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Yo no os aconsejo el trabajo, sino la lucha. Yo no os aconsejo la paz, sino la victoria. ¡Que vuestro trabajo sea una lucha!, ¡que vuestra paz sea una victoria!.

Friedrich Nietzsche, *Así hablaba Zaratustra*

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Resumen

Título: Descomposición del Operador Laplace-Beltrami sobre Variedades Riemannianas

En este trabajo estudiamos las condiciones geométricas requeridas sobre una variedad pseudo-Riemanniana M para garantizar la existencia de soluciones de una ecuación diferencial parcial de segundo orden planteada sobre M. Seguimos de cerca las ideas principales expuestas en [3], donde las consideraciones se llevan a cabo en presencia de una métrica Riemanniana y se usa la descomposición del operador Laplace-Beltrami dada por Helgason en [14]. A pesar de que en el caso indefinido la geometría de la variedad cambia significativamente y hay varios obstáculos a los que prestar atención, las condiciones para los resultados de Helgason se pueden encontrar naturalmente. En vista de esto, consideramos M como una variedad Lorentziana globalmente hiperbólica, pues en este contexto se puede identificar fácilmente una subvariedad unidimensional Σ transveral a las órbitas de una acción de grupo dada. Esto significa que M está dotada de una acción polar y así la ecuación se puede reducir sobre Σ como en el caso Riemanniano. Entonces las soluciones se obtienen de tal manera que resultan ser constantes a largo de las órbitas de la acción. Por último, proponemos una extensión de nuestras consideraciones a productos warped Lorentzianos.

Palabras clave: variedades con métricas indefinidas; problemas de existencia para EDPs; operador de Laplace-Beltrami; acciones polares.

Abstract

Title: Decomposition of the Laplace-Beltrami Operator on Riemannian Manifolds

The goal of this work is to study the geometric conditions required on a pseudo-Riemannian manifold M to guarantee the existence of solutions of a second-order partial differential equation posed on M. We closely follow the basic ideas in [3], where the considerations are carried out in the presence of a Riemannian metric and it is used the decomposition of the Laplace-Beltrami operator given by Helgason in [14]. In the indefinite case, the geometry of the manifold changes significantly and there are various pitfalls to watch out for. However, conditions for Helgason's results, in a certain sense, can be naturally set. Hence, we consider M as a globally hyperbolic Lorentzian manifold because a one-dimensional submanifold Σ transversal to the orbits of a given group action is easily recognisable. This means M is endowed with a polar action and thus the equation can be reduced on Σ as in the Riemannian case. Then the solutions are obtained in such a way that these are constant along the orbits of the action. Finally, we propose an extension of our considerations to Lorentzian warped products.

Keywords: manifolds with indefinite metrics; existence problems for PDEs; Laplace-Beltrami operator; polar actions.

Contents

1 Introduction

Differential operators are widely studied in Mathematics and their generalization to manifolds, rather than to Euclidean spaces, implies a geometric richness for their study from several points of view. One of the aforementioned points of view consists of studying them as invariant objects under an action of a topological group. We then consider differential operators defined on a pseudo-Riemannian manifold equipped with an action of a isometry group. Hence, such an operator is assumed to be invariant under isometries. In this specific setup, we focus on a differential operator defined in terms of the chosen pseudo-Riemannian structure: the *Laplace-Beltrami operator*. For instance, the Laplace-Beltrami operator on $\mathbb{R}^n$ is invariant under orthogonal transformations.

In this text, we study the geometric assumptions required to guarantee the existence of solutions of a second-order semi-linear problem posed on a pseudo-Riemannian manifold M and involving the Laplace-Beltrami operator. The notion of pseudo-Riemannian manifold extends that of Riemannian manifold by relaxing the requirement of positive definiteness. Our analysis is inspired by and closely follows that of Becerra, Galvis, and Martinez in [3]. In the aforementioned reference, M is supposed to be a Riemannian manifold provided with a *polar action* of a compact group H of isometries and it is shown the existence of constant solutions along the orbits of the given group action. The submanifold Σ transversal to the action is assumed to be one-dimensional. This allows to reduce the problem to a one-dimensional problem posed on Σ using the formula of the *radial part* of the Laplace-Beltrami operator given by Helgason in Theorem 3.2 in [14]. Existence and properties of solutions to the reduced problem are obtained by Theorem 1.1 in [25].

In our work, we follow a similar course. Nevertheless, additional conditions are necessary because in the indefinite case the geometry of the manifold changes significantly. Despite this, we can weaken the condition that the acting group of isometries H is compact and suppose more generally that it is closed. The reason for this chosen assumption is discussed shortly. We also choose to limit our considerations to a Lorentzian context because adapting the treatment of the problem to this case turns out to be more natural. Furthermore, Lorentzian Geometry has been as studied as Riemannian Geometry and this makes it easy to compare and contrast the results and techniques of the second one to those of the first one. The references which inspired our approach to the subject were [5, 8, 10, 14, 21]. Lorentzian manifolds are important in applications of General Relativity because describe mathematically the *space-time*. They are treated from the viewpoint of Differential Geometry and their study lies specially in three fundamental topics: geodesic and metric completeness, the Lorentzian distance function, and Morse index theory. We deal with two important examples of space-times: *Minkowski space-times* and *Robertson-Walker space-times*.

We give now a more precise description of the conditions required by us. Since the (geodesic) completeness of a Lorentzian manifold does not guarantee the existence of a maximal nonspacelike geodesic segment between any causally related pair of points, we assume that the manifold M is *globally hyperbolic*, see [5]. In fact, C. Bär, N. Ginoux and F. Pfäffle give a very complete understanding of wave equations on this class of manifolds in [8]. On the other hand, dealing with the *Minkowski space* $\mathbb{R}^n_1$, it is not possible to find polar actions of compact groups whose transversal submanifold is one-dimensional. But some results in [1] assert that the action of any closed group does imply this.

It was pointed out by Helgason that Theorem 3.2 in [14] remains true on a pseudo-Riemannian manifold, provided two additional conditions are satisfied: 1) each orbit under the action of H is a nondegenerate closed submanifold of M and 2) each isotropy subgroup of H is compact. It is well known that both closedness of the orbits and compactness of isotropy subgroups are a consequence of the manifold being equipped with a proper action. In [3] this condition is superfluous because of a compact group always acts properly. However, orbits whose induced metric is degenerated may occur. On each of them, a tangent vector is orthogonal to itself, then for reasons of dimensionality a polar action could not be considered.

Lorentzian examples that we use to describe the geometric treatment given to the Equation (5-1) showed to come from a broader class of pseudo-Riemannian manifolds, including products, called *warped products*. In our discussion of warped products we use Lorentzian warped products whose metric is defined on a product manifold $M = I \times N$, with I being a (possibly infinite) open interval in $\mathbb{R}^1_1$ and N a isotropic Riemannian manifold. Robertson-Walker space-times are modeled geometrically as those spaces which become of non-constant curvature. These space-times are foliated by a special set of spacelike hypersurfaces such that each hypersurface corresponds to an instant of time, see [5, Chapter 5]. A study of the basic geometric properties of M indicated that generalized polar coordinates parameterize also this space and thus the solution method to semi-linear problem can be successfully replicated in this context. This suggests that our analysis is more general than that developed by Helgason on isotropic manifolds and that developed by Galvis et al. to find solutions to the Equation (5-1).

We organise the different chapters as follows. In Chapter 2, we explore from scratch the collection of notions and basic results on pseudo-Riemannian Geometry useful for the development of our work. A remarkable goal of this part is to pay particular attention to the basic differences between pseudo-Riemannian Geometry and Riemannian Geometry. With a view to setting the properties of the Lorentzian spaces that concern us, in this chapter we also define global hyperbolicity, Cauchy surfaces, hyperquadrics, warped product spaces, and isometry groups.

In Chapter 3, we define quickly proper actions, isometric actions, and polar actions. In addition we explore substantially the integration theory and invariant measures on locally compact groups. This topic turns out to be purely technical for our purposes. Chapter 4 introduces the Laplace-Beltrami operator and is a summary of Helgason's results aimed at describing the radial part of the Laplace-Beltrami operator. Finally, in Chapter 5 we study with all the previous material the semi-liner problem on globally hyperbolic Lorentzian manifolds and Lorentzian warped products.

2 Generalities on Pseudo-Riemannian Geometry

The topics dealt below are about differential manifolds endowed with a pseudo-Riemannian metric structure. This means a smooth assigment of nondegenerated symmetric bilinear forms on tangent spaces. For the sake of self-completeness of our exposition, we introduce in this chapter definitions and main results of Riemannian and pseudo-Riemannian Geometry as well as establish our geometric context and notations for later use. A deeper treatment and proofs can be revised in O'neill [21], do Carmo [11] and Beem and Ehrlich [5], our principal references.

2.1 Smooth Manifolds and Connected Definitions

Let M be a Hausdorff and second countable topological space with the property that each point $p \in M$ has a neighbourhood U homeomorphic to an open subset $O \subset \mathbb{R}^n$. Then, owing to the image by φ of each point $q \in U$ is a point in $\mathbb{R}^n$, we can associate an $n-$tuple of real numbers to q:

$$\varphi(q) \mapsto (x_1(q), \ldots, x_n(q)).$$

This $n-$tuple is called *local coordinates* at q and φ is called a *coordinate map*. We shall denote local coordinates on any $U \subset M$ as $(x_1, \ldots, x_n)$ or (x_i) depending on the context. Also, $x_1, \ldots, x_n$ are called *coordinate functions* defined on U and the pair (U, φ) is called a *local chart*. Now, for two homeomorphisms $\varphi, \hat{\varphi}$ like the ones above, consider the homeomorphism

$$\Phi : \hat{\varphi} \circ \varphi^{-1} : O \to \widehat{O}.$$

Then Φ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ described as $\Phi : (\Phi_1, \ldots, \Phi_n)$ by n continuous functions Φ_i. If for an arbitrary point $p \in U \cap \widehat{U}$, we write the local coordinates $(x_1(q), \ldots, x_n(q))$ with respect to (U, φ) and $(y_1(q), \ldots, y_n(q))$ with respect $(\hat{U}, \hat{\varphi})$, there is a relation

$$y_i(p) = \Phi_i(x_1(q), \ldots, x_n(q))$$

between them. Thus Φ describes what is called a *coordinate change*. We shall say that M is a *smooth manifold of dimension* n or simply a *manifold*, if additionally the coordinate changes are all diffeomorphisms of class C^∞. Regarding the aim of this work, it is appropriate to present the following simple example.

Example 2.1. [Polar coordinates] By choosing a smooth local parametrization for an open subset U of $\mathbb{S}^{n-1}$, we can get polar coordinates for $\mathbb{R}^n$. Indeed, if such a parametrization is denoted by $\hat{\psi} : \hat{U} \to U$, where $\hat{U}$ is an open subset on $\mathbb{R}^{n-1}$, and we define $\hat{\Psi} : \mathbb{R}^+ \times \hat{U} \to \mathbb{R}^n$ by $\hat{\Psi}(r, \omega_1, \ldots, \omega_{n-1}) = r\hat{\psi}(\omega_1, \ldots, \omega_{n-1})$, it is possible to consider $P := \hat{\Psi}^{-1}$ as a smooth coordinate map on the open subset $\hat{\Psi}(\mathbb{R}^+ \times \hat{U}) \subseteq \mathbb{R}^n \setminus \{0\}$ owing to the fact that the differential of $\hat{\Psi}$ vanishes nowhere. Two classic examples of this are the ordinary polar coordinates in $\mathbb{R}^2$ and the spherical coordinates in $\mathbb{R}^3$. These have the special feature that the coordinate function r is defined as the norm of any $x \in \hat{\Psi}(\mathbb{R}^+ \times \hat{U})$.

Throughout this work, we shall only be dealing with C^∞ functions on manifolds, that means functions $f : M \to \mathbb{R}$ such that for any local chart (U, ϕ), the function $f \circ \phi^{-1} : U \to \mathbb{R}$ is a C^∞ function on the open set $U \subset \mathbb{R}$. The algebra of real valued indefinitely differentiable functions on M will be denoted by $C^\infty(M)$. For $f \in C^\infty(M)$ the set

$$\operatorname{supp} f := \overline{\{p \in M | f(p) \neq 0\}},$$

is called the *support* of f. We shall denote the set of all C^∞ functions on M with compact support by $C_c^\infty(M)$.

For any $p \in M$ the *tangent space* to M at p, denoted by T_pM, can be characterized as the vector space containing all velocity vectors at the point p of curves in M passing through p. Also, if we consider local coordinates (x_i) on a coordinate neighbourhood of p, these will give a basis for T_pM consisting of the parcial derivates $\frac{\partial}{\partial x_i}$. When coordinates are clear in the context, we shall frequently use ∂x_i or ∂_i instead of $\frac{\partial}{\partial x_i}$.

If M is a smooth manifold, a *submanifold* (or *immersed submanifold*) of M is a smooth manifold N that is a subset of M such that the inclusion $\iota : N \hookrightarrow M$ is an immersion, that is, its differential is injective at each point of N. We shall consider immersed submanifolds because they include embedded submanifolds as a special case and thus become the more general of the two types of submanifolds.

A *vector field* X on M is an assignment to each point $p \in M$ of a vector tangent $X_p \in T_pM$. A vector field X acts on any function $f \in C^\infty(M)$ as a directional derivative of f, that is, $(Xf)(p) = X_pf$ for each $p \in M$. We say a vector field X is smooth if and only for every smooth function f on M, the function Xf is smooth on M. We shall denote the set of all smooth vector fields on M by $\mathfrak{X}(M)$.

If $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, it makes sense to consider the functions $X(Yf)$ and $Y(Xf)$ and define the function $[X, Y]$ from $C^\infty(M)$ to $C^\infty(M)$ sending each f to $X(Yf) - Y(Xf)$. It is known $[X, Y]$ is a smooth vector field on M called the *bracket* of X and Y and it defines a map $[\,,\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the following properties: for all $a, b \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{X}(M)$,

(i) bilinearity

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y],$$

(ii) anticommutativity

$$[Y, X] = -[X, Y],$$

(iii) Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The bracket of vector fields equips the real vector space $\mathfrak{X}(M)$ with a structure of a *real Lie algebra*.

2.2 Pseudo-Riemannian Geometry

Next, we shall introduce a special class of smooth manifolds endowed with a metric structure. To this end we start by considering some preliminary definitions.

Let V be a finite dimensional real vector space and let g be a bilineal form on V. Then g is called a *scalar product* if the following conditions hold:

- g is *symmetric*; i.e., $g(u, v) = g(v, u)$ for all $u, v \in V$.
- g is *non-degenerate*; i.e., $g(u, v) = 0$ for all v implies that $u = 0$.

A *pseudo-Riemannian metric* on a smooth manifold M is a correspondence g which smoothly associates to each point p of M a scalar product g_p on the tangent space T_pM. In order to give a coordinate expression of g we take $(x_1, \ldots, x_n)$, local coordinates on a open subset U of M, then the corresponding components of g are given by

$$g_{ij} = g(\partial x_i, \partial x_j)$$

and g can be written

$$g = \sum g_{ij}\, dx^i \otimes dx^j,$$

where $\otimes$ denotes the tensor product. The notation can be shortened by using the symmetric product of two $1-$forms. If it is denoted by concatenation, the symmetry of g allows us to writte g as follows:

$$g = \sum g_{ij}\, dx^i dx^j. \tag{2-1}$$

The smoothness of g means each g_{ij} is a smooth function on U. Since g is non-degenerate, at each point $p \in U$ the symmetric matrix $(g_{ij}(p))$ is invertible and we shall denote its inverse matrix by $(g^{ij}(p))$ (the point will be omitted as long as it is not necessary to specify it). The usual formula for the inverse of a matrix shows that the functions g^{ij} are smooth on U and because of $g_{ij} = g_{ji}$ then $g^{ij} = g^{ji}$ for $1 \leq i, j \leq n$. Finally the next identity is clear by definition

$$\sum_j g^{ij} g_{jk} = \delta^i_k. \tag{2-2}$$

It is defined the *norm* $|v|$ of a vector $v \in T_pM$ to be $|g(v,v)|^{\frac{1}{2}}$. A *unit vector* u is a vector of norm 1, that is, $g(u,u) = \pm 1$. Also, we shall say that vectors $v, w \in T_pM$ are *orthogonal*, provided $g(v,w) = 0$. Lastly, a set of mutually orthogonal unit vectors is said to be *orthonormal* and for $n = \dim M$, any set of n orthonormal vectors in T_pM is necessarily a basis for T_pM. It is appropriate to mention now that the matrix of g relative to a orthonormal basis $\partial x_1, \dots, \partial x_n$ for T_pM is diagonal; in fact

$$g(\partial x_i, \partial x_j) = \delta_{ij}\varepsilon_j, \quad \text{where} \quad \varepsilon_j = g(\partial x_j, \partial x_j) = \pm 1.$$

There exists a way to distinguish two different pseudo-Riemannian metrics on a manifold M. If we fix a metric g and a point $p \in M$ we can construct an orthonormal basis $(e_1, \dots, e_n)$ for T_pM in such a way that g is of the form

$$g = -(\eta^1)^2 - \cdots - (\eta^\nu)^2 + (\eta^{\nu+1})^2 + \cdots + (\eta^n)^2, \tag{2-3}$$

for some integer $0 \le \nu \le n = \dim M$ and where $(\eta^1, \dots, \eta^n)$ is the corresponding dual basis, then this integer ν is called *index* of g and it is equal to the maximum dimension of any subspace of T_pM on which g is negative definite. We shall denote it by $\operatorname{ind} M$.

Definition 2.2. A *pseudo-Riemannian* manifold is a smooth manifold M furnished with a pseudo-Riemannian metric of constant index.

We denote a pseudo-Riemannian manifold by (M, g), but we shall use only M if the pseudo-Riemannian metric is obvious from the context. When $\nu = 0$, M is a *Riemannian manifold* and g is a positive definitive scalar product, i.e., $g_p(v,v) > 0$ for all $v \in T_pM$, $v \neq 0$. In the indefinite case, we highlight when $\nu = 1$ and $n \ge 2$, then M is called a *Lorentzian manifold.*

Notation 2.3. We shall use $\langle\ ,\ \rangle$ to denote g, then $g(v,w) = \langle v, w\rangle \in \mathbb{R}$ for tangent vectors, and $g(X,Y) = \langle X, Y\rangle \in C^\infty(M)$ for vector fields.

Example 2.4. In the sequel we define the basic examples of pseudo-Riemannian metrics: the Euclidean one and the Minkowski one.

i) The *Euclidean metric* on $\mathbb{R}^n$, denoted by $\hat{g}$, is just the usual inner product on each tangent space $T_p\mathbb{R}^n$ under the natural identification

$$(v_1, \dots, v_n) \in \mathbb{R}^n \mapsto v := \sum_i v_i \left.\frac{\partial}{\partial x_i}\right|_p \in T_p\mathbb{R}^n, \tag{2-4}$$

provided we take on $\mathbb{R}^n$ the standard coordinates $(x_1, \dots, x_n)$. Then $\hat{g}$ can be written as:

$$\hat{g} = (dx^1)^2 + \cdots + (dx^n)^2,$$

that applied to vectors $v, w \in T_p\mathbb{R}^n$ means

$$\hat{g}(v,w) = \sum_{i=1}^{n} v_i w_i. \tag{2-5}$$

Note that $\mathbb{R}^n$ equipped with this metric is a Riemannian manifold.

ii) Now, let $(x_1, \dots, x_n)$ be standard coordinates on $\mathbb{R}^n$. The *Minkowski metric* on $\mathbb{R}^n$, denoted by m, is defined as follows. Take $v, w \in T_p\mathbb{R}^n$ then

$$m(v,w) := -v_1 w_1 + \left(\sum_{i=2}^{n} v_i w_i\right),$$

by using the identification in (2-4). In coordinates m is written as

$$m = -(dx^1)^2 + \cdots + (dx^n)^2.$$

Endowed with this metric, $\mathbb{R}^n$ is the most important example of a Lorentzian manifold. Nonetheless, we can define a more general pseudo-Riemannian metric on $\mathbb{R}^n$: for an integer ν with $0 \leq \nu \leq n$, changing the first ν plus signs in (2-5) to minus gives a scalar product on each $T_p\mathbb{R}^n$

$$\langle v, w\rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{n} v_j w_j, \qquad v, w \in T_p\mathbb{R}^n.$$

This yields a pseudo-Riemannian metric of index ν, thus the previous cases correspond to $\nu = 0$ and $\nu = 1$ respectively. The resulting manifold endowed with the metric structure described above will be denoted by $\mathbb{R}^n_\nu$.

There exists a way to get new pseudo-Riemannian manifolds from old. In the Riemannian case, immersed manifolds turn out to be a representative example. In fact, if (M, g) is a Riemannian manifold, the pullback of g by the inclusion $\iota : N \hookrightarrow M$, i.e., $\iota^* g$, will be a Riemannian metric on the submanifold N of M (see [18]) defined by

$$\langle v, w\rangle = \langle d\iota_p(v), d\iota_p(w)\rangle, \qquad v, w \in T_pN.$$

For instance, the *standard $n-$sphere of radious* $r > 0$ defined by

$$\mathbb{S}^n(r) := \{p : |p| = r\} \subset \mathbb{R}^{n+1},$$

becomes a Riemannian submanifold with the metric induced from $(\mathbb{R}^{n+1}, \hat{g})$. Namely, if $\iota_{\mathbb{S}^n(r)} : \mathbb{S}^n(r) \to \mathbb{R}^{n+1}$ is the corresponding inclusion then the Riemannian metric on $\mathbb{S}^n(r)$, known as the *round metric*, is defined by $\iota^*_{\mathbb{S}^n(r)}\hat{g}$. However, if g is a pseudo-Riemannian metric on M, $\iota^* g$ does not need to be a metric on N. With this difficulty in sight, the definition of pseudo-Riemannian submanifold changes a bit.

Definition 2.5. Let S be a submanifold of a pseudo-Riemannian manifold (M, g) with inclusion $\iota : S \hookrightarrow M$. If $\iota^* g$ is a metric on S then S, endowed with this metric, is said to be a *pseudo-Riemannian submanifold* of (M, g). Moreover, the metric $\iota^* g$ is called the *induced metric* on S.

As an example of the previous definition we can consider the natural inclusion $\iota : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$, with $m < n$, into the first m coordinates of $\mathbb{R}^n$. When we endow the space $\mathbb{R}^n$ with the pseudo-Riemannian metric structure $\mathbb{R}^n_\nu$, the submanifold $\mathbb{R}^m$ becomes a pseudo-Riemannian submanifold of $\mathbb{R}^n_\nu$.

Definition 2.6. The submanifold N of (M, g) is said to be *nondegenerate* if for each $p \in N$ and nonzero $v \in T_pN$, there exists some $w \in T_pN$ with $g(v, w) \neq 0$.

For instance, we consider the space $\mathbb{R}^n_1$ generated by the canonical basis $\{e_1, ..., e_n\}$. We define the vector subspace $V = gen\{e_1 + e_2, e_3\}$, so we have that

$$\langle e_1 + e_2, e_1 + e_2 \rangle = \langle e_1 + e_2, e_3 \rangle = 0,$$

then V is a degenerate subspace of $\mathbb{R}^n_1$.

Definition 2.7. Let M and N be pseudo-Riemannian manifolds with metrics g_M and g_N. A diffeomorphism $\phi : M \to N$ is called an *isometry* if it preserves metrics, that is $\phi^*(g_N) = g_M$.

The definition explicitly means that $\langle v, w \rangle = \langle d_p\phi(v), d_p\phi(w) \rangle$ for all $v, w \in T_pM$, $p \in M$. Note that the identity map of a pseudo-Riemannian manifold is an isometry. Likewise, a composition of isometries is an isometry and the inverse map of an isometry is an isometry. In short, the set of all isometries of a pseudo-Riemannian manifold to itself is a group.

2.2.1 Tools from pseudo-Riemannian Geometry

In the following, we shall introduce briefly a collection of concepts that describe the geometry of the spaces equipped with a pseudo-Riemannian metric. In general, most of them and the related classic results are widely known in the Riemannian context and their extension to the pseudo-Riemannian setup is straightforward.

We shall begin with a way to take directional derivatives of vector fields on pseudo-Riemannian manifolds. On $\mathbb{R}^n_\nu$, this can be done naturally. Just consider $(x_1, \ldots x_n)$ the standard coordinates on $\mathbb{R}^n_\nu$ and if $X, Y \in \mathfrak{X}(\mathbb{R}^n_\nu)$, then the *natural covariant derivative* of W with respect to V is defined by

$$\nabla_V W := V(W_1)\frac{\partial}{\partial x_1} + \cdots + V(W_n)\frac{\partial}{\partial x_n}, \tag{2-6}$$

where $W = \sum W_i \partial_i$. This formula satisfies the properties below.

Definition 2.8. A *connection* ∇ on a smooth manifold M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

i) $\nabla_{fV_1+gV_2}W = f\nabla_{V_1}W + g\nabla_{V_2}W, \qquad$ for $f, g \in C^\infty(M)$.

ii) $\nabla_V(aW_1 + bW_2) = a\nabla_V W_1 + b\nabla_V W_2, \qquad$ for $a, b \in \mathbb{R}$.

iii) $\nabla_V(fW) = f\nabla_V W + V(f)W \qquad f \in C^\infty(M)$.

$\nabla_V W$ is called the *covariant derivative* of W in the direction V for the connection ∇.

As in the Riemannian case, the choice of a pseudo-Riemannian metric on a manifold uniquely singles out a certain connection. Two additional conditions are enough to do this. If g is a pseudo-Riemannian metric on a manifold M and ∇ its connection, one such condition is to require that

$$\nabla_X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle,$$

for all $X, V, W \in \mathfrak{X}(M)$. In this case ∇ is said to be *compatible with* g. However the uniqueness is guaranteed if ∇ is *torsion free*, that is to say, if it satisfies

$$\nabla_W V - \nabla_V W = [V, W], \tag{2-7}$$

with $V, W \in \mathfrak{X}(M)$. We now enunciate a vital theorem of pseudo-Riemmanian geometry.

Theorem 2.9. [21, Chapter 3, Theorem 11] On a pseudo-Riemannian manifold (M, g) there is a unique connection ∇, called the *Levi-Civita Connection*, that is symmetric and compatible with g.

Given $(x_1, \ldots, x_n)$ local coordinates on an open subset U of pseudo-Riemannian manifold M, the expression of the associated Levi-Civita conection ∇ with respect to these coordinates is given by the Christoffel symbols. These are real-valued functions Γ_{ij}^k defined on U as

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k. \tag{2-8}$$

Since $[\partial_i, \partial_j] = 0$ and owing to ∇ is free torsion, it follows that $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ and so $\Gamma_{ij}^k = \Gamma_{ji}^k$. Using the Kosul formula, it is possible to prove that

$$\Gamma_{ij}^k = \frac{1}{2}\sum_m g^{km}\left\{\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right\}. \tag{2-9}$$

Example 2.10. The natural connection (2-6) is the Levi-Civita connection of the pseudo-Riemannian manifold $\mathbb{R}_\nu^n$ for every $\nu = 0, 1, \ldots, n$. By definition of then metric,

$$g_{ij} = \delta_{ij}\varepsilon_j, \qquad \text{where } \varepsilon_j = \begin{cases} -1, & \text{for } 1 \le j \le \nu, \\ +1, & \text{for } \nu + 1 \le j \le n \end{cases} \tag{2-10}$$

and by the formula 2-9, $\Gamma_{ij}^k = 0$ for all $1 \le i, j, k \le n$ because the $g_{ij}s$ are constant.

Two concepts associated with connections are geodesic and parallel transport. To address them, it is convenient to mention that we shall always consider smooth curves and so simply say curve. A *vector field along a curve* $\alpha : I \to M$, with $I \subseteq \mathbb{R}$ an open interval, is a smooth map $V : I \to TM$ such that $V(t) \in T_{\alpha(t)}M$ for every $t \in I$. The space of vector fields along α will be denoted by $\mathfrak{X}(\alpha)$. The most representative example of a vector field along a curve α is clearly its velocity vector $\dot{\alpha} \in T_{\alpha(t)}M$ for each t. Now we can restrict the notion of covariant derivative of vector fields along a differentiable curve.

Proposition 2.11. Let $\alpha : I \to M$ be a curve in a pseudo-Riemannian manifold M. Then there exists a unique correspondence $V \mapsto \frac{DV}{dt}$ from $\mathfrak{X}(\alpha)$ to $\mathfrak{X}(\alpha)$, called the **covariant derivative of** V **along** α, such that

i) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$.

ii) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$ where f is a differentiable function on I.

iii) $\frac{DV}{dt} = \nabla_{\dot{\alpha}(t)}Y$, where $t \in I$ and $V(t) = Y(\alpha(t))$.

Furthemore,

iv) $\frac{d}{dt}\langle V_1, V_2\rangle = \langle \frac{DV_1}{dt}, V_2\rangle + \langle V_1, \frac{DV_2}{dt}\rangle$.

If (x_i) are local coordinates on M we denote γ_i by $x_i \circ \gamma$, such that $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ and $V(t) = \sum_k V_i(t)\partial_i|_{\gamma(t)}$, the covariant derivative will have the representation

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dV_k}{dt} + \sum_{i,j} V_j \dot{\gamma}_i \Gamma_{ij}^k \right\} \partial_k. \tag{2-11}$$

With this in mind, geodesics are nothing but curves with zero acceleration both in Riemannian geometry and pseudo-Riemannian geometry.

Definition 2.12. Let (M, g) be a pseudo-Riemannian manifold, $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$, be a curve. Then γ is said to be a *geodesic* if its velocity vector field satisfies

$$\frac{D\dot{\gamma}}{dt} = 0.$$

In local coordinates, if we write the components of γ as above, it will be geodesic if and only if

$$\ddot{\gamma}^k + \sum_{i,j} \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j = 0, \tag{2-12}$$

for $k = 1, \ldots, n$.

Observe that (2-12) is a system of second-order ordinary differential equations and it follows from the existence and uniqueness theorem that, for any given point $p \in M$ and any given tangent vector $v \in T_pM$, there exists a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

Besides, I is maximal in the sense that if $\alpha : J \to M$ is a geodesic with $\dot{\alpha}(0) = v$ then $J \subset I$ and $\alpha = \gamma|_J$. A geodesic with largest possible domain is called *maximal geodesic*.

Definition 2.13. Let M be a pseudo-Riemannian manifold and $\gamma : I \to M$ be a maximal geodesic in M. Then M is said to be *geodesically complete* -or merely *complete*- if $I = \mathbb{R}$.

Let M be manifold with ∇ its associated connection. A vector field V along a curve γ is said to be *parallel along* γ (with respect ∇) if $\frac{DV}{dt} = 0$. By using Christoffel symbols and the previous local coordinate representation for γ, this means that

$$\dot{V}_k(t) = -V_j(t)\dot{\gamma}_i(t)\Gamma^k_{ij}(\gamma(t)), \quad k = 1, \ldots, n.$$

Note that a geodesic can be characterized as a curve whose velocity vector field is parallel along the curve. In addition, a distinguished fact about parallel vector fields is that every tangent vector at any point on a curve can be uniquely extended to a parallel vector field along the entire curve. Actually, since the equation above is equivalent to a system of linear ordinary differential equations, the next theorem guarantees the statement.

Theorem 2.14. For a curve $\gamma : I \to M$, let $a \in I$ and $v \in T_{\gamma(a)}M$. Then there is a unique parallel vector field V on γ such that $V(a) = v$.

According to the theorem above, if $b \in I$ then the function

$$P^b_a(\gamma) : T_pM \to T_qM$$

sending each v to $V(b)$ is called *parallel translation along* γ from $p = \gamma(a)$ to $q = \gamma(b)$. We highlight not only this function is an isomorphism but also an isometry (see [21]).

We recall now parallel translation motives the notion of curvature. Because of several of our future considerations will depend on this, we make a brief introduction of this notion.

Definition 2.15 (curvature)**.** The *curvature* R of a pseudo-Riemannian manifold M with Levi-Civita connection ∇ is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for all vector fields X, Y and Z.

Example 2.16. By definition, it is clear that if $M = \mathbb{R}^n_\nu$, with $0 \leq \nu \leq n$, then $R(X,Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n_\nu)$. Indeed, if the vector field $Z = \sum Z_i\partial_i$ is given in standard coordinates of $\mathbb{R}^n$, we obtain

$$\nabla_Y Z = (YZ_1, \ldots, Yz_n),$$

hence

$$\nabla_X \nabla_Y Z = (XYZ_1, \ldots, XYZ_n).$$

By similar computation for $\nabla_Y \nabla_X Z$ and using the definition of $[X, Y]$, the assertion follows.

Just now, we define the sectional curvature which completely determines R. For ease of notation, if we consider vectors $v, w \in T_pM$ then we denote by $Q(v, w)$

$$\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2.$$

If Π represents a two-dimensional subspace of T_pM generated by an orthonormal basis $\{v, w\}$, $Q(v, w)$ will be positive if $g|_\Pi$ is definite, or negative if it is indefinite. In the next definition we consider R as a function on tangent vectors of the same tangent space.

Definition 2.17. Given a point $p \in M$ and a two-dimensional subspace $\Pi \subset T_pM$, the real number

$$K(v, w) = \frac{\langle R(v, w)v, w\rangle}{Q(v, w)},$$

where $\{v, w\}$ is any basis of Π, is called the *sectional curvature* of Π at p.

It is a fact that the last definition does not depend on the choice of the basis v, w for Π. On the other hand, also it can be proved the knowledge of sectional curvature for all two-dimensional subspace of T_pM determines the curvature R completely. The reciprocal is always true by definition (see [11], [21]).

We shall say a pseudo-Riemannian manifold M has *constant curvature* if its sectional curvature K is a constant function on the set of all nondegenerate tangent planes to M. These manifolds will be of utmost importance to us. For the nonce, we claim $\mathbb{R}^n_\nu$ is *flat*, that means K is identically zero. It follows from Example 2.16.

The Exponential Map

Just as it happens in $\mathbb{R}^n$, setting nice coordinates on a manifold simplifies calculations and expressions which depends on these. For example, let us think about rectangular and polar ones, which are mostly used in multivariable calculus. In what follows, we shall mention how the tangent space, being identified with $\mathbb{R}^n$, induces naturally such coordinates on a manifold. First, we give an essential definition.

Definition 2.18 (Exponential map)**.** Let M be a pseudo-Riemannian manifold, $p \in M$, $v \in T_pM$. The map

$$\begin{aligned} \exp_p : & V_p \to M \\ & v \mapsto \gamma_v(1), \end{aligned}$$

where γ_v is the geodesic with $\gamma_v(0) = p$, $\dot{\gamma_v}(0) = v$ and $V_p := \{v \in T_pM : \gamma_v \text{ is defined on } [0,1]\}$ is called *exponential map* of M at p.

In general, the exponential map is only locally defined. A point $q \in M$ is image of $v \in T_pM$ by $\exp_p$, if p and q are joined by the geodesic that goes in direction of v for a unit time. On the other hand, if we fix $\lambda \in \mathbb{R}$ and define the geodesic $t \mapsto \gamma_v(\lambda t)$, its initial tangent vector is $\lambda\dot{\gamma_v}(0) = \lambda v$. Hence $\gamma_{\lambda v}(t)$ coincides with $\gamma_v(\lambda t)$ for all λ and t where the both are defined. So that, it is clear that

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t) \tag{2-13}$$

The reason why it is possible to use the exponential map to define coordinates in M is given by the following theorem.

Theorem 2.19. The exponential map $\exp_p$ maps a neighborhood of $0 \in T_pM$ diffeomorphically onto a neighborhood of $p \in M$.

Proof. By identifying $T_0(T_pM)$ with T_pM itself, the goal is to show that the differential map

$$d\ \exp_0 : T_0(T_pM) \simeq T_pM \to T_pM$$

is the identity map and hence, it is nonsingular. Then the assertion is a consequence of Inverse Function Theorem. To compute $d\ \exp_p(0)(v)$ for an arbitrary vector $v \in T_pM$, we take a curve α in T_pM starting at 0 such that $\dot{\alpha}(0) = v$. The curve given by $\alpha(t) = tv$ is convenient and this yields

$$d\ \exp_p(0)(v) = d\ \exp_p(0)(\dot{\alpha}(0)) = (\exp_p \circ \alpha)'(0) = \dot{\gamma}_v(0) = v,$$

by the equality (2-13). $\Diamond$

A consequence of Theorem 2.19 is the possibility of using the exponential map to define coordinate maps for M at least locally. To enter into details we consider an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM at the fixed basepoint $p \in M$, then if $v \in T_pM$, we denote its components with respect to this basis by $(v_1, \dots, v_n)$ and obtain a map

$$\begin{aligned} \Theta : T_pM &\longrightarrow \quad \mathbb{R}^n \\ v = \sum v_i e_i &\mapsto (v_1, \dots, v_n), \end{aligned} \tag{2-14}$$

that identifies T_pM with $\mathbb{R}^n$, as well as gives Euclidean coordinates to T_pM. Since there exist a neighborhood $\mathcal{U}$ of p and a neighborhood $\mathcal{V}$ of $0 \in T_pM$ which are diffeomorphic via $\exp_p$, it is called *normal coordinates* $(x_1, \dots, x_n)$ on $\mathcal{U}$ to those defined on M by the chart

$$(\Theta \circ \exp_p^{-1}, \mathcal{U}). \tag{2-15}$$

This means that such coordinates assign to any point $q \in \mathcal{U}$ the coordinates of $\exp_p^{-1}(q)$ with respect to $\{e_1, \dots, e_n\}$, that is

$$\Theta \circ \exp_p^{-1}(q) = (x_1(q), \cdots, x_n(q)).$$

Usually $\mathcal{U}$ is called *normal neighbourhood* of p and it can be proved that this uniquely determines the neighbourhood $\mathcal{V}$, (see [21]). Normal coordinates have nice properties. For example, when considering $\{f^1, \dots, f^n\}$, the dual base to $\{e_1, \dots, e_n\}$, we have

$$x_i \circ \exp_p = f^i.$$

So that if $v \in T_pM$, we write $v = \sum v_i e_i$ and since $\exp_p(tv) = \gamma_v(t)$,

$$x_i(\gamma_v(t)) = x_i \circ \exp_p(tv) = f^i(tv) = tf^i(v) = tv_i.$$

As a consequence, the geodesic γ_v starting at p with initial velocity vector v is represented in normal coordinates by the *radial line segment*

$$\gamma_v(t) = (tv_1, \dots, tv_n), \tag{2-16}$$

provided that γ_v stays within $\mathcal{U}$ and so γ_v is said to be *radial geodesic*. Some common calculations with coordinates and the geodesic equation (2-12) also show that $g_{ij}(p) = \delta_{ij}\varepsilon_j$ and $\gamma_{ij}^k(p) = 0$ (see [21]).

Gauss' Lemma

Example 2.1 describes the polar coordinates for $\mathbb{R}^n$ and now we denote them by (r, ω), where $r \in (0, \infty)$ and $\omega = \hat{\psi}(\omega_1, \dots, \omega_{n-1})$ parametrizes the unit sphere $\mathbb{S}^{n-1}$. Let (M, g) be a pseudo-Riemannian manifold, $p \in M$ and let $\mathcal{U}$ be a normal neighbourhood of p. Given a choice of polar coordinates for $\mathbb{R}^n \setminus \{0\}$, we can obtain a coordinate map $\hat{P} : T_pM \to \mathbb{R}^+ \times \hat{U}$ via Θ again, where $\hat{U}$ is an open set of $\mathbb{R}^{n-1}$:

$$\begin{tikzcd} \mathbb{R}^n \setminus \{0\} \arrow[r, "P"] & \mathbb{R}^+ \times \hat{U} \\ T_pM \arrow[u, "\Theta"] \arrow[ur, "\hat{P} = P \circ \Theta"'] & \end{tikzcd}$$

with P as in the example. Therefore every point $w \in T_pM \setminus \{0\}$ such that $|w| \neq 0$, can be written in polar coordinates as $w = r_w \cdot v_w$ where $r_w = |w|$ (with respect to pseudo-Riemannian metric) and v_w is the unit vector $w/|w|$ of T_pM having the coordinates $(w_1, \dots, w_{n-1})$. The appropriate composition of $\hat{P}$ with $\exp_p^{-1}$ defines the *polar normal coordinates* on a open set $\mathcal{U} \setminus \{p\} \subset M$. By definition, for each point $q \in M$ close to p and with coordinates (r, ω), the radial distance r is the geodesic distance from p to q and the polar angle ω is the angle at p of the corresponding shortest geodesic curve connecting p and q.

For $p \in M$, we put

$$V_\epsilon(0) = \{v \in T_pM : 0 \leq |v| < \epsilon\},$$

where the radius ϵ is measured with respect to the norm defined by g_p. In what follows we shall call *geodesic ball* to the image set $\exp_p(V_\epsilon(0))$ and *geodesic sphere* to $\exp_p(\partial\overline{V}_\epsilon(0))$ as long as $\epsilon > 0$ is such that $\exp_p$ is a diffeomorphism on an open set of T_pM containing $\overline{V}_\epsilon(0)$. We shall denote these sets by $B_\epsilon(p)$ and $S_\epsilon(p)$ respectively. Note that for all $r \leq \epsilon$, $S_r(p)$ is a submanifold of M. Theorem 2.20 below is a geometric statement which asserts the transversality between geodesics spheres and their radius, analogous to the case of the ordinary Euclidean spheres.

Theorem 2.20 (Gauss' Lemma)**.** Let $\mathcal{U}$ be a geodesic ball centered at $p \in M$. Then every geodesic passing through p intersects the geodesic spheres in $\mathcal{U}$ orthogonally, (i.e., the velocity vector of the geodesic at the point of intersection is orthogonal to the tangent space to the geodesic spheres).

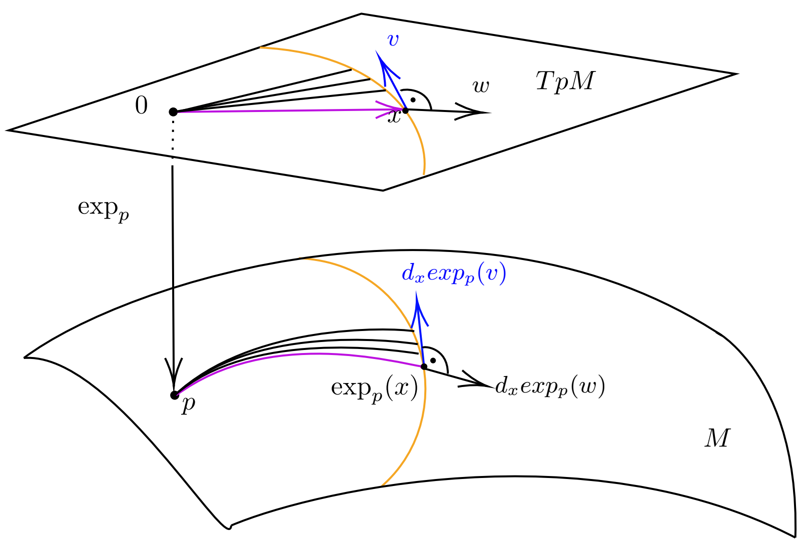

Figure **2-1**: Gauss' Lemma

Proof. Let $q \in \mathcal{U}$ and let $X \in T_qM$ be a vector tangent to the geodesic sphere through q. We recall that $\exp_p$ is a diffeomorphism onto $\mathcal{U}$ then there exist a vector $v \in T_pM$ such that $q = \exp_p(v)$, and there exists a vector $w \in T_v(T_pM) \approx T_pM$ such that $X = (D_v \exp_p)w$. If we define $r = |v|_g$, note that $v \in \partial B_r(0)$ and $w \in T_v(\partial B_r(0))$. Now, let $\gamma : [0,1] \to M$ be the geodesic from p to q with initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = v$, then $\gamma(t) = \exp_p(tv)$. Our goal is to prove that X and $\dot\gamma(1)$, the velocity vector of γ at q, are orthogonal.

Choose a curve $\sigma : (-\epsilon, \epsilon) \to T_pM$ such that $\sigma(0) = v$, $\dot\sigma(0) = w$ and $|\sigma(s)| = r$ for all $s \in (-\epsilon, \epsilon)$. Because $\exp_p v$ is defined, we can considerer $\Gamma : (-\epsilon, \epsilon) \times [0,1] \to M$ given by

$$\Gamma(s,t) = \exp_p(t\sigma(s)).$$

Observe that $\Gamma(0,t) = \gamma(t)$ and that (since $\exp_p$ maps the curve $t \mapsto t\sigma(s)$ in T_pM onto the geodesic $c_{\sigma(s)}$ in M with initial condition $\dot{c}_{\sigma(s)}(0) = \sigma(s)$) for each $s \in (-\epsilon, \epsilon)$, the curve $\Gamma_s(t) := \Gamma(s,t)$ is a geodesic in M whose tangent vector at $t = 0$ is $\dot{\gamma}_s(0) = (D_0 \exp_p)\sigma(s) = \sigma(s)$. Defining

$$S(s,t) = \frac{\partial \Gamma}{\partial s}(s,t) \qquad\qquad T(s,t) = \frac{\partial \Gamma}{\partial t}(s,t),$$

it follows from the definitions that

$$\begin{aligned}
S(0,0) &= \left.\frac{d}{ds}\right|_{s=0} \exp_p(0) = 0;\\
T(0,0) &= \left.\frac{d}{dt}\right|_{t=0} \Gamma(0,t) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = v;\\
S(0,1) &= \left.\frac{d}{ds}\right|_{s=0} \Gamma(s,1) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(\sigma(s)) = X;\\
T(0,1) &= \left.\frac{d}{dt}\right|_{t=1} \Gamma(0,t) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \dot{\gamma}(1).
\end{aligned}$$

Therefore $\langle S,T\rangle = 0$ when $(s,t) = (0,0)$ and $\langle S,T\rangle = \langle X, \dot{\gamma}(1)\rangle$ when $(s,t) = (0,1)$, so if we show $\langle S,T\rangle$ is independent of t, it turns out

$$\langle X, \dot{\gamma}(1)\rangle = \langle S,T\rangle(0,0) = 0.$$

So, we must verify that $\frac{\partial}{\partial t}\langle S,T\rangle = 0$. Since each curve Γ_s is a geodesic,

$$\frac{DT}{dt} = \frac{D}{dt}\left(\frac{\partial \Gamma}{\partial t}\right) = 0,$$

and for all $(s,t) \in (-\epsilon,\epsilon) \times [0,1]$,

$$\langle T,T\rangle(s,t) = \langle \sigma(s), \sigma(s)\rangle = r^2$$

because the length of the tangent vector is constant along any geodesic.

We compute

$$\begin{aligned}\frac{\partial}{\partial t}\langle S,T\rangle &= \langle \frac{DS}{dt},T\rangle + \langle S,\frac{DT}{dt}\rangle \\ &= \langle \frac{DT}{ds},T\rangle + 0 \\ &= \frac{1}{2}\frac{\partial}{\partial s}\langle T,T\rangle = 0.\end{aligned}$$

$\Diamond$

A consequence of Gauss's Lemma is that when M is Riemannian the expression in polar coordinates of the matrix (g_{ij}) of the metric becomes

$$g_{ij} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{\omega\omega}(r,\omega) & \\ 0 & & & \end{pmatrix},$$

where $g_{\omega\omega}(r,\omega)$ is the $(n-1)\times(n-1)$ matrix of the components of the metric with respect variables $(\omega_1,\dots,\omega_{n-1})$ parametrizing $\mathbb{S}^{n-1}$.

On the other side, when M is a Lorentzian manifold, Gauss' Lemma implies that the matirx of the metric described in polar coordinates becomes

$$g_{ij} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{\omega\omega}(r,\omega) & \\ 0 & & & \end{pmatrix},$$

where $g_{\omega\omega}(r,\omega)$ is the $(n-1)\times(n-1)$ matrix of the components of the metric with respect variables $(\omega_1,\dots,\omega_{n-1})$ parametrizing the set of points $S=\{v\in\mathbb{R}^n_1 : |v|=1\}$.

Jacobi Fields

At this very instant we introduce a useful notion for purporses of computations. Given a pseudo-Riemannian manifold M and a geodesic $\gamma:[a,b]\to M$, a vector field J along γ is said to be a *Jacobi field* if it safisties the equation

$$\frac{D^2J}{dt^2} + R(\dot\gamma(t),J(t))\dot\gamma(t) = 0, \tag{2-17}$$

where R is the curvature associated to the Levi-Civita conection of M. This equation can be written as a system of second-order linear ordinary differential equations, so it has a unique solution given initial values for J and $\frac{DJ}{dt}$ at one point [20, Proposition 10.2].

For metrics with constant sectional curvature there exists an explicit formula for Jacobi fields, this one expresses a Jacobi field as a sum of scalar multiple of parallel vector fields. To introduce this result, we make the following definition. Given a curve $\gamma : I \to M$, a *normal vector field along* γ is a vector field $V \in \mathfrak{X}(M)$ such that $\langle V(t), \dot{\gamma}(t)\rangle = 0$ for each $t \in I$ where $\dot{\gamma}(t) \neq 0$. We cite [21], [19] and [17] for the respective proof and treatment.

Proposition 2.21. Let γ be a geodesic in a manifold of constant curvature C, and let J be a normal Jacobi field along γ. Then, the Jacobi equation for J is

$$\frac{D^2 J}{dt^2} + C\langle\dot{\gamma}, \dot{\gamma}\rangle = 0$$

and if $c = |C\langle\dot{\gamma}, \dot{\gamma}\rangle|^{1/2}$, on γ there exist V, W, parallel normal vector fields along γ, such that

$$J(t) = \begin{cases} \cos(ct)V(t) + \sin(ct)W(t), & \text{if } C\langle\dot{\gamma},\dot{\gamma}\rangle > 0, \\ V(t) + tW(t), & \text{if } C\langle\dot{\gamma},\dot{\gamma}\rangle = 0, \\ \cosh(ct)V(t) + \sinh(ct)W(t), & \text{if } C\langle\dot{\gamma},\dot{\gamma}\rangle < 0. \end{cases} \tag{2-18}$$

The next theorem shows how the Jacobi fields are used to compute the derivative of the exponential map.

Theorem 2.22. [26, Proposition 3.4.3] Let $v, w \in T_pM$, $p \in M$. Let γ be the geodesic $\exp_p(tw)$. Then

$$D_{tv}\exp_p(tw) = J(t),$$

where J is the Jacobi field along $\gamma = \gamma_v(t)$ such that $J(t) = 0$ and $\frac{D}{dt}J(0) = w$.

Proof. The derivative of exp_p at tv is a map from $T_{tv}(T_pM)$ to $T_{\exp_p(tv)}M$. Due to $T_{tv}(T_pM) \cong T_pM$, it is convenient to consider tw as an element of $T_{tv}(T_pM)$ for this proof. As it was done in the proof of Gauss's Lemma, for $(s,t) \in (-\epsilon, \epsilon) \times [0,1]$ consider

$$\Gamma(s,t) = \exp_p(t(v + sw)).$$

By Chain Rule we obtain

$$\partial_s\Gamma(s,t) = D_{t(v+sw)}\exp_p(tw) = \gamma_{v+sw}(t). \tag{2-19}$$

Now, define the vector field $J(t) = \partial_s\Gamma(0,t)$ and let us see $J(t)$ defined in this way is a Jacobi field. Firstly note since each curve $t \mapsto \Gamma(s,t)$ is geodesic, using the notation in proof of Gauss' Lemma we have $\frac{DT}{dt}(s,t) = 0$. and taking covariant derivative with respect to s, it is clear that $\frac{D}{ds}\frac{DT}{dt}(s,t) = 0$. It follows by [21, Proposition 4.44] and symmetry lemma that:

$$\begin{aligned}\frac{D}{ds}\frac{D}{dt}(s,t) &= 0\\ \frac{D}{dt}\frac{DT}{ds}(s,t) + R(S,T)T &= 0\\ \frac{D}{dt}\frac{DT}{dt}(s,t) + R(S,T)T &= 0\end{aligned}$$

Evaluating at $s = 0$, where $S(0,t) = J(t)$ and $T(0,t) = \dot{\gamma}_v(t)$, we get (2-17). Besides J satisfies the initial condition because of

$$\frac{D}{dt}J(t) = \frac{D}{dt}D_{tv}\exp_p(tw) = D_{tv}\exp_p w + t\frac{D}{dt}(D_{tv}\exp_p u)$$

and thus

$$\frac{D}{dt}J(0) = D_0\exp_p w = w.$$

$\diamondsuit$

Theorem 2.23. [26, Proposition 3.4.5] Let $B_r(0) \subset M$ be the set $B_r(0) = \{v \in T_pM | 0 \leq |v| < r\}$ such that $\exp_p$ gives a diffeomorphism of $B_r(0)$ onto its image in M and let $q = \exp_p(tu) \in B_r(0)$. Here u is a vector in T_pM and $t \in (0,r)$. Let $(x_1, \ldots, x_n)$ the local normal coordinates of q. Then

$$\left.\frac{\partial}{\partial x_i}\right|_q = \frac{1}{t}J_i(t),$$

where J_i is the Jacobi field along the curve $\gamma(t) = \exp_p(tu)$ such that $J_i(0) = 0$ and $\frac{D}{dt}J(0) = e_i$, $i = 1, \ldots, n$ with $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_pM.

Proof. Observe that due to $D_{tu}\exp_p$ is invertible, the set

$$\{D_{tu}\exp_p(e_1), \ldots, D_{tu}\exp_p(e_n)\} \tag{2-20}$$

is a basis of T_qM. To prove that this is nothing but the local basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$, we have to show that, for any smooth function f on $B_r(0)$, $D_{tu}\exp_p(e_i)$ and $\frac{\partial}{\partial x_i}$ have the same action on f. By definition

$$\left.\frac{\partial f}{\partial x_i}\right|_q = \left[\left.\frac{d}{ds}\right|_{s=0}\exp_p(tu + se_i)\right]f = [D_{tu}\exp_p(e_i)]f,$$

and according to Theorem 2.22, for the geodesic $\gamma(t) = \exp_p(tu)$

$$D_{tu}\exp_p(te_i) = J_i(t),$$

where J_i is the Jacobi field along γ such that $J_i(0) = 0$ and $\frac{D}{dt}J(0) = e_i$, for $i = 1 \ldots, n$. Hence

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_q, \ldots, \frac{\partial}{\partial x_n}\bigg|_q \right\} = \left\{ \frac{1}{t} J_1(t), \ldots, \frac{1}{t} J_n(t) \right\}$$

is the canonical basis of $T_q M$. ♢

Volumen form and Riemannian measure

In a pseudo-Riemannian manifold there is a way to define the notion of volume. For this aim, we recall that for a pseudo-Riemannian manifold (M, g) of dimension n oriented by a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of local charts, there is a (global) unique up constant nowhere-vanishing $n-$form dV_g on M called *volume element*, such that in local coordinates $x = (x_1, \ldots, x_n)$ on any U_α, it is given by

$$dV_g|_{U_\alpha} = \sqrt{|\det(g_{ij}^x)|}\, dx^1 \wedge \cdots \wedge dx^n.$$

Picking a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$, we can define globally dV_g as

$$dV_g = \sum_{\alpha \in A} \rho_\alpha \sqrt{\bar{g}^x}\, dx^1 \wedge \cdots \wedge dx^n, \tag{2-21}$$

where we have used the notation by Helgason in [15]:

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), \qquad \bar{g}^x = |\det(g_{ij}^x)|. \tag{2-22}$$

The volume element allows us to define the integral of a function on the oriented pseudo-Riemannian manifold M. If f is a smooth, compactly supported function on $U_\alpha \subset M$ then we set

$$\int_{U_\alpha} f\, dV_g = \int_{\varphi_\alpha(U_\alpha)} f \circ \varphi_\alpha^{-1} \sqrt{\bar{g}^x}\, dx^1 \ldots dx^n. \tag{2-23}$$

Note that the integral above is defined on whole manifold via partition of unity using (2-21). Besides $y = (y_1, \ldots, y_n)$ are local coordinates on M given by another local chart (U_β, φ_β) in the same orientation for M, the change of coordinates $F = \varphi_\alpha \circ \varphi_\beta^{-1}$ is a diffeomorphism which preserves the orientation and represents a change of variables on $U_\alpha \cap U_\beta$ such that $x = F(y)$. we know that

$$\frac{\partial}{\partial y_j} = \sum_i \frac{\partial x_i}{\partial y_j} \frac{\partial}{\partial x_i},$$

where $\partial x_i / \partial y_j$ denotes the $(i, j)-$entry of the matrix $\mathcal{J}F$, the Jacobian matrix of F. Then, due to linearity of g, the matrix g^y is related to g^x by

$$g^y_{ij} = \sum_{k,l} g^x_{kl} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j}, \qquad \text{or} \qquad g^y = \left(\frac{\partial x}{\partial y}\right)^T g^x \left(\frac{\partial x}{\partial y}\right),$$

so that

$$\det g^y = \det g^x \left[\det\left(\frac{\partial x}{\partial y}\right)\right]^2. \tag{2-24}$$

Using the change of variables formula for multiple integrals and the equation (2-24), we see that if f has compact support inside the intersection $U_\alpha \cap U_\beta$, we obtain

$$\begin{aligned}
\int_{U_\alpha} f\sqrt{\overline{g}^x}\, dx^1 \wedge \cdots \wedge dx^n &= \int_{\varphi_\alpha(U_\alpha)} f \circ \varphi_\alpha^{-1} \sqrt{\overline{g}^x}\, dx^1 \dots dx^n \\
&= \int_{\varphi_\beta(U_\alpha \cap U_\beta)} f \circ \varphi_\beta^{-1} \sqrt{\overline{g}^x} |\det JF|\, dy^1 \dots dy^n \\
&= \int_{\varphi_\beta(U_\beta)} f \circ \varphi_\beta^{-1} \sqrt{\overline{g}^y}\, dy^1 \dots dy^n \\
&= \int_{U_\beta} f\sqrt{\overline{g}^y}\, dy^1 \wedge \cdots \wedge dy^n,
\end{aligned}$$

by which the integral (2-23) is invariant under coordinate changes.

In brief, we are able to introduce a functional on $C_c(M)$ determined by dV_g thanks to relation (2-24). Previously, we mention that a *measure* (see [15]) on M is a linear mapping $\Lambda : C_c(M) \to \mathbb{R}$ with the property that for each compact subset $K \subset M$ there is a constant C_K such that

$$|\Lambda(f)| = C_K \sup_{x\in M} |f(x)|,$$

for all $f \in C_c(M)$ whose support is contained in K. Saying that Λ is positive will mean $\Lambda(f) \geq 0$ whenever $f \geq 0$, (in fact, a positive linear map $\Lambda : C_c(M) \to \mathbb{R}$ is a measure on M so that in this context we shall say Λ is a *positive measure*.

Proposition 2.24. Let (M, g) be an oriented pseudo-Riemannian manifold and suppose $f \in C_c(M)$ satisfying on $f \geq 0$. Then $\int_M f\, dV_g \geq 0$, with equality if and only if $f \equiv 0$.

The proof basically is to observe that in local coordinates on $U_\alpha \subset M$, the equation (2-23) always is nonnegative under the hypothesis $f \geq 0$. Therefore the proposition states a volume element (or any $n-$form defined on M which does not vanish) give rise to a positive measure as follows. Let us consider an oriented pseudo-Riemannian manifold (M, g) and a local chart (U, φ) in the orientation of M as before. Also let us use the notation (2-22) again. Then, for each function $f \in C_c(U_\alpha)$ put

$$\Lambda(f) = \int_{\varphi(U)} (f \circ \varphi^{-1})(x_1, \dots, x_n)\sqrt{\overline{g}}\, dx^1 \dots dx^n.$$

Because of the integral is now well-defined, $\Lambda(f)$ can be extend for all $f \in C_c(M)$. The resulting positive measure

$$\Lambda : f \to \int_M f\, dV_g$$

is called *Riemannian measure* on M. In the next theorem we simply identify it with

$$\sqrt{\bar{g}}\, dx^1 \dots dx^n.$$

Theorem 2.25. [15, Chapter I, Lemma 1.2] Let M be a pseudo-Riemannian manifold, oriented by means of a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of local charts. Let $(x_1, \dots x_n)$ be local coordinates on U_α, and let $\bar{g}$ be defined as above. There exists a unique $n-$form ω on M such that

$$f \to \int_M f\omega$$

is the Riemannian measure. On U_α, ω is given by

$$\omega = \sqrt{\bar{g}}\, dx^1 \wedge \dots \wedge dx^n. \tag{2-25}$$

Recall that if a manifold of dimension n is oriented by a collection of local charts, there exists an $n-$tuple of continuous vector fields $(V_1, \dots, V_n)$ on all U_α such that for each $p \in U_\alpha$, $(V_1(p), \dots, V_n(p))$ represents an orientation of T_pM, (see [18, Proposition 15.6]).

Proof. For any point $p \in U_\alpha$ we choose any oriented orthonormal basis $\{X_1, \dots, X_n\}$ of T_pM, that is to say, if

$$\left.\frac{\partial}{\partial x_j}\right|_p = \sum_i a_{ij} X_i,$$

then the matrix of change of basis $A = (a_{ij})$ has positive determinant. Now, we define an $n-$form ω on M by

$$\omega_p(X_1, \dots X_n) = \frac{1}{n!}. \tag{2-26}$$

Clearly, ω_p does not depend on the choice of (X_i). So, we compute

$$\begin{aligned} g_{ij} &= \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \\ &= \left\langle \sum_k a_{ki} X_k, \sum_l a_{lj} X_l \right\rangle \\ &= \sum_k \sum_l a_{ki} a_{lj} \langle X_k, X_l \rangle \\ &= \sum_k a_{ki} a_{kj}, \end{aligned}$$

where the last expression is the $(i, j)-$entry of the matrix product $A^T A$. Thus

$$\det(g_{ij}) = \det(A^T A) = \det A^T \det A = (\det A)^2$$

and by the positivity, $\det A = \sqrt{g}$. Moreover,

$$\begin{aligned} \omega_p \left(\frac{\partial}{\partial x_i}, \dots, \frac{\partial}{\partial x_n} \right) &= \sum a_{1i_1} \cdots a_{mi_m} \omega_p(X_{i_1}, \dots, X_{i_n}) \\ &= \frac{1}{n!} \det A = \sqrt{g}\, (dx^1 \wedge \cdots \wedge dx^n)_p \left(\frac{\partial}{\partial x_i}, \dots, \frac{\partial}{\partial x_n} \right). \end{aligned}$$

It follows ω defined by (2-26) satisfies (2-25). Now, the change variable formula (2-27) below implies for $f \in C_c(U_\alpha)$

$$\int_{U_\alpha} f\omega = \int_{\varphi_\alpha(U_\alpha)} (f \circ \varphi_\alpha^{-1})(x_1, \dots, x_n) \sqrt{g}\, dx^1 \cdots dx^n,$$

so that ω gives the Riemannian measure. $\diamondsuit$

Example 2.26. In the Euclidean space $\mathbb{R}^n$, standard coordinates for a point $x = (x_1, x_2, \dots . x_n)$ can be expressed in polar coordinates by mean of the formulas

$$\begin{aligned} x_1 &= r \cos \omega_2 \\ x_2 &= r \sin \omega_2 \cos \omega_3 \\ x_3 &= r \sin \omega_2 \sin \omega_3 \cos \omega_4 \\ &\vdots \\ x_{n-1} &= r \sin \omega_2 \dots \sin \omega_{n-1} \cos \omega_n \\ x_n &= r \sin \omega_1 \dots \sin \omega_{n-1} \sin \omega_n, \end{aligned}$$

where r is positive, the angles $\omega_2, \omega_3, \ldots, \omega_{n-1}$ range over $[0, \pi]$ and ω_n ranges over $[0, 2\pi)$. Thus, in polar coordinates $(r, \omega_2, \omega_3, \ldots, \omega_n)$, the matrix g_{ij} is:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & r^2 & 0 & \ldots & 0 \\ 0 & 0 & r^2 \sin^2 \omega_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & r^2 \operatorname{sen}^2 \omega_1 \cdots \operatorname{sen}^2 \theta_{n-2} \end{pmatrix}.$$

It is easy to see that for $2 \leq i, j \leq n$, the resulting submatrix of the one above corresponds to the round metric of $\mathbb{S}^{n-1}(r)$. If we denote by $d\omega_r$ and $d\omega_1$ the volume elements of $\mathbb{S}^{n-1}(r)$ and $\mathbb{S}^{n-1}(1)$ respectively given by this metric, then

$$d\omega_r = r^{n-1}\, d\omega_1.$$

Now observe that to write the usual volume element on $\mathbb{R}^n$ given by $d\mu = dx^1 \wedge \cdots \wedge dx^n$, it is enough to set $0 \leq r < \infty$, so

$$d\mu = r^{n-1} dr\, d\omega_1.$$

Let $\Phi : M \to N$ be a diffeomorphism of manifolds of dimension n and h a pseudo-Riemannian metric on N. Let dV_h denote the volume element of N. Then the pullback $\Phi^* dV_h$ defines a volume element $dV_{\Phi^* h}$ on M and so we can define a Riemannian measure on M by the next change variable formula

$$\int_M f\, dV_{\Phi^* h} = \int_N (f \circ \Phi^{-1})\, dV_h, \qquad f \in C_c(M). \tag{2-27}$$

In fact, for local charts (U, φ), (V, ψ) of M, N respectively and f with support in U, we have $\sqrt{g} = |\det d(\psi \circ \Phi \circ \varphi) \circ \varphi| \sqrt{h} \circ \Phi$ where $g = \Phi^* h$. Then we apply again the usual change of variable formula. Note that in particular if $\Phi : M \to M$ is an isometry then $g = \Phi^* g$. This concludes that the Riemannian measure is invariant under isometries.

Next, we add that for any compact subset $A \subset (M, g)$, the integral of constant function 1,

$$\int_A dV_g,$$

is said to be the $n-$*dimensional volume of* A and we shall denote it by $\mathrm{Vol}_n(A)$. We assert that for a submanifold K of M with $k = \dim K < n$, the n-dimensional volume equals 0. In this case it is considered the $k-$dimensional volume $\mathrm{Vol}_k(K)$ of K, which is defined as the measure of K with respect to the Riemannian measure of the induced Riemannian metric via the embedding $\Phi : K \to M$. Note that $\mathrm{Vol}_k(K)$ is finite if K is compact, (see [22]).

It is usual take local charts to compute volumes or integrals because the computations reduce to multiple integrals in $\mathbb{R}^n$. For instance, we shall frequently use the local chart defined by the exponential map. So recall that if M and N are pseudo-Riemannian manifolds with volume elements dM and dN respectively and if $\phi : M \to N$ is a smooth function then

$$\phi^*(dN) = \det(\mathcal{J}(\phi))\, dM, \tag{2-28}$$

with $\mathcal{J}(\phi)$ the Jacobian matrix of ϕ. By definition, applying the formula to vectors $v_1, \ldots, v_n$ in T_pM gives

$$dN(d\phi(v_1), \ldots, d\phi(v_n)) = \det(\mathcal{J}(\phi))(p) dM(v_1, \ldots, v_n).$$

Provided ϕ is a local diffeomorphism $\det(\mathcal{J}(\phi))$ is nonvanishing, so that if $v_1, \ldots, v_n$ is a basis for T_pM, we will get

$$|\det(\mathcal{J}(\phi))(p)| = \frac{|dN(d\phi(v_1), \ldots, d\phi(v_n))|}{|dM(v_1, \ldots, v_n)|}.$$

This expression defines $|\det(\mathcal{J}(\phi))(p)|$ as the ratio of volume elements near p under the function ϕ.

Now we fix a pseudo-Riemannian manifold (M, g) and a point $p \in M$. Taking $\phi = \exp_p$ restricted to $V_r(0) \subset T_pM$ such that $\exp_p$ gives a diffeomorphism of $V_r(0)$ with its image $B_r(0) \subset M$, we can wonder about the absolute value of the determinant of $D_v \exp_p : T_v(T_pM) \simeq T_pM \to T_qM$, where q satisfies $q = \exp_p(v)$. Then it is enough to pick an orthonormal basis in each one of the vector spaces and the determinant of the matrix representing $D_v \exp_p$ with respect to these bases. The equation 2-28 says that if $d\boldsymbol{\omega_r}$ and $d\,\omega_r$ denote the volume elements of $B_r(0)$ and $V_r(0)$ respectively then

$$\exp_p^*(d\boldsymbol{\omega_r}) = \det(\mathcal{J})\, d\,\omega_r,$$

where $\mathcal{J} = \mathcal{J}(\exp_p)$. We shall use polar coordinates for calculating $\mathcal{J}$.

Let us fix any vector $v \in T_pM$, $v \neq 0$ whose representation in polar coordinates is $v = r \cdot u$, where u is a unit vector in the direction of v and $t \in [0, r]$ with $r > 0$. Let $\gamma : [0, r] \to M$ be the geodesic given by $\gamma(t) = \exp_p(tu)$. We can complete an orthonormal basis $\{u, e_2, \ldots, e_n\}$ to T_pM (related to the pseudo-Riemannian metric) and use parallel transport along γ to extend it to a frame $\{\dot\gamma(t), e_2(t), \ldots, e_n(t)\}$ of vector fields. Now we construct Jacobi fields $J_i(t)$ for $i = 2, \ldots, n$ along γ such that $J_i(0) = 0$ and $\frac{DJ_i}{dt}(0) = e_i$. It is clear that $D_0(exp_p)(u) = \dot\gamma(0) = u$ and also by Theorem 2.22 we know that

$$D_{tu} \exp_p(e_i) = \frac{1}{t} J_i(t).$$

Then we can write

$$\begin{aligned}\det(\mathcal{J}(\exp_p)(tu)) &= \det[(D_{tu}\exp_p(u), D_{tu}\exp_p(e_2)), \dots, D_{tu}\exp_p(e_n)] \\ &= \det[(\dot\gamma(t), \frac{1}{t}J_2(t), \dots, \frac{1}{t}J_n(t)] \\ &= \sqrt{\det([(\dot\gamma(t), \frac{1}{t}J_2(t), \dots, \frac{1}{t}J_n(t)][(\dot\gamma(t), \frac{1}{t}J_2(t), \dots, \frac{1}{t}J_n(t)]^T)}.\end{aligned}$$

But note that the expression inside the determinant is equal to

$$\begin{pmatrix} \langle\dot\gamma(t), \dot\gamma(t)\rangle & \langle\dot\gamma(t), \frac{1}{t}J_2(t)\rangle & \cdots & \langle\dot\gamma(t), \frac{1}{t}J_n(t)\rangle \\ \langle\frac{1}{t}J_2(t), \dot\gamma(t)\rangle & \langle\frac{1}{t}J_2(t), \frac{1}{t}J_2(t)\rangle & \cdots & \langle\frac{1}{t}J_2(t), \frac{1}{t}J_n(t)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\frac{1}{t}J_n(t), \dot\gamma(t)\rangle & \langle\frac{1}{t}J_n(t), \frac{1}{t}J_2(t)\rangle & \cdots & \langle\frac{1}{t}J_n(t), \frac{1}{t}J_n(t)\rangle \end{pmatrix}.$$

Because the parallel translation with respect to the Levi-Civita connection is an isometry, the vector $\dot\gamma(t)$ has the same length at each $t \in [0, r]$, which is equal to the length of the vector $\dot\gamma(0) : |\dot\gamma(t)| = |u|$. Also, Gauss's Lemma states that the orthogonality to radial directions, that is to the direction of u, is preserved. Consequently, we have $\langle\dot\gamma(t), \frac{1}{t}J_n(t)\rangle = 0$ and the matrix in the right side of the equation above turns to be:

$$\frac{1}{t^{2(n-1)}}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \langle J_2(t), J_2(t)\rangle & \cdots & \langle J_2(t), J_n(t)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \langle J_n(t), J_2(t)\rangle & \cdots & \langle J_n(t), J_n(t)\rangle \end{pmatrix}.$$

On the other hand, by the Chain Rule $\dot\gamma(t) = D_{tu}\exp_p(u)$ and if $q = \exp_p(tu)$ is represented in normal coordinates by $(x_1, \dots, x_n)$ then

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_q, \dots, \left.\frac{\partial}{\partial x_n}\right|_q \right\} = \left\{ \dot\gamma(t), \frac{1}{t}J_2(t), \dots, \frac{1}{t}J_n(t) \right\}$$

will be the canonical basis of T_qM using Theorem 2.23. We conclude that under the local polar normal coordinates (t, u) of q, the volume element at $q = \exp_p(tu)$ transforms to

$$dV_g = \sqrt{\det\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle}\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det\langle J_i(t), J_j(t)\rangle}\, dt \wedge d\omega_1$$

where $d\omega_1$ is the volume element of $\mathbb{S}^{n-1}(1)$, regarded as the unit sphere of T_pM. Also replacing in the equation 2.2.1, we get

$$
\begin{aligned}
\exp_p^*(d\boldsymbol{\omega_r}) &= \det(\mathcal{J}(\exp_p))\, d\,\omega_r \\
&= \frac{1}{t^{n-1}} \sqrt{\det\langle J_i(t), J_j(t)\rangle}\, d\,\omega_r \\
&= \sqrt{\det\langle J_i(t), J_j(t)\rangle}\, d\,\omega_1
\end{aligned}
$$

Example 2.27. For spaces of constant sectional curvature, for instance $K = 1, 0, -1$, it is possible to compute the ratio of the volume elements of $B_r(p)$ and $V_r(0)$ at $q = \exp_p(tu)$. By Proposition 2.21, $\det J(\exp_p)(tu)$ equals $\left(\frac{\sin t}{t}\right)^{n-1}, 1,$
$\left(\frac{\sinh t}{t}\right)^{n-1}$ in the positively curved, flat and negatively curved case respectively ([12]).

2.2.2 Some Geometric Aspects of pseudo-Riemannian Manifolds

The following section is about some geometric differences and facts which arise from consider specifically a pseudo-Riemannian ambient. The situation changes considerably and there are various pitfalls to watch out for in this case. For instance, when we pass to Lorentzian metric, "spheres" are no longer compact and good analogue for the notion of (geodesic) completeness of Riemannian manifolds does not exist, even more, the Hopf-Rinow Theorem has no satisfactory generalization.

Recall that a pseudo-Riemannian metric can be described by its index, besides each nonzero tangent vector can be classified according to the sign of the scalar product. That is, if $v \in T_pM$ for (M, g) a pseudo-Riemannian manifold and $p \in M$, v is called *timelike, null, spacelike* or *nonspacelike* whether $g_p(v, v) < 0, = 0, > 0$ or ≤ 0, respectively. The category into which a given tangent vector falls is called its *causal character*. The set C_p of all null vectors in T_pM is called the *nullcone* at p (Figure **2-2**).

A curve α in M is *spacelike* if all of its velocity vectors $\dot{\alpha}(t)$ are spacelike; similarly for *timelike* and *null*. An arbitrary curve does not need to have one of these causal characters, but a geodesic γ always does since $\dot{\gamma}$ is parallel, and parallel translations preserves causal character of vectors.

Globally Hyperbolicity

Suppose (M, g) a Lorentzian manifold (one whose metric g has index $\nu = 1$), thus all of its tangent spaces are defined to be a *Lorentz vector space*, that is, a vector space of dimension ≥ 2 furnished with the Minkowski metric. Each point of a Lorentzian manifold has tangent vectors of any causal character, so we shall call *nonspacelike curve* to one whose tangent vectors are all nonspacelike.

Let W be a subspace of a Lorentz vector space V, and let g be the scalar product of V. The causal character of W is naturally defined. We distinguish three mutually exclusive possibilities for the restricted product scalar $g|_W$:

1) $g|_W$ is positive definite, then W is said to be *spacelike*.
2) $g|_W$ is nondegenerate of index 1, then W is *timelike*.

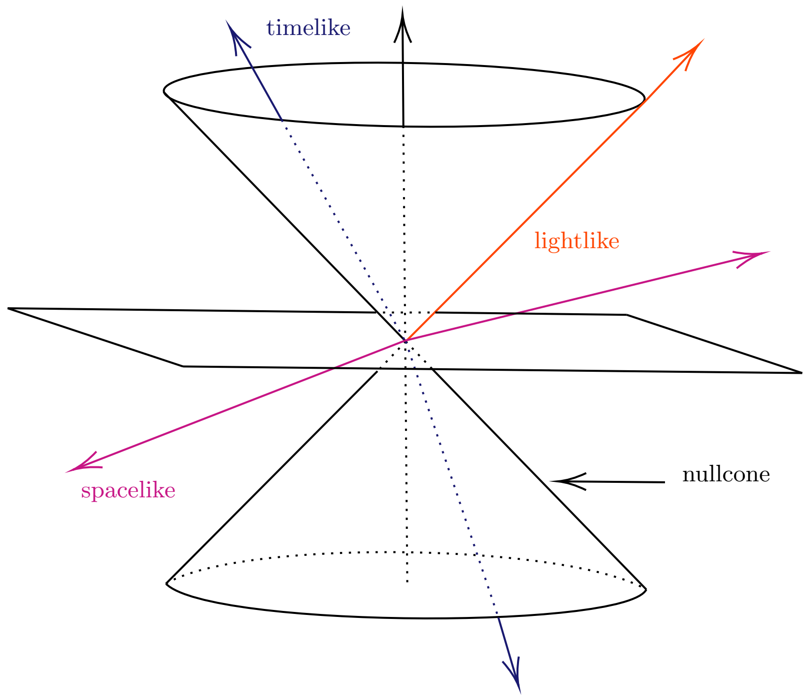

Figure **2-2**: Location of the causal character in $\mathbb{R}^3_1$.

3) $g|_W$ is degenerate, then W is *lightlike*.

In the next proposition, $\mathbb{R}v$ denotes the vector subspace generated by a vector $v \in V$.

Proposition 2.28. If z is a timelike vector in a Lorentz vector space, then the subspace $z^\perp = \{v \in V : g(z,v) = 0\}$ is spacelike and V is the direct sum $\mathbb{R}z + z^\perp$.

The proof given in [21] is simple since $\mathbb{R}z$ is nondegenerate with index 1, hence $z^\perp$ is also nondegenerate and the direct sum holds. Owing to $\operatorname{ind} V = \operatorname{ind} \mathbb{R}z + \operatorname{ind} z^\perp$, we obtain $\operatorname{ind} z^\perp = 0$ and so $z^\perp$ is spacelike.

With this justification at hand, we can generalize the assertion to any vector subspace of a Lorentz vector space. In fact, W is timelike if and only if $W^\perp$ is spacelike. It analogously occurs if we reverse the role of the words timelike and spacelike.

A smooth vector field V on a Lorentzian manifold (M, g) is *timelike* if $g_p(V_p, V_p) < 0$ for all points $p \in M$. If M admits a timelike vector field $V \in \mathfrak{X}(M)$, then M is said to be *temporally orientable* by V. When V is fixed, we shall say that M is *temporally oriented* by V. In this case, (M, g) has two distinct time orientations: the one defined by V and the one defined by $-V$.

Example 2.29. On the Minkoswki space $\mathbb{R}^n_1$, and in general on any Lorentzian manifold, the vector field which associates either the vector $(1, 0, \ldots, 0)$ or $(-1, 0, \ldots, 0)$ at each point is timelike. Then this defines a time orientation for the manifold.

Let M a Lorentzian manifold temporally oriented by a timelike vector field V. Note that the timelike vector field V divides all nonspacelike tangent vectors (and so the interior of each null cone

C_p of M) into two separate classes. A nonspacelike tangent vector $v \in T_pM$ is said to be *future* (respectively, *past*) *directed* if $g_p(V_p, v) \leq 0$ (respectively, $g_p(V_p, v) \geq 0$). The set

$$D(V_p) = \{v \in T : \langle V_p, v\rangle < 0\}$$

is the *solid future directed cone* containig V_p. Then the *solid past directed cone* contains $-V_p$ and is defined as

$$D(-V_p) = -C(V_p) = \{v \in T : \langle V_p, v\rangle > 0\}.$$

From now on M will denote a connected temporally-oriented Lorentzian manifold of dimension n. At this very instant, we define two *causality relations* on the points of M. It is standard to write $p \ll q$ if there is a future directed timelike curve in M from p to q, and $p \leq q$ if $p = q$ or if there is a future directed nonspacelike curve in M from p to q. The *chronological past* and *future* of p (see Figure 2-3) are then given respectively by

$$I^-(p) = \{q \in M : q \ll p\},$$
$$I^+(p) = \{q \in M : p \ll q\}.$$

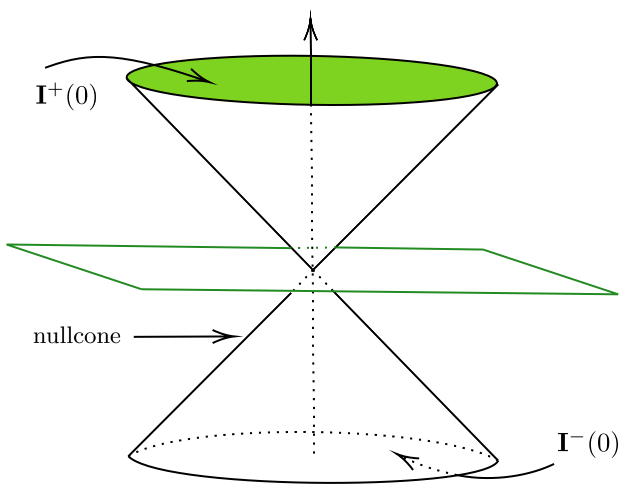

Figure **2-3**: Chronological past and future of $0 \in \mathbb{R}^3_1$.

The *causal past* and *future* of p are defined as

$$J^-(p) = \{q \in M : q \leq p\},$$
$$J^+(p) = \{q \in M : p \leq q\}.$$

Prior to one of the main definitions of this section, we mention that a Lorentzian manifold M is said to satisfy the *causality condition* if it does not contain any closed nonspacelike curve. Also,

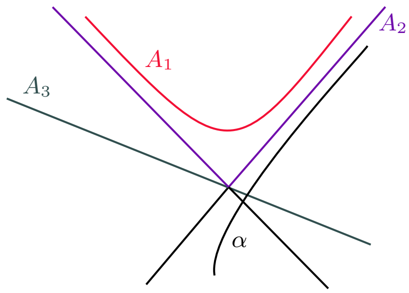

Figure **2-4**: Only the subset A_3 is a Cauchy hypersurface since, for instance, the inextendible timelike curve α does not hit A_1 or A_2 (see [7]).

it is said that M satisfies the *strong causality condition* if for each point $p \in M$ and for each open neighbourhood U of p, there exists an open neighbourhood $V \subset U$ of p such that each nonspacelike curve in M starting and ending in V is entirely contained in U. Obviously, the strong causality condition implies the causality condition.

Definition 2.30. A connected temporally-oriented Lorentzian manifold M is called *globally hyperbolic* if it satisfies the strong causality condition and if for all $p, q \in M$ the intersection $J^+(p) \cap J^-(q)$ is compact.

The globally hyperbolicity has the advantage of being easily identified in many cases. In order to state a characterization of globally hyperbolic manifolds which we shall significantly use later, we introduce the following concepts. A curve in a M is called *inextendible*, if no reparametrization of the curve can be continuously extended to any of the end points of the parameter interval. A subset S of M is a *Cauchy hypersurface* if each inextendible timelike curve in M meets S at exactly one point (see Figure **2-4**). Any two Cauchy hypersurfaces in M are homeomorphic. An important result of Lorentzian Geometry affirms that a Lorentzian globally hyperbolic manifold is topologically (although not always metrically) a product $\mathbb{R} \times S$, (see [5]). We omit the proof of the next theorem as it is composed of several parts. In [5], [8] and references therein more details can be found.

Theorem 2.31. Let M be a connected temporally oriented Lorentzian manifold. Then the following are equivalent:

i) M is globally hyperbolic.

ii) There is a Cauchy hypersurface in M.

iii) M is isometric to $\mathbb{R} \times S$ with metric $-\beta dt^2 + g_t$ where $\beta : \mathbb{R} \times S \to \mathbb{R}$ is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.

Example 2.32. In the Minkowski space $\mathbb{R}^n_1$, every spacelike hyperplane $\{t\} \times \mathbb{R}^{n-1}$ is a Cauchy hypersurface. Then we decompose $\mathbb{R}^n_1 = \mathbb{R}^1_1 \times \mathbb{R}^{n-1}$, where we consider the metric $-dt^2 + g_t$, being g_t the Euclidean metric on $\mathbb{R}^{n-1}$ which does not depend on t. Hence $\mathbb{R}^n_1$ is globally hyperbolic. In general, for an open interval $I \subset \mathbb{R}$ and (S, g) a Riemannian manifold, a product space $I \times S$ with metric $-dt^2 + g$ is globally hyperbolic if and only if (S, g) is complete. This applies in particular if (S, g) is compact. We shall return to this case later.

It is worthwhile making here a remarkable annotation. According to Hopf-Rinow Theorem for Riemannian manifolds (see [11, Chapter 7, Theorem 2.8]), any pair of points on a complete manifold may be joined by a geodesic of minimal length. A Lorentzian analogue arises in the sense of using the compactness of $J^+(p) \cap J^-(q)$ it can be shown that globally hyperbolic manifolds always contain maximal nonspacelike geodesics joining any two causally related points. This result and a background related to Lorentzian geometry can be seen in ([5, Chapter 6, Theorem 6.1]) and [21].

Hyperquadrics

Let $\tilde{M}$ be a pseudo-Riemannian submanifold of a pseudo-Riemannian manifold M. By definition, each tangent space $T_p\tilde{M}$ is a nondegenerated subspace of T_pM. Then for each $p \in \tilde{M}$, there is a well-defined subspace $(T_p\tilde{M})^\perp$ of T_pM consisting of all orthogonal vectors to $T_p\tilde{M}$. These vectors will be call *normal to* $\tilde{M}$ *at* p and thus $(T_p\tilde{M})^\perp$ is the *normal space at* p. It is clear that the direct sum

$$T_pM = T_p\tilde{M} + (T_p\tilde{M})^\perp$$

holds and $(T_p\tilde{M})^\perp$ is also nondegenerate. A vector field $Z \in \mathfrak{X}(M)$ is *normal* to $\tilde{M}$ provided each value $Z(p)$ is normal to $\tilde{M}$. The set of all vector fields normal to $\tilde{M}$ is often denoted by $\mathfrak{X}(\tilde{M})^\perp$.

A *pseudo-Riemannian hypersurface* N of M is just a pseudo-Riemannian submanifold of codimension 1. Then the one-dimensional normal subspace $(T_pN)^\perp$ has index at most one. Both cases are summarised as follow.

Definition 2.33. The *sign* ε of a pseudo-Riemannian manifold hypersurface N of M is

- $+1$ if the index of $N_p^\perp$ is 0, that is, $\langle z, z\rangle > 0$ for every normal vector $z \neq 0$;
- -1 if the index of $N_p^\perp$ is 1, that is, $\langle z, z\rangle < 0$ for every normal vector $z \neq 0$.

Note that in the indefinite case, $\operatorname{ind} M = \operatorname{ind} N$ if $\varepsilon = 1$, but $\operatorname{ind} M = \operatorname{ind} N - 1$ if $\varepsilon = -1$. It is now our interest to describe the *hyperquadrics* of $\mathbb{R}^{n+1}_\nu$, the analogous hypersurfaces to standard spheres of $\mathbb{R}^{n+1}$. As in the Riemannian case, the preimage of a regular value of a smooth real-valued function is a pseudo-Riemannian hypersurface. Namely, if b is a regular value of a smooth real-valued function $f : M \to \mathbb{R}$, for any $a \in f^{-1}(b)$ we have $\operatorname{Ker}(D_af) = T_a(f^{-1}(b))$ and owing to f is submersion $\dim(T_a(f^{-1}(b))) = n - 1$. Due to $D_af : T_aM \to T_b\mathbb{R} \cong \mathbb{R}$ is a linear functional, there exists $w \in T_aM$ such that $D_af(v) = \langle w, v\rangle$ for all $v \in T_aM$. In particular, since $\langle w, v\rangle = 0$ for $v \in T_a(f^{-1}(b))$, we have that $w \in (T_a(f^{-1}(b)))^\perp$. Also, f is submersion, then D_af is non-zero and we have that $D_af(w) = \langle w, w\rangle \neq 0$. So w can not belong to $T_a(f^{-1}(b))$. Due to $\dim(T_a(f^{-1}(b))) = n-1$, it span the vector space $(T_a(f^{-1}(b)))^\perp$. We conclude that $T_a(f^{-1}(b)) \oplus (T_a(f^{-1}(b)))^\perp = T_aM$. It implies that $f^{-1}(b)$ is a pseudo-Riemannian submanifold of M.

Proposition 2.34. Let f be a smooth real-valued function on a pseudo-Riemannian M and let c be a value of f. Then $N = f^{-1}(c)$ is a pseudo-Riemannian hypersurface of M if and only if $\langle \operatorname{grad} f, \operatorname{grad} f\rangle$ is > 0 or < 0 on N. In this case the sign of M is the (constant) sign of $\langle \operatorname{grad} f, \operatorname{grad} f\rangle$, and $U = \operatorname{grad} f / |\operatorname{grad} f|$ is a unit normal vector field on N.

Proof. Since the gradient satisfies

$$\langle \operatorname{grad} f, X\rangle = df(X) = X(f), \tag{2-29}$$

it is firstly noted that if $\langle \operatorname{grad} f, \operatorname{grad} f\rangle = 0$ at some point $p \in N$, then $\operatorname{grad} f(p) = 0$ and so df_p is not surjective. Conversely, suppose $\langle \operatorname{grad} f, \operatorname{grad} f\rangle$ is > 0 or < 0 at each point $p \in N$. If $\operatorname{Im}(df_p) = 0$, $\langle \operatorname{grad} f(p), v\rangle = 0$ for each $v \in N_p$ and so $\operatorname{grad} f(p) = 0$ which contradicts the hypothesis. It follows that df_p is surjective and N is a hypersurface by Regular Level Set Theorem (see [18, Corollary 5.14]). Now, without loss of generality assume N is connected, then the same condition for $\operatorname{grad} f$ ensures that the one-dimensional normal space at each point of N is nondegenerate and hence T_pN is nondegenerate. We conclude N is pseudo-Riemannian. Finally, owing to the fact that f is constant on M,

$$\langle \operatorname{grad} f, v\rangle = v(f) = v(f|_M) = 0$$

for any $v \in T_pN$, so $\operatorname{grad} f$ is normal to N. ◇

For instance the standard $n-$sphere $\mathbb{S}^n(r)$ is certainly a hypersurface in $\mathbb{R}^{n+1}$, since for $f = \Sigma(x^i)^2$, we have $\operatorname{grad} f$ is zero only at the origin and thus $\mathbb{S}^n(r) = f^{-1}(r^2)$, $r > 0$. Likewise, there is an important family of pseudo-Riemannian hypersurfaces in the Minkowski space $\mathbb{R}^{n+1}_\nu$ that is defined as it follows. Let q be the smooth function $q(v) := \langle v, v\rangle$ for each tangent vector v to $\mathbb{R}^{n+1}_\nu$. At each point $p \in \mathbb{R}^{n+1}_\nu$, q gives the bilineal form that defines the scalar product at p. Relative to standard coordinates,

$$q = -\sum_{i=1}^{\nu}(x_i)^2 + \sum_{j=\nu+1}^{n+1}(x_j)^2.$$

For the vector field $P = \sum x_i\partial_i$ of $\mathbb{R}^n_\nu$, $q = \langle P, P\rangle$. Let us take $V \in \mathfrak{X}(\mathbb{R}^n_\nu)$, by (2-29) and since ∇ is compatible with the metric,

$$\langle \operatorname{grad} q, V\rangle = Vq = V\langle P, P\rangle = 2\langle \nabla_V P, P\rangle,$$

where $\nabla_V P = V$ (see (2-6)). Thus

$$\langle \operatorname{grad} q, V\rangle = 2\langle V, P\rangle,$$

and by nondegeneracy of $\langle\,,\rangle$ it follows $\operatorname{grad} q = 2P$. Then we have $\langle \operatorname{grad} q, \operatorname{grad} q\rangle = 4\langle P, P\rangle = 4q$ and by Proposition 2.34, it concludes that for $r > 0$ and $\epsilon = \pm 1$, $Q := q^{-1}(\varepsilon r^2)$ is a pseudo-Riemannian hypersurface of $\mathbb{R}^{n+1}_\nu$. The two families $\varepsilon = 1$ and $\varepsilon = -1$ fill all of $\mathbb{R}^{n+1}_\nu$ except the set $q^{-1}(0)$, which consists of the nullcone $\Lambda = q^{-1}(0) - \{0\}$ and the origin 0. We assert without proof that Λ is diffeormorphic to $(\mathbb{R}^\nu - 0) \times S^{n-\nu}$ although it is not a pseudo-Riemannian manifold since there is a vector field of $\mathbb{R}^{n+1}_\nu$ which is both tangent and normal to Λ, (see [21, Chapter 4, Proposition 22]).

Definition 2.35. Let $n \geq 2$ and $0 \leq \nu \leq n$. Then

(1) The *pseudosphere* of radius $r > 0$ in $\mathbb{R}^{n+1}_\nu$ is the hypersurface

$$\mathbb{S}^n_\nu(r) = q^{-1}(r^2) = \{p \in \mathbb{R}^{n+1}_\nu : \langle p, p \rangle = r^2\}$$

with dimension n and index ν.

(2) The *pseudohyperbolic space* of radius $r > 0$ in $\mathbb{R}^{n+1}_{\nu+1}$ is the hypersurface

$$\mathbb{H}^n_\nu(r) = q^{-1}(-r^2) = \{p \in \mathbb{R}^{n+1}_{\nu+1} : \langle p, p \rangle = -r^2\},$$

with dimension n and index ν.

Since on pseudospheres $\langle \operatorname{grad} q, \operatorname{grad} q \rangle = 4r^2$, these have sign $+1$ and their index in $\mathbb{R}^n_\nu$ is ν. On pseudohyperbolic spaces $\langle \operatorname{grad} q, \operatorname{grad} q \rangle = -4r^2$, thus these have sign -1 and have index $\nu - 1$ in $\mathbb{R}^n_\nu$. Of course for $\nu = 0$, $\mathbb{S}^n_0(r)$ is just the standard sphere $\mathbb{S}^n(r)$ in Euclidean space $\mathbb{R}^{n+1}_0 = \mathbb{R}^{n+1}$. Then, we claim the only Riemannian manifolds of codimension one, aside from spheres, in Minkowski space $\mathbb{R}^{n+1}_1$ are the hypersurfaces $H^n_0(r)$. Additionally, it can be shown that $\mathbb{S}^n_\nu(r)$ is diffeomorphic to $\mathbb{R}^\nu \times S^{n-\nu}$ while $\mathbb{H}^n_\nu(r)$ is diffeomorphic to $S^\nu \times \mathbb{R}^{n-\nu}$. We cite [21] for this and more facts about hyperquadrics.

In the following example, we shall address our attention to the expression in coordinates of the metric on the pseudosphere $\mathbb{S}^n_1(r)$ induced by the Minkowski metric of the ambient space $\mathbb{R}^n_1$. The computations for the pseudohyperbolic space $\mathbb{H}^n_0(r)$ are similar, but we shall omit them owing to in this case the resulting metric is no longer Lorentzian (while the one of the pseudosphere is) and so it loses interest for our purposes.

Example 2.36. We shall proceed to get a Lorentzian metric on $\mathbb{S}^n_1$ as the induced metric via the inclusion map $\iota : \mathbb{S}^n_1 \hookrightarrow \mathbb{R}^{n+1}_1$. Then, we consider a parametrization of $\mathbb{S}^n_1$ given by

$$\begin{aligned}
x_1 &= r \sinh(t) \\
x_2 &= r \cosh(t) \cos(\theta_1) \\
x_3 &= r \cosh(t) \sin\theta_1 \cos(\theta_2) \\
&\ \vdots \\
x_{n+1} &= r \cosh(t) \cos(\theta_1) \dots \sin\theta_{n-1}
\end{aligned}$$

for suitable choice of the variables $t \in \mathbb{R}$, $\theta_1, ..., \theta_{n-1}$. Thus the metric on $\mathbb{S}^n_1(r)$ has the expression:

$$dh^2 = -r^2\, dt^2 + r^2 \cosh^2(t)\, ds^2, \tag{2-30}$$

where ds denotes the round metric on the standard sphere $\mathbb{S}^{n-1}(r)$. In the literature, it space is called *deSitter spacetime*. Note that this pseudo-Riemannian manifold has metric of index -1.

Warped Products

Warped product spaces are pseudo-Riemannian manifolds arising from interesting topics, mainly in theoretical physics (see for example [21, Chapter 12]). We shall focus on the case when warped product metrics are used to construct Lorentzian manifolds.

If (M, g) and (K, h) are pseudo-Riemannian manifolds, there is a natural product metric $\bar{g}$ on the product manifold $M \times K$ such that $(M \times K, \bar{g})$ is again a pseudo-Riemannian manifold. By using any smooth function $f : M \to (0, \infty)$, which we shall call *warping function*, we can obtain on $M \times K$ a rich class of metrics, including products. Throught this section, we shall let $\pi : M \times K \to M$ and $\sigma : M \times K \to K$ denote the smooth projection maps given by $\pi(p, q) = p$ and $\sigma(p, q) = q$ for $(p, q) \in M \times K$.

Definition 2.37. The *warped product* $M \times_f K$ is the product manifold $M \times K$ furnished with the metric

$$\bar{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(h).$$

Explicitly, if v, w are tangent vectors to $M \times K$ at (p, q), then

$$\bar{g}(v, w) = g(d\pi(v), d\pi(w)) + f^2(p) \cdot h(d\sigma(v), d\sigma(w)).$$

If $f = 1$, then $M \times_f K$ reduces to a pseudo-Riemannian product manifold. In addition, Lorentzian warped product spaces can be constructed from products of Lorentzian and Riemannian manifolds. We only deal with product manifolds such that the first factor is Lorentzian and the second one is Riemannian.

Example 2.38. *i)* The metric defined in Example 2.36 allows us to interpreted the pseudo-sphere $\mathbb{S}^n_1$ as a Lorentzian warped product space $\mathbb{R} \times_f \mathbb{S}^{n-1}$, where $f(t) = \cosh(t)$ and the metric on $\mathbb{S}^{n-1}$ is the round metric.

ii) In Example 2.32 the Minkowski space was written as $\mathbb{R}^1_1 \times \mathbb{R}^{n-1}$ and equipped with the metric $-dt^2 + g_t$. Clearly it is a Lorentzian warped metric with $f(t) = 1$.

In order to give some properties of warped products, we define a diffeomorphism $\Psi : (M_1, g_1) \to (M_2, g_2)$ of pseudo-Riemannian manifolds to be a *homothety* if $\Psi^*(g_2) = c\, g_1$, for some constant c. From [5] and [21], we mention that the warped metric is characterized by

(1) For each $q \in K$, the map $\pi|(M \times q)$ is an isometry onto M.

(2) For each $p \in M$, the map $\sigma|(p \times K)$ is a positive homothety onto K, with $c = 1/f(p)$.

(3) For each $(p, q) \in M \times K$, the submanifolds $\pi^{-1}(p)$ and $\sigma^{-1}(q)$ are nondegenerate and orthogonal at (p, q).

(4) If $\phi : K \to K$ is an isometry, then the map $\Phi = 1 \times \phi : M \times_f K \to M \times_f K$ given by $\Phi(m, k) = (m, \phi(k))$ is an isometry of $M \times_f K$.

(5) If $\psi : M \to M$ is an isometry such that $f \circ \psi = f$, then the map $\Psi = \psi \times 1 : M \times_f K \to M \times_f K$ given by $\Psi(m, k) = (\psi(m), k)$ is an isometry on $M \times_f K$.

Proposition 2.39. The leaves $\sigma^{-1}(q) = M \times q$ of a warped product are totally geodesic.

From the geometry of M and K, we can describe that of the pseudo-Riemannian product $M \times K$. To express this simply we now introduce some terminology. The projections π and σ allow us "lift" tangent vector, and so vector fields, from each factor to $M \times K$. For instance, if $v \in T_pM$ and $q \in K$ then the *lift* $\tilde{v}$ of v to (p, q) is the unique vector in $T_{(p,q)}M$ such that $d\pi(\tilde{v}) = v$. Also, if $V \in \mathfrak{X}(M)$ the *lift* of V to $M \times H$ is the smooth vector field $\tilde{V}$ whose value at each (p, q) is the lift of V_p to (p, q). We get the similar definition using the projection σ to lift vectors and vector fields on K to $M \times K$. Despite having different domains, we shall denote tangent vectors, vector fields and their lifts without tilde.

We can temporally orient a Lorentzian warped product space provided the Lorentzian factor is temporally oriented. Let $(M \times_f K, \bar{g})$ be as before with the Lorentzian manifolds M oriented by a timelike vector field V. For $p = (m, k) \in M \times K$, consider the isomorphism

$$T_p(M \times_f K) = T_p(M \times K) = T_mM \times T_kK.$$

Then, we identify $T_p(M \times K)$ with $T_mM \times T_kK$ to define the lift of V to a timelike vector field $\widetilde{V}$ on $M \times K$ such that $\widetilde{V}_p = V_m$. Thus we define $\widetilde{V}$ at p setting $\widetilde{V}_p = (V_m, 0_k)$. Note that $d\sigma(V_p) = 0_k$ and so, by definition of warped metric, $\bar{g}(\widetilde{V}, \widetilde{V}) = g(V, V) < 0$. Thus $\widetilde{V}$ temporally orients $M \times_f K$. Note that if $\dim M = 1$, namely M is diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$, we have consider g to be negative definite and take a smooth vector field W on M with $g(W, W) = -1$. Then the vector field $\widetilde{W}$ defined by $\widetilde{W}_p = (W_m, 0_k)$ as above, orients $M \times_f K$ temporally.

As far as we are concerned, we shall take $I \times_f K$ to be an $n-$dimensional Lorentzian manifold written in the form of a Lorentzian warped product, where $I = (a, b)$ with $-\infty < a < b \le \infty$ is given the negative definite metric $-dt^2$, (K, h) a Riemannian manifold of dimension $n-1$ (that is, a spacelike hypersurface) and $f : I \to (0, \infty)$ is a warping function. We shall write its metric as $\bar{g} = -dt^2 + (f(t))^2h$, as usual.

Let $U = \partial_t$ be the lift to $I \times_f K$ of the standard vector field d/dt on I. For each $p \in K$, $I \times \{p\}$ can be parametrised by the curve $\gamma(t) = (t, p)$. Observe that U gives the velocity of each such γ. Holding t constant gives the hypersurface $K_t = \{t\} \times K$. In summary, the geometry here satisfies: (i) $\langle U, U\rangle = -1$ and (ii) U is normal to K_t for all $t \in I$. We mention now other geometric properties having to do with future calculations. Their proof and their consequences can be read in [21] and [10]. We shall denote respectively by ∇ and R the Levi-Civita connection and the curvature of $I \times_f K$.

Proposition 2.40. If V is the lift of a vector field on K to $I \times_f K$, then

i) $\nabla_U U = 0$,

ii) $\nabla_U V = \nabla_V U = \dfrac{f'}{f} V$, *iii)* $R(V,U)U = \dfrac{f''}{f}V.$

A direct consequence of *i)* is each γ is a geodesic. By *iii)* and Definition 2.17, every plane containing a vector of U has sectional curvature f''/f.

Warped product spaces $M \times_f K$ fail in general to be globally hyperbolic. For instance, regardless of the chosen Riemannian metric h for K, no warped product $\mathbb{S}^1 \times_f K$ is globally hyperbolic, (see [5, Page 104]). Nevertheless, Theorem 2.31 guarantees that in spaces $(\mathbb{R} \times S, -dt^2 + h)$, with S a Cauchy hypersurface in $\mathbb{R} \times S$, this property is fullfied. This suggests there must be conditions in order to the warped product $M \times_f K$, where $\dim M = 1$, is globally hyperbolic. Conditions when $\dim M \geq 2$ also exist. Both results and the proof are in [5].

Theorem 2.41. Let (K,h) be a Riemannian manifold, and let $I = (a,b)$ with $-\infty \leq a < b \leq \infty$ be given the negative metric $-dt^2$. Then the Lorentzian warped product $(I \times_f K, \bar{g})$ with $\bar{g} = -dt^2 + (f(t))^2 h$ is globally hyperbolic if and only if (K,h) is complete.

In particular, consider K either as a compact space or as a closed submanifold of a complete manifold makes the warped product a globally hyperbolic space.

2.2.3 Isometry Groups

In this section we pick up on definition of isometry focusing on the pseudo-Riemannian context. If M is an arbitrary pseudo-Riemannian manifold, its pseudo-Riemannian metric makes each of its tangent spaces a $\mathbb{R}^n_\nu$ of the same dimension and index as M itself in the sense that a scalar product space of dimension n and index ν is isometric, as a pseudo-Riemannian, to $\mathbb{R}^n_\nu$. Thus an approach to the isometry group $I(\mathbb{R}^n_\nu)$ is interesting. Let us begin by studying functions that preserve the scalar product of $\mathbb{R}^n_\nu$.

Definition 2.42. A linear transformation $\Lambda : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is said to be a **pseudo-orthogonal transformation** if

$$\langle \Lambda v, \Lambda w \rangle = \langle v, w \rangle,$$

for all $v, w \in \mathbb{R}^n_\nu$.

By definition, any pseudo-orthogonal transformation is an isometry. Also, recall that a linear transformation has a matrix representation, then under such an identification the set of pseudo-orthogonal transformations is the same as the set of all matrices $\Lambda \in GL(n,\mathbb{R})$ that preserve the scalar product of $\mathbb{R}^n_\nu$. It is called the *pseudo-orthogonal group* and we shall denote it by $\mathbb{O}_\nu(n)$. When the considered metric on $\mathbb{R}$ is the Euclidean one or its negative, the group $\mathbb{O}_\nu(n)$ is the well known *orthogonal group* $\mathbb{O}(n)$. For $n \geq 2$, $\mathbb{O}_1(n)$ is called the *Lorentz group*.

Let $I_{\nu,n-\nu}$ denote the diagonal matrix (d_{ij}) with $d_{ii} = -1\,(1 \leq i \leq \nu)$, $d_{jj} = 1\,(\nu+1 \leq j \leq n)$. Then Λ with transpose $\Lambda^\top$ belongs to $\mathbb{O}_\nu(n)$ if and only if $\Lambda^\top I_{\nu,n-\nu} \Lambda = I_{\nu,n-\nu}$. It implies $\det \Lambda = \pm 1$

and $\mathbb{O}_\nu(n)$ is a closed subgroup of $GL(n,\mathbb{R})$. We shall discuss now some facts about compactness and connectivity of the groups $\mathbb{O}_\nu(n)$, the next example turns out to be illustrative.

Example 2.43. $\mathbb{O}(2)$. In $\mathbb{R}^2$, each number $\theta \in \mathbb{R}$ determines a orthogonal transformation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ represented by orthogonal matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In essence, R_θ is a rotation of $\mathbb{R}^2$ around de origin through (oriented) angle θ. The function $\theta \mapsto R_\theta$ is a smooth homomorphism from $\mathbb{R}$, under addition, into $\mathbb{O}(2)$. Its kernel is $2\pi\mathbb{Z}$ and its image is the rotation group $\mathbb{O}^+(2)$, the component of the identity in $\mathbb{O}(2)$. Thus the two components of $\mathbb{O}(2)$, $\mathbb{O}^+(2)$ and $\mathbb{O}^-(2)$, are diffeomorphic to circles.

$\mathbb{O}_1(2)$. Given $\varphi \in \mathbb{R}$, the pseudo-orthogonal matrix

$$\Lambda_\varphi = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$$

represents a pseudo-orthogonal transformation $\Lambda_\varphi : \mathbb{R}^2_1 \to \mathbb{R}^2_1$. As above, $\varphi \mapsto \Lambda_\varphi$ is a homomorphism, but in this case it is one-to-one. Also, note that Λ_φ preserves each of the four branches of the set of the hyperbolas $\{p \in \mathbb{R}^2_1 | \langle p,p\rangle \pm 1\}$. However, any $\Lambda \in \mathbb{O}_1(2)$ must carry each hyperbola into itself but may reverse the branches of each. These two choices split $\mathbb{O}_1(2)$ into four disjoint open subsets and the set $\mathcal{B}$ containing Λ_φ is precisely the component of the identity in $\mathbb{O}_1(2)$ and it is a subgroup diffeomorphic to $\mathbb{R}^1$. Of course this preserves all branches. The other three sets are cosets of $\mathcal{B}$, hence $\mathbb{O}_1(2)$ has four components each diffeomorphic to $\mathbb{R}^1$.

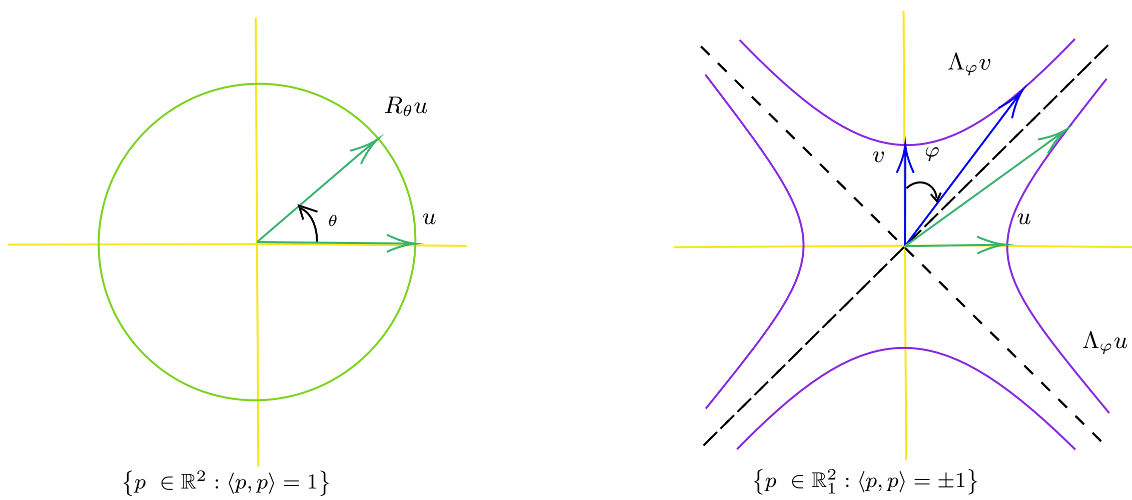

Figure **2-5**: Orthogonal and pseudo-orthogonal transformations.

We can realise with the example that $\mathbb{O}_1(2)$ has four components and is not compact in $\mathbb{R}^4$ (since it is unbounded) unlike $\mathbb{O}(2)$ that has two components and is compact in $\mathbb{R}^4$ (see Figure

2-5). These differences remain in the general case, namely for $\mathbb{O}_\nu(n)$ and $\mathbb{O}(n)$ with $0 < \nu < n$. For instance, elements of the form

$$\begin{pmatrix} \cosh\varphi & 0 & \sinh\varphi \\ 0 & I & 0 \\ \sinh\varphi & 0 & \cosh\varphi \end{pmatrix},$$

where I is the indentity matrix, constitute an unbounded subset of $\mathbb{R}^{n^2}$.

In order to describe the components of $\mathbb{O}_\nu(n)$ for ν fixed ($0 < \nu < n$), we consider the decomposition $\mathbb{R}^n_\nu = \mathbb{R}^\nu_\nu \times \mathbb{R}^{n-\nu}$ and use it to write $\Lambda \in \mathbb{O}_\nu(n)$ in block form as

$$\begin{pmatrix} \Lambda_T & b \\ c & \Lambda_S \end{pmatrix},$$

where Λ_T is a square matrix of order ν and Λ_S is a square matrix of order $n-\nu$. Here the *timelike part*, $\Lambda_T : \mathbb{R}^\nu_\nu \to \mathbb{R}^\nu_\nu$, is $\Lambda|_{\mathbb{R}^\nu_\nu}$ followed by orthogonal projection on $\mathbb{R}^\nu_\nu$, and similarly for the *spacelike part*, $\Lambda_S : \mathbb{R}^{n-\nu} \to \mathbb{R}^{n-\nu}$. Since Λ is invertible and preserve casual character, it follows that both Λ_T and Λ_S are invertible.

Theorem 2.44. [16, Chapter IV, Lemma 1.4] Let $\Lambda \in \mathbb{O}_\nu(n)$. Then

$$|\det(\Lambda_T)| \geq 1, \qquad |\det(\Lambda_S)| \geq 1.$$

The components of $\mathbb{O}_\nu(n)$ are obtained by

$$\det\Lambda_T \geq 1, \qquad \det\Lambda_S \geq 1; \tag{2-31}$$
$$\det\Lambda_T \geq 1, \qquad \det\Lambda_S \leq -1; \tag{2-32}$$
$$\det\Lambda_T \leq -1, \qquad \det\Lambda_S \geq 1; \tag{2-33}$$
$$\det\Lambda_T \leq -1, \qquad \det\Lambda_S \leq -1. \tag{2-34}$$

Thus $\mathbb{O}_\nu(n)$ has four components if $n \geq 1$, $n-\nu \geq 1$, two components if n or $n-\nu = 0$.

We adopt the notation of [21] for these disjoint sets. Respectively, we have

$$\mathbb{O}^{++}_\nu(n), \quad \mathbb{O}^{+-}_\nu(n), \quad \mathbb{O}^{-+}_\nu(n), \quad \mathbb{O}^{--}_\nu(n).$$

All of them are open, hence closed, sets of $\mathbb{O}_\nu(n)$. Moreover, it is possible to show that $\mathbb{O}^{++}_\nu(n)$, the identity component, is a normal subgroup of $\mathbb{O}_\nu(n)$ and it is connected if $0 \leq \nu \leq n$. Lastly, we mention that the *special pseudo-orthogonal group* is the subgroup of $\mathbb{O}_\nu(n)$ given by

$$\mathbb{SO}_\nu(n) = \{\Lambda \in \mathbb{O}_\nu(n) : \det\Lambda = 1\},$$

which is nothing but $\mathbb{O}_{\nu}^{++}(n) \cup \mathbb{O}_{\nu}^{--}(n)$.

Finally, we describe the group $I(\mathbb{R}^n_\nu)$. Let $a \in \mathbb{R}^n_\nu$ and $T_a : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ be the translation given by $v \mapsto v + a$. The set of all translations of $\mathbb{R}^n_\nu$ is an abelian subgroup of $I(\mathbb{R}^n_\nu)$. It can be proven there exists uniquely $a \in \mathbb{R}^n_\nu$ and $\Lambda \in \mathbb{O}_\nu(n)$ such that any isometry of $\mathbb{R}^n_\nu$ is written as $T_a \circ \Lambda$. On the other hand, the cartesian product $\mathbb{R}^n \times \mathbb{O}_\nu(n)$ furnished with the multiplication given by $(a, \Lambda_1)(b, \Lambda_2) = (T_{a+\Lambda_1(b)}, \Lambda_1\Lambda_2)$ becomes a group (the semi-direct product) denoted usually by $\mathbb{R}^n \rtimes \mathbb{O}_\nu(n)$. This group is isomorphic to $I(\mathbb{R}^n_\nu)$.

3 Action Groups and Integration

This chapter gives a brief description of polar actions on manifolds. This notion encloses geometric properties of a manifold described via its set of transformations. Contents of this chapter about topological groups and Lie groups were reviewed in Helgason [13] and Lee [18]. We also give an introduction to integration theory on locally compact groups in order to justify some definitions and computations in the following chapter. We present the main concepts and descriptions and detail some of the proofs. Our principal reference sources for the subject are Helgason [15] and Wijsman [23].

3.1 Polar Actions on Manifolds

We begin by considering the global viewpoint of the smooth actions on manifolds to introduce as general as we can the considerations and results. At the same time, we set a collection of needed definitions which will appear throughout the remainder of this work.

A topological group G is a set with the structure of both a group and a topological space such that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ given by

$$m(g, h) = gh, \qquad i(g) = g^{-1} \tag{3-1}$$

are continuous on G. Usually, the identity element is denoted by e. If $g \in G$ is fixed, then each of the maps $L_g : x \to gx$, $R_g : x \to xg$ and $x \to x^{-1}$ will be homeomorphisms of G.

It is said that G is a *Lie group* if it is also a smooth manifold in which the group operations 3-1 are smooth. Aditionally, it can be proved that for an element $g \in G$ the maps L_g and R_g are diffeomorphisms. It is common for a Lie group to be acting on some other manifold. Then if M is a manifold, a (*left*) *action* of G on M is a smooth map $(g, p) \mapsto g \cdot p$ of $\theta : G \times M \to M$ that satisfies $e \cdot p = p$ and $(g\hat{g}) \cdot p = g \cdot (\hat{g} \cdot p)$ for all $g, \hat{g} \in G$, $p \in M$. The *right action* $M \times G \to M$ of G on M is analogously defined. The smoothness here means that for each $g \in M$ the map $\theta_g : M \to M$ defined by $\theta_g(g, p) = g \cdot p$ is smooth. Certainly θ_g has smooth inverse map $\theta_{g^{-1}}$.

Consider a Lie group G acting on a manifold M, then the *orbit of* p under G is the set $G{\cdot}p = \{g{\cdot}p : g \in G\}$ in M and the *isotropy group* of p is the subgroup of G defined by $G_p = \{g \in G : g \cdot p = p\}$. We add that each orbit $G \cdot p$ is a immersed submanifold of M. In fact it is enough to observe that the *orbit map* defined by

$$\begin{aligned} \theta^{(p)} : G &\longrightarrow M \\ g &\mapsto g \cdot p, \end{aligned} \tag{3-2}$$

is smooth and has constant rank. Since it is inyective then it also is a immersion, thus its image, $\theta^{(p)}(G) = G \cdot p$, is an immersed submanifold of M.

It is said that the action of a Lie group G on a manifold M is *free* if every isotropy group is trivial. It is said to be *transitive* if for all $p, q \in M$ there exists $g \in G$ such that $g \cdot p = q$. Lastly the action is *proper* if the map $G \times M \to M \times M$ defined by $(g, p) \mapsto (g \cdot p, p)$ is proper, that is, if the inverse image of each compact set in $M \times M$ is also compact in $G \times M$. If the group acting is compact, it is known that the action becomes proper. Also it follows directly from the definition that each isotropy group of a proper action is compact, since the preimage of $(p, p) \in M \times M$ under such a map is $G_p \times \{p\}$. In addition, properness of an action implies properness of the orbit map $\theta^{(p)}$ for any $p \in M$, and owing to every proper continuous map is closed, its image $G \cdot p$ is closed in M.

Defining a equivalence relation on M by setting $p \sim q$ if there exists $g \in M$ such that $g \cdot p = q$, we can define the *orbit space* of the action as the topological quotient space M/G of the equivalence classes given by the relation. This space might have very unlikely topological properties. However this becomes a smooth manifold provided G acts smoothly, freely and properly. This is the case of the manifold G/H, where H is a closed subgroup of G. which acts on the left via the action $H \times G \to G$ such that $(h, g) \mapsto hg$.

Now, consider the orbit map (3-2) for a fixed $p \in M$ and let $g, \hat{g} \in G$ such that $\theta^{(p)}(g) = \theta^{(p)}(\hat{g})$. Then $g \cdot p = \hat{g} \cdot p$ which implies $\hat{g}^{-1} g \cdot p = p$. That means that $\hat{g}^{-1} g \in G \cdot p$ and since $g = \hat{g}(\hat{g}^{-1} g)$ we conclude that the orbit map $\theta^{(p)}$ can be passed throughout the quotient G/G_p as a map $\Theta : G/G_p \to G \cdot p$. Then we have the following theorem:

Theorem 3.1. Suppose G acts on M smoothly. Then the map $\Theta : G/G_p \to G \cdot p$, $gG_p = g \cdot p$ is a diffeomorphism.

Note that if the action of G on M is transitive, for any $p \in M$ we have $G \cdot p = M$. Then the identification of the points of M with the cosets gG_p is done regardless of the choice of $p \in M$ because the isotropy groups are all conjugate to each other, namely $G_{g \cdot p} = gG_p g^{-1}$. With this in mind, we define

Definition 3.2. A manifold M is a *homogeneous space* if a Lie group acts transitively on M.

Example 3.3.

- The Riemannian sphere $\mathbb{S}^n(r)$ is an homogeneous space. By the Graham-Smith procedure, we can prove there exists a transitive action from the group $\mathbb{S}O(n+1)$ on $\mathbb{S}^n(r)$. The stabilizer of any point is equivalent to the group $\mathbb{S}O(n)$ regarded as a subgroup of $\mathbb{S}O(n+1)$.
- The action of $I(\mathbb{R}^n_\nu) \cong \mathbb{R}^n \rtimes \mathbb{O}_\nu(n)$ on $\mathbb{R}^n_\nu$, given by $I(\mathbb{R}^n_\nu) \times \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$, $((a, \Lambda), p) \mapsto \Lambda(p) + a$, is isometric and transitive, while $\mathbb{O}_\nu(n)$ is the isotropy group of the point $0 \in \mathbb{R}^n_\nu$. This realises $\mathbb{R}^n_\nu$ as a homogeneous space and allows us to write $\mathbb{R}^n_\nu = \mathbb{R}^n \rtimes \mathbb{O}_\nu(n)/\mathbb{O}_\nu(n)$.

- Let G be topological group. Let H be a closed subgroup of G. We consider the action defined by the map $H \times G \to G$ such that $(h, g) \mapsto hg$. This left action is a free action on the topological space G, then the orbit space G/H inherits a topological structure from G. If G is a Lie group, we can prove the space G/H has a natural structure of smooth manifold. In that case, the smooth manifold G/H is endowed with a transitive action from the Lie group G, defined by the map $G \times G/H \to G/H$, via $(g, kH) \mapsto gkH$.

It is well known that an action of a group G on M is, formally, a group homomorphism from G into the group $\mathrm{Aut}(M)$ of all automorphism of M, namely each $g \in G$ determines a *transformation* of M which we have already denoted by $\theta_g : M \to M$ and defined to be diffeomorphism. However in the presence of a pseudo-Riemannian structure on M we wish to find actions which preserve it.

Definition 3.4. Let M be a pseudo-Riemmanian manifold and G be a Lie group acting smoothly on M. Either an action $\theta : G \times M \to M$ is an *isometric action* or the group G *acts by isometries* on M, if for each $g \in M$ the map $\theta_g : M \to M,\ p \mapsto g \cdot p$ is a bijective isometry from M to itself.

Observe that for pseudo-Riemannian manifolds, the automorphism group is nothing but the group of isometries which usually is denoted by $I(M)$. We claim $I(M)$, endowed with the compact-open topology, is a second-countable locally compact group (see definition below). Also it is Hausdorff owing to M is. In addition to this, there is a unique way to give to $I(M)$ a differentiable structure compatible with this topology such that $I(M)$ is a Lie group and the action $I(M) \times M \to M$ is smooth. According to the definition above, the group G acting can be assumed, without loss of generality, as a closed subgroup (and so a Lie subgroup) of $I(M)$. When the metric of M is positive definite, that is if M is a Riemanninan manifold, it is a fact that a subgroup $G \subset I(M)$ acts properly on M if and only if G is closed. The proof of this can be found in ([9]).

Remark 3.5. When a Riemannian manifold is a homogeneous space, that is $I(M)$ acts transitively on M, it can be proven that it is also complete. However this conclusion is false in general for indefinite homogeneous spaces [5],[21].

Let (M, g) be a Riemannian manifold. Then g defines the *Riemannian distance function*

$$d : M \times M \to [0, \infty)$$

as follows. Let $\Omega_{p,q}$ denote the set of smooth curves in M from p to q. Given a curve $\alpha \in \Omega_{p,q}$ with $\alpha : [a, b] \to M$, the Riemannian arc length of α with respect to g is defined as

$$L(\alpha) = \int_a^b |\dot{\alpha}(t)|\, dt.$$

The Riemannian distance $d(p, q)$ between p and q is then defined to be

$$d(p, q) = \mathrm{Inf}\{L(\alpha) : \alpha \in \Omega_{p,q}\} \geq 0.$$

Definition 3.6. A Riemannian manifold M is called *two-points homogeneous* if for any two point pairs $p_1, q_1 \in M$, $p_2, q_2 \in M$ satisfying $d(p_1, q_1) = d(p_2, q_2)$ there exists an isometry g of M such that $g \cdot p_1 = p_2$ and $g \cdot q_1 = q_2$.

The particular case $p_i = q_i$ proves M is homogeneous and hence complete. The most representative examples of two-points homogeneous spaces are the Euclidean space $\mathbb{R}^n$, the spheres $\mathbb{S}^n(r)$ and the hyperbolic spaces $\mathbb{H}^n(r)$. A more complete list of examples classified up to isometry can be viewed in [24, Theorem 8.12.2].

Although for any $p \in M$ the set $G \cdot p$ is an immersed submanifold of M, there are examples where the orbit has no a pseudo-Riemannian structure induced from that of the ambient manifold even for proper actions. It forces us to impose one such condition on the orbits of an action to give the next definition.

Definition 3.7. An action by isometries of a group G on a pseudo-Riemaniann manifold M is called *polar* if there exists a pseudo-Riemannian submanifold Σ satisfying the following conditions: for each $s \in \Sigma$

$$\Sigma \cap G \cdot s = \{s\}, \qquad T_s\Sigma \oplus T_s(G \cdot s) = T_sM \quad \text{(orthogonal direct sum)}.$$

Whenever such a submanifold Σ exists, it is said to be *transversal to the action of* G. Note that the second condition above implies that the vector subspaces $T_s\Sigma$ and $T_s(G \cdot s)$ are nondegenerate. Some examples of polar actions are given by isometric actions whose orbits have codimension one. In Chapter 5 we shall discuss a few important examples for our interests.

Remark 3.8. A polar action with $\dim \Sigma = 1$ yields a description by coordinates for M where to each point on M corresponds a point on Σ and a point on any orbit. Later, we shall call such a description *generalized polar coordinates*.

3.2 Integration on Locally compact groups

When the topology on G is Hausdorff and locally compact, G is called *locally compact group*. As before, let $C_c(G)$ be the set of real valued continuous functions on G with compact support. Then we shall understand any measure on G to be in the sense of a linear functional on $C_c(G)$ by means of a "invariant integral".

Definition 3.9. A measure μ on a locally compact group G is called *left-invariant*, or a *left Haar measure*, if

$$\mu(f \circ L_g) = \mu(f), \quad \text{for all} \quad g \in G,\ f \in C_c(G). \tag{3-3}$$

A non identically zero left Haar measure always exists on locally compact groups, besides it can be shown that it is unique up to multiplication by positive reals. A similar definition for *right-invariant* (or *right Haar*) measure also exists by replacing $f \circ L_g$ by $f \circ R_g$. We refer the reader to [15], [23] for more details in this topic. At this moment, we shall focus on measures on locally compact groups which have a structure of differentiable manifold.

In what follows we shall suposse $\dim G = n$. A vector field X on G is called *left invariant* if $DL_g X = X$ for all $g \in G$. Given a tangent vector $V_e \in T_eG$ there exists exactly one left invariant vector field $\tilde{V}$ on G such that $\tilde{V}_e = V_e$ and it is defined by $\tilde{V}_g = DL_g(V_e)$. If $V_e, W_e \in T_eG$, then the vector field $[\tilde{V}, \tilde{W}]$ is left invariant (see [18]) and we shall denote the tangent vector $[\tilde{V}, \tilde{W}]_e$ by $[V, W]$. The analogue case for a *right invariant* vector field is defined with DL_g replaced by DR_g. We denote the vector space T_eG together with the bracket operation $(V, W) \to [V, W]$ by $\mathfrak{g}$ and call it the *Lie algebra* of G.

In order to define the Haar measure on G, we need to integrate functions on Lie groups, so we shall begin by defining a volume form as we did in the case of peudo-Riemannian manifolds. Hence, we require that the manifold to be orientable (although this condition can be weakened) and indeed any Lie group is since its tangent bundle $TG \simeq G \times \mathfrak{g}$ is trivial (see [18, Corollary 8.39]).

We are going to define invariant differential forms on G. We shall write ω_g for the differential form ω at $g \in G$. A differential form ω on G will be called *left-invariant* if $L_g^*\omega = \omega$ for all $g \in G$ and *right-invariant* if $R_g^*\omega = \omega$. Note the condition $(L_g^*\omega)_x = \omega_x$ for all $x \in G$ is equivalent to

$$L_g^*(\omega_{gx}) = \omega_x, \tag{3-4}$$

so the pullback $L_g^*\omega$ is the form on G defined pointwise by $(L_g^*\omega)_x = L_g^*(\omega_{gx})$. Further (3-4) shows that a left-invariant form is uniquely defined on the whole of G by its value at the identity due to for any $g \in G$,

$$\omega_g = L_{g^{-1}}^*(\omega_e). \tag{3-5}$$

Let us consider $V_1, \ldots, V_n$ a basis of $\mathfrak{g}$ and its corresponding left-invariant basis denoted by $\widetilde{V}_1, \ldots, \widetilde{V}_n$. The equations $\omega^i(\widetilde{V}_j) = \delta_{ij}$ determine uniquely n $1-$forms ω^i on G which are also left-invariant since for any $x \in G$

$$\begin{aligned} L_g(\omega_{gx})(\widetilde{V}_x) &= L_g^* L_{g^-1}^*(\omega_x)(\widetilde{V}_x) \\ &= L_e^*(\omega_x)(\widetilde{V}_x) \\ &= \omega_x(D_x L_e(\widetilde{V}_x) \\ &= \omega_x(\widetilde{V}_x), \end{aligned}$$

where the last equality follows from $\widetilde{X}$ is a left-invariant field. Thus the wedge product of two differential forms given by (3-5) is again left-invariant because pullback distributes over wedge product. This is specially useful for building the left-invariant $n-$form $\omega = \omega^1 \wedge \cdots \wedge \omega^n$. Then each $n-$form can be written $f\omega$, where $f \in C^\infty(M)$.

Theorem 3.10. Left-invariant volume form exists on any Lie group G and is unique up to a multiplicative constant.

Proof. Let us take $\omega = \omega^1 \wedge \cdots \wedge \omega^n$ as above. Since the space of $n-$forms at e has dimension 1, ω_e is unique except for a multiplicative constant, and the same is then true for a left-invariant n-form (3-5). Indeed if ω' is another left-invariant volume form on G then there exists some constant $C \neq 0$ such that $\omega'_e = C\omega_e$. Thus the uniqueness follows from (3-5):

$$\omega'_g = L^*_{g^{-1}}(\omega'_e) = L^*_{g^{-1}}(C\omega_e) = CL^*_{g^{-1}}(\omega_e) = \omega_g,$$

for all $g \in G$. $\diamondsuit$

We now look for a coordinate expression for a left-invariant $n-$form. Let (U, ϕ) be a local chart for G about e and let $(x_1, \ldots, x_n)$ be the respective local coordinates valid on the open neighbourhood U of e. On U, ω has an expression

$$\omega_U = F(x_1, \ldots, x_n)\, dx^1 \wedge \cdots \wedge dx^n$$

with $F > 0$. Further if $g \in G$, the pair $(L_g U, \phi \circ L_{g^{-1}})$ is a local chart on a neighbourhood of g. We put $(\phi \circ L_{g^{-1}})(x) = (y_1(x), \ldots, y_n(x))$ and

$$\omega_{L_g U} = G(y_1, \ldots, y_n)\, dy^1 \wedge \cdots \wedge dy^n.$$

for all $x \in L_g U$. Note that the function L_g, which maps e to g, has a coordinate expression that describes each y_i to be a smooth function of $(x_1, \ldots, x_n)$ such that

$$(y_1, \ldots, y_n) = (x_1, \ldots, x_n),$$

because $y_i(gx) = x_i(x)$ for $x \in U \cap L_g U$. Besides, since ω is left-invariant, that is $\omega_x = L^*_g(\omega_{gx})$ for $x \in U \cap L_g U$, we get

$$G(y_1(x), \ldots, y_n(x))\, (dy^1 \wedge \cdots \wedge dy^n)_x = \\ G(x_1(x), \ldots, x_n(x))\, (dx^1 \wedge \cdots \wedge dx^n)_x.$$

Then $F(x_1(x), \ldots, x_n(x)) = G(x_1(x), \ldots, x_n(x))$ and the expressions for ω_x are related by means of

$$F(x_1(x), \ldots, x_n(x)) = F(y_1(x), \ldots, y_n(x)) \frac{\partial(y_1(x), \ldots, y_n(x))}{\partial(x_1(x), \ldots, x_n(x))}\, dx^1 \wedge \cdots \wedge dx^n \tag{3-6}$$

for $x \in U \cap L_g U$ and where $\frac{\partial(y_1(x), \ldots, y_n(x))}{\partial(x_1(x), \ldots, x_n(x))}$ is the Jacobian of the coordinate change $(\phi \circ L_g^{-1}) \circ \phi^{-1}$. For a right-invariant $n-$form the only change in the above is to take the function $R_{g^{-1}}$.

Invariant volume forms on Lie groups can be used to obtain Haar measures. It basically consists of an invariance of integral under a diffeomorphism.

Theorem 3.11. [23, Proposition 7.7.1] If ω is a left-invariant volume element on a Lie group G, then it defines a left Haar measure on G. The same holds with "left" replaced by "right."

Recall if $\phi : M \to N$ is a diffeomorphism of manifolds of dimension n and ω is an $n-$form on N, the pullback $\phi^*\omega$ defines an $n-$form on M. Then, for a continuous function $f : N \to \mathbb{R}$ we have that

$$\int_N f\omega = \int_M (f \circ \phi)\, \phi^*\omega. \tag{3-7}$$

Proof. Owing to the left translation L_g is a diffeomorphism from G to G, it is enough to take ϕ to be L_g in the formula above. Also, let us suppose ω is positive with respect to the orientation of G. Thus,

$$\int_G f\omega = \int_G (f \circ L_g)\, L_g^*\omega, \tag{3-8}$$

where $L_g^*\omega = \omega$. This being true for every $g \in G$, a left Haar measure μ_l on G can be defined in the sense of Definition 3.9 via

$$f \in C_c(G) \to \int_G f\,\omega. \tag{3-9}$$

This integral depends on ω but since ω is defined up to some positive constant, so is the integral. Also, Equation (3-8) shows that (3-9) is a left-invariant measure on G since

$$\mu_l(f \circ L_g) = \int_G (f \circ L_g)\,\omega = \int_G (f \circ L_g)\, L_g^*\omega = \int_G f\,\omega = \mu_l(f).$$

The proof for right-invariant ω is similar, considering R_g instead of L_g. ♢

We shall denote ω in Theorem 3.11 by dl_g. Then for a Lie group G we have a family $\{c\mu_l : c > 0\}$ of left Haar measures. A choice is usually made out if there is a distinguished compact neighbourhood V of e and we assume that $\int_V dl_g = 1$.

Definition 3.12. Let G be a compact Lie group. A left Haar measure is said to be *normalized* if

$$\mathrm{Vol}(G) = \int_G d_l g = 1$$

The volume of a compact Lie group is always finite and since any two volume elements differs by a multiplicative constant, we can conclude that there exists a unique normalized left Haar measure on any compact Lie group.

The foregoing can be considered for a right-invariant volume element $d_r g$ (which is unique except for a constant positive factor). So, it is defined a *right Haar measure* μ_r on G by

$$\mu_r(f) = \int_G f \, d_r g,$$

for any $f \in C_c(M)$.

In general a left Haar measure need not to be a right Haar measure, but we can easily construct a right Haar measure from a given left Haar measure by means of inversion operation $\iota : x \to x^{-1}$. Firstly, let us see ι interchanges left and right volume elements.

Theorem 3.13. Let G be a Lie group and dl_g a left-invariant volume element on G. Then $\iota^* dl_g$ is a right-invariant volume element on G.

Proof. Using the relation $\iota \circ R_y = L_{y^{-1}} \circ \iota$, we get

$$R_y^* \iota^* dl_g = (\iota \circ R_y)^* dl_g = (L_{y^{-1}} \circ \iota)^* dl_g = \iota^* L_{y^{-1}}^* dl_g = \iota^* dl_g. \tag{3-10}$$

$\diamondsuit$

Then we assert that given $d_l g$ on G, it is defined a right Haar measure μ_r on G by

$$\mu_r(f) = \int_G f \, d_l g^{-1},$$

for $f \in C_c(G)$ and where $d_l g^{-1} = \iota^* d_l g$.

Additionally, a left-invariant volume element also behaves suitably under right multiplications.

Theorem 3.14. For any $y \in G$ and any left-invariant volume element $d_l g$, $R_y^* d_l g$ is also left invariant. Then there exists a positive constant c such that $c\, R_y^* d_l g = d_l g$.

Proof. The key fact is that any left multiplication commutes with any right multiplication. So that,

$$L_x^*(R_y^* d_l g) = (R_y \circ L_x)^* d_l g = (L_x \circ R_y)^* d_l g = R_y^* L_x^* d_l g = R_y^* d_l g.$$

$\diamondsuit$

If denote by μ_l, μ_l' the left Haar measures given by $d_l g$, $R_y^* d_l g$ respectively, it follows that there exists a positive constant, $\Delta_r(y)$, such that

$$\mu_l = \Delta_r(y) \mu_l',$$

which can be rewritten as $\mu_l(f) = \Delta_r(y) \mu_l(f \circ R_y)$ or

$$\int f(g)d_lg = \Delta_r(y)\int_G f(gy)\,d_lg \tag{3-11}$$

The function $y \to \Delta_r(y)$ is called the *right modular function* of G and it can be proved that this defines a continuous homomorphism $\Delta_r : G \to \mathbb{R}^+$ (see [23]). The same result is true for the *left modular function* Δ_l defined by

$$\mu_r' = \Delta_l(y)\mu_r, \tag{3-12}$$

where μ_r is a right Haar measure on G.

Notation 3.15. As a matter of notation we shall use $\Delta_r(y)$ (or $\Delta_l(y)$) with volume forms and measures.

The next proposition establishes a link between a left and right measure.

Proposition 3.16. Let μ_l be a left Haar measure on a Lie group given by a left-invariant volume element d_lg and let Δ_r be the respective right modular function. Then for $f \in C_c(G)$,

$$\int_G f(g^{-1})\,d_lg^{-1} = \int_G f(g)\Delta_r(g)\,d_lg \tag{3-13}$$

Proof. It is enough to prove that $(\iota^* d_lg)(g) = \Delta_r(g)d_lg(g)$. According to Theorem 3.13, we are going to prove $\Delta_r(g)d_lg(g)$ is right invariant. Indeed

$$R_h^*(\Delta_r(g)d_lg(g)) = \Delta_r(gh)(R_h^*d_lg)(g) = \Delta_r(g)\Delta_r(h)(R_h^*d_lg)(g) = \Delta_r(g)d_lg(g),$$

where we have used Theorem 3.14 in the last equality.Then there exists a positive constant c such that

$$\Delta_r(g)d_lg(g) = c\,(\iota^* d_lg)(g).$$

However,

$$d_lg(g) = \Delta_r(g^{-1})(c\,(\iota^* d_lg)(g)) = c\,\iota^*(\Delta_r d_lg)(g) = c^2\,(\iota^*\iota^* d_lg)(g) = c^2 d_lg(g).$$

Since d_lg is positive, we get $c = 1$. $\diamondsuit$

Note that if μ_r is a right Haar measure, the Equation 3-13 can be written as

$$\mu_r = \Delta_r(g)\,\mu_l, \tag{3-14}$$

which allows us to know when a left Haar measure on a Lie group is also a right Haar measure.

Definition 3.17. A Lie group G is called *unimodular* if $\Delta_r = 1$ (or $\Delta_l = 1$) on G.

Observe that by definition a Lie group is unimodular if and only if every left Haar measure is also a right Haar measure. Such is the case of commutative Lie groups and compact Lie groups (since the image of a compact set under the left or right modular function is a compact subgroup of the multiplicative group $\mathbb{R}^+$, indeed the trivial subgroup). Therefore, for such groups we can refer to the *Haar measure*, without indicating left o right. In what follows, we shall refer exclusively to the left Haar measure or the Haar measure and we use the notation dg having clarity in the context.

Let G be a Lie group and H a closed subgroup that acts on G to the left. Let π be the orbit projection $G \to G/H$, which is continuous and open with the quotient topology defined on G/H. The closedness of H ensures that H is locally compact and not only that, but also the orbit space G/H, which is the space of left cosets, is locally compact and a manifold. Further, there is more structure here, the one of G as a group acting smoothly and transitively to the left on G/H, turning G/H into a homogeneous space. We shall denote such action by τ. Observe that each $x \in G$ gives rise to a diffeomorphism $\tau(x) : gH \to xgH$ of G/H to itself. It is smooth because it is equal to the composition

$$\{x\} \times G/H \hookrightarrow G \times G/H \xrightarrow{\tau} G/H,$$

and its smooth inverse is $\tau(x^{-1})$. If $h \in H$, we denote by $D\tau(h)$ the differential of $\tau(h)$ at $[e]$, the left coset of the identity element of G, which become an endomorphism of the tangent space $T_{[e]}(G/H)$. In [15] it is proven that

$$\det(D\tau(h)) = \frac{\Delta_r^G(h)}{\Delta_r^H(h)} \qquad (h \in H), \tag{3-15}$$

where the superscripts indicate the domain of each modular function. In the sequel we briefly present invariant measures on the manifold G/H. The aim is to start from a given invariant measure on G and define a measure on G/H. A differential form ω on G/H is said to be *G-invariant* if for all $g \in G$ we have $\tau(g)^*\omega = \omega$. The next result gives a sufficient and necessary condition for the existence of a $G-$invariant volume form on G/H.

Proposition 3.18. [15, Chapter I, Proposition 1.8] Let $m = \dim(G/H)$. The following conditions are equivalent:

(i) G/H has a nonzero $G-$invariant volume element ω;

(ii) $\Delta_r^G(h) = \Delta_r^H(h)$ for $h \in H$.

If these conditions are satisfied, the $G-$invariant volume element ω is unique up to a constant factor. Also, G/H is orientable and each $\tau(g)$ is orientation-preserving.

Proof. Firstly we assume (i) Then let ω be a $G-$invariant volume element on G/H, $\omega \neq 0$, and let $\phi : p \to (x_1(p), \ldots, x_m(p))$ be a system of coordinates on an open connected neighbourhood

U of $[e] \in G/H$. The pair $(\tau(g)U, \phi \circ \tau(g^{-1}))$ is a local chart on an connected neighbourhood of $g \cdot [e] \in G/H$. We put $(\phi \circ \tau(g^{-1}))(p) = (y_1(p), \ldots, y_m(p))$ for $p \in \tau(g)U$. Then note that $\tau(g) : U \to \tau(g)U$ has expression in coordinates $y_i(x_1, \ldots, x_m) = x_i$ for $i = 1, \ldots, m$. Consequently if we take $g = h \in H$, the relation $\tau(h)^*\omega = \omega$ at the point $[e]$ implies $\det(D\tau(h)) = 1$, so (ii) holds. On the other hand, let $X_1, \ldots, X_m$ be a basis of $T_{[e]}(G/H)$ and let $\omega^1, \ldots, \omega^n$ be its dual basis determined by $\omega^i(X_j) = \delta_{ij}$. Let us denote by $\omega_{[e]} = \omega^1 \wedge \cdots \wedge \omega^m$, the wedge product of the covectors ω^i on the tangent space $(G/H)_{[e]}$. Observe that

$$\omega_{[e]}(D\tau(h)X_1, \ldots, D\tau(h)X_m) = (\det D\tau(h))\, \omega_{[e]}(X_1, \ldots, X_m),$$

however condition (ii) implies that $\det(D\tau(h)) = 1$. Thus the element $\omega_{[e]}$ is invariant under the endomorphism $D\tau(h)$. Because of the value at identity is known, it follows that there exists a unique $G-$invariant $m-$form ω on G/H such that $\omega_{[e]} = \omega^1 \wedge \cdots \wedge \omega^m$. If ω^* is another $G-$invariant $m-$form on G/H, then $\omega^* = f\omega$ where $f \in C^\infty(G/H)$. Owing to the $G-$invariance, f is constant. $\diamondsuit$

Assuming (i) and notations in the proof above, ω has an expression on U

$$\omega_U = F(x_1, \ldots, x_m)\, dx^1 \wedge \cdots \wedge dx^m, \tag{3-16}$$

with $F > 0$. Likewise on $\tau(g)U$, ω has an expression

$$\omega_{\tau(g)U} = G(y_1, \ldots, y_m)\, dy^1 \wedge \cdots \wedge dy^m$$

and since the $G-$invariance condition means pointwise $\omega_q = \tau(g)^*\omega_{\tau(g)q}$, we have for $q \in U \cap \tau(g)U$

$$\begin{aligned} \omega_q &= G(y_1(q), \ldots, y_m(q))\, (dy^1 \wedge \cdots \wedge dy^m)_q \\ &= G(x_1(q), \ldots, x_m(q))\, (dx^1 \wedge \cdots \wedge dx^m)_q, \end{aligned} \tag{3-17}$$

hence $F(x_1(q), \ldots, x_m(q)) = G(x_1(q), \ldots, x_m(q))$. Recall that the relation between the expression of ω_q in both considered systems of coordinates is given by

$$F(x_1(q), \ldots, x_m(q)) = G(y_1(q), \ldots, y_m(q)) \frac{\partial(y_1(q) \ldots, y_m(q))}{\partial(x_1(q), \ldots, x_m(q))},$$

then from (3-17) and the coordinate expression of $\tau(g^{-1})$ we obtain

$$F(x_1(q), \ldots, x_m(q)) = F(y_1(q), \ldots, y_m(q)) \frac{\partial(y_1(q) \ldots, y_m(q))}{\partial(x_1(q), \ldots, x_m(q))},$$

which shows that the Jacobian of the mapping $(\phi \circ \tau(g^{-1})) \circ g^{-1}$ is positive. Consequently, the collection $(\tau(g)U, \phi \circ \tau(g^{-1}))_{g \in G}$ of local charts turns G/H into an oriented manifold and each $\tau(g)$ is orientation-preserving.

The $G-$invariant form ω now gives rise to an integral $\int f\omega$ and thus to a positive measure η

$$f \in C_c(G/H) \to \int_{G/H} f\omega. \tag{3-18}$$

We shall use dg_H rather than ω in this context. Such a measure is said to be $G-$*invariant* in the sense that

$$\eta(f \circ \tau(g)) = \eta(f) \qquad \text{for all} \quad g \in G.$$

Any relation between measures on G and measures on G/H is provided by a relation between functions in $C_c(G/H)$ and functions in $C_c(G)$. There is an useful linear map from $C_c(G)$ to $C_c(G/H)$ which describes it. For $f \in C_c(G)$, define f^H on G by

$$f^H(g) = \int_H f(gh)\, dh \quad \text{for} \quad g \in G, \tag{3-19}$$

where dh gives the left Haar measure on H. The right side is well-defined since $f(gh)$ as a function of h has compact support on H, so it is in $C_c(H)$. Also, due to the left invariance of dh, we have $f^H(gh) = f^H(g)$, for all $g \in G$ and $h \in H$. That is, f^H is constant on each orbit and defines therefore a unique function in $C_c(G/H)$, which it will be denote by $\bar{f}$, such that

$$\bar{f}(gH) = f^H(g) = \int_H f(gh)\, dh, \quad \text{for all} \quad g \in G.$$

Observe that indeed $\bar{f} \in C_c(G/H)$ because $f^H = \bar{f} \circ \pi$, thus if $\operatorname{supp} f \subset K \subset G$, K compact, then $\operatorname{supp} \bar{f} \subset \pi(K)$, with $\pi(K)$ compact subset of G/H. Now, the aim is to introduce the measure η on G/H by requiring

$$\int_{G/H} \bar{f}(gH)\, dg_H = \int_G f(g)\, dg, \tag{3-20}$$

with dg defining a measure μ on G. In order for this to make sense, we need to guarantee two facts: (i) for $g \in C_c(G/H)$ it has to be shown that there exists $f \in C_c(G)$ such that $\bar{f} = g$, that is the "bar" map is surjective, and (ii) if $\bar{f_1} = \bar{f_2} = g$, then $\mu(f_1) = \mu(f_2)$, so that $\eta(g)$ can be defined unambiguously. The next theorem and its proof can be seen in [15] and [23].

Theorem 3.19. Let G a Lie group and H a closed subgroup with a positive left Haar measure given by dh. The mapping $f \to \bar{f}$ is a linear mapping of $C_c(G)$ onto $C_c(G/H)$.

Concerning (ii), it is enough to show that the integral on the right of (3-20) vanishes whenever $\bar{f} = 0$. Here we highlight some details, in [23] there is a more complete proof. Let us suppose that $f \in C_c(G)$ gives rise, by Theorem 3.19, to the function $\bar{f} \in C_c(G/H)$. For each element $h_0 \in H$, the right translate of f, that is $f \circ R_{h_0}$ is another function in $C_c(G)$, thus if f is replaced by $f \circ R_{h_0}$, the right hand side of (3-19) becomes

$$\int_H f(ghh_0)\,dh = \Delta_r^H(h_0^{-1})\int_H f(gh)\,dh \tag{3-21}$$

owing to (3-11). That is, $(f \circ R_g)^H = \Delta_r^H(h_0^{-1})f^H$, and therefore

$$\overline{f \circ R_g} = \Delta_r^H(h_0^{-1})\bar{f},$$

which is more useful to write as

$$\Delta_r^H(h_0)\overline{f \circ R_g} = \bar{f}. \tag{3-22}$$

That means all functions of the form

$$f - \Delta_r^H(h_0)(f \circ R_g) \quad f \in C_c(G), h_0 \in H \tag{3-23}$$

belong to the kernel of the bar map. On the other hand

$$\int_G (f \circ R_{h_0})(g)\,d_l g = \Delta_r^H(h_0^{-1})\int_G f(g)\,d_l g \quad f \in C_c(G), h_0 \in H. \tag{3-24}$$

That is, functions of the form

$$f - \Delta_r^G(h_0)(f \circ R_{h_0}) \tag{3-25}$$

have integral zero over G.

Comparing (3-23) and (3-25), we see that the integral (3-20) cannot vanish on all functions with $\bar{f} = 0$ unless the modular function Δ_r^H is equal to the restriction of the modular function Δ_r^G to H.

Theorem 3.20. [15, Chapter I., Theorem 1.9] Let G be a Lie group and H a closed subgroup. The relation

$$\Delta_r^G(h) = \Delta_r^H(h) \quad \text{for} \quad h \in H, \tag{3-26}$$

is a neccesary and sufficient condition for the existence of a positive $G-$invariant measure on G/H. This measure η is unique (up to a constant factor) and

$$\int_G f(g)dg = \int_{G/H}\left(\int_H f(gh)\,dh\right)\,dg_H, \quad f \in C_c(G), \tag{3-27}$$

if the left Haar measures on G and H given by dg and dh respectively, are suitably normalized.

Remark 3.21. In the particular case of H being compact, Proposition 7.7.6 in [23] give us a relation between the volume forms dg and dg_H which define the measures on G and G/H respectively. Let us take $n = \dim G$ and $n - m = \dim H$ for $0 < m < n$. Let $x_1, \ldots, x_n$ be the coordinates of a local chart at e, of which $x_1, \ldots, x_m$ are coordinates of a chart on G/H at $[e]$ and $x_{m+1}, \ldots, x_n$ the coordinates of a chart on H at e. If

$$dg_e = dx^1 \wedge \cdots \wedge dx^n \qquad \text{and} \qquad dh_e = dx^{m+1} \wedge \cdots \wedge dx^n$$

are the value at e of dg and dh respectively, the unique $G-$invariant measure η on G/H given by (3-18) corresponds to a $m-$form whose value at $[e]$ is

$$\mathrm{Vol}(H)\, dx^1 \wedge \cdots \wedge dx^m,$$

in which $\mathrm{Vol}(H) = \beta(1)$ where β is the Haar measure on H. Similar to how it was done in the proof of Proposition 3.18, we can see that this $m-$form is $G-$invariant.

Remark 3.22. Assume again H is compact, then condition (3-26) is satisfied since $\Delta_r^G(H)$ and $\Delta_r^H(H)$ are compact subgroups of the multiplicative groups of the positive reals and so identically 1. Hence there exists a $G-$invariant measure on the space G/H for any choice of (left) Haar measure μ on G, (see [23, Corollary 7.4.4]). Using the same notation above, take Haar measure β on H normalized by $\int_H dg = 1$. Then the volume element dg_H giving η has value $dx^1 \wedge \cdots \wedge dx^m$ at $[e]$. We shall refer to a *normalization* of the measure μ in such a way that the Equation 3-27 is fulfilled and denote it by $dg = dh\, dg_H$ for simplicity.

Density function

This is appropiate point to introduce a function on a pseudo-Riemannian M which yields the ratio between invariant measures defined on the orbit of an element $x \in M$ under the action of a closed subgroup H of $I(M)$. Recall $H \cdot x$ is an immersed submanifold of M, then we shall asumme that it inherits a pseudo-Riemannian structure from that of M. We highlight the fact that the corresponding Riemannian measure

$$\int_{H\cdot x} u(s)\, ds, \tag{3-28}$$

where ds denotes the volume element of $H \cdot x$, is $H-$invariant due to the group H acts by isometries on M and for all $h \in H$ we have that $h(H \cdot x) = H \cdot x$. On the other hand, we assume that H is unimodular and fix an invariant Haar measure on it. Considering also H_x unimodular implies, by Theorem 3.20, the quotient H/H_x has an invariant measure determined up to a constant factor. We use the formula (3-27) above with $d\dot{h}$ giving such a $H-$invariant measure on H/H_x as follows

$$\int_H u(h)\, dh = \int_{H/H_x} \left(\int_{H_x} u(hk)\, dk \right) d\dot{h}, \quad u \in C_c(H). \tag{3-29}$$

Since $H \cdot x = H/H_x$ (by Theorem 3.1), the respective measures must be, by uniqueness, proportional. Hence, it is defined the *density function* as the function δ on M which relates the measures as follows

$$\int_{H\cdot x} u(s)\, ds = \delta(x) \int_{H/H_x} u'(h \cdot x)\, d\dot{h}, \quad x \in M, \tag{3-30}$$

where $u' : H/H_x \to \mathbb{R}$ denotes the function obtained by $u'(hH_x) = u'(h \cdot x) = u(hx)$. We shall write this formula as the relation between the volume elements, that is

$$ds = \delta(x)\, d\dot{h}.$$

Example 3.23. Consider the action of $H = \mathbb{O}(n)$ on $\mathbb{R}^n$, whose orbit for each $x \in \mathbb{R}^n$, $x \neq 0$, is $\mathbb{S}^{n-1}(r)$ with $r = |x|$. Also observe that the isotropy subgroup of each element is diffeomorphic to $\mathbb{O}(n-1)$, thus $H/H_x \cong \mathbb{S}^{n-1}$. Since H is compact, we know that H/H_x has a $H-$invariant measure given by a nonzero $H-$invariant volume element $d\dot{h}$. Let $d\omega_r$ be the volume element of $\mathbb{S}^{n-1}(r)$. By definition we have $d\omega_r = \delta(x)\, d\dot{h}$ but $d\omega_r = r^{n-1}\, d\omega_1$. Hence

$$\frac{1}{\delta(x)} r^{n-1}\, d\omega_1 = d\dot{h}.$$

If we normalize $d\dot{h}$ by $d\dot{h} = d\omega_1$, we get $\delta(x) = |x|^{n-1}$.

4 Decomposition of Differential Operators

In this chapter, we summarize the constructions made by Helgason in [16, 14, 12] about differential operators on a pseudo-Riemannian manifold. In the presence of a metric, a manifold always has one of particular interest, the so-called *Laplace-Beltrami operator*. We begin with the usual definition of this operator and a quick review of its main features. In Section 4.2, we explore its expression in polar coordinates on isotropic spaces, focusing our attention on spaces of constant curvature in the Lorentzian case. Finally, we consider in Section 4.3 pseudo-Riemannian manifolds endowed with a proper and polar isometric action so that it is possible to define a new operator on them: the *radial part* of the Laplace-Beltrami operator.

Even though the theory is introduced in a general setup, we are mainly interested in the close relation between the radial part and the polar actions on Lorentzian manifolds.

4.1 The Laplace-Beltrami Operator

It is well known what we mean by Laplace operator on Euclidean space, that is, on $\mathbb{R}^n$ equipped with the Euclidean metric. We shall use the notation by Helgason and write L to denote it. If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, the Laplacian of f is defined by

$$Lf = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}. \tag{4-1}$$

In this section we present the Laplace-Beltrami operator, the generalization of (4-1) to the case of functions defined on pseudo-Riemannian manifolds. However, in [17] it can be found also its generalization to differential forms (recall that a smooth function is a differential $0-$form). We begin by defining differential operators on a manifold M of dimension n. The symbol ∂_i will denote the partial differentiation $\partial/\partial x_i$ for all $i = 1, \ldots, n$ and if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a $n-$tuple of integers $\alpha_i \geq 0$, we shall put $D^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$.

Definition 4.1. A linear transformation $D : C_c^\infty(M) \to C_c^\infty(M)$ is called a *differential operator* on M if the following condition is satisfied: for each $p \in M$ and each local chart (U, φ) around p there exists a finite set of functions $a_\alpha \in C^\infty(U)$ such that for each $f \in C_c^\infty(M)$ with support contained in U,

$$
\begin{aligned}
&[Df](q) = \sum_{\alpha} a_\alpha(q)[D^\alpha(f \circ \varphi^{-1})](\varphi(q)) && \text{if } q \in U,\\
&[Df](q) = 0 && \text{if } q \notin U.
\end{aligned}
$$

It is clear that whenever V is an open set on M and $f \in C_c^\infty(M)$ vanishes on V then Df vanishes on V. Thus D decreases supports, namely

$$
\operatorname{supp}(Df) \subset \operatorname{supp}(f), \qquad f \in C_c^\infty(M). \tag{4-2}
$$

Conversely, a linear map $D : C_c^\infty(M) \to C_c^\infty(M)$ which decreases supports is a differential operator, (see [15]). The property (4-2) implies that D can be extended to a linear operator, also denoted by D, from $C^\infty(M)$ to $C^\infty(M)$ by means of the formula

$$
Df(x) = D\phi(x).
$$

Here $x \in M$ and $f \in C^\infty(M)$ are arbitrary and ϕ is any function in $C_c^\infty(M)$ which coincides with f in a neighbourhood of x. The value $D\phi(x)$ depends only on f but is independent of the choice of ϕ. Additionally observe that although a differential operator acts on globally defined functions, it is actually a local operator. Indeed, for $f \in C_c^\infty(M)$ and a point $x \in M$, for any function ψ of compact support identically 1 on a neighborhood of x

$$
Df(x) = D(\psi f)(x).
$$

Indeed $f = \psi f + (1-\psi)f$ and D is linear, we have

$$
Df = D(\psi f) + D((1-\psi)f).
$$

Since D decreases supports $D((1-\psi)f)(x) = 0$, holding the assert . This allows us to consider Df on open subsets of M without loss of generality.

We now consider a pseudo-Riemannian manifold M with metric g and take a local chart (U, φ) on M such that $\varphi(q) = (x_1(q), x_2(q), \ldots, x_n(q))$ are local coordinates for all $q \in U$. As before, the functions g_{ij}, g^{ij}, $\bar{g}$ are defined on U by

$$
g_{ij} = g\left(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\right), \qquad \textstyle\sum_j g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k,
$$
$$
\bar{g} = |\det(g_{ij})|.
$$

Each function $f \in C^\infty(M)$ gives rise to a vector field on M, *gradient of* f, whose expression in local coordinates on U is given by

$$
\nabla f := \sum_{i,j} g^{ij} \partial_i f \frac{\partial}{\partial x_j}, \tag{4-3}
$$

On the other hand, the *divergence* of a vector field $Z = \sum_{i=1}^n Z_i \partial_i$ is the function on M which on U is given by

$$\operatorname{div} Z := \frac{1}{\sqrt{\overline{g}}} \sum_i \partial_i(\sqrt{\overline{g}} Z_i). \tag{4-4}$$

Finally, the Laplace-Beltrami operator L on M given by

$$Lf := \operatorname{div} \nabla f, \quad f \in C^\infty(M), \tag{4-5}$$

defines a function $L : C^\infty(M) \to C^\infty(M)$ whose expression in local coordinates on U is

$$Lf = \frac{1}{\sqrt{\overline{g}}} \sum_k \frac{\partial}{\partial x_k} \left(\sum_i g^{ik} \sqrt{\overline{g}} \frac{\partial f}{\partial x_i} \right), \tag{4-6}$$

which shows that L is a differential operator on M.

Next we shall mention some properties of L, the details of the respective proofs can be seen in [15]. The first one is about symmetry with respect to the Riemannian measure in the sense that on a pseudo-Riemannian manifold M

$$\int_M u(x)(Lv)(x)\, dx = \int_M (Lu)(x)v(x)\, dx, \quad u \in C_c(M),\, v \in C^\infty(M) \tag{4-7}$$

if dx is the volume element on M which determines the measure. Also, for an arbitrary coordinate system on M, we have an expression for L that will be useful for us

$$Lf = \sum_{ij} g^{ij}(\partial_i \partial_j f - \sum_k \Gamma_{ij}^k \partial_k f). \tag{4-8}$$

Finally, we consider a diffeomorphism Φ of M onto itself. A differential operator D on M is said to be *invariant under* Φ if

$$Df = (D(f \circ \Phi)) \circ \Phi^{-1}$$

for all $f \in C^\infty(M)$.

Proposition 4.2. [15, Chapter II, Proposition 2.4] Let Φ be a diffeomorphism of the pseudo-Riemannian manifold M. Then Φ leaves the Laplace-Beltrami operator Δ invariant if and only if it is a isometry.

Proof.

$\Rightarrow$) Suppose the diffeomorphism Φ is an isometry. Let $p \in M$ and let (V, ψ) a local chart around p. Then Φ defines a local chart around $\Phi(p)$ given by $(\Phi(V), \psi \circ \Phi^{-1})$. Denote by $(x_1, \ldots, x_n)$ and by $(y_1, \dots, y_n)$ the local coordinates on V and on $\Phi(V)$ respectively. Then for $x \in V$, $y \in \Phi(V)$ such that $y = \Phi(x)$, the coordinate expression of Φ says that $y_i = x_i$ $(i = 1, 2, \dots, n)$. Thus

$$d\Phi_x \left(\frac{\partial}{\partial x_i} \right)_x = \left(\frac{\partial}{\partial y_i} \right)_{\Phi(x)} \qquad (1 \leq i \leq n).$$

For each $f \in C_c^\infty(M)$, we apply Equation (4-6)

$$(Lf \circ \Phi)(x) = (Lf)(\Phi(x)) = \frac{1}{\sqrt{\bar{g}(y)}} \sum_k \frac{\partial}{\partial y_k} \left(\sum_i g^{ik}(y) \sqrt{\bar{g}}(y) \frac{\partial (f \circ (\psi \circ \Phi^{-1}))}{\partial y_i} \right) \tag{4-9}$$

$$(L(f \circ \Phi))(x) = \frac{1}{\sqrt{\bar{g}(x)}} \sum_k \frac{\partial}{\partial x_k} \left(\sum_i g^{ik}(x) \sqrt{\bar{g}}(x) \frac{\partial ((f \circ \Phi) \circ \psi^{-1})}{\partial x_i} \right). \tag{4-10}$$

Owing to the choice of coordinates we also have for $1 \leq i, j \leq n$

$$\frac{\partial (f \circ (\psi \circ \Phi^{-1}))}{\partial y_i} = \frac{\partial ((f \circ \Phi) \circ \psi^{-1})}{\partial x_i},$$
$$\frac{\partial^2 (f \circ (\psi \circ \Phi^{-1}))}{\partial y_i \partial y_j} = \frac{\partial^2 ((f \circ \Phi) \circ \psi^{-1})}{\partial x_i \partial x_j},$$

and by hypothesis $g_{ij}(x) = g_{ij}(y)$ for all i, j. Then the expressions (4-9) and (4-10) are the same and $(Lf) \circ \Phi = L(f \circ \Phi)$.

$\Leftarrow$) Suppose the expressions (4-9) and (4-10) agree. Under the considerations about the chosen local coordinates before, $g_{ij}(x) = g_{ij}(y)$ for all i, j by equating coefficients. It shows that Φ is an isometry.

$\diamondsuit$

4.2 The Laplace-Beltrami Operator on Isotropic Spaces

We shall now study the expression in coordinates of Lapace-Beltrami operator on *isotropic spaces*. The reason for this is that they contain the two-point homogeneous spaces and their Lorentzian anologs in constant curvature. We refer to [24] for a more detailed description and classification of these spaces.

Definition 4.3. A pseudo-Riemannian manifold M is called *isotropic* if given $p \in M$ and nonzero vector $v, w \in T_pM$ with $|v|^2 = |w|^2$, there is an isometry ϕ of M with $\phi(p) = p$ and $d\phi_p(v) = w$.

A proof of the class of isotropic Riemannian manifolds and the class of two-point homogeneous are the same can be found in [5] and [24]. In the indefinite case, by contrast, it can be proven an isotropic pseudo-Riemannian manifold M is necessarily homogeneous under the transitive action of the isometry group $I(M)$, (see [16]). In the following, we shall firstly focus on the Riemannian setup.

There is also an explicit formula in geodesic polar coordinates for the Laplace-Beltrami operator defined on a two-point homogeneous space M of dimension n. When it is a Euclidean space, such expression coincides with the usual one for us. For each $p \in M$, recall $S_r(p)$ denote the geodesic sphere in M with center p and radius r. We know that $S_r(p)$ is for each r a submanifold of M and has a Riemannian metric induced by that of M. We have already defined before the Riemannian measure on $S_r(p)$ according to such a metric and we shall denote by $A(r)$ the $(n-1)-$dimensional volume of $S_r(p)$.

From now on we distinguish between $A(r)$ and $V(r)$, with $V(r)$ being the $n-$dimensional volume of $B_r(p)$ in M. Since M is homogeneous, that is, has a transitive action from its group of isometries, $A(r)$ and $V(r)$ are independent of p. It is consequence of the invariance of Riemannian measure under isometries and the formula (3-7). The following result can be seen in [13], [12]. In the proof we shall use the formula (4-6) for the Laplace-Beltrami operator.

Theorem 4.4. In geodesic polar coordinates at p, the Laplace-Beltrami operator Δ on M has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)}\frac{dA}{dr}\frac{\partial}{\partial r} + \Delta', \tag{4-11}$$

where Δ' is the Laplace-Beltrami operator on $S_r(p)$.

Proof. Let $n = \dim M$ and let $\{r, \theta_2, \dots, \theta_n\}$ be a system of geodesic polar coordinates at the point $p \in M$. Put $\theta_1 = r$ and let

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(\theta_1, \dots, \theta_n)\, d\theta^i\, d\theta^j \tag{4-12}$$

denote the Riemannian structure of M expressed in geodesic polar coordinates at p. Due to Gauss's Lemma, the geodesics emanating from p intersect $S_r(p)$ orthogonally. Hence $g_{11}(\theta_1, \dots, \theta_n) = 1$ and $g_{1j}(\theta_1, \dots, \theta_n) = 0$ for $j = 2, \dots n$, thus 4-12 reduces to

$$ds^2 = dr^2 + \sum_{i,j=1}^{n} g_{ij}(r, \dots, \theta_n)\, d\theta^i\, d\theta^j. \tag{4-13}$$

Now, let us consider the identity mapping of $S_r(p)$ into M,

$$\begin{aligned} I: \quad & S_r(p) \longrightarrow M \\ & (\theta_2, \dots, \theta_n) \mapsto (r, \theta_2, \dots, \theta_n). \end{aligned} \tag{4-14}$$

Note that $I^*(dr^2) = 0$ because $S_r(p)$ has constant radius in M, so the Riemannian structure of $S_r(p)$ is given by

$$\sum_{i,j=2}^{n} g_{ij}(r,\ldots,\theta_n)\, d\theta^i\, d\theta^j. \tag{4-15}$$

Using (4-13) in the the expression in coordinates (4-6) of Laplace-Beltrami operator we obtain

$$\frac{1}{\sqrt{\bar{g}}}\frac{\partial}{\partial r}\left(\sum_{i=1}^{n} g^{i1}\sqrt{\bar{g}}\frac{\partial}{\partial \theta_i}\right) + \frac{1}{\sqrt{\bar{g}}}\sum_{k=2}^{n}\frac{\partial}{\partial \theta_k}\left(\sum_{i=1}^{n} g^{ik}\sqrt{\bar{g}}\frac{\partial}{\partial \theta_i}\right), \tag{4-16}$$

where in the first term of the sum $g^{i1}(r,\theta_2,\ldots,\theta_n) = 0$ for $i = 2,\ldots,n$ and in the second one $g^{1k}(r,\theta_2,\ldots,\theta_n) = 0$ for $k = 2,\ldots,n$ and $g^{11}(r,\theta_2,\ldots,\theta_n) = 1$. Hence,

$$\Delta = \frac{1}{\sqrt{\bar{g}}}\frac{\partial}{\partial r}(\sqrt{\bar{g}}\frac{\partial}{\partial r}) + \frac{1}{\sqrt{\bar{g}}}\sum_{k=2}^{n}\frac{\partial}{\partial \theta_k}\left(\sum_{i=2}^{n} g^{ik}\sqrt{\bar{g}}\frac{\partial}{\partial \theta_i}\right) \tag{4-17}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{\sqrt{\bar{g}}}\frac{\partial\sqrt{\bar{g}}}{\partial r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{\bar{g}}}\sum_{k=2}^{n}\frac{\partial}{\partial \theta_k}\left(\sum_{i=2}^{n} g^{ik}\sqrt{\bar{g}}\frac{\partial}{\partial \theta_i}\right) \tag{4-18}$$

If A denotes the matrix $(g_{ij})_{1\le i,j\le n}$ and B denotes the matrix $(g_{ij})_{2\le i,j\le n}$, we have

$$A = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & B^{-1} \end{bmatrix},$$

with $\det A = \det B$. Consequently, the last term in (4-18) is the Laplace-Beltrami operator Δ' on $S_r(p)$.

The next step is to calculate $A(r)$. By definition

$$A(r) = \int_{S_r(p)} dV_g,$$

where dV_g is the volume element on $S_r(p)$ which is locally given by

$$\sqrt{\det B}\, d\theta^2 \cdots d\theta^n = \sqrt{\bar{g}}\, d\theta^2 \cdots d\theta^n.$$

Due to M is a two-point homogeneous space, if $p, q_1, q_2 \in M$ satisfy $d(p,q_1) = d(p,q_2)$, there exists an isometry φ of M such that $\varphi \cdot p = p$ y $\varphi \cdot q_1 = q_2$. That means that under the action of $I(M)$, the isotropy subgroup $I(M)_p$ at p acts transitively on the geodesic sphere $S_r(p) = \exp_p(S_r(0))$ centered at p for a suitable $r > 0$. Let $\phi \in I(M)_p$. Note the differential $d_p\phi$ maps T_pM onto T_pM since $\phi(p) = p$. Due to $\langle d_p\phi(v), d_p\phi(v)\rangle = \langle v, v\rangle$ for any $v \in T_pM$, the differential $d_p\phi$ acts transitively on the sphere $V_r(0) = \{v \in T_pM : |v| = r,\ v \neq 0\}$. Since $\exp_p$ maps each v to a geodesic, the group $I(M)_p$ acts transitively on the set of the geodesics emanating from p.

Using the expression in coordinates for Laplace-Beltrami operator, we have for the coordinate function r

$$\Delta r = \frac{1}{\sqrt{\bar{g}}}\frac{\partial \sqrt{\bar{g}}}{\partial r},$$

which is a function of r alone because if we equip T_pM with polar coordinates, the term $\bar{g}$ coincides with the term $\bar{g}_r$, where g_r is the metric on the set $V_r(0)$ and does not depend on the angular coordinates on $V_r(0)$. Hence

$$\ln \sqrt{\bar{g}} = \alpha(r) + \beta(\theta_2 \dots \theta_n) \tag{4-19}$$

and

$$\sqrt{\bar{g}} = e^{\alpha(r)} e^{\beta(\theta_2 \dots \theta_n)}$$

for suitable functions α and β. Because of $\alpha(r)$ is independent of the choice of $\{\theta_2, \cdots, \theta_n\}$, we have

$$A(r) = \int_{S_r(p)} e^{\alpha(r)} e^{\beta(\theta_2 \dots \theta_n)}\, d\theta^2 \dots d\theta^n = Ce^{\alpha(r)}, \tag{4-20}$$

with $C =$ constant and since by (4-19)

$$\alpha'(r) = \frac{1}{\sqrt{\bar{g}}}\frac{\partial \sqrt{\bar{g}}}{\partial r},$$

we use (4-20) to find

$$\frac{1}{\sqrt{\bar{g}}}\frac{\partial \sqrt{\bar{g}}}{\partial r} = \frac{1}{A(r)}\frac{dA}{dr}.$$

This concludes the proof. ♢

Example 4.5. We present the matrix of Euclidean metric on $\mathbb{R}^n$ in polar coordinates in Example 2.26. Now we want to compute the Laplace-Beltrami operator. To this end observe that

$$\ln \sqrt{\bar{g}} = \ln (r^{n-1}) + \ln (\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin \theta_{n-2}),$$

then we can take $\alpha(r) = \ln (r^{n-1})$ and by (4-20) we have

$$A(r) = Cr^{n-1} \qquad (C = \text{constant}),$$

so

$$\frac{1}{A(r)}\frac{dA}{dr} = \frac{n-1}{r}. \tag{4-21}$$

On the other hand, using (4-18) we obtain the Laplace-Beltrami operator on $S_r(p)$

$$\Delta' = \frac{1}{\sqrt{\bar{g}}}\left[\frac{\partial}{\partial\theta_1}(r^{n-3}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-1}\frac{\partial}{\partial\theta_1}) + \frac{\partial}{\partial\theta_2}(r^{n-3}\sin^{n-4}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-1}\frac{\partial}{\partial\theta_2}) + \cdots + \frac{\partial}{\partial\theta_{n-2}}(r^{n-3}\sin^{n-4}\theta_1\sin^{n-5}\theta_2\cdots\sin^0\theta_{n-3}\sin\theta_{n-2}\frac{\partial}{\partial\theta_{n-2}})\right] + \frac{1}{r^2\sin^2\theta_1\sin^2\theta_2\cdots\sin^2\theta_{n-3}\sin^2\theta_{n-2}}\frac{\partial^2}{\partial^2\theta_{n-1}}, \tag{4-22}$$

and substituting (4-21) and (4-22) into (4-11), we get the expression for L on $\mathbb{R}^n$. Now we suppose $n = 3$, then the Laplace-Beltrami operator on $\mathbb{R}^3$ is

$$\begin{aligned}\Delta &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin\theta_1\frac{\partial}{\partial\theta_1}\right) + \frac{1}{r^2\sin\theta_1}\frac{\partial}{\partial\theta_2}\left(\frac{1}{\sin\theta_1}\frac{\partial}{\partial\theta_2}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin\theta_1\frac{\partial}{\partial\theta_1}\right) + \frac{1}{r^2\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}.\end{aligned}$$

In the pseudo-Riemannian setup, isotropic Lorentzian manifolds turn out to be a fairly specific class of manifolds. In fact, whenever a isotropic manifold is Lorentzian, it has curvature constant (see [24, Chapter 12]). We shall introduce briefly these spaces with the development done by Helgason in [16] but describe the context using the notation of [21]. Then, according to Theorem 9 in [12], there are only three cases of index 1, all of them depending on the curvatures 0, 1 and -1.

- *Flat Lorentz spaces*: Let us consider the Minkowski space $\mathbb{R}^{n+1}_1$ endowed with the Lorentz metric

$$Q(v,v) = -v_1^2 + v_2^2 + \cdots + v_{n+1}^2,$$

where $v = (v_1,\ldots,v_n)$ is a vector of the tangent space $T_p\mathbb{R}^{n+1}_1 \cong \mathbb{R}^{n+1}_1$ at some point $p \in \mathbb{R}^{n+1}_1$. We said before that in $\mathbb{R}^{n+1}_1$ there are three distinguished sets depending on the sign of Q. In the positive and negative cases, we can describe a vector $v \in T_p\mathbb{R}^{n+1}_1$ as $v = |v|s_v$ where s_v is an element such that $Q(s_v,s_v) = 1$ or $Q(s_v,s_v) = -1$ respectively. Then we define parametrizations, similar to the polar one, for those vectors via the next formulas:
For $Q(v,v) > 0$:

$$\begin{aligned} v_1 &= r\sinh(t)\\ v_2 &= r\cosh(t)\cos(\theta_1)\\ v_3 &= r\cosh(t)\sin(\theta_1)\cos(\theta_2)\\ &\vdots\\ v_{n+1} &= r\cosh(t)\sin(\theta_1)\cdots\sin(\theta_{n-1}).\end{aligned} \tag{4-23}$$

For $Q(v,v) < 0$ there exist two choices for the parametrization:

$$\begin{aligned}
v_1 &= r\cosh(t) \text{ for } v_1 \geq 0 \qquad (\text{ or } v_1 = -r\cosh(t) \text{ for } v_1 \leq 0)\\
v_2 &= r\sinh(t)\cos(\theta_1)\\
v_3 &= r\sinh(t)\sin(\theta_1)\cos(\theta_2)\\
&\vdots\\
v_{n+1} &= r\sinh(t)\sin(\theta_1)\cdots\sin(\theta_{n-1}).
\end{aligned} \tag{4-24}$$

for a suitable choice of the parameters $(r,t,\theta_1,...,\theta_{n-1})$. For each case, such parametrizations give an expression of the induced metric on these sets by the global Lorentz metric. In summary, we have

$$-dr^2 + r^2(-dt^2 + \cosh^2(t)d\sigma^2)$$

and

$$-dr^2 + r^2(dt^2 + \sinh^2(t)d\sigma^2)$$

respectively, where $d\sigma^2$ denotes the Riemannian metric induced on the set $S \subset T_p\mathbb{R}^{n+1}_1$ parametrized as

$$S = \{(0,0,\cos(\theta_1),\sin(\theta_1)\cos(\theta_2),...,\sin(\theta_1)\cdots\sin(\theta_{n-1}))\}.$$

- Let Q_- be the subset of points $p = (x_1,...,x_{n+2}) \in \mathbb{R}^{n+2}_1$ determined by the equation:

$$-x_1^2 + x_2^2 + \cdots + x_{n+2}^2 = 1.$$

By using the Equation 4-24, we can obtain a parametrization of the points of Q_- as:

$$\begin{aligned}
x_1 &= \sinh(r)\cosh(t)\\
x_2 &= \sinh(r)\sinh(t)\cos(\theta_1)\\
x_3 &= \sinh(r)\sinh(t)\sin(\theta_1)\cos(\theta_2)\\
&\vdots\\
x_{n+1} &= \sinh(r)\sinh(t)\sin(\theta_1)\cdots\sin(\theta_{n-1})\\
x_{n+2} &= \cosh(r).
\end{aligned} \tag{4-25}$$

for a suitable choice of the parameters $(r,t,\theta_1,...,\theta_{n-1})$. Consequently, the induced metric on Q_- obtained by means of the parametrization is

$$-dr^2 + \sinh^2(r)(dt^2 + \sinh^2(t)d\sigma^2),$$

where $d\sigma$ denotes the metric induced on a subset S of $T_p\mathbb{R}^{n+1}_1$ parametrized by

$$S = \{(0,\cos(\theta_1),\sin(\theta_1)\cos(\theta_2),...,\sin(\theta_1)\cdots\sin(\theta_{n-1}),0)\}.$$

- Let $\mathbb{R}_2^{n+2}$ be the space $\mathbb{R}^{n+2}$ endowed with the pseudo-Riemannian metric of index $\nu = 2$ given by
$$Q_+(v,v) = -v_1^2 + v_2^2 + \cdots + v_{n+1}^2 - v_{n+2}^2,$$
where v is a vector in the tangent space $T_p\mathbb{R}_2^{n+2} \cong \mathbb{R}_2^{n+2}$ at some point $p \in \mathbb{R}_2^{n+2}$. Let Q_+ be the subset of points $p = (x_1, ..., x_{n+2}) \in \mathbb{R}_2^{n+2}$ determined by the equation
$$-x_1^2 + x_2^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = -1.$$
Therefore, the set Q_+ can be parametrized as follows:
$$\begin{aligned}
x_1 &= \sin(r)\cosh(t)\\
x_2 &= \sin(r)\sinh(t)\cos(\theta_1)\\
x_3 &= \sin(r)\sinh(t)\sin(\theta_1)\cos(\theta_2)\\
x_4 &= \sin(r)\sinh(t)\sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\\
&\vdots\\
x_{n+1} &= \sin(r)\sinh(t)\sin(\theta_1)\cdots\sin(\theta_{n-1})\\
x_{n+2} &= \cos(r),
\end{aligned} \tag{4-26}$$
for a suitable choice of the parameters $(r, t, \theta_1, \ldots, \theta_{n-1})$. We use this parametrization to obtain the induced metric on Q_+ which turns to be
$$-dr^2 + \sin^2(r)(dt^2 + \sinh^2(t)d\sigma^2)$$
where, as before, $d\sigma^2$ denotes the metric induced on the subset of $T_p\mathbb{R}_1^{n+1}$ given by the parametrization
$$S = \{(0, \cos(\theta_1), \sin(\theta_1)\cos(\theta_2), ..., \sin(\theta_1)\cdots\sin(\theta_{n-1}), 0)\}.$$

We point out that the expression of the metric in the previous spaces allows us to calculate the volume element induced by the resulting Lorentz metric. For the space $\mathbb{R}_1^n$, we have
$$\begin{aligned}
&r^n\cosh^{n-1}(t)\,drdtd\sigma \quad \text{if } Q(v,v) > 0,\\
&r^n\sinh^{n-1}(t)\,drdtd\sigma \quad \text{if } Q(v,v) < 0.
\end{aligned}$$
For the space Q_- the volume element turns to be:
$$\sinh^n(r)\sinh^{n-1}(t)\,drdtd\theta.$$
Finally, for the space Q_+ the volume element results:
$$\sin^n(r)\sinh^{n-1}(t)\,drdtd\sigma.$$

Theorem 1.3, Chapter IV in [16] states the fact that these spaces of index $\nu = 1$ are isotropic and thus homogeneous. In addition, these are connected and an arbitrary isotropic Lorentzian manifold is locally isometric to any of them. According to Helgason ([16, Chapter IV, Theorem 1.5]), $I(Q_-) = \mathbb{O}_1(n+2)$ and $I(Q_+) = \mathbb{O}_2(n+2)$ act transitively on Q_- and Q_+ respectively, and the isotropy subgroup at $o = (1, 0, \ldots, 0)$ is identified with $\mathbb{O}_1(n+1)$ in both cases (recall that $I(\mathbb{R}_1^{n+1}) = \mathbb{R}^{n+1} \rtimes \mathbb{O}_1(n+1)/\mathbb{O}_1(n+1)$, and here $o = (0, \ldots, 0)$). Nevertheless, the identity components also act transitively resulting connected isotropy subgroups [16, Chapter IV, Lemma 2.3]. Therefore we can write: $\mathbb{R}_1^{n+1} = \mathbb{R}^{n+1} \rtimes \mathbb{O}_1^{++}(n+1)/\mathbb{O}_1^{++}(n+1)$, $Q_- = \mathbb{O}_1^{++}(n+2)/\mathbb{O}_1^{++}(n+1)$ and $Q_+ = \mathbb{O}_2^{++}(n+2)/\mathbb{O}_1^{++}(n+1)$.

Let us consider T_oM where $M = G/H$ with $H = \mathbb{O}_1^{++}(n+1)$ and G is either $G^0 = \mathbb{R}^{n+1} \rtimes \mathbb{O}_1^{++}(n+1)$, $G^- = \mathbb{O}_1^{++}(n+2)$ and $G^+ = \mathbb{O}_2^{++}(n+2)$. Certainly T_oM is isometric to Minkowski space. Recall in Example 2.29 we said that the vector field $V = (-1, 0, \ldots, 0)$ is timelike. Let D_o denotes the solid future directed cone which contains the timelike vector $V_o = (-1, 0, \ldots, 0)$. The component of the set $\{v \in T_oM : Q(v,v) < -r^2\}$ which lies in D_o will be denote $S_r(o)$. Note that the elements in the set D_o can be parametrized following the parametrical description as in Equation (4-24). It implies the metric in this set has the form:

$$-dr^2 + r^2(dt^2 + \sinh^2(t)d\sigma^2).$$

In the same way as we gave the metric on D_o, we have that for a fixed r the metric on the set $S_r(o)$ is:

$$r^2(dt^2 + \sinh^2(t)d\sigma^2).$$

It implies the metric on $S_r(o)$ is Riemannian and we get the volume element which it defines

$$d\omega_r = r^n \sinh^{n-1}(t)\, dt d\sigma.$$

If $d\omega_1$ is the volume element on the set $S_1(o)$, by the equation above we conclude:

$$d\omega_r = r^n\, d\omega_1.$$

Now, take $g \in G$ and let $q = g \cdot o$ be any point of M. Because of H is connected, we have $h \cdot D_o = D_o$. Hence, it makes sense to define the null cone C_q, D_q, $S_r(q) \subset T_qM$ as

$$C_q = g \cdot C_o, \qquad D_q = g \cdot D_o. \qquad S_r(q) = g \cdot S_r(o).$$

If exp denotes the exponential map from T_qM into M we set

$$\mathbf{C}_q = \exp(C_o), \qquad \mathbf{D}_q = \exp(D_o). \qquad \mathbf{S}_r(q) = \exp(S_r(o)).$$

Next, we address the computation of the Laplace-Beltrami operator inside $\mathbf{D}_o$ (and in general on any $\mathbf{D}_q$). In order to do this, we need to find a suitable normalization of the invariant measure on the orbits under the action of H and thus compute the density function δ. The claims we shall use here can be seen in [16]. Firstly, we highlight the unimodular H acts transitively on $\mathbf{S}_r(o)$ and the isotropy group K of $x \in \mathbf{S}_r(o)$ is compact. Then $\mathbf{S}_r(o) = H/K$. Now fix a Haar measure μ_o on

H given by dh. Since K is compact it has a unique Haar measure given by dk and normalized by $\int_K dk = 1$. It follows that there is a $H-$invariant measure on the orbit $H \cdot x \cong H/K$ defined by that of H and K. We use the Equation (3-27) in Theorem 3.20 with $d\dot{h}$ giving this measure and computate

$$\begin{aligned}\int_H u(h\cdot x)\,dh &= \int_{H/K}\int_K u(hk\cdot x)\,dk\,d\dot{h} = \int_{H/K}\int_K u(h\cdot x)\,dk\,d\dot{h}\\ &= \int_{H/K} u(h\cdot x)\int_K dk\,d\dot{h} = \int_{H/K} u(h\cdot x)d\dot{h}.\end{aligned} \tag{4-27}$$

Here $u \in C_c(M)$ and the integral on the right side converges as a consequence of compactness of K. Now take an arbitrary point $q \in M$ and an element $g \in G$ such that $q = g \cdot o$. Since M is homogeneous, we know the isotropy group of q is conjugate to H, namely it is gHg^{-1}. Then via the conjugation $z \mapsto gzg^{-1}$, $z \in H$, we can carry dh over to gHg^{-1} to obtain a form dh_q which will define a measure μ_q on gHg^{-1}. By the bi-invariance of μ_o, the measure μ_q does not depend on the choice of g and is also bi-invariant. Then by the compactness of the subset $gKg^{-1} = \tilde{K} \subset gHg^{-1}$ fixing any point of $\mathbf{S}_r(q)$, we obtain an $H-$invariant measure on $gHg^{-1}/\tilde{K}$ as before.

On the other hand, we assert that $\mathbf{S}_r(o)$ has a Riemannian structure induced by that of $M = G/H$. Let $d\boldsymbol{\omega}_r$ its volume element given by this metric and consider the Riemannian measure which it defines. It should be noted that this measure is $H-$invariant since H acts by isometry on M. Therefore, by definition of the density function we have

$$\int_{H/K} u(h\cdot x)\,d\dot{h} = \frac{1}{\delta(r)}\int_{\mathbf{S}_r(o)} u(s)\,d\boldsymbol{\omega}_r(s). \tag{4-28}$$

Theorem 4.6. On the solid future directed cone $\mathbf{D}_q \subset M$, for $M = \mathbb{R}^{n+1}_1$, Q_-, Q_+, the Laplace-Beltrami operator can be expressed as:

$$\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{A(r)}\frac{dA}{dr}\frac{\partial}{\partial r} + \Delta', \tag{4-29}$$

where Δ' denotes the Laplace-Beltrami operator on $\mathbf{S}_r(q)$

Proof. Due to H acts transitively on the space H/K, it is enough to suppose $q = o$. The characterization of geodesics in M (see [16, Chapter IV, Proposition 2.1]) implies the fact that $\exp : D_o \to \mathbf{D}_o$ is a diffeomorphism and maps $S_r(o)$ onto $\mathbf{S}_r(o)$. As a result we have polar normal coordinates $(r, t, \theta_1, \dots, \theta_{n-1})$ on $\mathbf{D}_o$. So the ratio of the volume elements of $\mathbf{S}_r(o)$ and $S_r(o)$ is given by $|\det(D_o \exp)|$. That means

$$d\boldsymbol{\omega}_r = |\det(D_o\exp)|d\omega_r.$$

It is handy here the Proposition 2.21. For a geodesic γ in M and $E(r)$ any parallel normal vector field along γ, the normal Jacobi fields along γ can be considered as

$$J(r) = \begin{cases} \sinh(r)E(r), & \text{if } M = Q_-, \\ rE(r), & \text{if } M = \mathbb{R}_1^{n+1}, \\ \sin(r)E(r), & \text{if } M = Q_+, \end{cases}$$

Because of $\det(D_o \exp)$ depends on these Jacobi fields (see Example 2.27), we get

$$|\det(D_o \exp)| = \begin{cases} \left(\frac{\sinh(r)}{r}\right)^n & \text{for the space } Q_- \\ 1 & \text{for the space } \mathbb{R}_1^{n+1} \\ \left(\frac{\sin(r)}{r}\right)^n & \text{for the space } Q_+. \end{cases} \tag{4-30}$$

Now, we denote by $dh = \delta(r)^{-1}\, d\boldsymbol{\omega}_r\, dk$ the result of equating (4-27) and (4-28). But the relation $d\omega_r = r^n\, d\omega_1$ holds. Thus we can write in the three respective cases

$$dh = \frac{\sinh^n r}{\delta(r)}\, d\omega_1\, dk, \qquad \frac{r^n}{\delta(r)}\, d\omega_1\, dk, \qquad \frac{\sin^n(r)}{\delta(r)}\, d\omega_1\, dk.$$

However, dh can be normalized by $dh = d\omega_1\, dk$ (see Remark 3.22) and for this choice it follows

$$\delta(r) = \begin{cases} \sinh^n(r) & \text{for } Q_- \\ r^n & \text{for } \mathbb{R}_1^{n+1} \\ \sin^n(r) & \text{for } Q_+. \end{cases}$$

In addition, recall that the parametrizations given before resulted in the Lorentzian structure on $\mathbf{D}_o$:

$$\begin{aligned} -dr^2 + r^2 d\theta^2, r \in \mathbb{R} \text{ for the space } \mathbb{R}_1^{n+1} \\ -dr^2 + \sinh^2(r) d\theta^2, r \in \mathbb{R} \text{ for the space } Q_-, \\ -dr^2 + \sin^2(r) d\theta^2, r \in [0,\pi] \text{ for the space } Q_+. \end{aligned} \tag{4-31}$$

Here $r^2\, d\theta^2$, $\sinh^2(r)d\theta^2$, $\sin^2(r)d\theta^2$ denote the induced metric $\mathbf{S}_r(o)$ in the space $\mathbb{R}_1^{n+1}$, Q_-, Q_+ respectively. The metric $d\theta^2$ was explicitly described before. Finally, (4-29) holds by calculations. $\diamondsuit$

We remark that the decomposition of the Laplace-Beltrami operator for isotropic manifolds introduced in this section is consequence of the homogeneity of these spaces. Nevertheless, there exists a theorem which gives a natural decomposition of the Laplace-Beltrami operator when the pseudo-riemannian manifold is endowed with an isometric polar action. We are going to establish this result in the next section.

4.3 The Radial Part of the Laplace-Beltrami Operator

Projections of Differential Operators

Let us consider a pseudo-Riemannian manifold (M, g) and $S \subset M$, any pseudo-Riemannian submanifold. For each $s \in S$, the union of all the geodesics in M starting at s which are transversal to S makes up a submanifold $S_s^\perp$ of M. Indeed, if we have an open ball $V(0) \subseteq T_sM$ on which $\exp_s$ is diffeomorphism and for T_sM its orthogonal decomposition $T_sM = T_sS \oplus N$, then $S_s^\perp = \exp_s(B(0) \cap N)$. Also we can assume, by fixing $s_0 \in S$, that all of $S_s^\perp$ are disjoint taking s in a suitable neighborhood S_0 of s_0 in S. Then their union

$$V_0 := \bigcup_{s \in S_0} S_s^\perp \tag{4-32}$$

is a neighborhood of s_0 in M. Now given $u \in C_c^\infty(S)$, we define $\tilde{u}$ on V_0 as constant on each $S_s^\perp$ and equal to u on S_0. Let D be a differential operator on M. We define D', the *projection on S of D*, by

$$(D'u)(s_0) := (D\tilde{u})(s_0). \tag{4-33}$$

Note that the righ-hand side makes sense because $\tilde{u}$ is C^∞ and $D'u$ turns out a $C^\infty-$function on S since (4-32) can still be used for slight variations of s_0 and so (4-33) varies smoothly with s_0. Also, D' decreases supports owing to D decreases supports. Thus D' is differential operator on S.

Example 4.7. In Example 4.5 we gave the Laplace-Beltrami operator on $\mathbb{R}^3$ expressed on polar coordinates (r, θ, ϕ). This is:

$$L_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi}\right). \tag{4-34}$$

Let S denote the unit sphere $x^2 + y^2 + z^3 = 1$. Here each $S_s^\perp$ is parameterized by r. Then the operator in parentheses is the projection $L'_{\mathbb{R}^3}$ of $L_{\mathbb{R}^3}$ on S by construction of $\tilde{u}$. Note that L' is the Laplace-Beltrami operator on S. The next theorem asserts that in general this fact is always true.

Theorem 4.8. [15, Chapter II, Theorem 3.2] Let M be a pseudo-Riemannian manifold of dimension n, $S \subset M$ a pseudo-Riemannian submanifold, and L_M and L_S the corresponding Laplace-Beltrami operators. Then L_S equals the projection of L_M on S :

$$L'_M = L_S$$

Proof. Let $s_0 \in S$ be arbitrary point and choose the local coordinates $(x_1, \ldots, x_n)$ on a neighborhood of s_0 in M in accordance with (4-32) as follows:

i) The mapping

$$s \mapsto (x_1(s), \ldots, x_r(s), 0, \ldots, 0)$$

is a local coordinate system near s_0 on S.

ii) For each $s \in S$ sufficiently close to s_0 and any constants $a_{r+1}, \ldots, a_n$, not all 0, the curve

$$t \mapsto (x_1(s), \ldots, x_r(s), a_{r+1}t, \ldots, a_n t) \tag{4-35}$$

is a geodesic in M starting at s orthogonal to S.

In order to express (4-33) on this coordinate system, we adopt the next notation of indices:

$$1 \leq i, j, k \leq r, \qquad r+1 \leq \alpha, \beta, \gamma \leq n.$$

Due to (4-35) is a geodesic, it satisfies the second order system

$$\frac{d^2 x_i}{dt^2} + \sum_{1 \leq p,q \leq n} \Gamma^i_{pq} \frac{dx_p}{dt} \frac{dx_q}{dt} = 0,$$

which at the point s reduces to

$$\sum_{\alpha,\beta} \Gamma^i_{\alpha\beta} a_\alpha a_\beta = 0. \tag{4-36}$$

Since $\Gamma^i_{\alpha\beta}$ is symmetric in α and β,

$$\Gamma^i_{\alpha\beta} = 0. \tag{4-37}$$

Since the geodesic is orthogonal S on s, we have

$$g_{i\alpha}(s) = 0, \qquad g^{i\alpha} = 0. \tag{4-38}$$

Now, we consider the coordinate representation of L_M given by the equation (4-8) and u and $\tilde{u}$ related as above. Then

$$(L'_M u)(s) = (L_M \tilde{u})(s) = \sum_{p,q} g^{pq} \left(\partial_p \partial_q \tilde{u} - \sum_t \Gamma^t_{pq} \partial_t \tilde{u} \right)(s), \tag{4-39}$$

but by definition $\tilde{u}$ is constant in the $n-r$ last variables, i.e. $\tilde{u}(x_1(s), \ldots, x_r(s), a_{r+1}t, \ldots, a_n t)$ is constant in t so, differentiating with respect to t, we obtain

$$\partial_\alpha \tilde{u} = \partial_{\alpha\beta} \tilde{u} = 0$$

at $(x_1(s), \ldots, x_r(s), 0, \ldots, 0)$. Using (4-37) and (4-38), the right-hand side of (4-39) reduces to

$$\sum_{i,j} g^{ij} \left(\partial_i \partial_j \tilde{u} - \sum_k \Gamma^k_{ij} \partial_k \tilde{u} \right)(s), \tag{4-40}$$

where each Christoffel symbol Γ^k_{ij} is definided on S because by (2-9) and (4-38), these are the restriction to S of the corresponding symbols for M. Besides $\tilde{u} = u$ on S. We conclude (4-40) equals $(L_S u)(s)$. $\diamondsuit$

Transversal part for differential operators

We now consider a pseudo-Riemannian manifold M and a closed subgroup $H \subset I(M)$ acting polar and properly on M. Then each orbit under the action is closed and each isotropy subgroup is compact. A submanifold $B \subset H$ is called a *local cross section* over an open set $U \subset H/H_p$ if the natural map $\pi : H \to H/H_p$ gives by restriction a diffeomorphism of B onto U. In this case, note that for all $p \in M$, $\dim B \cdot p = \dim B = \dim H/H_p$ and since $H/H_p \cong H \cdot p$, it follows that $B \cdot p$ is an open set in the orbit $H \cdot p$.

Lemma 4.9. [14, Lemma 2.1] Suppose W is a pseudo-Riemannian submanifold of M transversal to the polar action of H, that is, for each $w \in W$

$$T_w M = T_w(H \cdot w) \oplus T_w W. \tag{4-41}$$

Fix $w_0 \in W$. Then there exists an open relatively compact neighborhood W_0 of w_0 in W and a relatively compact submanifold B of H forming a cross-section through e over an open neighborhood U_0 of eH_{w_0} in H/H_{w_0} such the mapping

$$\eta : (b, w) \mapsto b \cdot w$$

is a differmorphism of $B \times W_0$ onto a neighborhood of w_0 in M.

Proof. Let $\mathfrak{h}^0$ denote de Lie algebra of H_{w_0} and $\mathfrak{n} \subset \mathfrak{h}$ any subspace such that $\mathfrak{h} = \mathfrak{h}^0 \oplus \mathfrak{n}$. If $\mathfrak{n}_0$ is a sufficiently small neighborhood of 0 in $\mathfrak{n}$, then exp is a diffeomorphism of $\mathfrak{n}_0$ onto a submanifold $B \subset H$ which becomes a local cross section in H over a neighborhood of eH_{w_0} in H/H_{w_0} (see [13, Chapter II, Lemma 4.1]). Therefore, it suffices to prove that the mapping $\phi : (X, w) \mapsto \exp X \cdot w$ of $\mathfrak{n} \times W$ into M is bijective at $(0, w_0)$.

First note that in any point $(0, w) \in \mathfrak{n} \times W$, $\phi(0, w) = w$, so the restriction of ϕ to $\{0\} \times W$ is the identity map of W, then

$$(d\phi)_{(0,w_0)}(0, W_{w_0}) = W_{w_0}. \tag{4-42}$$

Furthermore if we consider the orbit map $\theta^{(w_0)} : h \mapsto h \cdot w_0$, which maps H onto $H \cdot w_0$, then its differential at $e \in H$ is a bijection of $\mathfrak{n}$ onto $(H \cdot w_0)$. Indeed $d_{(e)}\theta^{(w_0)}$ has kernel $\mathfrak{h}^0$ and

$d_{(e)}\theta^{(w_0)}(\mathfrak{h}) = d_{(e)}\theta^{(w_0)}(\mathfrak{h}^0 \oplus \mathfrak{n}) = (H \cdot w_0)_{w_0}$ because $\theta^{(w_0)}$ has constant rank. Since $\mathfrak{h}^0 \cap \mathfrak{n} = \{0\}$ and $d_{(e)}\theta^{(w_0)}(\mathfrak{n}) = (H \cdot w_0)_{w_0}$, we have the bijection. Now observe that

$$(d\phi)_{(0,w_0)}(\mathfrak{n}, 0) = d_{(e)}\theta^{(w_0)}(\mathfrak{n}) = (H \cdot w_0)_{w_0}. \tag{4-43}$$

By (4-42), (4-43) and by the condition (4-41), $(d\phi)_{(0,w_0)}$ is bijective. Then there exist an open neighborhood $V = \tilde{\mathfrak{n}}_0 \times \widetilde{W}_0$ of $(0, w_0)$ such that $\phi|_V : V \to \phi(V)$ is a diffeomorphism. Finally, we choose B such that $\bar{B}$ is compact and $\bar{B} \subset \exp_0(\mathfrak{n}_0 \cap \tilde{\mathfrak{n}})$ and W_0 such that $\overline{W}_0$ is compact and $\overline{W}_0 \subset \widetilde{W}_0$. ♢

By the same construction as before, for $x \in M$ and $S = H \cdot x$, we have the submanifold $S_x^{\perp}$ and may assume that it is transversal to the action of H. We apply the previous lemma, taking $w_0 = x$, (and W_0, B as it was done there) to construct the neighbourhood $V_0 = \cup_{w \in W_0} B \cdot w$ of x in M. Given a differential operator D on M, we now define a new one which acts transversally to the orbits. Let u be a C^∞ function on M. We restrict u to $W_0 \subset S_x^{\perp}$ and then we extend this restriction to a function u_x on V_0 given by

$$u_x(b \cdot w) = u(w), \qquad b \in B,\ w \in W_0.$$

Then D_T, the *transversal part of* D, is defined by

$$D_T(u)(x) := D(u_x)(x)$$

Observe that since u_x is smooth near x the definition of D_T makes sense. Also it does not depend on B because $B \cdot w$ is a open set in the orbit $H \cdot w$ that contains w. Now take an open set U of M such that $u = 0$ on U. Since for all $x \in U$, $u_x = 0$ then $D_T u = 0$. Hence D_T decreases supports and it is a differential operator. When D is the Laplace-Beltrami operator on M, that is $D = L_M$, its transversal part is involved in a decomposition of variables for L_M.

In the next theorem we assume any orbit $S \subset M$ under the action of a Lie group inherits a pseudo-Riemannian structure from that of the ambient space M. This assumption is in order to be able to define the Laplace-Beltrami operator on S.

Theorem 4.10. [14, Theorem 2.2] Let M be a pseudo-Riemannian manifold, H a closed Lie group of $I(M)$ acting polar and properly on M. Let S be any orbit under H and $\bar{f}$ the restriction to S of a function $f \in C^\infty(M)$. Then the Laplace-Beltrami operators L_S on S and $L = L_M$ on M satisfy

$$\overline{Lf} = L_S \bar{f} + \overline{L_T f} \tag{4-44}$$

Proof. Let $s_0 \in S$ and let $S_s^{\perp}, B, W_0$ and V_0 be as in the proof above. Let $b \mapsto (y_1(b), \ldots, y_r(b))$ be any coordinate system on B such that $y(e) = \cdots = y_r(e) = 0$, and let $w \mapsto (z_{r+1}(w), \ldots, z_n(w))$ be a coordinate system on W_0 such that the geodesics forming $S_{s_0}^{\perp}$ correspond to the straight lines through 0. Then we define a coordinate system $(x_1, \ldots, x_n)$ on V_0 by

$$(x_1(b \cdot w), \ldots, x_r(b \cdot w), x_{r+1}(b \cdot w), \ldots, x_n(b \cdot w)) \\ = (y_1(b), \ldots, y_r(b), z_{r+1}(w), \ldots, z_n(w)). \quad (4\text{-}45)$$

Since isometries map geodesics to geodesics, note that from one $S_s^\perp$ we can obtain another one of these that contains any given point in M by using the group action. Thus this coordinate system has properties $i)$ and $ii)$ from the proof of Theorem 4.8 for each $s \in S$. Now, assume $\psi \in C^\infty(V_0)$ satisfies the condition

$$\psi(x_1, \ldots, x_n) = \psi(0, \ldots, 0, x_{r+1}, \ldots x_n), \quad (4\text{-}46)$$

or equivalently

$$\psi(b \cdot w) = \psi(w), \quad b \in B, \quad w \in W_0.$$

Then, by definition

$$\psi_{s_0} = \psi, \quad \bar{\psi} = constant, \quad (4\text{-}47)$$

so

$$(L_T\psi)(s_0) = (L\psi)(s_0), \quad L_S\bar{\psi}(s_0) = 0. \quad (4\text{-}48)$$

On the other hand, if $\phi \in C^\infty(V_0)$ satisfies

$$\phi(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_r, 0, \ldots, 0) \quad (4\text{-}49)$$

or equivalently,

$$\phi(b \cdot w) = \phi(b \cdot s_0), \quad b \in B, \quad w \in W_0,$$

and since H permutes the various $S_s^\perp (s \in S)$ we have, in a neighbourhood of s_0,

$$(\bar{\phi})^\sim = \phi, \quad \phi_{s_0} = constant, \quad (4\text{-}50)$$

where $\tilde{}$ is defined in connection with Theorem 4.8. From Theorem 4.8 we therefore deduce

$$(L_S\bar{\phi})(s_0) = (L'\bar{\phi}) = (L\phi)(s_0), \quad (L_T\phi)(s_0) = (L\phi_{s_0}) = 0. \quad (4\text{-}51)$$

Recall

$$L(\phi\psi) = \phi L\psi + 2\langle\nabla\phi, \nabla\psi\rangle + \psi L\phi, \quad (4\text{-}52)$$

then, using the orthogonality (4-38) and by definition of ∇, we see that for $1 \leq i, j \leq r$; $r+1 \leq \alpha, \beta \leq n$,

$$\nabla\phi = \sum_{i,j} g^{ij} \partial_i \phi \frac{\partial}{\partial x_j} + \sum_{\alpha,\beta} g^{\alpha,\beta} \partial_\alpha \phi \frac{\partial}{\partial x_\beta},$$

and likewise for ψ. But $\partial_i \psi = 0$ and $\partial_\alpha \phi = 0$, so the middle term in (4-52) is zero at the point s_0. Moreover, since $\phi_{s_0}(s_0) = \phi(s_0)$ and by (4-50) ϕ_{s_0} is a constant function, then by (4-47) and by definition of L_T, we see that the first term on the right in (4-52)

$$\phi(s_0)(L\psi)(s_0) = L(\phi_{s_0}\psi)(s_0) = (L(\phi\psi))_{s_0}(s_0) = (L_T(\phi\psi))(s_0).$$

Similarly, since by (4-47) $\bar{\psi}$ is a constant function and using (4-51), in the last term at s_0 we have

$$\psi(s_0)(L\phi)(s_0) = \bar{\psi}(s_0)(L\phi)(s_0) = \bar{\psi}(s_0)(L_S\phi)(s_0) = L_S(\bar{\phi}\bar{\psi})(s_0).$$

Consequetly,

$$L(\phi\psi)(s_0) = L_S(\bar{\phi}\bar{\psi})(s_0) + L_T(\phi\psi)(s_0).$$

By using the fact that the finite linear combinations of functions $\phi\psi$, where ϕ and ψ respectively satisfy (4-49) and (4-46), form a dense subspace of $C_c^\infty(V_0)$, the expected result is obtained by continuity and linearity of differential operators in equation above.

$\diamondsuit$

Example 4.11. Let us consider the action of $H = \mathbb{O}(3)$ by rotations on $M = \mathbb{R}^3 - \{0\}$. The expression (4-34) of the Laplace-Beltrami on $\mathbb{R}^3$ corresponds to the decomposition (4-44) for this case. Indeed,

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

is the transversal part of $L_{\mathbb{R}^3}$ and

$$\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi}\right)$$

is the Laplacian on the sphere $\mathbb{S}^2(r)$.

Formula for the radial part of the Laplace-Beltrami operator

Now, we want to study the decomposition in Lemma 4.10 by supposing some additional properties of the action. As in Lemma 4.9, we take $W \subset M$ is a pseudo-Riemannian submanifold transversal to the polar action of closed subgroup $H \subset I(M)$. Let $\mathfrak{h}$ be the Lie algebra of H. If $X \in \mathfrak{h}$, the vector field on M *induced by* X is defined as

$$(X^+ f)(p) = \left.\frac{d}{dt}\right|_{t=0} f(\exp tX \cdot p),$$

where $f \in C^\infty(M)$ and $p \in M$. A smooth function f on a open subset of M is said to be *locally invariant* if $X^+ f = 0$ for each $X \in \mathfrak{h}$.

Theorem 4.12. [14, Lemma 3.1.] Assume the transversality condition

$$T_w M = T_w(H \cdot w) \oplus T_w W$$

for each $w \in W$. Let D be a differential operator on M. Then there exists an unique differential operator $\Delta(D)$ on W such that

$$(Df)|_W = \Delta(D)(f|_W), \tag{4-53}$$

for each locally invariant function f on an open subset of M.

Definition 4.13. The operator $\Delta(D)$ is called the *radial part* of D.

We conclude this chapter with the key theorem in the remainder of this work. It gives a general formula for the radial part of the Laplace-Beltrami operator on pseudo-Riemannian manifolds endowed with a polar action of a closed subgroup of isometries. We strengthened some hypothesis so that the proof is the same as that given by Helgason in [15] and [14]. Then in order to obtain closed orbits and compact isotropy groups under the action, we shall think about proper actions. So, for instance there will be no problem considering invariant measures. Recall that properness is not guaranteed in the presence of an acting closed group of isometries in the indefinite case.

Theorem 4.14. [14, Theorem 3.2] Let M be a pseudo-Riemannian manifold, H a closed unimodular subgroup of the isometry group $I(M)$. Assume that H acts properly on M and that a submanifold $W \subset M$ is transversal to the action of H, that is for each $w \in W$,

$$(H \cdot w) \cap W = \{w\}, \qquad T_w V = T_w(H \cdot w) \oplus T_w W. \tag{4-54}$$

Then the radial part of L_M is given by

$$\Delta(L_M) = \delta^{-1/2} L_W \circ \delta^{1/2} - \delta^{-1/2} L_W(\delta^{1/2}), \tag{4-55}$$

where $\circ$ denotes composition of differential operators and δ is the density function.

Proof. Put $V^* = H \cdot W$ and consider the surjective mapping $\pi : H \times W \to V^*,\ (h, w) \mapsto h \cdot w$. Fix $w \in W$ and observe that the restriction of π to $H \times \{w\}$ is a mapping $\pi^{(w)} : H \to V^*$ which coincides the orbit map and whose image is $H \cdot w$, then $\pi^{(w)}$ has constant rank and so it is a submersion. Also, for any $w \in W$ and fixing $h \in H$, we have a diffeomorphism from $H \cdot w$ to itself with inverse h^{-1}. Thus π has a differential satisfying

$$\begin{aligned} d\pi_{(h,w)} &= d\pi_{(h,w)}(H_h \times \{0\}) + d\pi_{(h,w)}(\{0\} \times W_w) \\ &= (H \cdot w)_{h \cdot w} + (dh)_w(W_w) = (dh)_w((H \cdot w)_w) + (dh)_w(W_w) \\ &= (dh)_w((H \cdot w)_w + W_w). \end{aligned}$$

Because of (4-54) this image equals $T_{(h \cdot w)}M$, which means $d\pi$ is surjective at each point (h, w).This implies that π is an open mapping and thus V^* is open in M.

Let dx and dw be the volume elements defining the Riemannian measures on M and W respectively, we shall prove that there is a function $\delta \in C^\infty(W)$ such that

$$\int_{V^*} F(x)\, dx = \int_W \delta(w) \left(\int_{H \cdot w} F(h \cdot w)\, d\dot{h} \right) dw, \qquad F \in C_c(V^*). \tag{4-56}$$

In order to give a coordinate expression for dx and dw, take $w_0 \in W$ and select B, W_0, V_0, and η with the properties of Lemma 4.9. Assuming further that $b \to (y_1(b), \cdots, y_r(b))$ is a coordinate system on B with $y_1(e) = \cdots = y_r(e) = 0$ and $w \to (z_{r+1}(w), \cdots, z_n(w))$ is a coordinate system on W_0, we have a coordinate system $(x_1, \cdots x_n)$ on V_0 by putting

$$(x_1(b \cdot w), \cdots, x_r(b \cdot w), x_{r+1}(b \cdots), \cdots, x_n(b \cdots w)) = \tag{4-57}$$

$$(y_1(b), \cdots, y_r(b), z_{r+1}(w), \cdots z_n(w)). \tag{4-58}$$

In the formulas below let $1 \le i, j, k \le r,\ \ r + 1 \le \alpha, \beta, \gamma \le n$. Then we have the following coordinate expressions for the Riemannian measures:

$$dx = \sqrt{\bar{g}}\ dx^1 \cdots dx^n, \qquad dw = \sqrt{\bar{\gamma}}\ dx^{r+1} \cdots dx^n, \tag{4-59}$$

where

$$\bar{g} = |\det(g_{pq})_{1 \le p,q \le n}|, \qquad \bar{\gamma} = |\det g_{\alpha\beta}|. \tag{4-60}$$

Due to the orthogonality condition in (4-54) we have

$$g_{i\alpha}(w) = 0, \qquad w \in W_0. \tag{4-61}$$

Let $b \in B$ and let $b : V_0 \to b \cdot V_0$. Denoting by $\tau : V_0 \to \mathbb{R}^n$ the coordinate map, we have by definition, for $1 \leq p \leq n$,

$$\left.\frac{\partial}{\partial x_p}\right|_v f = \left.\frac{\partial}{\partial x_p}\right|_{\tau(v)} (f \circ \tau^{-1}), \qquad v \in V_0,\, f \in C^\infty(V_0).$$

Since the mapping $w \mapsto b \cdot w$ has coordinate expression

$$(x_{r+1}, \cdots, x_n) \to (x_{r+1}, \cdots, x_n),$$

it follows that

$$\begin{aligned}
\left. db\left(\frac{\partial}{\partial x_\alpha}\right)\right|_w f &= \left.\frac{\partial}{\partial x_\alpha}\right|_w (f \circ b) \\
&= \left(\frac{\partial}{\partial x_\alpha}(f \circ b \circ \tau^{-1})\right) \tau(w) \\
&= \left(\frac{\partial}{\partial x_\alpha}(f \circ b \circ \tau^{-1})\right) (x_1(e \cdot w), \dots x_r(e \cdot w), x_{r+1}(e \cdot w), \dots, x_n(e \cdot w)) \\
&= \left(\frac{\partial}{\partial x_\alpha}(f \circ \tau^{-1})\right) (x_1(b \cdot w), \dots x_r(b \cdot w), x_{r+1}(b \cdot w), \dots, x_n(b \cdot w)),
\end{aligned}$$

then

$$\left. db\left(\frac{\partial}{\partial x_\alpha}\right)\right|_w = \left.\frac{\partial}{\partial x_\alpha}\right|_{b \cdot w} \tag{4-62}$$

and so $g_{\alpha\beta}(b \cdot w) = g_{\alpha\beta}(w)$. On the other hand, the submanifold of V_0 which we obtain by fixing the coordinates x_α lies on an orbit of H. Then the differential of b maps $(H \cdot w)_w$ onto $(H \cdot w)_{b \cdot w}$ and it follows that

$$\left. db\left(\frac{\partial}{\partial x_i}\right)\right|_w = \sum_{j=1}^{r} a_{ij} \left. db\left(\frac{\partial}{\partial x_j}\right)\right|_{b \cdot w}, \tag{4-63}$$

where $a_{ij} \in \mathbb{R}$. By (4-61) it implies that $g_{i\alpha}(b \cdot w) = 0$. Consequently

$$\bar{g}(b \cdot w) = |\det(g_{ij}(b \cdot w))||\det(g_{\alpha\beta})(b \cdot w)| = |\det(g_{ij}(b \cdot w))|\bar{\gamma}(w). \tag{4-64}$$

Now, note that the Riemannian volume element $d\sigma_w$ on the orbit has coordinate expression

$$d\sigma_w = \sqrt{|\det(g_{ij})(b \cdot w)|}\, dx^1 \cdots dx^r (b \cdot w). \tag{4-65}$$

Thus, if $F \in C_c^\infty(V_0)$, we obtain from Fubini Theorem and (4-64) that

$$\begin{aligned}\int_V F(x)\,dx &= \int_V F(x)\,\sqrt{|\det(g_{ij}(b\cdot w))|\bar{\gamma}(w)}\,dx^1\cdots dx^n\\ &= \int_W \bar{\gamma}^{1/2}(w)\left(\int_{H\cdot w} F(p)\,d\sigma_w(p)\right)\,dx^{r+1}\cdots dx^n(w).\end{aligned}$$

But $d\sigma_w$ is invariant under H, so it must be a scalar multiple of $\dot{d}h$. Then the definition $(3-30)$ of δ proves (4-56) for all $F \in C_c(V_0)$. Besides it also holds for F having support inside $h \cdot V_0$ for some h. As w_0 runs through W the sets $h\cdot V_0\,(h\in H)$ form a covering of V^*. Passing to a locally finite refinement and a corresponding partition of unity, (4-56) follows for all $F \in C_c(V^*)$.

To make sense of the formula (4-56), we have to prove, for $f \in C_c(V^*)$, the mapping $f \to \dot{f}$ is surjective, where $\dot{f} \in C_c(W)$ is determined by

$$\dot{f}(w) = \int_{H\cdot w} f(h\cdot w)\,\dot{d}h.$$

In fact, let $F \in C_c(W)$ and let $C \subset W$ be a compact subset outset of which F vanishes. Let $\widetilde{C}$ be a compact subset of V^* such that $(H\cdot w)\cap\widetilde{C} = \emptyset$ for $w \notin C$ and such that for each $w \in C$, $(H\cdot w)\cap\widetilde{C}$ has measure > 0 with respect to the Riemannian measure on the orbit $H\cdot w$. For example, take $\widetilde{C} = C_H \cdot C$, where C_H is a compact neighbourhood of e in H. Select $f_1 \in C_c(V^*)$ be such that $f_1 \geq 0$ on V^* and $f_1 > 0$ on $\widetilde{C}$. Then $\dot{f}_1 > 0$ on C and the function

$$f(v) = \begin{cases} f_1(v)F(w)/\dot{f}_1(w) & \text{if } v \in H\cdot w,\ w\in C\\ 0 & \text{otherwise}\end{cases} \tag{4-66}$$

belongs to $C_c(V^*)$ and satisfies $\dot{f} = F$, proving the surjectivity

$$C_c(V^*)^{\cdot} = C_c(W). \tag{4-67}$$

Now we consider a basis $X_1,\ldots,X_l$ of Lie algebra $\mathfrak{h}$ of H and let $Y_i = X_i^+ (1 \leq i \leq l)$ denote the induced vector fields on V. Since the vector fields ∂x_i $(1 \leq i \leq r)$ are tangential to the $H-$orbits, we have on V_0

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{l} b_{ij}Y_j, \qquad b_{ij} \in C^\infty(V_0). \tag{4-68}$$

Using the coordinate expression of L_M (4-8) in Section 4.1 and taking its restriction to W, we obtain

$$\Delta(L_M) = \sum_{\alpha,\beta} g^{\alpha\beta}\left(\partial_\alpha\partial_\beta - \sum_k \Gamma^k_{\alpha\beta}\partial_k\right) + \sum_{i,j} g^{ij}\left(\partial_i\partial_j - \sum_k \Gamma^k_{ij}\partial_k\right) \tag{4-69}$$

by $(4-61)$ and if f is a locally invariant function on V_0,

$$\partial_i \, \partial_j \, (f|_W) = \partial_i \left(\sum_{m=1}^{l} b_{jm} Y_m \right) (f|_W) = 0,$$

Thus,

$$\Delta(L_M) = L_W + \text{lower-order terms.} \tag{4-70}$$

On the other hand, we have by the symmetry property (4-7) in Section 4.1

$$\int_{V^*} (L_M f_1)(x) f_2(x) \, dx = \int_{V^*} f_1(x)(L_M f_2)(x) \, dx, \tag{4-71}$$

for $f_1, f_2 \in C_c(V^*)$. But then this relation holds for all $f_2 \in C^\infty(V^*)$. In particular, let us take f_2 invariant under H. Applying (4-56) to the left hand side of (4-71) we obtain

$$\int_W \delta(w) f_2(w) \left(\int_{H \cdot w} (L_M f_1)(h \cdot w) d\dot{h} \right) dw. \tag{4-72}$$

We now use the H-invariante measure on the orbit (3-29) to consider the equation

$$\int_H (L_M f_1)(h \cdot w) dh = \int_{H/H_w} \int_{H_w} (L_M f_1)(hk \cdot w) \, dk \, d\dot{h}.$$

As the group H_w acts trivially on the element w, we have $(L_M f)(hk \cdot w) = (L_M f)(h \cdot w)$ for all $k \in H_w$, then

$$\begin{aligned} \int_{H/H_w} \int_{H_w} (L_M f_1)(hk \cdot w) \, dk \, d\dot{h} &= \int_{H/H_w} \int_{H_w} (L_M f_1)(h \cdot w) dk d\dot{h} \\ &= \int_{H/H_w} (L_M f_1)(h \cdot w) \left(\int_{H_w} dk \right) d\dot{h}. \end{aligned}$$

But for each $x \in M$ the isotropy subgroup H_x is compact and dk is normalized then $\int_{H_w} dk = 1$. This allows to write for the inner integral of (4-72)

$$\int_{H \cdot w} (L_M f_1)(h \cdot w) d\dot{h} = \int_{H/H_w} (L_M f_1)(h \cdot w) \, d\dot{h} = \int_H (L_M f_1)(h \cdot w) \, dh. \tag{4-73}$$

Again by the compactness of H_x and the invariance of L_M

$$(L_M)_x \left(\int_H f_1(h \cdot x) \, dh \right) = \int_H (L_M f_1)(h \cdot x) \, dh. \tag{4-74}$$

The unimodularity of H implies that the function $x \mapsto \int_H f_1(h \cdot x)\, dh$ is invariant under H, so putting $x = w$ in (4-74) we get

$$\int_H (L_M f_1)(h \cdot w)\, dh = (\Delta(L_M)\dot{f}_1)(w),$$

and the left-hand side of (4-71) reduces to

$$\int_W (\Delta(L_M)\dot{f}_1)(w)\bar{f}_2(w)\delta(w)\, dw.$$

Once again, formula (4-56), the $H-$invariance of $L_M f_2$ and the definition of $\Delta(L_M)$, the right-hand side of (4-71) reduces to

$$\int_W \dot{f}_1(w)(\Delta(L_M)\bar{f}_2)(w)\, dw.$$

But in view of (4-67) the functions $\dot{f}_1$ (and also $\bar{f}_2$) fill up $C_c(W)$, so the equality of the last expressions means that $\Delta(L_M)$ is symmetric with respect to Riemannian measure given by $\delta(w)\, dw$. But since L_W is symmetric with respect to Riemannian measure given by dw, it can be proven that the composition $\delta^{-\frac{1}{2}} L_W \circ \delta^{\frac{1}{2}}$ is symmetric with respect to Riemannian measure given by $\delta(w)\, dw$ and it agrees with L_W up to lower-order terms. Thus by (4-70) the symmetric operators $\Delta(L_M)$ and $\delta^{-\frac{1}{2}} L_W \circ \delta^{\frac{1}{2}}$ agree up to an operator of order ≤ 1. But this operator, being symmetric, must be a function, and now we evaluate on the function 1 to obtain:

$$\Delta(L_M) - \delta^{-\frac{1}{2}} L_W \circ \delta^{\frac{1}{2}} = -\delta^{-\frac{1}{2}} L_W(\delta^{\frac{1}{2}}).$$

It concludes the proof. $\diamondsuit$

Example 4.15. Consider the polar action of $H = \mathbb{O}(n)$ on $M = \mathbb{R}^n$ with $W = \mathbb{R}^+ - \{0\}$. Here the orbit for each $x \in \mathbb{R}^n$, $x \neq 0$, is $\mathbb{S}^{n-1}(r)$ with $r = |x|$. Also observe that the isotropy subgroup of each element is diffeomorphic to $\mathbb{O}(n-1)$, thus $\mathbb{O}(n)/\mathbb{O}(n-1) \cong \mathbb{S}^{n-1}$. Since $\delta(x) = |x|^{n-1}$ we get

$$\begin{aligned}\Delta(L_{\mathbb{R}^n}) &= r^{-(1/2)(n-1)}\frac{d^2}{dr^2} \circ r^{(1/2)(n-1)} - r^{-(1/2)(n-1)}\frac{d^2}{dr^2}\left(r^{(1/2)(n-1)}\right) \\ &= \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}.\end{aligned}$$

5 Solutions to a Semi-linear PDE via Polar coordinates

The purpose of this section is to study the following equation on a pseudo-Riemannian manifold:

$$\Delta_M u(x) + b(x) f(u(x)) = 0, \quad x \in M, \tag{5-1}$$

for smooth functions $u, b : M \to \mathbb{R}$, with u being non-negative and the function $f : \mathbb{R} \to \mathbb{R}$ representing the nonlinearity of (5-1). We shall follow the analysis done in [3] where it was considered M to be a Riemannian manifold endowed with a polar action of a compact Lie group $G \subset I(M)$ whose transversal submanifold Σ is one-dimensional. There, the hypothesis of Theorem 4.14 are fullfied since a compact group is unimodular and the isometric action of a compact Lie group on a Riemannian manifold is always proper.

Additionally, some technical assumptions were considered. On the one side, the solution to the problem posed by (5-1) is constructed by supposing it is constant on the orbits of the action. Then Theorem 4.14 works with the goal of reducing the problem to a one-dimensional problem posed on Σ. A priori, Σ depends on the choice of the points where it meets each orbit. It is easy to make examples where we have infinite number of choices for the transversal manifold Σ for a fixed polar action [3, Example 3.1]. For the latter ODE defined on any Σ, initial conditions were set to obtain a $C^\infty(\Sigma)$ solution u_Σ which depends on both Σ and such initial conditions. Note that in this way Σ gives a boundary condition to (5-1) via translation on the corresponding orbit. Using the polar action, the function u_Σ defines a function u on M and the geometrical setting implies that the solution u becomes continuous.

Nevertheless, the pseudo-Riemannian case has some drawbacks. So we shall proceed analogously except for some changes in the considerations. For the sake of simplicity, we shall emphasise mostly in the Lorentzian context. The generalization of Theorem 11 in [3] to Lorentzian manifolds turns out to be the most natural. We begin by assuming that the group acting polarly is closed and its action is proper. Properness in [3] is automatic owing to the compactness of G. On the other side, we realised that the continuity of solutions will depend on the existence of geodesics joining any pair of points in the manifold M. If M is a complete Riemannian manifold, we can always find such a geodesic because of the Hopf-Rinow Theorem. Unfortunately, the completeness is no longer enough on a Lorentzian manifold. To ensure the continuity of the global solution u in the Lorentzian case, a stronger condition is necessary. It is the *global hyperbolicity* condition (see Definition 2.30), which provides the existence of nonspacelike geodesics joining any pair of causally related points of M.

In this chapter, we take advantage of the fact that a globally hyperbolic Lorentzian manifold M can be parameterized via generalized polar coordinates, in such a way that the solution method to (5-1) used in [3] can be replicated on M. Polar actions that induce these coordinates are of cohomogeneity one and in this chapter are defined. Then we give some examples on the Minkowski space $\mathbb{R}^n_1$ based on some results of the papers [1] and [2]. In Section 5.2 we present the main theorem of this work which guarantees the existence of solutions on the setup we propose. Section 5.3 is an appropriate extension of our considerations and results to Lorentzian warped products.

5.1 Cohomogeneity one Polar Actions

An action of a Lie group G on a connected manifold M is called *of cohomogeneity one* if it has at least one orbit of codimension 1. Assuming M to be a Riemannian manifold and G to be a connected closed Lie subgroup of $I(M)$ acting by isometries yields nice consequences. Cohomogeneity one means in this case that the orbit space M/G is one-dimensional (and so up to rescaling, one of $\mathbb{R}$, $[0, +\infty)$, $\mathbb{S}^1$ and $[-1, 1]$). Undoubtedly the action turns out to be proper and if G also acts polarly, the orthogonal sum in Definition 3.7 says that the dimension of the transversal submanifold Σ coincides with the cohomogeneity of the action, (see [6]). Then looking for one-dimensional submanifolds transversal to the action is in general looking for cohomogeneity one polar actions. A greater depth in aspects about this topic on Riemannian manifolds can be reviewed in [4], [6].

When the metric structure on M is indefinite, that is M is a pseudo-Riemannian manifold, the same considerations do not allow us to conclude that the action is proper and the polarity of the action will depend on whether the metric is degenerated or not on the orbits of each point of M. Then if an arbitrary orbit inherits a degenerate metric structure, then there is no a pseudo-Riemannian submanifold transversal to the action. We shall focus in this aspect below.

Let $M \to M/G$ be the projection map from M onto the orbit space M/G. Given a point $p \in M$, the orbit $G \cdot p$ is *principal* (resp. *singular*) if the corresponding image in the orbit space is an interior (resp. boundary) point. All principal orbits are diffeomorphic to each other, each singular orbit is the dimension less than or equal to $n - 1$. Note that when $M/G = \mathbb{R}$, $\mathbb{S}^1$ all orbits are principal, if $M/G = [0, +\infty)$ there is one singular orbit and in the case $M/G = [-1, 1]$ there are two singular orbits.

From now on, we shall take the most classic Lorentzian space into account, namely the Minkowski space $\mathbb{R}^n_1$. The papers [1] and [2] present us interesting results regarding cohomogeneity one actions on this space in the general case and when $n = 4$. Also these provide us quite illustrative examples.

Theorem 5.1. [1, Lemma 3.3] Let $G \subset I(\mathbb{R}^n_\nu)$ be a connected, closed Lie group for $1 \leq \nu \leq n-1$. If the action of G on $\mathbb{R}^n_\nu$ is of cohomogeneity one, then G is not compact.

The essential observation for the proof is that in $\mathbb{R}^n_\nu$ there are no compact pseudo-Riemannian hypersurfaces (see [21]), thus assuming G to be compact is a contradiction. With this result at hand we follow [2], where the connected Lie groups $G \subset I(\mathbb{R}^4_1) \cong \mathbb{R}^4 \rtimes \mathbb{O}_1(4)$ of cohomogeneity one which acts by isometry on $\mathbb{R}^4_1$ are determined up conjugancy. Theorem 3.4 therein exhibits a list of all possible groups, of which those that act properly are of interest to us.

Theorem 5.2. [1, Theorem 3.1] Let G be a connected, closed Lie subgroup of $I(\mathbb{R}^n_1)$ whose action on $\mathbb{R}^n_1$ is proper and of cohomogeneity one. If there is a spacelike principal orbit, then each orbit is spacelike and isometric to $\mathbb{R}^{n-1}$. In particular there is no singular orbit.

The proof is a consequence of previous results in [1], including the fact that under the same hypotheses the orbit space M/G, for $M = \mathbb{R}^n_1$, is homeomorphic to $\mathbb{R}$ or $[0, +\infty)$. The latter will be noted below. Closed groups whose actions on $\mathbb{R}^4_1$ are of cohomogeneity one, proper and isometric are, up to conjugacy, of five types. We shall introduce three of them in order to illustrate our future considerations in the remainder of this chapter. The complete classification of cohomogeneity one actions on $\mathbb{R}^4_1$ in both cases, proper and non-proper, can be seen in [2]. We begin by fixing some notation. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of $\mathbb{R}^4_1$. The timelike line $\mathcal{L} \subset \mathbb{R}^4_1$ is defined by $\mathcal{L} = \mathbb{R}(e_2 - e_1)$ and the degenerate hyperplane $\mathbb{W}^3 \subset \mathbb{R}^4_1$ by $\mathbb{W}^3 = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathcal{L}$. Also, we shall use $\mathcal{O}_H$ to denote the collection of orbits of the action of H on $\mathbb{R}^4_1$.

Type I: *The Lie group H acts by translations.* Then $\mathcal{O}_H$ is invariant under a three-dimensional translation group. Here $H \in \{\mathbb{R}^3, \mathbb{R}^3_1, \mathbb{W}^3\}$, where $\mathbb{R}^3 = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4$, $\mathbb{R}^3_1 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$. Then the orbits in $\mathcal{O}_H$ are affine hyperplanes in $\mathbb{R}^4_1$ that are parallel to $\mathbb{R}^3, \mathbb{R}^3_1$ and $\mathbb{W}^3$ respectively. In all three cases every orbit is principal and totally geodesic, besides the orbit space is diffeomorphic to $\mathbb{R}$. By Theorem 5.2, it is clear that there are no singular orbits.

a) $\mathcal{O}_{\mathbb{R}^3} = \bigcup_{t\in\mathbb{R}}(te_1 + \mathbb{R}^3)$. In this case the orbits are hyperplanes in $\mathbb{R}^4_1$ parallel to $\mathbb{R}^3$. Each of them is spacelike and picking a point p_t in any of them, we can find a submanifod Σ transversal to the action defined by $\Sigma = \{(t, p_t) \in \mathbb{R}^1_1 \times \mathbb{R}^3 : -\infty < t < \infty\}$. Note that we have used the standard decomposition $\mathbb{R}^4_1 = \mathbb{R}^1_1 \times \mathbb{R}^3$, thus Σ is parallel at each p_t to $\mathbb{R}e_1$ and can be parametrised by the curve $\gamma(t) = (t, p_t)$. By causality of the orbits, Σ is timelike and for each $p = (t, p_t) \in \mathbb{R}^4_1$ the tangent space at p can be decomposed as $T_p\mathbb{R}^4_1 = T_p\Sigma \oplus T_p(te_1 + \mathbb{R}^3)$. Hence the action is polar.

b) $\mathcal{O}_{\mathbb{R}^3_1} = \bigcup_{t\in\mathbb{R}}(te_4 + \mathbb{R}^3_1)$. Namely the orbtis are hyperplanes in $\mathbb{R}^4_1$ parallel to $\mathbb{R}^3_1$. An argument analogous to the previus one allows to find at each point p_t of each timelike orbit a submanifold transversal to the action parallel to $\mathbb{R}e_4$. This submanifold becomes spacelike. Then H acts polarly.

c) $\mathcal{O}_{\mathbb{W}^3} = \bigcup_{t\in\mathbb{R}}(te_1 + te_2 + \mathbb{W}^3_1)$. Here we get as orbits parallel hyperplanes parallel to $\mathbb{W}^3$. Owing to each of them is degenerated, an orthogonal direct sum decomposition of $T_p\mathbb{R}^4_1$, at each point $p = (t, p_t)$ of an orbit, is no possible. Hence H does not act polarly.

Type II: $H = \mathbb{R}^2_1 \times \mathbb{SO}(2)$. *Then $\mathcal{O}_H$ is invariant under the two-dimensional translation group $\mathbb{R}^2_1$.* Observe that the action of $\mathbb{R}^2$ by translations generates orbits $\rho e_1 + \varsigma e_2 + \mathbb{R}^2_1$, for fixed $\rho, \varsigma \in \mathbb{R}$. On each of these orbits, $\mathbb{SO}(2)$ acts and its orbits consist of the single point $\{\rho e_1 + \varsigma e_2\}$ and $\rho e_1 + \varsigma e_2 + \mathbb{S}^1(r)$, where $r \in \mathbb{R}^+$. Thus the set of the orbits given by the action of H consists of the plane $\mathbb{R}^2_1$ and the pseudo-hyperbolic cylinders $\mathbb{R}^2_1 \times \mathbb{S}^1(r) \subset \mathbb{R}^2_1 \times \mathbb{R}^2$, where $r \in \mathbb{R}^+$:

$$\mathcal{O}_{\mathbb{R}^2_1\times\mathbb{SO}(2)} = \bigcup_{r\geq 0}(\mathbb{R}^2_1 \times \mathbb{S}^1(r)).$$

The orbit of the origin, namely if $r = 0$, is the unique singular orbit and is timelike. The latter is also true for the principal orbits, i.e. if $r > 0$. The orbit space is homeomorphic to $[0, \infty)$. We assert that for each point $p = te_1 + se_2 + p_r$ of a singular orbit, there exists a spacelike submanifold Σ passing through p transversaly and which is parallel to either $\mathbb{R}^+e_3$ or $\mathbb{R}^+e_4$. Since these meet $\mathbb{S}^1(r)$ orthogonally and by geometrical settings, the action is polar.

Type III $H = \mathbb{R}e_1 \times \mathbb{SO}(3)$. *Then* $\mathcal{O}_H$ *is invariant under the one-dimensional timelike translation group* $\mathbb{R}e_1$. Here, the orbits under the action by translation of $\mathbb{R}^3$ are $\rho e_1 + \mathbb{R}^3$. The action of $\mathbb{SO}(3)$ leaves these orbits invariant and generates on each one of them the collection of orbits given by the single point ρe_1 and the spheres centered at that point. This implies that H generates the singular timelike orbit $\mathbb{R}e_1$ and the cylinders $\mathbb{R}e_1 \times \mathbb{S}^2(r) \subset \mathbb{R}^1_1 \times \mathbb{R}^3$, where $r \in \mathbb{R}_+$:

$$\mathcal{O}_{\mathbb{R}e_1 \times \mathbb{SO}(3)} = \mathbb{R}e_1 \bigcup_{r \in \mathbb{R}_+} \left(\mathbb{R}e_1 \times \mathbb{S}^2(r)\right)$$

The orbit space is homeomorphic to $[0, \infty)$. By analogy with the action of a group Type II, we can find a spacelike submanifold Σ passing through each point p of each principal orbit. In fact, Σ can be choosen parallel to $\mathbb{R}^+e_2$, $\mathbb{R}^+e_3$, or $\mathbb{R}^+e_4$. As before, the action is polar.

Remark 5.3. Observe that in the cases where the orbit space is $[0, \infty)$, we have omitted the singular orbit in order to consider the polar action.

Remark 5.4. It is also clear that for each of these actions it can be found a subgroup of the acting group which is transitive on any orbit. In general, proper cohomogeneity one actions on $\mathbb{R}^n_1$ have this behavior, see [1] and references therein.

5.2 Existence of Solutions for Cohomogeneity one Polar Actions

Example 5.5 (Preliminary example)**.** Here we present the method to obtain solutions to (5-1) for the Euclidean space $\mathbb{R}^3$. It is well know what we mean by Laplace operator on it using standard coordinates:

$$L_{\mathbb{R}^3} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Step 1: Standard coordinates for a point $p = (x, y, z)$ can be expressed in polar coordinates by mean of the formulas

$$x = r\cos\theta_1 \quad y = r\sin\theta_1\cos\theta_2 \quad z = r\sin\theta_1\sin\theta_3.$$

Then, the Laplace operator becomes:

$$L_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin\theta_1\frac{\partial}{\partial\theta_1}\right) + \frac{1}{r^2\sin\theta_1}\frac{\partial}{\partial\theta_2}\left(\frac{1}{\sin\theta_1}\frac{\partial}{\partial\theta_2}\right).$$

We search for *radial* solutions, that is, functions of $r = |p|$. Suppose then that $u = u(r), b = b(r)$ are radial on (r, θ_1, θ_2) and thus

$$L_{\mathbb{R}^3} = u''(r) + \frac{2}{r}u'(r).$$

We get an equivalent problem to (5-1):

$$u''(r) + \frac{2}{r}u'(r) + b(r)f(u(r)) = 0.$$

Step 2: Under the change of variable $s = \int_1^r t^{-2}\, dt$ and if $z(s) = u(r)$, we obtain the following one-dimensional problem

$$z''(s) + (r(s))^4 b(r(s))f(z(s)) = 0,$$

which is posed on $\mathbb{R}$ and whose solution can be guaranteed. Here, it is used the fact that

$$\frac{dr}{ds} = r^2 \qquad \text{and} \qquad \frac{dz}{ds} = u'(r)r^2.$$

Henceforth, M will denote a temporally oriented and globally hyperbolic Lorentzian manifold. Thus there is a maximal nonspacelike geodesic joining any two causally related points of M. Consider the Equation (5-1) on M and assume the existence of a one-dimensional submanifold Σ of M transversal to the action of a closed subgroup $G \subset I(M)$. We remarked before that in the presence of a indefinite metric a submanifold might not inherit a metric structure from that of the ambient manifold. Then, we have to suppose that both the submanifold Σ and the orbits are endowed with a suitable pseudo-Riemannian metric induced by M. We shall firstly deal with case of Σ being timelike and hence each orbit is spacelilke.

A function u satisfying $u(x) = u(g \cdot x)$ for all $x \in M$ and $g \in G$ is said to be a *radial function*. Clearly this function is such that for all $s \in \Sigma$ and $g \in G$ we have $u(g \cdot s) = u(s) = u|_\Sigma$. Thus $u = \tilde{u}$ on M, where $\tilde{u}$ is defined as at the beginning of Section 4.3, and then we can apply Theorem 4.14 to obtain the explicit formula for the Laplace-Beltrami operator applied on u :

$$\begin{aligned} L_M(u) &= L_M(\tilde{u}) = \Delta(L_M)(u|_\Sigma) \\ &= \frac{1}{\sqrt{\delta(r)}} L_\Sigma(\sqrt{\delta(r)}u|_\Sigma) - \frac{1}{\sqrt{\delta(r)}} L_\Sigma(\sqrt{\delta(r)})u|_\Sigma \\ &= \frac{2\langle \nabla_\Sigma \sqrt{\delta(r)}, \nabla_\Sigma(u)\rangle}{\sqrt{\delta(r)}} + L_\Sigma(u), \end{aligned} \tag{5-2}$$

where we have used the identity $L(uv) = u(Lv) + 2\langle \nabla u, \nabla v\rangle + v(Lu)$, with $u, v \in C^\infty(M)$ and have considered r as the parameter along Σ. We now take Σ as a one-dimensional submanifold of M in Equation 5-2, together its character causal to obtain

$$
\begin{aligned}
L_M(u) &= -\frac{2}{\sqrt{\delta(r)}}\langle \nabla_\Sigma \sqrt{\delta(r)}, \nabla_\Sigma(u)\rangle - L_\Sigma(u) \\
&= -\langle \nabla_\Sigma \log \delta(r), \nabla_\Sigma(u)\rangle - L_\Sigma(u) \\
&= -(\ln(\delta))'u' - u''.
\end{aligned} \tag{5-3}
$$

Remark 5.6. We point out that in a general case of polar actions where the dimension of Σ is greater than 1, the PDE in (5-1) is equivalent to

$$\Delta_\Sigma u + 2\frac{\langle \nabla_\Sigma \sqrt{\delta(r)}, \nabla_\Sigma u\rangle}{\sqrt{\delta(r)}} - b(r)f(u) = 0. \tag{5-4}$$

Here the symbols Δ_Σ and ∇_Σ denote respectively the Laplace-Beltrami and gradient operator on Σ, defined by reversing the metric that Σ inherits of that M.

The Equation (5-4) holds for a locally invariant function u and we get

$$\Delta_\Sigma u + \langle \nabla_\Sigma \log \delta(r), \nabla_\Sigma u\rangle - b(r)f(u) = 0, \tag{5-5}$$

after using the formula

$$\frac{2}{\sqrt{\delta}}\nabla\sqrt{\delta} = \frac{1}{\delta}\nabla\delta = \nabla \ln \delta.$$

We have reduced the Equation (5-1) to a simpler equation defined on the submanifold Σ

$$(\delta(r)u'(r))' - \delta(r)b(r)f(u(r)) = 0, \tag{5-6}$$

assuming u, b to be radial functions. Therefore the semilinear equation becomes an ordinary differential equation for which we could study several interesting questions related to the existence of infinitely many polar action invariant solutions. Since the reduced dimensional problem might be parametrized by some suitable boundary data, we can consider different initial conditions for the original problem. With this in mind, we shall focus now the attention to study the existence of solutions to (5-1). In order to do this, we start by stating the theorem.

Theorem 5.7. Suposse M is a temporally oriented and globally hyperbolic Lorentzian manifold and H a closed unimodular subgroup of $I(M)$ whose action on M is proper, polar and of cohomogeneity one. Let Σ be a timelike submanifold of M transversal to the action of H and let $\delta : \Sigma \to [0, \infty)$ be the density map. Furthermore, suposse that

i) for $r_0 \in \Sigma$ the function $s = J(r) := \int_{r_0}^{r} \delta(t)^{-1}dt$ maps Σ to $\mathbb{R}$,

ii) the linear problem

$$z''(s) - b(r(s))\delta(r(s))^2 = 0, \quad s \in \mathbb{R}$$

with $b(r(\cdot))$ an α-regular negative function, has a unique positive $C_0(\mathbb{R}) \cap C^{2+\alpha}_{loc}(\mathbb{R})$ solution, *iii)* $f \in C^1((-\infty, 0), (0, \infty))$ satisfies $\lim_{q \to 0^-} f(q)/q = \infty$ and $\lim_{q \to -\infty} f(q)/q = 0$.

Then the problem (5-1) has at least one non-negative solution $u \in C^\infty(M)$.

The hypotheses in the theorem include the geometric setting where we are seeking a solution to the problem as well as the conditions which guarantee its existence. Our assumptions about the group acting are by virtue of getting closed orbits and compact isotropy groups under the action of H and thus Theorem 4.14 is fulfilled. The condition $i)$ refers to the non-integrability of the function $\delta(r)^{-1}$. Note that when Σ is parametrized by arch-length $r \in [0, R]$, this condition is related to the points where $\delta(r) = 0$ because these become singularities of the equation posed on Σ. Nevertheless this case does not concern us mainly because we are not considering fixed points of the action. The analytical conditions $ii), iii)$ have to do with the principal result given in [25] and these will be used as in [3].

Proof. Let u and b be radial functions, namely these only depend on the parameter describing Σ and which is defined by the global polar coordinates defined by the action of G on M. That means that the functions u, b satisfy that for all $g \in G$ and $x \in M$ we have $u(g \cdot x) = u(x)$ and $b(g \cdot x) = b(x)$. By (5-3), the problem (5-1) turns to be

$$\begin{cases} (\delta(r)u'(r))' - b(r)\delta(r)f(u(r)) = 0, \\ u(r_0) = d. \end{cases} \tag{5-7}$$

where d is a real number. Since the function $\delta(r)$ is positive for $r \neq 0$, we define $s = J(r) = \int_{r_0}^{r} \delta(t)^{-1} dt$ with $r_0 > 0$ and $z(s) = u(J^{-1}(s))$. By the Chain Rule, we get

$$\frac{d(r)}{ds} = \frac{d(J^{-1}(s))}{ds} = \frac{1}{J'(J^{-1}(s))} = \delta(r),$$

and

$$\frac{dz}{ds} = u'(r)\delta(r),$$

so that

$$\frac{d^2z}{ds^2} = \delta(r)\frac{d(u'(r)\delta(r))}{ds}.$$

Hence the problem 5-7 is equivalent to

$$\begin{cases} \dfrac{d^2z}{ds^2} - b(r(s))\delta(r(s))^2 f(z(s)) = 0, \\ z(r_0) = d, z(r(s)) \geq 0, \end{cases} \tag{5.7$'$}$$

where d is any real number.

Theorem 1.1 in [25] claims that if $\dfrac{d^2z}{ds^2} - b(r(s)\delta(r(s))^2 = 0$, with b an α-regular positive function, has a unique positive solution which is also in $C_0(\mathbb{R}) \cap C^{2+\alpha}_{loc}(\mathbb{R})$ then the problem (5.7′) has solution if in addition we suppose that $f \in C^1((-\infty,0),(0,\infty))$ satisfies $\lim_{q\to 0^-} f(q)/q = \infty$ and $\lim_{q\to -\infty} f(q)/q = 0$.

Note that in this case $u(r) = z(J(r))$ is a solution to (5-1). $\diamondsuit$
At this very instant, we proceed to fix a possible gap concerning the continuity of the solution in the proof of Theorem 5.7. Formally, the solution described above is defined on the transversal pseudo-Riemannian manifold Σ, but we can extend it to the whole manifold M by setting $\tilde{u}(x) = u(s)$, where for each $s \in \Sigma$, we have $x = g \cdot s$ for some $g \in G$. Namely, the function $\tilde{u}$ is constant on the orbits of the action of G. It implies that each orbit can be viewed as level sets $\tilde{u}^{-1}(c)$ for each $c \in \mathbb{R}$. Due to the transversality condition in Definition 3.7, we can suppose that for each $x \in M$ the gradient $\nabla\tilde{u}(x)$ is timelike because Σ does. Let $\alpha : (a,b) \to M$ be a future directed nonspacelike curve with nonvanishing tangent vector $\dot{\alpha}(t)$. Then,

$$g(\nabla(\tilde{u}(\alpha(t))), \dot{\alpha}(t)) = \dot{\alpha}(t)(\tilde{u})$$

must be always positive or always negative. It implies $\tilde{u}$ must be strictly increasing or strictly decreasing along γ. In addition to this, for all pair of points $p, q \in M$ the collection of open sets $I^+(p) \cap I^-(q)$ define a topology on M called the *Alexandrov Topology* . In general, it needs not coincide with the given manifold topology on M but if M is strongly causal, these two are the same [5, Proposition 3.11].

Lemma 5.8. Let $h : M \to \mathbb{R}$ a strictly increasing function on all nonspacelike curve and let $x \in M$. Then h is continuous at x.

Proof. Due to we are assuming that M globally hyperbolic, we have that the basic open sets are of the form $I^-(p) \cap I^+(q)$ for some pair of points $p, q \in M$. On the other side (also due to the global hyperbolicity) for all pair of points $p, q \in M$ such that $x \in I^-(p) \cap I^+(q)$ there exists a nonspacelike curve $\alpha : [-t,t] \to M$ with $\alpha(0) = x$, $\alpha(-t) = q$, $\quad \alpha(t) = p$ such that $h(\alpha)$ is continuous at 0. Now let $(y_i)_{i\in\mathbb{N}} \subset M$ be a sequence of points such that $y_i \to x$. For each $i \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$ such that for all $j > k_i$

$$y_j \in I^-(\alpha(-\frac{1}{i})) \cap I^+(\alpha(\frac{1}{i})).$$

Since h is strictly increasing function on all nonspacelike curve this implies that for all $j > k_i$

$$h(\alpha(-\frac{1}{i})) \leq h(y_j) \leq h(\alpha(\frac{1}{i})).$$

As $h(\alpha)$ is continuous at 0, we have that $h(\alpha(-\frac{1}{i})) \to h(\alpha(0)) = h(x)$ and $h(\alpha(\frac{1}{i})) \to h(\alpha(0)) = h(x)$. It implies $h(y_i) \to h(x)$, as required. $\diamondsuit$

Remark 5.9. It has been shown the continuity relative to the Alexandrov topology. However, a globally hyperbolic manifold is always by definition strongly causal, then we use what was said before the theorem. Despite being the most restrictive condition, we deal with global hyperbolicity because it is easier to verify on our examples.

Example 5.10. Let us use the description done in Section 5.1 of some proper, cohomogeneity one polar actions on $\mathbb{R}^4_1$ to find a solution to (5-1) on said space. In order to verify the assumptions of Theorem 5.7 and since $\mathbb{R}^4_1$ is globally hyperbolic, we write $\mathbb{R}^4_1 = \mathbb{R}^1_1 \times \mathbb{R}^3$ and of course, consider the metric $-dt^2 + g_t$, being g_t the Euclidean metric on $\mathbb{R}^3$ (see Example 2.32). Then we need to choose an action whose orbits are isometric to $\mathbb{R}^3$. Let us take the closed subgroup $H = \mathbb{R}^3$ of $I(\mathbb{R}^4_1)$. It is of Type I and acts by translations on $\mathbb{R}^4_1$. Commutativity of H implies that it is unimodular.

Next we calculate the density function δ. Let $p = (t, p_t) \in \mathbb{R}^1_1 \times \mathbb{R}^3$ be an arbitrary point. Note that the orbit of p is $H \cdot p = te_1 + \mathbb{R}^3$ and its isotropy subgroup is $H_p = \{e\}$. It follows from Remark 3.21 that volume elements giving the induced Riemannain measure on $H \cdot p$ and the $H-$invariant measure on H/H_p coincide. Hence $\delta(p) = 1$. It remains to find the solutions. Let $t_0 \in \Sigma$ be a fixed point of a transversal Σ to the action of H. A solution to the problem (5-1) is given by

$$u(r) = z\left(\int_{t_0}^{t} ds.\right).$$

Note that we can proceed similarly to solve (5-1) posed on $\mathbb{R}^n_1$.

Remark 5.11. We now point out the role of the causal character of the submanifold transversal to the polar action. Observe that the action group of Type I item b), Type II and Type III generate spacelike transversal submanifolds. This implies that (5-3) becomes

$$L_{\mathbb{R}^4_1}(u) = (\ln(\delta))'u' + u'',$$

as concluded in [3]. Then a solution to (5-1) is

$$u(r) = z\left(\int_{r_0}^{r} \frac{1}{\delta(t)}\, dt\right).$$

Since all the considered actions are proper, we always get compact isotropy subgroup H_p for all $p \in \mathbb{R}^4_1$. Hence H/H_p has a $H-$invariant measure. We now describe the computation of δ in each case.

- Action of $H = \mathbb{R}^3_1$ by translations: Here the orbit of an arbitrary point $p = (t, p_t) \in \mathbb{R}^4_1$ is $H \cdot p = te_4 + \mathbb{R}^3_1$, which inherits the Minkowski metric from that of $\mathbb{R}^4_1$. Besides $H_p = \{e\}$. By the identification $H/H_p \cong H \cdot p$ and by definition of δ, again the Riemannian measure on $H \cdot p$ and the $H-$invariante measure on H/H_p coincide and thus $\delta(p) = 1$.
- Action of $H = \mathbb{R}^2_1 \times \mathbb{SO}(2)$: For fixed $r > 0$ and $\rho, \varsigma \in \mathbb{R}$, the orbit of a point $p \neq \rho e_1 + \varsigma e_2$ of $\mathbb{R}^4_1$ under the action of H is $H \cdot p = \mathbb{R}^2_1 \times \mathbb{S}^1(r)$. Note that if (t, x, y, z) are coordinates for p then y, z satisfy $y^2 + z^2 = r^2$. As submanifold, $H \cdot p$ has a pseudo-Riemanian structure induced by $\mathbb{R}^4_1$ which is given by $-dt^2 + dx^2 + r^2\, d\theta^2$, where t, x are coordinates for $\mathbb{R}^2_1$ and θ for $\mathbb{S}^1(r)$.

Hence the volume element of each orbit is $d\omega_r = r\,dx \wedge dy \wedge d\theta$. Note that $d\omega_r = r\,d\omega_1$. On the other hand, owing to $H_p \cong \mathbb{SO}(1)$ we have $H/H_p \cong \mathbb{R}^2_1 \times \mathbb{S}^1$. Let $d\dot{h}$ be the $H-$invariant volume element of H/H_p. We can normalize $d\dot{h}$ by $d\dot{h} = d\omega_1$, and by definition of δ we obtain $\delta(p) = r$.

- Action of $H = \mathbb{R}e_1 \times \mathbb{SO}(3)$: Here we proceed analogously to the previous case. Take a fixed $r > 0$ and $t \in \mathbb{R}$. For a point $p \neq \rho e_1$ of $\mathbb{R}^4_1$, the orbit under the action of H is $H \cdot p = \mathbb{R}e_1 \times \mathbb{S}^2(r)$. Note that if (t,x,y,z) are coordinates for p then x,y,z satisfy $x^2 + y^2 + z^2 = r^2$. Each orbit has a structure pseudo-Riemannian inherited from $\mathbb{R}^4_1$. It is given by $-dx^2 + r^2\,d\theta_1^2 + r^2\sin^2\theta_1 d\theta_2^2$, where x is the coordinate for $\mathbb{R}^1_1$ and θ_1, θ_2 for $\mathbb{S}^2(r)$. Thus its volume element is $d\omega_r = r^2\sin\theta_1\,dx \wedge d\theta^1 \wedge d\theta^2$. Let $d\dot{h}$ denote the $H-$invariant volume element of H/H_p. Then $d\dot{h}$ can be normalized by $d\dot{h} = d\omega_1$ as we did before and thus $\delta(p) = r^2$.

Due to the pseudo-Riemannian structure of $\mathbb{R}^4_1$ is degenerate on the orbits of the action of $H = \mathbb{W}^3$, Theorem 4.14 by Helgason fails and so it is not possible to find the decomposition of Laplace-Beltrami operator.

5.3 Polar invariant Solutions on Warped Lorentzian Spaces

Warped product spaces have very favorable properties to define polar actions on them and to use the powerful machinery provided by Helgason [15] which we presented in Chapter 4. We introduced them in Section 2.2 with special interest in those which are Lorentzian spaces. We shall only deal with these cases. At the end of the section, we will have obtained a rich range of examples where our main theorem works to obtain solutions to the semi-linear problem (5-1).

We begin by showing how Theorem 5.7 can be used to prove the existence of polar invariant solutions to (5-1) posed on the Lorentzian space $\mathbb{S}^n_1(r)$.

Example 5.12. According to Example 2.36, the expression of the metric induced by the inclusion $i : \mathbb{S}^n_1(r) \hookrightarrow \mathbb{R}^{n+1}_1$ on the pseudosphere $\mathbb{S}^n_1(r)$ is

$$dh^2 = -r^2 dt^2 + r^2\cosh^2(t) ds^2,$$

with ds being the round metric defined on the Riemannian sphere. This metric allows us to see the pseudosphere $\mathbb{S}^n_1(r)$ as a Lorentzian manifold diffeomorphic to the product space

$$\mathbb{R} \times_f \mathbb{S}^n,$$

for $f(t) = \cosh(t)$. Note that according to Example 2.32, $\mathbb{S}^n_1$ is globally hyperbolic. Now, we can consider the action of the group $\mathbb{SO}(n)$ on the space $\mathbb{R} \times_f \mathbb{S}^n$ by rotation on the second variable. The transitivity of the action of the group $\mathbb{SO}(n)$ on the spheres $S_t = \{(t,s) \in \mathbb{R} \times_f \mathbb{S}^n | t \text{ fixed}\}$ implies $S_t = \mathbb{SO}/K_p$ for K_p the isotropy group of any point $p = (t, s_p)$. By fixing the value of s_p, we have that the timelike submanifold

$$\Sigma_p = \{(t, s_p) \in \mathbb{R} \times_f \mathbb{S}^n | t \in \mathbb{R}\}$$

is transversal to the action of the compact group $\mathbb{SO}(n)$ on $\mathbb{S}^n_1(r)$. By cause of the causal character of Σ_p, the action is polar.

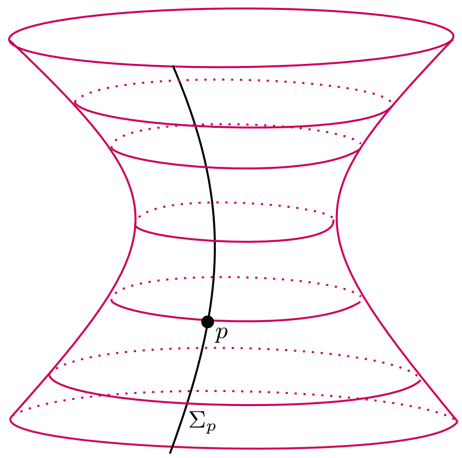

Figure **5-1**: Polar action of $\mathbb{SO}(2)$ on $\mathbb{S}^2_1 \subset \mathbb{R}^3_1$.

We can now apply Theorem 5.7 to the semi-linear equation

$$L(u(x)) + b(x)f(u(x)) = 0, \quad x \in \mathbb{S}^n_1(r), \tag{5-8}$$

where L stands for the Laplace-Beltrami operator on $\mathbb{S}^n_1(r)$. The idea boils down to describing explicitly the function $\delta(t)$ for $t \in \mathbb{R}$. Thus by straighforward calculations we get that:

$$\delta(t) = (\cosh(t))^{n-1},$$

and the solution to the semi-linear problem (5-8) takes the form:

$$u(r) = z\left(\int_{r_0}^{r} \frac{1}{(\cosh(t))^{n-1}} dt\right).$$

Remark 5.13. Here we are using the important fact that the underlying manifold $S_t \cong S^n$ has a Riemannian structure. So the compactness of S^n implies that its isometry group $I(S^n) = SO(n)$ is also a compact group.

Remark 5.14. The preceding calculations can also be done on the disconnected pseudohyperbolic space $\mathbb{H}^n_0(r)$. There, the metric induced by the inclusion $i : \mathbb{H}^n_0(r) \hookrightarrow \mathbb{R}^{n+1}_1$ is given by

$$dh^2 = r^2 dt^2 + r^2 \sinh^2(t) ds^2.$$

Since $\mathbb{H}^n_0(r)$ is a Riemannian manifold, we can consider (5-1) posed on it for arbitrary $n \in \mathbb{N}$. However we have skipped the calculations because it is possible to proceed as stated in [3]. In fact, all our considerations work in Riemannian warped product spaces as well as in the Lorentzian ones. Just take $\mathbb{R} \times_f M$ furnished with the metric $dh^2 = dt^2 + f(t)^2 g$, for $(\mathbb{R}, dt^2)$ being the real line with the usual metric and (M, g) a Riemannian manifold, and a cohomogeneity one polar action such that $\mathbb{R}$ is the submanifold transversal to the action.

We now want to consider a general warped product. We fix a setup as given in [10] and so the computations hold. Let (N, h) be a isotropic Riemannian manifold of dimension $n-1$ and suppose $I = (a, b)$ with $-\infty \leq a < b \leq \infty$ together with a warping function $w : I \to (0, \infty)$. We define the Lorentzian warped product

$$M := I \times_w N$$

equipped with the warped product metric

$$\bar{g} = -dt^2 + (w(t))^2 h. \tag{5-9}$$

In the literature, for instance see [5, Chapter 5], [21, Chapter 12], such a space is called *Robertson-Walker space-time*. By virtue of Proposition 2.40, for a chosen function $\mu : \mathbb{R} \to \mathbb{R}$ at least continuous, the function w may be found so that w is a $C^2(I)$ solution to the ordinary differential equation

$$w''(t) + \mu(t) w(t) = 0,$$

with initial conditions $w(t_0) = w_0$ and $w'(t_0) = w_1$. Thus μ may be regarded as describing the sectional curvature of the warped product $I \times_w N$.

We need to guarantee some assumptions in order to show existence of polar action invariant solutions to the problem (5-1) on a warped product space $(M, \bar{g})$ defined as above. Here we shall use several of the facts described in Subsection 2.2.2. First note that due to the vector field $U = \partial_t$ on M satisfies $\bar{g}(U, U) < -1$, then M is temporally oriented by U. We also ensure that, by Theorem 2.41, M is globally hyperbolic since (N, h) is an isotropic Riemannian manifold and thus it is complete.

Our task is now to describe a polar action on M. Let $I(N)$ be the group of isometries of N[1]. Let G be a unimodular closed subgroup of $I(N)$ which acts transitively on N. Then G acts properly on N. Furthermore, G can be identified with a subgroup $\bar{G}$ of $I(M)$ as follows. Given $\phi \in G$, define $\bar{\phi} \in \bar{G}$ by $\bar{\phi}(t, s) = (t, \phi(s))$ for all $(t, s) \in I \times N$. With this definition, the action of $\bar{G}$ on M is proper, polar and, restricted to each $N_t = \{t\} \times N$, is transitive. For a fixed $s \in N$, the timelike submanifold $\Sigma_s = I \times \{s\}$ is transversal to the orbits of the action due to each N_t is spacelike. With this at hand, we are ready to invoke Theorem 5.7 and find solutions to the semi-linear problem

$$L_M u(x) + b(x) f(u(x)) = 0, \quad x \in M. \tag{5-10}$$

[1]Note that if N is compact then $I(N)$ is compact as well.

Observe that our aim reduces to calculate the function $\delta(t)$. We know that for all $s \in N$, $N \cong G/G_s$ and that by compactness of G_s, this quotient has a $G-$invariant measure. Then we can assume without loss of generality that the $G-$invariant Riemannian measure given by h on N and the measure on G/G_s are the same. Next, for each $t \in I$, we consider the homothety $\phi_t : N \to N_t$ defined by $\phi_t(s) = (t, s)$ for each $s \in N$. Fixing $(t, s) \in M$, choose a base $\{V_2, \dots, V_n\}$ of $T_s N$ whose image under the isomorphism $d_s\phi : T_s N \to T_{(t,s)}N_t$ is the set $\{(0, V_2), \dots, (0, V_n)\}$. By definition of the warped metric on M these elements of $T_{(t,s)}N_t$ are spacelike. Let dN be the G-invariant volume element of N and dN_t the Riemannian volume element of the submanifold N_t induced by the warped metric of M. Then

$$\begin{aligned} dN_t((0, V_2) \wedge (0, V_3) \wedge ... \wedge (0, V_n)) &= |\det(\bar{g}((0, V_i), (0, V_j)))|^{1/2} \\ &= w(t)^{n-1} |\det(h(V_i, V_j)|^{1/2} \\ &= w(t)^{n-1} dN(V_2 \wedge V_3 \wedge ... \wedge V_n). \end{aligned}$$

Thus we conclude that

$$dN(V_2 \wedge V_3 \wedge ... \wedge V_n) = \frac{dN_t((0, V_2) \wedge (0, V_2) \wedge ... \wedge (0, V_n))}{w(t)^{n-1}}.$$

Note that for each $t \in I$ the group $\bar{G}$ induces an action from G on the submanifold N_t. Due to each N_t is the orbit of the action of $\bar{G}$ on M, the volume element dN_t is invariant by the induced action of G on N_t. We point out that it holds for any basis of the vector space $T_{(t,s)}N_t$. Now, if dN is normalized by $dN = dN_t$ it follows that

$$\delta(t) = w(t)^{n-1}.$$

Finally we obtain that the solutions to (5-10) are given by

$$z(r) = z\left(\int_{r_0}^{r} \frac{1}{w(t)^{n-1}} dt\right).$$

We end this section with an interesting comment about Robertson-Walker space-time. It is known that any three-dimensional isotropic Riemannian manifold (N, h) has constant sectional curvature N. Then all four-dimensional Robertson-Walker space-times are topologically either $\mathbb{R}^4$, $\mathbb{R} \times \mathbb{S}^3$, or $\mathbb{R} \times \mathbb{R}P^3$. If N is nonzero, the metric may be rescaled to be of the form $\bar{g} = -dt^2 + (w(t))^2 h$ on M so that N is either identically 1 or -1, (see [5, Section 5.4]). We have found polar invariant solutions to (5-1) for choices of N with both flat and positive curvature.

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