Dupin cyclides

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Introduction

We will study the problem of finding a Dupin cyclide given three contact conditions. A contact condition is a point m in $\mathbb{R}^3$ and an orientation at that point, i.e., a vector with initial point m, this vector can be seen as the normal vector of an oriented plane through m. A Dupin cyclide that, historically, is defined as the envelope of a family of spheres which are simultaneously tangent to three given spheres, can be defined in the context of the projective five-dimensional space as 2-plane section of a quadric in this projective space. The points of this quadric correspond to oriented spheres in 3D.

In chapter one we define the projective space and introduce the concepts of Minkowski product and Möbius hypersphere. Then we set up the pole-polar relationship. In particular for points outside the Möbius quadric, and using the widely known stereographic projection we construct the Möbius representation of the space of unoriented spheres. We will consider conditions of tangency between spheres, planes and points in $\mathbb{R}^3$ in terms of Möbius geometry. We will study general cyclides, a generalization of Dupin cyclides, defined in [14]. These are surfaces generated as envelopes of 1-parameter families of spheres/planes in the three-dimensional space $\mathbb{R}^3$. The set of spheres and planes is latter identified with points in a certain subset of $\mathbb{P}^4$, that will be called Ψ^+. We consider families of spheres depending on one parameter that can be given as conics in Ψ^+, and these will be written as Bézier curves of degree two. This condition yields general cyclides to be algebraic surfaces of degree at most four.

In our approach the Möbius space is not enough for the study of contact conditions in $\mathbb{R}^3$, because we work within the context of oriented spheres, which simplifies the description of Dupin cyclides. In chapter two we consider the space of oriented spheres, which will correspond to the Lie quadric. Having the space of oriented spheres the notion of oriented contact is determined, and with it a relationship involving the Lie scalar product, to establish when two oriented spheres are in oriented contact. Using a slightly different approach, the space of oriented spheres can be characterized as a one sheeted hyperboloid Λ^4, this is shown to be the affine part of the Lie quadric.

In chapter three we define Dupin cyclide and establish their relationship with the space of oriented spheres. Finally in chapter four we prove the main theorem of this dissertation, as given by [12], that gives necessary and sufficient conditions for a Dupin cyclide to satisfy three prescribed independent contact conditions. We construct three very particular functions using the contact conditions and prove them to be homographies from $\mathbb{P}^1$ to $\mathbb{P}^1$, using them later to express the above mentioned conditions.

In Appendix A, polynomial and rational Bézier curves are briefly introduced and several references are given for further details on the subject. The wonderfully simple, but powerful De Casteljau

algorithm is also introduced in Appendix A. In the last section we develop the blossoming principle and its relation with the tangent lines to the Bézier curve is provided.
Appendix B deals with the theory of Sylvester's resultant which is useful to calculate the envelope of a 1-parameter family of spheres. The last section covers some results of inversive geometry, including the interesting remark that the stereographic projection is a restriction of an inversion.

Chapter 1

Möbius Geometry

1.1 Projective Geometry

The Projective Space $\mathbb{P}^4$

We construct the projective space $\mathbb{P}^4$ by defining an equivalence relation on $\mathbb{R}^5 - \{0\}$ by setting $\mathbb{x} \sim \mathbb{y}$ if $\mathbb{x} = \lambda \mathbb{y}$ for $\lambda \neq 0$. The four-dimensional projective space is the quotient space of $\mathbb{R}^5 - \{0\}$ by this relation, i.e, $\mathbb{P}^4 = \mathbb{R}^5 - \{0\}/\sim$, which is the space of all lines through the origin in $\mathbb{R}^5 - \{0\}$. We denote by $\mathbf{x}$ the equivalence class determined by the vector $\mathbb{x} = (x_0, x_1, x_2, x_3, x_4)$ whose coordinates are referred to as the *homogeneous coordinates* of the point $\mathbf{x} = [(x_0, x_1, x_2, x_3, x_4)]$. Note that the homogeneous coordinates are defined up to a non-zero multiplicative constant.

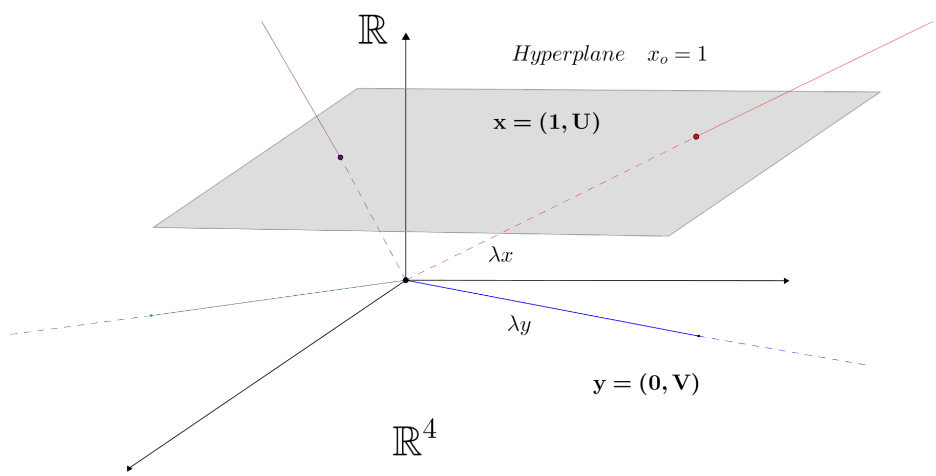


Figure 1.1: Projective Space $\mathbb{P}^4$.

The directions determined by the points in the hyperplane $x_0 = 1$ are in bijective correspondence with $\mathbb{R}^4$. The map $\psi(\mathbf{x}) = (\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0})$ from $\mathbb{P}^4 - \{x_0 \neq 0\}$ into $\mathbb{R}^4$, with inverse $\psi^{-1}(x, y, z, w) = [(1, x, y, z, w)]$ establishes this bijection. Let us note that $\mathbb{P}^4$ is the disjoint union of $\{\mathbf{x} : x_0 = 1\}$ and $\{\mathbf{x} : x_0 = 0\}$ which will be referred to as *affine part of* $\mathbb{P}^4$ and the *hyperplane at infinity*, respectively. See Figure 1.1.

Projective Subspaces

Let $\mathbb{V}$ be a vector subspace of $\mathbb{R}^5$, then $\mathbf{V} = \{\mathbf{v} \in \mathbb{P}^4 \mid (v_0, v_1, v_2, v_3, v_4) \in \mathbb{V} - \{0\}\}$ is called a *projective subspace* where $(v_0, v_1, v_2, v_3, v_4)$ are the homogeneous coordinates of the point $\mathbf{v} = [(v_0, v_1, v_2, v_3, v_4)]$.

A projective line, denoted $\mathbf{a} \wedge \mathbf{b}$ is the projective subspace generated by two projective points, namely $\mathbf{a}$ and $\mathbf{b}$, which represent two lines passing through the origin in $\mathbb{R}^5$, therefore a projective line is the projectivization of a 2-plane through the origin. Similarly for projective subspaces of any dimension we say:

$\mathbf{V}$ has (projective) dimension n *if and only if* $\mathbb{V}$ is a subspace of dimension $n+1$.

We define the *polar hyperplane* of $\mathbf{p} = [(p_0, p_1, p_2, p_3, p_4)]$ in $\mathbb{P}^4$ by the equation $-p_0x_0 + p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 = 0$.

Proposition 1.1. *Let* $\mathbf{p} = [(p_0, p_1, p_2, p_3, p_4)]$ *be a point in* $\mathbb{P}^4$*, the affine part of the polar hyperplane of* $\mathbf{p}$ *correspond to the affine 3-plane in* $\mathbb{R}^4$ *of equation*

$$p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 = p_0.$$

Proof. Let $\psi(\mathbf{x}) = (y_1, y_2, y_3, y_4)$, for $\mathbf{x} = [(x_0, x_1, x_2, x_3, x_4)]$, then:

$$\begin{aligned} -p_0x_0 + p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 &= 0 \\ \frac{-p_0x_0}{x_0} + \frac{p_1x_1}{x_0} + \frac{p_2x_2}{x_0} + \frac{p_3x_3}{x_0} + \frac{p_4x_4}{x_0} &= 0 \\ p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 &= p_0 \end{aligned}$$

Analogously, since $\psi^{-1}(y_1, y_2, y_3, y_4) = (1, y_1, y_2, y_3, y_4)$ the equation $p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 = p_0$ can be written as

$$-p_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 = 0.$$

■

1.2 The Möbius Space of Unoriented Spheres

We begin by defining the basic notions of Möbius geometry in order to establish the relationship between spheres/planes of $\mathbb{R}^3$ and a subset of the 4-dimensional projective space, as stated by M. Paluszny and M. Paluszny and W. Boehm in [13] and [14].

Definition 1.2. *The Minkowski product (or Lorentz metric) is the symmetric bilinear form*

$$\mathcal{M}(\mathbb{x}, \mathbb{y}) = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

where $\mathbb{x} = (x_0, x_1, x_2, x_3, x_4)$ *and* $\mathbb{y} = (y_0, y_1, y_2, y_3, y_4)$ *are in* $\mathbb{R}^5$. *The space* $\mathbb{R}^5$ *with the Minkoswki product is denoted by* $\mathbb{R}^5_1$ *and it is referred to as Lorentz space.*

Definition 1.3. *The Möbius hypersphere is defined by:*

$$\Psi = \left\{[(x_0, x_1, x_2, x_3, x_4)] \in \mathbb{P}^4 \mid \mathcal{M}((x_0, x_1, x_2, x_3, x_4), (x_0, x_1, x_2, x_3, x_4)) = 0\right\}.$$

Notice that $\psi(\Psi) = \mathbb{S}^3$, we will use this to establish a relationship between the space $\mathbb{R}^3$ and points in Ψ. To do so, recall that the stereographic projection from $\mathbb{R}^3 \cong \{(x, y, z, w) \in \mathbb{R}^4 \mid w = 0\}$ to $\mathbb{S}^3 - \{(0, 0, 0, 1)\}$ is given by

$$s(x, y, z, 0) = \left(\frac{2x}{x^2 + y^2 + z^2 + 1}, \frac{2y}{x^2 + y^2 + z^2 + 1}, \frac{2z}{x^2 + y^2 + z^2 + 1}, \frac{x^2 + y^2 + z^2 - 1}{x^2 + y^2 + z^2 + 1}\right).$$

To a point $(x, y, z) \in \mathbb{R}^3$ corresponds the point $\mathbf{x}$ with homogeneous coordinates:

$$\begin{aligned} x_0 &= x^2 + y^2 + z^2 + 1 \\ x_1 &= 2x \\ x_2 &= 2y \\ x_3 &= 2z \\ x_4 &= x^2 + y^2 + z^2 - 1, \end{aligned} \tag{1.1}$$

To justify the expressions above for the homogeneous coordinates x_i of $\mathbf{x} \in \mathbb{P}^4$ we notice that (x, y, z) in $\mathbb{R}^3$ corresponds to $(x, y, z, 0) \in \mathbb{R}^4$. And by the stereographic projection to the point $(x, y, z, 0) \in \mathbb{R}^4$ corresponds :
$s(x, y, z, 0) = \left(\frac{2x}{x^2+y^2+z^2+1}, \frac{2y}{x^2+y^2+z^2+1}, \frac{2z}{x^2+y^2+z^2+1}, \frac{x^2+y^2+z^2-1}{x^2+y^2+z^2+1}\right)$. The bijection ψ^{-1} maps $(x, y, z, 0)$ into

$$\psi^{-1}(s(x, y, z, 0)) = \left[\left(1, \frac{2x}{x^2 + y^2 + z^2 + 1}, \frac{2y}{x^2 + y^2 + z^2 + 1}, \frac{2z}{x^2 + y^2 + z^2 + 1}, \frac{x^2 + y^2 + z^2 - 1}{x^2 + y^2 + z^2 + 1}\right)\right],$$

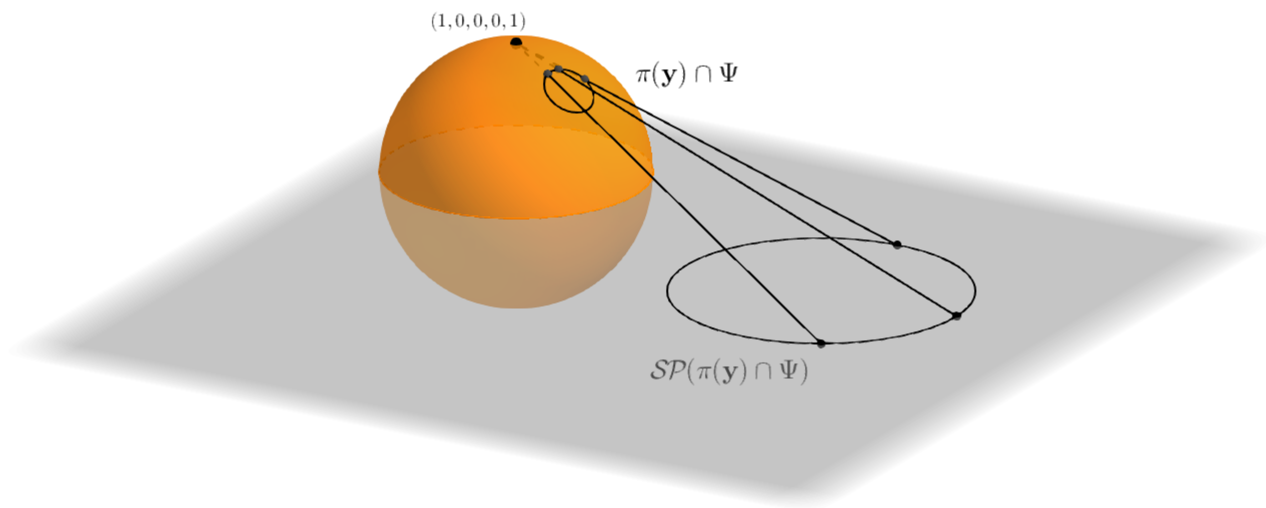


Figure 1.2: Stereographic Projection.

whose equivalence class is that of the point described in 1.1.
The function $\mathcal{SP} = \psi^{-1} \circ s(x, y, z) : \mathbb{R}^3 \to \Psi - \{\mathbb{n}\}$ (described in (1.1)) will be referred to as *generalized stereographic projection*. The point $[(1, 0, 0, 0, 1)]$ is usually denoted by $\mathbf{n}$ and it is called the north pole. We define the exterior of Ψ as the set of points in $\mathbb{P}^4$ whose Lorentz pseudonorm is positive and we denote it by Ψ^+.

Theorem 1.4. *Given a point* $\mathbf{y} = [(y_0, y_1, y_2, y_3, y_4)]$ *in* Ψ^+ *the intersection of its polar hyperplane* $\pi(\mathbf{y})$ *with* Ψ *is a sphere or a plane in* $\mathbb{R}^3$ *under the generalized stereographic projection.*

Proof. Let us note that the Möbius hypersphere does not intersect the hyperplane at infinity, indeed, if $\mathbf{x} \in \Psi$ and $x_0 = 0$, it would be necessary that $x_1 = x_2 = x_3 = x_4 = 0$, thus $\mathbf{x}$ would not correspond to a point of $\mathbb{P}^4$. Thus it is only needed to consider the affine part of $\mathbb{P}^4$, i.e, $\mathbb{R}^4$, and by Proposition 1.1, the affine part of $\pi(\mathbf{y})$ can be seen as the 3-plane in $\mathbb{R}^4$ given by the equation $y_1x_1+y_2x_2+y_3x_3+y_4x_4 = y_0$. Let $p \in \mathbb{S}^3 \cap \{\mathbb{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid y_1x_1+y_2x_2+y_3x_3+y_4x_4 = y_0\}$. Since $p = (p_1, p_2, p_3, p_4) \in \mathbb{S}^3$, we have that $p = s(x, y, z, 0)$. Thus there exists $t \neq 0$ in $\mathbb{R}$ such that $p - (0, 0, 0, 1) = t\left((x, y, z, 0) - (0, 0, 0, 1)\right)$ then $p_1 = tx$, $p_2 = ty$, $p_3 = tz$ and $p_4 = 1 - t$.
Let us define $q = \frac{1+p_4}{1-p_4}$, knowing that $1-p_4 \neq 0$ (because $p_4 = 1$ can only ocur when $p = (0, 0, 0, 1)$) and $s(\mathbb{R}^3) = \mathbb{S}^3 - \{(0, 0, 0, 1)\}$. It is easy to verify that $p_4 = \frac{q-1}{q+1}$, $t = \frac{2}{q+1}$ and $q = x^2 + y^2 + z^2$. Since $p \in \{y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4 = y_0\}$

$$\begin{aligned} y_0 &= y_1(tx) + y_2(ty) + y_3(tz) + y_4\frac{q-1}{q+1} \\ &= \frac{2y_1x + 2y_2y + 2y_3z + y_4(q-1)}{q+1} \\ y_0q + y_0 &= 2y_1x + 2y_2y + 2y_3z + y_4q - y_4 \\ y_0 + y_4 &= q(y_4 - y_0) + 2y_1x + 2y_2y + 2y_3z. \end{aligned}$$

Let $q = x^2 + y^2 + z^2$ then

$$(x^2 + y^2 + z^2)(y_4 - y_0) + 2y_1x + 2y_2y + 2y_3z = y_0 + y_4 \tag{1.2}$$

If $y_0 - y_4 = -\mathcal{M}(\mathbf{y}, \mathbf{n}) = 0$ in equation 1.2 we obtain

$$y_1x + y_2y + y_3z = y_0,$$

which is the equation of a plane in $\mathbb{R}^3$.
If $\mathcal{M}(\mathbf{y}, \mathbf{n}) \neq 0$ multiplying by $y_0 - y_4$ in equation 1.2 and completing squares we have that

$$\left(x - \frac{y_1}{y_0 - y_4}\right)^2 + \left(y - \frac{y_2}{y_0 - y_4}\right)^2 + \left(z - \frac{y_3}{y_0 - y_4}\right)^2 = \frac{\mathcal{M}(\mathbf{y}, \mathbf{y})}{(\mathcal{M}(\mathbf{y}, \mathbf{n}))^2},$$

the equation of a sphere in $\mathbb{R}^3$.

■

Corollary 1.5. *A point in Ψ^+ corresponds to a plane in $\mathbb{R}^3$ if it lies in the north hyperplane, i.e., lies in $\pi(\mathbf{n})$*

Conversely to the plane of equation $ax + by + cz = d$ corresponds the point in Ψ^+ with homogeneous coordinates $[(d, a, b, c, d)]$. For a sphere C centered in $c = (c_1, c_2, c_3)$ and radius r, the point $[(y_0, y_1, y_2, y_3, y_4)]$ in Ψ^+ which corresponds to C is: $[(||c||^2 - r^2 + 1, 2c_1, 2c_2, 2c_3, ||c||^2 - r^2 - 1)]$. Table 1.1 summarizes the above.

Notice that if we only consider the space $\mathbb{R}^4$ the construction of the space of spheres/planes of $\mathbb{R}^3$ can still be done, but not completely, since a sphere centered at c and radius r with the property that $||c||^2 - r^2 + 1 = 0$ (in particular the unit sphere centered at the origin) would have been missed. A point $[(p_0, p_1, p_2, p_3, 0)]$, which lies at infinity is required. That is why it is necessary to add some points to represent completely the space of spheres, namely the points at infinity whose polar hyperplanes generate such spheres.

Euclidean Space	Möbius Space
points: $u \in \mathbb{R}^3$	$[(\|\|u\|\|^2+1, 2u, \|\|u\|\|^2-1)]$
∞	$\mathbf{n} = [(1,0,0,0,1)]$
spheres: center c, radius r	$[(\|\|c\|\|^2 - r^2 + 1, 2c_1, 2c_2, 2c_3, \|\|c\|\|^2 - r^2 - 1)]$
planes: $ax + by + cz = d$	$[(d, a, b, c, d)]$

Table 1.1: Correspondance between points in Ψ and Ψ^+ with points, planes and spheres in $\mathbb{R}^3$.

Linear Subspaces of the Möbius Space

A one parameter family of spheres/planes can be viewed as a curve in $\mathbb{P}^4$. Let us consider the simplest case, the case of a line in $\mathbb{P}^4$. We will refer to such a linear family a *pencil of spheres/planes*. We will relate such families to circles, including circles of zero radius, of infinite radius and imaginary radius, i.e., points, lines and empty sets, respectively. The relationship between lines in $\mathbb{P}^4$ and circles in $\mathbb{R}^3$ is justified as follows: Let $\mathbf{a} \wedge \mathbf{b}$ be a line in $\mathbb{P}^4$. This way $\mathbf{a}$ and $\mathbf{b}$ are spheres or planes of the pencil, according to Theorem 1.4 these spheres correspond to 3-plane sections of $\mathbb{S}^3$, the intersection of these 3-planes is a 2- plane, which under stereographic projection can be seen as a circle in $\mathbb{R}^3$. Thus the spheres $\mathbf{a}$ and $\mathbf{b}$ have a common circle, known as *carrying circle*, precisely the image of their intersection 2-plane with $\mathbb{S}^3$, see Figures 1.3 and 1.4.

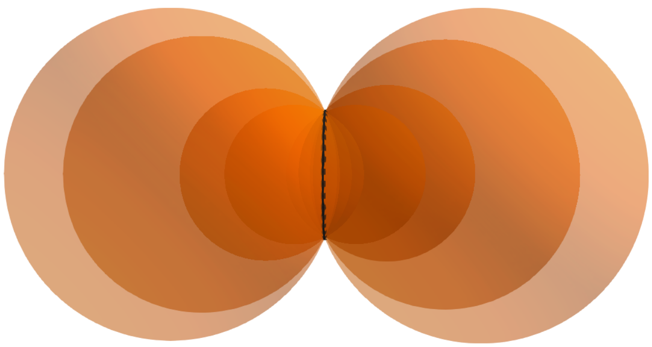
A line in $\mathbb{P}^4$ has at least one intersection point with every hyperplane, the point corresponding to its intersection with the north hyperplane, which represents the planes in $\mathbb{R}^3$, will be referred to as the *radical plane* of the pencil.

Another important point of a pencil is the *equatorial sphere*, whose center and radius coincide with those of the pencil's carrying circle. According to the parametrization of the pencil given above, the radical plane and the equatorial sphere correspond to the points $[\mathcal{M}(\mathbb{a},\mathbb{n})\mathbb{b} - \mathcal{M}(\mathbb{b},\mathbb{n})\mathbb{a}]$ and $[((\mathcal{M}(\mathbb{a},\mathbb{b})\mathbb{b} - \mathcal{M}(\mathbb{b},\mathbb{b})\mathbb{a})\cdot\mathbb{n})\,\mathbb{a} + ((\mathcal{M}(\mathbb{a},\mathbb{b})\mathbb{a} - \mathcal{M}(\mathbb{a},\mathbb{a})\mathbb{b})\cdot\mathbb{n})\,\mathbb{b}]$, respectively. Where $\mathbf{a} = [\mathbb{a}]$, $\mathbf{b} = [\mathbb{b}]$, $\mathbb{n} = (1,0,0,0,1)$ and $\cdot$ denotes the euclidean dot product.

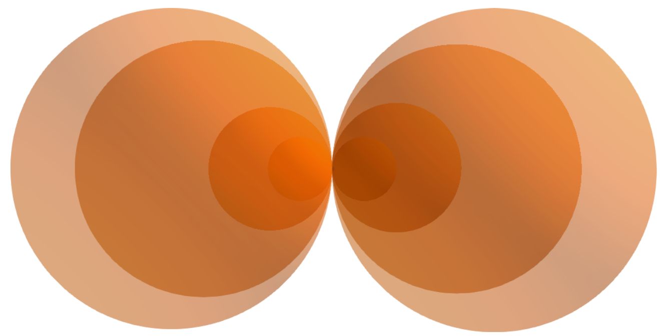
Let us consider a line not contained in the north hyperplane then:

- If the line does not intersects Ψ, then its points correspond to spheres of positive radius. Therefore the carrying circle will have positive radius. *Pencil of intersecting spheres*, see Figure 1.3a.

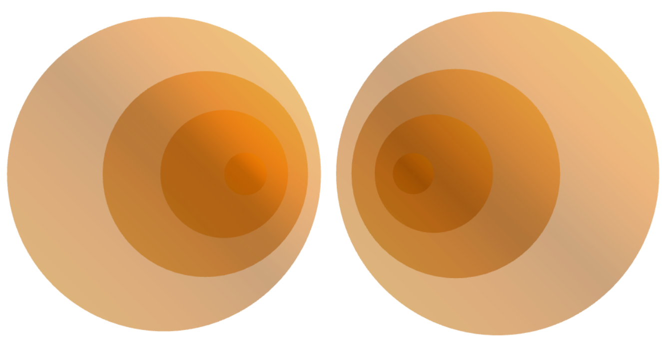
- If it is tangent to Ψ, then the pencil contains a point and the intersection of the spheres corresponding to the pencil elements is a circle of zero radius, we refer to such a pencil as a *pencil of tangent spheres*, see Figure 1.3b.



(a) Pencil of Intersecting Spheres.



(b) Pencil of Tangent Spheres.



(c) Pencil of non Intersecting Spheres.

Figure 1.3: Pencils of Spheres.

(a) Parallel Pencil of Planes

(b) Intersecting Pencil of Planes

Figure 1.4: Pencils of Planes

- If it intersects the Möbius hypersphere (not tangentially), there are spheres of imaginary radius in the pencil, corresponding to the points inside Ψ. In this case the pencil is refer to as *pencil of non intersecting spheres*, see Figure 1.3c.

Let us now consider a line lying in the north hyperplane:

- If the line contains the north pole, their carrying line lies at infinity, i.e, every plane on the pencil is parallel to every other. We refer to this line as *pencil of parallel planes*, see Figure 1.4a.

- If it does not contain the north pole, the carrying circle of the pencil is a line, in this case the pencil is known as a *pencil of intersecting planes*, see Figure 1.4b.

1.3 General Cyclides

In the previous section we studied linear families of spheres/planes[1]. In this section we turn our attention to quadratic families of spheres/planes, which are defined by quadratic polynomials[2] in $\mathbb{P}^4$, which will be seen as Bézier curves (see Appendix A) of degree two $b(t) = \mathbb{b}_0(1-t)^2 + 2\mathbb{b}_1 t(1-$

[1]The expression "spheres/planes" refers to any member of the complete space of spheres. There includes the planes which can be thought of as sphere of infinite radius. So, "sphere/plane" is a plane a or a sphere of finite radius.

[2]Polynomials in $\mathbb{P}^4$ are functions of the form $t \to (x_0(t), x_1(t), x_2(t), x_3(t), x_4(t))$ from $\mathbb{R}$ to $\mathbb{R}^5$, where $x_i(t)$ is a polynomial.

$t) + \mathbb{b}_2 t^2$ in $\mathbb{R}^5$.

A quadratic family of spheres defines as its envelope (see Appendix B) an algebraic surface of degree at most four. To study regular curves of higher degree and their corresponding algebraic surfaces, see [15].

Definition 1.6. *A general cyclide is the envelope of a quadratic family of spheres.*

Proposition 1.7. *Let $\mathbb{b}_0$, $\mathbb{b}_1$ and $\mathbb{b}_2$ in $\mathbb{R}^5_1$, corresponding to points in Ψ^+ the general cyclide associated with the Bézier conic*

$$b(t) = \mathbb{b}_0(1-t)^2 + 2\mathbb{b}_1 t(1-t) + \mathbb{b}_2 t^2$$

has as its implicit equation:

$$\mathcal{M}(\mathbb{b}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{b}_0, \mathbb{x})\mathcal{M}(\mathbb{b}_2, \mathbb{x}) = 0,$$

where $\mathbb{x}$ is given by the stereographic projection, i.e., $\mathbb{x} = (x^2+y^2+z^2+1, 2x, 2y, 2z, x^2+y^2+z^2-1)$.

Proof. By abusing notation we refer $b(t)$ in Ψ^+ as the sphere corresponding to it. Thus, if (x, y, z) is a point of the cyclide, by definition we have that $(x, y, z) \in b(t)$ and $(x, y, z) \in b(t + \Delta t)$ when $\Delta t \to 0$. Let $\mathbb{x} \in \Psi$ be its image under the stereographic projection. Note also that $\mathbb{x}$ belongs to the polar hyperplanes $\mathcal{M}(b(t), \mathbb{y}) = 0$ and $\mathcal{M}(b(t + \Delta t), \mathbb{y}) = 0$ when $\Delta t \to 0$, therefore $\mathbb{x}$ is in the intersection of the envelope of such hyperplanes and Ψ, to obtain the equation of the envelope in $3D$ we compute the envelope in Ψ and pull it back to $\mathbb{R}^3$ via stereographic projection. Namely the envelope in Ψ is given by the solution of the system

$$\begin{aligned} \mathcal{M}(b(t), \mathbb{x}) &= 0 \\ \mathcal{M}(b'(t), \mathbb{x}) &= 0 \end{aligned} \tag{1.3}$$

where:

$$\mathcal{M}(b(t), \mathbb{x}) = (\mathcal{M}(\mathbb{b}_0, \mathbb{x}) - 2\mathcal{M}(\mathbb{b}_1, \mathbb{x}) + \mathcal{M}(\mathbb{b}_2, \mathbb{x}))t^2 + 2(\mathcal{M}(\mathbb{b}_1, \mathbb{x}) - \mathcal{M}(\mathbb{b}_0, \mathbb{x}))t + \mathcal{M}(\mathbb{b}_0, \mathbb{x})$$

Thus by Silvester's resultant (see Appendix B):

$$\mathcal{M}(\mathbb{b}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{b}_0, \mathbb{x})\mathcal{M}(\mathbb{b}_2, \mathbb{x}) = 0. \tag{1.4}$$

The envelope in $3D$ is obtained writing $\mathbb{x}$ in terms of (x, y, z) in the equation above. ■

Theorem 1.8. *The equation of the envelope of a family of spheres/planes does not depend on the control points of its Bézier conic in $\mathbb{P}^4$.*

Proof. Let $\mathbb{b}_0, \mathbb{b}_1$ and $\mathbb{b}_2$ and $\mathbb{c}_0, \mathbb{c}_1$ and $\mathbb{c}_2$ two different sets of control points of a given Bézier conic in $\mathbb{P}^4$, we need to see that the equations of the envelope as given by 1.4 for the $\mathbb{b}_i' s$ and the $\mathbb{c}_i' s$ are the same equation

$$\begin{aligned} \mathcal{M}(\mathbb{b}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{b}_0, \mathbb{x})\mathcal{M}((\mathbb{b}_2, \mathbb{x}) = 0 \\ \mathcal{M}(\mathbb{c}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{c}_0, \mathbb{x})\mathcal{M}(\mathbb{c}_2, \mathbb{x}) = 0 \end{aligned} \tag{1.5}$$

Due to the properties of Bézier curves we have that for some t_1 and t_2

$$\mathbb{c}_0 = \mathbb{b}_0(1-t_1)^2 + 2\mathbb{b}_1 t_1(1-t_1) + \mathbb{b}_2 {t_1}^2$$

$$\mathbb{c}_2 = \mathbb{b}_0(1-t_2)^2 + 2\mathbb{b}_1 t_2(1-t_2) + \mathbb{b}_2 {t_2}^2.$$

Using the blossoming technique (see Appendix A, Section A.3) we get:

$$\mathbb{c}_1 = \mathbb{b}_0(1-t_1)(1-t_2) + \mathbb{b}_1 \left[t_1(1-t_2) + t_2(1-t_1)\right] + \mathbb{b}_2 t_1 t_2$$

Substituting the $\mathbb{c}_i' s$ in the second Equation in B.2 we obtain:

$$\begin{aligned} \mathcal{M}(\mathbb{c}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{c}_0, \mathbb{x})\mathcal{M}(\mathbb{c}_2, \mathbb{x}) = \left[((1-t_1)t_2 + (1-t_2)t_1)^2 - 4t_1t_2(1-t_1)(1-t_2)\right] \mathcal{M}(\mathbb{b}_1, \mathbb{x})^2 \\ - \left[(1-t_1)^2 {t_2}^2 - 2t_1t_2(1-t_1)(1-t_2) + (1-t_2)^2 {t_1}^2\right] \mathcal{M}(\mathbb{b}_0, \mathbb{x})\mathcal{M}(\mathbb{b}_2, \mathbb{x}). \end{aligned}$$

Therefore

$$\mathcal{M}(\mathbb{c}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{c}_0, \mathbb{x})\mathcal{M}(\mathbb{c}_2, \mathbb{x}) = [(1-t_1)t_2 + (1-t_2)t_1]^2 \left[\mathcal{M}(\mathbb{b}_1, \mathbb{x})^2 - \mathcal{M}(\mathbb{b}_0, \mathbb{x})\mathcal{M}(\mathbb{b}_2, \mathbb{x})\right]$$

It follows that the solution set (the envelope of the family of spheres/planes) of the two equations in (B.2) is the same. ■

Let G be a general cyclide and $b(t)$ be an associated Bézier curve in $\mathbb{P}^4$. The points $b(t)$ and $b(t+\Delta t)$ represent two spheres/planes in the quadratic family associated with G. The line $b(t) \wedge b(t+\Delta t)$ is secant to the Bézier conic through this points, furthermore as seen before these line corresponds to a circle in $\mathbb{R}^3$. See Figure 1.5.

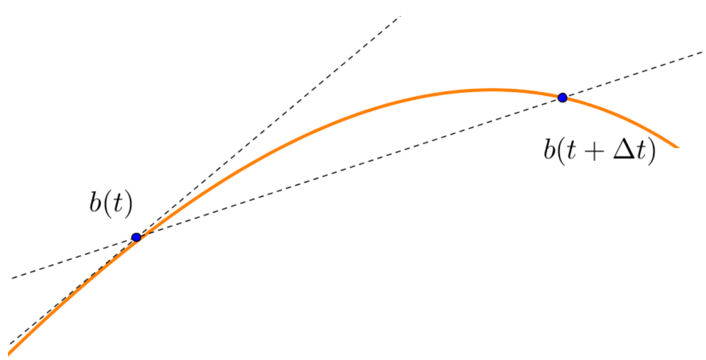


Figure 1.5: Tangent Line to the Quadratic Family at $b(t)$.

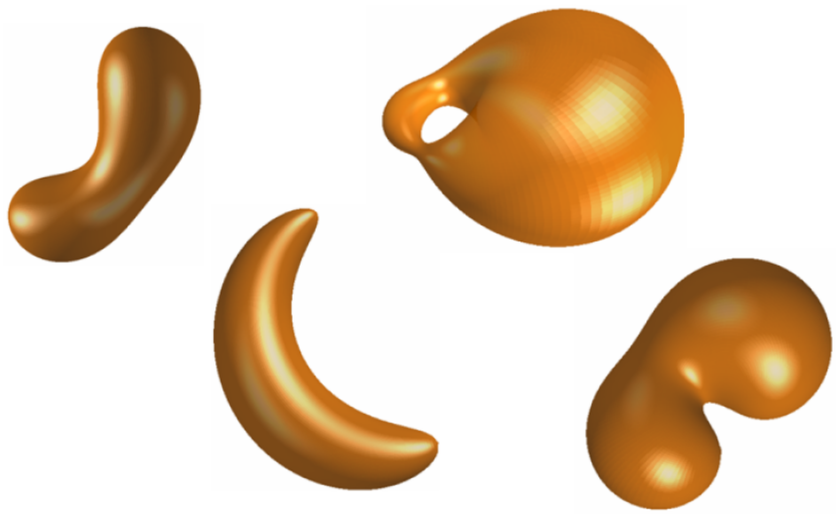


Figure 1.6: Examples of General Cyclides.

When $\Delta t \rightarrow 0$ the secant line tends to the tangent line to the conic at $b(t)$. Meanwhile the points in the circle defined by the line tend to points in the general cyclide. It follows that the tangent lines to the Bézier curve provide the general cyclide G, with circles (they could be imaginary or have infinite radius), to those we will refer to as *composing circles*. Figure 1.6 shows some examples of general cyclides[3].

[3]Graphics taken from [8].

Chapter 2

Lie Sphere Geometry

In this chapter we build a representation of the space of oriented spheres. This is constructed via a generalization of the Möbius space, and an equivalent but non-projective construction using a light cone. This construction will involve $\mathbb{R}^6$ as opposed to the Möbius representation which was based on $\mathbb{R}^5$.

2.1 The Space of Oriented Spheres

In this section we will describe two equivalent constructions of the space of oriented spheres, given in [2] and in [5], [6] and [12][1].

Definition 2.1. *The Lie metric or Lie scalar product is the symmetric bilinear form*

$$\mathcal{L}(\mathbb{x}, \mathbb{y}) = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 - x_5 y_5.$$

for $\mathbb{x} = (x_0, x_1, x_2, x_3, x_4, x_5)$*,* $\mathbb{y} = (y_0, y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^6$*. The space* $\mathbb{R}^6$ *with the Lie metric will be denoted by* $\mathbb{R}^6_2$*.*

As we did in the Möbius representatio of the space of (unoriented) spheres we will not consider the space $\mathbb{R}^6_2$ itself, but its projective compatification $\mathbb{P}^5$. The space of (oriented) spheres is connected to an extension of the embedding of the Möbius hypersphere into $\mathbb{P}^5$. We know that the dimension of the space $\Psi^+ \cup \Psi$ is four. Our extension of it will be a four-dimensional space as well, but, being a subspace of $\mathbb{P}^5$ will have an "extra" coordinate, which will allow us to determine the orientation of the spheres.

Definition 2.2. *The set* $\mathbb{Q}^4 = \{[(x_0, x_1, x_2, x_3, x_4, x_5)] \in \mathbb{P}^5 \mid \mathcal{L}((x_0, x_1, x_2, x_3, x_4, x_5), (x_0, x_1, x_2, x_3, x_4, x_5) = 0\}$ *is called the* **Lie Quadric***.*

[1] In this chapter $\mathbb{x} = (x_0, x_1, x_2, x_3, x_4, x_5)$ is a point of $\mathbb{R}^6$ as opposed to the previous where we used $\mathbb{x} \in \mathbb{R}^5$.

Let $\mathbf{x} = [(x_0, x_1, x_2, x_3, x_4)]$ be a point outside the Möbius hypersphere, i.e.,

$$\mathcal{M}((x_0, x_1, x_2, x_3, x_4, x_5), (x_0, x_1, x_2, x_3, x_4, x_5)) > 0.$$

We know that the affine points $\mathbb{x} = (x_0, x_1, x_2, x_3, x_4, x_5)$ and $\frac{1}{\mathcal{M}(\mathbb{x},\mathbb{x})}\mathbb{x}$ represent the projective point $\mathbf{x}$. We will suppose without loosing generality that $\mathcal{M}(\mathbb{x}, \mathbb{x}) = 1$. If $x_0 - x_4 \neq 0$ in Möbius geometry $\mathbf{x}$ represents a sphere in $\mathbb{R}^3$ centered at p and radius $r > 0$, i.e., $\mathbf{x} = \left[\pm\left(\frac{p\cdot p - r^2+1}{2r}, \frac{p}{r}, \frac{p\cdot p-r^2-1}{2r}\right)\right]$. Naturally there are two points of the Lie Quadric whose projections onto the Möbius space correspond to $\mathbf{x}$, namely, the points $[(x_0, x_1, x_2, x_3, x_4, x_5, \pm 1)]$ or equivalently $\left[\left(\frac{p\cdot p-r^2+1}{2}, p, \frac{p\cdot p-r^2-1}{2}, \pm r\right)\right]$. Thus the points $\mathbf{x}^+$ where the last coordinate is $+r$ and $\mathbf{x}^-$ where the last coordinate is $-r$ give us the signed radii of the oriented spheres corresponding to $\mathbf{x} \in \Psi^+$.
Now if $x_0 - x_4 = 0$ we have that $\mathbf{x} = \pm[(h, N, h)]$ which corresponds to the plane $u \cdot N = h$, with $||N|| = 1$. Then the points in the Lie Quadric $[(h, N, h, \pm 1)]$ project onto $\mathbf{x} \in \Psi^+$. Where $[(h, N, h, 1)]$ and $[(h, N, h, -1)]$ will represent the two orientations of the plane.
Finally, if we take $\mathbf{x} = [(x_0, x_1, x_2, x_3, x_4)] \in \Psi$ the point $[(x_0, x_1, x_2, x_3, x_4, 0)]$ lies in the Lie quadric and represents a point in the 3-dimensional space $\mathbb{R}^3$ or the point at infinity. See Table 2.1.

Euclidean Space	**Lie Sphere Space**
points: $u \in \mathbb{R}^3$	$\left[\left(\frac{u\cdot u+1}{2}, u, \frac{u\cdot u-1}{2}, 0\right)\right]$
∞	$[(1, 0, 0, 0, 1, 0)]$
spheres: center p signed radius r	$\left[\left(\frac{p\cdot p-r^2+1}{2}, p, \frac{p\cdot p-r^2-1}{2}, r\right)\right]$
planes: $u \cdot N = h$, with $\|\|N\|\| = 1$	$[(h, N, h, 1)]$

Table 2.1: Correspondence between points in $\mathbb{Q}^4$ and oriented spheres in $\mathbb{R}^3$

Now let $k = [(k_0, k_1, k_2, k_3, k_4, k_5)]$ be in the Lie quadric with the properties of $k_0 - k_4$ and k_5 being non-zero, hence we define the point $k^* = \frac{1}{k_0-k_4}k$ will belong to the same equivalent class and they will represent an oriented sphere centered at (k_1^*, k_2^*, k_3^*) and signed radius k_5^*.
Now taking a point k in the Lie quadric with $k_5 \neq 0$ and $k_0 - k_4 = 0$, we have that $||(k_1, k_2, k_3)||^2 = k_5^2$ and the point $k^* = \frac{k}{||(k_1,k_2,k_3)||}$ represents an oriented plane of equation $u \cdot N = k_0^*$ where $N = \frac{(k_1,k_2,k_3)}{||(k_1,k_2,k_3)||}$. On the other hand if $k_5 = 0$ the point k, as seen before, determines a point in $\mathbb{R}^3$ or the point at infinity.
From now on we will refer to oriented spheres, planes and points (spheres of zero radius) as *Lie spheres*, and their corresponding points in $\mathbb{Q}^4$ as their *Lie coordinates*. As far as orientation is concerned we will adopt the convention that a positive radius corresponds to the inward field

of unit normals, and a negative radius corresponds to the outward field of unit normals. When working with planes, the last coordinate equal one means the orientation given by N, while -1 will mean the orientation given by $-N$.

Definition 2.3. *Oriented Contact*

1. *Two spheres are in oriented contact if they are tangent to each other and have the same orientation at that point.*

2. *Two planes are in oriented contact if their unit normals are the same.*

3. *A sphere and a plane are in oriented contact if they are tangent and they have the same orientation at their tangency point.*

4. *A finite point is in oriented contact with a sphere or a plane if the point lies on it.*

5. *The point at infinity (this is the north pole of* $\mathbb{S}^3$*, i.e.,* $(0,0,0,1)$*, or equivalently* $[(1,0,0,0,1,0)]$ *in the Lie quadric) is in oriented contact with every plane.*

It is easily seen (see Figure 2.1) that two Lie spheres $[k_1]$ and $[k_2]$ are in oriented contact if and only if $\mathcal{L}(k_1, k_2) = 0$.

Theorem 2.4. *Let* $[k_1]$*,* $[k_2] \in \mathbb{Q}^4$*. The line determined by these points lies in the Lie quadric if* $[k_1]$ *and* $[k_2]$ *are in oriented contact and it consists of the Lie spheres in oriented contact with* $[k_1]$ *and* $[k_2]$*.*

Proof. A point in the line is of the form $[\alpha k_1 + \beta k_2]$, where α, $\beta \in \mathbb{R}$ with at least one of them nonzero. Then:

$$\begin{aligned}\mathcal{L}(\alpha k_1 + \beta k_2, \alpha k_1 + \beta k_2) &= \mathcal{L}(\alpha k_1, \alpha k_1) + \mathcal{L}(\alpha k_1, \beta k_2) + \mathcal{L}(\alpha k_2, \alpha k_2)\\ &= \alpha\beta\mathcal{L}(k_1, k_2)\\ &= 0.\end{aligned}$$

It is easy to see that $\mathcal{L}(\alpha k_1 + \beta k_2, k_1) = \mathcal{L}(\alpha k_1 + \beta k_2, k_2) = 0$, which means that $[\alpha k_1 + \beta k_2]$ is in oriented contact with $[k_1]$ and $[k_2]$ respectively, as desired. ■

The theorem above leads us to the definition of a *parabolic pencil of spheres* which will be the set of Lie spheres corresponding to a line in $\mathbb{Q}^4$, see Figure 2.2.

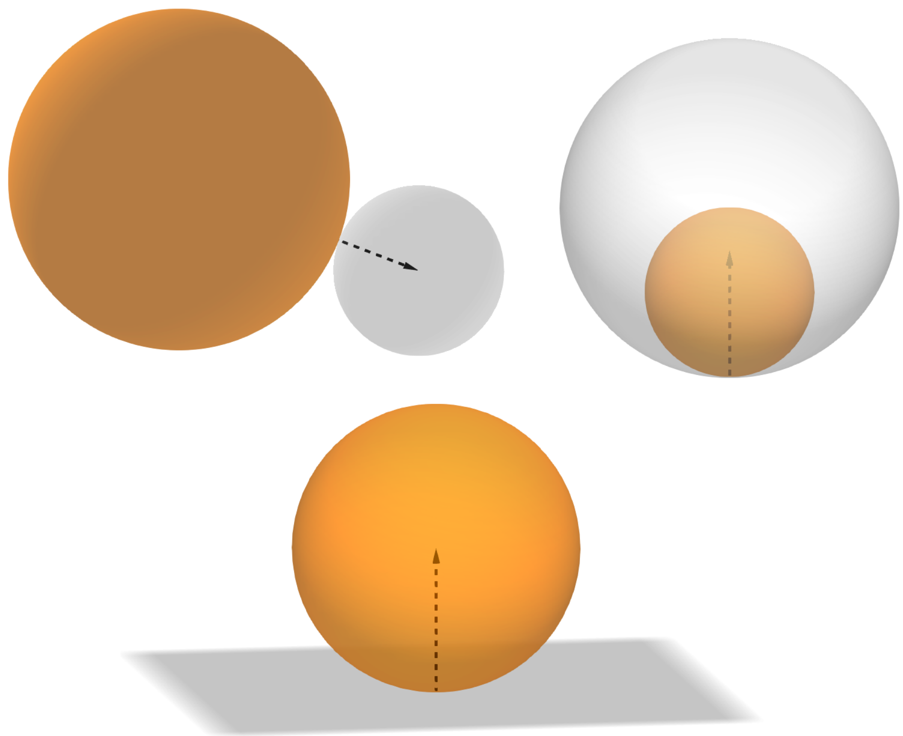


Figure 2.1: Oriented Contact of Lie Spheres.

Note that a parabolic pencil of spheres is equivalent to a pencil of tangent spheres, as seen in Chapter 1, but this notion takes into account their orientation. Finally, we state a theorem involving light-like subspaces. To see a proof see [2, page 21].

Theorem 2.5. *The Lie quadric contains projective lines, but no linear subspaces of higher dimension.*

2.2 Λ^4 and the Paraboloid Π

In this section we will consider the space of oriented spheres without using the projective space.

Definition 2.6. *We refer to as Light Cone to the subset of* $\mathbb{R}^5_1$*, given by the equation* $\mathcal{M}(\mathbb{x},\mathbb{x}) = 0$ *which is equivalent to*

$$x_0^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Definition 2.7. *The space* Λ^4 *is defined to be*

$$\Lambda^4 = \{\mathbb{x} \in \mathbb{R}^5_1 \mid \mathcal{M}(\mathbb{x},\mathbb{x}) = 1\}.$$

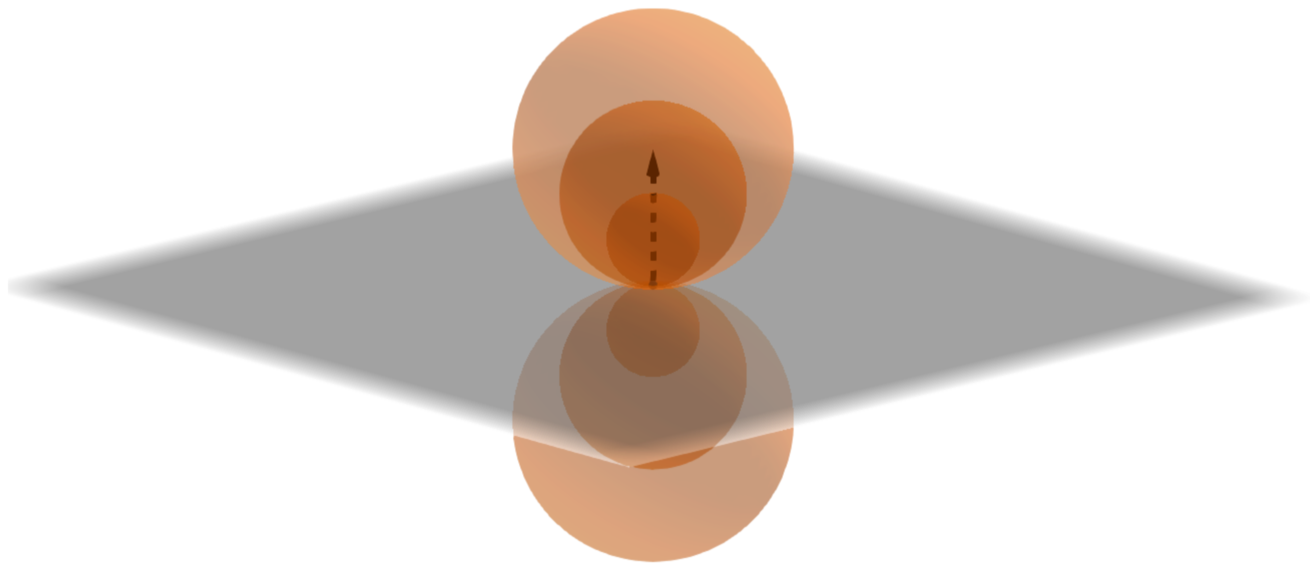


Figure 2.2: Parabolic Pencil of Spheres.

We will use the terminology of relativity theory to refer to points, vectors ($\mathbb{x}$) and planes (P) in $\mathbb{R}^5_1$ as space-like, time-like and light-like. See Table 2.2.

Type	Vector $\mathbb{x}$	Plane P
Space-like	$\mathcal{M}(\mathbb{x}, \mathbb{x}) > 0$	Every vector in the plane is space-like
Time-like	$\mathcal{M}(\mathbb{x}, \mathbb{x}) < 0$	If it contains exactly two light-like directions
Light-like	$\mathcal{M}(\mathbb{x}, \mathbb{x}) = 0$	Parallel plane to a tangent plane at a point in the light cone

Table 2.2: Types of Vectors and Planes of $\mathbb{R}^5_1$.

Proposition 2.8. *Let $\mathcal{H}$ be the affine light hyperplane of equation $x_0 - x_4 = 1$ or equivalently $\mathcal{M}(\mathbb{x}, (1, 0, 0, 0, 1)) = 1$. The paraboloid Π, defined as the intersection set of the light cone with $\mathcal{H}$, is in bijective correspondence with the space $\mathbb{R}^3$.*

Proof. For a point $\mathbb{x}$ to be in Π, its coordinates $(x_0, x_1, x_2, x_3, x_4)$ must satisfy the system:

$$x_0 - x_4 = 1$$
$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

Solving for the variables x_0 and x_4 we get:

$$x_0 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + 1),$$
$$x_4 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 - 1).$$

Then, the function ϕ from $\mathbb{R}^3$ to Π given by

$$\phi(x_1, x_2, x_3) = \begin{bmatrix} \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + 1) \\ x_1 \\ x_2 \\ x_3 \\ \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 - 1) \end{bmatrix},$$

is a bijective function from $\mathbb{R}^3$ to Π.

■

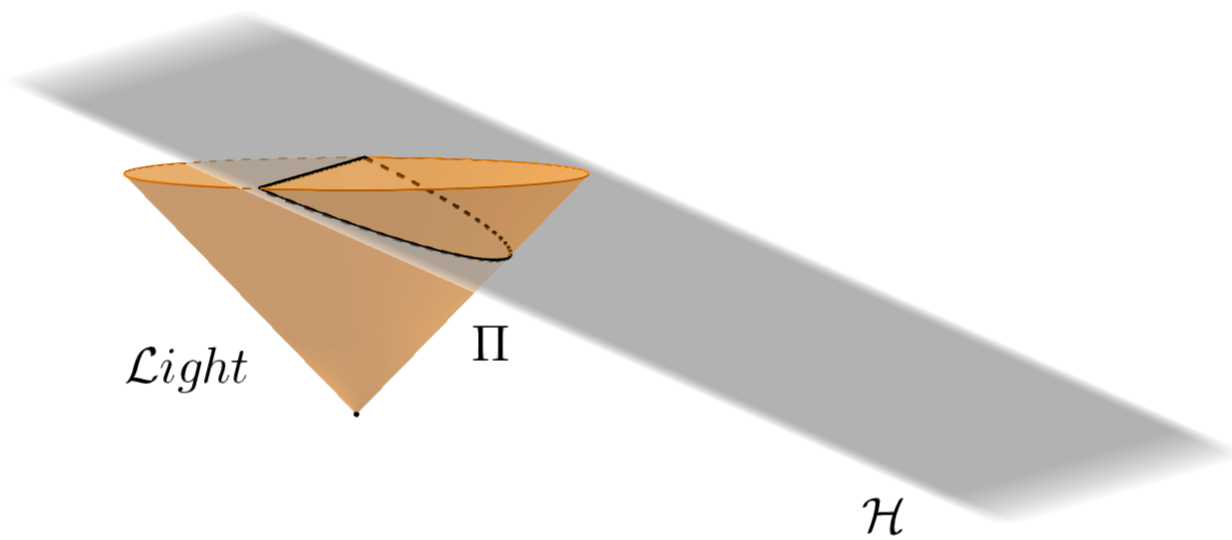


Figure 2.3: Model of the Euclidean Space $\mathbb{R}^3$ in the Light Cone.

The proposition above states that every light direction represents a point in $\mathbb{R}^3$ except for the light direction given by the vector $\mathbb{n}$, the "pole" of $\mathcal{H}$ (it is actually the pole of the vectorial light hyperplane parallel to $\mathcal{H}$), which represents the point at infinity. Let $\mathbb{x}$ be a light-like vector in $\mathbb{R}^5_1$, it is clear that this light direction can be written as $\mathbb{x} = (1, u)$ for $u \in \mathbb{R}^4$, thus the equation $\mathcal{M}(\mathbb{x}, \mathbb{x}) = 0$ becomes $u \cdot u = 1$, where $\cdot$ denotes the euclidean dot product. This means that the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ is diffeomorphic to the set of light-like directions in $\mathbb{R}^5_1$.

Hence the function ϕ defined in the proof of Proposition 2.8 is nothing more than the generalizad stereographic projection $\mathcal{SP}$ defined in Chapter 1, see Section 1.1.
For the purpose of interpreting points in Λ^4 as oriented spheres or oriented planes in $\mathbb{R}^3$, let us consider the polar hyperplane of $\sigma \in \Lambda^4$, with equation $\mathcal{M}(\sigma, \mathbb{x}) = 0$. Using Proposition 1.1 and in an analogous way as in Theorem 1.4 we calculate its intersection with the paraboloid Π. Doing so we get the correspondence illustrated in Table 2.3.

Euclidean Space	Λ^4 and Π
points: $u \in \mathbb{R}^3$	$\left(\frac{u \cdot u+1}{2}, u, \frac{u \cdot u-1}{2}\right)$
∞	$(1, 0, 0, 0, 1)$
spheres: center p, signed radius r	$\left(\frac{p \cdot p-r^2+1}{2r}, \frac{p}{r}, \frac{p \cdot p-r^2-1}{2r}\right)$
planes: $u \cdot N = h$, with $\|\|N\|\| = 1$	(h, N, h)

Table 2.3: Correspondence Between Points in Λ^4 and Π with Points, Spheres and Planes in $\mathbb{R}^3$

Note that this correspondence is almost the same as the one obtained from Lie's approach (compare Tables 2.3 and 2.1), the difference is that in the affine setting given by Table 2.3, but separating in two "disjoint" sets the points representing spheres and planes with the ones representing points in $\mathbb{R}^3$, namely, Λ^4 and Π. Both constructions projective ($\mathbb{Q}^4$) and non-projective (Λ^4 and Π) are equivalent.

Chapter 3

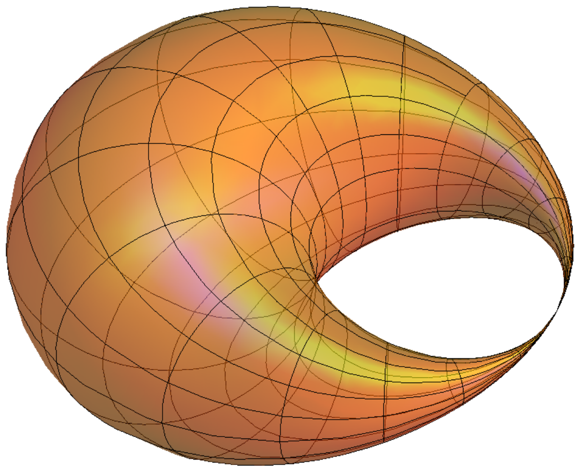
Dupin Cyclides

Definition 3.1. *Dupin Cyclide The envelope of a family of spheres/planes tangent to three prescribed spheres.*

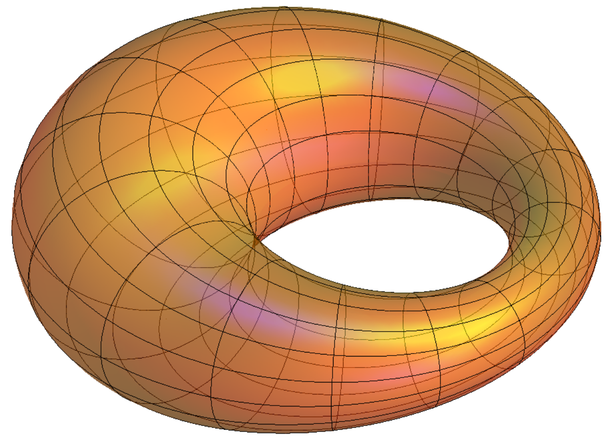
The above definition as well as the name cyclide are given in the classical Dupin's book [4]. There are another equivalent definitions of Dupin cyclides (see [14] and [5]) but the one we have chosen will come at handy when working within the space of spheres. It has been shown, in relation to the classical Apollonius problem of finding circles tangent to three fixed circles (see [13] and the references therein), that the set of spheres tangent to three given spheres split into four one-parameter families of spheres. Taking a Dupin cyclide as the envelope of a family of Lie spheres in oriented contact with three prescribed Lie spheres, we eliminate three of these four families. The oriented Dupin cyclide is therefore defined inheriting its orientation from the unique family of Lie spheres which defines it. A equation for Dupin cyclides is given by

$$Cyclide(\alpha, \beta) = \begin{pmatrix} \frac{\mu(c - a\cos\alpha\cos\beta) + \left(a^2 - c^2\right)\cos\alpha}{a - c\cos\alpha\cos\beta} \\ \\ \frac{\sqrt{a^2 - c^2}\sin\alpha(a - \mu\cos\beta)}{a - c\cos\alpha\cos\beta} \\ \\ \frac{\sqrt{a^2 - c^2}\sin\beta(c\cos\alpha - \mu)}{a - c\cos\alpha\cos\beta} \end{pmatrix}. \tag{3.1}$$

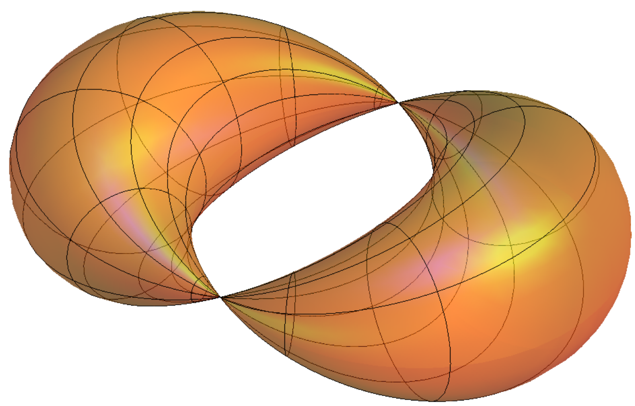
The parameter μ will determine the form of the cyclide. See Figures 3.1a, 3.1b and 3.1c for several kinds of Dupin cyclides: a spindle cyclide, a ring cyclide and a horn cyclide respectively. To see an extensive study on how the parameter μ affects the shape of the cyclide see [6].



(a) Spindle Cyclide.



(b) Ring Cyclide.



(c) Horn Cyclide.

Figure 3.1: Dupin Cyclides.

3.1 Dupin Cyclides as 2-Plane Sections of $\mathbb{Q}^4$

Let $[\sigma]$[1] be a representation in $\mathbb{Q}^4$ of an oriented sphere in $\mathbb{R}^3$. We know from Theorem 2.4 that the three-dimensional cone $\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma, \mathbb{x}) = 0\} \cap \mathbb{Q}^4$ represents all the spheres in oriented contact with $[\sigma]$.

Given three different oriented spheres $[\sigma_1], [\sigma_2]$ and $[\sigma_3]$, the set of Lie spheres in oriented contact with them is given by

$$\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_1, \mathbb{x}) = 0\} \cap \{\mathbb{x} \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_2, \mathbf{x}) = 0\} \cap \{\mathbb{x} \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_3, \mathbb{x}) = 0\} \cap \mathbb{Q}^4.$$

[1]We will refer to $[\sigma]$ as the sphere as well.

Where the set $\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_1, \mathbb{x}) = 0\} \cap \{\mathbb{x} \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_2, \mathbb{x}) = 0\} \cap \{\mathbb{x} \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_3, \mathbb{x}) = 0\}$ is a 2-plane in $\mathbb{P}^5$ (or equivalently a 3-plane through the origin in $\mathbb{R}^6_2$). We have built a curve (knowing that the intersection of every 2-plane with $\mathbb{Q}^4$ is a one-dimensional set) in the space of oriented spheres, i.e., a monoparametric family of oriented spheres, which are in oriented contact with three fixed spheres. Constructing the envelope of this family, we have constructed an **oriented Dupin cyclide**.

It is known that two different families of spheres define the same oriented Dupin cyclide (see [13]). The defining characteristic of these two families is that of being in oriented contact with each other, i.e., every sphere belonging to the first family is in oriented contact with those belonging to the other family. This leads to the following definition.

Definition 3.2. *Let P be a 2-plane. We define its brother 2-plane to be the set*

$$P^* = \{[\mathbb{y}] \in \mathbb{P}^5 \mid \mathcal{L}(\mathbb{x}, \mathbb{y}) = 0, \ \textit{for all} \ [\mathbb{x}] \in P\}.$$

Note that the brother 2-plane of P^* is P itself, i.e., $P^{**} = P$. We know that a 2-plane in $\mathbb{P}^5$ is determined by three points (in the same manner as a 3-plane in $\mathbb{R}^6_2$ is defined by three independent points). Then for a 2- plane P, its brother plane P^* is completely determined by three points lying in P, i.e., taking $[\tau_1], [\tau_2]$ and $[\tau_3]$ belonging to P we have that

$$P^* = \{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\tau_1, \mathbb{x}) = 0\} \cap \{\mathbb{x} \in \mathbb{P}^5 \mid \mathcal{L}(\tau_2, \mathbb{x}) = 0\} \cap \{\mathbb{x} \in \mathbb{P}^5 \mid \mathcal{L}(\tau_3, \mathbb{x}) = 0\}.$$

Where the choice of $[\tau_i] \in P$ for $i = 1, 2, 3$ is arbitrary.

Note that a line L joining any point of $[\sigma]$ of $P \cap \mathbb{Q}^4$ to a point $[\sigma^*]$ in $P^* \cap \mathbb{Q}^4$, by Theorem 2.4, lies in $\mathbb{Q}^4$, i.e., is a parabolic pencil of spheres. The point where the contact condition holds, belongs to the Dupin cyclide.

Types of Dupin Cyclides and Pencils of Spheres

Let us explore the posibilities that appear when choosing three Lie spheres in order to define a Dupin cyclide. On the assumption that one of the oriented spheres is a point of $\mathbb{R}^3$ not contained in the other two Lie spheres, its envelope will produce a spindle cyclide (see Figure 3.1a). Since it will have one singular point. If we pick two points of $\mathbb{R}^3$ and an oriented sphere not containing them, we will have a horn cyclide (see Figure 3.1c), because we have two singular points.

Let us choose instead of three oriented spheres, three points in $\mathbb{R}^3$ and calculate their corresponding 2-plane and the envelope of the family of spheres in oriented contact with them. By Definition

2.1, a sphere is in oriented contact with a point if the point belongs to the sphere. Then when calculating its envelope, we are finding the envelope of a family of Lie spheres passing through three given points, i.e., an oriented circle. Recall that a circle in $\mathbb{R}^3$ is represented by a pencil of intersecting spheres in the space of unoriented spheres Ψ^+. See Figure 1.3, page 14. On the other hand we know that if one of the points is the point at infinity, the set of Lie spheres in oriented contact with it, is the set of planes, namely, the hyperplane $\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}((1,0,0,0,1,0), \mathbb{x}) = 0\}$, and the intersection 2-plane of the three point's hyperplanes, will lie on this hyperplane, i.e., it will be a family of planes passing through two finite points and the point at infinity, in other words, their envelope is a line in $\mathbb{R}^3$, so we have a pencil of intersecting (oriented) planes.

We have seen that a pencil of tangent spheres can be seen as a line in $\mathbb{Q}^4$ (see Theorem 2.4, page 21), taking the line representing the other orientation at the point of tangency, we get two lines, i.e., a 2-plane section of $\mathbb{Q}^4$. We have just proved that a pencil of (oriented) spheres can be seen as a 2-plane section of $\mathbb{Q}^4$ using Dupin cyclides.

Chapter 4

Three Contact Conditions

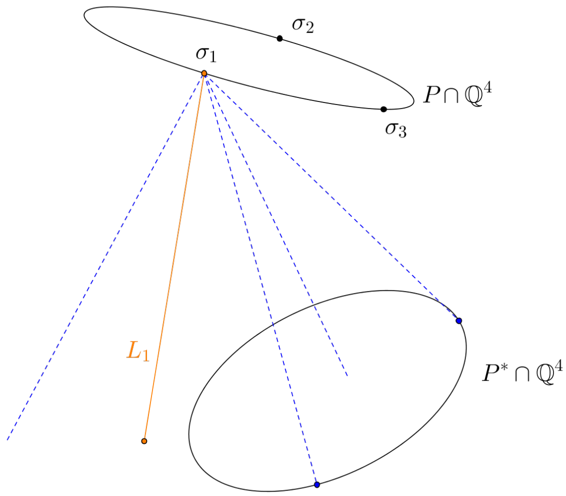
In this chapter we will prove the main theorem of this dissertation, which states under which circumstances we can find a Dupin cyclide fulfilling three given contact conditions. In order to do so we will define what a contact condition is, and its relationship with the space of Lie spheres $\mathbb{Q}^4$. We will introduce the concept of homography and highlight some of the properties necessary to define the maps *pang*, *pong* and *ping*. The fundamental tool used in the proof of the theorem.

4.1 Contact Conditions

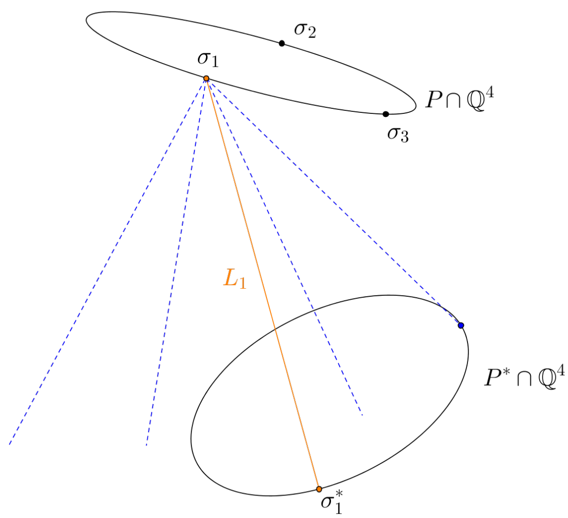
Definition 4.1. *Let m be a point in $\mathbb{R}^3$ and h an oriented plane containing m, the pair (m, h) is referred to as a contact condition in $\mathbb{R}^3$ at m.*

A contact condition generates a pencil of spheres, namely, the Lie spheres in oriented contact with h at m. We have seen that such a pencil is a line L in $\mathbb{Q}^4$ (a 2-plane through the origin 0_6 in $\mathbb{R}^6_2$), the so-called parabolic pencil of spheres. Recall Figure 2.2.

An initial approach to solve the three contact conditions problem would be to take three spheres $[\sigma_1], [\sigma_2]$ and $[\sigma_3]$ one on each line $L_i,\ i = 1, 2, 3$ representing the contact conditions and consider the Dupin cyclide defined by the 2-plane $P = \{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_1, \mathbb{x}) = 0\} \cap \{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_2, \mathbb{x}) = 0\} \cap \{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_3, \mathbb{x}) = 0\}$. The contact point m_1, as being part of the cyclide should belong to one of the spheres, say $[\sigma_1^*]$, in the brother 2-plane P^* since the spheres $[\sigma_1]$ and $[\sigma_1^*]$ must be in oriented contact at that point. Thus if the Dupin cyclide defined by the curve $P \cap \mathbb{Q}^4$ satisfies the contact condition L_1, $[\sigma_1^*]$ must belong to L_1, i.e., L_1 must intersect P^*. Figure 4.1 shows two situations, one when the curve defining a Dupin cyclide satisfies a contact condition, and a second when it does not. The dot lines (blue in the web version) are contact conditions at the point $[\sigma_1]$, and the straight line (orange in the web version) is the contact condition L_1. They all are part of the three-dimensional cone $\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_1, \mathbb{x}) = 0\} \cap \mathbb{Q}^4$.



(a) Not Satisfying the Contact Condition.



(b) Satisfying the Contact Condition.

Figure 4.1: A Necessary Condition for a Dupin Cyclide to Satisfy a Contact Condition.

Homographies

Definition 4.2. *A homography H is a function from $\mathbb{P}^1$ to $\mathbb{P}^1$ given by*

$$H \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $ad - bc \neq 0$.

An affine point of $\mathbb{P}^1$ is of the form $\begin{pmatrix} x \\ 1 \end{pmatrix}$[1] thus

$$H\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix},$$

and if $x \neq -\frac{d}{c}$ we have induced a real-valued function from $\mathbb{R} - \{-\frac{d}{c}\}$ given by the formula

$$f(x) = \frac{ax+b}{cx+d}.$$

Usually a homography is defined as the real-valued function given by the formula above with $ab - cd \neq 0$ and extended to the projective line $\mathbb{P}^1$ as given by Definition 4.2.
We will be dealing with homographies which exchange points in $\mathbb{P}^1$, that is to say linear transformation exchanging lines in $\mathbb{R}^2$. They will be characterized as involutions, i.e., as maps whose square transformation is the identity map in $\mathbb{P}$, in the following proposition.

Proposition 4.3. *A homography H exchanges two points of $\mathbb{P}$ if and only if H is a fixed-point-free involution.*

Proof. For a linear transformation H of $\mathbb{R}^2$ to be an involution of $\mathbb{P}^1$, its square transformation needs to be a multiple of the identity matrix, i.e., $H^2 = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where k is a nonzero real number. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in the standard basis of $\mathbb{R}^2$ for H, let $\begin{pmatrix} e \\ f \end{pmatrix}$ and $\begin{pmatrix} g \\ h \end{pmatrix}$ be two direction vectors for two points (lines in $\mathbb{R}^2$) exchanged by H, i.e., $H\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$ and $H\begin{pmatrix} g \\ h \end{pmatrix} = k\begin{pmatrix} e \\ f \end{pmatrix}$ for some $k \in \mathbb{R} - \{0\}$.
Let $\{\begin{pmatrix} e \\ f \end{pmatrix}, \begin{pmatrix} g \\ h \end{pmatrix}\}$ be a basis[2] of $\mathbb{R}^2$ consisting of the two directions given by the points exchanged by H. The matrix of H in this new basis is given by the product

$$\begin{pmatrix} e & g \\ f & h \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}.$$

[1] Given that we can choose where the point at infinity would be, an affine point could have the form $\begin{pmatrix} 1 \\ y \end{pmatrix}$. Although it would change in an algebraic way, geometrically would be no change, since those constructions of the projective line $\mathbb{P}^1$ are equivalent.

[2] If it is not a basis of $\mathbb{R}^2$ the vectors would be linearly dependent, i.e., they would represent the same line through the origin and hence represent the same point in $\mathbb{P}^1$.

To see that H is an involution we calculate the matrix of H^2 to get

$$\begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}^2 = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which proofs that H is an involution.
If H is an involution it is clear that exchanges points since $H^2 = Id_{\mathbb{P}^1}$. ■

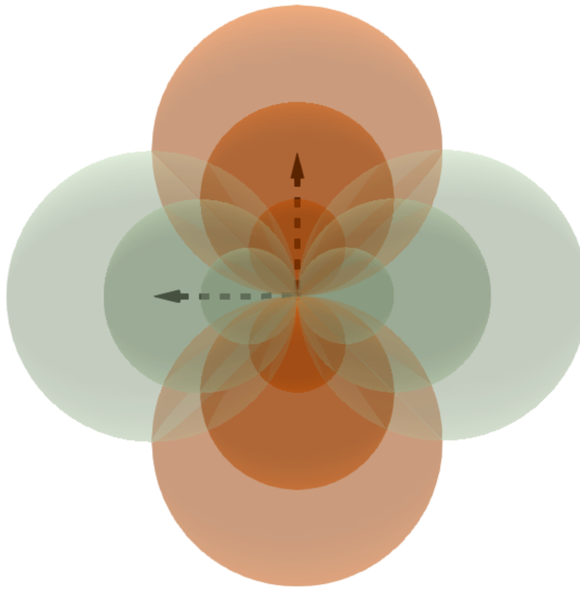
Note that for a homography which exchanges points, we have that its matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and has zero trace. The eigenvalues of such a map are $\pm\sqrt{-(ad-bc)}$ and $det(A) < 0$ cannot happen. Otherwise we would have two real eigenvalues and the lines given by their eigenvectors would be fixed. Therefore we can characterize the linear maps that induce involutions as maps of zero trace without real eigenvalues, or equivalently linear maps of zero trace and positive determinant.
See the classical, but beautifully written, Coxeter's book [3, pages 242-246] for further details on homographies.

4.2 The Homographies *pang*, *pong* and *ping*

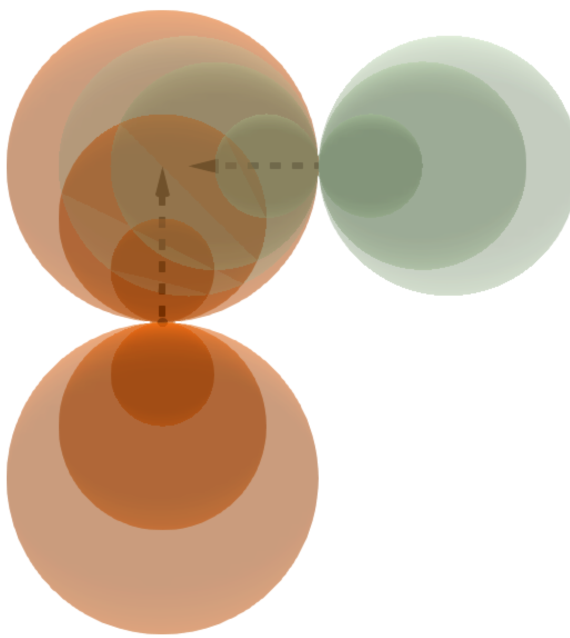
Let us explore some difficulties encountered while solving the problem of finding a Dupin cyclide given three contact conditions. Let L_1, L_2 and L_3 be three contact conditions at three different points m_1, m_2 and m_3 respectively. If some of the points are equal we will have more than one contact condition at the same point which leads us to a contradiction because we are demanding different orientations at the same point, see Figure 4.2a. Another "pathology" arises when one of the points, say m_1, belongs to one of the spheres of another contact conditions, say L_2 (see Figure 4.2b), i.e., we have two different contact conditions on the same sphere. In terms of the geometry of the Lie quadric for these two situations we have that L_1 is contained in a hyperplane $\{[\mathrm{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_2, \mathrm{x}) = 0\}$ for $[\sigma_2] \in L_2$.
To avoid these "pathologies" and to be able to define the homographies *pang*, *pong* and *ping*, we introduce the concept of independent contact conditions.

Definition 4.4. *Let L_1, L_2 and L_3 be three projective lines given by three contact conditions at finite points. We say that they are independent if and only if The smallest projective subspace containing them is $\mathbb{P}^5$ itself.*

The above definition states that three contact conditions are independent if the direct sum of their corresponding 2-planes in $\mathbb{R}^6_2$ spans $\mathbb{R}^6_2$. In terms of the contact condition at $\mathbb{R}^3$, the



(a) Two Contact Conditions at the Same Point.



(b) Two Contact Conditions on the Same Sphere.

Figure 4.2: "Pathological" Pairs of Contact Conditions.

independence of the 2- planes in $\mathbb{R}^6_2$ rules out the "pathological" cases seen in Figure 4.2.
Let L_1, L_2 and L_3 be three independent contact conditions let us define now the maps *ping*, *pong* and *pang* as follows:

$$pang : L_1 \to L_2;\ pang([\sigma_1]) = \Big\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_1, \mathbb{x}) = 0\Big\} \cap L_2, \tag{4.1}$$

$$pong : L_2 \to L_3;\ pang([\sigma_2]) = \Big\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_2, \mathbb{x}) = 0\Big\} \cap L_3, \tag{4.2}$$

$$ping : L_3 \to L_1;\ pang([\sigma_3]) = \Big\{[\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_3, \mathbb{x}) = 0\Big\} \cap L_1. \tag{4.3}$$

They are well defined since every projective line has exactly one point of intersection with every hyperplane which does not contain it. The map *pong* "finds" the sphere corresponding to the parabolic pencil of spheres L_2, which is in oriented contact with the sphere σ_1 corresponding to L_1, since it is the intersection of the space of all spheres in oriented contact with $[\sigma_1]$, with L_2. In a analogous way if $[\sigma_2]$ and $[\sigma_3]$ belong to L_2 and L_3 respectively, the maps *pang* and *ping* find the spheres in L_3 and L_1 in oriented contact with $[\sigma_2]$ and $[\sigma_3]$, respectively.

Proposition 4.5. *The maps pang, pong and ping are homographies.*

Proof. Let us consider the affine part of these three lines, abusing notation we will refer to them as L_i for $i = 1, 2, 3$ and we can parametrize these lines as follows

$$[\sigma_1] = [\sigma_{12} + t_1 m_1],\ t_1 \in \mathbb{R}$$
$$[\sigma_2] = [\sigma_{23} + t_2 m_2],\ t_2 \in \mathbb{R}$$
$$[\sigma_3] = [\sigma_{31} + t_3 m_3],\ t_3 \in \mathbb{R},$$

where $[\sigma_{jk}]$ is the sphere contained in L_j which contains the point m_k, and the symbols m_i and $[m_i]$ for $i = 1, 2, 3$ will represent the points in $\mathbb{R}^3$ and the light-like vectors corresponding to those points.
To prove that the map *pang* is a homography from $\mathbb{R}$ to $\mathbb{R}$ (i.e., $t_1 \to t_2$) we will take the parametrization $[\sigma_{23} + t_2 m_2^1]$ of the line L_2, where the direction $[m_2^1]$ is the same as $[m_2]$ but with the property that $\mathcal{M}(m_1, m_2^1) = -1$. The point $pang([\sigma_1])$ belongs to the hyperplane of equation $\mathcal{M}(\mathbb{x}, \sigma_1) = 1$ and to the line L_2, therefore it satisfies the equation $\mathcal{M}(\sigma_{12} + t_1 m_1, \sigma_{23} + t_2 m_2^1) = 1$, thus solving for t_2 we have

$$1 = \mathcal{M}(\sigma_{12} + t_1 m_1, \sigma_{23} + t_2 m_2^1)$$
$$1 = \mathcal{M}(\sigma_{12}+, \sigma_{23} + t_2 m_2^1) + \mathcal{M}(t_1 m_1, \sigma_{23} + t_2 m_2^1)$$
$$1 = \mathcal{M}(\sigma_{12}, \sigma_{23}) + t_2 \mathcal{M}(\sigma_{12}, m_2^1) + t_1 \mathcal{M}(m_1, \sigma_{23}) + t_1 t_2 \mathcal{M}(m_1, m_2^1).$$

Since $[\sigma_{12}]$ contains $[m_2]$ and m_2^1 is an scalar multiple of m_2, the product $\mathcal{M}(\sigma_{12}, m_2^1)$ is zero, and by construction we know that $\mathcal{M}(m_1, m_2^1) = -1$ then we get

$$t_2 = \frac{-1 + \mathcal{M}(\sigma_{12}, \sigma_{23}) + t_1 \mathcal{M}(m_1, \sigma_{23})}{t_1}. \tag{4.4}$$

Thus if we called $\alpha_2 = \mathcal{M}(m_1, \sigma_{23})$ and $\beta_1 = -1 + \mathcal{M}(\sigma_{12}, \sigma_{23})$ we have that

$$pang(t_1) = \frac{\alpha_2 t_1 + \beta_1}{t_1}.$$

Whence *pang* can be viewed as a homography from $\mathbb{P}^1 \cong L_1$ to $\mathbb{P}^1 \cong L_2$ with matrix

$$pang = \begin{pmatrix} \alpha_2 & \beta_1 \\ 1 & 0 \end{pmatrix}. \tag{4.5}$$

To prove that the map *pong* is a homography we proceed in an analogous fashion. Let us now fix m_2 and choose m_3^2 such that $\mathcal{M}(m_2, m_3^2) = -1$ and parametrizing L_3 by the equation $[\sigma_{31} + t_3 m_3^2]$. Performing as before we get

$$t_3 = \frac{-1 + \mathcal{M}(\sigma_{31}, \sigma_{23}) + t_2 \mathcal{M}(m_2, \sigma_{31})}{t_2}.$$

And defining $\alpha_3 = \mathcal{M}(m_2, \sigma_{31})$ and $\beta_2 = -1 + \mathcal{M}(\sigma_{31}, \sigma_{23})$ we obtain that the matrix of the map is given by

$$pong = \begin{pmatrix} \alpha_3 & \beta_2 \\ 1 & 0 \end{pmatrix}. \tag{4.6}$$

Finally the map *ping* has as its matrix

$$ping = \begin{pmatrix} \alpha_1 & \beta_3 \\ 1 & 0 \end{pmatrix}. \tag{4.7}$$

Where $\alpha_1 = \mathcal{M}(m_3, \sigma_{12})$ and $\beta_3 = -1 + \mathcal{M}(\sigma_{12}, \sigma_{31})$ constructed by choosing m_1^3 such that $\mathcal{M}(m_3, m_1^3) = -1$ and parametrizing L_1 using the equation $[\sigma_{12} + t_3 m_1^3]$. ■

The choice of the points $m_1^3, m_2^1, m_3^2, \sigma_{12}, \sigma_{32}$ and σ_{31} in the proof of Propositon 4.5 is just for calculations purposes, to simplify the arising formulas. Although the choice is arbitrary, if we change them, we change the homographies matrices, but not their final outcome.

Let us choose another point $m_1' = \mathbf{s} m_1$, for $\mathbf{s} \in \mathbb{R} - \{0\}$. We construct a point $\bar{m}_2^1$ such that $\mathcal{M}(m_1', \bar{m}_2^1) = -1$, then using the parametrizations $\sigma_{12} + t_1' m_1'$ of L_1 and $\sigma_{23} + t_2' \bar{m}_2^1$ of L_2 we have that $t_2' = \frac{-1 + \mathcal{M}(\sigma_{12}, \sigma_{23}) + t_1' \mathcal{M}(m_1', \sigma_{23})}{t_1'}$, and since $m_1' = \mathbf{s} m_1$ it follows that

$$t_2' = \frac{\beta_1 + \mathbf{s} t_1' \alpha_2}{t_1'},$$

where α_2 and β_1 are defined as in Equation 4.5. This latter equation is equivalent to

$$t_2' = \mathbf{s}\left(\frac{\beta_1 + \mathbf{s}t_1'\alpha_2}{\mathbf{s}t_1'}\right).$$

Note that we have performed a change of variables on both lines, namely, $t_1 = \frac{t_1'}{\mathbf{s}}$ and $t_2 = \frac{t_2'}{\mathbf{s}}$, induced by the equations $m_1' = \mathbf{s}m_1$ and $\bar{m}_2^1 = \mathbf{s}m_2^1$. The homography *pang* given by this new basis is, therefore, a dilation on both variables (the same one on both) of the original. In an analogous way we prove that the maps *pong* and *ping* are independent of the choice of m_2 and m_3, respectively.

Now we define the *ping* ∘ *pong* ∘ *pang* map, which maps L_1 onto itself, and being a composition of homographies is a homography as well. The *ping* ∘ *pong* ∘ *pang* matrix is given by the product of the matrices 4.7 , 4.6 and 4.5 (in the same order as listed)

$$ping \circ pong \circ pang = \begin{pmatrix} \alpha_1\alpha_2\alpha_3 + \alpha_1\beta_2 + \alpha_2\beta_3 & \alpha_1\alpha_3\beta_1 + \beta_1\beta_3 \\ \alpha_2\alpha_3 + \beta_2 & \alpha_3\beta_1 \end{pmatrix}. \tag{4.8}$$

The Main Theorem

Theorem 4.6. *There is a family of Dupin cyclides satisfying three independent contact conditions* L_1, L_2 *and* L_3 *if and only if the map ping* ∘ *pong* ∘ *pang constructed from them is a fixed-point-free-involution.*

Proof. Let us first show that if the *ping* ∘ *pong* ∘ *pang* is a fixed-point-free-involution we will have a family of Dupin cyclides satisfying the three contact conditions.

Let $[\sigma_1]$ be a point in L_1, and consider the points $[\sigma_2] = pang([\sigma_1])$, $[\sigma_3] = pong([\sigma_2])$, $[\sigma_4] = ping([\sigma_3])$, $[\sigma_5] = pang([\sigma_4])$ and $[\sigma_6] = pang([\sigma_5])$. Since *ping* ∘ *pong* ∘ *pang* is an involution we have that $(ping \circ pong \circ pang)^2([\sigma_1]) = [\sigma_1]$, hence $ping([\sigma_6]) = [\sigma_1]$ (see Figure 4.3).Next we show that

$$\mathcal{L}(\sigma_i, \sigma_j) = 0 \quad \text{for} \quad i = 1, 3, 5 \quad \text{and} \quad j = 2, 4, 6. \tag{4.9}$$

By definition of the *pang* and *ping* maps, we have that $\mathcal{L}(\sigma_1, \sigma_2) = 0$ and $\mathcal{L}(\sigma_1, \sigma_6) = 0$ respectively. On the other hand $[\sigma_1]$ and $[\sigma_4]$ are both points of L_1 then it follows that $\mathcal{L}(\sigma_1, \sigma_4) = 0$. In an analogous way we prove Equation 4.9 for $[\sigma_3]$ and $[\sigma_5]$.

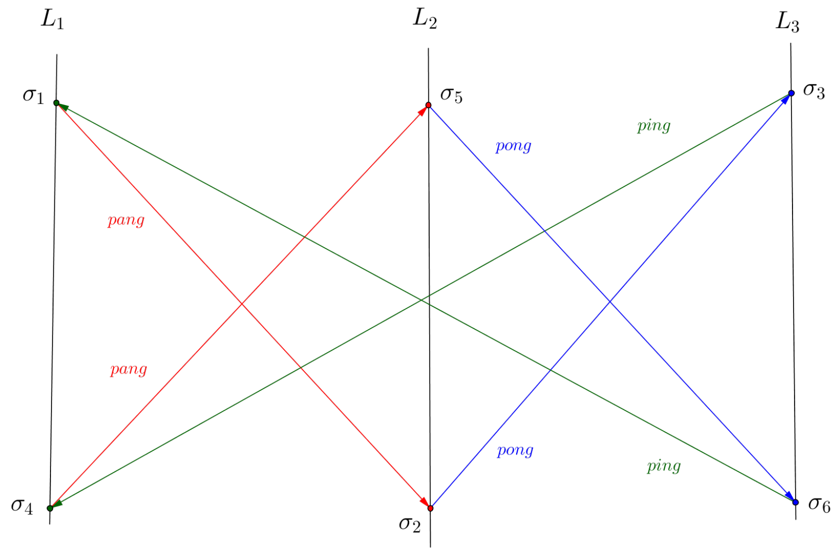


Figure 4.3: The *ping* $\circ$ *pong* $\circ$ *pang* map.

Let us define a 2-plane P as follows

$$P = \Big\{ [\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_1, \mathbb{x}) = 0 \Big\} \cap \Big\{ [\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_3, \mathbf{x}) = 0 \Big\} \cap \Big\{ [\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_5, \mathbb{x}) = 0 \Big\},$$

note that $[\sigma_2], [\sigma_4]$ and $[\sigma_6]$ belong to P, and by the remark following Definition 3.2 (see page 28) we have that its brother 2-plane P^* is given by

$$P^* = \Big\{ [\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_2, \mathbb{x}) = 0 \Big\} \cap \Big\{ [\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_4, \mathbb{x}) = 0 \Big\} \cap \Big\{ [\mathbb{x}] \in \mathbb{P}^5 \mid \mathcal{L}(\sigma_6, \mathbb{x}) = 0 \Big\}.$$

Finally the Dupin cyclide defined by these 2-planes satisfies the three contact conditions, because $[\sigma_1]$ and $[\sigma_4]$, $[\sigma_2]$ and $[\sigma_5]$ and $[\sigma_3]$ and $[\sigma_6]$ belong to L_1, L_2 and L_3 respectively. This gives us a family of Dupin cyclides as the point $[\sigma_1]$ is arbitrary.

Now let us prove that if there is a family of Dupin cyclides satisfying L_1, L_2 and L_3, the map *ping* $\circ$ *pong* $\circ$ *pang* is a fixed-point-free-involution. As seen before (once again see Figure 4.3), if a Dupin cyclide satisfies the three contact conditions *ping* $\circ$ *pong* $\circ$ *pang* must exchange the points of intersection of L_1 with the two brother 2-planes P and P^* defining the cyclide.

We must therefore prove that this map has no fixed points. Let us suppose that it does, i.e., that there exists $[\varsigma_1] \in L_1$ such that *ping* $\circ$ *pong* $\circ$ *pang*$([\varsigma_1]) = [\varsigma_1]$. Hence we will have exactly three points belonging to L_1, L_2 and L_3 respectively, $[\varsigma_1], [\varsigma_2] =$ *pang*$([\varsigma_1])$ and $[\varsigma_3] =$ *pong*$([\varsigma_2])$, because *ping*$([\varsigma_3]) = [\varsigma_1]$. See Figure 4.4.

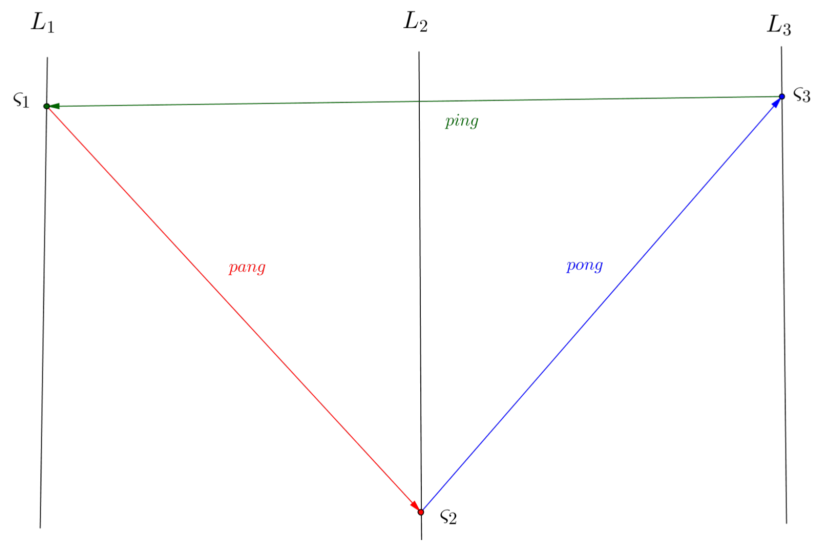


Figure 4.4: The *ping* ∘ *pong* ∘ *pang* Map Having a Fixed Point.

As seen above (see Figure 4.3) the functions *ping*, *pong* and *pang* map a Lie sphere belonging to the family defined by P to another Lie sphere belonging to the brother family of spheres defined by P^*. Then if $[\varsigma_1] \in P$ we would have that $[\varsigma_2] \in P^*$, $[\varsigma_3] \in P$ and $[\varsigma_1] = ping([\varsigma_3]) \in P^*$, which contradicts the fact that brother 2-planes are disjoint sets.
Thus we have that the *ping*∘*pong*∘*pang* map is a homography exchanging points and by Proposition 4.3 (see page 32) is a fixed-point-free-involution. ■

The above theorem states that an, a priori, geometrical requirement, such as finding a Dupin cyclide satisfying oriented tangency at three given points, can be solved by calculating some algebraic conditions of a 2×2 matrix, such as having positive determinant and zero trace.
If the matrix as given by Equation 4.8 satisfies

$$det \begin{pmatrix} \alpha_1\alpha_2\alpha_3 + \alpha_1\beta_2 + \alpha_2\beta_3 & \alpha_1\alpha_3\beta_1 + \beta_1\beta_3 \\ \alpha_2\alpha_3 + \beta_2 & \alpha_3\beta_1 \end{pmatrix} > 0,$$

and

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 = 0,$$

we will be able to find, not just one cyclide, but a family of Dupin cyclides passing through the three given points and having as tangent planes at those points, the three planes that define their orientation.
We will present an algorithm which will allow us to find, if it exists, a family of Dupin cyclides, tangent to three given contact conditions (m_1, h_1), (m_2, h_2) and (m_3, h_3) without having to compute (at least not directly) the homography matrices and therefore the trace and determinant given in

the equations above.

We know that these contact conditions are represented by three lines L_1, L_2 and L_3 in the projective space. From the proof of Proposition 4.5, page 35, we also know that the contact conditions L_i can be parametrized as $\Sigma_i + tM_i$, with $t \in \mathbb{R}$. Here Σ_i is an arbitrary point of L_i (an arbitrary Lie sphere in oriented contact with h_i at m_i) and M_i represents the point m_i in $\mathbb{R}^3$ where the oriented contact is being held (both obtained using Table 2.1, page 20), for $i = 1, 2, 3$. As in the proof of Theorem 4.6 let us choose an arbitrary Lie sphere σ_1 in L_1, i.e., an arbitrary real number t_1 such that $\sigma_1 = \Sigma_1 + t_1 M_1$, then $\sigma_2 = pang(\sigma_1) = \Sigma_2 + t_2 M_2,\ \sigma_3 = pong(\sigma_2) = \Sigma_3 + t_3 M_3,\ \sigma_4 = ping(\sigma_3) = \Sigma_1 + t_4 M_1,\ \sigma_5 = pang(\sigma_4) = \Sigma_2 + t_5 M_2,\ \sigma_6 = pong(\sigma_5) = \Sigma_3 + t_6 M_3$ and $\sigma_7 = ping(\sigma_6) = \Sigma_1 + t_7 M_1$ where

$$
\begin{aligned}
t_2 &= \frac{1 - \mathcal{M}(\Sigma_1, \Sigma_2) - t_1 \mathcal{M}(M_1, \Sigma_2)}{\mathcal{M}(\Sigma_1, M_2) + t_1 \mathcal{M}(M_1, M_2)}, \\
t_3 &= \frac{1 - \mathcal{M}(\Sigma_2, \Sigma_3) - t_2 \mathcal{M}(M_2, \Sigma_3)}{\mathcal{M}(\Sigma_2, M_3) + t_2 \mathcal{M}(M_2, M_3)}, \\
t_4 &= \frac{1 - \mathcal{M}(\Sigma_1, \Sigma_3) - t_3 \mathcal{M}(M_1, \Sigma_3)}{\mathcal{M}(\Sigma_1, M_3) + t_3 \mathcal{M}(M_1, M_3)}, \\
t_5 &= \frac{1 - \mathcal{M}(\Sigma_2, \Sigma_3) - t_4 \mathcal{M}(M_2, \Sigma_3)}{\mathcal{M}(\Sigma_2, M_3) + t_4 \mathcal{M}(M_2, M_3)}, \\
t_6 &= \frac{1 - \mathcal{M}(\Sigma_2, \Sigma_3) - t_5 \mathcal{M}(M_2, \Sigma_3)}{\mathcal{M}(\Sigma_2, M_3) + t_5 \mathcal{M}(M_2, M_3)}, \\
t_7 &= \frac{1 - \mathcal{M}(\Sigma_1, \Sigma_3) - t_6 \mathcal{M}(M_1, \Sigma_3)}{\mathcal{M}(\Sigma_1, M_3) + t_6 \mathcal{M}(M_1, M_3)}.
\end{aligned}
$$

By theorem 4.6 we know that there exists a family of Dupin cyclides if and only if $\sigma_7 = \sigma_1$ (see Figure 4.3), which is equivalent to the equation

$$t_7 = t_1.$$

Appendix A

Bézier Curves and Blossoms

The techniques described in this chapter are among the most important tools in computer aided geometric design (CAGD).
First of all we will study Bézier curves, defining them using Bernstein polynomials, and studying some of their properties. In the following section we introduce the De Casteljau algorithm and its relation with Bézier curves. If executing the De Casteljau algorithm, we perform each step using a different variable, we get to the concept of blossoming, studied in the last section.
For further studies in CAGD techniques, see [7], along with the vast number of references therein.

A.1 Bézier Curves

Polynomial Bézier Curves

Let $n \in \mathbb{N}$, we define the Bernstein polynomial of degree n by the equation:

$$B_i^n(t) = \left(\begin{array}{c} n \\ i \end{array} \right) t^i (1-t)^{n-i}.$$

Where $\left(\begin{array}{c} n \\ i \end{array} \right) = \frac{n!}{i!(n-i)!}$. Among the properties of the Bernstein polynomials is that of being a partition of unity, i.e, $\sum_{j=1}^{n} B_j^n(t) \equiv 1$ for every t. This fact follows from the Newton binomial theorem

$$1 = (t + (1-t))^n = \sum_{j=1}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) t^i (1-t)^{n-i} = \sum_{j=1}^{n} B_j^n(t).$$

Polynomials in Bernstein form have been widely studied in the CAGD context, see for example [7]. For our purposes Bernstein polynomials will be useful as a basis for Bézier curves.

Definition A.1. *Given $n + 1$ points $\mathbb{b}_0, \mathbb{b}_1, \ldots, \mathbb{b}_n$ in $\mathbb{R}^n$, that we will call control points, the Bézier curve defined by them is:*

$$\mathbf{B}^n(t) = \sum_{i=0}^{n} B_i^n(t)\mathbb{b}_i,$$

with $t \in [0, 1], \quad 0 \leq i \leq n.$

Note that the degree of a Bézier curve with $n+1$ control points is n, Bézier curves also have the property called endpoint interpolation, this is, they pass through $\mathbb{b}_0$ and $\mathbb{b}_n$, it is only necessary to evaluate the function in $t = 0$ and in $t = 1$ to see this. If we consider the polygon formed by the line segments joining $\mathbb{b}_i$ with $\mathbb{b}_{i+1}$ for $i = 0, n - 1$, called *Control Polygon*, the Bézier curve lies in the convex hull, defined by it, i.e., the set of all the barycentric combinations of the control points. This follows from the property that the Bernstein polynomials form a partition of the unity. In Figure A.1 we display four Bézier curves, using 4, 5, 6 and 7 control points, with their respective control polygons.

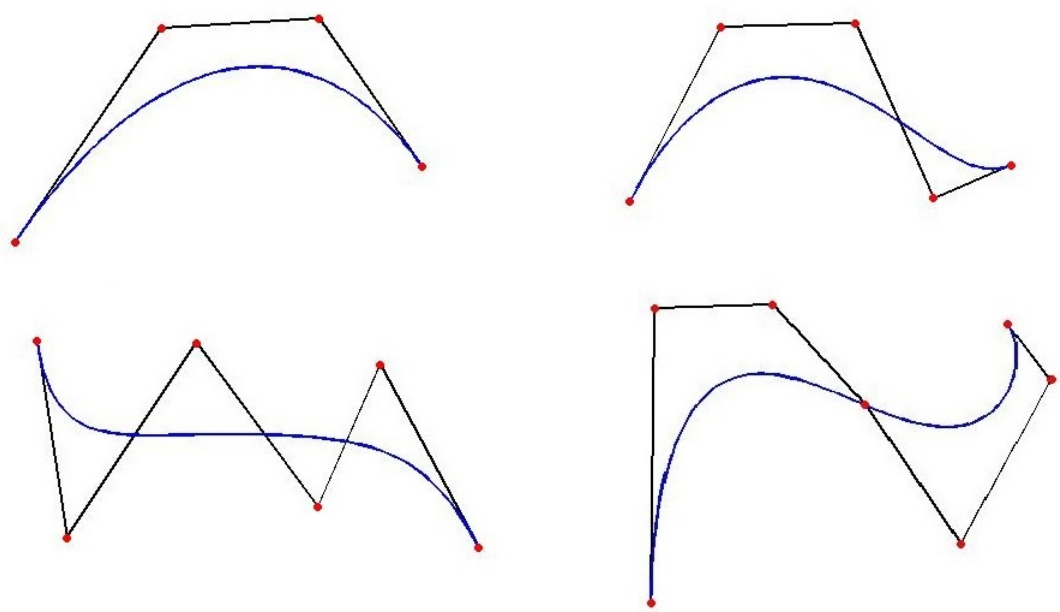


Figure A.1: Bézier Curves, in Blue, with 4, 5, 6 and 7 Control Points, in Red, and Their Respective Control Polygons, in Black.

Rational Bézier Curves

We need to be able to control the shape of the Bézier curve without adding control points. which would increase the degree of the polynomial. We also want to represent arbitrary conic sections as Bézier curves, but using the polynomial representation, the only conic section we can model is the parabola. In order to manipulate the Bézier curve without changing its degree we scale the Bernstein polynomials using weights w_i, i.e, we build new curves in terms of $w_i B_i^n(t)$. But if we build the Bézier curve using this new basis, we would loose important properties, such as the affine invariance. This is because the new basis is no longer a partition of unity. Hence we define the rational functions:

$$R_i^n(t) = \frac{w_i B_i^n(t)}{\sum_{i=0}^n w_i B_i^n(t)},\ t \in [0,1],\ 0 \leq i \leq n.$$

These constitute a partition of unity, and we define the rational Bézier curves as follows:

$$\mathbf{B}^n(t) = \sum_{i=0}^n R_i^n(t) \mathbb{b}_i,$$

or

$$\mathbf{B}^n(t) = \frac{w_0 \mathbb{b}_0 B_0^n(t) + w_1 \mathbb{b}_1 B_1^n(t) + \cdots + w_n \mathbb{b}_n B_n^n(t)}{w_0 B_0^n(t) + w_1 B_1^n(t) + \cdots + w_n B_n^n(t)}. \tag{A.1}$$

Equation A.1 is nothing but the projection of the curve in $\mathbb{R}^{n+1}$ (or $\mathbb{P}^n$) given by:

$$[w_0 \mathbb{b}_0 B_0^n(t) + w_1 \mathbb{b}_1 B_1^n(t) + \cdots + w_n \mathbb{b}_n B_n^n(t), w_0 B_0^n(t) + w_1 B_1^n(t) + \cdots + w_n B_n^n(t)]$$

into $\mathbb{R}^n$.

A.2 The De Casteljau Algorithm

We will describe the algorithm, studied by De Casteljau in 1959[1], this is perhaps the most fundamental tool in the field of CAGD, and it is so, probably for its amazing simplicity but powerful outcome.

Let us consider the simplest scenario, building a parabola using three points, $\mathbb{b}_0$, $\mathbb{b}_1$ and $\mathbb{b}_2$, for that purpose we define the curves

$$\mathbb{b}_0^1(t) = (1-t)\mathbb{b}_0 + t\mathbb{b}_1,$$
$$\mathbb{b}_1^1(t) = (1-t)\mathbb{b}_1 + t\mathbb{b}_2,$$

which are simply the lines joining $\mathbb{b}_0$ and $\mathbb{b}_1$, and $\mathbb{b}_1$ and $\mathbb{b}_2$, respectively, and then we join the points $\mathbb{b}_0^1(t)$ and $\mathbb{b}_1^1(t)$ by a line segment, see Figure A.2,

$$\mathbb{b}_2^0(t) = (1-t)\mathbb{b}_0^1 + t\mathbb{b}_1^1,$$

and this is the desired parabola.

[1]Paul De Casteljau worked at Citroën, hence the results he achieved where written in technical reports for his company, and are not accessible to everyone. It was only until 1975 when W. Boehm could read some of them, and were presented to the academic community.

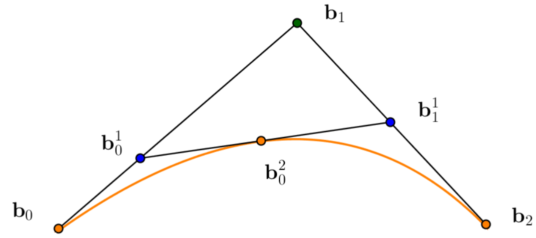


Figure A.2: De Casteljau Algorithm for Three Control Points.

For an arbitrary number n of control points the algorithm works exactly the same way as before, performing at first a linear interpolation between the control points, and then at each step a linear interpolation of the points found in the previous step.

IN: $\mathbb{b}_0, \mathbb{b}_1, \ldots, \mathbb{b}_n$ in $\mathbb{R}^n$
Set in the r-th step:

$$\mathbb{b}_k^i(t) = (1-t)\mathbb{b}_{k-1}^i(t) + t\mathbb{b}_{k-1}^{i+1}(t), \quad \begin{cases} k = 1, \cdots, n \\ i = 0, \cdots, n-k \end{cases}$$

OUT: $\mathbb{b}_0^n(t)$.

The superscript and the subscript denote the degree of the polynomial and the position in the auxiliary "control polygon" at each step, respectively. The very interesting fact about the De Casteljau algorithm is that it delivers the Bézier curve of degree n. The De Casteljau scheme shown below, corresponds to the cubic case, where $\mathbb{b}_0^3(t)$ is the Bézier curve with control points $\mathbb{b}_0$, $\mathbb{b}_1$, $\mathbb{b}_2$ and $\mathbb{b}_3$.

$$\begin{array}{llll} \mathbb{b}_0 & & & \\ \mathbb{b}_1 & \mathbb{b}_0^1 & & \\ \mathbb{b}_2 & \mathbb{b}_1^1 & \mathbb{b}_0^2 & \\ \mathbb{b}_3 & \mathbb{b}_2^1 & \mathbb{b}_1^2 & \mathbb{b}_0^3. \end{array}$$

A.3 Blossoms

The principle of blossoming is closely related to the De Casteljau algorithm, since it consists of performing the k-th step of the De Casteljau scheme with respect to a variable t_k instead of doing so for the variable t. The object obtained from this process will be called the blossom of degree n, we show below the De Casteljau scheme for a cubic blossom:

$$
\begin{array}{llll}
\mathbb{b}_0 & & & \\
\mathbb{b}_1 & \mathbb{b}_0^1(t_1) & & \\
\mathbb{b}_2 & \mathbb{b}_1^1(t_1) & \mathbb{b}_0^2(t_1,t_2) & \\
\mathbb{b}_3 & \mathbb{b}_2^1(t_1) & \mathbb{b}_1^2(t_1,t_2) & \mathbb{b}_0^3(t_1,t_2,t_3).
\end{array}
$$

Menelao's theorem states that blossoms are symmetric with respect to their variables, i.e, if $p(t_1, t_2, t_3)$ is a permutation of the variables t_1, t_2, t_3 we have that:

$$\mathbb{b}_0^n(t_1, t_2, t_3) = \mathbb{b}_0^n(p(t_1, t_2, t_3)).$$

And this can be generalized for n variables, performing the De Casteljau algorithm for $t_1, \cdots, t_n$ we get $\mathbb{b}_0^n(t_1, \cdots, t_n) = \mathbb{b}_0^n(p(t_1, \cdots, t_n))$.

From its definition it is clear that the curve acquired by substituting every variable $t_i,\ i = 1, \cdots, n$ for the variable t, it is nothing more than the Bézier curve of degree n. Another property of blossoms is multiaffinity, i.e., they are affinely invariant with respect to all of their variables, this follows from the property of affinely invariance of linear interpolation.

Let us consider three control points $\mathbb{b}_0$, $\mathbb{b}_1$ and $\mathbb{b}_2$, they form a quadratic blossom with equation:

$$\mathbb{b}_0^2(t_1, t_2) = \mathbb{b}(1 - t_1)(1 - t_2) + \mathbb{b}_{\mathbb{1}} \left[t_1(1 - t_2) + t_2(1 - t_1)\right] + \mathbb{b}_{\mathbb{2}} t_1 t_2$$

Evaluating in the points $(0, 0)$, $(1, 0)$ (which coincides with evaluating at $(0, 1)$) and $(1, 1)$ we recover the control points $\mathbb{b}_0$, $\mathbb{b}_1$ and $\mathbb{b}_2$. Furthermore if we restrict ourselves to the one dimension subset $(0, t_2)$ weobtain the tangent line to the Bézier curve at $\mathbb{b}_0$, and restricting to the subset $(t_1, 1)$ the tangent line at $\mathbb{b}_2$, and $\mathbb{b}_1$ is the intersection point of these two tangent lines (see, once again, figure A.2). In general the lines $\mathbb{b}_0^n(t, t_0)$ and $\mathbb{b}_0^n(s_0, t)$ are the tangent lines at the points $\mathbb{b}_0^n(t_0, t_0)$ and $\mathbb{b}_0^n(s_0, s_0)$, and their intersection is exactly the point $\mathbb{b}_0^n(s_0, t_0)$.

Appendix B

Envelopes and Inversion in $\mathbb{R}^3$

Generally, finding the envelope of a family of surfaces is not an easy problem, since it envolves solving a system, which is more often than not, nonlinear. We give a brief but concise study of the technique of Sylvester resultant, and as an example, we find the envelope's equation where the family of spheres is quadratic.
The properties of inversion (see [16]) are explored in the last section, and some important results involving it are proven.

B.1 Envelopes

Definition B.1. *Let $f(x, y, z, t) = 0$ be a 1-parameter family of surfaces with f a differentiable function. We called the envelope of the family to the surface $F(x, y, z) = 0$, where $F(x, y, z)$ is the solution, if exists, of the system*

$$\begin{aligned} f(x, y, z, t) &= 0 \\ \tfrac{df}{dt}(x, y, z, t) &= 0. \end{aligned} \tag{B.1}$$

Sylvester resultant is an important tool for finding the envelope of a family of surfaces when the dependence on the parameter t is given in a polynomial fashion, i.e., $f(x, y, z, t) = a_0(x, y, z) + a_1(x, y, z)t + \cdots + a_n(x, y, z)t^n$. Since it finds the common root of two given polynomials[1].
Let us find the common root of two given polynomials. This root shall be the solution of the system

$$\begin{aligned} a_0 + a_1 t + \cdots + a_n t^n &= 0 \\ b_0 + b_1 t + \cdots + b_m t^m &= 0. \end{aligned} \tag{B.2}$$

[1]For an extensive study of Sylvester resultants see the book *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Conmutative Algebra,* David Cox, John Little and Donald O'Shea, pp. 152-155

The theory of Sylvester resultants states that this system has a solution *if and only if* the equation $detA = 0$ has a solution, where A is the $n+m$ square matrix

$$\begin{pmatrix} a_0 & \cdots & a_n & 0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_n & 0 & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & 0 & 0 & a_0 & \cdots & a_n \\ b_0 & \cdots & b_m & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_m & 0 & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & 0 & 0 & b_0 & \cdots & b_m \end{pmatrix}.$$

It is also known that the solution of the equation $det(A) = 0$ gives us conditions over the coefficients in B.2 for the system to have a solution.

Let us now suppose that we have a quadratic monoparametric family of surfaces, this means, a family that depends quadratically on the parameter t, $f(x, y, z, t) = a(x, y, z)t^2 + 2b(x, y, z)t + c(x, y, z)$. If we assume that $a(x, y, z), b(x, y, z)$ and $c(x, y, z)$ are constant (or simply fix a generic point (x, y, z)), we can find an equation to its envelope, given by the Sylvester resultant of the system B.1

$$det \begin{pmatrix} a & b & c \\ 2a & 2b & 0 \\ 0 & 2a & 2b \end{pmatrix} = 0.$$

Calculating this determinant we have the equation $-4a(b^2 - ac) = 0$, whose solutions are $a = 0$ and $b^2 - ac = 0$, the first one is just the trivial solution (since $a = 0$ makes the matrix singular, and $b = c = 0$), therefore the geometrically meaningful part of the equation of the envelope is:

$$[b(x, y, z)]^2 - a(x, y, z)c(x, y, z) = 0. \tag{B.3}$$

B.2 Inversion in $\mathbb{R}^3$

Given a sphere Σ with center c and radius r, and a point p in $\mathbb{R}^3 \setminus \{c\}$ we define the **inverse of** p **in** Σ, or **inverse of** p **with respect to** Σ, denoted by $T_\Sigma(p)$ to be the point p' on the ray from c to p such that:

$$||cp|| \cdot ||cp'|| = r^2.$$

We call Σ the circle of inversion and c the center of inversion.

Note that any point p in $\mathbb{R}^3 \setminus \{c\}$ is the inverse in Σ of $T_\Sigma(p)$, which means that $T_\Sigma^2 = I_{\mathbb{R}^3\setminus\{c\}}$, note also that the points belonging to the sphere Σ are invariant under T_Σ.

An inversion in an arbitrary sphere is conjugate under dilation and traslation in $3D$ to any other inversion, therefore we will restrict ourselves, without losing generality, to the inversion with respect to S^2 and call $T_{S^2}(p)$ the inverse of p if there is no room for confusion.

Theorem B.2. *$T_{\mathbb{S}^2}$ preserves angles but not their orientation.*

Proof. A map preserves angles if its jacobian matrix is a scalar multiple of an orthogonal matrix and reverses orientation if it has negative determinant (see [11, pages 130 and 271]). Since $T_{\mathbb{S}^2}(\mathbb{x}) = \frac{\mathbb{x}}{||\mathbb{x}||^2}$ its jacobian matrix at a point $\mathbb{x} = (x, y, z)$ is given by

$$\mathcal{J} = \frac{1}{||\mathbb{x}||^4}\begin{pmatrix} ||\mathbb{x}||^2 - 2x^2 & -2xy & -2xz \\ -2xy & ||\mathbb{x}||^2 - 2y^2 & -2yz \\ -2xz & -2yz & ||\mathbb{x}||^2 - 2z^2 \end{pmatrix}.$$

Therefore $\mathcal{J}\mathcal{J}^T = \frac{1}{||\mathbb{x}||^4} I_3$ and $det(\mathcal{J}) = -\frac{1}{||\mathbb{x}||^2}$, which proves the result. ∎

Theorem B.3. *T_{S^2} maps spheres/planes into spheres/planes.*

Proof. Let C be the sphere with center $\mathbb{a}$ and radius r.
Case 1: Inverting the sphere C:
a) O belongs to C.
If $x \in C$ then $||\mathbb{x} - \mathbb{a}|| = r$ in particular we have that $||\mathbb{a}|| = ||O - \mathbb{a}|| = r$:

$$\begin{aligned} r^2 &= ||\mathbb{x} - \mathbb{a}||^2 \\ &= (\mathbb{x} - \mathbb{a}) \cdot (\mathbb{x} - \mathbb{a}) \\ &= \mathbb{x} \cdot \mathbb{x} - 2\mathbb{x} \cdot \mathbb{a} + \mathbb{a} \cdot \mathbb{a} \\ &= ||\mathbb{x}||^2 - 2\mathbb{x} \cdot \mathbb{a} + ||\mathbb{a}||^2 \\ 0 &= ||\mathbb{x}||^2 - 2\mathbb{x} \cdot \mathbb{a} \\ ||\mathbb{x}||^2 &= 2\mathbb{x} \cdot \mathbb{a} \\ \frac{\mathbb{x} \cdot \mathbb{a}}{||\mathbb{x}||^2} &= 1/2 \\ \frac{\mathbb{x}}{||\mathbb{x}||^2} \cdot \mathbb{a} &= 1/2. \end{aligned}$$

But $T_{S^2}(\mathbb{x}) = \frac{\mathbb{x}}{||\mathbb{x}||^2}$ therefore if $\mathbb{x} \in C$, $\quad T_{S^2}(\mathbb{x}) \cdot \mathbb{a} = 1/2$ which is the equation of a plane (see figure B.1).

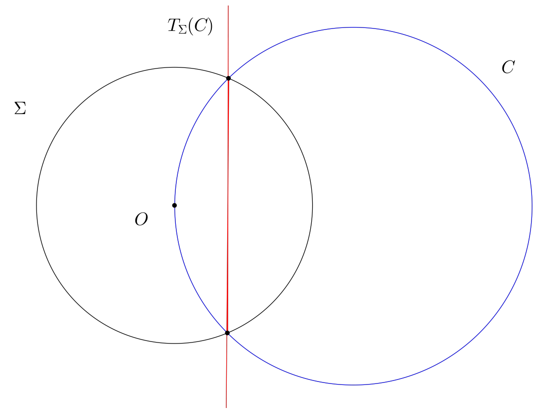


Figure B.1: Inverse of a Sphere Through O.

b) O does not belong to C.

Let $R = (x, y, z)$ be a point lying in a sphere C not passing through the origin and let (u, v, w) be its image under $T_{\mathbb{S}^2}$, i.e., $(u, v, w) = \frac{1}{x^2+y^2+z^2}(x, y, z)$. Knowing this we have that $||(u, v, w)|| = \frac{1}{||R||}, ||R|| = \frac{1}{||(u,v,w)||}$ and $||R||^2(u, v, w) = (x, y, z)$. On the other hand, we also know that C is given by the equation

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0,$$

where d is a nonzero real number. The above equation in terms of u, v and w transforms into

$$\frac{1}{u^2 + v^2 + w^2} + 2||R||^2(au + bv + cw) + d = 0,$$

multiplying by $||(u, v, w)||^2$ we get

$$1 + 2(au + bv + cw) + d(u^2 + v^2 + w^2) = 0,$$

which represents a sphere of equation

$$u^2 + v^2 + w^2 + 2a'u + 2b'v + 2c'w + d' = 0,$$

with $a' = \frac{a}{d}$, $b' = \frac{b}{d}$, $c' = \frac{c}{d}$ and $d' = \frac{1}{d}$.

Case 2: Now we invert the plane π with normal vector $\mathbb{a}$:

a) O belongs to π.

Since $T_{S^2}(\mathbb{x})$ is in the ray from O to $\mathbb{x}$ and both points lie in π, $T_{S^2}(\mathbb{x}) \in \pi$ hence T_{S^2} leaves π invariant, but not in a pointwise fashion.

b) O does not belong to π.

Note that this case is covered by case 1a, knowing that an inversion is its own inverse. ■

Remark B.4. *Let Σ be the sphere in $\mathbb{R}^n$ centered at the north pole of $\mathbb{S}^{n-1}$ with radius 2, a straightfoward computation shows that the stereographic projection is the inversion with respect to Σ restricted to $\mathbb{S}^{n-1}$. see Figure B.2.*

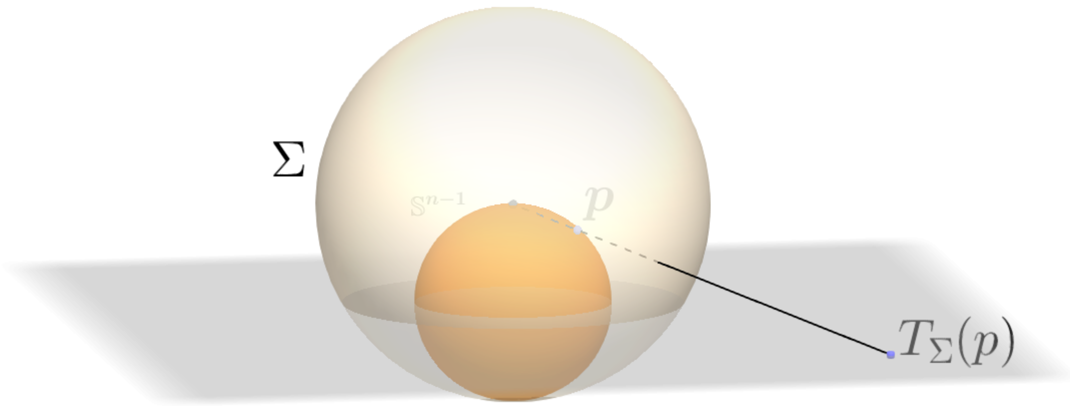


Figure B.2: The Stereographic Projection as an Inversion with Respect to Σ.

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