (σ,δ)-compatible and P is a right prime ideal of R such that $P = \text{ann}_R(M/N)$ for some submodule N_R of M_R, then*

$$PA = \text{ann}_A(M[x^{-1}]/N[x^{-1}]).$$

Proof. By Propositions 4.1.8 and 4.1.10, if M_R is completely (σ,δ)-compatible then $C_{mf} \subseteq N$ for all $m(x) \in M[x^{-1}]_A$, $f(x) \in PA$, and thus $m(x)f(x) \in N[x^{-1}]_A$ which shows that PA is a subset of $\text{ann}_A(M[x^{-1}]/N[x^{-1}])$. Now, if $f(x) \notin PA$ then there exists a monomial $r_l x^l$ of $f(x)$ such that $r_l \notin P$. So there exists $m \in M_R$ such that $mr_l \notin N_R$, which implies that $mf(x) \notin N[x^{-1}]_A$ and so $f(x) \notin \text{ann}_A(M[x^{-1}]/N[x^{-1}])$ whence $\text{ann}_A(M[x^{-1}]/N[x^{-1}]) \subseteq PA$. □

Theorem 4.1.13 shows that right ideals of A generated by attached prime ideals of M_R are attached prime ideals of $M[x^{-1}]_A$, and extends [Ann11, Theorem 2.1].

Theorem 4.1.13. *If M_R is completely (σ,δ)-compatible then*

$$\text{Att}(M[x^{-1}]_A) \supseteq \{PA \mid P \in \text{Att}(M_R)\}.$$

Proof. If P is an attached prime ideal of M_R and $(M/N)_R$ is the quotient coprime of M_R such that $P = \text{ann}_R(M/N)$ for some submodule N_R of M_R, it follows that $PA = \text{ann}_A(M[x^{-1}]/N[x^{-1}])$ by Lemma 4.1.12. Let us prove that $(M[x^{-1}]/N[x^{-1}])_A$ is a quotient coprime. If $M[x^{-1}]/N[x^{-1}] \neq 0$

then there exists a submodule Q_A of $M[x^{-1}]_A$ such that $Q \supsetneq N[x^{-1}]$. Let C_Q be the subset of M that consists of all coefficients of the elements of Q and consider Q'_R the submodule of M_R generated by C_Q. Notice that if $Q \neq M[x^{-1}]$ then $Q' \neq M$, and if $(M/N)_R$ is a coprime module then $P = \mathrm{ann}_R(M/N) = \mathrm{ann}_R(M/Q')$.

If $g(x) = r_0 + \cdots + r_j x^j \in \mathrm{ann}_A(M[x^{-1}]/Q)$ then $f(x)g(x) \in Q_A$ and hence $C_{fg} \subseteq Q'_R$ for all $f(x) = m_0 + m_1 x^{-1} + \cdots + m_k x^{-k} \in M[x^{-1}]_A$. By Propositions 4.1.8 and 4.1.10, if M_R is completely (σ,δ)-compatible then $m_i r_j \in Q'_R$, whence $m_i r_j \in N_R$ for all i, j. Since $m_i r_j \in N_R$, we have that $m(x) f(x) \in N[x^{-1}]_A$ by Propositions 4.1.8 and 4.1.10, and thus $M[x^{-1}] f(x) \subseteq N[x^{-1}]$. Therefore $f(x) \in \mathrm{ann}_A(M[x^{-1}]/N[x^{-1}])$ proving that $\mathrm{ann}_A(M[x^{-1}]/Q) \subseteq \mathrm{ann}_A(M[x^{-1}]/N[x^{-1}])$.

If $f \in \mathrm{ann}_A(M[x^{-1}]/N[x^{-1}])$ then $M[x^{-1}] f \subseteq N[x^{-1}]$, and since $N[x^{-1}] \subsetneq Q$ we have that $M[x^{-1}] f \subsetneq Q$ showing that $\mathrm{ann}_A(M[x^{-1}]/N[x^{-1}]) \subseteq \mathrm{ann}_A(M[x^{-1}]/Q)$, and so $(M[x^{-1}]/N[x^{-1}])_A$ is a coprime module. □

Corollary 4.1.14 ([Ann11, Theorem 2.1])**.** *If M_R is completely σ-compatible then*

$$\mathrm{Att}(M[x^{-1}]_S) \supseteq \{P[x] \mid P \in \mathrm{Att}(M_R)\}.$$

Under compatibility conditions, the following lemma proves that a coprime module over A is a coprime module over R, and extends [Ann11, Lemma 2.4].

Lemma 4.1.15. *If P_A is a coprime module and P_R is a completely (σ,δ)-compatible module then P_R is coprime.*

Proof. Let Q_R be a submodule of P_R. If $r \in \mathrm{ann}_R(P)$ then $Pr = 0 \in Q_R$, and so $r \in \mathrm{ann}_R(P/Q)$. For the other inclusion, assume that $r \in \mathrm{ann}_R(P/Q)$ and $N_A := \sum Q'_A$ where Q'_A is any submodule over A with $Q' \subseteq Q$. So $N_A \subseteq Q_R \subsetneq P_A$, and if P_A is coprime then $\mathrm{ann}_A(P) = \mathrm{ann}_A(P/N)$. Let $p \in P_A$ and denotes by prA_A the module generated by pr. Let us prove that $prA \subseteq N$. Since $x^{-1}r := \sigma'(r)x^{-1} + \delta'(r)$, we have that $rx = x\sigma'(r) + x\delta'(r)x$ for all $r \in R$, and thus if $f(x) = r_0 + \cdots + r_l x^l \in A$ and $Prr_j \subseteq Q$ for every $0 \leq j \leq l$, then $prf(x) \in Q_A$ by Propositions 4.1.8 and 4.1.10. Hence $prA \subseteq Q$ and so $prA \subseteq N$ by definition of N_A. In this way, $Pr \subseteq N$ which implies that $r \in \mathrm{ann}_A(P/N) = \mathrm{ann}_A(P)$ proving that $\mathrm{ann}_R(P) = \mathrm{ann}_R(P/Q)$ for every proper submodule Q_R of P_R, that is, P_R is a coprime module. □

If P_R is a submodule of $M[x^{-1}]_R$, we set $P_k := \{m \in M \mid mx^{-k} \in P\}$ for each $k \in \mathbb{N}$ and denote by $\langle P_k \rangle$ the submodule of M_R generated by P_k. Lemma 4.1.16 guarantees the existence of certain maximal submodules of M_R and generalizes [Ann11, Lemma 2.5].

Lemma 4.1.16. *If P_R is a maximal submodule of $M[x^{-1}]_R$, we either have $\langle P_k \rangle = M$ or else $\langle P_k \rangle$ is a maximal submodule of M_R for each $k \in \mathbb{N}$. Additionally, there exists $k \in \mathbb{N}$ for which the latter holds.*

Proof. Assume that there exists a submodule M'_R such that $\langle P_k \rangle \subsetneq M' \subseteq M$ and let us see that $M' = M$. If $m' \in M'_R$ and $m \notin \langle P_k \rangle$ then $m'x^{-k} \notin P_R$, and since P_R is a maximal submodule of $M[x^{-1}]_R$ we obtain that $M[x^{-1}]_R = P_R + m'x^{-k}R_R$, that is, for every $f(x) \in M[x^{-1}]_R$ there exist $p \in P_R$ and $r \in R$ such that $f(x) = p + m'x^{-k}r$. If $m'x^{-k}r = m\sigma'^k(r)x^{-k} + mp_{k,r}$ where

$p_{k,r} := \sum_{i=0}^{k-1} f_k^i(r)x^{-i}$, consider the element $f'(x) = mx^{-k} + mp_{k,r}$. So $f'(x) = p + m'x^{-k}r$ for some $p \in P_R$ which implies that $p = (m - m'\sigma'^k(r))x^{-k} \in P_R$, whence $m - m'\sigma'^k(r) \in \langle P_k \rangle \subseteq M'$ and thus $m \in M'_R$. Therefore $M \subseteq M'$ and so $M' = M$ proving that $\langle P_k \rangle$ is a maximal submodule of M_R. If $\langle P_k \rangle = M$ for all $k \in \mathbb{N}$, then $P = M[x^{-1}]$ which is a contradiction. In this way, there is $k \in \mathbb{N}$ such that $\langle P_k \rangle$ is a maximal submodule of M_R. □

Under compatibility conditions, the following theorem characterizes the attached prime ideals of $M[x^{-1}]_A$ and extends [Ann11, Theorem 3.2].

Theorem 4.1.17. *If $M[x^{-1}]_R$ is a completely (σ,δ)-compatible Bass module then*

$$\mathrm{Att}(M[x^{-1}]_A) = \{PA \mid P \in \mathrm{Att}(M_R)\}.$$

Proof. In view of Theorem 4.1.13, we only need to prove that

$$\mathrm{Att}(M[x^{-1}]_A) \subseteq \{PA \mid P \in \mathrm{Att}(M_R)\}.$$

Let $I \in \mathrm{Att}(M[x^{-1}]_A)$ and Q_A be a submodule of $M[x^{-1}]_A$ such that $(M[x^{-1}]/Q)_A$ is a coprime module with $I = \mathrm{ann}_A(M[x^{-1}]/Q)$. It is clear that $I \cap R$ is equal to $\mathrm{ann}_R(M[x^{-1}]/Q)$. By Lemma 4.1.15, $(M[x^{-1}]/Q)_R$ is a coprime module and since $M[x^{-1}]_R$ is a Bass module, $(M[x^{-1}]/Q)_R$ contains a maximal submodule such that $P/Q \subseteq M[x^{-1}]/Q$. By the coprimality of $(M[x^{-1}]/Q)_R$, $(M[x^{-1}]/P)_R$ is coprime and $I \cap R = \mathrm{ann}_R(M[x^{-1}]/P)$. Let us prove that $I \cap R \in \mathrm{Att}(M_R)$ and $I = (I \cap R)A$.

If P_R is a maximal submodule of $M[x^{-1}]_R$, then there exists $k \in \mathbb{N}$ such that $\langle P_k \rangle$ is a maximal submodule of M_R by Lemma 4.1.16, and we can set the smallest k that satisfies this hypothesis. If $\langle P_k \rangle$ is maximal then there exists $m_k \in M$ such that $m_k \notin \langle P_k \rangle$, and so $mx^{-k} \notin P_R$ whence $mx^{-k} + P$ is a cyclic generator of the simple module $(M[x^{-1}]/P)_R$. Let φ be the map of $(M/\langle P_k \rangle)_R$ over $(M[x^{-1}]/P)_R$ given by $\varphi(m_k + \langle P_k \rangle) := m_k x^{-k} + P$. By the complete (σ,δ)-compatibility of M_R and Propositions 4.1.8 and 4.1.10, if $m_k r \in \langle P_k \rangle$ for some $r \in R$ then $m_k \sigma'^k(r) \in \langle P_k \rangle$ whence $m_k \sigma'^k(r)x^{-k} \in P_R$. By minimality of k, $m_k f_k^i(r)x^{-i} \in P_R$ for all $0 \le i \le k-1$, which implies that $m_k x^{-k} r \in P_R$ and hence φ is well defined.

Let us see that φ is surjective. If $m_k x^{-k} + P$ generates the module $(M[x^{-1}]/P)_R$ and $\varphi(m_k + \langle P_k \rangle) := m_k x^{-k} + P$ then φ is surjective. Let ψ be the homomorphism of M_R over $(M/\langle P_k \rangle)_R$ defined by $\psi(m) := m + \langle P_k \rangle$ for every $m \in M_R$. If $\varphi \circ \psi$ is a surjective homomorphism of M_R over $(M[x^{-1}]/P)_R$ and $I \cap R \in \mathrm{Att}((M[x^{-1}]/P)_R)$, then $I \cap R \in \mathrm{Att}(M_R)$.

We need to prove that $I = (I \cap R)A$. Since I is an ideal of A we have $(I \cap R)A \subseteq I$. For the other inclusion, take an element $f(x) = r_0 + \cdots + r_j x^j \in I$ and let us see by induction that $r_i \in I \cap R$ for all $1 \le i \le j$. Notice that $(m_k x^{-k} + P)f(x) = m_k x^{-k} r_0 + \text{lower terms} \in Q \subseteq P$, and since every monomial of the “lower terms” belongs to the submodule P_R by minimality of k, we have $m_k x^{-k} r_0 \in P_R$ which implies that $r_0 \in I \cap R$. Assume that $r_0, \ldots, r_i \in I \cap R$ for some $i \le j$, and let us prove that $r_{i+1} \in I \cap R$. If $r_0, \ldots, r_i \in I \cap R$ then $r_0 + \cdots + r_i x^i \in I$, whence $r_{i+1}x^{i+1} + \cdots + r_j x^j \in I$.

$$(m_k x^{-k-i-1})(r_{i+1}x^{i+1} + \cdots + r_j x^j) = m_k \sigma'^{k+i+1}(r_{i+1})x^{-k} + \text{lower terms} \in Q \subseteq P.$$

By minimality of k, every monomial of the "lower terms" belongs to the submodule P_R and thus $m_k\sigma'^{k+i+1}(r_{i+1})x^{-k} \in P_R$, and by relation $x^{-1}r := \sigma'(r)x^{-1} + \delta'(r)$ it follows that

$$m_k\sigma'^{k+i+1}(r_{i+1})x^{-k} = m_k x^{-k}\sigma'^{i+1}(r_{i+1}) + \text{lower terms} \in Q \subseteq P,$$

where every monomial of the "lower terms" belongs to P_R by minimality of k. So, if $M[x^{-1}]_R$ is a completely (σ,δ)-compatible module and $m_k x^{-k}\sigma'^{i+1}(r_{i+1}) \in P_R$, then $m_k x^{-k} r_{i+1} \in P_R$ and thus $r_{i+1} \in I \cap R$ whence $f(x) = r_0 + \cdots + r_j x^j$ belongs to $(I \cap R)A$. Therefore $I = (I \cap R)A$. □

Corollary 4.1.18 ([Ann11, Theorem 3.2])**.** *If $M[x^{-1}]_R$ is a completely σ-compatible Bass module then*

$$\text{Att}(M[x^{-1}]_S) = \{P[x] \mid P \in \text{Att}(M_R)\}.$$

We present some examples that illustrate Theorems 4.1.13 and 4.1.17.

Example 4.1.19.

(i) If A is the Jordan plane $\mathcal{J}(\Bbbk)$, $M_{\Bbbk[x]}$ is a right module such that $M[y^{-1}]_{\Bbbk[x]}$ is a completely (σ,δ)-compatible module Bass, then the characterization of the attached prime ideals of $M[y^{-1}]_A$ is obtained from Theorems 4.1.13 and 4.1.17.

(ii) If $M[x^{-1}]_{\mathbb{C}[y]}$ is a completely (σ,δ)-compatible Bass module, then the characterization of the attached primes of $M[x^{-1}]_A$ where $A := MW_q$ is obtained from Theorems 4.1.13 and 4.1.17. Following the change of variable presented by Lopes (Example 4.1.2 (4)), MW_q can be interpreted as the quantum plane $\mathbb{C}_q[x,y]$ with $yx = qxy$. In this way, if $M[x^{-1}]_{\mathbb{C}[y]}$ is a completely σ-compatible Bass module with $\sigma(y) := q^{-1}y$, then the description of the attached primes of $M[x^{-1}]_A$ follows from Theorems 4.1.13 and 4.1.17 or Corollary 4.1.18.

(iii) If $M[y^{-1}]_{\Bbbk[x]}$ is a completely (σ,δ)-compatible Bass module, then Theorems 4.1.13 and 4.1.17 described the attached prime ideals of $M[y^{-1}]_A$, where A is the algebra of skew Ore polynomials of higher order $Q(0,b,c)$. In a similar way, we get the characterization of these ideals over $M[x^{-1}]_A$ when A is the algebra $Q(a,b,0)$.

(iv) If A is the skew Ore polynomial defined by Smits [Smi68] with δ_1 an automorphism of D and $\{\delta_2,\dots,\delta_k\}$ a set of left D-independent endomorphism and $M[x^{-1}]_D$ is completely $(\delta_1,\delta_1^{-1}\delta_2)$-compatible Bass, then the characterization of the attached primes of $M[x^{-1}]_A$ follows from Theorems 4.1.13 and 4.1.17.

(v) If $M[y_1^{-1}]_{\Bbbk[y_2]}$ is completely (σ,δ)-compatible Bass, then the description of the attached prime ideals of $M[y_2^{-1}]_A$ follows from Theorems 4.1.13 and 4.1.17, where A is the trimmed double extension $R_P[y_1,y_2;\sigma]$.

4.2 Future work

If Γ is an Abelian group, then R is Γ-*graded* if there exists a collection of subgroups $\{R_\gamma\}_{\gamma\in\Gamma}$ of R^+ such that $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for all $\alpha,\beta \in \Gamma$. Richardson [Ric03] studied secondary modules and the attached primes of representable modules over Γ-graded rings.

Richardson [Ric03] investigated the graded-secondary and graded Artinian modules defined by Sharp [Sha86], and introduced the terms *honest* or *honestly* to refer to properties of modules in the original sense. According to Richardson, M_R is Γ-*graded Artinian* if it satisfies the descending chain condition on Γ-graded submodules of M_R [Ric03, p. 2606]. The following proposition shows that if R is Γ-graded, then the notions of honestly Artinian and Γ-graded-Artinian coincide.

Proposition 4.2.1 ([Ric03, Proposition 1.3])**.** *M_R is Γ-graded Artinian if and only if M_R is honestly Artinian.*

If every homogeneous element of R either divides or is nilpotent on M_R, then M_R is called Γ-*graded secondary* [Ric03, p. 2606]. If M_R is honestly secondary then M_R is graded-secondary, but the other implication does not hold. For example, if $R = \Bbbk[x, x^{-1}]$ and $M_R = R_R$ then M_R is graded-secondary, but not honestly secondary [Ric03, p. 2606]. The following proposition gives conditions to ensure that every Γ-graded secondary module is honestly secondary.

Proposition 4.2.2 ([Ric03, Theorem 1.5])**.** *If R is a positively Γ-graded ring and M_R is Γ-graded secondary Artinian, then M_R is honestly secondary.*

As a consequence of Proposition 4.2.2, it follows that any Γ-graded secondary representation of a Γ-graded Artinian module is an honest secondary representation, and so $\mathrm{Att}_R(M)$ consists of homogeneous ideals [Ric03, Corollary 1.6]. Since there exist algebras that cannot be expressed as Γ-graded rings, some more general results concerning secondary modules and attached prime ideals for these rings contributes to to the properties of graded rings. In this way, we think as future work to study the secondary modules and attached prime ideals of semi-graded rings.

The theory of attached prime ideals of finitely generated modules over Noetherian local rings is related to *Matlis duality*. Alberola [Alb14] showed how this duality can also be applied to the study injective envelopes of local rings and the classification of Artin-Gorenstein modules.

Definition 4.2.3 ([Alb14, Definition 2.1])**.** If R is local with maximal ideal $\mathfrak{m}$ and $k := R/\mathfrak{m}$, the *Matlis dual* of M_R is defined as $M^\vee = \mathrm{Hom}_R(M, E_R(k))$, where $E_R(k)$ is the enveloping injective of k. We can write $(-)^\vee = \mathrm{Hom}_R(-, E_R(k))$, which is a contravariant exact functor from the category of R-modules to itself.

A local ring R is called *complete* if the natural embedding into its completion with respect to the maximal ideal $\mathfrak{m}$ is an isomorphism. Following Alberola [Alb14], if R is a complete Noetherian local ring, then $R^\vee \cong E_R(k)$, $E_R(k)^\vee \cong R$, $R \cong R^{\vee\vee}$ and $E_R(k) \cong E_R(k)^{\vee\vee}$ [Alb14, Lemma 2.5]. In this way, characterizing the injective envelope of k, we can know the Matlis dual of R. Alberola calculated the enveloping injective $E_R(k)$ of the ring of formal series $R = \Bbbk[[x_1, \ldots, x_n]]$ with maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$. The polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ can be seen as R-module with two linear structures: *derivation* and *contraction*.

If $\mathrm{char}(\Bbbk) = 0$ then the R-module structure of S by *derivation* is given by

$$x^\beta \cdot x^\alpha := \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha}, \text{ for all } x^\beta \in S,\ x^\alpha \in R,$$

with $\alpha! = \prod_{i=1}^n \alpha_i!$, $\beta \geq \alpha$ and $\beta, \alpha \in \mathbb{N}^n$, and 0 otherwise. The right module S_R with the structure defined by derivation is denoted by (S, der) [Alb14, p. 15]. If $\mathrm{char}(\Bbbk) \geq 0$ then the R-module structure of S by *contraction* is defined by $x^\beta \cdot x^\alpha := x^{\beta-\alpha}$, for all $x^\beta \in S$, $x^\alpha \in R$, with $\beta \geq \alpha$ and $\beta, \alpha \in \mathbb{N}^n$, and 0 otherwise. The module S_R with the structure given by contraction is denoted by $(S, cont)$ [Alb14, p. 16]. The following proposition calculated the enveloping injective of the residue field of a power series ring.

Proposition 4.2.4 ([Alb14, Theorem 2.10])**.** *Let $R = \Bbbk[[x_1, \dots, x_n]]$ be the power series ring over $\Bbbk$. If $\Bbbk$ is of characteristic zero then*

$$E_R(\Bbbk) \cong (S, der) \cong (S, cont).$$

If $\Bbbk$ is of positive characteristic then $E_R(\Bbbk) \cong (S, cont)$.

In some sense, the theory of secondary representations is dual to primary decompositions and this duality can be made explicit, at least for local rings, via the Matlis duality. More exactly, if R is a Noetherian complete local ring and M_R is finitely generated, then $\operatorname{Att}_R(M^\vee) = \operatorname{Ass}_R(M)$ [BS13, Corollary 10.2.20]. By Proposition 4.2.4, if $R = \Bbbk[[x_1, \dots, x_n]]$ and $\Bbbk$ is of characteristic zero, then $\operatorname{Att}_R(R) = \operatorname{Ass}_R((S, der)) = \operatorname{Ass}_R((S, cont))$, and if $\Bbbk$ is of positive characteristic, then $\operatorname{Att}_R(R) = \operatorname{Ass}_R((S, cont))$. Following these results, we think as future work to explicitly describe the injective envelope of noncommutative rings and study Matlis duality to characterize the attached prime ideals of inverse polynomial modules over different noncommutative rings.

CHAPTER 5

Uniform and couniform dimensions

In this chapter, we study and present original results about the uniform dimension of induced modules over skew PBW extensions and inverse polynomial modules over skew Ore polynomials. In addition, we investigate the couniform dimension of $M[x^{-1}]_A$.

Section 5.1 contains some results concerning the essential and uniform modules of induced modules over skew PBW extensions (Lemma 5.1.1 and Theorem 5.1.2) and describes the uniform dimension of these modules (Theorem 5.1.4). We extend several results formulated for skew polynomial rings by Leroy and Matczuk [LM04]. Additionally, we study the essential submodules of $M[x^{-1}]_A$, where $A = R(x;\sigma,\delta)$ (Lemma 5.1.7), and prove that if N_R is a uniform submodule of M_R, then $N[x^{-1}]_A$ is a uniform submodule of $M[x^{-1}]_A$ (Lemma 5.1.8). We show that M_R and $M[x^{-1}]_A$ have the same uniform dimension (Theorem 5.1.9).

In Section 5.2, we investigate the small and hollow modules of $M[x^{-1}]_A$ (Lemmas 5.2.3 and 5.2.6) and show that the couniform dimensions of M_R and $M[x^{-1}]_A$ are equal when $M[x^{-1}]_R$ is right Bass (Theorem 5.2.7). Our results extend those corresponding to skew polynomial rings of automorphism type presented by Annin [Ann05].

Finally, Section 5.3 presents some ideas for future work.

5.1 Uniform dimension

5.1.1 Induced modules over skew PBW extensions

In this section, we study the essential modules and the uniform dimension of induced modules over skew PBW extensions. The following lemma extends [LM04, Lemma 4.1].

Lemma 5.1.1. *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a bijective skew PBW extension of R and N_R is a submodule M_R, then:*

(1) *$N\langle X\rangle_R$ is essential in $M\langle X\rangle_R$ if and only if $N\langle X\rangle_A$ is essential in $M\langle X\rangle_A$.*

(2) *$Z(M_R) = 0$ if and only if $Z(M\langle X\rangle_R) = 0$.*

(3) *If* $Z(M_R)=0$ *then* $Z(M\langle X\rangle_A)=0$.

Proof. (1) If $N\langle X\rangle_R$ is essential in $M\langle X\rangle_R$, then $0\neq mR\cap N\langle X\rangle_R\subseteq mA\cap N\langle X\rangle_A$ for each $m\in M\langle X\rangle_A$, and thus $N\langle X\rangle_A$ is essential in $M\langle X\rangle_A$. Suppose that $N\langle X\rangle_A$ is essential in $M\langle X\rangle_A$ and consider the A-module homomorphism of $M\langle X\rangle_A$ over $(M/N)\langle X\rangle_A$ defined by $\phi(m_1x^{\alpha_1}+\cdots+m_kx^{\alpha_k}):=\overline{m_1}x^{\alpha_1}+\cdots+\overline{m_k}x^{\alpha_k}$. It is not difficult to see that ϕ induces an isomorphism $\overline{\phi}$ between $M\langle X\rangle_A/N\langle X\rangle_A$ and $(M/N)\langle X\rangle_A$. Let $\overline{f}\in(M/N)\langle X\rangle_A$ be the image of f by $\overline{\phi}$, for some $f\in M\langle X\rangle_A/N\langle X\rangle_A$. By Lemma 3.4.3 (1), there exists $r\in R$ such that $\overline{f}r$ is a good polynomial and since $N\langle X\rangle_A$ is an essential submodule of $M\langle X\rangle_A$, we have that $frA\cap N\langle X\rangle_A\neq 0$, that is, there exists $g=b_1x^{\beta_1}+\cdots+b_tx^{\beta_t}\in A$ such that $0\neq frg\in N\langle X\rangle_A$. By Lemma 3.4.3 (2) if $frg\in N\langle X\rangle_A$, then $g\in\operatorname{ann}_A(\overline{f}r)=\operatorname{ann}_R(\overline{f}r)A$ whence $b_i\in\operatorname{ann}_R(\overline{f}r)$ for all $1\leq i\leq t$. Thus, $\overline{f}rb_i=0$ and so $frb_i\in N\langle X\rangle_R$. Additionally if $frg\neq 0$ then $0\neq frb_i\in N\langle X\rangle_R$, for some $1\leq i\leq t$. This shows that $fR\cap N\langle X\rangle_R\neq 0$ and hence $N\langle X\rangle_R$ is an essential submodule of $M\langle X\rangle_R$.

(2) Since $Z(M_R)=Z(M\langle X\rangle_R)\bigcap M_R$, it follows that $Z(M\langle X\rangle_R)=0$ implies $Z(M_R)=0$. Assume that $Z(M\langle X\rangle_R)\neq 0$ and let $f=m_1x^{\alpha_1}+\cdots+m_kx^{\alpha_k}\in Z(M\langle X\rangle_R)$ with leading coefficient $m_k\neq 0$ and minimal monomial leading, that is, $\operatorname{lm}(f)\preceq\operatorname{lm}(g)$, for all $g\in fR$. By Lemma 3.4.2 (2) and (5), f is a good polynomial and $\operatorname{ann}_R(f)=\sigma^{-\alpha_k}(\operatorname{ann}_R(m_k))$ is an essential ideal of R which implies that $0\neq m_k\in Z(M\langle X\rangle_R)\bigcap M_R=Z(M_R)$.

(3) Assume that $Z(M_R)=0$ and $Z(M\langle X\rangle_A)\neq 0$. Let $f=m_1x^{\alpha_1}+\cdots+m_kx^{\alpha_k}\in Z(M\langle X\rangle_A)$ with leading coefficient $m_k\neq 0$ and minimal monomial leading, that is, $\operatorname{lm}(f)\preceq\operatorname{lm}(g)$ for all $g\in fA$. By Lemma 3.4.2 (3) f is a good polynomial and by Lemma 3.4.3 (2), we have that $\operatorname{ann}_A(f)=\sigma^{-\alpha_k}(I)A$ where $I=\operatorname{ann}_R(a_k)$. Since $\sigma^{-\alpha_k}(I)A\cap m'A\neq 0$ for all $m'\in M\langle X\rangle_A$, it follows that $\sigma^{-\alpha_k}(I)\cap m'M_R\neq 0$ and so $\sigma^{-\alpha_k}(I)$ is an essential ideal of R. Therefore $0\neq a_k\in Z(M_R)$, which is a contradiction. This proves that $Z(M\langle X\rangle_A)=0$ as desired.

□

Theorem 5.1.2 characterizes the essential and uniform submodules of $M\langle X\rangle_A$.

Theorem 5.1.2. *If* $A=\sigma(R)\langle x_1,\ldots,x_n\rangle$ *is a bijective a skew PBW extension of* R, N_R *is a submodule of* M_R *and* $N\langle X\rangle_A$ *is a good module, then:*

(1) N_R *is essential in* M_R *if and only if* $N\langle X\rangle_A$ *is essential in* $M\langle X\rangle_A$.

(2) N_R *is uniform if and only if* $N\langle X\rangle_A$ *is uniform.*

Proof. (1) If T_R is a submodule of M_R such that $T_R\cap N_R=0$ then $T\langle X\rangle_A\cap N\langle X\rangle_A=0$, which proves that if $N\langle X\rangle_A$ is essential in $M\langle X\rangle_A$ then N_R is essential in M_R. For the other implication, suppose that N_R is an essential submodule of M_R. By Lemma 5.1.1 (1) we need to show that $N\langle X\rangle_R$ is an essential submodule in $M\langle X\rangle_R$, that is, $mR\cap N\langle X\rangle_R\neq 0$ for all $m\in M\langle X\rangle_R$. We proceed by induction on the monomials. By Lemma 3.4.3 we may assume that $m=m_1x^{\alpha_1}+\cdots+m_kx^{\alpha_k}\in M\langle X\rangle_R$ is a good polynomial. If $\alpha_k=0$ then $m=m_k\in M_R$, and $mR\cap N_R\neq 0$ because N_R is essential. Suppose the statement is true for any leading monomial x^{β} such that $x^{\beta}\prec x^{\alpha_k}$ with $\beta\in\mathbb{N}^n$. If N_R is an essential submodule in M_R then $m_k R \cap N_R \neq 0$, whence $m_k\sigma^{\alpha_k}(r) \in N_R$ for some $r \in R$. The element $m_k\sigma^{\alpha_k}(r) \in N \subseteq N\langle X\rangle_A$ is a good polynomial, and since $N\langle X\rangle_A$ is a good module there exists a good polynomial $g \in m_k\sigma^{\alpha_k}(r)A \subseteq N\langle X\rangle_A$ such that $\mathrm{lm}(g) = \mathrm{lm}(mr)$. The leading coefficient of g belongs to $m_k\sigma^{\alpha_k}(r)R$, so there exists $w \in R$ such that mrw and g have the same leading coefficient, and then g and mrw have the same leading term. If $g = mrw$ it follows that $g \in mR \cap N\langle X\rangle_R$ which proves that $N\langle X\rangle_R$ is essential in $M\langle X\rangle_R$. If $g \neq mrw$, then $mrw - g \neq 0$ and since mrw and g have the same leading term, the leading monomial of $mrw - g$ is x^β for some $\beta \in \mathbb{N}^n$ with $x^\beta \prec x^{\alpha_k}$. Thus, by the inductive hypothesis, there exists $s \in R$ such that $h := (mrw - g)s \in N\langle X\rangle_R$ with $h \neq 0$. Since mrw and g are good polynomials of the same leading term, they have the same annihilator in R. In this way, if $mrws = 0$ then $gs = 0$ and so $h = 0$, which is a contradiction. Therefore $mrws \neq 0$ and $0 \neq mrws = gs + h \in mR \cap N\langle X\rangle_R$.

(2) It is clear that if $N\langle X\rangle_A$ is a uniform module then N_R is a uniform module. For the other implication, suppose that N_R is a uniform module. If $N\langle X\rangle_A$ is not uniform then there exist non-zero polynomials $f, g \in N\langle X\rangle_R$ such that $fA \cap gA = 0$ with $f = n_1x^{\alpha_1} + \cdots + n_kx^{\alpha_k}$ and $g = n'_1x^{\beta_1} + \cdots + n'_lx^{\beta_l}$. By Lemma 3.4.3 (1) we may assume that f and g are good polynomials with $\mathrm{lm}(g) \preceq \mathrm{lm}(f)$. Since $N\langle X\rangle_A$ is a good submodule of $M\langle X\rangle_A$, there is a good polynomial $h \in gA$ with $\mathrm{lm}(h) = \mathrm{lm}(f)$. Let n_k and n''_t be the leading coefficients of the polynomials f and h, respectively. Since $n_k, n''_t \in N_R$ and N_R is uniform, there exist $r, s \in R$ such that $n_kr = n''_ts \neq 0$. Consider the polynomial

$$z = f\sigma^{-\alpha_k}(r) - h\sigma^{-\alpha_k}(s) \in N\langle X\rangle_A$$

with leading monomial x^β for some $\beta \in \mathbb{N}$ where $x^\beta \prec x^{\alpha_k}$. If $f\sigma^{-\alpha_k}(r) = h\sigma^{-\alpha_k}(s) \in gA$, then $f\sigma^{-\alpha_k}(r) = 0$ since $fA \cap gA = 0$. Thus, $f\sigma^{-\alpha_k}(r) \neq h\sigma^{-\alpha_k}(s)$ whence $z \neq 0$. In addition, if f and g are good polynomials with $\mathrm{lm}(g) \preceq \mathrm{lm}(f)$, then $zA \cap gA \neq 0$. Thus, there exist $v_1, v_2 \in A$ such that

$$gv_2 = zv_1 = f\sigma^{-\alpha_k}(r)v_1 - h\sigma^{-\alpha_k}(s)v_1.$$

Since $f\sigma^{-\alpha_k}(r)$ and $h\sigma^{-\alpha_k}(s)$ are good polynomials of the same leading term and

$$f\sigma^{-\alpha_k}(r)v_1 = gv_2 + h\sigma^{-\alpha_k}(s)v_1 \in fA \cap gA = 0,$$

then they have the same annihilators in A, and so $h\sigma^{-\alpha_k}(s)v_1 = 0$, which is a contradiction. Therefore $N\langle X\rangle_A$ is a uniform submodule of $M\langle X\rangle_A$.

□

Corollary 5.1.3 ([LM04, Theorem 4.6])**.** *If N_R is a submodule of M_R such that $\widehat{N}_S$ is good, then:*

(1) *N_R is essential in M_R if and only if $\widehat{N}_S$ is essential in $\widehat{M}_S$.*

(2) *N_R is uniform if and only if $\widehat{N}_S$ is uniform.*

We recall that M_R has *finite uniform dimension* if there exist uniform submodules $U_1, \dots, U_n$ of M_R such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M_R [Lam98, Definition 6.2], and the *uniform dimension* of M_R is denoted by $\mathrm{rudim}(M_R) = n < \infty$. According to Lam [Lam98], M_R has infinite uniform dimension if and only if M_R contains an infinite direct sum of non-zero submodules [Lam98, Proposition 6.4].

Theorem 5.1.4 establishes sufficient conditions to ensure that M_R and $M\langle X\rangle_A$ have the same uniform dimension (c.f. [Rey14, Proposition 4.10]).

Theorem 5.1.4. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R. If $M\langle X\rangle_A$ is a good module then* $\mathrm{rudim}(M\langle X\rangle_A) = \mathrm{rudim}(M_R)$.

Proof. Assume that $\mathrm{rudim}(M_R) = k$. There exist $N_1, \dots, N_k$ uniform submodules of M_R such that $N_1 \oplus \cdots \oplus N_k$ is an essential submodule of M_R. By Theorem 5.1.2 we have that $N_1\langle X\rangle_A, \dots, N_k\langle X\rangle_A$ are uniform submodules of $M\langle X\rangle_A$, and the submodule $N_1\langle X\rangle_A \oplus \cdots \oplus N_k\langle X\rangle_A$ is essential in $M\langle X\rangle_A$, whence $\mathrm{rudim}(M\langle X\rangle_A) = k$.

If $\mathrm{rudim}(M_R) = \infty$, there exist non-zero submodules $N_1, N_2, \dots$ of M_R such that $N_1 \oplus N_2 \oplus \cdots$ is a submodule of M_R. Thus, every $0 \neq N_i\langle X\rangle_A$ is a submodule of $M\langle X\rangle_A$ for each $i \geq 1$, and $N_1\langle X\rangle_A \oplus N_2\langle X\rangle_A \oplus \cdots$ is a submodule of $M\langle X\rangle_A$, which implies that $\mathrm{udim}(M\langle X\rangle_A) = \infty$. Therefore $\mathrm{rudim}(M\langle X\rangle_A) = \mathrm{rudim}(M_R)$. □

Corollary 5.1.5 ([LM04, Theorem 4.9])**.** *If $\widehat{M_S}$ is good, then* $\mathrm{udim}(\widehat{M_S}) = \mathrm{udim}(M_R)$.

We present some examples that illustrate Theorem 5.1.4.

Example 5.1.6. (i) If A is the diffusion algebra, M_R is a right module where $R = \Bbbk[x_1, \dots, x_n]$ and $M\langle D_1, \dots, D_n\rangle_A$ is a good module, then Theorem 5.1.4 shows that $M\langle D_1, \dots, D_n\rangle_A$ and M_R have the same uniform dimension.

(ii) If A is the enveloping algebra $U(\mathfrak{so}(5, \mathbb{C}))$ and $M\langle \mathbf{J}_{\alpha\beta}\rangle_A$ is a good module, for some right module $M_{\mathbb{C}}$, then Theorem 5.1.4 characterizes the uniform dimension of $M\langle \mathbf{J}_{\alpha\beta}\rangle_A$.

(iii) If $M_{\mathbb{C}}$ is a right module over $\mathbb{C}$ and $M\langle I_1, I_2, I_3\rangle_A$ is a good module, then Theorem 5.1.4 characterizes the uniform dimension of $M\langle I_1, I_2, I_3\rangle_A$, where A is the algebra $U'_q(\mathfrak{so}_3)$.

(iv) If A is the Askey-Wilson algebra $AW(3)$, $M_{\mathbb{R}}$ is a right module over $\mathbb{R}$ and $M\langle K_0, K_1, K_2\rangle_A$ is a good module, then $\mathrm{rudim}(M\langle K_0, K_1, K_2\rangle_A) = \mathrm{rudim}(M_{\mathbb{R}})$ by Theorem 5.1.4.

5.1.2 Inverse polynomial modules over skew Ore polynomials

In this section, we study the uniform dimension of $M[x^{-1}]_A$. The following lemma treats on the property of being essential and its passage from M_R to $M[x^{-1}]_A$.

Lemma 5.1.7. *If M_R is a completely (σ, δ)-compatible module and N_R is an essential submodule of M_R, then $N[x^{-1}]_A$ is an essential submodule of $M[x^{-1}]_A$.*

Proof. Suppose that N_R is an essential submodule of M_R and let $m(x) \in M[x^{-1}]_A$ with negative degree k and leading coefficient m_k. If N_R is essential, then there exists $r \in R$ such that $m_k r \in N_R$ with $m_k r \neq 0$. Notice that $m(x)\left(x^{k-1}\sigma(r)\right) = m_k r x^{-1} - m_k\delta(r)$. In this way, if $m_k r \in N_R$ and M_R is completely (σ,δ)-compatible, then $m_k\delta(r) \in N_R$ and hence $m(x)x^{k-1}\sigma(r) \in N[x^{-1}]_A$. Therefore $N[x^{-1}]_A$ is essential submodule of $M[x^{-1}]_A$. □

The following lemma studies the property uniform and its passage from M_R to $M[x^{-1}]_A$.

Lemma 5.1.8. *If N_R is a uniform submodule of M_R then $N[x^{-1}]_A$ is a uniform submodule of $M[x^{-1}]_A$.*

Proof. Assume that N_R is a uniform submodule of M_R, and let $n(x), n'(x)$ be two non-zero polynomials of $N[x^{-1}]_A$ with leading coefficients m and m', respectively. If N_R is uniform then $0 \neq mR \cap m'R \subseteq n(x)A \cap n'(x)A$, and therefore $N[x^{-1}]_A$ is a uniform submodule of $M[x^{-1}]_A$. □

The following theorem shows that M_R and $M[x^{-1}]_A$ have the same right uniform dimension and generalizes [Ann02a, Theorem 4.7].

Theorem 5.1.9. *If M_R is completely (σ,δ)-compatible, then*

$$\operatorname{rudim}\left(M[x^{-1}]_A\right) = \operatorname{rudim}(M_R).$$

Proof. Assume that $\operatorname{rudim}(M_R) = n$. So there exist uniform submodules $N_1, \ldots, N_n$ of M_R such that $N_1 \oplus \cdots \oplus N_n$ is an essential submodule of M_R. By Lemmas 5.1.7 and 5.1.8, we obtain that $N_1[x^{-1}], \ldots, N_n[x^{-1}]$ are uniform submodules of $M[x^{-1}]_A$, and $N_1[x^{-1}] \oplus \cdots \oplus N_n[x^{-1}]$ is an essential submodule of $M[x^{-1}]_A$, proving that $\operatorname{rudim}\left(M[x^{-1}]_A\right) = n$.

If $\operatorname{rudim}(M_R) = \infty$, there exist non-zero submodules $N_1, N_2, \ldots$ of M_R such that $N_1 \oplus N_2 \oplus \cdots$ is a submodule of M_R. Thus, we get that $N_i[x^{-1}]_A$ is a non-zero submodule of $M[x^{-1}]_A$ for all $i \geq 1$ and $N_1[x^{-1}] \oplus N_2[x^{-1}] \oplus \cdots$ is a submodule of $M[x^{-1}]_A$, which implies that $\operatorname{rudim}\left(M[x^{-1}]_A\right) = \infty$. Therefore $\operatorname{rudim}\left(M[x^{-1}]_A\right) = \operatorname{rudim}(M_R)$. □

We present some examples that illustrate Theorem 5.1.9.

Example 5.1.10.

(i) If A is the Jordan plane $\mathcal{J}(\Bbbk)$ and $M_{\Bbbk[x]}$ is a module such that $M[y^{-1}]_{\Bbbk[x]}$ is a completely (σ,δ)-compatible module, then $M_{\Bbbk[x]}$ and $M[y^{-1}]_A$ have the same uniform dimension by Theorem 5.1.9.

(ii) If $M[x^{-1}]_{\Bbbk[y]}$ is completely (σ,δ)-compatible then the uniform dimensions of $M_{\Bbbk[y]}$ and $M[x^{-1}]_A$ are equal by Theorem 5.1.9, where A is the q-meromorphic Weyl algebra MW_q.

(iii) If $M[y^{-1}]_{\Bbbk[x]}$ is completely (σ,δ)-compatible, for some right module $M_{\Bbbk[x]}$, then $M_{\Bbbk[x]}$ and $M[y^{-1}]_A$ have the same uniform dimension by Theorem 5.1.9, where A is the ring of skew Ore polynomials of higher order $Q(0,b,c)$.

(iv) If A is the skew Ore polynomial defined by Smits [Smi68] with δ_1 an automorphism of D and $\{\delta_2,\ldots,\delta_k\}$ a set of left D-independent endomorphism and $M[x^{-1}]_D$ is completely $(\delta_1, \delta_1^{-1}\delta_2)$-compatible then $\operatorname{rudim}(M_D) = \operatorname{rudim}(M[x^{-1}]_A)$ by Theorem 5.1.9.

(v) If $M[y_1^{-1}]_{\Bbbk[y_2]}$ is completely (σ,δ)-compatible, then $\text{rudim}(M_{\Bbbk[y_2]}) = \text{rudim}(M[y_1^{-1}]_A)$ by Theorem 5.1.9, where A is the trimmed double extension $R_P[y_1, y_2;\sigma]$.

5.2 Couniform dimension

In this section, we study the couniform dimension of the inverse polynomial module $M[x^{-1}]_A$.

Definition 5.2.1 ([Var79, Definition 1.8])**.** The *couniform dimension* of M_R is defined by

$$\text{corank}(M_R) = \sup\left\{k \mid M_R \text{ surjects onto a direct sum of } k \text{ non-zero modules}\right\}.$$

In particular, $\text{corank}(\{0\}) = 0$.

A submodule N_R of M_R is *small* if $N'_R + N_R = M_R$ implies that $N'_R = M_R$ for every submodule N'_R of M_R; if N_R is a small submodule of M_R, we write $N_R \subseteq_s M_R$ [Lam98, p. 74]. If every proper submodule N_R of M_R is small, then M_R is called *Hollow* [Var79, Definition 1.10]. M_R is hollow if the sum of any two proper submodules remains proper [Ann05, Definition 1.4]. According to Varadarajan [Var79], M_R is hollow if and only if $\text{corank}(M_R) = 1$ [Var79, Proposition 1.11].

Proposition 5.2.2. (a) [Ann05, Proposition 1.3] *The following assertions hold:*

(1) *If N_R is a submodule of M_R then* $\text{corank}((M/N)_R) \leq \text{corank}(M_R)$.

(2) *If N_R is a small submodule of M_R then* $\text{corank}(M_R) = \text{corank}((M/N)_R)$. *The converse holds if* $\text{corank}(M_R) < \infty$.

(3) *If $M_1, M_2, \dots, M_n$ are right modules then*

$$\text{corank}\left(\bigoplus_{i=1}^{n} M_i\right) = \sum_{i=0}^{n} \text{corank}(M_i).$$

(b) [Var79, Theorem 1.20] *For any M_R, we have that* $\text{corank}(M_R) < \infty$ *if and only if there exist $H_1, \dots, H_k$ hollow R-modules and a surjective homomorphism $\varphi : M \to H_1 \oplus \cdots \oplus H_k$ such that* $\ker\varphi \subseteq_s M$.

If P_R is a submodule of $M[x^{-1}]_R$, we set $P_k := \left\{m \in M \mid mx^{-k} \in P\right\}$ for each $k \in \mathbb{N}$ and denote by $\langle P_k\rangle$ the submodule of M_R generated by P_k. Lemma 5.2.3 extends [Ann05, Lemma 2.8].

Lemma 5.2.3. *If $M[x^{-1}]_R$ is right Bass, then $N_R \subseteq_s M_R$ if and only if $N[x^{-1}]_A \subseteq_s M[x^{-1}]_A$.*

Proof. If N_R is not a small submodule of M_R, there exists a submodule L_R of M_R such that $M_R = L_R + N_R$ whence $M[x^{-1}]_A = L[x^{-1}]_A + N[x^{-1}]_A$. Thus $N[x^{-1}]_A$ is not a small submodule of $M[x^{-1}]_A$.

If $N[x^{-1}]_A$ is not a small submodule of $M[x^{-1}]_A$, then there exists a proper submodule Q_A of $M[x^{-1}]_A$ with $M[x^{-1}]_A = Q_A + N[x^{-1}]_A$. Since $M[x^{-1}]_R$ is right Bass, there exists a maximal submodule P_R of $M[x^{-1}]_R$ such that $Q_R \subseteq P_R$ whence $M[x^{-1}]_R = P_R + N[x^{-1}]_R$. It is easy to show that $N[x^{-1}]_R \nsubseteq P_R$, and so there exists $nx^{-k} \in N[x^{-1}]_R$ such that $nx^{-k} \notin P_R$. Hence, we have that $n \in N_R$ and $n \notin \langle P_k \rangle$ which proves that $N \nsubseteq P_k$ and thus $M_R = \langle P_k \rangle + N_R$ and $\langle P_k \rangle \neq M_R$ by Lemma 4.1.16. Therefore, N_R is not a small submodule of M_R. □

Remark 5.2.4. Lemma 5.2.3 holds if we change the condition that $M[x^{-1}]_R$ is Bass and assume that R is right perfect. By [Lam91, Exercise 24.7], if R is right perfect then every non-zero module M_R has a maximal submodule, and so the proof follows the same arguments.

Lemma 5.2.5 shows that under certain conditions, $M[x^{-1}]_A$ is a hollow module.

Lemma 5.2.5. *If M_R is simple and Q_A is a submodule of $M[x^{-1}]_A$ which contains polynomials of arbitrarily negative degree, then $Q_A = M[x^{-1}]$. In particular, $M[x^{-1}]_A$ is a hollow module.*

Proof. We show by induction on k that Q_A must contain all inverse monomials of any negative degree. If $m(x) \in Q_A$ with leading coefficient $m_k \neq 0$, for some $k \in \mathbb{N}$, then $m_k = m(x)x^k \in Q_A$ whence $m_k \in Q \cap M \subseteq M$. Since M_R is simple, we have that $Q \cap M = M$. Assume that Q_A contains all monomials of any degree less than k. If $m(x)$ is a polynomial of Q_A of negative degree at most k and with leading coefficient $m_l \neq 0$ and $l \geq k$, then $m(x)x^{l-k} \in Q_A$ with leading term $m_l x^{-k}$. By induction hypothesis, all non-leading terms of $m(x)x^{l-k}$ belong to Q_A whence $m_l x^{-k} \in Q_A$, and thus Q_A contains all inverse monomials of any negative degree k. □

Lemma 5.2.6 is important to prove our main result and extends [Ann05, Lemma 2.9].

Lemma 5.2.6. *If $M[x^{-1}]_R$ is right Bass, then M_R is hollow if and only if $M[x^{-1}]_A$ is hollow.*

Proof. Suppose that M_R is hollow and $M[x^{-1}]_A = N_A + N'_A$ for some proper submodules N_A, N'_A of $M[x^{-1}]_A$. If R is right perfect, then there exists a maximal submodule Q_R of M_R, and since M_R is hollow, we get that $Q \subseteq_s M$. Lemma 5.2.3 implies that $Q[x^{-1}] \subseteq_s M[x^{-1}]$, and thus $Q[x^{-1}] + N$ and $Q[x^{-1}] + N'$ are both proper submodules of $M[x^{-1}]_A$ where

$$\left(Q[x^{-1}] + N\right) + \left(Q[x^{-1}] + N'\right) = M[x^{-1}].$$

Since $Q[x^{-1}]_A$ is a small submodule of $M[x^{-1}]_A$, it follows that $N_A, N'_A \subsetneq Q[x^{-1}]_A$. So, the images of these two modules in $M[x^{-1}]/Q[x^{-1}] \cong (M/Q)\,[x^{-1}]$ are non-zero and proper, and they sum to the whole module $(M/Q)[x^{-1}]_A$, i.e. $(M/Q)[x^{-1}]_A$ is not hollow. Now, if $(M/Q)_R$ is simple then $(M/Q)[x^{-1}]_A$ is hollow by Lemma 5.2.5, which is a contradiction. Hence $M[x^{-1}]_A$ is hollow. □

Lemma 5.2.6 is also true if R is right perfect (Remark 5.2.4). Theorem 5.2.7 shows that the couniform dimensions of M_R and $M[x^{-1}]_A$ are equal and generalizes [Ann05, Theorem 2.10].

Theorem 5.2.7. *If $M[x^{-1}]_R$ is a right Bass module then*

$$\operatorname{corank}\left(M[x^{-1}]_A\right) = \operatorname{corank}\left(M_R\right).$$

Proof. By Proposition 5.2.2, if corank$(M_R) = n$ then there exist $H_1, \ldots, H_n$ hollow R-modules and a surjective homomorphism φ of M_R over $\overline{M}_R := H_1 \oplus \cdots \oplus H_n$ such that $\ker\varphi \subseteq_s M$. In addition, the map φ induces a homomorphism ψ of $M[x^{-1}]_A$ over $\overline{M[x^{-1}]}_A := H_1[x^{-1}] \oplus \cdots \oplus H_n[x^{-1}]$ defined by $\psi\left(mx^{-k}\right) := \varphi(m)x^{-k}$ for all $mx^{-k} \in M[x^{-1}]_A$. If $m'x^{-i} \in \overline{M[x^{-1}]}_A$ and φ is surjective, then there exists $m \in M_R$ such that $\varphi(m) = m'$ and thus $\psi\left(mx^{-i}\right) = \varphi(m)x^{-i} = m'x^{-i}$. Therefore, we have that ψ is surjective.

Note that if $mx^{-i} \in \ker\psi$ then $\varphi(m)x^{-i} = 0$ whence $\varphi(m) = 0$, and so $\varphi(m)x^{-i} \in \left(\ker\varphi\right)[x^{-1}]$. It is not difficult to show that $\left(\ker\varphi\right)[x^{-1}] \subseteq \ker\psi$, and hence $\ker\psi = \left(\ker\varphi\right)[x^{-1}]$. By Lemma 5.2.3, $\ker\psi = \left(\ker\varphi\right)[x^{-1}] \subseteq_s M[x^{-1}]$, and by Proposition 5.2.2 (2) we have that

$$\operatorname{corank}\left(M[x^{-1}]_A\right) = \operatorname{corank}\left((M[x^{-1}]/\ker\psi)_A\right) = \operatorname{corank}\left(\overline{M[x^{-1}]}_A\right).$$

If H_i is hollow then $H_i[x^{-1}]_A$ is a hollow module, and so $\operatorname{corank}\left(H_i[x^{-1}]_A\right) = 1$ for all $1 \le i \le n$ by Lemma 5.2.6. This implies the equalities

$$\operatorname{corank}\left(M[x^{-1}]_A\right) = \sum_{i=1}^{n} \operatorname{corank}\left(H_i[x^{-1}]_A\right) = n.$$

If $\operatorname{corank}(M_R) = \infty$, then there exists a surjective map φ_k of M_R over $\overline{M}_R := N_1 \oplus \cdots \oplus N_k$ with $N_i \neq 0$ and $k \in \mathbb{N}$ arbitrarily large. This homomorphism φ_k induces a surjective homomorphism ψ_k of $M[x^{-1}]_A$ over $\overline{M[x^{-1}]}_A$ for each $k \in \mathbb{N}$, which shows that $\operatorname{corank}\left(M[x^{-1}]_A\right) = \infty$. □

As a consequence of Remark 5.2.4, we have the following corollaries.

Corollary 5.2.8. *If R is right perfect then*

$$\operatorname{corank}\left(M[x^{-1}]_A\right) = \operatorname{corank}(M_R).$$

Corollary 5.2.9 ([Ann05, Theorem 2.10])**.** *If R is right perfect then*

$$\operatorname{corank}\left(M[x^{-1}]_S\right) = \operatorname{corank}(M_R).$$

We present some examples that illustrate Theorem 5.2.7.

Example 5.2.10.

(i) If A is the Jordan plane $\mathscr{J}(\Bbbk)$ and $M[y^{-1}]_{\Bbbk[x]}$ is a Bass module, then $M_{\Bbbk[x]}$ and $M[y^{-1}]_A$ have the same couniform dimension by Theorem 5.2.7.

(ii) If $M[x^{-1}]_{\Bbbk[y]}$ is a Bass module then couniform dimensions of $M_{\Bbbk[y]}$ and $M[x^{-1}]_A$ are equal by Theorem 5.2.7, where A is the q-meromorphic Weyl algebra MW_q.

(iii) If $M[y^{-1}]_{\Bbbk[x]}$ is a Bass module then $\operatorname{corank}(M_{\Bbbk[x]}) = \operatorname{corank}(M[y^{-1}]_A)$ by Theorem 5.2.7, where A is the ring of skew Ore polynomials of higher order $Q(0,b,c)$.

(iv) If A is the skew Ore polynomial defined by Smits [Smi68] with δ_1 an automorphism of D and $\{\delta_2,\dots,\delta_k\}$ a set of left D-independent endomorphism and $M[x^{-1}]_D$ is Bass then M_D and $M[x^{-1}]_A$ have the same couniform dimension by Theorem 5.2.7.

(v) If $M[x^{-1}]_{\Bbbk[y]}$ is a Bass module then couniform dimensions of $M_{\Bbbk[y]}$ and $M[x^{-1}]_A$ are equal by Theorem 5.2.7, where A is the trimmed double extension $R_P[y_1,y_2;\sigma]$.

5.3 Future work

According to Bell and Goodearl [BG88], "when studying modules over a prime Noetherian ring, often one is interested in the torsionfree rank of a module, as opposed to its uniform rank. Over general Noetherian rings, this leads to the notion of reduced rank, which turns out to behave better than uniform rank when inducing up to differential operator rings"[BG88, p. 32].

R is *right Goldie* if R_R has finite uniform dimension and R has the ascending chain condition on right annihilators [GJ04, p. 115]. On the other hand, the *torsion submodule* of M_R is given by

$$t(M) = \{m \in M \mid mr = 0 \text{ for some regular element } r \in R\},$$

and M_R is *torsion* (resp., *torsionfree*) if $t(M) = M$ (resp., $t(M) = 0$) [GJ04, p. 126].

Definition 5.3.1 ([GJ04, p. 187]). If R is semiprime right Goldie, the *reduced rank* (or *torsionfree rank*) of M_R is defined as $\rho_R(M) = \text{udim}(M/t(M))$ and it is denoted by $\rho(A)$ if R is understood.

A *composition series* for M_R is a chain of submodules

$$M = M_0 \geq M_1 \geq M_1 \geq \cdots \geq M_k = 0$$

such that each of the factors M_i/M_{i-1} is simple. The number k is the *length* of the composition series, and the factors M_i/M_{i-1} are called the *composition factors* of M_R corresponding to this composition series [GJ04, p. 73]. It is well-known that $M/t(M)$ has a finite uniform dimension if and only if $M \otimes_R Q(R)$ has finite length as right $Q(R)$-module [GJ04, p. 130]. So, Definition 5.3.1 can be rewritten as follows: if $Q(R)$ is the right quotient ring of R, then

$$\rho_R(M) = \text{length}(M \otimes_R Q(R)).$$

Following the concept of *length* for Artinian modules that are not semisimple, the definition of reduced rank is extended (see [GJ04, p. 188] for more details).

Definition 5.3.2 ([GJ04, p. 189]). If R is a right Noetherian with prime radical $N_*(R)$, and

$$M = M_0 \geq M_1 \geq M_1 \geq \cdots \geq M_k = 0$$

is a chain of submodules of M_R such that $M_{i-1}N_*(R) \subseteq M_i$ for all $0 \leq i \leq n-1$, then *reduced rank* of M_R is defined as the sum of the reduced ranks of the right R/N-modules M_{i-1}/M_i, that is,

$$\rho_R(M) = \sum_{i=0}^{n-1} \rho_{R/N}(M_i/M_{i-1}).$$

It is not hard to see that this definition is independent of the chain [GJ04, Lemma 11.1].

Bell and Goodearl [BG88] considered Definition 5.3.2 and proved that if R is right Noetherian and $A = R\langle x_1, \dots, x_n\rangle$ is a PBW extension of R, then A and R have the same reduced rank [BG88, Theorem 6.4]. Since there exist algebras that cannot be regarded as PBW extensions (Chapter 1), a more general result on reduced rank for these rings contributes to the theory of ring-theoretical properties of noncommutative rings of polynomial type. In this way, we think as future work to study the reduced rank of skew PBW extensions and semi-graded rings.

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