NEUTRINO PHENOMENOLOGY IN A LITTLE HIGGS MODEL WITH GAUGE SYMMETRY $SU(3)_c \otimes SU(4)_L \otimes U(1)_X$

MASTER OF SCIENCE THESIS

by
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To my parents.

A mis padres.

“A person who never made a mistake never tried anything new”
Albert Einstein

Abstract

We carry out a search for possible mechanisms that explain the smallness of neutrino masses in the context of an anomaly-free Little Higgs model with electroweak gauge symmetry $SU(4)_L \otimes U(1)_X$. By the introduction of one neutral right-handed neutrino per generation, it is shown that the leading order contribution to the lightest neutrino masses comes at the one loop level. The model also leads to small corrections on the Higgs mass, that are negligible in comparison with the one-loop log-divergent contribution coming from the gauge bosons. Our model leads to unsuppressed Yukawa couplings in the neutral leptonic sector, providing an answer for the observable neutrino masses without fine-tuning of Yukawa couplings. The Majorana mass for the new neutral lepton is suppressed by the energy scale ~ 10 TeV which represents the UV cut-off.

Resumen

Llevamos a cabo una búsqueda de posibles mecanismos para explicar la pequeñez de la masa de neutrinos en el contexto de un modelo Little Higgs libre de anomalías con simetría gauge electrodébil $SU(4)_L \otimes U(1)_X$. Introduciendo un neutrino derecho por generación, se muestra que la contribución dominante a la masa de neutrinos livianos es dada a orden de un loop. El modelo da lugar a pequeñas correciones a la masa del Higgs, las cuales son despreciables en comparación con las contribuciones logaritmicamente divergentes provenientes de los bosones gauge. Adicionalmente no existe supresión sobre los acoples de Yukawa del sector leptónico neutro, proporcionando una respuesta a las masas observadas para los neutrinos sin hacer ajustes sobre los acoples de Yukawa. Las masas de Majorana para el neutrino de quiralidad derecha es suprimida por el cut-off de la teoría.

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Content

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Introduction

The understanding of the physics behind neutrinos masses and mixing has been, and still remains, one of the major focus of research in particle physics. In the Standard Model (SM) neutrinos are massless for two independent reasons: first, the absence of right-handed neutrinos avoids to build a Dirac mass term for neutrinos, and second, as Lepton Number is exactly conserved then a Majorana mass term is forbidden. Despite the SM has been very successful in describing most of elementary particle phenomenology up to energies that has been probed so far, there are both theoretical and experimental reasons that suggest it is not the ultimate theory of Nature. In addition to the neutrino problem, shortcomings such as the replications of fermions in Nature and the fact that quadratically divergent corrections to the Higgs boson mass m_H^2 destabilize the electroweak scale, remain as puzzles to be solved. Even after considering the Planck scale as the natural cut-off of the SM, the current experimental data suggest that the SM remains valid up to ~ 5 TeV [1], however, in order to avoid large radiative corrections to the Higgs mass, new physics is expected at or below ~ 1 TeV. This latter issue is known as the *little hierarchy* problem. Such an inconvenience has been one of the motivations to look for new physics beyond the SM.

Supersymmetric extensions of the SM [2] arise as theories in which the dangerous radiative contributions to the Higgs mass are cancelled between the SM particles and their super-partners. Additionally, lead to gauge coupling unification and also account for a dark matter candidate in the Universe when R-parity conservation is imposed. Despite of the theoretical success of Supersymmetry (SUSY), nothing new has been found in the Large Hadron Collider (LHC) until now and, with the current lower limit[1] on the sparticles masses [3], if SUSY is a fundamental symmetry of Nature, fine-tuning is required to stabilize the electroweak scale.

A theory that provides an answer for both the *little hierarchy* problem and the replication of fermions in Nature is the Simplest Little Higgs Model (SLHM)[2] based on the approximate global $[SU(4)/SU(3)]^4$ symmetry [6]. In such a model, the Higgs arises as a Pseudo-Nambu-Goldstone Boson(PNGB) after the spontaneous breaking of the global $[SU(4)]$ symmetry. In order to trigger the Electroweak Symmetry Breaking (EWSB), a special breaking pattern

[1] Many of the supersymmetry searches rely on the missing energy signature as an indication of new physics. The analysis of the data accumulated until now is still ongoing.

[2] Little Higgs models have also a high degree of fine-tuning, what leaves SUSY and this new approach on the same ground. An analysis of fine-tuning in LH models has been carried out [4], and a strong critique against LH models was done by H. Georgi [5].

called *collective symmetry breaking* is implemented [7]. The one-loop quadratic divergences to the squared Higgs mass m_H^2 are cancelled between particles of the same spin, and the two-loop divergences are negligible. In this SLHM the $SU(2)_L \otimes U(1)_Y$ electroweak gauge symmetry is extended to $SU(4)_L \otimes U(1)_X$ [6]. This electroweak extension can provide an explanation for the replication of fermions in Nature when the cancellation of anomalies takes place between families [8] (this is achieved by embedding the two first generations of quarks into the anti-fundamental representation of $SU(4)_L$ while the third generation of quarks and all the three generations of leptons are embedded into the fundamental representation).

Neutrino physics is one of the most rapidly developing areas of particle physics. Solar [9–12], atmospheric[13, 14] and reactor experiments [15–18] have shown compelling evidences that support the idea of neutrino oscillations (neutrinos transform one into another). If such a phenomenon happens the neutrinos must be massive particles, and models to explain their masses and mixing are required. As a first attempt, in 1979 Steven Weinberg built up, in the context of the SM, the unique non-renormalizable dimension five effective operator [19] which potentially could explain the tiny value of neutrino masses assuming, first, that lepton number is not conserved and, second, that neutrino masses appear as low energy effects of a high energy scale associated to one (or several) unknown field(s)[3]. The tree-level realization of the Weinberg operator yields to the well-known seesaw mechanisms. The Type I [20], Type II [21], and Type III [22] seesaw mechanisms corresponds to the inclusion of a right-handed neutrino, a scalar triplet and a fermion triplet, respectively. Even thought the Type I seesaw mechanism gives an explanation for the masses of neutrinos, such a theory predicts the existence of new particles far beyond the electroweak scale with energies $\sim 10^{15}$ GeV, too heavy to be observed experimentally with any (current and future) accelerators. There are also different scenarios where the neutrinos acquire their mass radiatively: loop suppression factors and a *natural* Yukawa coupling allow to explain the smallness of neutrino masses in comparison with the charged leptons and, in addition, new heavy fields with energies at the TeV scale are predicted, what makes this kind of scenarios testable in the forthcoming experiments [See for instance Refs [23–27]].

With the recent measurement of θ_{13} at more than 5 σ C.L. by DAYA-BAY [28] and RENO [29] collaborations, a new window for a better understanding of neutrino mixing is open, as well as the possibility of testing CP violation in the lepton sector.

Taking into consideration the success of the approximate global $[SU(4)/SU(3)]^4$ non-linear sigma model both in describing physics at low-energies and in providing a *solution* to the two theoretical difficulties of the SM mentioned above, the exploration of neutrino masses and mixing is the next step.

This thesis is organized as follows. In Chapter I, some of the best known mechanisms studied in literature for neutrino mass generation are reviewed. A general introduction of the Simplest Little Higgs Models based on the global $[SU(4)/SU(3)]^4$ symmetry is given in Chapter II. In Chapter III, the mechanism for neutrino mass generation in the context of this SLH Model is studied. Finally, the conclusions are outlined in Chapter IV. Two appendices

[3]Depending whether this new interaction has a tree- or loop-level realization

are also included; the first one contains the mathematical details of the loop calculations associated to the analysis in Chapter III, while in the second one a study of the issue of gauge coupling unification in the $SU(3)c \otimes SU(4)_L \otimes U(1)_X$ and the $SU(4)c \otimes SU(2)_L \otimes U(1)_X$ extensions of the SM is presented. This last study was done in parallel to the main goal of this thesis.

Chapter 1

Models for Neutrino Masses

Solar, atmospheric and reactor neutrino experiments have indicated that neutrinos do have masses. Since the birth of the Standard Model (SM) the quest for the understanding the neutrino began and by now it is well known from the current experimental data that only the left-handed (LH) neutrinos ν_L (as well as right-handed (RH) antineutrinos $\overline{\nu}_L$) are produced in weak interaction processes. In the SM neutrinos are massless due to the absence of RH neutrinos ν_R. However, if RH neutrinos (as well as LH antineutrinos $\overline{\nu}_R$) exist in Nature, their interaction with matter should be much weaker than the weak interaction of left-handed neutrinos. As a consequence, RH neutrinos would not *feel* weak interactions and also do not possess color charge, then they must transform as a singlet under the SM $G_{SM} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge group. In other words, it means that they have not gauge interaction[1]. If in addition to the SM particle content it is assumed the existence of hypothetical new fields (right-handed neutrinos, a 4th generation of fermions, new scalars *etc.*), these could play a crucial role in the neutrino mass generation (all models that include massive neutrinos, are of this type).

For any Dirac particle ψ, a mass term is given by [30]:

$$\begin{aligned} -\mathcal{L}^{D}_{Mass} &= m\overline{\psi}\psi \\ &= m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L), \end{aligned} \tag{1.1}$$

where the relation $\psi = \psi_L + \psi_R$, has been used. However such a term is not invariant under G_{SM}, and therefore forbidden. The way to explain the mass generation in the SM relies in the Higgs Mechanism [31–33], where a single Higgs doublet $H = (H_1^+, H_1^0)^T$ is responsible for generating all fermion masses through Yukawa couplings. All known fermions (except neutrinos) acquire masses after the electroweak spontaneous symmetry breaking (EWSB) takes place:

[1] They would not couple to the weak $W^{\pm}$, Z^0, gluons and photon bosons.

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \stackrel{v\,\sim\,256\,\text{GeV}}{\longrightarrow} SU(3)_C \otimes U(1)_Q. \tag{1.2}$$

The interaction between fermions and the fundamental scalar, known as the *Yukawa interaction* has the form:

$$-\mathcal{L}_{Yuk} = \lambda \overline{\psi}_R H^\dagger \psi_L + h.c, \tag{1.3}$$

being λ the Yukawa coupling that measures the strength of the interaction between the scalar and the fermion. When the Higgs acquire a vacuum expectation value, $\langle H \rangle = v/\sqrt{2}$, the previous equation becomes

$$-\mathcal{L}_{Yuk} = \lambda \frac{v}{\sqrt{2}} \overline{\psi}_R \psi_L + h.c, \tag{1.4}$$

Comparing Eq. (1.1) with Eq. (1.4), we find that after the EWSB, the fermion ψ acquire a mass: $m_D = (\lambda v)/\sqrt{2}$. However this mass term require the existence both of the right-handed and the left-handed components of the fermion field ψ. This mass term is called a *Dirac mass term.* This mechanism generates a mass term for each fermion in the SM, except for the neutrino because there are not right-handed neutrinos in the SM. Do RH neutrinos exist in Nature?; many extensions of the SM provides an answer to the neutrino puzzle as well as many other shortcomings (electroweak hierarchy problem, dark matter, gauge couplings unification, *etc.*) just by extending either the fermion or the scalar particle content. Since a Majorana, ψ, particle can be its own antiparticle, it must have zero electric charge [34]. This implies that $\psi^c = \psi$ (here, c stands for the charge conjugation operator), where the phase term has been neglected. We can write again Eq. (1.1) for a Majorana field, in the next form

$$-\mathcal{L}^M_{Mass} \;=\; \frac{1}{2} m \overline{\psi^c} \psi + h.c., \tag{1.5}$$

this is called a *Majorana mass term.*
At this stage, neutrinos can be either Dirac or Majorana particles. However in the Standard Model [34].

- Dirac mass terms are forbidden due to the absence of right-handed neutrinos.
- If lepton number conservation is imposed, then Majorana mass term is also forbidden.

The field content of the SM consists of three types of particles: fermions (spin-1/2 fields), gauge bosons (spin-1 fields), and scalars (spin-0 fields). The particle content is summarized

		$(SU(3)_C, SU(2)_L, U(1)_Y)$	$U(1)_Q$
Quarks	$Q_L^\alpha = \begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L$	$(\mathbf{3}, \mathbf{2}, 1/3)$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$
	u_R^α	$(\bar{\mathbf{3}}, \mathbf{1}, 4/3)$	$2/3$
	d_R^α	$(\bar{\mathbf{3}}, \mathbf{1}, 2/3)$	$-1/3$
Leptons	$L_L^\alpha = \begin{pmatrix} \nu^\alpha \\ l^\alpha \end{pmatrix}_L$	$(\mathbf{1}, \mathbf{2}, -1)$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	l_R^α	$(\mathbf{1}, \mathbf{1}, -2)$	-1
Higgs	$H = \begin{pmatrix} H_1^+ \\ H_1^0 \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, 1)$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Gauge bosons	G_μ^α	$(\mathbf{8}, \mathbf{1}, 0)$	0
	W_μ^i	$(\mathbf{1}, \mathbf{3}, 0)$	$(0, \pm 1)$
	B_μ	$(\mathbf{1}, \mathbf{1}, 0)$	0

Table 1.1: The particle content of the SM. α corresponds to a generation index and each up-type quark u^α and down-type one d^α, carries also color charge.

in Table 1.1. Fermions come in three generations and are naturally massless due to a chiral symmetry.

All charged fermions (matter fields) acquire masses after a complex scalar field (The Higgs boson) develops a non zero vacuum expectation value (VEV).
In order to explain neutrino masses, it is necessary go beyond the SM. In the next sections we review several mechanisms widely studied, which allow neutrinos to be massive particles.

1.1 Seesaw mechanism

The first attempt to explain the smallness of the neutrino mass relies in the idea of extend either the fermion or the scalar content of the Standard Model. First of all, let us add a RH neutrino N_R per generation [20].
The Yukawa Lagrangian involving the usual LH doublet $L_L = (\nu_L, l_L)^T$, the scalar field $H = (H_1^+, H_1^0)^T$ and the new RH neutrino looks like

$$-\mathcal{L}_{Yuk} = \lambda \overline{L}_L \widetilde{H} N_R + h.c, \tag{1.6}$$

where $\widetilde{H} = i\tau_2 H^*$ (being τ_2 the second Pauli matrix). After the EWSB, Eq. (1.6) leads to a Dirac mass term

$$-\mathcal{L}^D_{Mass} = \overline{\nu}_L m_D N_R + h.c. \tag{1.7}$$

being $m_D = \lambda v/\sqrt{2}$. Now, as pointed out before, we can also add a Majorana mass term for N_R,

$$- \mathcal{L}^M_{Mass} = \frac{1}{2}\overline{N^c_R} m_R N_R + h.c.. \tag{1.8}$$

Denoting $n_L = (\nu_L, N^c_R)^T$, the full Lagrangian is written as

$$\begin{aligned} \mathcal{L}_{Mass} &= \mathcal{L}^M_{Mass} + \mathcal{L}^D_{Mass} \\ &= \frac{1}{2}\overline{n^c_L} M n_L, \end{aligned} \tag{1.9}$$

where

$$M = \begin{pmatrix} 0 & m_D \\ m_D^T & m_R \end{pmatrix}. \tag{1.10}$$

After diagonalization we find

$$m_{2,1} = \frac{1}{2}\Big(m_R \pm \sqrt{m_R^2 + 4m_D^2}\Big). \tag{1.11}$$

If we demand that m_1 is comparable to the charged leptons masses then, in order to explain the low experimental upper limit on the neutrino mass, m_2 must be close to the GUT scale. This means $m_2 \sim 10^{15}$ GeV. With this mechanism, the smallness of neutrino masses is a consequence of the heaviness of the right-handed neutrinos. Such a mechanism is called *the type I seesaw mechanism*.

There exist three realizations of the seesaw mechanism at tree-level [23] which are shown in Fig 1.1. These are based on the fact that two $SU(2)$ doublets can be decomposed into a singlet and a triplet ($\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$).

In 1979 Weinberg proved that the symmetries of the SM allow only one (unique) dimension-five effective operator [19]:

$$\mathcal{L}_\Lambda = \frac{1}{2} f_{\alpha\beta}\Big(\overline{L^c_{L\alpha}}\widetilde{H}^*\Big)\Big(\widetilde{H}^\dagger L_{L\beta}\Big) + h.c., \tag{1.12}$$

where $f_{\alpha\beta}$ is a coefficient suppressed by an energy scale Λ (associated to the existence of new heavy fields) and its calculation depends of the degree of realization of the operator (depending whether the realization is either at tree-level or at loop-level).

The seesaw mechanisms (type I [20], type II [21] and type III [22]) are the three realizations of the dimension-five effective operator at tree-level. This effective theory has an analogy

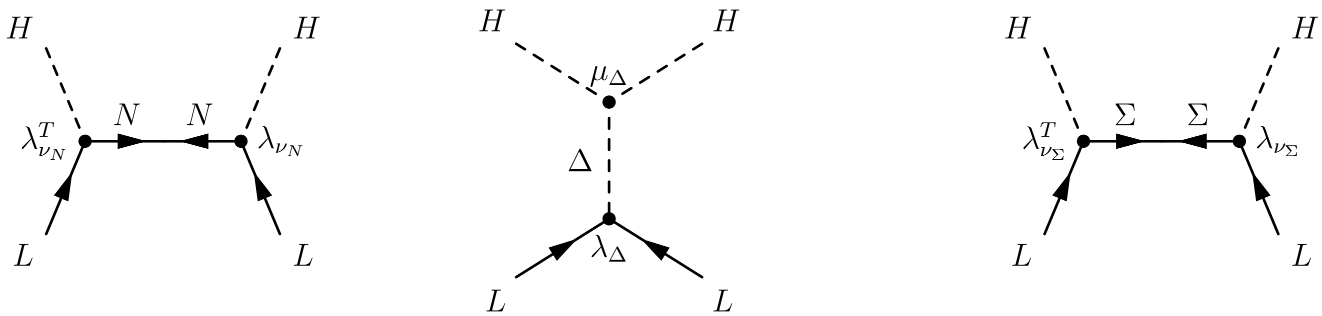

Figure 1.1: The three realizations of the seesaw mechanism: Type I (left), Type II (in the middle) and Type III (right). The massive particle exchanged corresponds to a fermion singlet $N_R \sim (\mathbf{1}, \mathbf{1}, 0)$, a scalar triplet $\Delta \sim (\mathbf{1}, \mathbf{3}, -2)$ and a fermion triplet $\Sigma \sim (\mathbf{1}, \mathbf{3}, 0)$, respectively.

with the *four-fermion point interaction* proposed by Fermi in the early 30's to explain the β-decay. This proposal describes, at low energies, the *weak interaction* without $W^\pm$ or Z bosons. In Fig. 1.2 we show both cases: on the left the *Fermi's four-point interaction*, and on the right the Feynman diagram associated to the effective operator given in Eq. (1.12). To date we know that the intermediate particles responsible for the weak interaction (and, as a consequence, for the β-decay) are the $W^\pm$ or Z bosons; however is still experimentally unknown if there exist a heavy field that mediates the interaction drawn on the right of Fig. 1.2. If we assume that the realization of the Weinberg operator is at tree-level, then there exist only three different types of fields that could mediate the *interaction*, these are shown in the Feynman diagram in Fig. 1.1.

In what follows, we discuss briefly each one of the tree-level realizations of the Weinberg effective Lagrangian.

Figure 1.2: Effective four-point theories: Fermi's four-point diagram (left), Weinberg's four-point diagram (right)

1.1.1 Type I/III Seesaw

We can compute the *"effective mass"*[2] directly from the Feynman diagram or by using the effective Lagrangian.

$$\mathcal{L}_\Lambda = \frac{1}{2} f_{\alpha\beta} \left(\overline{L^c_{L\alpha}} \widetilde{H}^* \right) \left(\widetilde{H}^\dagger L_{L\beta} \right) + h.c.. \tag{1.13}$$

After the EWSB, the Higgs field acquires a non-zero VEV: $\langle H \rangle = v/\sqrt{2}$, the previous equation becomes:

$$\mathcal{L}_\Lambda = \frac{1}{2} (\mathcal{M}_\nu)_{\alpha\beta} \overline{\nu^c_L} \nu_L, \tag{1.14}$$

with

$$(\mathcal{M}_\nu)_{\alpha\beta} = f_{\alpha\beta} v^2, \tag{1.15}$$

that is, a Majorana mass. A simple evaluation shows that, in order to obtain $m_\nu < 1$ eV, then $(f_{\alpha\beta})^{-1} > 10^{15}$ GeV. However, what is $f_{\alpha\beta}$?; with the aim of giving an answer, let us considerer the most general Yukawa Lagrangian already written in Eq. (1.9) for the type I seesaw.

$$-\mathcal{L}_{yuk} = \lambda \overline{L}_L \widetilde{H} N_R + \frac{1}{2} \overline{N^c_R} m_R N_R + h.c. \tag{1.16}$$

After the EWSB, and in matrix form:

$$-\mathcal{L}_{yuk} = \frac{1}{2} \begin{pmatrix} \overline{\nu_L} & \overline{N^c_R} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D^T & m_R \end{pmatrix} \begin{pmatrix} \nu^c_L \\ N_R \end{pmatrix} + h.c. \tag{1.17}$$

Now if $m_D << m_R$, by block diagonalization we obtain:

$$\begin{aligned} \mathcal{M}_{\nu_L} &\simeq -m_D m_R^{-1} m_D^T, \\ \mathcal{M}_{N_R} &\simeq m_R. \end{aligned} \tag{1.18}$$

From Eq. (1.15) and Eq. (1.18), we find that $f_{\alpha\beta}$ has the form:

$$f_{\alpha\beta} = -\frac{1}{2} \frac{\lambda \lambda^T}{m_R}, \tag{1.19}$$

[2] I give it this name because it comes from an effective Lagrangian.

is found that in order to obtain $M_{\nu L} \sim 1$ eV, $m_R \sim 10^{15}$ for Yukawa couplings $\mathcal{O}(\lambda) \sim 1$.
The same procedure should be done for the type III seesaw. By introducing a fermion triplet $\Sigma_R \sim (\mathbf{1}, \mathbf{3}, 0)$ to the SM, the most general Yukawa Lagrangian involving this new field and the neutrino has the form

$$- \mathcal{L}_{yuk} = \overline{L}_L \lambda_\Sigma \big(\overrightarrow{\Sigma}.\overrightarrow{\tau}\big) \widetilde{H} + \frac{1}{2} \overline{\overrightarrow{\Sigma}^c} m_\Sigma \overrightarrow{\Sigma} + h.c. \tag{1.20}$$

After the EWSB and in matrix form the previous equation yields:

$$- \mathcal{L}_{yuk} = \frac{1}{2} \begin{pmatrix} \overline{\nu_L} & \overline{\Sigma_3^c} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D^T & m_{\Sigma_3} \end{pmatrix} \begin{pmatrix} \nu_L^c \\ \Sigma_3 \end{pmatrix} + h.c., \tag{1.21}$$

with $\overrightarrow{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$. This mechanism leads to the same result of type I seesaw after we block diagonalize the mass matrix. Besides, adding a fermion triplet instead a fermion singlet, yields additional phenomenology. For instance, this model predicts the existence of two charged fermions $\Sigma^+ \equiv (\Sigma_1 - i\Sigma_2)/\sqrt{2}$ and $\Sigma^- \equiv (\Sigma_1 + i\Sigma_2)/\sqrt{2}$ with masses $M_{(\Sigma^+, \Sigma^-)} >$ 100.8 GeV [3] (lower experimental bound on its mass). The fact that they have interaction with the gauge bosons, can lead to new processes like $\Sigma^\pm \to l^\pm \nu$.

1.1.2 Type II Seesaw

Including the scalar triplet $\Delta \sim (\mathbf{1}, \mathbf{3}, 2)$:

$$\Delta = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta_0 & -\Delta^+/\sqrt{2} \end{pmatrix}, \tag{1.22}$$

the relevant Lagrangian is written as:

$$- \mathcal{L}_\Delta = \Big(\overline{\widetilde{L}}_L \lambda_\Delta \Delta L_L + h.c.\Big) + V(H, \Delta), \tag{1.23}$$

with

$$V(H, \Delta) = m_\Delta^2 tr\{\Delta \Delta^\dagger\} + (\mu_\Delta \widetilde{H}^\dagger \Delta^\dagger H + h.c.). \tag{1.24}$$

In order to guarantee Lepton Number (LN) conservation in Eq. (1.23), we assign LN $= -2$ to Δ, but this implies the LN is violated explicitly by the *μ-term* in Eq. (1.24).
After the EWSB, and allowing to the scalar triplet to develop a non-vanishing VEV in the neutral direction $\langle \Delta \rangle = v_\Delta$, the term relevant for neutrino masses given in Eq. (1.23) acquires the form:

$$-\mathcal{L}_{Mass} \quad = \quad \lambda_\Delta v_\Delta \overline{\nu_L^c} \nu_L, \tag{1.25}$$

then

$$\mathcal{M}_\nu \quad = \quad 2\lambda_\Delta v_\Delta. \tag{1.26}$$

From Eq. (1.24), after the two scalars acquire a non-zero VEV and ensuring a minimum value for $V(H, \Delta)$, we find

$$v_\Delta \quad = \quad -\frac{\mu_\Delta v^2}{4m_\Delta^2}, \qquad \text{for} \quad m_\Delta >> m_H. \tag{1.27}$$

Taking into account the two previous equations, the expression for the neutrino mass looks like

$$\mathcal{M}_\nu \quad = \quad -\frac{\lambda_\Delta \mu_\Delta v^2}{2m_\Delta^2}, \tag{1.28}$$

Doing a comparison between this result and the outcome from the effective dimension-five operator in Eq. (1.12), we find:

$$f_{\alpha\beta} \quad = \quad -\frac{\lambda_\Delta \mu_\Delta}{m_\Delta^2}. \tag{1.29}$$

As it happens in the type III seesaw, the type II seesaw has a very rich phenomenology. First of all, the existence of the new field Δ, still undiscovered, modifies the ρ *parameter* of the SM putting a strong constraint on the VEV of this new scalar: $v_\Delta \leq 3$ GeV [3]. An interesting feature of this seesaw is that leads to lepton number violating processes in the charged sector such as $\mu \to e\gamma$, $\tau \to \mu\gamma$ *etc.* Additionally, the existence of a new field Δ^{++} with exotic electric charge[3] has very low theoretical motivation because until now all known elementary particles in Nature have ordinary electric charges.
In the next section we are going to review briefly the most famous *radiative* models that account for neutrino masses.

[3]By exotic I means the fact that all other charges for the elementary particles are either 0 or $\pm|Q_e|$ for leptons, scalars and vectors fields, and $(1/3)|Q_e|$, $(2/3)|Q_e|$ for up-type and down-type quarks, respectively.

1.2 Radiative Models

One of the greatest mysteries yet to be unravelled in the SM is associated to the hierarchy between the masses of all known fermions. All charged fermions acquire mass through Yukawa couplings with the Higgs boson after the EWSB takes place, however the neutrino remains massless in the SM. The type I seesaw mechanisms discussed previously lead naturally to a massive neutrino by the introduction of heavy fields (at the GUT scale) with masses around 10^{15} GeV, making these theories very difficult (with rare decays under special conditions) to be tested in colliders. Neutrino masses, however, could be originated from a *radiative* mechanism. This kind of scenario are very attractive due, first, to the ability of this type of model to explain the smallness of the neutrino mass as a consequence of *loop factors* suppression in the Yukawa couplings and, second, to the prediction of the existence of new particles that can be found at the Large Hadron Collider (LHC) in the coming years. In what follows we review the most famous *radiative* neutrino mass models in the literature.

1.2.1 The Zee's Model

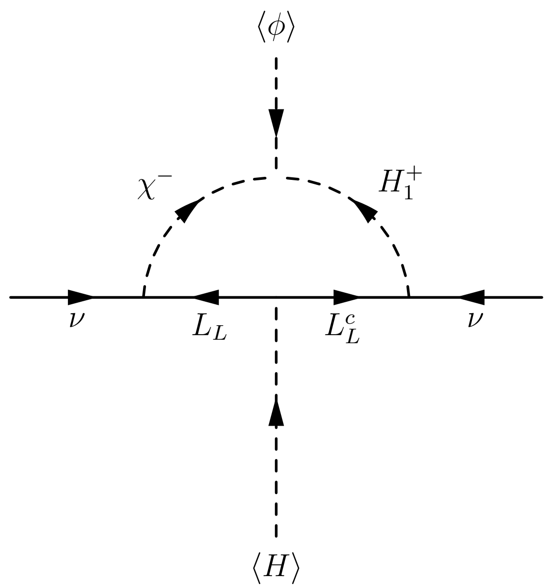

Figure 1.3: One-loop diagram in the Zee model

An interesting mechanism to generate masses for neutrinos is given by the Zee model [26] in which the masses are generated at one-loop order. In this model the scalar sector of the SM is extend. In addition to the Higgs doublet $H = (H_1^+, H_1^0)^T$, it is included another doublet scalar field $\phi \sim (\mathbf{1}, \mathbf{2}, 1)$ and an $SU(2)_L$ singlet(scalar) $\chi^{(+)} \sim (\mathbf{1}, \mathbf{1}, 2)$. [The numbers in the parentheses stand for the quantum numbers $(SU(3)_C, SU(2)_L, U(1)_Y)$]
The relevant Lagrangian of the Zee model is:

$$\mathcal{L}_{Zee} = \overline{\widetilde{L}_{L\alpha}} h_{\alpha\beta} L_{L\beta} \chi^{(+)} + \mu \chi^{(+)} H^{\dagger} \widetilde{\phi} + h.c., \quad (1.30)$$

where only the SM doublet H couples to leptons and the $SU(2)_L$ singlet $\chi^{(+)}$ carries lepton number -2 in order to ensure lepton number conservation in the Yukawa sector. Fermi-Dirac statistic makes the coupling $h_{\alpha\beta}$ an antisymmetry matrix, which implies a neutrino mass matrix with zeros in its diagonal. The Zee model allows to generate Majorana mass for the neutrino at one-loop order. The relevant Feynman diagram is shown in Fig. 1.3. In this model the neutrino mass matrix has the form:

$$\mathcal{M}_{\nu} \sim \begin{pmatrix} 0 & h_{\mu e}(m_{\mu}^2 - m_e^2) & h_{\tau e}(m_{\tau}^2 - m_e^2) \\ h_{\mu e}(m_{\mu}^2 - m_e^2) & 0 & h_{\tau\mu}(m_{\tau}^2 - m_{\mu}^2) \\ h_{\tau e}(m_{\tau}^2 - m_e^2) & h_{\tau\mu}(m_{\tau}^2 - m_{\mu}^2) & 0 \end{pmatrix}. \quad (1.31)$$

This model has a rich phenomenology: processes that lead to LFV such as $\mu \to e\gamma$, $\tau \to e\mu$, *etc*, are allowed and the existence of the new scalar field ϕ can be looked for at the LHC. However, the original version of Zee model does not match with the experimental data: predicts a maximum value for the solar neutrino mixing angle $\theta_{\odot}(\theta_{12})$, and does not reproduce the spectrum of neutrino masses. For these reasons the simplest version of Zee model has been rule out [35]. However, a general version of Zee model [36], in where, both scalar doublets couple to leptons is still a viable model for neutrino masses and mixing.

1.2.2 The Babu model

In this model [27] the scalar sector of the SM is extend with a single charged field $h^{(+)} \sim (\mathbf{1}, \mathbf{1}, 2)$ and a double charged field $k^{(++)} \sim (\mathbf{1}, \mathbf{1}, 4)$. The renormalizable Lagrangian (Yukawa sector and scalar sector) associated with these new fields read:

$$\mathcal{L}_{Babu} = \left(\overline{L^c_{L\alpha}} f_{\alpha\beta} L_{L\beta} h^{(+)} + \overline{L^c_{R\alpha}} y_{\alpha\beta} L_{R\beta} k^{(++)} + h.c. \right) - V(H, h^+, k^{++}), \quad (1.32)$$

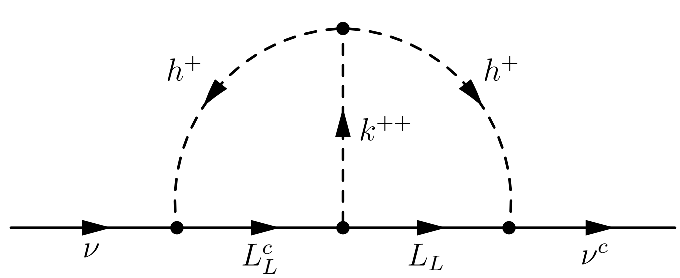

Figure 1.4: Two-loop neutrino mass generation in the Babu model

with the potential term given by:

$$V(H, h^+, k^{++}) \;=\; \mu h^- h^- k^{++} + h.c. \tag{1.33}$$

Fermi statistic implies antisymmetry of $f_{\alpha\beta}$ and symmetry of $y_{\alpha\beta}$. The new trilinear interaction shown in Eq. (1.33) violates lepton number by two units. Small Majorana neutrino masses contributions appear at two-loop level as it is shown in Fig. 1.4.
The neutrino masses are calculated from the Feynman diagram, and the mass matrix has the form:

$$\mathcal{M}_{\alpha\beta} \;=\; 8\mu f_{\alpha\gamma} y'_{\gamma\delta} m_\gamma m_\delta \mathcal{I}_{\gamma\delta} (y^\dagger)_{\gamma\beta}, \tag{1.34}$$

with $y'_{\alpha\beta} = \zeta y'_{\alpha\beta}$ where $\zeta = 1$ for $\alpha = \beta$ and $\zeta = 2$ for $\alpha \neq \beta$, being $m_{(\gamma,\delta)}$ the charged lepton masses.
The term $I_{\gamma\beta}$ is a two-loop integral

$$\mathcal{I}_{\gamma\delta} \;=\; \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(p^2 - m_h^2)} \frac{1}{(p^2 - m_\gamma^2)} \frac{1}{(q^2 - m_h^2)} \frac{1}{(q^2 - m_\delta^2)} \frac{1}{(p-q)^2 - m_k^2}. \tag{1.35}$$

This integral has been evaluated in Ref. [37].
Because $\det(\mathcal{M}_\nu) = 0$, the model matches with the current experimental data [3], and predicts one of the neutrinos to be massless. The Babu model also leads to decay processes that violate lepton number such as $\mu \to eee$, $\tau \to \mu\mu\mu$ which occurs at tree-level via $k^{(++)}$ exchange.

1.2.3 The Radiative Seesaw Model

An interesting model that can account for neutrino masses and also provides a dark matter candidate in the Universe is the so-called *Radiative Seesaw* [38]. This model is based on the two-Higgs Doublet (2HDM) model [39] where, in addition to the Standard Model gauge group G_{SM} is assumed an exact discrete Z_2 symmetry, and a minimal particle extension: a hypothetical new scalar doublet and three right-handed neutrinos. Under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes Z_2$, the new particle content transforms as:

$$\eta \;=\; (\eta^+, \eta^0)^T \sim (\mathbf{1}, \mathbf{2}, 1, -), \quad N_{R\alpha} \sim (\mathbf{1}, \mathbf{1}, 0, -), \tag{1.36}$$

The new particles, *i.e.* $N_{R\alpha}$ and the scalar doublet $(\eta^+, \eta^0)^T$ are odd under Z_2. The remaining particles of the SM are even under Z_2.
The relevant Lagrangian reads:

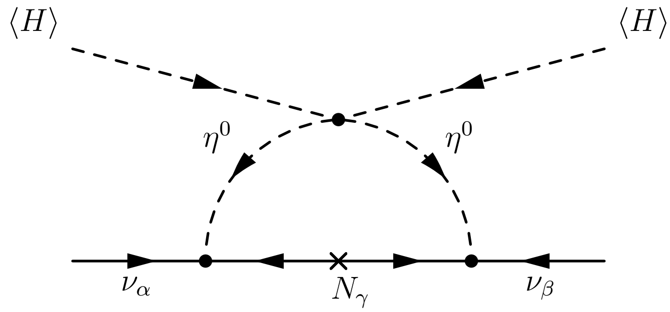

Figure 1.5: One-loop neutrino mass generation in the Ma model

$$\mathcal{L}_{Ma} = h_{\alpha\beta}(\nu_\alpha \eta^0 - l_\beta \eta^+)N_{R\beta} + \frac{1}{2}\overline{N^c_{R\alpha}} M_{R_{\alpha\beta}} N_{R\beta} + \frac{1}{2}\lambda_5 (H^\dagger \eta)^2 + h.c. \quad (1.37)$$

In order to ensure that Z_2 is exact, η can not develop a VEV. Also, as a consequence of the exact Z_2 symmetry, this model has a neutral lightest stable particle (LSP).
Neutrino mass is generated at one-loop level as it is shown in Fig. 1.5. The mass matrix takes the form:

$$(\mathcal{M}_\nu)_{\alpha\beta} = \sum_\gamma \frac{h_{\alpha\gamma} h_{\beta\gamma} M_\gamma}{16\pi^2} \left[\frac{m_R^2}{m_R^2 - M_\gamma^2} \ln\left(\frac{m_R^2}{M_\gamma^2}\right) - \frac{m_I^2}{m_I^2 - M_\gamma^2} \ln\left(\frac{m_I^2}{M_\gamma^2}\right) \right], \quad (1.38)$$

being m_R and m_I the masses of $\sqrt{2}Re(\eta^0)$ and $\sqrt{2}Im(\eta^0)$, respectively.
An important feature of this model is that can be verifiable at the LHC: in order to generate neutrino masses $m_\nu \sim 1$ eV and assuming $\lambda_5 \sim 10^{-4}$ is found that $M_\gamma \sim 1$ TeV. On the other hand, there will be observable dacays[4] such as $\eta^{(\pm)} \to l^{(\pm)} N_{1,2,3}$.

1.2.4 Neutrinos in the Simplest Little Higgs

In the Simplest Little Higgs model [40], the SM electroweak gauge group is enlarged to $SU(3)_L \otimes U(1)_X$. Each SM generation is embedded into $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ as follows:

[4] Assuming that the mass of the scalar fields are greater than M_γ.

$$\begin{aligned}
Q_{\alpha L} &= (u_\alpha, d_\alpha, U_\alpha)^T_L \sim (3,3,1/3), \\
d^c_\alpha &\sim (3^*,1,1/3), \quad u^c_\alpha \sim (3^*,1,-2/3), \quad U^c_\alpha \sim (3^*,1,-2/3), \\
\psi_{\alpha L} &= (-i\nu_\alpha, -il_\alpha, N_\alpha)^T_L \sim (1,3,-1/3), \\
l^c_\alpha &\sim (1,1,1), \quad n^c_\alpha \sim (1,1,0),
\end{aligned} \tag{1.39}$$

where α corresponds to a generation index. The symmetry breaking is triggered by the VEVs of two triplets $\phi_{1,2}$ (of a global $[SU(3)]^2$ symmetry) which transform as $(\mathbf{3},-1/3)$ under $SU(3)_L \otimes U(1)_X$:

$$\begin{aligned}
\phi_1 &\rightarrow e^{i\theta\cot\beta/f} \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}, \\
\phi_2 &\rightarrow e^{-i\theta\tan\beta/f} \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix},
\end{aligned} \tag{1.40}$$

where $\tan\beta = f_1/f_2$, $f = \sqrt{f_1^2 + f_2^2}$, and θ has the matrix form

$$\theta = \frac{\eta}{\sqrt{2}} + \begin{pmatrix} 0 & 0 & H_1^0 \\ 0 & 0 & H_1^+ \\ H_1^0 & H_1^- & 0 \end{pmatrix}. \tag{1.41}$$

For the sake of simplicity identical VEVs for both triplets are assumed ($f_1 = f_2 \sim f$). Masses for the neutral leptons arise from the interactions of the form (considering just one generation):

$$-\mathcal{L}_{yuk} = \lambda_\nu \phi_1^\dagger \psi_L n^c + h.c. \tag{1.42}$$

This Lagrangian respects the global $[SU(3)]^2$ symmetry and does not generate radiative contribution to the Higgs mass m_h^2. In the basis (ν, N, n^c), the mass matrix takes the form:

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & -\lambda_\nu v \\ 0 & 0 & \lambda_\nu f \\ \lambda_\nu v & \lambda_\nu f & 0 \end{pmatrix}, \tag{1.43}$$

where v is the VEV of the $SU(2)$ Higgs doublet H, and f is the scale at which the two condensate sigma fields develop VEV. At this stage the lightest neutrino remains massless, however it can acquire mass when one-loop level contribution are taken into account.

Allowing a Majorana mass term for n^c of the form $(1/2)Mn^cn^c$, at lowest order the neutrino mass matrix would read

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & -\lambda_\nu v \\ 0 & 0 & \lambda_\nu f \\ \lambda_\nu v & \lambda_\nu f & M \end{pmatrix}. \quad (1.44)$$

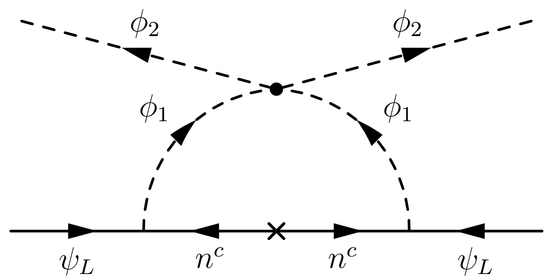

Figure 1.6: One-loop contribution to $(\phi_2^\dagger\psi_L)(\phi_2^\dagger\psi_L)$

In this model neutrinos acquire mass from the *unique* non-renormalizable dimension-five effective Lagrangian [41][5]

$$\mathcal{L}_{eff} = \frac{1}{2\Lambda_\nu}(\phi_2^\dagger\psi_L)(\phi_2^\dagger\psi_L) + h.c. \quad (1.45)$$

The effective operator $(\phi_2^\dagger\psi_L)(\phi_2^\dagger\psi_L)$ has the one-loop realization shown in Fig. 1.6. Taking into consideration this new contribution, the complete neutrino mass matrix reads:

$$\mathcal{M} = \begin{pmatrix} v^2/\Lambda_\nu & vf/\Lambda_\nu & -\lambda_\nu v \\ vf/\Lambda_\nu & f^2/\Lambda_\nu & \lambda_\nu f \\ \lambda_\nu v & \lambda_\nu f & M \end{pmatrix}. \quad (1.46)$$

The term $(1/\Lambda_\nu)$ is directly calculable from the Feynman diagram. Assuming $M < f$ and $m_h^2 < (\lambda_\nu f)^2$ we have [41]:

$$\frac{1}{\Lambda_\nu} = \frac{\lambda M}{16\pi^2}\left[\frac{x - 1 - \ln x}{(x-1)^2}\right], \quad (1.47)$$

[5]The vertex involving the scalar self-interaction(λ) breaks the global $[SU(3)]^2$ symmetry, however the correction to m_h^2 is logarithmically dependent and no fine-tuning is required.

with $x = m_h^2/(\lambda_\nu f)^2$ and λ the Higgs quartic coupling.
After diagonalizing Eq. (1.44), the lightest neutrino acquire mass which is given by

$$m_\nu \simeq v^2 \frac{\lambda M}{4\pi^2 f^2} \ln\left(\frac{(\lambda_\nu f)^2}{m_h^2}\right). \tag{1.48}$$

A interesting feature of this model is that neutrino mass depends only logarithmically on the Yukawa coupling constant, and predicts the existence of a yet to be observed right-handed neutrino with Majorana mass at the KeV scale[6].

[6]Demanding $m_\nu \sim 1$ eV.

Chapter 2

Little Higgs Model

Even with the remarkable success of the Standard Model (SM), there are theoretical reasons for believing that it is not the ultimate theory. One of the greatest mysteries still to be solved concerns the fact that the mass-squared parameter for the Higgs m_H^2 gets large one-loop quadratic corrections. By assuming a light Higgs mass ($m_H \sim 125$ GeV) as the recent discovery at ATLAS and CMS [42, 43] suggest[1], then new physics is required at the TeV scale or below in order to cancel the quadratically divergent contributions. The most dangerous of these contributions to m_H^2 come from loops of the top quark, the W-boson and the Higgs itself (Figure 2.1). From the Feynman diagrams displayed in Fig. 2.1 the total correction Δm_H^2 to the Higgs mass is given by [44]:

$$\Delta m_H^2 = \frac{1}{16\pi^2}\Big(\lambda + \frac{3}{2}g^2 - 6\lambda_t^2\Big)\Lambda_{UV}^2, \tag{2.1}$$

being λ the self-interacting Higgs coupling, λ_t the top quark Yukawa coupling, g the $SU(2)_L$ gauge coupling and Λ_{UV} an ultraviolet momentum cutoff.

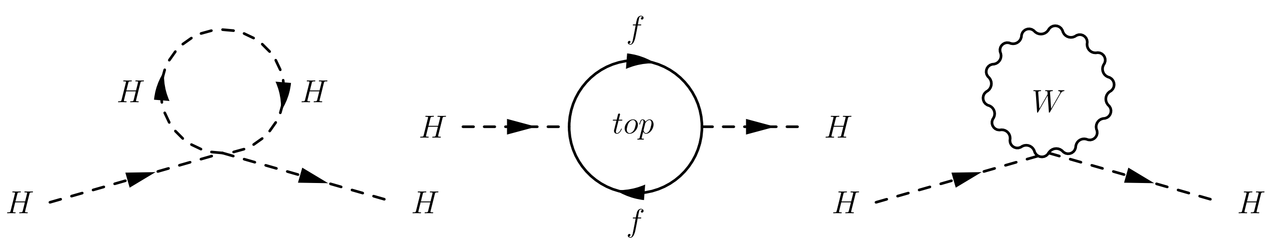

Figure 2.1: One-loop radiative corrections to the Higgs mass m_H^2. From left to right, contributions from the Higgs, the top quark and the W-boson.

[1]Until now, the data suggest that the new boson match with the SM Higgs boson

Assuming that the SM is valid up to $M_{Planck} \sim 2 \times 10^{18}$ GeV, a scale at which gravitational effects spoil the renormalizability of the theory, then what protects the Higgs mass from radiative corrections?. These corrections push up the Higgs mass to energies far beyond its bare mass destabilizing the electroweak symmetry breaking scale. This shortcoming is known as the *hierarchy* problem. From an experimental point of view, the current data suggest that SM, including radiative corrections, is a successful theory up to 5 TeV, scale at which, is expected that a fundamental theory should manifest in Nature (The UV completion of the SM) [47]. This scale is still too high, and radiative corrections to the Higgs mass would require fine-tuning. On the other hand, from a theoretical point of view, in order to stabilizes the electroweak scale, then new physics (elusive particles that are expected to be found in the current- and forthcoming experiments) is required at ~ 1 TeV. This issue is known as the *little hierarchy* problem [1, 47].

Several solutions [2, 7, 49, 50] have been intensively studied in the last decade, one is Supersymmetry (SUSY) in which the quadratic divergences to the Higgs mass coming from particles and sparticles cancel between each other. Also are the Little Higgs Models(LHMs) [7, 45, 46, 48, 51] in which the Higgs is thought as a Pseudo-Nambu Goldstone Boson (PNGB), massless at tree-level, which is allowed to acquire small mass radiatively. In this latter framework the radiative contributions to the Higgs mass are cancelled between particles of the same spin.

The trick behind Little Higgs models, the so-called (*collective symmetry breaking*), lies in fact that no single coupling explicitly breaks the global symmetry. The Higgs mass is protected by a global symmetry which is spontaneously broken. After the breaking the Higgs arises as a massless PNGB, and acquires mass logarithmically at one-loop order or quadratically at two-loop order (this latter contribution is negligible).

The Little Higgs model based on the approximate $[SU(4)/SU(3)]^4$ global symmetry has a strong theoretical motivation because of its ability to reproduce the low-energy phenomenology with a set of minimal parameters and to generate the Higgs quartic self-coupling without fine-tuning (unlike the model based on the approximate $[SU(3)/SU(2)]^2$ in which fine-tuning is required). Recently a study was carried out in order to determinate whether the Higgs-like particle discovered at CERN is the pseudo-Goldstone Boson of the Little Higgs Models or not [52]. By now the Simplest Little Higgs model [53] matchs with the current experimental data. In what follows we described briefly the Simplest Little Higgs (SLH) Model based on the approximate $[SU(4)/SU(3)]^4$ global symmetry.

2.1 The model

In the SLH Model based on the $SU(4)$ global symmetry, the electroweak $SU(2)_L \otimes U(1)_Y$ SM gauge group is enlarged to $SU(4)_L \otimes U(1)_X$ [6, 53]. This symmetry group has been proposed as an electroweak extension of the SM [54] and, among its features, the most remarkable

Table 2.1: Anomaly-free fermion content.

$Q_{iL} = \begin{pmatrix} d_i \\ u_i \\ U_i \\ D_i \end{pmatrix}_L$	d^c_{iL}	u^c_{iL}	U^c_{iL}	D^c_{iL}
$[3, 4^*, \frac{1}{6}]$	$[3^*, 1, \frac{1}{3}]$	$[3^*, 1, -\frac{2}{3}]$	$[3^*, 1, -\frac{2}{3}]$	$[3^*, 1, \frac{1}{3}]$
$Q_{3L} = \begin{pmatrix} u_3 \\ d_3 \\ D_3 \\ U_3 \end{pmatrix}_L$	u^c_{3L}	d^c_{3L}	D^c_{3L}	U^c_{3L}
$[3, 4, \frac{1}{6}]$	$[3^*, 1, -\frac{2}{3}]$	$[3^*, 1, \frac{1}{3}]$	$[3^*, 1, \frac{1}{3}]$	$[3^*, 1, -\frac{2}{3}]$
$L_{\alpha L} = \begin{pmatrix} \nu^0_{e\alpha} \\ e^-_\alpha \\ E^-_\alpha \\ n^0_\alpha \end{pmatrix}_L$	$e^+_{\alpha L}$	$E^+_{\alpha L}$		
$[1, 4, -\frac{1}{2}]$	$[1, 1, 1]$	$[1, 1, 1]$		

one is that if the cancellation of anomalies takes place between families[2] and not family by family as in the SM, then this model can account for the number of generations of fermions in Nature. Models with anomaly-free embedding are more attractive than the anomalous ones in the sense that do not leave the cancellation of anomalies to new physics at the UV scale and, as a consequence, offer an easier way to build up the UV completion theory (an example of such a situation can be found in [55]). Two independent anomaly free embeddings[3] can be built up in the $SU(4)_L \otimes U(1)_X$ extension. Here we considerer the model E [56], with the fermion content displayed in Table 2.1, where the numbers in parentheses stand for the quantum numbers ($SU(3)_C$, $SU(4)_L$, $U(1)_X$).

In the following we review briefly some important aspects of the model (for a complete review see [6] and references therein). The symmetry breaking is triggered by the Vacuum Expectation Values (VEVs) of four quadruplets Φ_i,Ψ_i, $i = 1, 2$.

$$\Phi_1 = e^{i\mathcal{H}_d \frac{f_2}{f_1}} \begin{pmatrix} 0 \\ 0 \\ f_1 \\ 0 \end{pmatrix}, \quad \Phi_2 = e^{-i\mathcal{H}_d \frac{f_1}{f_2}} \begin{pmatrix} 0 \\ 0 \\ f_2 \\ 0 \end{pmatrix},$$

[2]Under $SU(4)_L$ the third generation of quarks transforms as the fundamental **4** while the other two transforms as $\mathbf{4}^*$ or vice versa.

[3]that lead to the fermion content of the so-called three-family models [54].

$$\Psi_1 = e^{i\mathcal{H}_u \frac{f_4}{f_3}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_3 \end{pmatrix}, \quad \Psi_2 = e^{-i\mathcal{H}_u \frac{f_3}{f_4}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_4 \end{pmatrix}. \tag{2.2}$$

These sigma model fields transform under $SU(4)_L \otimes U(1)_X$ as $\Phi_1 \sim (\mathbf{4}, 1/2)$, $\Phi_2 \sim (\mathbf{4}, 1/2)$, $\Psi_1 \sim (\mathbf{4}, -1/2)$, $\Psi_2 \sim (\mathbf{4}, -1/2)$;
with

$$\mathcal{H}_d = \frac{1}{f_{12}} \begin{pmatrix} 0 \;\; 0 & h_d & 0 \\ 0 \;\; 0 & & 0 \\ h_d^\dagger & 0 & 0 \\ 0 \;\; 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{H}_u = \frac{1}{f_{34}} \begin{pmatrix} 0 \;\; 0 & 0 & h_u \\ 0 \;\; 0 & 0 & \\ 0 \;\; 0 & 0 & 0 \\ h_u^\dagger & 0 & 0 \end{pmatrix}, \tag{2.3}$$

being $f_{ij}^2 = f_i^2 + f_j^2$. Additionally, h_d and h_u are $SU(2)$ doublets with hypercharges 1/2 and $-1/2$, respectively, and acquire VEVs:

$$\langle h_d \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_d \end{pmatrix} \qquad \text{and} \qquad \langle h_u \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_u \\ 0 \end{pmatrix}. \tag{2.4}$$

At low-energies this model reproduce one Two-Higgs-Doublet Model (2HDM). One of the main features of this model (unlike the one based on global $SU(3)$ symmetry) is that it allows to generate automatically (at tree-level) a self-interacting quartic coupling for the scalar field which stabilizes the Higgs VEV.

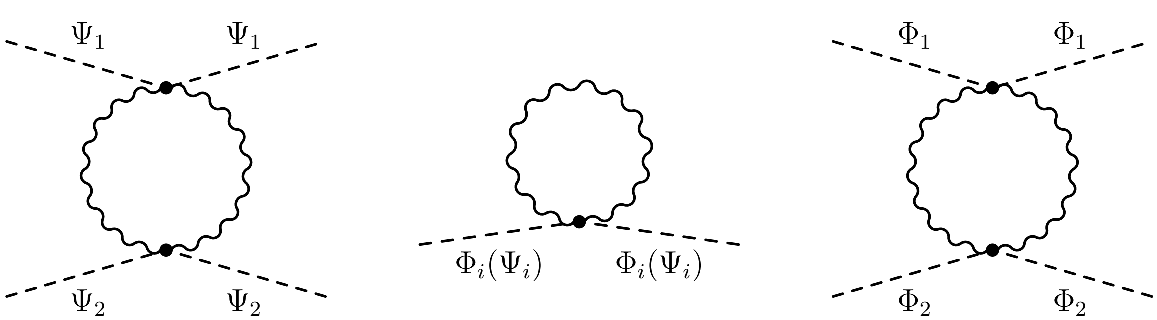

Figure 2.2: Gauge boson contribution to the Higgs potential

Contributions to the scalar potential arise at one-loop level from the realization of the operators $|\Phi_i^\dagger \Phi_i|$, $|\Psi_i^\dagger \Psi_i|$, $|\Phi_1^\dagger \Phi_2|^2$ and $|\Psi_1^\dagger \Psi_2|^2$ (Figure 2.2). The first and last Feynman diagrams in Fig. 2.2 explicitly violate $[SU(4)]^4$ and do contribute to the PNGBs masses:

$$\begin{aligned}\Delta\mathcal{L}_{up} &\sim \frac{g^4}{16\pi^2}|\Psi_2^\dagger \Psi_1|^2 \ln\left(\frac{\Lambda^2}{f_{34}^2}\right) \sim -\frac{f_{34}^2}{16\pi^2} h_u^\dagger h_u, \\ \Delta\mathcal{L}_{down} &\sim \frac{g^4}{16\pi^2}|\Phi_2^\dagger \Phi_1|^2 \ln\left(\frac{\Lambda^2}{f_{12}^2}\right) \sim -\frac{f_{12}^2}{16\pi^2} h_d^\dagger h_d,\end{aligned} \tag{2.5}$$

where g is the gauge coupling associated to $SU(4)_L$. The diagram in the middle of Fig. 2.2 is quadratically divergent but preserves the global $[SU(4)]^4$ symmetry and, as consequence, do not contribute to the PNGBs masses. Diagrams involving more than two $\Phi_i(\Psi_i)$ insertions will be finite (and also contribute to the Higgs mass). In what follows we concentrate our discussion in the top quark contribution to the Higgs mass.

2.1.1 Top Yukawa Coupling

The one-loop quadratic divergences to the Higgs mass cancel between particles of the same spin. As an example we verify it in the Yukawa Lagrangian associated to the Top quark[4].

$$-\mathcal{L}_{Y_3}^{Q} = \Big(\lambda_1^{u_3} i u_{3L}^{1c} \Psi_1^\dagger + \lambda_2^{u_3} i u_{3L}^{2c} \Psi_2^\dagger + \lambda_1^{d_3} i d_{3L}^{1c} \Phi_1^\dagger + \lambda_2^{d_3} i d_{3L}^{2c} \Phi_2^\dagger\Big) Q_{3L}, \tag{2.6}$$

where $u_{3L}^{(1c,2c)}(d_{3L}^{(1c,2c)})$ are linear combinations of up-type (down-type) conjugate quarks $u_{3L}^c(d_{3L}^c)$ and $U_{3L}^c(D_{3L}^c)$. Expanding the scalar fields in Eq. (2.2) to second order, the Top Yukawa Lagrangian becomes:

$$\begin{aligned}-\mathcal{L}_{Y_3}^{Q} = &\Bigg[\lambda_1^{u_3}\left(\frac{f_4}{f_{34}}\right)\langle h_u\rangle u_{3L}^{1c} u_{3L} - \lambda_2^{u_3}\left(\frac{f_3}{f_{34}}\right)\langle h_u\rangle u_{3L}^{2c} u_{3L} + \lambda_1^{u_3} i f_3 u_{3L}^{1c} U_{3L} \\ &+ \lambda_2^{u_3} i f_4 u_{3L}^{2c} U_{3L} - \lambda_1^{u_3} i\left(\frac{f_4^2}{f_3 f_{34}^2}\right)\langle h_u\rangle^2 u_{3L}^{1c} U_{3L} - \lambda_2^{u_3} i\left(\frac{f_3^2}{f_4 f_{34}^2}\right)\langle h_u\rangle^2 u_{3L}^{2c} U_{3L} \\ &+ \lambda_1^{d_3}\left(\frac{f_2}{f_{12}}\right)\langle h_d\rangle d_{3L}^{1c} d_{3L} - \lambda_2^{d_3}\left(\frac{f_1}{f_{12}}\right)\langle h_d\rangle d_{3L}^{2c} d_{3L} + \lambda_1^{d_3} i f_1 d_{3L}^{1c} D_{3L} \\ &+ \lambda_2^{d_3} i f_2 d_{3L}^{2c} D_{3L} - \lambda_1^{d_3} i\left(\frac{f_2^2}{f_1 f_{12}^2}\right)\langle h_d\rangle^2 d_{3L}^{1c} D_{3L} - \lambda_2^{d_3} i\left(\frac{f_1^2}{f_2 f_{12}^2}\right)\langle h_d\rangle^2 d_{3L}^{2c} D_{3L}\Bigg]\end{aligned} \tag{2.7}$$

[4]As pointed out above, here is assumed that the third generation of quarks transforms differently from the other two.

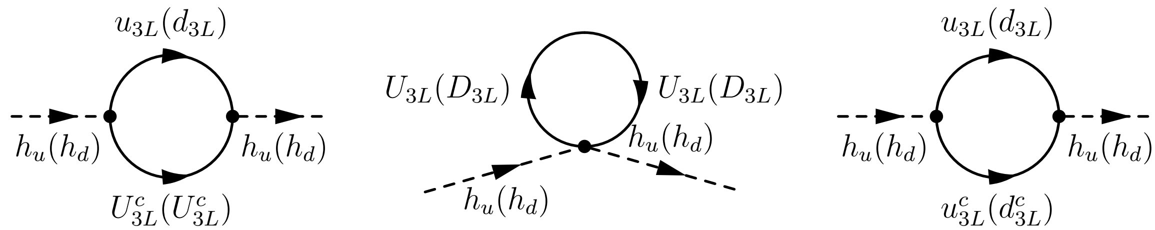

Figure 2.3: Top(Bottom) contribution to the Higgs mass.

In the mass eigenstates basis the previous equation acquires the form

$$\begin{aligned} -\mathcal{L}^Q_{Y_3} &= \lambda_{u_3}\langle h_u\rangle u^c_{3L}u_{3L} + \lambda_{U_3u_3}\langle h_u\rangle U^c_{3L}u_{3L} + \frac{\lambda_{U_3}}{2M_{U_3}}\langle h_u\rangle^2 U^c_{3L}u_{3L} \\ &+ \lambda_{d_3}\langle h_d\rangle d^c_{3L}d_{3L} + \lambda_{D_3d_3}\langle h_d\rangle D^c_{3L}d_{3L} + \frac{\lambda_{D_3}}{2M_{D_3}}\langle h_d\rangle^2 D^c_{3L}d_{3L}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \lambda_{u_3} &= \frac{\lambda_1^{u_3}\lambda_2^{u_3}f_{34}}{\sqrt{2}M_{U_3}}, \quad \lambda_{U_3u_3} = \frac{\left[(\lambda_1^{u_3})^2-(\lambda_2^{u_3})^2\right]f_3f_4}{\sqrt{2}f_{34}M_{U_3}}, \quad \lambda_{U_3} = \frac{(\lambda_1^{u_3}f_4)^2+(\lambda_2^{u_3}f_3)^2}{2f_{34}^2} \\ \lambda_{d_3} &= \frac{\lambda_1^{d_3}\lambda_2^{d_3}f_{12}}{\sqrt{2}M_{D_3}}, \quad \lambda_{D_3d_3} = \frac{\left[(\lambda_1^{d_3})^2-(\lambda_2^{d_3})^2\right]f_1f_2}{\sqrt{2}f_{12}M_{D_3}}, \quad \lambda_{D_3} = \frac{(\lambda_1^{d_3}f_2)^2+(\lambda_2^{d_3}f_1)^2}{2f_{12}^2}, \end{aligned} \tag{2.9}$$

$M_U = \sqrt{(\lambda_1^{u_3}f_3)^2+(\lambda_2^{u_3}f_4)^2}$ and $M_D = \sqrt{(\lambda_1^{d_3}f_1)^2+(\lambda_2^{d_3}f_2)^2}$.
The couplings between the up-type (down-type) Higgs and the top (bottom) quarks in Eq. (2.8) do not contribute to the Higgs mass due to a *miraculous* cancellation of the quadratic divergences. In Fig. 2.3 we display the one loop diagrams involving these interactions. The quadratic divergences coming from the first and last diagrams are cancelled by the contribution from the diagram in the middle. Contributions at two-loop level (from the top quark and gauge bosons) to the Higgs mass are also present, but are negligible.[5]

[5]Under the assumption that the ultraviolet cutoff is ~ 10 TeV.

2.1.2 Yukawa Lagrangian for the Neutral Leptons

With the particle content of the model under consideration it is not possible to generate masses for neutral leptons unless we introduce right-handed neutrinos $N_R \sim (\mathbf{1}, \mathbf{1}, 0)$ in the model. The most general gauge invariant Yukawa Lagrangian looks like:

$$-\mathcal{L}_Y^{neutro} = \lambda_1 \overline{N}_{\alpha R} \Psi_1^\dagger L_{\beta L} + \lambda_2 \overline{N}_{\alpha R} \Psi_2^\dagger L_{\beta L} + h.c,$$

where α and β are generation indexes. This Lagrangian does not respect the global $[SU(4)]^4$ symmetry and, as a consequence, also gives contribution to the PNGBs masses. Two solutions to this problem have been proposed: the first one is to assume only a Yukawa coupling: forbidding, for example, λ_2 and leaving the interaction that involves λ_1[6]; the second one is assuming that the global $[SU(4)]^4$ symmetry is *approximate* [57]. In this case there exist a strong suppression on the coupling $\lambda_2 << 1$ while λ_1 remains unsuppressed. In this way the contribution to the PNGBs coming from the leptonic sector remains negligible due to the smallness of λ_2. In this scenario the collective symmetry breaking is achieved only when $\lambda_2 \to 0$. This latter alternative is the option that is explored in this thesis. Further details concerning the generation of neutrino masses will be considered in Chapter 3.

[6]This idea is applied by Lee [58] and F. del Águila et al. [41] to generate masses in the Kaplan-Schmaltz model [53]

Chapter 3

Radiative Neutrino Mass Generation

3.1 Model

As pointed out in Chapter 2, the Simplest Little Higgs model based on the approximate $[SU(4)/SU(3)]^4$ global symmetry "solve" the electroweak hierarchy problem (little hierarchy problem) of the SM. The key of these theories is the implementation of the so-called collective symmetry breaking, which avoids large radiative corrections at one loop level to the Higgs mass. The Schmaltz's model [40] has been extended to the approximate $[SU(4)/SU(3)]^4$ non-liner sigma model [6], with the new top quark partners cancelling the divergences coming from the ordinary quark top. The quadratic divergences at one loop order to the Higgs mass are suppressed, and at two loops are negligible. In this work, we are going to considerer a more generic version of the Simplest Little Higgs model, known as the *minimal little higgs model* already studied in [57] for the $[SU(3)/SU(2)]^2$ non-liner sigma model. Here, it is assumed that the global $SU(4)$ symmetry which protects the Higgs mass is *approximate*. Then the Higgs mass receives quadratic divergences at one loop, but these new contributions are negligible mainly by the suppression imposed by the global symmetry on the new Yukawa interactions. Assuming an anomaly free embedding in the $SU(4)_L \otimes U(1)_X$ gauge symmetry, we have the following lepton content[1]

$$L_{\alpha L} = \big(-i\nu^0_{e\alpha}, -ie^-_\alpha, E^-_\alpha, n^0_\alpha \big)^T_L \sim (\mathbf{1}, \mathbf{4}, -1/2),\ e^+_{\alpha L} \sim (\mathbf{1}, \mathbf{1}, 1),\ E^+_{\alpha L} \sim (\mathbf{1}, \mathbf{1}, 1), \qquad (3.1)$$

with α being a generation index and the numbers inside the parentheses correspond to the way the lepton field transform under $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ (3-4-1 symmetry).

[1]Exactly the same lepton content displayed in Table 2.1, where a $-i$ phase is included in the SM lepton doublet.

The scalar sector in the model is given by,

$$\Phi_1 = e^{i\mathcal{H}_d \frac{f_2}{f_1}} \begin{pmatrix} 0 \\ 0 \\ f_1 \\ 0 \end{pmatrix}, \quad \Phi_2 = e^{-i\mathcal{H}_d \frac{f_1}{f_2}} \begin{pmatrix} 0 \\ 0 \\ f_2 \\ 0 \end{pmatrix}, \tag{3.2}$$

$$\Psi_1 = e^{i\mathcal{H}_u \frac{f_4}{f_3}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_3 \end{pmatrix}, \quad \Psi_2 = e^{-i\mathcal{H}_u \frac{f_3}{f_4}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_4 \end{pmatrix}. \tag{3.3}$$

These sigma model fields transform under the $SU(4)_L \otimes U(1)_Y$ symmetry as $\Phi_1 \sim (\mathbf{4}, 1/2)$, $\Phi_2 \sim (\mathbf{4}, 1/2)$, $\Psi_1 \sim (\mathbf{4}, -1/2)$, $\Psi_2 \sim (\mathbf{4}, -1/2)$.
The model studied in [6] does not provide a mechanism for neutrino mass generation. As a first step, following the idea behind the seesaw mechanisms we extend the particle content of the model but keeping in mind that it is forbidden to break the anomaly-free structure of the model. The most simple choice is to add a right-handed neutrino $N^0_{iR} \sim (\mathbf{1}, \mathbf{1}, 0)$ per generation. Since neutrinos do not possess electric charge, they can be Majorana particles.

3.2 One generation case

The Yukawa Lagrangian for neutral leptons is given by:

$$-\mathcal{L}_{yuk} = \lambda^n \overline{N}_R \Psi_1^\dagger L_L + \lambda^m \overline{N}_R \Psi_2^\dagger L_L + h.c. \tag{3.4}$$

Assuming that $L_L \sim (4,1,1,1)$, $\Psi_1 \sim (4,1,1,1)$ and $\Psi_2 \sim (1,4,1,1)$ under $SU(4)_1 \otimes SU(4)_2 \otimes SU(4)_3 \otimes SU(4)_4([SU(4)]^4)$, we note that the coupling λ^n is unsuppressed, but λ^m is suppressed by the global symmetry, therefore, we impose the hierarchy condition $\lambda^n >> \lambda^m$.
After the Spontaneous Symmetry Breaking (SSB) takes place, the scalar fields develop a VEV, which is expressed by:

$$\langle \Psi_1 \rangle = e^{i\beta_1 A} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_3 \end{pmatrix}, \qquad \langle \Psi_2 \rangle = e^{-i\beta_2 A} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_4 \end{pmatrix}, \tag{3.5}$$

with

$$\beta_1 = \frac{f_4}{f_3}\frac{v_\mu}{f_{34}\sqrt{2}}, \qquad \beta_2 = \frac{f_3}{f_4}\frac{v_\mu}{f_{34}\sqrt{2}} \quad \text{and} \quad A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

From here on we are going to do some mathematical procedure that will allow us to write down the Yukawa Lagrangian in terms of the neutrino mixing angle.
Expanding the exponential function

$$e^{i\beta_1 A} = \mathbb{1} + i\beta_1 A + \frac{1}{2!}(i\beta_1)^2 A^2 + \frac{1}{3!}(i\beta_1)^3 A^3 + ... \tag{3.7}$$

Is easy to show that:

$$A^m = \begin{cases} C, & m \text{ odd} \\ A, & m \text{ even.} \end{cases} \tag{3.8}$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

From Eq. (3.7), and using the results from Eq. (3.8)

$$\begin{aligned} e^{i\beta_1 A} &= \mathbb{1} + iA\Big[\beta_1 - \frac{1}{3!}\beta_1^3 + \frac{1}{5!}\beta_1^5 - \frac{1}{7!}\beta_1^7 \pm ...\Big] + C\Big[-\frac{1}{2!}\beta_1^2 + \frac{1}{4!}\beta_1^4 - \frac{1}{6!}\beta_1^6 \mp ...\Big] \\ &= \mathbb{1} - C + iA\sin\beta_1 + C\cos\beta_1 \\ &= \begin{pmatrix} \cos\beta_1 & 0 & 0 & i\sin\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\sin\beta_1 & 0 & 0 & \cos\beta_1 \end{pmatrix}. \end{aligned} \tag{3.10}$$

In order to calculate the masses of the neutrinos at tree-level, going from weak eigenstates to mass eigenstates, we need to compute the VEV of the scalar fields, which can be evaluated directly using the calculation of $e^{i\beta_1 A}$. After such a calcutation is obtained:

$$\langle\Psi_1\rangle^\dagger = e^{-i\beta_1 A}\begin{pmatrix} 0 & 0 & 0 & f_3 \end{pmatrix} = \begin{pmatrix} -if_3\sin\beta_1 & 0 & 0 & f_3\cos\beta_1 \end{pmatrix}, \tag{3.11}$$

$$\langle\Psi_2\rangle^\dagger \;=\; e^{i\beta_2 A}\begin{pmatrix}0 & 0 & 0 & f_4\end{pmatrix} = \begin{pmatrix}if_4\sin\beta_2 & 0 & 0 & f_4\cos\beta_2\end{pmatrix}. \quad (3.12)$$

Where, the Eq. (3.12) is easily obtained from Eq. (3.11) by making the changes $\beta_1 \to \beta_2$, $f_3 \to f_4$ and $i \to -i$.
Taking into account the previous relations, the Yukawa Lagrangian in Eq. (3.4) is expressed by:

$$-\mathcal{L}_{yuk} = \lambda^n \overline{N}_R\big(f_3\sin\beta_1\nu^0_{eL} + f_3\cos\beta_1 n^0_L\big) + \lambda^m\overline{N}_R\big(-f_4\sin\beta_2\nu^0_{eL} + f_4\cos\beta_1 n^0_L\big) + h.c.$$

Now, assuming that $f_i \sim 1$ TeV; with f_i being the energy scale at which the global $SU(4)$ symmetry is spontaneously broken, and taking into consideration that $\beta_1 = \frac{f_4}{f_3}\frac{\nu_\mu}{f_{34}\sqrt{2}} \sim \frac{1}{f_i}$ and $\beta_2 = \frac{f_3}{f_4}\frac{\nu_\mu}{f_{34}\sqrt{2}} \sim \frac{1}{f_i}$, it follows:

$$\sin\beta_i \simeq \beta_i \quad \text{and} \quad \cos\beta_i \simeq 1, \quad \text{for} \quad i = 1, 2. \quad (3.13)$$

After that, and parametrizing $\tan\alpha = \frac{f_3}{f_4}$, we reach the following Yukawa Lagrangian for neutral leptons

$$-\mathcal{L}_{yuk} = f_{_{34}}\overline{N}_R\left(\Big[\lambda^m\sin\alpha - \lambda^n\cos\alpha\Big]\frac{v_\mu}{f_{_{34}}\sqrt{2}}\nu^0_e + \Big[\lambda^n\sin\alpha + \lambda^m\cos\alpha\Big]n^0_L\right) + h.c. \quad (3.14)$$

Now, rotating from the weak eigenstates (n^0_L, ν^0_e) to mass eigenstates $(\hat{n}^0_L, \hat{\nu}^0_e)$, we have

$$\begin{pmatrix}\hat{n}^0_L \\ \hat{\nu}^0_e\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}n^0_L \\ \nu^0_e\end{pmatrix}, \quad (3.15)$$

where θ is the mixing angle.
From Eqs. (3.14) and (3.15) is straightforward to obtain the Lagrangian in the mass eigenstate basis

$$-\mathcal{L}^{tree-level}_{yuk} = f_{34}\left(\sqrt{\big(\lambda^n\sin\alpha + \lambda^m\cos\alpha\big)^2 + \big(\lambda^n\cos\alpha - \lambda^m\sin\alpha\big)^2\frac{v^2_\mu}{2f^2_{34}}}\right)\overline{N}_R\hat{n}^0_L + h.c.$$

With the mixing angle determined by:

$$\begin{aligned}\sin\theta &= \frac{\big(\lambda^m\sin\alpha - \lambda^n\cos\alpha\big)\frac{v_\mu}{\sqrt{2}f_{34}}}{\sqrt{\big(\lambda^n\sin\alpha + \lambda^m\cos\alpha\big)^2 + \big(\lambda^m\sin\alpha - \lambda^n\cos\alpha\big)^2\frac{v^2_\mu}{2f^2_{34}}}},\\ \cos\theta &= \frac{\big(\lambda^n\sin\alpha + \lambda^m\cos\alpha\big)}{\sqrt{\big(\lambda^n\sin\alpha + \lambda^m\cos\alpha\big)^2 + \big(\lambda^m\sin\alpha - \lambda^n\cos\alpha\big)^2\frac{v^2_\mu}{2f^2_{34}}}},\end{aligned} \quad (3.16)$$

and the mass-term from the Yukawa Lagrangian in Eq. (3.16) is:

$$m_N = f_{34}\left(\sqrt{\left(\lambda^n \sin\alpha + \lambda^m \cos\alpha\right)^2 + \left(\lambda^n \cos\alpha - \lambda^m \sin\alpha\right)^2 \frac{v_\mu^2}{2 f_{34}^2}}\right). \tag{3.17}$$

In order to compare the latter expression with the neutrino mass reported in [58], we must to expand and suppress completely either λ^n or λ^m. For instance, making $\lambda^m = 0$ and $\lambda^n \neq 0$ is reproduced the results in Ref. [58].

At this stage, two heavy neutrinos acquire masses of the order of the mass term given in Eq. (3.17), but the SM neutrino (the lightest neutrino) remains massless (this results holds even if we considerer the three generations of fermions).

From now on, we will concentrate in the study of another mechanism to generate neutrino masses. As a first attempt we explore the so-called radiative seesaw mechanism.

From Eq. (3.14), we can write down the Lagrangian in matrix form (where the super-index in the neutral leptons had been suppressed):

$$-\mathcal{L}_{yuk}^{Tree-level} = \frac{1}{2}\left(\overline{\nu_{eL}^c} \quad \overline{n_L^c} \quad \overline{N_R}\right)\mathcal{M}\begin{pmatrix}\nu_{eL}\\ n_L\\ N_R^c\end{pmatrix} + h.c., \tag{3.18}$$

with $\mathcal{M}$, the tree-level mass matrix which is given by:

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & \left[(\lambda^m)^\dagger \sin\alpha - (\lambda^n)^\dagger \cos\alpha\right]\frac{v_\mu}{\sqrt{2}} \\ 0 & 0 & f_{34}\left[(\lambda^n)^\dagger \sin\alpha + (\lambda^m)^\dagger \cos\alpha\right] \\ \left[\lambda^m \sin\alpha - \lambda^n \cos\alpha\right]\frac{v_\mu}{\sqrt{2}} & f_{34}\left[\lambda^n \sin\alpha + \lambda^m \cos\alpha\right] & 0 \end{pmatrix} \tag{3.19}$$

.

If we introduce a Majorana mass term for the right-handed neutrino N_R, then we must to add to the tree-level Lagrangian the term $\frac{1}{2}M\overline{N_R}N_R^c + h.c$, being M the mass of the particle. At the lowest order the neutrino mass has the structure

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & \left[(\lambda^m)^\dagger \sin\alpha - (\lambda^n)^\dagger \cos\alpha\right]\frac{v_\mu}{\sqrt{2}} \\ 0 & 0 & f_{34}\left[(\lambda^n)^\dagger \sin\alpha + (\lambda^m)^\dagger \cos\alpha\right] \\ \left[\lambda^m \sin\alpha - \lambda^n \cos\alpha\right]\frac{v_\mu}{\sqrt{2}} & f_{34}\left[\lambda^n \sin\alpha + \lambda^m \cos\alpha\right] & M \end{pmatrix} \tag{3.20}$$

Diagonalizing this last matrix it is found that the SM neutrino remains massless. In order to give it a non-zero mass, we are going to introduce a dimension-five effective operator, which are a generalization of the Weinberg operator in the SM [19].

3.3 Radiative Seesaw Mechanism

In order to explain the smallness of neutrino mass in nature, we can introduce higher dimensional operators which will automatically generate tiny neutrino masses. The *Minimal Little Higgs Model* based on an electroweak symmetry $SU(4)_L \otimes U(1)_X$ can be thought as an effective theory that is valid up to an energy scale Λ.
The Lagrangian is written as:

$$\mathcal{L}_{yuk} = \mathcal{L}_{yuk}^{tree-level} + \mathcal{L}_{yuk}^{loop-level} \tag{3.21}$$

.
After expanding $\mathcal{L}^{loop-level}$, it follows:

$$\mathcal{L}_{yuk} = \mathcal{L}_{yuk}^{tree-level} + \mathcal{L}_5 + \mathcal{L}_6 + \dots \tag{3.22}$$

In the SM the Weinberg's dimension-five effective operator [19] provides an answer for neutrino masses puzzle (this operator violate lepton number, and, neutrinos are Majorana particles).
$\mathcal{L}_5 = \frac{1}{\Lambda}\mathcal{O}_5$ with $\mathcal{O}_5 \sim LLHH$
Here,we are going to build up all the possible effective operators of dimension $d = 5$ in our model:

$$\mathcal{O}_5 \supset \left\{ (\overline{L_L^c}\Psi_1^*)(\Psi_1^\dagger L_L),\ (\overline{L_L^c}\Psi_2^*)(\Psi_2^\dagger L_L),\ (\overline{L_L^c}\Psi_1^*)(\Psi_2^\dagger L_L),\ (\overline{L_L^c}\Psi_2^*)(\Psi_1^\dagger L_L) \right\} \tag{3.23}$$

All of these terms generate one-loop contributions[2] (see Fig. 3.1).
From the dimension-five effective operator we have that the effective Lagrangian is giving by:

$$\begin{aligned}\mathcal{L}_5 &= \sum_{i,j}^{2} \frac{1}{2\Lambda_{ij}} \left(\overline{L_L^c}\Psi_i^*\right)\left(\Psi_j^\dagger L_L\right) \\ &= \frac{1}{2\Lambda_{11}} \left(\overline{L_L^c}\Psi_1^*\right)\left(\Psi_1^\dagger L_L\right) + \frac{1}{2\Lambda_{22}} \left(\overline{L_L^c}\Psi_2^*\right)\left(\Psi_2^\dagger L_L\right) + \frac{1}{2\Lambda_{12}} \left(\overline{L_L^c}\Psi_1^*\right)\left(\Psi_2^\dagger L_L\right) + \frac{1}{2\Lambda_{21}} \left(\overline{L_L^c}\Psi_2^*\right)\left(\Psi_1^\dagger L_L\right),\end{aligned}$$

[2]These effective operator differ slightly from studied in [41], where is not clear the Lorentz invariance of the effective operator.

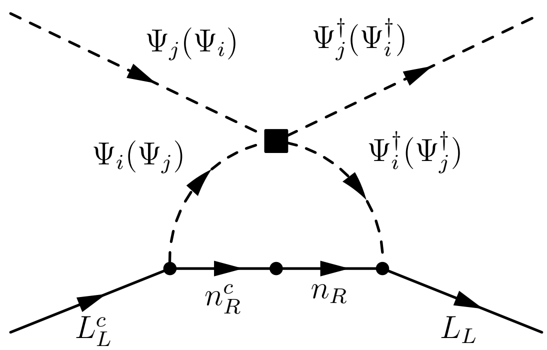

Figure 3.1: One loop contribution to $\left(\overline{L_L^c}\Psi_i^*\right)\left(\Psi_j^\dagger L_L\right)$

where the relation $\Lambda_{12}=\Lambda_{21}$ is used[3].
Let us label $\Lambda_{11}=\Lambda_{mm}$, $\Lambda_{22}=\Lambda_{nn}$ and $\Lambda_{12}=\Lambda_{mn}$, where we have implicitly taken into account the dependence of Λ_{ij} on the Yukawa couplings λ_n and λ_m (see, for instance, the Feynman diagram in Appendix A).

Let us compute the contribution to the mass matrix from each effective Lagrangian.
After the SSB, Eq. (3.24) becomes:

$$\begin{aligned}\mathcal{L}_5 &= \frac{1}{2\Lambda_{11}}\overline{L_L^c}\langle\Psi_1^*\rangle\langle\Psi_1\rangle^\dagger L_L+\frac{1}{2\Lambda_{22}}\overline{L_L^c}\langle\Psi_2^*\rangle\langle\Psi_2\rangle^\dagger L_L\\ &+ \frac{1}{2\Lambda_{12}}\overline{L_L^c}\langle\Psi_1^*\rangle\langle\Psi_2\rangle^\dagger L_L+\frac{1}{2\Lambda_{21}}\overline{L_L^c}\langle\Psi_2^*\rangle\langle\Psi_1\rangle^\dagger L_L.\end{aligned}$$

Using the results obtained in Eqs. (3.11) and (3.12), we get

$$\begin{aligned}\frac{1}{2\Lambda_{nn}}\overline{L_L^c}\langle\Psi_2^*\rangle\langle\Psi_2\rangle^\dagger L_L &= \frac{1}{2\Lambda_{nn}}\Big(f_4^2\beta_2^2\overline{\nu_{eL}^c}\nu_{eL}+f_4^2\beta_2\overline{\nu_{eL}^c}n_{eL}+f_4^2\beta_2\overline{n_{eL}^c}\nu_{eL}+f_4^2\overline{n_{eL}^c}n_{eL}\Big),\\ \frac{1}{2\Lambda_{mm}}\overline{L_L^c}\langle\Psi_1^*\rangle\langle\Psi_1\rangle^\dagger L_L &= \frac{1}{2\Lambda_{mm}}\Big(f_3^2\beta_1^2\overline{\nu_{eL}^c}\nu_{eL}-f_3^2\beta_1\overline{\nu_{eL}^c}n_{eL}-f_3^2\beta_1\overline{n_{eL}^c}\nu_{eL}+f_3^2\overline{n_{eL}^c}n_{eL}\Big),\\ \frac{1}{2\Lambda_{nm}}\overline{L_L^c}\langle\Psi_1^*\rangle\langle\Psi_2\rangle^\dagger L_L &= \frac{1}{2\Lambda_{nm}}\Big(-f_3f_4\beta_1\beta_2\overline{\nu_{eL}^c}\nu_{eL}-f_3f_4\beta_1\overline{\nu_{eL}^c}n_{eL}+f_3f_4\beta_2\overline{n_{eL}^c}\nu_{eL}+f_3f_4\overline{n_{eL}^c}n_{eL}\Big),\\ \frac{1}{2\Lambda_{nm}}\overline{L_L^c}\langle\Psi_2^*\rangle\langle\Psi_1\rangle^\dagger L_L &= \frac{1}{2\Lambda_{nm}}\Big(-f_3f_4\beta_1\beta_2\overline{\nu_{eL}^c}\nu_{eL}-f_3f_4\beta_1\overline{\nu_{eL}^c}n_{eL}+f_3f_4\beta_2\overline{n_{eL}^c}\nu_{eL}+f_3f_4\overline{n_{eL}^c}n_{eL}\Big).\end{aligned}$$

[3]Is straightforward proof that $\Lambda_{12}=\Lambda_{21}$, from the symmetry properties of the Feynman diagram displayed in Fig. 3.1.

In the basis (ν_{eL}, n_{eL},N_R^c), the effective Lagrangian reads:

$$-\mathcal{L}_5^{one-loop} \;=\; \frac{1}{2}\left(\overline{\nu_{eL}^c} \quad \overline{n_{eL}^c} \quad \overline{N_R}\right)\mathcal{M}^{one-loop}\begin{pmatrix}\nu_{eL}\\ n_{eL}\\ N_R^c\end{pmatrix} + h.c., \tag{3.24}$$

where $\mathcal{M}^{one-loop}$ has the form:

$$\mathcal{M}^{one-loop} = \begin{pmatrix} \left[\frac{f_4^2\beta_2^2}{\Lambda_{nn}} + \frac{f_3^2\beta_1^2}{\Lambda_{mm}} - \frac{2\left(f_3 f_4\beta_1\beta 2\right)}{\Lambda_{nm}}\right] & \left[\frac{f_4^2\beta_2}{\Lambda_{nn}} - \frac{f_3^2\beta_1}{\Lambda_{mm}} + \frac{\left(f_3 f_4\beta_2 - f_3 f_4\beta_1\right)}{\Lambda_{nm}}\right] & 0 \\ \left[\frac{f_4^2\beta_2}{\Lambda_{nn}} - \frac{f_3^2\beta_1}{\Lambda_{mm}} + \frac{\left(f_3 f_4\beta_2 - f_3 f_4\beta_1\right)}{\Lambda_{nm}}\right] & \left[\frac{f_4^2}{\Lambda_{nn}} + \frac{f_3^2}{\Lambda_{mm}} + \frac{2\left(f_3 f_4\right)}{\Lambda_{nm}}\right] & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.25}$$

Now, the full mass matrix is giving by $\mathcal{M} = \mathcal{M}^{tree-level} + \mathcal{M}^{one-loop}$.
From the one-loop and tree-level contributions to the mass matrix, we build up the full mass matrix as follows:

$$\mathcal{M} = \begin{pmatrix} A & B^\dagger & C^\dagger \\ B & D & E^\dagger \\ C & E & F \end{pmatrix}, \tag{3.26}$$

where:

$$\begin{aligned} A &= \left[\frac{f_4^2\beta_2^2}{\Lambda_{nn}} + \frac{f_3^2\beta_1^2}{\Lambda_{mm}} - \frac{2\left(f_3 f_4\beta_1\beta_2\right)}{\Lambda_{nm}}\right], \\ B &= \left[\frac{f_4^2\beta_2}{\Lambda_{nn}} - \frac{f_3^2\beta_1}{\Lambda_{mm}} + \frac{\left(f_3 f_4\beta_2 - f_3 f_4\beta_1\right)}{\Lambda_{nm}}\right], \\ C &= \left[\lambda^m \sin\alpha - \lambda^n \cos\alpha\right]\frac{v_\mu}{\sqrt{2}}, \\ D &= \left[\frac{f_4^2}{\Lambda_{nn}} + \frac{f_3^2}{\Lambda_{mm}} + \frac{2\left(f_3 f_4\right)}{\Lambda_{nm}}\right], \\ E &= f_{34}\left[\lambda^n \sin\alpha + \lambda^m \cos\alpha\right], \\ F &= M. \end{aligned} \tag{3.27}$$

.

In order to deal more easily with the mass matrix and since, as we have argued before, the global symmetry impose the hierarchy $\lambda^n >> \lambda^m$, let us introduce the ρ parameter defined as: $\lambda^n = \rho\lambda^m$, where $\rho >> 1$.

Now, as it is shown in Appendix A,

$$\begin{aligned}
\frac{1}{\Lambda_{nn}} &= (\lambda^n)^2 \frac{\lambda M}{16\pi^2}\left[\frac{-(m_n^c)^2 + m_\psi^2 - (m_n^c)^2 \ln\left(\frac{m_\psi^2}{(m_n^c)^2}\right)}{\left[(m_n^c)^2 - m_\psi^2\right]^2}\right], \\
\frac{1}{\Lambda_{mm}} &= (\lambda^m)^2 \frac{\lambda M}{16\pi^2}\left[\frac{-(m_n^c)^2 + m_\psi^2 - (m_n^c)^2 \ln\left(\frac{m_\psi^2}{(m_n^c)^2}\right)}{\left[(m_n^c)^2 - m_\psi^2\right]^2}\right], \\
\frac{1}{\Lambda_{nm}} &= \lambda^n\lambda^m \frac{\lambda M}{16\pi^2}\left[\frac{-(m_n^c)^2 + m_\psi^2 - (m_n^c)^2 \ln\left(\frac{m_\psi^2}{(m_n^c)^2}\right)}{\left[(m_n^c)^2 - m_\psi^2\right]^2}\right].
\end{aligned} \tag{3.28}$$

From the last expressions, the following relations are obtained:

$$\frac{1}{\Lambda_{mm}} = \frac{1}{\rho^2}\frac{1}{\Lambda_{nn}} \quad \text{and} \quad \frac{1}{\Lambda_{nm}} = \frac{1}{\rho}\frac{1}{\Lambda_{nn}}. \tag{3.29}$$

Also, taking into account that $\tan\alpha = \frac{f_3}{f_4}$, $\beta_1 = \frac{f_4}{f_3}\frac{v_\mu}{f_{34}\sqrt{2}}$ and $\beta_2 = \frac{f_3}{f_4}\frac{v_\mu}{f_{34}\sqrt{2}}$,

the matrix elements in Eq. (3.27) are re-written as:

$$
\begin{aligned}
A &= \frac{1}{\Lambda_{nn}}\left[\sin^2\alpha - \frac{2\sin\alpha\cos\alpha}{\rho} + \frac{\cos^2\alpha}{\rho^2}\right]\frac{v_\mu^2}{2}, \\
B &= \frac{1}{\Lambda_{nn}}\left[\sin\alpha + \frac{\sin\alpha(\tan\alpha - \cot\alpha)}{\rho} - \frac{\cos\alpha}{\rho^2}\right]\frac{f_4 v_\mu}{\sqrt{2}}, \\
C &= \left[\frac{\sin\alpha}{\rho} - \cos\alpha\right]\frac{\lambda^n v_\mu}{\sqrt{2}}, \\
D &= \frac{1}{\Lambda_{nn}}\left[1 + \frac{2\tan\alpha}{\rho} + \frac{\tan^2\alpha}{\rho^2}\right] f_4^2, \\
E &= \left[\tan\alpha + \frac{1}{\rho}\right]\lambda^n f_4, \\
F &= M.
\end{aligned} \tag{3.30}
$$

From here on, with the aim of diagonalizing the mass matrix and get a mass term for the SM neutrinos, let us express all matrix elements in terms of physical parameters (couplings, energy scales, scalar field mass, and Majorana mass terms). To this purpose, we are going to deal first with the term Λ_{nn}.
From Eq. (3.28)

$$
\begin{aligned}
\frac{1}{\Lambda_{nn}} &= (\lambda^n)^2 \frac{\lambda M}{16\pi^2}\left[\frac{-(m_n^c)^2 + m_\psi^2 - (m_n^c)^2 \ln\left(\frac{m_\psi^2}{(m_n^c)^2}\right)}{\left[(m_n^c)^2 - m_\psi^2\right]^2}\right] \\
&= (\lambda^n)^2 \frac{\lambda M}{16\pi^2}\frac{(m_n^c)^2}{(m_n^c)^4}\left[\frac{-1 + \left(\frac{m_\psi}{m_n^c}\right)^2 - \ln\left(\left(\frac{m_\psi}{m_n^c}\right)^2\right)}{\left[1 - \left(\frac{m_\psi}{m_n^c}\right)^2\right]^2}\right] \\
&= (\lambda^n)^2 \frac{\lambda M}{16\pi^2}\frac{1}{(m_n^c)^2}\left[\frac{x - 1 - \ln(x)}{(x-1)^2}\right],
\end{aligned} \tag{3.31}
$$

where $x = \dfrac{m_\psi}{m_n^c}$ and m_n^c is the mass of the heavy neutrino. We can give an approximate value of m_n^c from the diagonalization of the matrix in Eq. (3.19).

$$\begin{aligned}(m_n^c)^2 &= \frac{(2f_4^2+\cos^2\alpha v_\mu^2)\lambda_n^2(\rho-\tan\alpha)^2}{2\rho^2} \\ &= \frac{\left(2+\left(\frac{v_\mu}{f_4}\right)^2\cos^2\alpha\right)\left(1-\frac{\tan\alpha}{\rho}\right)^2}{2}(f_4\lambda^n)^2. \end{aligned} \quad (3.32)$$

Now, in the limit $\rho\to\infty$ and $f_4 >> v_\mu$, we have $(m_n^c)^2 = (f_4\lambda^n)^2$.

For simplicity, let us label:

$$\begin{aligned} Q^{-1} &= \frac{\left(2+\left(\frac{v_\mu}{f_4}\right)^2\cos^2\alpha\right)\left(1-\frac{\tan\alpha}{\rho}\right)^2}{2}, \\ L &= \frac{\lambda}{16\pi^2}\left[\frac{x-1-\ln(x)}{(x-1)^2}\right], \\ w_{11} &= \left[\sin^2\alpha-\frac{2\sin\alpha\cos\alpha}{\rho}+\frac{\cos^2\alpha}{\rho^2}\right], \\ w_{12} &= \left[\sin\alpha+\frac{\sin\alpha(\tan\alpha-\cot\alpha)}{\rho}-\frac{\cos\alpha}{\rho^2}\right], \\ w_{13} &= \left[\frac{\sin\alpha}{\rho}-\cos\alpha\right], \\ w_{22} &= \left[1+\frac{2\tan\alpha}{\rho}+\frac{\tan^2\alpha}{\rho^2}\right], \\ w_{23} &= \left[\tan\alpha+\frac{1}{\rho}\right]. \end{aligned} \quad (3.33)$$

Considering the definitions in Eqs. (3.31) and (3.33), the matrix entries in Eq. (3.30) acquire the form:

$$
\begin{aligned}
A &= \frac{1}{2} L Q w_{11} M \Big(\frac{v_\mu}{f_4}\Big)^2, \\
B &= \frac{1}{\sqrt{2}} L Q w_{12} M \Big(\frac{v_\mu}{f_4}\Big), \\
C &= \frac{1}{\sqrt{2}} w_{13} \big(\lambda^n v_\mu\big), \\
D &= L Q w_{22} M, \\
E &= w_{23} \big(\lambda^n f_4\big), \\
F &= M.
\end{aligned}
\tag{3.34}
$$

and the mass matrix is given by:

$$
\mathcal{M} = \begin{pmatrix}
\frac{1}{2} L Q w_{11} M \Big(\frac{v_\mu}{f_4}\Big)^2 & \frac{1}{\sqrt{2}} L Q w_{12} M \Big(\frac{v_\mu}{f_4}\Big) & \frac{1}{\sqrt{2}} w_{13} \big(\lambda^n v_\mu\big) \\
\frac{1}{\sqrt{2}} L Q w_{12} M \Big(\frac{v_\mu}{f_4}\Big) & L Q w_{22} M & w_{23} \big(\lambda^n f_4\big) \\
\frac{1}{\sqrt{2}} w_{13} \big(\lambda^n v_\mu\big) & w_{23} \big(\lambda^n f_4\big) & M
\end{pmatrix}. \tag{3.35}
$$

Extracting f_4 and introducing the parameters $a = \frac{v_\nu}{f_4}$ and $b = \frac{M}{f_4}$, where the hierarchy imposes that $a << 1$, $b << 1$, we have:

$$
\mathcal{M} = f_4 \begin{pmatrix}
\frac{1}{2} L Q w_{11} a^2 b & \frac{1}{\sqrt{2}} L Q w_{12} a b & \frac{1}{\sqrt{2}} w_{13} a \lambda^n \\
\frac{1}{\sqrt{2}} L Q w_{12} a b & L Q w_{22} b & w_{23} \lambda^n \\
\frac{1}{\sqrt{2}} w_{13} a \lambda^n & w_{23} \lambda^n & b
\end{pmatrix}, \tag{3.36}
$$

where $w_{ij} \sim Q \simeq 1$ for $\{i,j\} = 1,2$. We perform the calculations of $\mathcal{M}$ eigenvalues to third order in perturbation theory. The mass for the lightest neutrino reads:

$$
m_{\nu_e} = L Q \Big(\frac{1}{2} w_{11} - \frac{w_{12} w_{13}}{w_{23}} + \frac{1}{2} \frac{w_{13}^2 w_{22}}{w_{23}^2}\Big) a^2 b f_4. \tag{3.37}
$$

Expanding Eq. (3.37) and keeping only terms $\mathcal{O}(1/\rho)$, we reach:

$$
\begin{aligned}
m_{\nu_e} &= La^2bf_4\left(2+\frac{\cos^2\alpha v_\mu^2}{f_4^2}\right)^{-1}\Bigg[2\cos^2\alpha+\cos^2\alpha\cot^2\alpha+\sin^2\alpha+...\\
&+ \frac{1}{\rho}\Big(2\sin^2\alpha\tan\alpha+2\cos\alpha\sin\alpha-2\cos^2\alpha\cot^2\alpha-2\cos^2\alpha\cot\alpha\Big)\Bigg], \qquad (3.38)
\end{aligned}
$$

expression that can be re-written as

$$
m_{\nu_e} = La^2bf_4\left(2+\frac{\cos^2\alpha v_\mu^2}{f_4^2}\right)^{-1}\gamma, \qquad (3.39)
$$

with

$$
\begin{aligned}
\gamma &= \Bigg[2\cos^2\alpha+\cos^2\alpha\cot^2\alpha+\sin^2\alpha+...\\
&+ \frac{1}{\rho}\Big(2\sin^2\alpha\tan\alpha+2\cos\alpha\sin\alpha-2\cos^2\alpha\cot^2\alpha-2\cos^2\alpha\cot\alpha\Big)\Bigg]. \qquad (3.40)
\end{aligned}
$$

recalling that $a = v_\mu/f_4$ and $b = M/f_4$ we have:

$$
m_{\nu_e} = L\frac{v_\mu^2}{f_4^2}M\left(2+\frac{\cos^2\alpha v_\mu^2}{f_4^2}\right)^{-1}\gamma, \qquad (3.41)
$$

Since $v_\mu^2/f_4^2 << 1$ a binomial expansion produces

$$
m_{\nu_e} = \frac{1}{2}LM\frac{v_\mu^2}{f_4^2}\gamma, \qquad (3.42)
$$

Now, from Eq. (3.33)

$$
\begin{aligned}
L &= \frac{\lambda}{16\pi^2}\left[\frac{x-1-\ln(x)}{(x-1)^2}\right]\\
&\simeq \frac{\lambda}{16\pi^2}\Big(-\ln(x)\Big), \qquad (3.43)
\end{aligned}
$$

where $x = m_\psi/m_n^c$ and where is assumed $x << 1$, so that

$$m_{\nu_e} = \frac{\lambda M}{32\pi^2} \ln\left(\frac{m_n^c}{m_\psi}\right) \gamma \frac{v_\mu^2}{f_4^2}, \tag{3.44}$$

Taking back the γ definition given in Eq. (3.40), we finally get

$$\begin{aligned} m_{\nu_e} &= \frac{\lambda M}{32\pi^2} \ln\left(\frac{m_n^c}{m_\psi}\right) \Big[2\cos^2\alpha + \cos^2\alpha\cot^2\alpha + \sin^2\alpha + ... \\ &+ \frac{1}{\rho}\Big(2\sin^2\alpha\tan\alpha + 2\cos\alpha\sin\alpha - 2\cos^2\alpha\cot^2\alpha - 2\cos^2\alpha\cot\alpha \Big)\Big] \frac{v_\mu^2}{f_4^2} \end{aligned} \tag{3.45}$$

Let us compare our results with the ones in [41]. Before going on, three remarks must be done:

- In [41], the VEV of the Higgs-like field is written as $\langle h^0 \rangle = (0, v)^T$, which differs from our notation where $\langle h_\mu \rangle = \frac{1}{\sqrt{2}}(v_\mu, 0)^T$.
- Also in [41], the tree-level Lagrangian contains a VEV for the scalar field that is written as $\langle \phi_1^\dagger \rangle = (-\langle h^\dagger \rangle, f)$, being f the scale at which the global symmetry breaks down, while in our case $\langle \psi_1^\dagger \rangle = (\pm\langle h^\dagger \rangle, 0, \beta_{(1,2)} f)$.
- The coupling λ^m is absent in [41], which means $\rho \to \infty$.

Then, with the goal of doing a direct comparison[4], $v_\mu = \sqrt{2} v_\mu^D$ and $f \sin\beta_i \sim 1$ for $i = 1, 2$, which in our notation means $f_3/f_4 = f_3/f_{34} = f_4/f_{34} \sim 1$. This implies $\sin\alpha = \cos\alpha = \tan\alpha = 1$.
With the clarifications above, the neutrino mass is given by:

$$\begin{aligned} m_{\nu_e} &= \frac{\lambda M}{32\pi^2} \ln\left(\frac{m_n^c}{m_\psi}\right) \Big[1 + 2 + 1 \Big] \frac{\left(\sqrt{2} v_\mu^D\right)^2}{f_4^2} \\ &= \frac{\lambda M}{4\pi^2} \ln\left(\frac{m_n^c}{m_\psi}\right) \frac{{v_\mu^D}^2}{f_4^2}, \end{aligned} \tag{3.46}$$

that exactly coincides with the result in [41].

[4] With v_μ^D we refers to the value v_μ used in [41]

Now, setting $f_3 = f_4$ in Eq. (3.40) (where $\tan\alpha = f_3/f_4$), we obtain:

$$\begin{aligned} m_{\nu_e} &= \frac{\lambda M}{32\pi^2}\ln\left(\frac{m_n^c}{m_\psi}\right)\left[1+1+1+\frac{1}{\rho}\Big(1+1-1-1\Big)\right]\frac{v_\mu^2}{f_4^2} \\ &= \frac{\lambda M}{32\pi^2}\ln\left(\frac{m_n^c}{m_\psi}\right)\Big[3\Big]\frac{v_\mu^2}{f_4^2} \end{aligned} \tag{3.47}$$

A very interesting consequence of this result is that with $f_3 = f_4$, the diagram (see Fig. A.3 in Appendix A) involving the couplings λ^m and λ^n do not contributes to the neutrino mass. Such a situation occurs independently of the suppression imposed by the global symmetry on λ^m. As we can see from Eq. (3.46), when $f_3 \neq f_4$ contribution from this new diagram takes place and give small corrections. Notice that imposing a Z_2 symmetry that completely forbids the digram associated to the unsuppressed Yukawa coupling while allows the new diagram to survive, would allow us in turn to explain in a natural way the smallness of the neutrino mass. This idea, however, does not work because a Z_2 discrete symmetry destroys the collectively symmetry breaking which is a keystone in the Little Higgs model.

3.4 The three generation case

The generalization to three generations is straightforward from the analysis already done for one family. From Eq. (3.4), with α and β as flavour indexes, we have:

$$-\mathcal{L}_{yuk} = \lambda^n_{\alpha\beta}\overline{N}_{R\alpha}\Psi_1^\dagger L_{L\alpha} + \lambda^m_{\alpha\beta}\overline{N}_{R\alpha}\Psi_2^\dagger L_{L\beta} + h.c., \tag{3.48}$$

which can be transformed, analogously to Eq. (3.14), as:

$$-\mathcal{L}_{yuk} = f_{34}\overline{N}_{R\alpha}\left(\Big[\lambda^m_{\alpha\beta}\sin\alpha - \lambda^n_{\alpha\beta}\cos\alpha\Big]\frac{v_\mu}{f_{34}\sqrt{2}}\nu^0_{e\beta} + \Big[\lambda^n_{\alpha\beta}\sin\alpha + \lambda^m_{\alpha\beta}\cos\alpha\Big]n^0_{L\beta}\right) + h.c.$$

This expression can be easily reduced to a better form

$$-\mathcal{L}_{yuk} = (\overline{\mathbb{V}^c_L}, \overline{\mathbb{N}^c_L}, \overline{\mathbb{N}}_R)\mathcal{M}\begin{pmatrix}\mathbb{V}_L \\ \mathbb{N}_L \\ \mathbb{N}_R\end{pmatrix} + h.c., \tag{3.49}$$

where $\mathbb{V}_L = (\nu^0_{eL},\ \nu^0_{\mu L},\ \nu^0_{\tau L})^T$, $\mathbb{N}_L = (n^0_{eL},\ n^0_{\mu L},\ n^0_{\tau L})^T$, $\mathbb{N}_R = (N^0_{eR},\ N^0_{\mu R},\ N^0_{\tau R})^T$, and:

$$\mathcal{M} = \begin{pmatrix} [0] & [0] & \Big([\lambda^m]^\dagger\sin\alpha - [\lambda^n]^\dagger\cos\alpha\Big)\frac{v_\mu}{\sqrt{2}} \\ [0] & [0] & f_{34}\Big([\lambda^n]^\dagger\sin\alpha + [\lambda^m]^\dagger\cos\alpha\Big) \\ \Big([\lambda^m]\sin\alpha - [\lambda^n]\cos\alpha\Big)\frac{v_\mu}{\sqrt{2}} & f_{34}\Big([\lambda^n]\sin\alpha + [\lambda^m]\cos\alpha\Big) & [0] \end{pmatrix} \tag{3.50}$$

Here $[\lambda^m]$ and $[\lambda^n]$ are 3×3 matrices (where the family indexes have been omitted[5]), and $[0]$ are also 3×3 matrices with all the entries equal to zero.
Now, in the study of the one-generation case the parameter $\rho = \lambda^n/\lambda^m \gg 1$ makes explicit the suppression on λ^m due to the global symmetry. Since in the present case we are dealing with matrices, we are going to assume that $[\lambda^m]_{\alpha\beta} = [\epsilon]_{\alpha\gamma}[\lambda^m]_{\gamma\beta}$ being $\epsilon_{\alpha\beta} << 1$. Let us also assume that $[\epsilon]$ is diagonal with all the non-zero entries equal, then $[\epsilon] = \rho[1]$, with $[1]$ the identity matrix.
After some simplifications the mass matrix take the form:

$$\mathcal{M} = \begin{pmatrix} [0] & [0] & \left(\epsilon \sin\alpha - \cos\alpha\right) \frac{v_\mu}{\sqrt{2}}[\lambda^n]^\dagger \\ [0] & [0] & \left(\tan\alpha + \epsilon\right) f_4 [\lambda^m]^\dagger \\ \left([\epsilon \sin\alpha - \cos\alpha\right) \frac{v_\mu}{\sqrt{2}}[\lambda^n] & \left(\tan\alpha + \epsilon\right) f_4[\lambda^n] & [0] \end{pmatrix} \tag{3.51}$$

Let us now call $\mathbb{U}$ the unitary matrix that diagonalizes $\mathcal{M}$

$$\mathcal{M}^{Diag} = \mathbb{U}^\dagger \mathcal{M} \mathbb{U}, \tag{3.52}$$

where

$$\mathbb{U} = \begin{pmatrix} \cos\theta[1] & \sin\theta[1] & [0] \\ -\sin\theta[1] & \cos\theta[1] & [0] \\ [0] & [0] & [1] \end{pmatrix} \begin{pmatrix} [1] & [0] & [0] \\ [0] & \mathbb{V}_L & [0] \\ [0] & [0] & \mathbb{V}_R \end{pmatrix} \begin{pmatrix} [1] & [0] & [0] \\ [0] & \frac{-1}{\sqrt{2}}[1] & \frac{1}{\sqrt{2}}[1] \\ [0] & \frac{1}{\sqrt{2}}[1] & \frac{1}{\sqrt{2}}[1] \end{pmatrix}. \tag{3.53}$$

The mixing angles are given by Eq. (3.16), and can be re-written in terms of the trigonometric expressions w_{ij} in Eq. (3.33) where we have defined $\rho = \epsilon^{-1}$

$$\sin\theta = \frac{a}{\sqrt{2}} \frac{w_{13}}{\sqrt{w_{23}^2 + (w_{13}^2)\frac{a^2}{2}}} \;,\; \cos\theta = \frac{w_{23}}{\sqrt{w_{23}^2 + (w_{13}^2)\frac{a^2}{2}}}.$$

The matrices $\mathbb{V}_L$ and $\mathbb{V}_\mathbb{R}$ allow to diagonalize $[\lambda^n]$.

$$\mathbb{V}_R^\dagger [\lambda^n] \mathbb{V}_L = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}. \tag{3.54}$$

Assuming that the eigenvalues of $[\lambda^n]$ are real, Eq. (3.54) yields:

[5]That is, the matrix entries of $[\lambda^m]$ are $[\lambda^m]_{\alpha\beta}$

$$\mathbb{V}_L^\dagger[\lambda^n]^\dagger\mathbb{V}_R = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}.$$

After some calculations, the diagonal mass matrix in Eq. (3.52) take the form

$$\mathcal{M}^{Diag} = \begin{pmatrix} [0] & [0] & [0] \\ [0] & \delta_1 & [0] \\ [0] & [0] & \delta_2 \end{pmatrix}, \tag{3.55}$$

where

$$\begin{aligned} \delta_1 &= -\left((\epsilon\sin\alpha - \cos\alpha)\frac{v_\mu}{\sqrt{2}}\sin\theta + (\tan\alpha + \epsilon)f_4\cos\theta\right)\mathrm{diag}(\lambda_1^n, \lambda_2^n, \lambda_3^n). \\ \delta_2 &= \left((\epsilon\sin\alpha - \cos\alpha)\frac{v_\mu}{\sqrt{2}}\sin\theta + (\tan\alpha + \epsilon)f_4\cos\theta\right)\mathrm{diag}(\lambda_1^n, \lambda_2^n, \lambda_3^n). \end{aligned} \tag{3.56}$$

Finally we have

$$\mathcal{M}^{Diag} = \begin{pmatrix} \mathrm{diag}(0,0,0) & & \\ & -\mathrm{diag}(M_{N1}, M_{N2}, M_{N3}) & \\ & & \mathrm{diag}(M_{N1}, M_{N2}, M_{N3}) \end{pmatrix}, \tag{3.57}$$

with

$$M_{Ni} = \sqrt{w_{23}^2 + w_{13}^2\frac{a^2}{2}}(f_4\lambda_i^n). \tag{3.58}$$

At tree-level the U_{PMNS} mixing matrix[6] is given by Eq. (3.53). We can also write down the mass eigenstates $(\widehat{\mathbb{V}}_L, \widehat{\mathbb{N}}_L, \widehat{\mathbb{N}}_R^C)^T$ as a linear combination of weak eigenstates $(\mathbb{V}_L, \mathbb{N}_L, \mathbb{N}_R^C)^T$:

$$\begin{aligned} \begin{pmatrix} \mathbb{V}_L \\ \mathbb{N}_L \\ \mathbb{N}_R^C \end{pmatrix} &= \begin{pmatrix} \cos\theta[1] & \sin\theta[1] & [0] \\ -\sin\theta[1] & \cos\theta[1] & [0] \\ [0] & [0] & [1] \end{pmatrix} \begin{pmatrix} [1] & [0] & [0] \\ [0] & \mathbb{V}_L & [0] \\ [0] & [0] & \mathbb{V}_R \end{pmatrix} \begin{pmatrix} [1] & [0] & [0] \\ [0] & \frac{-1}{\sqrt{2}}[1] & \frac{1}{\sqrt{2}}[1] \\ [0] & \frac{1}{\sqrt{2}}[1] & \frac{1}{\sqrt{2}}[1] \end{pmatrix} \begin{pmatrix} \widehat{\mathbb{V}}_L \\ \widehat{\mathbb{N}}_L \\ \widehat{\mathbb{N}}_R^C \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta[1] & -\frac{\mathbb{V}_L}{\sqrt{2}}\sin\theta & \frac{\mathbb{V}_L}{\sqrt{2}}\sin\theta \\ -\sin\theta[1] & -\frac{\mathbb{V}_L}{\sqrt{2}}\cos\theta & \frac{\mathbb{V}_L}{\sqrt{2}}\cos\theta \\ [0] & \frac{1}{\sqrt{2}}\mathbb{V}_R & \frac{1}{\sqrt{2}}\mathbb{V}_R \end{pmatrix} \begin{pmatrix} \widehat{\mathbb{V}}_L \\ \widehat{\mathbb{N}}_L \\ \widehat{\mathbb{N}}_R^C \end{pmatrix} \end{aligned} \tag{3.59}$$

[6]Pontecorvo-Maki-Nakagawa-Sakata mixing matrix of the nine neutral leptons

This mixing matrix is incomplete in the sense that the lightest neutrino does not mixes with the right-handed ones. This feature will change after introducing a Majorana mass term for the three right-handed neutrinos. The radiative corrections will appear as a consequence of the one-loop level contribution to the mass matrix. Under these conditions the full mass matrix acquires the form:

$$\mathbb{M} = \mathcal{M}_{tree} + \mathcal{M}_{loop}, \tag{3.60}$$

where $\mathcal{M}_{tree}$ is the tree-level mass matrix which contains both Dirac and Majorana mass terms, and $\mathcal{M}_{loop}$ are the terms that arise at loop level.
Analogously to the one-generation case in Eq. (3.35), the Majorana mass term and the radiative correction to the mass matrix entries, are given by:

$$\mathcal{M}_M = \begin{pmatrix} \frac{1}{2}LQw_{11}[M]\left(\frac{v_\mu}{f_4}\right)^2 & \frac{1}{\sqrt{2}}LQw_{12}[M]\left(\frac{v_\mu}{f_4}\right) & 0) \\ \frac{1}{\sqrt{2}}LQw_{12}[M]\left(\frac{v_\mu}{f_4}\right) & LQw_{22}[M] & 0 \\ 0 & 0 & [M] \end{pmatrix}. \tag{3.61}$$

where the w_{ij} are re-defined in terms of ϵ by making the change $\epsilon \to 1/\rho$, and $[M] = diag(M_1, M_2, M_3)$ is the Majorana mass term for the three right handed neutrinos in a basis in which is already diagonal.
From Eq. (3.59) the tree-level PNMS mixing matrix is given by:

$$U_{PNMS} = \begin{pmatrix} \cos\theta[1] & -\frac{\mathbb{V}_L}{\sqrt{2}}\sin\theta & \frac{\mathbb{V}_L}{\sqrt{2}}\sin\theta \\ -\sin\theta[1] & -\frac{\mathbb{V}_L}{\sqrt{2}}\cos\theta & \frac{\mathbb{V}_L}{\sqrt{2}}\cos\theta \\ [0] & \frac{1}{\sqrt{2}}\mathbb{V}_R & \frac{1}{\sqrt{2}}\mathbb{V}_R \end{pmatrix}. \tag{3.62}$$

From Eq. (3.59) we can build up an operator $\mathbb{P}$ that projects only into the three lights neutrino states of the nine neutral leptons states. Easily is obtained:

$$\mathbb{P} = \begin{pmatrix} \cos^2\theta[1] & -\cos\theta\sin\theta[1] & [0] \\ -\cos\theta\sin\theta[1] & \sin^2\theta[1] & [0] \\ [0] & [0] & [0] \end{pmatrix}. \tag{3.63}$$

The mass eigenstates of the SM neutrinos arise from the solution of:

$$\det[\mathbb{P}\mathcal{M}_M\mathbb{P} - \Omega\mathbb{I}] = 0, \tag{3.64}$$

being Ω the mass eigenvalues that, at first order in perturbation theory, are:

$$\Omega_k = LQ\Big(\frac{1}{2}w_{11} - \frac{w_{12}w_{13}}{w_{23}} + \frac{1}{2}\frac{w_{13}^2 w_{22}}{w_{23}^2}\Big)a^2 M_k. \tag{3.65}$$

This latter expression is exactly the same that was previously found for the one-generation case.

Conclusions

In this work we have explored a mechanism for neutrino mass generation in a slight variation of the Simplest Little Higgs Model based on the approximate $[SU(4)/SU(3)]^4$ global symmetry. Since the highest scale of the theory is of the order of 10 TeV, and as the type I-like seesaw mechanism is not enough for neutrino mass generation, it was explore an alternative that allows the right-handed neutrino mass be at or below the TeV scale and account for neutrino masses. In this Little Higgs Model we have shown that one loop diagrams involving two scalar 4-plets naturally lead to small Majorana masses for the neutral leptons, included the SM neutrinos. We have found that the masses for the lightest neutrinos depend logarithmically on the Yukawa couplings, leading to unsuppressed Yukawa couplings with a theoretical cut-off of up to 10 TeV. The latter result is similar to the obtained by [41], but is also different in the sense that applies for a SLHM based on a different electroweak gauge symmetry, and is also more generic for two independent reasons: first, the global symmetry that protects the Higgs mass is assumed to be approximate in the neutral lepton sector, then an small (negligible) quadratic divergent contribution to the Higgs mass arises at one loop and, second, as the scalar fields VEV's are different, the neutrino masses also receive contributions from the operator $(\overline{L_L^c}\Psi_i^*)(\Psi_j^\dagger L_L)$. The analysis done in this thesis yields masses for the heavy Majorana neutrinos of the order of KeV; the light neutrino mass hierarchy (normal or inverted) is exactly the same for the heavy neutral leptons.

An analysis for the three generation case was also done. The results differ slightly from the ones in the one-generation case. Further exploration on the neutral lepton mixing are required in order to restrict the heavy leptons masses by using the currently known data on neutrino oscillations. On the other hand, the study of various decay modes of the heavy neutral leptons could lead to restrictions on the mixing between the heavy and light neutrinos. The origin of the Majorana mass for the right-handed neutrinos could be justified after to build up the UV completion of the model.

High precision experiments as WMAP soon will provide an answer for the number of neutrinos (active plus sterile) in Nature. Also, with the future International Linear Collider (ILC), the mass hierarchy of the heavy neutral leptons can be revealed. We expect that in the next coming years the experiments give us better insights of the nature of neutral leptons.

Appendices

Appendix A

Neutrino Loop Calculation

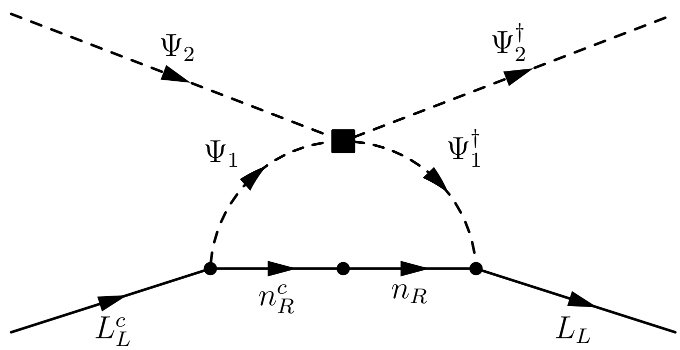

Figure A.1: One loop-contribution to $\left(\overline{L_L^c}\Psi_2^*\right)\left(\Psi_2^\dagger L_L\right)$.

From this diagram

$$\frac{i}{\Lambda_{nn}} = (-i\lambda)(i\lambda^n)^2(-iM)\int \frac{d^4k}{(2\pi)^4}\frac{i}{(p-k)^2-m^2}\frac{i}{(p-k)^2-m^2}\frac{i}{(\not{k})-m_{n^c}}\frac{i}{(-\not{k})-m_{n^c}} \quad (A.1)$$

The result must be independent of the momentum of the incoming lepton (L_L), so we can set this momentum to zero without loss of generality[1]. The propagator for the Majorana neutrino in momentum space is given by:

$$\begin{aligned} S_f &= \frac{i}{\not{k}-m_{n^c}} \\ &= \frac{i(\not{k}+m_{n^c})}{k^2-m_{n^c}^2}, \end{aligned} \quad (A.2)$$

[1]A non-zero momentum only shifts $\not{p}$ but not the neutrino mass

where we have used the field equation in momentum space $(\not{k} - m_{n^c})\psi = 0$[2].
Making $p = 0$, Eq. (A.1) becomes:

$$\frac{i}{\Lambda_{nn}} = -\lambda(\lambda^n)^2 M \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2]^2} \frac{1}{k^2 - m_{n^c}^2}. \tag{A.3}$$

In order to solve the integral we introduce the *Feynman parameters* [30]

$$\frac{1}{a_1^2 a_2 ... a_n} = n! \int_0^1 \frac{x_1 dx_1 dx_2 ... dx_n}{a_1 x_1 + a_2 x_2 ... a_n x_n} \delta\Big(1 - \sum_{i=1}^{n} x_i\Big), \tag{A.4}$$

Using this identity the loop integral (A.3) becomes

$$\frac{i}{\Lambda_{nn}} = -\lambda(\lambda^n)^2 M \int \frac{d^4k}{(2\pi)^4} 2 \int_0^1 \frac{x dx dy}{\big[(k^2 - m^2)x + (k^2 - m_{n^c}^2)y\big]^3} \delta(1 - (x + y)). \tag{A.5}$$

After using the δ function we have

$$\begin{aligned} \frac{i}{\Lambda_{nn}} &= -2\lambda(\lambda^n)^2 M \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{x dx}{\big[(k^2 - m^2)x + (k^2 - m_{n^c}^2)(1 - x)\big]^3} \\ &= -2\lambda(\lambda^n)^2 M \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{x dx}{\big[(k^2 - m^2)x + k^2 - m_{n^c}^2 - x(k^2 - m_{n^c}^2)\big]^3} \\ &= -2\lambda(\lambda^n)^2 M \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{x dx}{\big[k^2 - x(m^2 - m_{n^c}^2) - m_{n^c}^2\big]^3}. \end{aligned}$$

Performing now the Wick rotation $k^0 \to ik_e^0$ we can transform the integration over a 4D-Euclidean space[3]. So, $k^2 = k^\mu k_\mu = k^{02} - \vec{k}^2 = -k_e^{02} - \vec{k}^2 = -k_e^2$. Changing $dk^0 \to idk_e^0$ and $k^2 \to -k_e^2$ the loop integral acquires the form

$$\begin{aligned} \frac{i}{\Lambda_{nn}} &= -2i\lambda(\lambda^n)^2 M \int \frac{d^4k_e}{(2\pi)^4} \int_0^1 \frac{x dx}{\big[-k_e^2 - x(m^2 - m_{n^c}^2) - m_{n^c}^2\big]^3} \\ &= 2i\lambda(\lambda^n)^2 M \int \frac{d^4k_e}{(2\pi)^4} \int_0^1 \frac{x dx}{\big[k_e^2 - x(m^2 - m_{n^c}^2) - m_{n^c}^2\big]^3}. \end{aligned} \tag{A.6}$$

---

[2]Here ψ is a Dirac field

[3]A integral defined in a 4D hypersphere

Let us now consider the integral:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+c^2)^3}. \tag{A.7}$$

In general, the volume of a sphere of radius k in D-dimensional Euclidean space is given by $V_d(k) = C_d k^d$, where $C_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$, being Γ the gamma function which is defined by

$$\Gamma z = \int_0^\infty e^{-t} t^{z-1} dt. \tag{A.8}$$

For a 4-dimensional Euclidean space we have:

$$\begin{aligned} V &= \frac{\pi^2 k^4}{2}, \\ dV &= 2\pi^2 k^3 dk, \\ dk^4 &= 2\pi^2 k^3 dk, \end{aligned} \tag{A.9}$$

Using the previous result, Eq. A.7 yields

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+c^2)^3} &= \int_0^\infty \frac{2\pi^2 k^3 dk}{(2\pi)^4} \frac{1}{(k^2+c^2)^3} \\ &= \int_0^\infty \frac{2dkk^3}{16\pi^2(k^2+a^2)^3} \\ &= \int_0^\infty \frac{(2kdk)k^2}{16\pi^2(k^2+a^2)^3} \\ &= \int_0^\infty \frac{(dk^2)k^2}{16\pi^2(k^2+a^2)^3}. \end{aligned} \tag{A.10}$$

Using the fact that

$$\int_0^\infty \frac{t^{m-1}dt}{(t+c^2)^n} = \frac{1}{(c^2)^{(n-m)}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}, \tag{A.11}$$

and setting $m = 2$ and $n = 3$, Eq. (A.10) becomes equal to Eq. (A.11). It is now straightforward the calculation of the 4-dimensional integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+c^2)^3} = \frac{1}{32\pi^2 c^2}. \tag{A.12}$$

With this result we can solve the 4-dimensional integral over k_e in Eq. (A.6). We get

$$\frac{i}{\Lambda_{nn}} = \frac{i\lambda(\lambda^n)^2 M}{16\pi^2}\int_0^1 \frac{xdx}{\left[x(m^2 - m_{n^c}^2) + m_{n^c}^2\right]}, \tag{A.13}$$

Let us, for simplicity, label $a = m^2 - m_{n^c}^2$ and $b = m_{n^c}^2$, and to make the change of variables, $u = ax + b$. After some direct calculations we obtain

$$\frac{i}{\Lambda_{nn}} = \frac{i\lambda(\lambda^n)^2 M}{16\pi^2 a^2}\left[a - b\ln\left(\frac{a}{b} + 1\right)\right]. \tag{A.14}$$

Finally the loop calculation yields the expression

$$\frac{i}{\Lambda_{nn}} = \frac{i\lambda(\lambda^n)^2 M}{16\pi^2}\left[\frac{m^2 - m_{n^c}^2 - m_{n^c}^2\ln\left(\frac{m^2}{m_{n^c}^2}\right)}{(m^2 - m_{n^c}^2)^2}\right]. \tag{A.15}$$

For the other one-loop diagrams the calculation is similar. The first one is

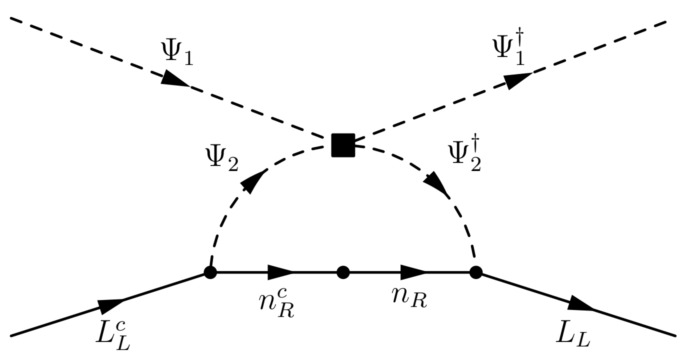

Figure A.2: One loop-contribution to $\left(\overline{L_L^c}\Psi_1^*\right)\left(\Psi_1^\dagger L_L\right)$.

In this case the loop contribution is given by

$$\frac{i}{\Lambda_{mm}} = \frac{i\lambda(\lambda^m)^2 M}{16\pi^2}\left[\frac{m^2 - m_{n^c}^2 - m_{n^c}^2\ln\left(\frac{m^2}{m_{n^c}^2}\right)}{(m^2 - m_{n^c}^2)^2}\right]. \tag{A.16}$$

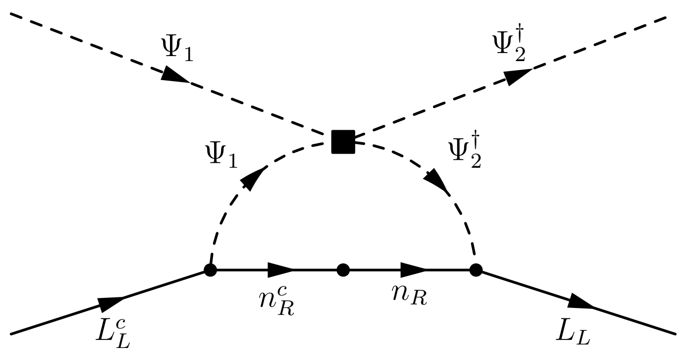

Figure A.3: One loop-contribution to $\left(\overline{L_L^c}\Psi_1^*\right)\left(\Psi_2^\dagger L_L\right)$.

such a contribution is very small due to the suppression on the coupling λ^m imposed by the global $SU(4)$ symmetry and can be considered negligible.
Finally there are another diagram (see Fig. A.3), this diagram gives the contribution:

$$\frac{i}{\Lambda_{nm}} = \frac{i\lambda(\lambda^m\lambda^n)M}{16\pi^2}\left[\frac{m^2 - m_{n^c}^2 - m_{n^c}^2 \ln\left(\frac{m^2}{m_{n^c}^2}\right)}{(m^2 - m_{n^c}^2)^2}\right]. \tag{A.17}$$

This contribution is also small as a consequence of the suppression on the coupling λ^m and is considered in this work as a contribution to the mass matrix.

Appendix B

Preprint: Unification of gauge coupling constants

In this appendix is included a work that is still under construction associated to the unification of gauge coupling in two gauge extensions of the Standard Model.

International Journal of Modern Physics A

GAUGE COUPLINGS UNIFICATION IN THE 3-4-1 AND 4-2-1 EXTENSIONS OF THE STANDARD MODEL

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We examine the gauge couplings unification of the $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ and $SU(4) \otimes SU(2)_L \otimes U(1)_X$ extensions of Standard Model (SM). Based on the Renormalization Group Equations (RGEs), we compared the behaviour of the couplings constants with the energy scale and is shown in which cases is possible to achieve unification depending on the model under consideration.

Keywords: unification; Renormalization Group Equation; electroweak extension; color extension.

PACS numbers:11.10Hi, 11.10.Gh, 11.15.-q , 11.15.Ex, 11.15.-q, 12.40.-y, 12.60.-i

1. Introduction

The Standard Model (SM) of elementary particles physics remains as the best theory which, at present, explain almost all physical processes in Nature. Is based on gauge group $G_{SM} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ which breaks down to $SU(3)_C \otimes U(1)_Q$ after the Spontaneous Symmetry Breaking(SSB) take place. Recently, the CMS and ATLAS collaboration at CERN announced the discovery of a scalar particle[1,2] that seems to be the Higgs boson of the SM with a mass around 126 GeV; additionally new measurements in the ratio of the branching fractions $\mathcal{R}(D^{(*)}) = \mathcal{B}r(\overline{B} \to D^{(*)}\tau^-\overline{\nu}_\tau)/\mathcal{B}r(\overline{B} \to D^{(*)}l^-\overline{\nu}_l)$(where l is either e or μ) reported by BaBar collaboration[3] are in not accordance with SM predictions; giving us a strong clue of new physics beyond the electroweak scale. From a theoretical point of view there are also reasons to believe that the SM is not the final theory. Lots of question remains unanswered such as the electroweak hierarchy problem, neutrino masses, electric charge quantization, the unification of gauge couplings. Among all the theories that go beyond the SM one of the best perspectives is the unification idea, where in principle all the fundamental forces between elementary particles (strong, weak and electromagnetic) are different manifestations of the same fundamental interaction involving a single coupling strength[4]. This idea makes sense when all the gauge

couplings constants meet in a single point at some energy scale.
Since the birth of the SM, $SU(5)$ was proposed as the first attempt to unify all the fundamental interactions in Nature[4]. When the SM gauge group G_{SM} is embedded into $SU(5)$ new interesting phenomenology appear, and the model itself is able to go further and explain charge quantization, introduces a new symmetry in Nature at high energies which consist in B-L[5] conservation, add news intermediate vector bosons called lepto-quarks which allow for a quark transforms into a lepton and, as as a consequence, lead to the proton decay. However $SU(5)$ is rule out because their theoretical prediction are in not agreements with the experimental data, in particular the place where all the couplings constants meet is an entire region, and not a single point. As a requirement for embed the G_{SM} into an Unification Group G_{GUT} we must guarantee that all the low-energy generators be normalized in the same way, leading to a well-normalized structure to the hypercharge operator, and such normalization will depends completely of choice of the GUT group[a].
With the aim of solve the SM puzzles, it has to be extended in different ways, specifically the quest for unification require the inclusion of additional particle content that modifies the Callan-Symanzik β-functions leading to the unification of gauge couplings. Several attempts have been done, such as; a polychromatic extension [6], the inclusion of vector-like particles[7,8] with masses between $1-100$ TeV , and the inclusion of a color sextet scalar at the TeV scale and a lepton triplet[9] . Other theories beyond the SM include the possibility of extending either the electroweak or color sector. Electroweak extensions has been widely study and the most famous are the models with gauge group $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$[10] and $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ [11] known as the 331 and 341 extension, respectively. The other scenario concerns to the possibility of extend the color group, where $SU(3)_C$ is considered as a low-energy remnant symmetry of a larger group G_C. In Ref. 12 it was shown that in order to reproduce the low-energy phenomenology and assuming that G_C is a simple group which breaks down to $SU(3)_C$ by introducing only one colored Higgs multiple, then the only possible choices are $G_C = SU(k)$ for $k = 4, 5$. An analysis of the Renormalization Group Equations (RGEs) for the 331 extension was done[13] and was shown in which cases there are unification of gauge coupling constants. Here, by analysing the RGEs, we carry out a study of the running of gauge coupling constants in: The 341 extension, such extension can account for the number of generations of fermions when cancellation of anomalies take places between families and not family by family as in the SM, and to the $SU(4) \otimes SU(2)_L \otimes U(1)_X$ color extension. In this paper we do not pretend to embed the extensions into a GUT group; then for a sake of simplicity, we introduce a free parameter to define the well-normalize hypercharge generator at the different scales of energy and explore the possibility of unification at some energy scale.
This letter is organized as follows. In section 2 we review how evolves the couplings constants at one loop order. In section 3 we introduce the 341 extension, establish

[a]For instance, you can review the $SU(5)$ gauge theory in Ref. 5

the matching condition and evaluate the unification of gauge couplings for two different models. In section 4 we review the 421 color extension, evaluate the matching condition and carry out an analysis on the running of couplings constants, finally our conclusion is outline in section 5.

2. Evolutions of Couplings Constants

The evolution of the running couplings constants at one-loop order is given by the Renormalization Group Equations (RGEs), and can be written as

$$\frac{1}{\alpha_i(\mu_2)} = \frac{1}{\alpha_i(\mu_1)} - \frac{b_i}{2\pi} ln\left(\frac{\mu_2}{\mu_1}\right), \tag{1}$$

with $\alpha_i \equiv \frac{g_i^2}{4\pi}$ and b_i the coefficients which depend on the particle content of the model under consideration. They are the coefficient of the β-function in the Callan-Symanzik equation[14], and are given by[15]

$$b_i = \frac{2}{3}\sum_f T_{Ri}(f) + \frac{1}{3}\sum_s T_{Ri}(s) - \frac{11}{3}C_{2i}(G). \tag{2}$$

Here the summations over $T_{Ri}(f)$, $T_{Ri}(s)$ and $C_{2i}(G)$ are the contribution to the b_i coming from Weyl fermions, scalars and gauge bosons respectively. T_{Ri} are the Dynking index[b] and $C_2(G)$ is the quadratic Casimir for the adjoint representation. For fundamental representations of $SU(N)$ we have: $T_R = \frac{1}{2}$ and $C_2(G) = d(R) = N$, being $d(R)$ the dimension of the representation. For the abelian group $U(1)$ is clear that: $C_2(G) = 0$ and $T_R = \sum Y^2$, with Y, the hypercharge generator. Given the situation of have particle content in the 2nd rank antisymmetry irreducible representation of $SU(N)$, then $T_R = (N-2)/2$ and to the 2nd rank symmetry irreducible representation, $T_R = (N+2)/2$.

3. $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ extension

The gauge theory based on $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$(hereafter 341) is an electroweak extension which arises as an alternative in the quest for new physics beyond the SM. This extension has a stronger theoretical motivation based on its ability to explain the replication of fermions in the Nature when anomaly cancellation take place between families. Also increase the amount of known fermions, what bring a new phenomenology still under exploration. By Demanding particle content without exotic electric charge [20], there are in principle few realistic models, the so called one-family models(three-family model) where the cancellation of anomalies take place family by family(between families) which are the phenomenological accepted

[b]These coefficients are group theory factors, and do not dependent on the spin of particles

models.

In the 341 extension, the electroweak sector of the SM $SU(2)_L \otimes U(1)_Y$ is enlarged to $SU(4)_L \otimes U(1)_X$[17,18], and the following symmetry breaking pattern is assumed:

$$3-4-1 \xrightarrow{V'} 3-3-1 \xrightarrow{V} 3-2-1 \xrightarrow{v+v'} 3-1, \tag{3}$$

where 3-1 refers to the $SU(3)_C \otimes U(1)_Q$ symmetry.

The scalar sector responsible for the patter breaking given in Eq. (3) is expressed by:

$$\begin{aligned}
\langle \phi_1^T \rangle &= \langle (\phi_1^0, \phi_1^+, \phi_1'^+, \phi_1'^0) \rangle = (v,0,0,0) \sim [1,4^*,1/2]\,,\\
\langle \phi_2^T \rangle &= \langle (\phi_2^-, \phi_2^0, \phi_2'^0, \phi_2'^-) \rangle = (0,v',0,0) \sim [1,4^*,-1/2]\,,\\
\langle \phi_3^T \rangle &= \langle (\phi_3^-, \phi_3^0, \phi_3'^0, \phi_3'^-) \rangle = (0,0,V,0) \sim [1,4^*,-1/2]\,,\\
\langle \phi_4^T \rangle &= \langle (\phi_4^0, \phi_4^+, \phi_4'^+, \phi_4'^0) \rangle = (0,0,0,V') \sim [1,4^*,1/2]\,.
\end{aligned} \tag{4}$$

Where we impose the hierarchy condition $V \sim V' >> v \sim v' \simeq 174 \quad \text{GeV}$.

3.1. *Matching conditions*

The charge generator in the 341 extension is written as a linear combination of the diagonal generators[17].

$$Q = T_{3L} + \frac{1}{\sqrt{3}} b T_{8L} + \frac{1}{\sqrt{6}} c T_{15L} + X_{341} I_4, \tag{5}$$

where $T_{iL} = \lambda_{iL}/2$, being the λ_{iL} the Gell-Mann matrices for $SU(4)_L$ normalized as $Tr(\lambda_i \lambda_j) = 2\delta_{ij}$, $I_4 = Dg(1,1,1,1)$ is the diagonal 4×4 matrix, and b and c are free parameter to be chosen according to the model under consideration[11]. Using the Gell-Mann Nishijima formula $Q = T_{3L} + Y$, the SM hypercharge can be written as

$$Y = \frac{1}{\sqrt{3}} b T_{8L} + \frac{1}{\sqrt{6}} c T_{15L} + X_{341} I_4. \tag{6}$$

In order to achieve unification the hypercharges must be well-normalized. Let us assume that $Y = a\widetilde{Y}$ and $X_{341} = d\widetilde{X}_{341}$, here a and d ensures the well-normalization of the hypercharges. Finally the hypercharge is written as

$$a\widetilde{Y} = \frac{1}{\sqrt{3}} b T_{8L} + \frac{1}{\sqrt{6}} c T_{15L} + d\widetilde{X}_{341}, \tag{7}$$

assuming that all the generator are normalized in the same way, it follows

$$a^2 = \frac{b^2}{3} + \frac{c^2}{6} + d^2. \tag{8}$$

At the energy scale M_X, where the 331 symmetry breaks down to the 321 gauge symmetry of the SM, the condition $\alpha^{-1}_{X_{331}} = \alpha^{-1}_Y$ must be satisfied(the first matching condition). From Eq. (5) is possible to establish a relationship between the hypercharges of the 331 extension and the SM, so

$$Y = \frac{1}{\sqrt{3}} bT_{8L} + X_{331}. \tag{9}$$

The well-normalized operator associated to the 3-3-1 hypercharge is given by: $X_{331} = f\widetilde{X}_{331}$, which leads to:

$$a\widetilde{Y} = \frac{1}{\sqrt{3}} bT_{8L} + f\widetilde{X}_{331}, \tag{10}$$

$$a^2\widetilde{\alpha}^{-1}_Y = \frac{b^2}{3}\alpha^{-1}_{3L} + f^2\widetilde{\alpha}^{-1}_{X_{331}}, \tag{11}$$

thus we get

$$\widetilde{\alpha}^{-1}_{X_{331}} = \frac{1}{a^2 - \frac{b^2}{3}}\Big(\alpha^{-1}_Y - \frac{b^2}{3}\alpha^{-1}_{3L}\Big). \tag{12}$$

Then, at M_X where $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ breaks down to G_{SM} we have

$$\alpha^{-1}_{X_{331}}(M_X) = \alpha^{-1}_Y(M_X) - \frac{b^2}{3}\alpha^{-1}_{3L}(M_X). \tag{13}$$

The second matching condition arise at $M_{X'}$ where 341 breaks to the 331 gauge symmetry. In this case the hypercharges relation is given by:

$$X_{331} = \frac{1}{\sqrt{6}} cT_{15L} + X_{341}, \tag{14}$$

and taking into account the well-normalized structure of the hypercharge generator, Eq. (14) becomes

$$f\widetilde{X}_{331} = \frac{1}{\sqrt{6}} cT_{15L} + d\widetilde{X}_{341}. \tag{15}$$

With those conditions, the behaviour of the hypercharge generators at different energy scales is determinate.

3.2. *Couplings Constants*

Using the two matching conditions and Eq. (1) we find that the coupling constants at one loop order are expressed by

$$\widetilde{\alpha}^{-1}_{X_{341}} = \left(\frac{1}{a^2 - \frac{b^2}{3} - \frac{c^2}{6}}\right)\Bigg[\alpha^{-1}_Y(M_Z) - \Big(\frac{b^2}{3} + \frac{c^2}{6}\Big)\alpha^{-1}_{2L}(M_Z) - \frac{(b_Y - (\frac{b^2}{3} + \frac{c^2}{6})b_{2L})}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{(\overset{331}{b_X} - \frac{c^2}{6}b_{3L})}{2\pi} ln\left(\frac{M'_X}{M_X}\right) - \frac{\overset{341}{b_X}}{2\pi} ln\left(\frac{M_U}{M'_X}\right)\Bigg] \tag{16}$$

$$\alpha^{-1}_{4L} = \alpha^{-1}_{2L}(M_Z) - \frac{b_{2L}}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{b_{3L}}{2\pi} ln\left(\frac{M'_X}{M_X}\right) - \frac{b_{4L}}{2\pi} ln\left(\frac{M_U}{M'_X}\right) \tag{17}$$

$$\alpha^{-1}_{3C} = \alpha^{-1}_S(M_Z) - \frac{b_S}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{\overset{331}{b_{3C}}}{2\pi} ln\left(\frac{M'_X}{M_X}\right) - \frac{\overset{341}{b_{3C}}}{2\pi} ln\left(\frac{M_U}{M'_X}\right). \tag{18}$$

Where b_S, b_{2L} and b_Y are the b_i coefficients of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ respectively, and are calculate for energies at the range $M_Z \leq \mu \leq M_X$. The coefficients b_{3C}^{331}, b_{3L} and b_X^{331} are related with $SU(3)_C$, $SU(3)_L$ and $U(1)_X$, and they are calculated for energies at the range $M_X \leq \mu \leq M_{X'}$. The coefficients b_{3C}^{341}, b_{4L} and b_X^{341} are related with $SU(3)_C$, $SU(4)_L$ and $U(1)_X$, and they are calculated for energies at the range $M_{X'} \leq \mu \leq M_U$.
The input parameters at the electroweak scale are [19]

$$\begin{aligned} \alpha_Y^{-1}(M_Z) &= 127.934 \pm 0.027, \\ \alpha_{2L}^{-1}(M_Z) &= 29.56938 \pm 0.00068, \\ \alpha_s(M_Z) &= 0.1172 \pm 0.00068, \\ \sin^2\theta_w(M_Z) &= 0.23113 \pm 0.00015. \end{aligned} \tag{19}$$

From the equations for α_{3C}^{-1} and α_{4L}^{-1} we find that the unification mass M_U is given by:

$$M_U = M_X' \left(\frac{M_X}{M_Z}\right)^{-\dfrac{b_{2L}-b_S}{b_{4L}-b_{3C}^{341}}} \left(\frac{M_X'}{M_X}\right)^{-\dfrac{b_{3L}-b_{3C}^{331}}{b_{4L}-b_{3C}^{341}}} \exp\left(2\pi \frac{\alpha_{2L}^{-1}(M_Z) - \alpha_S^{-1}(M_Z)}{b_{4L}-b_{3C}^{341}}\right) \tag{20}$$

The hierarchy condition $M_{X'} \leq M_U \leq M_{Planck}$, must be satisfied. We impose the stronger condition $M_U \leq 10^{18} GeV$ in order to avoid gravitational effects at that scale of energy. The hierarchy condition becomes

$$M_{X'} \leq M_U \leq 10^{18} GeV. \tag{21}$$

From α_{4L}^{-1} and $\widetilde{\alpha}_{X_{341}}^{-1}$ at the unification scale and using the previous result, we express the parameter a^2 as a function of M_x and $M_{x'}$

$$\left(\frac{1}{a^2 - \frac{b^2}{3} - \frac{c^2}{6}}\right)\alpha_{X_{341}}^{-1} = \alpha_{4L}^{-1}, \tag{22}$$

$$\begin{aligned} a^2 = \frac{b^2}{3} + \frac{c^2}{6} + &\Bigg\{\alpha_Y^{-1}(M_Z) - \Big(\frac{b^2}{3} + \frac{c^2}{6}\Big)\alpha_{2L}^{-1}(M_Z) - \frac{(b_Y - (\frac{b^2}{3}+\frac{c^2}{6})b_{2L})}{2\pi} ln\left(\frac{M_X}{M_Z}\right) \\ &- \frac{b_X^{331} - \frac{c^2}{6}b_{3L}}{2\pi} ln\left(\frac{M_X'}{M_X}\right) - \frac{b_X^{341}}{2\pi}\Bigg[-\left(\frac{b_{2L}-b_S}{b_{4L}-b_{3C}^{341}}\right) ln\left(\frac{M_X}{M_Z}\right) \\ &- \left(\frac{b_{3L}-b_{3C}^{331}}{b_{4L}-b_{3C}^{341}}\right) ln\left(\frac{M_{X'}}{M_X}\right) + 2\pi\left(\frac{\alpha_{2L}^{-1}(M_Z) - \alpha_S^{-1}(M_Z)}{b_{4L}-b_{3C}^{341}}\right)\Bigg]\Bigg\} \\ &\times \Bigg\{\alpha_{2L}^{-1}(M_Z) - \frac{b_{2L}}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{b_{3L}}{2\pi} ln\left(\frac{M_X'}{M_X}\right) - \frac{b_{4L}}{2\pi}\Bigg[-\left(\frac{b_{2L}-b_S}{b_{4L}-b_{3C}^{341}}\right) ln\left(\frac{M_X}{M_Z}\right) \\ &- \left(\frac{b_{3L}-b_{3C}^{331}}{b_{4L}-b_{3C}^{341}}\right) ln\left(\frac{M_{X'}}{M_X}\right) + 2\pi\left(\frac{\alpha_{2L}^{-1}(M_Z) - \alpha_S^{-1}(M_Z)}{b_{4L}-b_{3C}^{341}}\right)\Bigg]\Bigg\}^{-1} \end{aligned} \tag{23}$$

what follows now is to considerer different models of the 341 extension of the SM.

3.3. *Application I: One-Family model*

Here we considerer the Model D[11] in which the cancellation of chiral anomalies take place family by family. By demanding models without exotic electric charge there are few possibilities of values that can be taken by b and c in the electric charge operator given in Eq. (5). In particular by fixing $b = 1$ and $c = 1$ a set of models is reach, among them, the so-called model D.

Its fermion content is

$$\begin{aligned}
\psi_{1L} &= (e^-, \nu_e^0, N^0, N'^0)_L^T \sim (1, 4^*, -1/4), \\
\psi_{2L} &= (E_0^-, N_1^0, N_2^0, N_3^0)_L^T \sim (1, 4^*, -1/4), \\
\psi_{3L} &= (N_4^0, E_1^+, e^+, E_2^+)_L^T \sim (1, 4^*, 3/4), \\
Q_L &= (u, d, D, D')_L^T \sim (3, 4, -1/12), \\
u_L^c &\sim (3^*, 1, -2/3), d_L^c \sim (3^*, 1, 1/3), D_L^c \sim (3^*, 1, 1/3), \\
D_L'^c &\sim (3^*, 1, 1/3), E_{2L}^- \sim (1, 1, -1)
\end{aligned} \tag{24}$$

there are other two generations which are copies of the first one.

Due to the patter breaking, before obtain the particle content of the SM, first we must considerer as intermediate step the 331 extension. From Eq. (14) is possible by making a hypercharge analysis, identify which model is reach in the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ extension when the Model D of $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ breaks down to it. Such model called as Model A[21] and is also an one-family model, with the next fermion content.

$$\begin{aligned}
\psi_{1L} &= (e^-, \nu_e^0, N^0)_L^T \sim (1, 3^*, -1/3), \\
\psi_{2L} &= (E_0^-, N_1^0, N_2^0)_L^T \sim (1, 3^*, -1/3), \\
\psi_{3L} &= (N_4^0, E_1^+, e^+)_L^T \sim (1, 3^*, 2/3), \\
Q_L &= (u, d, D)_L^T \sim (3, 3, 0), \\
u_L^c &\sim (3^*, 1, -2/3), d_L^c \sim (3^*, 1, 1/3), D_L^c \sim (3^*, 1, 1/3)
\end{aligned} \tag{25}$$

By using the expression to calculate the coefficients b_i giving in Eq. (2) , and taking into consideration the quantum number assigned for each representation in this model, at energies below of M_X the coefficients take the next form:

$$\begin{aligned}
b_Y &= \frac{20}{9} N_g + \frac{1}{6} N_H, \\
b_{2L} &= \frac{4}{3} N_g + \frac{1}{6} N_H - \frac{22}{3}, \\
b_S &= \frac{4}{3} N_g - 11,
\end{aligned} \tag{26}$$

where N_g is the number of families, N_H the number of scalar doublets in $SU(2)_L$. Here at low energies, the 341 extension leads to a two Higgs double model, then

$N_H = 2$.
For energies in the range $M_X \leq \mu \leq M_{X'}$ the b_i's coefficient are giving by:

$$\begin{aligned} b_X^{331} &= \frac{8}{3}N_g + \frac{2}{3}, \\ b_{3L} &= 2N_g + \frac{1}{2} - 11, \\ b_{3C}^{331} &= 2N_g - 11. \end{aligned} \tag{27}$$

And finally for energies above the $M_{X'}$,the b_i's are expressed by:

$$\begin{aligned} b_X^{341} &= \frac{62}{18}N_g + \frac{4}{3}, \\ b_{4L} &= 2N_g - \frac{42}{3}, \\ b_{3C}^{341} &= \frac{8}{3}N_g - 11. \end{aligned} \tag{28}$$

Then for the spectrum showed in Eq. (24) we obtain for energies above $M_{X'}$

$$\left(b_X^{341}, b_{4L}, b_{3C}^{341}\right) = \left(\frac{70}{6}, -8, -4\right) \tag{29}$$

Here $b_{4L} < b_{3C}^{341}$ and as a consequence, α_{4L} and α_{3C} diverges leading a scenario where unification of gauge couplings in not accomplish. To solve this threat, we extend the scalar extension of the model, in an analogous way to the procedure carried out in Ref. 22, where, by the introduction of an additional scalar field transforming as a 2nd symmetry representation of $SU(4)$, can explain the smallness of neutrino particle. Here we considerer a set of m decuplets $\Delta \sim (1, 10, z)$, being z its hypercharge. By choosing $z = 0$, then this new scalar couples only to right-handed neutrinos $N_R \sim (1, 1, 0)$, which can be added into the model without loss of generality [c]. The scalar Δ modifies the b_{4L} coefficient:

$$\delta b_{4L} = 1. \tag{30}$$

In order to make b_{4L} a little bigger than b_{3C}^{341} we considerer a set of $m = 5$. that is, 5 scalar $\Delta \sim (1, 10, 0)$ are include into the model. The new values of the b_i coefficients are:

$$\left(b_X^{341}, b_{4L}, b_{3C}^{341}\right) = \left(\frac{70}{6}, -3, -4\right). \tag{31}$$

For energies in the range $M_X \leq \mu \leq M_{X'}$ where the particle content is the spectrum showed in Eq. (25) we have:

$$\left(b_X^{331}, b_{3L}, b_{3C}^{331}\right) = \left(\frac{26}{3}, -\frac{9}{2}, -5\right) \tag{32}$$

and

$$\left(b_Y, b_{2L}, b_S\right) = \left(\frac{22}{3}, -3, -7\right). \tag{33}$$

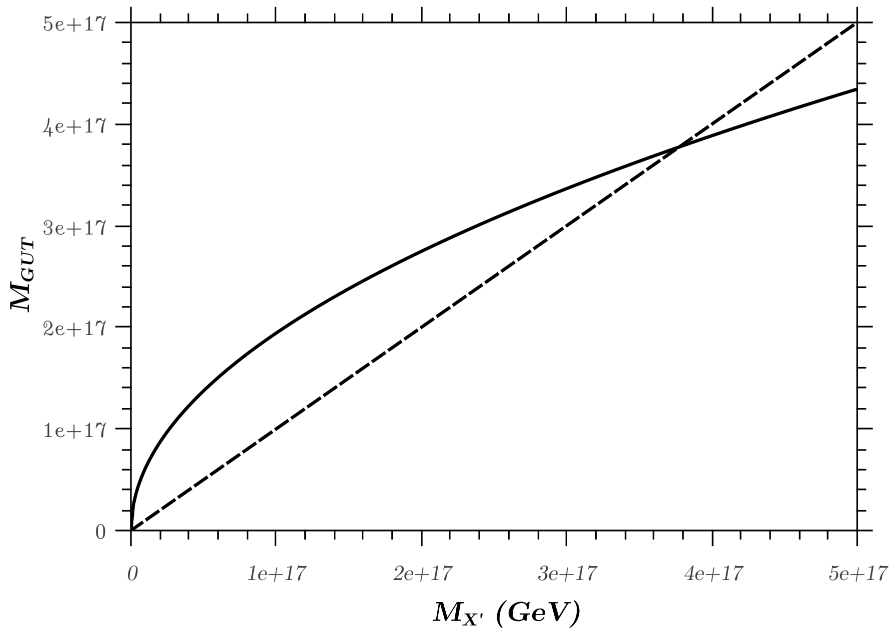

Fig. 1. Unification mass as a function of $M_{X'}$.

for energies below M_X.

Using the hierarchy condition we plot the allowed interval for $M_{X'}$. In Fig. 1, the solid line represent M_U as a function of $M_{X'}$ and the dashed line represents the functions $M_U = M_{X'}$. From this plot we fixed $M_{X'} = 2 \times 10^{17}$ GeV, which is a value that match with the hierarchy.
For sake of simplicity, we fixed the scale $M_X = 1.5 \times 10^{16}$, and obtain an Unification mass $M_U \sim 2.7 \times 10^{17}$ GeV, at that energy scale the couplings are going to met, as we are going to show below. The normalization parameter satisfies $a^2 \geq 0.5$, leading the next restriction $M_{X'} \geq 1.1 \times 10^{12}$ GeV and as a consequence the previous choice for $M_{X'}$ match perfectly. We found that $a^2 = 1.83135$.
After fixing the free parameter, we evaluate the running couplings constants in Fig. 2, and as is pointed out above, the unification scale appears at $M_U \sim 2.7 \times 10^{17}$ GeV.

[c]The right-handed neutrinos do not affect the b_i coefficients

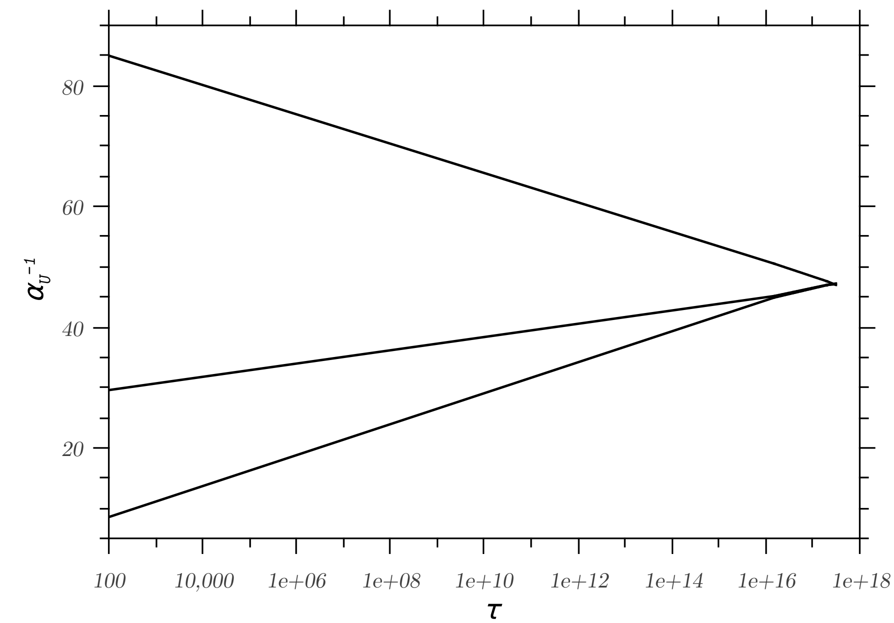

Fig. 2. Runnnig of gauge couplings constants .

3.4. *Application II: Three-Family model*

Let us now considerer a model in which the cancellation of anomalies take place between families. In Ref. 11, by choosing $b = 1$ and $c = 1$ we find two three-family structures, among them the so called model A with the next spectrum

$$\begin{aligned}
Q_{iL} &= (u, d, D, D')^T_L \sim (3, 4, -1/12), u^c_{iL} \sim (3^*, 1, -2/3), \\
d^c_{iL} &\sim (3^*, 1, 1/3),\ D^c_{iL} \sim (3^*, 1, 1/3),\ D'^c_{iL} \sim (3^*, 1, 1/3), \\
Q_{3L} &= (d, u, U, U')^T_L \sim (3, 4^*, 5/12), u^c_{3L} \sim (3^*, 1, -2/3), \\
d^c_{3L} &\sim (3^*, 1, 1/3),\ U^c_{3L} \sim (3^*, 1, -2/3),\ U'^c_{3L} \sim (3^*, 1, -2/3), \\
\psi_{\alpha L} &= (e^-, \nu^0_e, N^0, N'^0)^T_L \sim (1, 4^*, -1/4),\ e^+_{\alpha L} \sim (1, 1, 1).
\end{aligned} \tag{34}$$

In an analogous way to the previous application, in the 331 extension by making a hypercharge analysis, we identify that model A (in 341 extension) breaks down to model A in the 331 extension[10]. Such model is also a three-family model, with the next fermion content.

$$
\begin{aligned}
Q_{iL} &= (u,d,D)_L^T \sim (3,3,0), u_{iL}^c \sim (3^*,1,-2/3),\\
d_{iL}^c &\sim (3^*,1,1/3),\ D_{iL}^c \sim (3^*,1,1/3),\\
Q_{3L} &= (d,u,U)_L^T \sim (3,3^*,1/3), u_{3L}^c \sim (3^*,1,-2/3),\\
d_{3L}^c &\sim (3^*,1,1/3),\ U_{3L}^c \sim (3^*,1,-2/3),\\
\psi_{\alpha L} &= (e^-,\nu_e^0,N^0)_L^T \sim (1,3^*,-1/3),\ e_{\alpha L}^+ \sim (1,1,1).
\end{aligned} \tag{35}
$$

The b_i coefficients are calculate by using the expression in Eq. (2) , and taking into consideration the quantum number assigned for each representation in this model. At energies below of M_X the coefficients take the next form:

$$
\begin{aligned}
b_Y &= \frac{20}{9}N_g + \frac{1}{6}N_H,\\
b_{2L} &= \frac{4}{3}N_g + \frac{1}{6}N_H - \frac{22}{3},\\
b_S &= \frac{4}{3}N_g - 11,
\end{aligned} \tag{36}
$$

where N_g is the number of families, N_H the number of scalar doublets in $SU(2)_L$. Here at low energies, the 341 extension leads to a two Higgs double model, then $N_H = 2$.

For energies in the range $M_X \le \mu \le M_{X'}$ the b_i's coefficient are $(b_X^{331}, b_{3L}, b_{3C}^{331}) = (\frac{26}{3}, -45/6, -5)$, Finally for energies above the $M_{X'}$,the b_i's are $(b_X^{341}, b_{4L}, b_{3C}^{341}) = (\frac{34}{3}, -2, -3)$ where we also extended the scalar sector of the 341, by adding a set of 8 decuplets $\Delta \sim (1,10,0)$ previously introduced to the one-family case.

4. $SU(4) \otimes SU(2)_L \otimes U(1)_X$ Extension

It is not clear what lies beyond the SM, many models beyond it had been studied and among them polychromatic extension. In Ref. 12 was considered the possibility that the $SU(3)_c$ color group is a remnant symmetry of $SU(4)$. Here we assume that the quarks transform under the fundamental representation of $SU(4)$, and the anomalies cancel in each family as in the SM.

It was realize in Ref. 23 that a Higgs field transforming under the irrep(irreducible representation) $[N(N+1)/2]_S$ of $SU(N)$ can break $SU(N)$ down to $SU(N-1)$, while the antisymmetry irrep $[N(N-1)/2]_A$ can breaks $SU(N)$ down to $SU(N-2) \otimes SU(2)$. In order to obtain a phenomenologically consistent extension, and assuming that G_C is a simple group; then the simplest realizations are $SU(4)$ and $SU(5)$. For sake of simplicity here we are considering the extension where $G_C = SU(4)$.

The breaking of the gauge group goes in two stages:

$$\begin{aligned} SU(4)\otimes SU(2)_L\otimes U(1)_{Y'} &\xrightarrow{\chi} SU(3)_C\otimes SU(2)_L\otimes U(1)_Y \\ &\xrightarrow{\phi} SU(3)_C\otimes U(1)_Q \end{aligned} \tag{37}$$

Being χ and ϕ a Higgs multiplet and the Higgs doublet of the SM transforming as $(10, 1, 1/2)$ and $(1, 2, 1/2)$ respectively.

4.1. *Matching Condition*

The electric charge generator is defined as:

$$\mathcal{Q} = T_{3L} + \frac{1}{\sqrt{6}} a T_{15C} + \frac{Y^{'}}{2} I_4, \tag{38}$$

with $T_{i\alpha} = \lambda_i/2$ (where α stands for L and C), being the λ_i the generalized Gell-mann matrices normalized as $Tr(\lambda_i\lambda_j) = 2\delta_{ij}$, $I_4 = Dg(1, 1, 1, 1)$ and a a free parameter to be chosen.
Let us label for simplicity $X = Y^{'}/2$, where X is the hypercharge operator. The SM hypercharge has the form:

$$Y = \frac{1}{\sqrt{6}} a T_{15C} + X, \tag{39}$$

assuming the well-normalized hypercharge operator $Y = b\widehat{Y}$ and $X = f\widehat{X}$, the previous equation becomes

$$b\widehat{Y} = \frac{1}{\sqrt{6}} a T_{15C} + f\widehat{X}, \tag{40}$$

now, considering that the all generators are normalized in the same way, then is straightforward obtain the next relations:

$$b^2 = f^2 + \frac{a^2}{6}, \tag{41}$$

$$b^2\widetilde{\alpha}_Y^{-1} = f^2\widetilde{\alpha}_X^{-1} + \frac{a^2}{6}\widetilde{\alpha}_{4C}^{-1}. \tag{42}$$

From Eq. (42) and taking into account Eq. (41) we find out the form of the well-normalized hypercharge coupling, which takes the form:

$$\widetilde{\alpha}_X^{-1} = \frac{1}{b^2 - \frac{a^2}{6}}\left(b^2\widetilde{\alpha}_Y^{-1} - \frac{a^2}{6}\alpha_{4C}^{-1}\right). \tag{43}$$

The matching condition arises at M_X, when $SU(4)$ breaks down to $SU(3)_C$ we must warranty

$$\alpha_X^{-1}(M_X) = \alpha_Y^{-1}(M_X) - \frac{a^2}{6}\alpha_{4C}^{-1}(M_X), \tag{44}$$

by using Eq. (1), we propose an expansion for α_X^{-1}

$$\alpha_X^{-1} = \alpha_X^{-1}(M_Z) - \frac{b_{Y'}}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{b_X}{2\pi} ln\left(\frac{M_U}{M_X}\right). \quad (45)$$

Now, evaluating $\mu = M_X$ in Eq. (44) and comparing this result with the condition in Eq. (45), it follows

$$\alpha_X^{-1}(M_Z) = \alpha_Y^{-1}(M_Z) - \frac{a^2}{6}\alpha_{4C}^{-1}(M_Z). \quad (46)$$

$$b_{Y'} = b_Y - \frac{a^2}{6} b_{3C}.$$

4.2. *Couplings Constants*

With those conditions and taking into account Eq. (1), the evaluation of the running couplings at one loop order are giveng by

$$\widetilde{\alpha}_X^{-1} = \left(\frac{1}{b^2 - \frac{a^2}{6}}\right)\left[\alpha_Y^{-1}(M_Z) - \left(\frac{a^2}{6}\right)\alpha_{3C}^{-1}(M_Z) - \frac{(b_Y - \frac{a^2}{6}b_{3C})}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{b_X}{2\pi} ln\left(\frac{M_U}{M_X}\right)\right] \quad (47)$$

$$\alpha_{2L}^{-1} = \alpha_{2L}^{-1}(M_Z) - \frac{b_{2L}}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{{b_{2L}}'}{2\pi} ln\left(\frac{M_U}{M_X}\right) \quad (48)$$

$$\alpha_{4C}^{-1} = \alpha_S^{-1}(M_Z) - \frac{b_S}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{b_{4C}}{2\pi} ln\left(\frac{M_U}{M_X}\right). \quad (49)$$

Where b_S, b_{2L} and b_Y are the b_i coefficients of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ respectively, and are calculate for energies at the range $M_Z \leq \mu \leq M_X$. The coefficients b_{4C}, $b_{2L}^{'}$ and b_X are related with $SU(4)$, $SU(2)_L$ and $U(1)_X$, and they are calculated for energies at the range $M_X \leq \mu \leq M_U$.
From Eq. (48) and Eq. (49) we calculate the unification mass

$$M_U = M_X \left(\frac{M_X}{M_Z}\right)^{-\frac{b_{2L} - b_S}{b_{2L}^{'} - b_{4C}}} \exp\left(2\pi \frac{\alpha_{2L}^{-1}(M_Z) - \alpha_S^{-1}(M_Z)}{b_{2L}^{'} - b_{4C}}\right) \quad (50)$$

Is possible also to calculate the free parameter b^2, by using the previous result and Eqs (47, 48).

$$b^2 = \frac{a^2}{6} + \Bigg\{\alpha_Y^{-1}(M_Z) - \frac{a^2}{6}\alpha_{3C}^{-1}(M_Z) - \frac{(b_Y - \frac{a^2}{6}b_S)}{2\pi} ln\left(\frac{M_X}{M_Z}\right)$$
$$- \frac{b_X}{2\pi}\left[-\left(\frac{b_{2L}-b_S}{b_{2L}^{'}-b_{4C}}\right) ln\left(\frac{M_X}{M_Z}\right) + 2\pi\left(\frac{\alpha_{2L}^{-1}(M_Z)-\alpha_S^{-1}(M_Z)}{b_{2L}^{'}-b_{4C}}\right)\right]\Bigg\}$$
$$\times \Bigg\{\alpha_{2L}^{-1}(M_Z) - \frac{b_{2L}}{2\pi} ln\left(\frac{M_X}{M_Z}\right) - \frac{b_{2L}^{'}}{2\pi}\left[-\left(\frac{b_{2L}-b_S}{b_{2L}^{'}-b_{4C}}\right) ln\left(\frac{M_X}{M_Z}\right)\right.$$
$$\left.+ 2\pi\left(\frac{\alpha_{2L}^{-1}(M_Z)-\alpha_S^{-1}(M_Z)}{b_{2L}^{'}-b_{4C}}\right)\right]\Bigg\}^{-1}$$

4.3. *Application III*

Making $a = 1/2$ in the electric charge operator, the fermion content of the 421 extension is

$$\begin{aligned} f_L &= (\nu_e, e^-)_L \sim (1,2,-1), \quad e_R \sim (1,1,-2), \\ Q_L &= (u,d)_L \sim (4,2,1/2), u_R, \sim (4,1,5/4), d_R \sim (4,1,-3/4). \end{aligned} \tag{51}$$

By using Eq. (2) the b_i coefficients are calculate to the different scales of energy as follows: For energies in the range $M_X < \mu < M_U$

$$\begin{aligned} b_{4C} &= \frac{4}{3}N_g - \frac{41}{3}, \\ b_{2L} &= \frac{10}{6}N_g - \frac{43}{6}, \\ b_X &= \frac{10}{4}N_g + \frac{3}{8}. \end{aligned} \tag{52}$$

Considering that the number of generations $N_g = 3$ we find the next values for the b_i coefficients $(b_{4C}, b_{2L}^{'}, b_X) = (-29/3, -13/6, 63/8)$. On the other hand, for energies in the range $M_Z < \mu < M_X$ the particle content correspond to the SM, but the scalar sector is compose only per one doublet Higgs model, and not per two like in the 341 extension. The values of the b_i coefficients are $(b_S, b_{2L}, b_Y) = (-7, -19/6, 41/6)$.

Using the hierarchy condition we plot the allowed interval for M_X. In Fig. 3, the solid line represent M_U as a function of M_X and the dashed line represents the

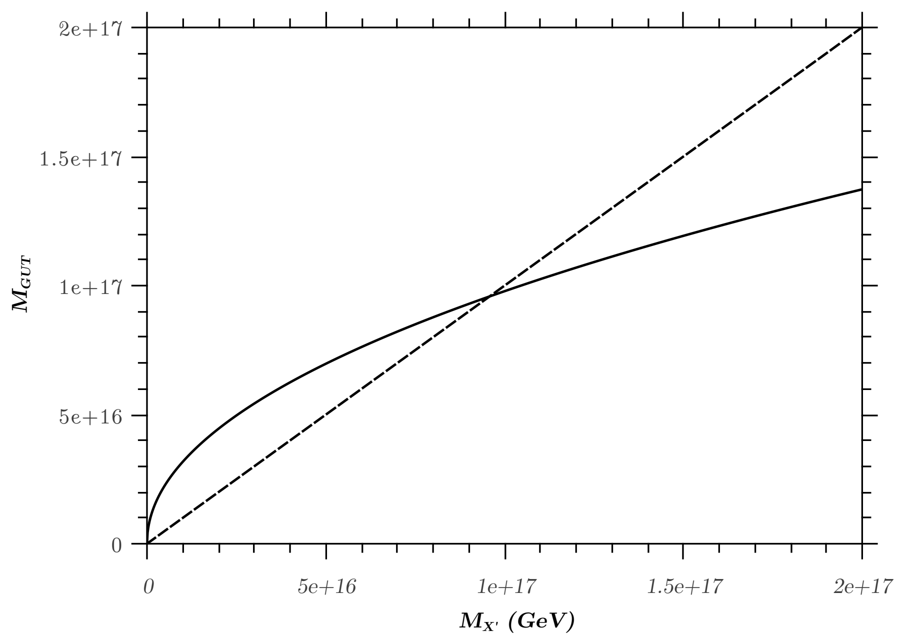

Fig. 3. Unification mass as a function of M_X.

functions $M_U = M_X$. From this plot we fixed $M_X = 6.9 \times 10^{13}$ GeV, which is a value that match with the hierarchy.
We obtain an unification mass $M_U \sim 2.7 \times 10^{15}$ GeV, at that energy scale the couplings are going to met, as we are going to show below. The normalization parameter satisfies $b^2 \geq 1/24$, putting the next restriction $M_X \leq 7.0 \times 10^{13}$ GeV and as a consequence the the previous choice for M_X match perfectly. We found that $b^2 = 2.1106$.
After fixing the free parameter, we evaluate the running couplings constants in Fig. 4, and as was point out below, the unification scale appears at $M_U \sim 2.7 \times 10^{15}$ GeV.

5. Conclusions

In the present paper, we show that unification of gauge couplings constants is achieve two extensions of the SM. In spite...

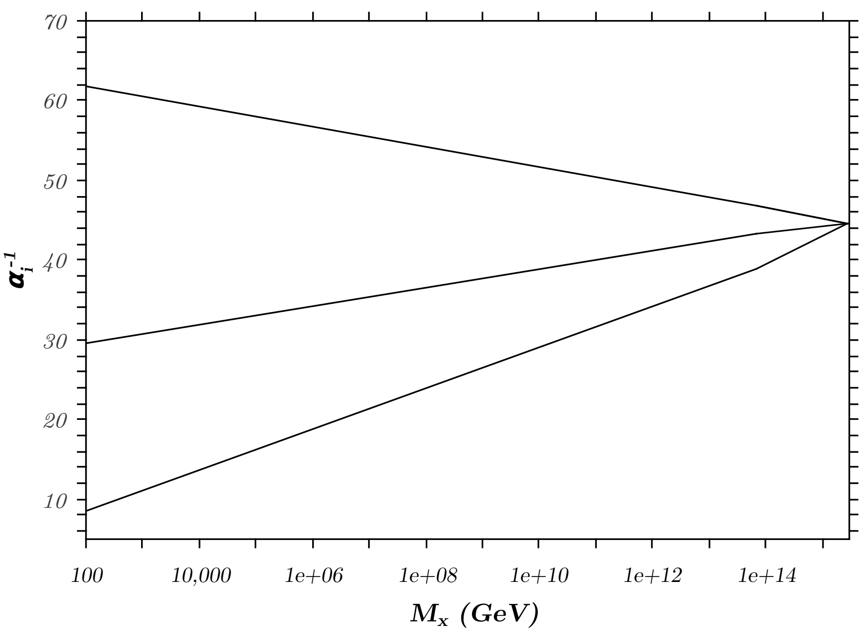

Fig. 4. Runnnig of gauge couplings constants .

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