Morphisms between semi-graded rings and polynomial applications

María Camila Ramírez Cubillos

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
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María Camila Ramírez Cubillos
Mathematician, Master of Science in Mathematics

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Advisor
Armando Reyes, Ph. D.
Associate Professor

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
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TITLE

MORPHISMS BETWEEN SEMI-GRADED RINGS AND POLYNOMIAL APPLICATIONS

TÍTULO

MORFISMOS ENTRE ANILLOS SEMI-GRADUADOS Y APLICACIONES POLINOMIALES

ABSTRACT: In this thesis, we study several kinds of morphisms between families of semi-graded rings with their corresponding polynomial applications. First, we present some ring-theoretical notions of these objects that are necessary throughout the thesis. With the aim of showing the generality of these rings in areas such as ring theory and noncommutative geometry, we include a non-exhaustive list of noncommutative algebras that are particular examples of these rings. Second, we develop a theory of cv-polynomials for iterated Ore extensions over arbitrary rings extending some results in the literature. We characterize these polynomials by using inner derivations of the coefficient ring, and also consider the problem of isomorphisms between these extensions. We illustrate our treatment with several noncommutative algebras. Third, for double Ore extensions introduced by Zhang and Zhang in the problem of classification of Artin-Schelter regular algebras of dimension four, we propose a theory of homomorphisms between them by introducing an adequate notion of cv-polynomial, and show that the computation of homomorphisms corresponding to these polynomials is non-trivial. Since there are no inclusions between the classes of all double Ore extensions of an algebra and of all length two iterated Ore extensions of the same algebra, we also present a comparison between theories of cv-polynomials between both families of algebras. We illustrate our results with Nakayama automorphisms of trimmed double Ore extensions. Finally, motivated by the research on maps between noncommutative projective spaces over $\mathbb{N}$-graded rings in the sense of Rosenberg and Van den Bergh, and the notion of schematicness introduced by Van Oystaeyen and Willaert to $\mathbb{N}$-graded rings with the aim of formulating a noncommutative scheme theory à la Grothendieck, we investigate maps in the setting of noncommutative projective spaces over schematic semi-graded rings, and extend different results from the category of schematic $\mathbb{N}$-graded rings to the category of schematic semi-graded rings.

RESUMEN: En esta tesis, estudiamos diversas clases de morfismos entre familias de anillos semi-graduados con sus correspondientes aplicaciones polinomiales. En primer lugar, presentamos algunas nociones de la teoría de anillos de estos objetos que son necesarios a lo largo de la tesis. Con el propósito de mostrar la generalidad de estos anillos en áreas como la teoría de anillos y la geometría no conmutativa, incluimos una lista no exhaustiva de álgebras no conmutativas que son ejemplos particulares de estos anillos. En segundo lugar, desarrollamos una teoría de cv-polinomios para extensiones de Ore iteradas sobre anillos arbitrarios extendiendo algunos resultados en la literatura. Caracterizamos estos polinomios utilizando derivaciones internas, y también consideramos el problema de isomorfismos entre estas extensiones. Ilustramos nuestro tratamiento con diversas álgebras no conmutativas. Tercero, para las extensiones dobles de Ore introducidas por Zhang y Zhang en el problema de clasificación de álgebras regulares de Artin-Schelter de dimensión cuatro, proponemos una teoría entre estas introduciendo una noción adecuada de cv-polinomio, y mostramos que el cálculo de homomorfismos correspondientes a

estos polinomios es no trivial. Teniendo en cuenta que no hay inclusiones entre las clases de extensiones dobles de Ore de un álgebra y extensiones de Ore iteradas de longitud dos sobre la misma álgebra, también presentamos una comparación entre las teorías de cv-polinomios entre ambas familias de álgebras. Ilustramos nuestros resultados con automofismos de Nakayama de extensiones dobles de Ore cortadas. Finalmente, motivados por la investigación sobre morfismos entre espacios proyectivos no connmutativos sobre anillos $\mathbb{N}$-graduados en el sentido de Rosenberg and Van den Bergh, y la noción de esquematicidad introducida por Van Oystaeyen y Willaert para anillos $\mathbb{N}$-graduados con el propósito de formular una teoría de esquemas no conmutativa a la Grothendieck, investigamos morfismos en contexto de espacios proyectivos no conmutativos sobre anillos semi-graduados esquemáticos, y extendemos diversos resultados de la categoría de anillos $\mathbb{N}$-graduados esquemáticos a la categoría de anillos semi-graduados esquemáticos.

KEYWORDS: Semi-graded ring, quantum algebra, Ore extension, double Ore extension, schematic ring, cv-polynomial, inner derivation, Nakayama automorphism, closed immersion, noncommutative projective geometry.

PALABRAS CLAVE: *Anillo semi-graduado, álgebra cuántica, extensión de Ore, extensión doble de Ore, anillo esquemático, cv-polinomio, derivación interna, automorfismo de Nakayama, inmersión cerrada, geometría proyectiva no conmutativa.*

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Jury
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DEDICATORY

To Alejandro, my brother

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I want to thank Professor Samuel Lopes[2] for contacting me and introducing me to several of his articles related to the study of automorphisms and isomorphisms of noncommutative polynomial extensions. He pointed out a few related things to my research, which helped me exemplify and compare several of the results I obtained here with the corresponding ones in his papers.

[1] Miami University, USA.
[2] Universidade do Porto, Portugal.

Last but not least, this thesis is not just my individual effort but a collective result of the support and encouragement from these wonderful individuals and institutions. Thank you all for being an integral part of this academic process.

CONTENTS

INTRODUCTION

The study of morphisms and isomorphisms of polynomial extensions has been an active and interesting area of research. For instance, Gilmer [Gil68] determined all the automorphisms of the commutative polynomial ring $R[x]$ (R commutative) that restrict to the identity on R. The case where R is any ring has been handled by Coleman and Enochs [CE71]. Later, for the *Ore extensions*, also known as *skew polynomial rings*, introduced by Ore [Ore31, Ore33] (some earlier works related to the differential case already appeared in Landau [Lan02], while Noether and Schmeidler [NS20] were interested in some kind of differential operator rings), Parmenter [Par] considered the case $R[x;\sigma]$ where R is commutative and σ is any automorphism of R. Rimmer [Rim78, Theorem 1] generalized all these results. He determined all automorphisms of the skew polynomial ring of automorphism type $R[x;\sigma]$ which fix R elementwise, where R is any ring. He also extended his research to isomorphisms between different skew polynomial rings that preserve an underlying ring isomorphism [Rim78, Theorem 3]. Ferrero and Kishimoto [FK80] studied automorphisms of skew polynomial rings of derivation type. They discussed conditions on $f(x) \in R[x;\delta]$ for the R-linear map $R[x;\delta] \to R[x;\delta]$ defined by $x^k \mapsto f(x)^k$ to be a R-ring automorphism (c.f. Kikumasa [Kik90] characterized automorphisms of a certain skew polynomial ring of derivation type). Surely, the most remarkable example of this kind of skew polynomials is *the first Weyl algebra*, and precisely, one of the most famous open problems is the conjecture posed by Dixmier [Dix68]: any algebra endomorphism of the first Weyl algebra is an automorphism. Tsuchimoto [Tsu03] and Belov-Kanel and Kontsevich [BKK07] proved independently that the Dixmier conjecture is stably equivalent to the Jacobian conjecture formulated by Keller [Kel39]. Some works in the literature exhibit algebras related to the first Weyl algebra with the property that every endomorphism is an automorphism (e.g., [KL14, KL19] and references therein).

For skew polynomial rings of mixed type over division rings, Lam and Leroy [LL92] studied "transformations" from one Ore extension to another[3]. They formalized the notion of "transformation" and investigated homomorphisms between this kind of Ore extensions. Briefly, if we consider two Ore extensions $D[x;\sigma,\delta]$ and $D[x';\sigma',\delta']$ over a division ring D, a D-homomorphism ϕ from $D[x';\sigma',\delta']$ to $D[x;\sigma,\delta]$ is determined by $p(x) := \phi(x') \in D[x;\sigma,\delta]$ which satisfies the rule $p(x)r = \sigma'(r)p(x) + \delta'(r)$ in $D[x;\sigma,\delta]$, for every element $r \in D$. The polynomial $p(t)$ allows to make a "change of variables" from x' to x, so this is called a *change-of-variable*

[3]Surprisingly, Lam and Leroy do not cite the works of Parmenter, Rimmer, Ferrero, Kishimoto and Kikumasa mentioned above, so according to the search we carried out in the literature, this is the first place where all these treatments are cited and presented, in order, to extend them partially to other families of noncommutative rings.

polynomial (cv-polynomial for short). Notice that whenever such a polynomial is given, we obtain a unique D-homomorphism $\phi : D[x';\sigma',\delta'] \to D[x;\sigma,\delta]$ sending x' to $p(x)$, say, for $g(x') = \sum r_i x'^i \in D[x';\sigma',\delta']$, we get $\phi(g) = \sum r_i p(x)^i$. In their paper, Lam and Leroy obtained several characterizations of these cv-polynomials by using inner derivations. Isomorphisms of noncommutative polynomial extensions have been an interesting topic of research (e.g., [AKP87, RR92, LM92, Chu93, Ber03, RS06, CL09, Chu13, BLO15, SAV15, LR22], and references therein).

On the other hand, *Artin-Schelter* (*AS* for short) *regular algebras* introduced by Artin And Schelter [AS87] are considered as noncommutative analogues of commutative polynomial rings due to its important role in noncommutative geometry. As one can appreciate in the literature, these algebras have been extensively studied. With the aim of presenting new examples of Artin-Schelter regular algebras of dimension four, Zhang and Zhang [ZZ08, ZZ09] introduced algebra extensions which they called *double Ore extensions* and constructed 26 families of these algebras. Many AS regular of these algebras are new and are not isomorphic to either a normal extension or an Ore extension of an Artin-Schelter regular algebra of global dimension three. In opposition of what happens with Ore extensions, very few properties are known to be preserved under double Ore extensions. Several researchers have investigated different relations of double Ore extensions with Poisson, Hopf, Koszul and Calabi-Yau algebras (e.g. [GS20, Li22, LOW20, LOWY18, JLZ15, SCR21, LRS17, ZvOZ17]). From the definition of double Ore extensions it is possible appreciate some similarities to that of a two-step iterated Ore extensions. Nevertheless, there are no inclusions between the classes of all double Ore extensions of an algebra and of all length two iterated Ore extensions of the same algebra. Carvalho et al. [CLM11] formulated necessary and sufficient conditions for a double Ore extension to be presented as a two-step iterated Ore extension.

In the setting of noncommutative projective geometry developed by Rosenberg [Ros95, Ros98] and Van den Bergh [VdB01], Smith [Smi03, Smi16] investigated maps between noncommutative spaces of the form $\mathrm{Proj}_{nc}A$. More exactly, he takes a *Grothendieck category* as the basic non-commutative geometric object, where we think of a Grothendieck category $\mathsf{Mod}X$ (the idea for the notation $X = \mathsf{Mod}X$ is Van den Bergh's) as "the quasi-coherent sheaves on an imaginary non-commutative space X". The standard commutative example is the category $\mathsf{Qcoh}X$ of quasi-coherent sheaves on a quasi-separated, quasi-compact scheme X. The two non-commutative models are $\mathsf{Mod}R$, the category of right modules over a ring, and $\mathsf{Proj}A$, the non-commutative projective spaces having a (not necessarily commutative) graded ring A as homogeneous coordinate ring, which were defined by Verevkin [Ver92b, Ver92a] and Artin and Zhang [AZ94] (for more details, see [Smi02, Definition 2.2]). A map $g : Y \to X$ between two spaces is an adjoint pair of functors (g^*, g_*) with $g_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ and g^* left adjoint to g_*. If g_* is faithful and has a right adjoint, then g is called affine. An immediate example is a ring homomorphism $\varphi : R \to S$ that induces an affine map $g : Y \to X$ between the affine spaces defined by $\mathsf{Mod}Y := \mathsf{Mod}S$ and $\mathsf{Mod}X := \mathsf{Mod}R$. In his papers, Smith considered the question on maps between noncommutative projective spaces. He called a map $g : Y \to X$ a closed immersion if it is affine and the essential image of $\mathsf{Mod}Y$ in $\mathsf{Mod}X$ under g^* is closed under submodules and quotients. For J a graded ideal in a not necessarily commutative $\mathbb{N}$-graded $\Bbbk$-algebra $A = A_0 \oplus A_1 \oplus \cdots$ in which $\dim_{\Bbbk} A_i < \infty$ for all i, Smith [Smi03, Theorem 3.2], [Smi16, Theorem 1.2] showed that a surjective homomorphism $A \to A/J$ of graded rings induces

a closed immersion $i : \mathrm{Proj}_{nc} A/J \to \mathrm{Proj}_{nc} A$ between the non-commutative projective spaces with homogeneous coordinate rings A and A/J.

Motivated by all treatments above on maps between noncommutative algebraic and geometric objects, in this thesis we are interested in the characterization of several kinds of morphisms of *semi-graded* rings introduced by Lezama and Latorre [LL17]. These rings are a generalization of $\mathbb{N}$-graded rings and several families of polynomial extensions non-$\mathbb{N}$-graded (of course not in a trivial way). Semi-graded rings include and extend several families of algebras appearing in ring theory, module theory, noncommutative algebra and noncommutative geometry (see [CR22, FGL$^+$20, Lez20, Lez21, LR14] for more details). From the algebraic point of view, we formulate several characterizations of morphisms and cv-polynomials of iterated Ore extensions and double Ore extensions. Next, by considering this theory of cv-polynomials, we propose an analogous theory of homomorphisms between double Ore extensions. Finally, from the geometric point of view, we consider maps in the Smith's sense but now in the more general setting of noncommutative projective spaces over semi-graded rings.

On the structure of the thesis, this is based on a collection of papers. Chapter 1 recalls some ring-theoretical notions of semi-graded rings that are necessary in the next chapters. With the aim of showing their generality in areas such as ring theory and noncommutative geometry, we include a non-exhaustive list of noncommutative algebras that are particular examples of these rings. Next, Chapter 2 presents the original results on theories of homomorphisms and cv-polynomials for iterated Ore extensions and double Ore extensions. We illustrate our treatment with several noncommutative algebras and show that the computation of corresponding homomorphisms is non-trivial. First, we develop a theory of morphisms and cv-polynomials of two-step, three-step and n-step iterated Ore extensions by using inner derivations. We organize information in different tables with the aim of presenting different possibilities for cv-polynomials. It is important to say that we have found some mistakes in particular cases of Lam and Leroy's paper [LL92] that we illustrate and present their corresponding versions. Second, we propose a theory of homomorphisms and cv-polynomials between double Ore extensions. We obtain results on the characterizations of cv-polynomials and their relations with inner derivations of the ring of coefficients of the double algebra. We illustrate our results with Nakayama automorphisms of trimmed double Ore extensions. Finally, in Chapter 3, we study morphisms between schematic semi-graded rings. Our interest in this topic is due to Professor Jason Gaddis who asked us in the evaluation of our doctoral project if we can use geometric tools to study families of semi-graded rings. More exactly, he considered relevant to the present project Smith's papers above. For him, it would be reasonable to try and define a notion of Proj for some families of semi-graded rings. Well, then in this chapter we give a first answer to the question of Professor Gaddis in the setting of schematic semi-graded rings.

Notation and some terminology

Symbol	Meaning
$\mathbb{N}$	The set of natural numbers including zero
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
R, R'	Associative ring (not necessarily commutative) with identity
R^*	The non-zero elements of the ring R
K	Commutative ring with identity
D	Division ring
$\Bbbk$	Field
$Z(R)$	The center of R
$\mathrm{Idem}(R)$	The set of idempotent elements of R
$N(R)$	The set of nilpotent elements of R
$I \lhd_l R$	I is a left ideal of R
$M_{r\times c}(R)$	The ring of matrices of size $r \times c$ with entries in R

Throughout this thesis, the term ring means an associative (not necessarily commutative) ring with identity, and the term module means a left unital module. All localizations are considered by the left side.

Statement of contributions

The chapters two and three in this thesis correspond to the following papers containing original results.

- **Chapter 2. Ramírez, M. C.** and Reyes, A. On homomorphisms and cv-polynomials between iterated Ore extensions (2024) [Manuscript submitted for publication]. Available online at https://arxiv.org/abs/2401.11516
- **Chapter 2. Ramírez, M. C.** and Reyes, A. A view toward homomorphisms and cv-polynomials between double Ore extensions. *Algebra Colloquium* (2024). To appear. Available online at https://arxiv.org/abs/2401.14162
- **Chapter 3.** Chacón, A., **Ramírez, M. C.**, Reyes, A. Maps between schematic semi-graded rings. *Beiträge zur Algebra und Geometrie* (2024). To appear. Available online at https://arxiv.org/abs/2401.15631

CHAPTER 1

SEMI-GRADED RINGS

In this chapter, we present the algebraic structures of interest in this thesis: the *semi-graded rings*. Also, we formulate some ring-theoretical notions that are necessary in the next chapters.

More exactly, Section 1.1 contains definitions and some key properties of semi-graded rings, finitely semi-graded rings and modules over these rings. Next, in Section 1.2 we present a list (not exhaustive) of noncommutative algebraic structures that are particular examples of semi-graded rings. Our aim in this section is to show explicitly the generality of these rings in ring theory and noncommutative geometry.

1.1 PRELIMINARIES AND KEY PROPERTIES

Lezama and Latorre [LL17] introduced the *semi-graded rings* as a generalization of $\mathbb{N}$-graded rings and several families of noncommutative rings of polynomial type non-$\mathbb{N}$-graded (not in a trivial way). In that paper, they considered some notions of noncommutative algebraic geometry for semi-graded rings such as the Hilbert series, Hilbert polynomial, Gelfand-Kirillov dimension, and the notion of noncommutative projective scheme in the setting of semi-graded rings generalizing the Serre-Artin-Zhang-Verevkin theorem (see also [Lez21, Cha22, CR23]). In Section 3.2 we present more details about this theorem.

For the moment, we recall briefly some definitions and results about semi-graded rings which are key in the following chapters.

DEFINITION 1.1 ([LL17, DEFINITION 2.1]). Let R be a ring. R is said to be *semi-graded* (SG) if there exists a collection $\{R_n\}_{n\in\mathbb{Z}}$ of subgroups of the additive group R^+ such that the following conditions hold:

(i) $R = \bigoplus_{n\in\mathbb{Z}} R_n$.

(ii) For every $m, n \in \mathbb{Z}$ we have that $R_m R_n \subseteq \bigoplus_{k\leq m+n} R_k$.

(iii) $1 \in R_0$.

The collection $\{R_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* R and we say that the elements of R_n are *homogeneous of degree* n. We say that R is *positively semi-graded* if $R_n = 0$ for every $n < 0$. If R and S are semi-graded rings and $f : R \to S$ is a ring homomorphism, then we say that f is *homogeneous* if $f(R_n) \subseteq S_n$ for every $n \in \mathbb{Z}$.

Definitions 1.2 and 1.3 present the notion of finitely semi-graded ring and finitely semi-graded algebra, respectively.

DEFINITION 1.2 ([LL17, DEFINITION 2.4]). A ring R is called *finitely semi-graded* (FSG) if it satisfies the following conditions:

(i) R is SG.

(ii) There exist finitely many elements $x_1, \ldots, x_n \in R$ such that the subring generated by R_0 and $x_1, \ldots, x_n$ coincides with R.

(iii) For every $n \geq 0$, we have that R_n is a left free R_0-module of finite rank.

DEFINITION 1.3 ([LG19, DEFINITION 10]). A $\mathbb{k}$-algebra R is said to be *finitely semi-graded* (FSG) if the following conditions hold:

(i) R is an FSG ring with positive semi-graduation given by $R = \bigoplus_{n\geq 0} R_n$.

(ii) For every $m, n \geq 1$, we have that $R_m R_n \subseteq R_1 \oplus \cdots \oplus R_{m+n}$.

(iii) R is connected, i.e., $R_0 = \mathbb{k}$.

(iv) R is generated in degree 1, that is, R is generated as a $\mathbb{k}$-algebra by finitely elements of R_1.

From Definition 1.3, it follows that if R is a FSG $\mathbb{k}$-algebra, then $R_+ := \bigoplus_{n\geq 1} R_n$ is a maximal ideal of R.

Notice that graded rings are SG. Finitely generated graded $\mathbb{k}$-algebras, PBW extensions [BG88], 3-dimensional skew polynomial rings [BS90], down-up algebras [Ben99, BR98], diffusion algebras [IPR01] and skew PBW extensions [GL11] are examples of FSG rings. Definitions of these families of algebras and others are presented in Section 1.2. Semi-graded rings and finitely semi-graded rings have been studied recently in the literature. For instance, Lezama et al. [Lez21, LG19] computed the set of point modules of finitely semi-graded rings. By considering the parametrization of the point modules for the quantum affine n-space, Lezama obtained the set of point modules for some important examples of non $\mathbb{N}$-graded quantum algebras [Lez20, Theorem 5.3].

Next, we present some results about modules in the setting of semi-graded rings.

DEFINITION 1.4 ([LL17, DEFINITION 2.1]). Let R be an SG ring and let M be an R-module. We say that M is *semi-graded* (SG) if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups M_n of the additive group M^+ such that the following conditions hold:

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) For every $m, n \in \mathbb{Z}$, $R_m M_n \subseteq \bigoplus_{k \leq m+n} M_k$.

The collection $\{M_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* M, and we say that the elements of M_n are *homogeneous of degree* n.

M is said to be *positively semi-graded* if $M_n = 0$, for every $n < 0$. Let $f : M \to N$ be a homomorphism of R-modules, where M and N are semi-graded R-modules. We say that f is *homogeneous* if $f(M_n) \subseteq N_n$, for every $n \in \mathbb{Z}$.

DEFINITION 1.5 ([LL17, DEFINITION 2.3]). Let R be an SG ring, M an SG R-module, and N a submodule of M. We say that N is a *semi-graded* (SG) *submodule of* M if $N = \bigoplus_{n\in\mathbb{Z}} N_n$, where $N_n = M_n \cap N$. In this case, N is an SG R-module.

PROPOSITION 1.1 ([LL17, PROPOSITION 2.6]). *If R is an* SG *ring, M is an* SG *R-module, and N is a submodule of M, then the following conditions are equivalent:*

(1) *N is a semi-graded submodule of M.*

(2) *For every $z \in N$, the homogeneous components of z belong to N.*

(3) *M/N is an* SG *R-module with semi-graduation given by*

$$(M/N)_n = (M_n + N)/N, \ n \in \mathbb{Z}.$$

REMARK 1. Let R be an SG ring and M be an SG R-module. Then:

(i) If N is an SG submodule of M, then the canonical map $M \to M/N$ is a homogeneous homomorphism.

(ii) If $\{M_i\}_{i\in I}$ is a family of SG submodules of M, then $\bigcap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are SG submodules of M.

Let N be a subset of M. We define the SG *submodule generated by* N as the intersection of all SG submodules of M containing N, and we denote it as $\langle N\rangle^{\mathsf{SG}}$. If $N = \{n_1, \ldots, n_l\}$, then we write $\langle N\rangle^{\mathsf{SG}} = \langle n_1, \ldots, n_l\rangle^{\mathsf{SG}}$. We will say that M is a *finitely generated* SG R-module if there exist finitely elements $m_1, \ldots, m_t$ such that $M = \langle m_1, \ldots, m_t\rangle^{\mathsf{SG}}$. If M is simultaneously a module over different kinds of rings and there is risk of confusion, we write $\langle -\rangle_R^{\mathsf{SG}}$ to indicate the ring R we are considering. If N is an SG submodule of M, the notion of *finitely generated* SG submodule is defined in the natural way.

EXAMPLE 1.1. Consider the first Weyl algebra $A_1(\Bbbk) = \Bbbk\{x, y\}/\langle yx - xy - 1\rangle$ over $\Bbbk$. By using that $yx = xy + 1$, the graded condition for multiplication does not hold since we do not obtain a product of pure degree one, so $A_1(\Bbbk)$ is not an $\mathbb{N}$-graded ring. On the other hand, the canonical semi-graduation of $A_1(\Bbbk)$ [Lez21, Example 4.5] is given by

$$A_1(\Bbbk) = \Bbbk \oplus {}_{\Bbbk}\langle x, y\rangle \oplus {}_{\Bbbk}\langle x^2, xy, x^2\rangle \oplus \cdots$$

Then:

(i) $A_1(\Bbbk)y$ is an SG submodule of $A_1(\Bbbk)$, whence $\langle y\rangle^{\mathsf{SG}} = A_1(\Bbbk)y$.

(ii) $1 \notin A_1(\Bbbk)x$, but due to the relation $yx = xy+1$, it follows that $1 \in \langle x\rangle^{\mathsf{SG}}$.

DEFINITION 1.6. If R is a positively SG ring, for $t \in \mathbb{N}$ we define $R_{\geq t}$ as the intersection of all two-sided ideals that are SG submodules containing $\bigoplus_{k\geq t} R_k$.

Different properties of modules over families of semi-graded rings have been investigated by some mathematicians [Art15, GL17, HR23, LG19, Lez21, LR20, NRR20, Rey19, TRS20].

1.2 SOME FAMILIES OF EXAMPLES

Semi-graded rings extend several kinds of noncommutative rings of polynomial type such as

- Ore extensions [Ore31, Ore33],
- families of differential operators generalizing Weyl algebras and universal enveloping algebras of finite dimensional Lie algebras [Bav92, BG88, Smi91],
- algebras appearing in mathematical physics [IPR01, RS22, Zhe91],
- down-up algebras [Ben99, BR98, KMP99],
- ambiskew polynomial rings [Jor00, JW96],
- 3-dimensional skew polynomial rings [BS90, Red99, RS22, Ros95],
- Poincaré-Birkhoff-Witt extensions [BG88], and
- skew PBW extensions [GL11].

Ring-theoretical, algebraic and geometrical properties of semi-graded rings have been investigated in the literature (e.g., [Art15, Bav23, CR22, HKA17, RR21, RS19, RS20, Sei10, SCR22, SRS23, TRS20] and references therein).

In this section, we present families of noncommutative rings that are particular examples of semi-graded rings with the aim of showing the generality of these objects, and the scope of the results presented in Chapters 2 and 3. For the completeness of the thesis, we include detailed references for every family of rings.

1.2.1 ORE EXTENSIONS

Ore extensions (also known as *skew polynomial rings*) were introduced by Ore [Ore31, Ore33]. Briefly, for σ an endomorphism of a ring R, a σ-derivation on R is any additive map $\delta : R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$, for all $r, s \in R$ (strictly speaking, this is the definition of *left σ-derivation*). If σ is the identity map on R, then σ-derivations are just ordinary derivations. Notice that

$$\delta(1) = \delta(1\cdot 1) = \sigma(1)\delta(1) + \delta(1)1 = 2\delta(1), \quad \text{whence} \quad \delta(1) = 0.$$

The pair (σ,δ) is called a *quasi-derivation on* R [BGTV03, Definition 3.1].

PROPOSITION 1.2 ([BGTV03, PROPOSITION 3.2]). *For any pair of maps $\sigma,\delta : R \to R$, the following assertions are equivalent:*

(1) (σ,δ) *is a quasi-derivation;*

(2) *the map*

$$\begin{aligned} \phi : R &\to M_{2\times 2}(R) \\ r &\mapsto \begin{bmatrix} \sigma(r) & \delta(r) \\ 0 & r \end{bmatrix} \end{aligned}$$

is a ring homomorphism.

DEFINITION 1.7 ([GJ04, P. 34]). Let R be a ring, σ a ring endomorphism of R and δ a σ-derivation on R. We will write $R[x;\sigma,\delta]$ provided

(i) $R[x;\sigma,\delta]$ is a ring, containing R as a subring;

(ii) x is an element of $R[x;\sigma,\delta]$

(iii) $R[x;\sigma,\delta]$ is a free left R-module with basis $\{1,x,x^2,\ldots\}$;

(iv) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

Such a ring $R[x;\sigma,\delta]$ is called a *skew polynomial ring over* R or an *Ore extension of* R *of mixed type*. If σ is an injective map of R, then we call it an *Ore extension of injective type*, while if σ is the identity of R, then we write $R[x;\delta]$ and call it an *Ore extension of derivation type*. On the other hand, if δ is the zero map, then we write $R[x;\sigma]$ which is known as an *Ore extension of endomorphism type*.

The construction of Ore extensions can be iterated. Indeed, if we consider $S_1 = R[x_1;\sigma_1,\delta_1]$ for a quasi-derivation (σ_1,δ_1) on R, and we assume that (σ_2,δ_2) is a quasi-derivation on S_1, then one may associate to it the Ore extension

$$S_2 = S_1[x_2;\sigma_2,\delta_2] = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2].$$

Repeating this procedure, one obtains the so-called *iterated Ore extensions*. Morphisms between these extensions will be of our topic of interest in Chapter 2.

Since its introduction to the present, a lot of papers and books have been published concerning ring-theoretical, homological and geometrical properties of Ore extensions (e.g., [BG02, GJ04, FGL$^+$20, FLP$^+$24, MR01], and references therein). Ore extensions are one of the most important techniques to define noncommutative algebras. Next, we illustrate this situation with Weyl algebras, some of its deformations, the q-Heisenberg algebra, and the quantum matrix algebra.

- The *Jordan's plane* $\mathscr{J}(\Bbbk)$ introduced by Jordan [Jor01] is the free $\Bbbk$-algebra defined by $\mathscr{J}(\Bbbk) := \Bbbk\{x,y\}/\langle yx - xy - y^2\rangle$. An easy computation shows that $\mathscr{J}(\Bbbk)$ can be expressed as $\Bbbk[y][x;\delta_2]$ with $\delta_2(y) = -y^2$. Shirikov [Shi05, Theorem 1.4] proved that this algebra is not isomorphic to the *Quantum plane* $\mathscr{O}_q(\Bbbk) = \Bbbk\{x,y\}/\langle yx - qxy\rangle$, $q \in \Bbbk^*$.

- Díaz and Pariguan [DP09] defined the *q-meromorphic Weyl algebra* (also known as the *q-skew Jordan's plane*) $\mathscr{J}_q(\Bbbk)$ as the unital associative algebra with generators x, y and defining relation $qyx - xy - y^2 = 0$ for an element $q \in \Bbbk^*$. This algebra can be written as the Ore extension $\Bbbk[y][x;\sigma_2,\delta_2]$ with $\sigma_2(y) = qy$ and $\delta_2(y) = -y^2$. In a personal communication, Professor Lopes told us that using the generator $x' := x + (q-1)^{-1}y$ instead of x we find that the q-skew Jordan's plane is the quantum plane in disguise. For more details, see [Lop24, Example 3.1].

- About the family of Weyl algebras $A_n(\Bbbk)$, in the literature it is common to find characterizations of these algebras as rings of differential operators. Following Coutinho [Cou95], the nth *Weyl algebra* $A_n(\Bbbk)$ over $\Bbbk$ is the $\Bbbk$-algebra generated by the $2n$ indeterminates $x_1,\dots,x_n,y_1,\dots,y_n$ where

$$\begin{aligned} &x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, && 1 \le i < j \le n,\\ &y_jx_i = x_iy_j + \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker's delta}, && 1 \le i,j \le n. \end{aligned}$$

  From the relations defining Weyl algebras, it follows that these cannot be expressed as skew polynomial rings of automorphism type (since the algebra is simple) but are skew polynomial rings with non-trivial derivations.

- Based on Wess and Zumino's paper [WZ90], for an element $q \in \Bbbk \setminus \{0\}$, Giaquinto and Zhang [GZ95] defined the $\Bbbk$-algebra $A_1^q(\Bbbk)$ generated by two indeterminates x and y subject to the relation $xy - qyx = 1$, which is known as a *quantized Weyl algebra* over $\Bbbk$. Note that $A_1^q(\Bbbk) = A_1(\Bbbk) = \Bbbk[y][x;d/dy]$, when $q = 1$. If $q \neq 1$, then $A_1^q(\Bbbk) = \Bbbk[y][x;\sigma,\delta]$, where σ is the $\Bbbk$-algebra automorphism given by $\sigma(f(y)) = f(qy)$, and δ is the *q-difference operator* (also known as *Eulerian derivative*)

$$\delta(f(y)) = \frac{f(qy) - f(y)}{qy - y} = \frac{\alpha(f) - f}{\alpha(y) - y},$$

  as it can be seen in [GJ04, Exercise 2N].

- A generalization of $A_1^q(\Bbbk)$ is given by the *additive analogue of the Weyl algebra* $A_n(q_1,\dots,q_n)$. This algebra was introduced in Quantum Physics by Kuryshkin [Kur80] and studied by Jannussis et al. [JBS81] (see [Li02] for more details). For non-zero elements $q_1,\dots,q_n \in \Bbbk$, this algebra is generated by the indeterminates $x_1,\dots,x_n$ and $y_1,\dots,y_n$ satisfying the relations

$$x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, \quad \text{for every } 1 \le i,j \le n, \quad \text{and}$$

$$y_ix_j = x_jy_i, \quad \text{for all } i \neq j, \quad \text{and} \quad y_ix_i = q_ix_iy_i + 1, \quad \text{for} \quad 1 \le i \le n.$$

- Another deformation of Weyl algebras was introduced by Giaquinto and Zhang [GZ95] with the aim of studying the Jordan Hecke symmetry as a quantization of the usual second

Weyl algebra. By definition, the *quantum Weyl algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the indeterminates $x_1, x_2, \partial_1, \partial_2$, with relations (depending on parameters $a, b \in \Bbbk$)

$$\begin{aligned} x_1x_2 &= x_2x_1 + ax_1^2, & \partial_2\partial_1 &= \partial_1\partial_2 + b\partial_2^2 \\ \partial_1x_1 &= 1 + x_1\partial_1 + ax_1\partial_2, & \partial_1x_2 &= -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2 \\ \partial_2x_1 &= x_1\partial_2, & \partial_2x_2 &= 1 - bx_1\partial_2 + x_2\partial_2. \end{aligned}$$

Note that if $a = b = 0$, then $A_2(J_{0,0})$ is precisely the second Weyl algebra $A_2(\Bbbk)$.

- By definition, the *q-Heisenberg algebra* $\mathbf{H}_n(q)$ is the $\Bbbk$-algebra generated over $\Bbbk$ by the indeterminates x_i, y_i, z_i, for $1 \le i \le n$, subject to the relations

$$\begin{aligned} x_ix_j = x_jx_i, \quad y_iy_j = y_jy_i, \quad z_jz_i = z_iz_j, & \qquad 1 \le i < j \le n, \\ x_iz_i - qz_ix_i = z_iy_i - qy_iz_i = x_iy_i - q^{-1}y_ix_i + z_i = 0, & \qquad 1 \le i \le n, \\ x_iy_j = y_jx_i, \quad x_iz_j = z_jx_i, \quad y_iz_j = z_jy_i, & \qquad i \ne j. \end{aligned}$$

It can be seen that $\mathbf{H}_n(q)$ can be expressed as an iterated skew polynomial ring.

- The *single parameter quantum matrix algebra* $O_p(M_n(\Bbbk))$ for $p \in \Bbbk^*$ has generating set $\{x_{ij} \mid 1 \le i, j \le n\}$, subject to the relations given by

$$x_{ij}x_{lm} = \begin{cases} px_{lm}x_{ij}, & i > l, j = m, \\ px_{lm}x_{ij}, & i = l, j > m, \\ x_{lm}x_{ij}, & i > l, j < m, \\ x_{lm}x_{ij} + (p - p^{-1})x_{im}x_{lj}, & i > l, j > m. \end{cases}$$

It is straightforward to show that $O_p(M_2(\Bbbk)) \cong \Bbbk[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}][x_{22};\sigma_{22},\delta_{22}]$, with the maps given by

$$\begin{aligned} \sigma_{12}(x_{11}) &= px_{11}, & \sigma_{21}(x_{11}) &= px_{11}, \\ \sigma_{22}(x_{21}) &= px_{21}, & \sigma_{22}(x_{12}) &= px_{12}, \\ \delta_{22}(x_{11}) &= (p - p^{-1})x_{21}x_{12}. \end{aligned}$$

Otherwise, the value of σ_{ij} is the identity map, and for $1 \le i, j \le 2$, the maps δ_{ij} are zero [GJ04, Exercise 2V].

- Jordan [Jor95] introduced a certain class of iterated Ore extensions called *ambiskew polynomial rings*. These structures have been studied by Jordan et al. [Jor00, JW96] at various levels of generality that contain different examples of noncommutative algebras. Next, we recall briefly its definition.

  Consider a commutative $\Bbbk$-algebra B, a $\Bbbk$-automorphism of B, and elements $c \in B$ and $p \in \Bbbk^*$. Let S be the Ore extension $B[x;\sigma^{-1}]$ and extend σ to S by setting $\sigma(x) = px$. By [Coh85, p. 41], there is a σ-derivation δ of S such that $\delta(B) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $R = R(B, \sigma, c, p)$ is the Ore extension $S[y;\sigma,\delta]$, whence the following

relations hold:

$$yx - pxy = c, \quad \text{and, for all } b \in B, \quad xb = \sigma^{-1}(b)x \quad \text{and} \quad yb = \sigma(b)y. \tag{1.1}$$

Equivalently, R can be presented as $R = B[y;\sigma][x;\sigma^{-1},\delta']$ with $\sigma(y) = p^{-1}y$, $\delta'(B) = 0$, and $\delta'(y) = -p^{-1}c$, so that $xy - p^{-1}yx = -p^{-1}c$. If we consider the relation $xb = \sigma^{-1}(b)x$ as $bx = x\sigma(b)$, then we can see that the definition involves twists from both sides using σ; this is the reason for the name of the objects.

1.2.2 DOUBLE ORE EXTENSIONS

In the setting of noncommutative algebras appearing in noncommutative geometry, *Artin-Schelter regular algebras* introduced by Artin And Schelter [AS87] are considered as noncommutative analogues of commutative polynomial rings due to its important role in noncommutative geometry. As one can appreciate in the literature, these algebras have been extensively studied (see the excellent survey on these algebras carried out by Rogalski [Rog24]). With the aim of presenting new examples of Artin-Schelter regular algebras of dimension four, Zhang and Zhang [ZZ08, ZZ09] introduced algebra extensions which they called *double Ore extensions* (*double extensions* for short) and constructed 26 families of these algebras. Many AS regular of these algebras are new and are not isomorphic to Ore extensions or extensions of an Artin-Schelter regular algebra of global dimension three. Several researchers have investigated different relations of double extensions with Poisson, Hopf, Koszul and Calabi-Yau algebras (e.g. [GS20, Li22, LOW20, LOWY18, JLZ15, SAR21, SCR21, LRS17, ZvOZ17]). From the definition of double extensions it is possible appreciate some similarities to that of a two-step iterated Ore extensions. Nevertheless, there are no inclusions between the classes of all double extensions of an algebra and of all length two iterated Ore extensions of the same algebra. Precisely, Carvalho et al. [CLM11] formulated necessary and sufficient conditions for a double extension to be presented as two-step iterated Ore extensions.

Next, we recall the definition of double extension introduced by Zhang and Zhang [ZZ08] with some of its properties. Since some typos ocurred in their papers [ZZ08, p. 2674] and [ZZ09, p. 379] concerning the relations that the data of a double extension must satisfy, we follow the corrections and results presented by Carvalho et al. [CLM11] (all of them will be important in Section 2.5).

DEFINITION 1.8 ([ZZ08, DEFINITION 1.3]; [CLM11, DEFINITION 1.1]). Let R be a subalgebra of a $\Bbbk$-algebra B.

(a) B is called a *right double extension* of R if the following conditions hold:

(i) B is generated by R and two new indeterminates y_1 and y_2;

(ii) y_1 and y_2 satisfy the relation

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0, \tag{1.2}$$

for some $p_{12}, p_{11} \in \Bbbk$ and $\tau_1, \tau_2, \tau_0 \in R$;

(iii) B is a free left R-module with basis $\left\{y_1^i y_2^j \mid i,j \geq 0\right\}$.

(iv) $y_1R + y_2R + R \subseteq Ry_1 + Ry_2 + R$.

(b) A right double extension B of R is called a *double extension* if

(i) $p_{12} \neq 0$;

(ii) B is a free right R-module with basis $\left\{y_2^i y_1^j \mid i,j \geq 0\right\}$;

(iii) $y_1R + y_2R + R = Ry_1 + Ry_2 + R$.

Notice that Condition (a)(iv) from Definition 1.8 is equivalent to the existence of two maps

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : R \to M_{2\times 2}(R) \quad \text{and} \quad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} : R \to M_{2\times 1}(R),$$

such that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} r = \sigma(r) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(r), \quad \text{for all } r \in R. \tag{1.3}$$

If B is a right double extension of R, we write $B = R_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = \{p_{12}, p_{11}\} \subseteq \Bbbk$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq R$, and σ, δ are as above. The set P is called a *parameter* and τ a *tail*.

For $R_p[y_1, y_2; \sigma, \delta, \tau]$ a right double extension, all maps σ_{ij} and δ_i are endomorphisms of the $\Bbbk$-vector space R. From [ZZ08, Lemma 1.7] we know that σ must be a homomorphism of algebras, and δ is a σ-derivation in the sense that δ is $\Bbbk$-linear and satisfes $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$, for all $r, r' \in R$. It is straightforward to see that if the matrix $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is triangular, then both σ_{11} and σ_{22} are algebra homomorphisms.

Recall that a map $d : R \to R$ is a σ-derivation, where σ is an endomorphism of R, if and only if the map from R to $M_{2\times 2}(R)$ sending r onto $\begin{bmatrix} \sigma(r) & d(r) \\ 0 & r \end{bmatrix}$ is a homomorphism of algebras. This, in particular, implies that for any algebra endomorphism σ of the commutative polynomial ring $\Bbbk[x]$ and any polynomial $w \in \Bbbk[x]$, there exists a (unique) σ-derivation d of $\Bbbk[x]$ such that $d(x) = w$ [CLM11, p. 2840].

In the case that $\tau \subseteq \Bbbk$, the subalgebra of $R_P[y_1, y_2; \sigma, \delta, \tau]$ generated by y_1 and y_2 is the double extension $\Bbbk_P[y_1, y_2; \sigma', \delta', \tau']$, where $\sigma' = \sigma|_{\Bbbk}$ is the canonical embedding of $\Bbbk$ in $M_{2\times 2}(\Bbbk)$ and $\delta' = \delta|_{\Bbbk} = 0$ is the zero map. Carvalho et al., [CLM11] proved that the latter is always an iterated Ore extension.

PROPOSITION 1.3 ([CLM11, PROPOSITION 1.2]). *Let $B = \Bbbk_P[y_1, y_2; \sigma', \delta', \tau']$. Then B is an iterated Ore extension $\Bbbk[x_1][x_2; \sigma_2, d_2]$, where σ_2 is the algebra endomorphism of the polynomial ring $\Bbbk[x_1]$ defined by $\sigma_2(x_1) = p_{12}x_1 + \tau_2$ and d_2 is the σ_2-derivation of $\Bbbk[x_1]$ given by $d_2(x_1) = p_{11}x_1^2 + \tau x_1 + \tau_0$. Moreover, B is a double extension of $\Bbbk$ if and only if $p_{12} \neq 0$.*

REMARK 2 ([CLM11, REMARK 1.3]). Let $C = \Bbbk[y_1][y_2; \sigma_2, \delta_2]$ be as in Proposition 1.3. Then, for

any $\Bbbk$-algebra R, we have $R \otimes_{\Bbbk} C = R_P[y_1, y_2; \sigma, \delta, \tau]$, where

$$\sigma = \begin{bmatrix} \mathrm{id}_R & 0 \\ 0 & \mathrm{id}_R \end{bmatrix}, \quad \delta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P = \{p_{12}, p_{11}\}, \quad \text{and} \quad \tau = \{\tau_0, \tau_1, \tau_2\}$$

are as in Proposition 1.3.

PROPOSITION 1.4 ([CLM11, PROPOSITION 1.4]). *Given $P = \{p_{12}, p_{11}\}$ and $\tau = \{\tau_0, \tau_1, \tau_2\}$ subsets of $\Bbbk$, $\sigma : R \to M_{2\times 2}(R)$ an algebra homomorphism, and $\delta : R \to M_{2\times 1}(R)$ a σ-derivation, let $C = \Bbbk[y_1][y_2; \sigma_2, d_2]$ be as in Proposition 1.3. Then, the following conditions are equivalent:*

(1) *The right double extension $R_P[y_1, y_2; \sigma, \delta, \tau]$ exists.*

(2) *One can extend the multiplications from R and C to a multiplication in the vector space $R \otimes_{\Bbbk} C$, satisfying $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} r = \sigma(r) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(r)$, for all $r \in R$.*

PROPOSITION 1.5 ([ZZ08, LEMMA 1.10 AND PROPOSITION 1.11]; [CLM11, PROPOSITION 1.5]). *Given a $\Bbbk$-algebra R, let σ be a homomorphism from R to $M_{2\times 2}(R)$, δ a σ-derivation from R to $M_{2\times 1}(R)$, $P = \{p_{12}, p_{11}\}$ a set of elements of $\Bbbk$, and $\tau = \{\tau_0, \tau_1, \tau_2\}$ a set of elements of R. Then, the associative $\Bbbk$-algebra B generated by R, y_1 and y_2 subject to the relations (1.2) and (1.3), is a right double extension if and only if the maps σ_{ij} and ρ_k, $i \in \{1,2\}$, $j, k \in \{0,1,2\}$, satisfy the six relations (1.4) - (1.9) below, where $\sigma_{i0} = \delta_i$ and ρ_k is a right multiplication by τ_k:*

$$\sigma_{21}\sigma_{11} + p_{11}\sigma_{22}\sigma_{11} = p_{11}\sigma_{11}^2 + p_{11}^2\sigma_{12}\sigma_{11} + p_{12}\sigma_{11}\sigma_{21} + p_{11}p_{12}\sigma_{12}\sigma_{21}, \tag{1.4}$$

$$\sigma_{21}\sigma_{12} + p_{12}\sigma_{22}\sigma_{11} = p_{11}\sigma_{11}\sigma_{12} + p_{11}p_{12}\sigma_{12}\sigma_{11} + p_{12}\sigma_{11}\sigma_{22} + p_{12}^2\sigma_{12}\sigma_{21}, \tag{1.5}$$

$$\sigma_{22}\sigma_{12} = p_{11}\sigma_{12}^2 + p_{12}\sigma_{12}\sigma_{22}, \tag{1.6}$$

$$\begin{aligned} \sigma_{20}\sigma_{11} + \sigma_{21}\sigma_{10} + \underline{\rho_1\sigma_{22}\sigma_{11}} &= p_{11}(\sigma_{10}\sigma_{11} + \sigma_{11}\sigma_{10} + \rho_1\sigma_{12}\sigma_{11}) \\ &\quad + p_{12}(\sigma_{10}\sigma_{21} + \sigma_{11}\sigma_{20} + \rho_1\sigma_{12}\sigma_{21}) + \tau_1\sigma_{11} + \tau_2\sigma_{21}, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \sigma_{20}\sigma_{12} + \sigma_{22}\sigma_{10} + \underline{\rho_2\sigma_{22}\sigma_{11}} &= p_{11}(\sigma_{10}\sigma_{12} + \sigma_{12}\sigma_{10} + \rho_2\sigma_{12}\sigma_{11}) \\ &\quad + p_{12}(\sigma_{10}\sigma_{22} + \sigma_{12}\sigma_{20} + \rho_2\sigma_{12}\sigma_{21}) + \tau_1\sigma_{12} + \tau_2\sigma_{22}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} \sigma_{20}\sigma_{10} + \underline{\rho_0\sigma_{22}\sigma_{11}} &= p_{11}(\sigma_{10}^2 + \rho_0\sigma_{12}\sigma_{11}) + p_{12}(\sigma_{10}\sigma_{20} + \rho_0\sigma_{12}\sigma_{21}) \\ &\quad + \tau_1\sigma_{10} + \tau_2\sigma_{10} + \tau_0\mathrm{id}_R. \end{aligned} \tag{1.9}$$

REMARK 3 ([CLM11, REMARK 1.6]). (1) Proposition 1.4 can be used to obtain a direct proof of Proposition 1.5.

(2) Proposition 1.5 implies the uniqueness, up to isomorphism, of a right double extension of R, with given σ, δ, P and τ, provided such an extension exists. Indeed, assume $\overline{B} = R_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of R. Then, by [ZZ08, Lemmas 1.7 and 1.10(b)], the data σ, δ, P and τ satisfy the conditions of Proposition 1.5. Let B be as in this proposition. Then, there is an algebra homomorphism from B to $\overline{B}$ which restricts to the identity on R and maps $y_i \in B$ to the corresponding element $y_i \in \overline{B}$, $i = 1, 2$. Since B is

a free left R-module with basis $\left\{y_1^i y_2^j \mid i,j \geq 0\right\}$ and the same holds for $\overline{B}$, this map is an isomorphism, thus proving uniqueness.

Zhang and Zhang [ZZ08, Remark 1.4] noticed that by choosing a suitable basis of the vector space $\Bbbk y_1 + \Bbbk y_2$, the following result holds.

PROPOSITION 1.6 ([CLM11, LEMMA 1.7]). *Let $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension.*

(1) *If $p_{11} \neq 0$ and $p_{12} = 1$, then*

$$B \cong R_{\{1,1\}}\left[\overline{y}_1, \overline{y}_2; \begin{bmatrix} \sigma_{11} & p_{11}\sigma_{12} \\ p_{11}^{-1}\sigma_{21} & \sigma_{22} \end{bmatrix}, \begin{bmatrix} p_{11}\delta_1 \\ \delta_2 \end{bmatrix}, \overline{\tau}\right],$$

where $\overline{\tau} = \{p_{11}\tau_0, \tau_1, p_{11}\tau_2\}$, $\overline{y}_1 = p_{11}y_1$ and $\overline{y}_2 = y_2$.

(2) *If $p_{12} \neq 1$, then*

$$B \cong A_{\{p_{12},0\}}\left[\overline{y}_1, \overline{y}_2; \begin{bmatrix} \sigma_{11} - q\sigma_{12} & \sigma_{12} \\ \sigma_{21} + q(\sigma_{11} - \sigma_{22}) - q^2\sigma_{12} & \sigma_{22} + q\sigma_{12} \end{bmatrix}, \begin{bmatrix} \delta_1 \\ \delta_2 + q\delta_1 \end{bmatrix}, \overline{\tau}\right],$$

where $q = \frac{p_{11}}{p_{12}-1}$, $\tau = \{\tau_0, \tau_1 - q\tau_2, \tau_2\}$, $\overline{y}_1 = y_1$ and $\overline{y}_2 = y_2 + qy_1$.

Following [CLM11, p. 2842], let $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension and suppose that $p_{12} \neq 1$. Then, from the ideas above, by choosing adequate generators $\overline{y}_i$ and (possibly) modifying the data σ, δ, τ one can assume that $p_{11} = 0$. If $\overline{B} = R_P[\overline{y}_1, \overline{y}_2; \overline{\sigma}, \overline{\delta}, \overline{\tau}]$ is a right double extension with $p_{11} = 0$, then $\overline{B}$ has a natural filtration, given by setting $\deg R = 0$ and $\deg \overline{y}_1 = \deg \overline{y}_2 = 1$. It can be seen that in view of relations 1.2 and 1.3, that the associated graded algebra $G(B)$ is isomorphic to $R_P[\overline{y}_1, \overline{y}_2; \overline{\sigma}, \{0,0,0\}]$. In this way, we have the following assertion:

PROPOSITION 1.7 ([CLM11, COROLLARY 1.8]). *Suppose that $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of R, with $p_{12} \neq 1$. Then, there exists a filtration on B such that the associated graded algebra $G(B)$ can be presented as follows: $G(B)$ is generated over R by indeterminates z_1, z_2; it is free as a left R-module with basis $\{z_1^i z_2^j \mid i,j \geq 0\}$; multiplication in $G(B)$ is given by left multiplication in R and the conditions $z_2 z_1 = p_{12} z_1 z_2$ and $z_1 R + z_2 R \subseteq R z_1 + R z_2$, with $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} r = \overline{\sigma}(r) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, where $\overline{\sigma}$ is obtained from σ and $q = \frac{p_{11}}{p_{12}-1}$ as in Proposition 1.6 (2).*

Furthermore, in case B is a double extension, then $G(B)$ is also free as a right R-module with basis $\{z_2^i z_1^j \mid i,j \geq 0\}$ and $z_1 R + z_2 R = R z_1 + R z_2$.

Next, we recall the key results formulated by Carvalho et al. [CLM11] about relations between double extensions and two-step iterated Ore extensions (c.f. Zhang and Zhang [ZZ09, Proposition 3.6]).

PROPOSITION 1.8 ([CLM11, THEOREM 2.2]). *Let R, B be $\Bbbk$-algebras such that B is an extension of R. Assume $P = \{p_{12}, p_{11}\} \subseteq \Bbbk$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq R$, σ is an algebra homomorphism from R to $M_2(R)$ and δ is a σ-derivation from R to $M_{2\times 1}(R)$.*

(1) *The following conditions are equivalent:*

(a) $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ *is a right double extension of* R *which can be presented as an iterated Ore extension* $R[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$.

(b) $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ *is right double extension of* R *with* $\sigma_{12} = 0$.

(c) $R[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ *is and iterated Ore extensions such that*

$$\sigma_2(R) \subseteq (R), \quad \sigma_2(y_1) = p_{12} y_1 + \tau_2,$$
$$d_2(R) \subseteq R y_1 + R, \quad d_2(y_1) = p_{11} y_1^2 + \tau_1 y_1 + \tau_0,$$

for some $p_{ij} \in \Bbbk$ *and* $\tau_i \in R$. *The maps* σ, δ, σ_i *and* δ_i, $i = 1, 2$, *are related by*

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_{21} & \sigma_2|_R \end{bmatrix}, \quad \delta(a) = \begin{bmatrix} d_1(a) \\ d_2(r) - \sigma_{21}(a) y_1 \end{bmatrix}, \quad \text{for all } a \in R.$$

(2) *If one of the equivalent statements from* (1) *holds, then* B *is a double extension of* R *if and only if* $\sigma_1 = \sigma_{11}$ *and* $\sigma_2|_R = \sigma_{22}$ *are automorphisms of* R *and* $p_{12} \neq 0$.

PROPOSITION 1.9 ([CLM11, THEOREM 2.4]). *Let* $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ *be a right double extension of the* $\Bbbk$*-algebra* R, *where* $P = \{p_{12}, p_{11}\} \in \Bbbk$, $\tau = \{\tau_2, \tau_1, \tau_0\} \subseteq R$, $\sigma : R \to M_2(R)$ *is an algebra homomorphism and* $\delta : R \to M_{2\times 1}(R)$ *is a* σ*-derivation. Then,* B *can be presented as an iterated Ore extension* $R[y_2; \sigma_2', d_2'][y_1; \sigma_1', d_1']$ *if and only if* $\sigma_{21} = 0$, $p_{12} \neq 0$ *and* $p_{11} = 0$. *In this case,* B *is a double extension if and only if* $\sigma_2' = \sigma_{22}$ *and* $\sigma_1'|_R = \sigma_{11}$ *are automorphisms of* R.

Let us see an example of a double extension that can be expressed as a two-step iterated Ore extension.

EXAMPLE 1.2 ([ZZ09, SUBCASE 4.1.1]). Let $B = (\Bbbk_Q[x_1, x_2])_P[y_1, y_2; \sigma, \delta, \tau]$ be the right double extension generated by x_1, x_2, y_1, y_2 subject to the relations

$$x_2 x_1 = x_1 x_2 + x_1^2, \qquad y_2 y_1 = y_1 y_2 + y_1^2,$$
$$y_1 x_1 = f x_1 y_1, \qquad y_1 x_2 = g x_1 y_1 + f x_2 y_1,$$
$$y_2 x_1 = h x_1 y_1 + f x_1 y_2, \qquad y_2 x_2 = m x_1 y_1 + h x_2 y_1 + g x_1 y_2 + f x_2 y_2,$$

where $f, g, h, m \in \Bbbk$ and $f \neq 0$. Note that in the relations

$$y_1 x_1 = \sigma_{11}(x_1) y_1 + \sigma_{12}(x_1) y_2, \quad \text{and} \quad y_1 x_2 = \sigma_{11}(x_2) y_1 + \sigma_{12}(x_2) y_2,$$

we have $\sigma_{12} = 0$. Hence, in this case $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of R. By Proposition 1.8(ii), B can be presented as the two-step iterated Ore extension $R[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$.

1.2.3 UNIVERSAL ENVELOPING ALGEBRAS AND PBW EXTENSIONS

If $\mathfrak{g}$ is a finite dimensional Lie algebra over a commutative ring K with basis $\{x_1, \ldots, x_n\}$, then by the *Poincaré-Birkhoff-Witt theorem*, the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the algebra generated by $x_1, \ldots, x_n$ subject to the relations $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g}$, where $[x_i, x_j] \subseteq K x_1 + \ldots + K x_n$, for all $1 \leq i, j \leq n$. As is well-known, in general these algebras are not

skew polynomial rings even including non-zero trivial derivations. Some enveloping algebras can be expressed as skew polynomial rings using derivations that are non-trivial. Let us see an example.

The standard basis for the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ is labelled $\{e, f, h\}$, where

$$[e, f] = h, \quad [h, e] = 2e, \quad \text{and} \quad [h, f] = -2f.$$

In this way, the enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ is the $\Bbbk$-algebra presented by three generators e, f, h and three relations

$$ef - fe = h, \quad he - eh = 2e, \quad \text{and} \quad hf - fh = -2f.$$

If R is the subalgebra of $U(\mathfrak{sl}_2(\Bbbk))$ generated by e and h, then $R = \Bbbk[e][h;\delta_1] = \Bbbk[h][e;\sigma_1]$, where $\Bbbk[e]$ and $\Bbbk[h]$ are commutative polynomial rings, $\delta_1(e) = 2e$, and σ_1 is the $\Bbbk$-algebra automorphism of $\Bbbk[h]$ with $\sigma_1(h) = h-2$. Thus, $U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk[e][h;\delta_1][f;\sigma_2,\delta_2] = \Bbbk[h][e;\sigma_1][f;\sigma_2,\delta_2]$, where $\sigma_2(e) = e$, $\sigma_2(h) = h+2$, $\delta_2(e) = -h$, and $\delta_2(h) = 0$ [GJ04, Exercise 2S]. Other examples of universal enveloping algebras known as *parafermionic* and *parabosonic algebras* are presented in Section 1.2.8.

Universal enveloping algebras above are PBW extensions over K in the sense of Bell and Goodearl [BG88] (note that these authors presented other examples of enveloping rings introduced by Passman [Pas87] related to enveloping universal algebras). In Remark 7 (iv), we will say some words about these extensions.

1.2.4 3-DIMENSIONAL SKEW POLYNOMIAL ALGEBRAS

Another kind of noncommutative rings which includes the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$, the Dispin algebra $U(\mathfrak{osp}(1,2))$ and Woronowicz's algebra $W_\nu(\mathfrak{sl}_2(\Bbbk))$ [Wor87], is the family of *3-dimensional skew polynomial algebras*. These algebras were introduced by Bell and Smith [BS90] and are very important in noncommutative algebraic geometry and noncommutative differential geometry (e.g., [Red96, Red99, RS22, Ros95] and references therein). Next, we recall their definition and classification.

DEFINITION 1.9 ([ROS95, DEFINITION C4.3]). *A 3-dimensional skew polynomial algebra A* is a $\Bbbk$-algebra generated by the indeterminates x, y, z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that

(i) $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$;

(ii) standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\Bbbk$-basis of the algebra.

PROPOSITION 1.10 ([ROS95, THEOREM C.4.3.1]). *If A is a 3-dimensional skew polynomial $\Bbbk$-algebra, then A is one of the following algebras:*

(1) *if $|\{\alpha, \beta, \gamma\}| = 3$, then A can be presented using the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$.*

(2) *if* $|\{\alpha,\beta,\gamma\}| = 2$ *and* $\beta \neq \alpha = \gamma = 1$, *then* A *can be presented as:*

(i) $yz - zy = z, \quad zx - \beta xz = y, \quad xy - yx = x;$

(ii) $yz - zy = z, \quad zx - \beta xz = b, \quad xy - yx = x;$

(iii) $yz - zy = 0, \quad zx - \beta xz = y, \quad xy - yx = 0;$

(iv) $yz - zy = 0, \quad zx - \beta xz = b, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x;$

(vi) $yz - zy = z, \quad zx - \beta xz = 0, \quad xy - yx = 0,$

where a, b *are any elements of* $\Bbbk$. *All non-zero values of* b *give isomorphic algebras.*

(3) *If* $|\{\alpha,\beta,\gamma\}| = 2$ *and* $\beta \neq \alpha = \gamma \neq 1$, *then* A *can be presented using the relations of the following form:*

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$

(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

In this case, b *is an arbitrary element of* $\Bbbk$. *Again, any non-zero values of* b *give isomorphic algebras.*

(4) *If* $\alpha = \beta = \gamma \neq 1$, *then* A *is the algebra defined by the relations* $yz - \alpha zy = a_1x + b_1$, $zx - \alpha xz = a_2y + b_2$, $xy - \alpha yx = a_3z + b_3$. *If* $a_i = 0$ $(i = 1,2,3)$, *then all non-zero values of* b_i *give isomorphic algebras.*

(5) *If* $\alpha = \beta = \gamma = 1$, *then* A *can be presented as*

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$

(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$

(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$

(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

Parameters $a, b \in \Bbbk$ *are arbitrary, and all non-zero values of* b *generate isomorphic algebras.*

1.2.5 BI-QUADRATIC ALGEBRAS ON 3 GENERATORS WITH PBW BASES

Related to algebras generated by three indeterminates, recently Bavula [Bav23] defined the *skew bi-quadratic algebras* with the aim of giving an explicit description of bi-quadratic algebras on 3 generators with PBW basis.

For a ring R and a natural number $n \geq 2$, a family $M = (m_{ij})_{i>j}$ of elements $m_{ij} \in R$ $(1 \leq j < i \leq n)$ is called a *lower triangular half-matrix* with coefficients in R. The set of all such matrices is denoted by $L_n(R)$.

If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is an n-tuple of commuting endomorphisms of R, $\delta = (\delta_1, \ldots, \delta_n)$ is an n-tuple of σ-endomorphisms of R (that is, δ_i is a σ_i-derivation of R for $i = 1, \ldots, n$), $Q = (q_{ij}) \in$

$L_n(Z(R))$, $\mathbb{A} := (a_{ij,k})$ where $a_{ij,k} \in R$, $1 \le j < i \le n$ and $k = 1, \dots, n$, and $\mathbb{B} := (b_{ij}) \in L_n(R)$, the *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1, \dots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ is a ring generated by the ring R and elements $x_1, \dots, x_n$ subject to the defining relations

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \quad \text{for } i = 1, \dots, n, \text{ and every } r \in R, \tag{1.10}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij}, \quad \text{for all } j < i. \tag{1.11}$$

In the particular case when $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for $i = 1, \dots, n$, the ring A is called the *bi-quadratic algebra* (*BQA*) and is denoted by $A = R[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$. A has *PBW basis* if $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x^\alpha$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The following result classifies (up to isomorphism) the bi-quadratic algebras on three generators of Lie type, i.e., when $q_1 = q_2 = q_3 = 1$.

PROPOSITION 1.11 ([BAV23, THEOREM 1.4]). *Let A be an algebra of Lie type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the algebra A is isomorphic to one of the following (pairwise non-isomorphic) algebras:*

(1) $P_3 = \Bbbk[x_1, x_2, x_3]$, *a polynomial in three indeterminates.*

(2) $U(\mathfrak{sl}_2(\Bbbk))$, *the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$.*

(3) $U(\mathfrak{H}_3))$, *the universal enveloping algebra of the Heisenberg Lie algebra $\mathfrak{H}_3$.*

(4) $U(\mathcal{N})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = z, [x, z] = 0, [y, z] = 1\rangle$, *and the algebra $U(\mathcal{N})/\langle c-1\rangle$ is a tensor product $A_1 \otimes \Bbbk[x']$ of its subalgebras, the Weyl algebra $A_1(\Bbbk) = \Bbbk\{y, z\}/\langle [y, z] = 1\rangle$ and the polynomial algebra $[x']$ where $x' = x + \frac{1}{2}z^2$.*

(5) $U(\mathfrak{n}_2 \times \Bbbk z) \cong \Bbbk\{x, y, z\}/\langle [x, y] = y\rangle$, *and z is a central element.*

(6) $U(\mathcal{M})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = y, [x, z] = 1, [y, z] = 0\rangle$ *and the algebra $U(\mathcal{M})/\langle c-1\rangle$ is a skew polynomial algebra $A_1(\Bbbk)[y; \sigma]$ where $A_1(\Bbbk) = \Bbbk\{x, z\}/\langle [x, z] = 1\rangle$ is the Weyl algebra and σ is an automorphism of $A_1(\Bbbk)$ given by the rule $\sigma(x+1)$ and $\sigma(z) = z$.*

PROPOSITION 1.12 ([BAV23, THEOREM 2.1]). *Up to isomorphism, there are only five bi-quadratic algebras on two generators:*

(1) *The polynomial algebra $\Bbbk[x_1, x_2]$.*

(2) *The Weyl algebra $A_1(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_1 x_2 - x_2 x_1 = 1\rangle$.*

(3) *The universal enveloping algebra of the Lie algebra $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$, $U(\mathfrak{n}_2) = \Bbbk\{x_1, x_2\}/\langle x_2 x_1 - x_1 x_2 = x_1\rangle$.*

(4) *The quantum plane $\mathcal{O}_q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2 x_1 - q x_1 x_2\rangle$, where $q \in \Bbbk \setminus \{0, 1\}$.*

(5) *The quantum Weyl algebra $A_1(q) = \Bbbk\{x_1, x_2\}/\langle x_2 x_1 - q x_1 x_2 = 1\rangle$, where $q \in \Bbbk \setminus \{0, 1\}$.*

1.2.6 DIFFUSION ALGEBRAS

Diffusion algebras were introduced by Isaev et al. [IPR01] as quadratic algebras that appear as algebras of operators that model the stochastic flow of motion of particles in a one dimensional discrete lattice. However, its origin can be found in Krebs and Sandow [KS97].

DEFINITION 1.10. ([IPR01, p. 5817]) The *diffusion algebras type 1* are affine algebras $\mathscr{D}$ that are generated by n indeterminates $D_1,\dots,D_n$ over $\Bbbk$ that admit a PBW basis of ordered monomials of the form $D_{\alpha_1}^{k_1}D_{\alpha_2}^{k_2}\cdots D_{\alpha_n}^{k_n}$ with $k_j\in\mathbb{N}$ and $\alpha_1>\alpha_2>\cdots>\alpha_n$, and there exist elements $x_1,\dots,x_n\in\Bbbk$ such that for all $1\leq i<j\leq n$, there exist $\lambda_{ji}\in\Bbbk^*$ such that

$$\lambda_{ij}D_iD_j-\lambda_{ji}D_jD_i=x_jD_i-x_iD_j. \tag{1.12}$$

Notice that a diffusion algebra in one indeterminate is precisely a commutative polynomial ring in one indeterminate. A diffusion algebra with $x_t=0$, for all $t=1,\dots,n$, is a *multiparameter quantum affine n−space*.

Fajardo et al. [FGL+20] studied ring-theoretical properties of a graded version of these algebras.

DEFINITION 1.11. ([FGL+20, Section 2.4]) The *diffusion algebras type 2* are affine algebras $\mathscr{D}$ generated by $2n$ variables $\{D_1,\dots,D_n,x_1,\dots,x_n\}$ over a field $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $B_{\alpha_1}^{k_1}B_{\alpha_2}^{k_2}\cdots B_{\alpha_n}^{k_n}$ with $B_{\alpha_i}\in\{D_1,\dots,D_n,x_1,\dots,x_n\}$, for all $i\leq 2n$, $k_j\in\mathbb{N}$, and $\alpha_1>\alpha_2>\cdots>\alpha_n$, such that for all $1\leq i<j\leq n$, there exist elements $\lambda_{ij}\in\Bbbk^*$ satisfying the relations

$$\lambda_{ij}D_iD_j-\lambda_{ji}D_jD_i=x_jD_i-x_iD_j. \tag{1.13}$$

Different physical applications of algebras type 1 and 2 have been studied in the literature. From the point of view of ring-theoretical, homological and computational properties, several thesis and papers have been published (e.g., [FGL+20, HHR20, Hin05, Lev05, Twa02]). For instance, notice that a diffusion algebra type 1 generated by n indeterminates has Gelfand-Kirillov dimension n since because of the PBW basis, the vector subspace consisting of elements of total degree at most l is isomorphic to that of a commutative polynomial ring in n indeterminates. Similarly, diffusion algebras type 2 have Gelfand-Kirillov dimension $2n$.

REMARK 4. About the above definitions of diffusion algebras, we have the following facts:

(i) Isaev et al. [IPR01] and Pyatov and Twarok [PT02] defined diffusion algebras type 1 by taking $\Bbbk=\mathbb{C}$. Nevertheless, for the results obtained in this thesis we can take any field not necessarily $\mathbb{C}$.

(ii) Following Krebs and Sandow [KS97], the relations (1.12) are consequence of subtracting (quadratic) operator relations of the type

$$\Gamma_{\gamma\delta}^{\alpha\beta}D_\alpha D_\beta=D_\gamma X_\delta-X_\gamma D_\delta,\ \text{ for all }\ \gamma,\delta=0,1,\dots,n-1,$$

where $\Gamma_{\gamma\delta}^{\alpha\beta}\in\Bbbk$, and D_i's and X_j's are operators of a particular vector space such that not necessarily $[D_i,X_j]=0$ holds [KS97, p. 3168].

(iii) Hinchcliffe in his PhD thesis [Hin05, Definition 2.1.1] considered the following notation for diffusion algebras. Let R be the algebra generated by n indeterminates $x_1, x_2, \ldots, x_n$ over $\mathbb{C}$ subject to relations $a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j$, whenever $i < j$, for some parameters $a_{ij} \in \mathbb{C} \setminus \{0\}$, for all $i < j$ and $b_{ij}, r_i \in \mathbb{C}$, for all $i < j$. He defined the *standard monomials* to be those of the form $x_n^{i_n}x_{n-1}^{i_{n-1}} \cdots x_2^{i_2}x_1^{i_1}$. R is called a *diffusion algebra* if it admits a *PBW basis of these standard monomials*. In other words, R is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for R. If all the elements $q_{ij} := \frac{b_{ij}}{a_{ij}}$'s are non-zero, then the diffusion algebras have a PBW basis in any order of the indeterminates [Hin05, Remark 2.1.6].

Diffusion algebras of n generators (also called *n-diffusion algebras*) are constructed in such a way that the subalgebras of three generators are also diffusion algebras. As we can see in Proposition 1.13, diffusion algebras type 1 of three generators can be classified into 4 families, A, B, C, and D, and these in turn are divided into classes as shown below (notice that this classification reflects the number of coefficients x_s, $s \in \{i, j, k\}$, being zero in comparison with the expression (1.12)).

PROPOSITION 1.13 ([PT02, P. 3270]). *If $\mathscr{D}$ is a diffusion algebra type 1 generated by the indeterminates D_i, D_j and D_k with $i < j < k$, and $\Lambda \in \Bbbk$, then $\mathscr{D}$ is isomorphic to a diffusion algebra of one of the following clases:*

(1) The case of A_I:

$$\begin{aligned} gD_iD_j - gD_jD_i &= x_jD_i - x_iD_j, \\ gD_iD_k - gD_kD_i &= x_kD_i - x_iD_k, \\ gD_jD_k - gD_kD_j &= x_kD_j - x_jD_k, \end{aligned}$$

where $g \in \Bbbk$ is a non-zero element.

(2) The case of A_{II}:

$$\begin{aligned} g_{ij}D_iD_j &= x_jD_i - x_iD_j, \\ g_{ik}D_iD_k &= x_kD_i - x_iD_k, \\ g_{jk}D_jD_k &= x_kD_j - x_jD_k, \end{aligned}$$

where $g_{st} := g_s - g_t$ with $g_s \neq g_t$, for all $s < t$, and $s, t \in \{i, j, k\}$, are elements belonging to $\Bbbk$.

(3) The case of $B^{(1)}$:

$$\begin{aligned} g_jD_iD_j - (g_j - \Lambda)D_jD_i &= -x_iD_j, \\ gD_iD_k - (g - \Lambda)D_kD_i &= x_kD_i - x_iD_k, \\ g_jD_jD_k - (g_j - \Lambda)D_kD_j &= x_kD_j, \end{aligned}$$

where $g, g_j \in \Bbbk^*$.

(4) The case of $B^{(2)}$:

$$\begin{aligned} g_{ij}D_iD_j &= -x_iD_j,\\ g_{ik}D_iD_k - \lambda_{ki}D_kD_i &= x_kD_i - x_iD_k,\\ g_{jk}D_jD_k &= x_kD_j, \end{aligned}$$

where g_{ij}, g_{ik}, g_{jk} are non-zero elements of $\Bbbk$.

(5) The case of $B^{(3)}$.

$$\begin{aligned} gD_iD_j - (g-\Lambda)D_jD_i &= x_jD_i - x_iD_j,\\ g_kD_iD_k &= -x_iD_k,\\ (g_k-\Lambda)D_jD_k &= -x_jD_k, \end{aligned}$$

where $g \neq 0$ and $g_k \neq 0, \Lambda$, are elements belonging to $\Bbbk$.

(6) The case of $B^{(4)}$:

$$\begin{aligned} (g_i-\Lambda)D_iD_j &= x_jD_i,\\ g_iD_iD_k &= x_kD_i,\\ gD_jD_k - (g-\Lambda)D_kD_j &= x_kD_j - x_jD_k, \end{aligned}$$

where $g \neq 0$ and $g_i \neq 0, \Lambda$.

(7) The case of $C^{(1)}$:

$$\begin{aligned} g_jD_iD_j - (g_j-\Lambda)D_jD_i &= -x_iD_j,\\ g_kD_iD_k - (g_k-\Lambda)D_kD_i &= -x_iD_k,\\ g_{jk}D_jD_k - g_{kj}D_kD_j &= 0, \end{aligned}$$

where $g_j, g_k, g_{j,k} \in \Bbbk^*$.

(8) The case of $C^{(2)}$:

$$\begin{aligned} g_{ij}D_iD_j - g_{ji}D_jD_i &= -x_iD_j,\\ g_{ik}D_iD_k - g_{ki}D_kD_i &= -x_iD_k,\\ D_jD_k &= 0, \end{aligned}$$

where $g_{ij}, g_{ik} \in \Bbbk^*$.

(9) The case of D: *With $q_{st} := \frac{g_{ts}}{g_{st}}$, where $s,t \in \{i,j,k\}$ (recall that $g_{st} \neq 0$, for $s < t$), we have*

$$\begin{aligned} D_iD_j - q_{ji}D_jD_i &= 0,\\ D_iD_k - q_{ki}D_kD_i &= 0,\\ D_jD_k - q_{kj}D_kD_j &= 0. \end{aligned}$$

About the relationship between diffusion algebras and skew polynomial rings, if we consider the notation in Remark 4 (3), then a 3-diffusion algebra generated by the indeterminates x_1, x_2, x_3 is a skew polynomial ring over its 2-diffusion subalgebra generated by x_2 and x_3 [Hin05, Lemma 2.2.1], where it is easy to see that a 2-diffusion algebra is a skew polynomial ring over the polynomial subalgebra generated by x_2. In general an n-diffusion algebra (generated by the indeterminates $x_1, \ldots, x_n$) is a skew polynomial ring over its $(n-1)$ diffusion subalgebra generated by $x_2, \ldots, x_n$ [Hin05, Remark 2.2.2].

Since a diffusion algebra on $n \geq 2$ generators is left Noetherian if and only if $q_{ij} \neq 0$, for all $i < j$ [Hin05, Proposition 2.2.5], where q_{ij} is given in Remark 4 (3), then every Noetherian 2-diffusion algebra is isomorphic to one of the following three types of algebra [Hin05, Proposition 3.3.1]:

- *The quantum affine plane*, that is, the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1x_2 - qx_2x_1 = 0$, for some $q \in \mathbb{C}^*$ (allowing the possibility $q = 1$).
- *The quantized Weyl algebra*, i.e., the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1x_2 - qx_2x_1 = 1$, for some $q \in \mathbb{C} \setminus \{0, 1\}$.
- *The universal enveloping algebra of the 2-dimensional soluble Lie algebra*, that is, the free algebra generated by the indeterminates x_1 and x_2 subject to the relation $x_1x_2 - x_2x_1 = x_1$.

Related to Proposition 1.13, Hinchcliffe [Hin05] proved the following result about classification of diffusion algebras assuming certain conditions on the coefficients of commutation of the indeterminates.

PROPOSITION 1.14 ([HIN05, PROPOSITION 3.1.4]). *If $q_{ij} \notin \{0, 1\}$, for all i, j, then a diffusion algebra R is isomorphic either to multiparameter quantum affine n-space or to the $\mathbb{C}$-algebra generated by the indeterminates $x_1, x_2, x_3, \ldots, x_n$ subject to relations*

$$\begin{aligned} x_1x_2 - q_{12}x_2x_1 &= 1, \text{ where } q_{12} \neq 1, \\ x_1x_i - q_{1i}x_ix_1 &= 0, \text{ where } q_{1i} \neq 1, \\ x_2x_i - q_{2i}^{-1}x_ix_2 &= 0, \\ x_ix_j - q_{ij}x_jx_i &= 0, \text{ for all } 3 \leq i < j. \end{aligned}$$

1.2.7 GENERALIZED WEYL ALGEBRAS AND DOWN-UP ALGEBRAS

Other algebraic structures that illustrate the results obtained in this thesis are the *generalized Weyl algebras* and *down-up algebras*. We briefly present the definitions and some relations between these algebras (see [Jor95, Jor00, JW96] for a detailed description).

Given an automorphism σ and a central element a of a ring R, Bavula [Bav92] defined the *generalized Weyl algebra* $R(\sigma, a)$ as the ring extension of R generated by the indeterminates X^- and X^+ subject to the relations $X^-X^+ = a$, $X^+X^- = \sigma(a)$, and, for all $b \in R$, $X^+b = \sigma(b)X^+$, $X^-\sigma(b) = bX^-$. This family of algebras includes the classical Weyl algebras, primitive quotients of $U(\mathfrak{sl}_2)$, and ambiskew polynomial rings. Generalized Weyl algebras have been

extensively studied in the literature by various authors (see [Bav92, Bav23, Jor00], and references therein).

On the other hand, the *down-up algebras* $A(\alpha,\beta,\gamma)$, where $\alpha,\beta,\gamma \in \mathbb{C}$, were defined by Benkart and Roby [Ben99, BR98] as generalizations of algebras generated by a pair of operators, precisely, the "down" and "up" operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered set P. More exactly, consider a partially ordered set $(P,\prec)$ and let $\mathbb{C}P$ be the complex vector space with basis P. If for an element p of P, the sets $\{x \in P \mid x \succ p\}$ and $\{x \in P \mid x \prec p\}$ are finite, then we can define the "down" operator d and the "up" operator u in $\mathrm{End}_{\mathbb{C}}\ \mathbb{C}P$ as $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$, respectively (for partially ordered sets in general, one needs to complete $\mathbb{C}P$ to define d and u). For any $\alpha,\beta,\gamma \in \mathbb{C}$, the *down-up algebra* is the $\mathbb{C}$-algebra generated by d and u subject to the relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$ and $du^2 = \alpha udu + \beta u^2 d + \gamma u$. A partially ordered set P is called (q,r)-*differential* if there exist $q,r \in \mathbb{C}$ such that the down and up operators for P satisfy both relations, and $\alpha = q(q+1), \beta = -q^3$, and $\gamma = r$. From [BR98], we know that for $0 \neq \lambda \in \mathbb{C}$, $A(\alpha,\beta,\gamma) \simeq A(\alpha,\beta,\lambda\gamma)$. This means that when $\gamma \neq 0$ we can assume without loss of generality that in fact $\gamma = 1$. For more details about the combinatorial origins of down-up algebras, see [Ben99, Section 1].

Remarkable examples of down-up algebras include the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and some of its deformations introduced by Witten [Wit90] and Woronowicz [Wor87]. Related to the theoretical properties of these algebras, Kirkman et al. [KMP99] proved that a down-up algebra $A(\alpha,\beta,\gamma)$ is Noetherian if and only if β is non-zero. As a matter of fact, they showed that $A(\alpha,\beta,\gamma)$ is a generalized Weyl algebra and that $A(\alpha,\beta,\gamma)$ has a filtration for which the associated graded ring is an iterated Ore extension over $\mathbb{C}$.

According to [Ben99, p. 32], if $\mathfrak{g}$ is a 3-dimensional Lie algebra over $\mathbb{C}$ with basis $x, y, [x,y]$ such that $[x,[x,y]] = \gamma x$ and $[[x,y],y] = \gamma y$, then in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ these relations are given by $x^2y - 2xyx + yx^2 = \gamma x$ and $xy^2 - 2yxy + y^2x = \gamma y$. Notice that $U(\mathfrak{g})$ is a homomorphic algebra of the down-up algebra $A(2,-1,\gamma)$ via the mapping $\phi : A(2,-1,\gamma) \to U(\mathfrak{g})$, $d \mapsto x, u \mapsto y$, and the mapping $\psi : \mathfrak{g} \to A(2,-1,\gamma)$, $x \mapsto d, y \mapsto u$, $[x,y] \mapsto du - ud$, extends by the universal property of $U(\mathfrak{g})$ to an algebra homomorphism $\psi : U(\mathfrak{g}) \to A(2,-1,\gamma)$ which is the inverse of ψ. Hence, $U(\mathfrak{g})$ is isomorphic to $A(2,-1,\gamma)$.

It is straightforward to see that $U(\mathfrak{sl}_2(\mathbb{C})) \cong A(2,-1,-2)$. Also, for the Heisenberg Lie algebra $\mathfrak{h}$ with basis x, y, z where $[x,y] = z$ and $[z,x] = [z,y] = 0$, $U(\mathfrak{h}) \cong A(2,-1,0)$.

Now, with the aim of providing an explanation of the existence of quantum groups, Witten [Wit90, Wit91] introduced a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$. By definition, Witten's deformation is a unital associative algebra over a field $\Bbbk$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\underline{\xi} = (\xi_1,\ldots,\xi_7)$ of elements of $\Bbbk$. This algebra, denoted by $W(\underline{\xi})$, is generated by the indeterminates x, y, z subject to the defining relations $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4$, and $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$. From [Ben99, Section 2], we know that a Witten's deformation algebra $W(\xi)$ with

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4, \tag{1.14}$$

is isomorphic to one down-up algebra. Notice that any down-up algebra $A(\alpha,\beta,\gamma)$ with not both α and β equal to 0 is isomorphic to a Witten deformation algebra $W(\underline{\xi})$ whose parameters satisfy

(1.14).

Since algebras $W(\underline{\xi})$ are filtered, Le Bruyn [Bru94, Bru95] studied the algebras $W(\underline{\xi})$ whose associated graded algebras are Auslander regular. He determined a 3-parameter family of deformation algebras which are said to be *conformal* $\mathfrak{sl}_2$ *algebras* that are generated by the indeterminates x, y, z over a field $\Bbbk$ subject to the relations given by $zx - axz = x$, $zy - ayz = y$, and $yx - cxy = bz^2 + z$. In the case $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with these three defining relations is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1+ac)$, $\beta = -ac^{-1}$ and $\gamma = -c^{-1}$. Notice that if $c = b = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}, \beta = 0$, and $\gamma = -a^{-1}$. As one can check, conformal $\mathfrak{sl}_2$ algebras are not Ore extensions.

Kulkarni [Kul99] showed that under certain assumptions on the parameters, a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2(\Bbbk)$ algebra or to an iterated Ore extension. More exactly, following [Kul99, Theorem 3.0.3] if $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $W(\underline{\xi})$ is isomorphic to one of the following algebras: (i) a conformal $\mathfrak{sl}_2$ algebra with generators x, y, z and relations given above or (ii) an iterated Ore extension whose generators satisfy

- $xz - zx = x$, $zy - yz = \zeta y$, $yx - \eta xy = 0$, or
- $xw = \theta wx$, $wy = \kappa yw$, $yx = \lambda xy$, for parameters $\zeta, \eta, \theta, \kappa, \lambda \in \Bbbk$.

Notice that iterated Ore extensions above are defined in the following way:

(i) The Witten deformation algebra is isomorphic to $\Bbbk[z][y, \sigma_1][x, \sigma_2]$ where σ_1 is the automorphism of $\Bbbk[z]$ defined as $\sigma_1(z) = z - \zeta$, with $zy - yz = \zeta y$; σ_2 is the automorphism of $\Bbbk[z][y, \sigma_1]$ defined as $\sigma_2(y) = \eta^{-1}y$, $\sigma_2(z) = z + 1$, which satisfies $xz - zx = x$ and $yx - \eta xy = 0$.

(ii) The Witten deformation algebra is isomorphic to $\Bbbk[w][y, \sigma_1][x, \sigma_2]$ where σ_1 is the automorphism of $\Bbbk[w]$ defined as $\sigma_1(w) = \kappa^{-1}w$ with $wy = \kappa yw$, and σ_2 is the automorphism of $\Bbbk[w][y, \sigma_1]$ defined as $\sigma_2(w) = \theta w$, $\sigma_2(y) = \lambda^{-1}y$ such that $wy = \kappa yw$ and $yx = \lambda xy$.

1.2.8 OTHER FAMILIES OF QUANTUM ALGEBRAS

In this section, we recall the definitions of some examples of noncommutative rings known in the literature as *quantum algebras* or *quantized algebras*.

- Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \ldots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The *homogenized enveloping algebra of* $\mathfrak{g}$ is $\mathscr{A}(\mathfrak{g}) := T(\mathfrak{g} \oplus \Bbbk z)/\langle R \rangle$, where $T(\mathfrak{g} \oplus \Bbbk z)$ denotes the tensor algebra, z is a new indeterminate, and R is spanned by the union of sets $\{z \otimes x - x \otimes z \mid x \in \mathfrak{g}\}$ and $\{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathfrak{g}\}$.
- From [GJ04, p. 41], for q an element of $\Bbbk$ with $q \neq \pm 1$, the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ corresponding to the choice of q is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ presented by the generators E, F, K, K^{-1} and the relations

$$KK^{-1} = K^{-1}K = 1, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad KE = q^2EK, \quad \text{and} \quad KF = q^{-2}FK.$$

From [GJ04, Exercise 2T], we know that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][K^{\pm 1};\sigma_1][F;\sigma_2,\delta_2]$ [GJ04, Exercise 2T].

- Following Yamane [Yam89], if $q \in \mathbb{C}$ with $q^8 \neq 1$, the complex algebra A generated by the indeterminates $e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2$ subject to the relations

$$
\begin{aligned}
&e_{13}e_{12} = q^{-2}e_{12}e_{13}, && f_{13}f_{12} = q^{-2}f_{12}f_{13}, && \\
&e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, && f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13}, && \\
&e_{23}e_{13} = q^{-2}e_{13}e_{23}, && f_{23}f_{13} = q^{-2}f_{13}f_{23}, && \\
&e_{12}f_{12} = f_{12}e_{12} + \frac{k_1^2 - l_1^2}{q^2 - q^{-2}}, && e_{12}k_1 = q^{-2}k_1e_{12}, && k_1f_{12} = q^{-2}f_{12}k_1, \\
&e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, && e_{12}k_2 = qk_2e_{12}, && k_2f_{12} = qf_{12}k_2, \\
&e_{12}f_{23} = f_{23}e_{12}, && e_{13}k_1 = q^{-1}k_1e_{13}, && k_1f_{13} = q^{-1}f_{13}k_1, \\
&e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, && e_{13}k_2 = q^{-1}k_2e_{13}, && k_2f_{13} = q^{-1}f_{13}k_2, \\
&e_{13}f_{13} = f_{13}e_{13} - \frac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, && e_{23}k_1 = qk_1e_{23}, && k_1f_{23} = qf_{23}k_1, \\
&e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, && e_{23}k_2 = q^{-2}k_2e_{23}, && k_2f_{23} = q^{-2}f_{23}k_2, \\
&e_{23}f_{12} = f_{12}e_{23}, && e_{12}l_1 = q^{2}l_1e_{12}, && l_1f_{12} = q^{2}f_{12}l_1, \\
&e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, && e_{12}l_2 = q^{-1}l_2e_{12}, && l_2f_{12} = q^{-1}f_{12}l_2, \\
&e_{23}f_{23} = f_{23}e_{23} + \frac{k_2^2 - l_2^2}{q^2 - q^{-2}}, && e_{13}l_1 = ql_1e_{13}, && l_1f_{13} = qf_{13}l_1, \\
&e_{13}l_2 = ql_2e_{13}, && l_2f_{13} = qf_{13}l_2, && e_{23}l_1 = q^{-1}l_1e_{23}, \\
&l_1f_{23} = q^{-1}f_{23}l_1, && e_{23}l_2 = q^{2}l_2e_{23}, && l_2f_{23} = q^{2}f_{23}l_2, \\
&l_1k_1 = k_1l_1, && l_2k_1 = k_1l_2, && k_2k_1 = k_1k_2, \\
&l_1k_2 = k_2l_1, && l_2k_2 = k_2l_2, && l_2l_1 = l_1l_2,
\end{aligned}
$$

  is necessary to define the *quantized enveloping algebra of* $\mathfrak{sl}_3(\mathbb{C})$.

- With the aim of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi [Hay90] considered the free algebra $\mathbf{U}$. Following Berger [Ber92, Example 2.7.7], $\mathbf{U}$ is a $\Bbbk$-algebra generated by the indeterminates $\omega_1,\ldots,\omega_n,\psi_1,\ldots,\psi_n$, and $\psi_1^*,\ldots,\psi_n^*$, subject to the relations

$$
\begin{aligned}
\psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* = 0, \quad & 1 \leq i < j \leq n, \\
\omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* = 0, \quad & 1 \leq i, j \leq n, \\
\psi_i^*\psi_i - q^2\psi_i\psi_i^* = -q^2\omega_i^2, \quad & 1 \leq i \leq n.
\end{aligned}
$$

- The *Non-Hermitian realization of a Lie deformed* defined by Jannussis et al. [JLM95] is an important example of a non-canonical Heisenberg algebra considering the case of non-Hermitian (i.e., $\hbar = 1$) operators A_j, B_k, where the following relations are satisfied:

$$
\begin{aligned}
A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j = i\delta_{jk}, & \\
[A_j, B_k] = 0 \ \ (j \neq k), & \\
[A_j, A_k] = [B_j, B_k] = 0, &
\end{aligned}
$$

and,

$$\begin{aligned} A_j^+(1+i\lambda_{jk})B_k^+ - B_k^+(1-i\lambda_{jk})A_j^+ &= i\delta_{jk}, \\ [A_j^+, B_k^+] &= 0 \ (j \neq k), \\ [A_j^+, A_k^+] = [B_j^+, B_k^+] &= 0, \end{aligned} \tag{1.15}$$

with $A_j \neq A_j^+$, $B_k \neq B_k^+$ $(j,k=1,2,3)$. If the operators A_j, B_k are in the form $A_j = f_j(N_j+1)a_j$, $B_k = a_k^+ f_k(N_k+1)$, where a_j, a_j^+ are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator ($N_j = a_j^+ a_j$, $\langle N_j \mid n_j\rangle = \langle n_j | n_j\rangle$), and the structure functions $f_j(N_j+1)$ complex, then it is showed that A_j and B_k are given by

$$A_j = \sqrt{\frac{i}{1+i\lambda_j}}\left(\frac{[(1-i\lambda_j)/(1+i\lambda_j)]^{N_j+1}-1}{(1-i\lambda_j)/(1+i\lambda_j)-1}\frac{1}{N_j+1}\right)^{\frac{1}{2}} a_j,$$
$$B_k = \sqrt{\frac{i}{1+i\lambda_k}}\, a_k^+\left(\frac{[(1-i\lambda_k)/(1+i\lambda_k)]^{N_k+1}-1}{(1-i\lambda_k)/(1+i\lambda_k)-1}\frac{1}{N_k+1}\right)^{\frac{1}{2}}.$$

- Following Havlíček et al. [HKP00, p. 79], the $\mathbb{C}$-algebra $U_q'(\mathfrak{so}_3)$ is generated by the indeterminates I_1, I_2, and I_3, subject to the relations given by

$$I_2I_1 - qI_1I_2 = -q^{\frac{1}{2}}I_3, \quad I_3I_1 - q^{-1}I_1I_3 = q^{-\frac{1}{2}}I_2, \quad I_3I_2 - qI_2I_3 = -q^{\frac{1}{2}}I_1,$$

where q is a non-zero element of $\mathbb{C}$. It is straightforward to show that $U_q'(\mathfrak{so}_3)$ cannot be expressed as an iterated Ore extension.

- Zhedanov [Zhe91, Section 1] introduced the *Askey-Wilson algebra* $AW(3)$ as the algebra generated by three operators K_0, K_1, and K_2, that satisfy the commutation relations

$$[K_0,K_1]_\omega = K_2, [K_2,K_0]_\omega = BK_0 + C_1K_1 + D_1, \text{ and } [K_1,K_2]_\omega = BK_1 + C_0K_0 + D_0,$$

where B, C_0, C_1, D_0, and D_1 are the structure constants of the algebra, which Zhedanov assumes are real, and the q-commutator $[-,-]_\omega$ is given by $[\square,\triangle]_\omega := e^\omega\square\triangle - e^{-\omega}\triangle\square$, where $\omega \in \mathbb{R}$. Notice that in the limit $\omega \to 0$, the algebra AW(3) becomes an ordinary Lie algebra with three generators (D_0 and D_1 are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as

$$\begin{aligned} e^\omega K_0K_1 - e^{-\omega}K_1K_0 &= K_2, \\ e^\omega K_2K_0 - e^{-\omega}K_0K_2 &= BK_0 + C_1K_1 + D_1, \\ e^\omega K_1K_2 - e^{-\omega}K_2K_1 &= BK_1 + C_0K_0 + D_0. \end{aligned}$$

- With the purpose of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [Gre53] and Greenberg and Messiah [GM65] introduced by means of generators and relations the *parafermionic* and *parabosonic algebras*. For the

completeness of the thesis, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [KD09]. Let $[\square,\triangle] := \square\triangle - \triangle\square$ and $\{\square,\triangle\} := \square\triangle + \triangle\square$.

Consider the $\Bbbk$-vector space V_F freely generated by the elements f_i^+, f_j^-, with $i,j = 1,\ldots,n$. If $T(V_F)$ is the tensor algebra of V_F and I_F is the two-sided ideal I_F generated by the elements $[[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon-\eta)^2\delta_{jk}f_i^\xi + \frac{1}{2}(\varepsilon-\xi)^2\delta_{ik}f_j^\eta$, for all values of $\xi,\eta,\varepsilon = \pm 1$, and $i,j,k = 1,\ldots,n$, then the *parafermionic algebra* in $2n$ generators $P_F^{(n)}$ (n parafermions) is the quotient algebra of $T(V_F)$ with the ideal I_F, that is,

$$P_F^{(n)} = \frac{T(V_F)}{\langle [[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon-\eta)^2\delta_{jk}f_i^\xi + \frac{1}{2}(\varepsilon-\xi)^2\delta_{ik}f_j^\eta \mid \xi,\eta,\varepsilon = \pm 1, i,j,k = 1,\ldots,n\rangle}.$$

It is well-known (e.g., [KD09, Section 18.2]) that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$, i.e., $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$.

Similarly, if V_B denotes the $\Bbbk$-vector space freely generated by the elements b_i^+, b_j^-, $i,j = 1,\ldots,n$, $T(V_B)$ is the tensor algebra of V_B, and I_B is the two-sided ideal of $T(V_B)$ generated by the elements $[\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon] - (\varepsilon-\eta)\delta_{jk}b_i^\xi - (\varepsilon-\xi)\delta_{ik}b_j^\eta$, for all values of $\xi,\eta,\varepsilon = \pm 1$, and $i,j = 1,\ldots,n$, then the *parabosonic algebra* $P_B^{(n)}$ in $2n$ generators (n parabosons) is defined as the quotient algebra $P_B^{(n)}/I_B$, that is,

$$P_B^{(n)} = \frac{T(V_B)}{\langle [\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon] - (\varepsilon-\eta)\delta_{jk}b_i^\xi - (\varepsilon-\xi)\delta_{ik}b_j^\eta \mid \xi,\eta,\varepsilon = \pm 1, i,j = 1,\ldots,n\rangle}.$$

It is known that the parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0,n)$, that is, $P_B^{(n)} \cong U(B(0,n))$. For more details about *parafermionic* and *parabosonic algebras*, see [KD09, Proposition 18.2], and references therein.

1.2.9 Ore polynomials of higher order generated by homogeneous quadratic relations

For a ring R, as we saw in Section 1.2.1, the *Ore extensions* introduced by Ore [Ore31, Ore33] consist of the uniquely representable elements $r_0 + r_1x + \cdots + r_kx^k$, $k = k(r) = 0,1,2,\ldots, r_i \in R$, with the commutation relation $xr = \sigma(r)x + \delta(r)$, where σ is an endomorphism of R and δ is a σ-derivation of R. Different generalizations, called *skew Ore polynomials*, have been introduced and studied by Cohn [Coh61, Coh85], Dumas [Dum91], and Smits [Smi68], considering the commutation relation $xr = \Psi_1(r)x + \Psi_2(r)x^2 + \cdots$, where the Ψ's are endomorphisms of R. Nevertheless, there are cases of quadratic algebras such as Clifford algebras, Weyl-Heisenberg algebras, and Sklyanin algebras, in which this commutation relation is not sufficient to define the noncommutative structure of the algebras since a free non-zero term Ψ_0 is required (e.g., Ostrovskii and Samoilenko [OS92]). Precisely, *skew Ore polynomials of higher order* with commutation relation with this free term, that is, $xr = \Psi_0(r) + \Psi_1(r)x + \cdots + \Psi_n(r)x^n + \cdots$, were studied

by Maksimov [Mak00], where, for every $r, s \in R$, the free term Ψ_0 satisfies the relation

$$\Psi_0(rs) = \Psi_0(r)s + \Psi_1(r)\Psi_0(s) + \Psi_2(r)\Psi_0^2(s) + \cdots,$$

or the equivalent operator equation $\Psi_0 r = \Psi_0(r) + \Psi_1(r)\Psi_0 + \Psi_2(r)\Psi_0^2$, where r is considered as the operator of left multiplication by r on R. Notice that one may consider Ψ_0 as a singular differentiation operator with respect to $\Psi_1, \Psi_2, \ldots$, but where Ψ_1 need not be an endomorphism of R.

Later, Golovashkin and Maksimov [GM05] investigated the representation of algebras $Q(a,b,c)$ over a field $\Bbbk$ of characteristic zero defined by a *quadratic relation* in two generators x, y given by

$$yx = ax^2 + bxy + cy^2, \tag{1.16}$$

as an algebra of Ore polynomials of higher degree with commutation relation (1.16) with a, b, c belong to $\Bbbk$. As one can check, the algebra generated by the relation is represented in the form of an algebra of skew Ore polynomials of higher order if the elements $\{x^m y^n\}$ form a linear basis of the algebra. Hence, this algebra can be defined by a system of linear mappings $\Psi_0, \Psi_1, \Psi_2, \ldots$ of the algebra of polynomials $\Bbbk[x]$ into itself such that for an arbitrary element $p(x) \in \Bbbk[x]$, $yp(x) = \Psi_0(p(x)) + \Psi_1(p(x))y + \cdots + \Psi_k(p(x))y^k$, $k = k(p(x))$, $k = 0, 1, 2, \ldots$ If this representation exists, then one can obtain the relations between the operators $\Psi_0, \Psi_1, \Psi_2, \ldots$ They found conditions for such an algebra $Q(a,b,c)$ to be expressed as a skew polynomial with generator y over the polynomial ring $\Bbbk[x]$ (cf. [GM98]), and proved that these conditions are equivalent to the existence of a PBW basis, i.e., basis of the form $\{x^m y^n\}$. Notice that this kind of algebras have been previously studied in the literature where its Poincaré series was calculated by Ufnarovskii [Ufn90].

Next, we recall briefly some of the results presented in [GM05] about PBW bases of these algebras which are useful in Chapter 3.

First of all, Golovashkin and Maksimov [GM05, Section 1] distinguished three types of algebras that can be occur from relation (1.16):

(i) Algebras in which the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ form a PBW basis.

(ii) Algebras in which the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ are linearly dependent (for instance, the algebra determined by the relation $yx = x^2 + xy + y^2$).

(iii) Algebras in which the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$, are linearly independent, but do not form a PBW basis (for instance, the algebra subject to the relation $yx = x^2 - xy + y^2$).

Case (i) is of interest since in this situation the quadratic algebra is determined by the structural constants that arise when expanding the products $(x^k y^r)(x^l y^s)$ in terms of the basis $\{x^m y^n\}$. Nevertheless, it is more useful to use special linear mappings of the ring of polynomials $\Bbbk[x]$ rather than structural constraints. Let us see the details.

If the monomials $\{x^m y^n\}$ form a basis, then for every power $x^n \in \Bbbk[x]$, yx^n has a unique

expression given by

$$yx^n = \Psi_{0,n}(x) + \Psi_{1,n}(x)y + \cdots + \Psi_{m(n),n}(x)y^{m(n)}, \tag{1.17}$$

where $\Psi_{k,n}(x)$, for each k, are polynomials from $\Bbbk[x]$. Precisely, for $k = 0, 1, \ldots$, it can be defined a linear mapping $\Psi_k : \Bbbk[x] \to \Bbbk[x]$ given by $\Psi_k(x^n) = \Psi_{k,n}(x)$. If we define $x^0 = y^0 = 1_{\Bbbk}$, then $yx^0 = y \cdot 1 = 1 \cdot y + 0 \cdot y^2 + \cdots$. By (1.17),

$$\Psi_0(1) = 0, \quad \Psi_1(1) = 1, \quad \Psi_k(1) = 0, \quad k = 2, 3, \ldots \tag{1.18}$$

which shows that for every element $p(x) \in \Bbbk[x]$, there is a unique expresion given by

$$yp(x) = \Psi_0(p(x)) + \Psi_1(p(x))y + \cdots + \Psi_{m(p(x))}(p(x))y^{m(p(x))}. \tag{1.19}$$

Having in mind that $y^n(p(x)) = y(y^{n-1}p(x)) = \cdots = y(y(\cdots y(yp(x))))$, it follows that the values of the operators Ψ_k, $k = 0, 1, \ldots$, uniquely determine the algebra of skew polynomials generated by (1.16) in the case that $\{x^m y^n \mid m, n \in \mathbb{N}\}$.

REMARK 5. A general algebra of skew polynomials with indeterminate x over R is also determined by certain linear operators $\Psi_k : R \to R$ such that for each $r \in R$, there is a unique representation $xr = \Psi_0(r) + \Psi_1(r)x + \cdots + \Psi_{m(r)}x^{m(r)}$. Of course, if R is a ring with one generator, then the algebra has a PBW basis. If $m(0) = 1$, then we obtain the classical Ore extensions [Ore33]. It it important to say that the associativity and uniqueness of the representation of the product xr guarantee conditions on the set of operators $\{\Psi_k\}_{k\in\mathbb{N}}$ which are necessary and sufficient to determinate the algebra of skew polynomials (for more details, see [Mak00, Mak05]).

Example 1.3 contains examples of skew polynomials with PBW basis over the ring $\Bbbk[x]$ generated by the quadratic homogeneous relation (1.16).

EXAMPLE 1.3 ([GM05, SECTION 1.3]). The following two cases arise in the study of operators in functional analysis [OS92, VK95]:

(i) $a = 0$ and $c \neq 0$. Here, expression (1.16) turns out to be $yx = bxy + cy^2$. If $x_1 := x$, $y_1 := cy$, then we obtain $y_1x_1 = bx_1y_1 + y_1^2$.

(ii) $a \neq 0$ and $c = 0$. Again, if $x_1 := x$, $y_1 := cy$, then (1.16) is equivalent to $y_1x_1 = bx_1y_1 + x_1^2$.

Denote by T the operator of multiplying by x a polynomial $f(x)$, that is, $Tf(x) = xf(x)$. Let D be the ordinary operator of differentiation, that is, $Dx^n = nx^{n-1}$, and D_q be the operator of q-differentiation given by $D_q f(x) = \frac{f(x)-f(qx)}{x-qx}$, for every $q \in \Bbbk$. Of course, for $q = 1$, we consider $D_1 := D$, while for $q = 0$ we get the operator $D_0 = \overline{D}$ is the operator of difference quotient given by $\overline{D}f(x) = \frac{f(x)-f(0)}{x}$. If one consider the operator of integration J, $Jx^n = \frac{1}{n+1}x^{n+1}$, the operator of q-integration J_q defined by $J_q x^n = \frac{1-q}{1-q^{n+1}}x^{n+1}$, the Dirac operator V_0 given by $V_0 f(x) = f(0)$, and the identity operator I, then the following relations hold at the basis $\{x^n \mid n \in \mathbb{N}\}$:

$$\overline{D}T = I, \quad T\overline{D} = I - V_0, \quad D_q J_q = I, \quad J_q D_q = I - V_0, D_q X - qXD_q = I,$$

and

$$\overline{D}D_q = qD_q\overline{D} + \overline{D}^2, \text{ and } TJ_q = qJ_qT + J_q^2,$$

which are precisely the Cases (i) and (ii) considered in Example 1.3. If $q = 1$, we obtain the Weyl relation $DT - TD = I$, and the equivalent relations $\overline{D}D - D\overline{D} = \overline{D}^2$ and $TJ - JT = J^2$.

It is straightforward to see that the sets of operators $\left\{T^m D_q^n\right\}$ and $\left\{D_q^m T^n\right\}$, for $m, n \in \mathbb{N}$, are PBW bases of the algebra generated by the operators T and D_q. Similarly, the sets $\left\{T^m J_q^n\right\}$ and $\left\{J_q T^n\right\}$ ($\left\{\overline{D}^m D^n\right\}$ and $\left\{D^m \overline{D}^n\right\}$), where $m, n \in \mathbb{N}$, are PBW bases of the algebras generated by the operators T and J_q (D and $\overline{D}$). It follows that for both Cases (i) and (ii) in Example 1.3, the sets $\left\{a^m b^n \mid m, n \in \mathbb{N}\right\}$ and $\left\{b^m a^n \mid m, n \in \mathbb{N}\right\}$ are bases of the algebra (1.16).

Notice that if $a = 0$ in (1.16), then the quadratic algebra is given by $yx = bxy + cy^2$. From [GM05, Expression (11)], we know that this is an algebra of skew polynomials determined by the operators $\Psi_0 = 0, \Psi_1(x^n) = b^n x^n, \Psi_2(x^n) = \Psi_1 c D_b(x^n), \Psi_3(x^n) = \Psi_1 c^2 D_b^2(x^n)$, that is, $\Psi_k = \Psi_1 c^{k-1} D_b^{k-1}$, for $k = 1, 2, \ldots$

EXAMPLE 1.4 ([GM05, SECTION 1.5]). When two of the three coefficients a, b, and c are equal to zero, the resulting quadratic algebras are given by three types: (iii) $yx = ax^2$, (iv) $yx = cy^2$, and (v) $yx = bxy$. The set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ is a PBW basis for algebras (iii) and (iv), while the algebra (v) is an Ore extension.

Examples 1.3 and 1.4 guarantee that if $ab = 0$, then the quadratic algebra defined by (1.16) is an algebra of skew polynomials over $\Bbbk[x]$ and has a PBW basis $\{x^m y^n \mid m, n \in \mathbb{N}\}$ [GM05, Proposition 1].

A useful tool in the study of the PBW bases for quadratic algebras defined by (1.16) is the matrix given by

$$M := \begin{pmatrix} b & a \\ -c & 1 \end{pmatrix},$$

which is called the *companion* matrix for (1.16) [GM05, Section 2.2]. If the lower-right elements of the matrices M^l does not vanish for all $l \in \mathbb{N}$, then $Q(a, b, c)$ has a PBW basis of the form $\{x^m y^n \mid m, n \in \mathbb{N}\}$ [GM05, Proposition 4]. Equivalently, $Q(a, b, c)$ is the ring of skew polynomials over $\Bbbk[x]$ determined by the sequence of operators Ψ_k, $k = 0, 1, \ldots$, satisfying the infinite relations

$$\begin{aligned} \Psi_0 X &= aX^2 + bX\Psi_0 + c\Psi_0^{(2)}, \\ \Psi_1 &= \quad bX\Psi_1 + c\Psi_1^{(2)}, \\ &\vdots \\ \Psi_k X &= \quad bX\Psi_k + c\Psi_k^{(2)}, \end{aligned}$$

where

$$\Psi_0^{(2)} = \Psi_0^2, \qquad \Psi_k^{(2)} = \Psi_0\Psi_k + \Psi_1\Psi_{k-1} + \cdots + \Psi_k\Psi_0,$$

see [GM05, Lemma 1]. General relations between $\Psi_j^{(k)}$ were formulated in [Mak00, Mak05, Smi68].

If $b + ac \neq 0$ a necessary and sufficient condition for the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ form a

basis of $Q(a,b,c)$ is precisely that the lower-right elements of the matrices M^l does not vanish for all $l \in \mathbb{N}$. If $a = c = 1$ and $b = -1$, then the elements $\{x^m y^n \mid m,n \in \mathbb{N}\}$ are linearly independent but do not form a PBW basis of $Q(a,b,c)$ [GM05, Propositions 5 and 10].

1.2.10 SKEW POINCARÉ-BIRKHOFF-WITT EXTENSIONS

Skew Poincaré-Birkhoff-Witt extensions were defined by Gallego and Lezama [GL11] with the aim of generalizing Poincaré-Birkhoff-Witt extensions introduced by Bell and Goodearl [BG88] (Section 1.2.3) and Ore extensions of injective type defined by Ore [Ore31, Ore33] (Section 1.2.1). Over the years, several authors have shown that skew PBW extensions also generalize families of noncommutative algebras such as those mentioned in the previous sections. Due to their generality, in Chapters 2 and 3 we prove several results about these objects that can be illustrated with algebras described in the previous sections. Precisely, the importance of skew PBW extensions is that they do not assume that the coefficients commute with the variables, and the coefficients do not necessarily belong to fields. As a matter of fact, skew PBW extensions contain well-known groups of algebras such as some types of G-algebras in the sense of Apel [Ape88], Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, families of differential operator rings, and many other algebras of interest in noncommutative algebraic geometry and noncommutative differential geometry. Ring, theoretical and geometrical properties of skew PBW extensions have been presented in several works [AT24, Art15, Faj18, Gal15, HHR20, HR22, LG19, Rey13, RS16, Sua17, SR17, SRS23, Ven20].

DEFINITION 1.12 ([GL11, DEFINITION 1]). Let R and A be rings. We say that A is a *skew PBW extension* over R (also called a *σ-PBW extension of R*) if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist elements $x_1,\dots,x_n \in A \setminus R$ such that A is a left free R-module with basis given by the set $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n\}$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i,j \leq n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n,$$

i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)},\dots,r_n^{(i,j)} \in R$ with

$$x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \textstyle\sum_{k=1}^n r_k^{(i,j)} x_k.$$

From now on, we use freely the notation $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ to denote a skew PBW extension A over a ring R in the indeterminates $x_1,\dots,x_n$. R will be called the *ring of coefficients* of the extension A.

REMARK 6 ([GL11, REMARK 2]). (i) Since $\text{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ in Definition 1.12 are unique.

(ii) If $r = 0$, it follows that $c_{i,0} = 0$. In fact, from $0 = x_i 0 = c_{i,0} + r_i$, with $r_i \in R$, we obtain $c_{i,0} = 0 = r_i$ for all i.

(iii) In Definition 1.12 (iv), $d_{i,i} = 1$. This follows from $x_i^2 - d_{i,i}x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - d_{i,i} = 0 = s_i$.

(iv) Let $i < j$. From Definition 1.12 it follows that there exist elements $d_{j,i}, d_{i,j} \in R$ such that $x_i x_j - d_{j,i} x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$, and hence $1 = d_{j,i} d_{i,j}$, that is, for each $1 \leq i < j \leq n$, $d_{i,j}$ has a left inverse and $d_{j,i}$ has a right inverse.

(v) Every element $f \in A \setminus \{0\}$ has a unique representation as $f = \sum_{i=0}^{t} r_i X_i$, with $r_i \in R \setminus \{0\}$ and $X_i \in \text{Mon}(A)$ for $0 \leq i \leq t$ with $X_0 = 1$. When necessary, we use the notation $f = \sum_{i=0}^{t} r_i Y_i$.

PROPOSITION 1.15 ([GL11, PROPOSITION 3]). *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over R. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$.*

We use the notation $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$ for the families of injective endomorphisms and σ_i-derivations, respectively, established in Proposition 1.15. For a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ over R, we say that the pair (Σ, Δ) is a *system of endomorphisms and Σ-derivations* of R with respect to A. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^\alpha := \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes the classical composition of functions.

The next definition presents some particular examples of skew PBW extensions.

DEFINITION 1.13 ([GL11, DEFINITION 4], [LAR15, DEFINITION 2.3 (II)]). Consider a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ over R.

(a) A is called *quasi-commutative* if the conditions (iii) - (iv) in Definition (1.12) are replaced by the following:

(iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For every $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = d_{i,j} x_i x_j$.

(b) A is *bijective* if σ_i is bijective, for every $1 \leq i \leq n$, and $d_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(c) If σ_i is the identity map of R for each $i = 1, \ldots, n$, then we say that A is a skew PBW extension of *derivation type*. Similarly, if δ_i is zero, for every i, then A is called a skew PBW extension of *endomorphism type*.

(d) A is said to be *semi-commutative* if it is quasi-commutative and $x_i r = r x_i$, for each i and every $r \in R$.

DEFINITION 1.14 ([GL11, DEFINITION 6]). Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over R.

(i) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) For $X = x^{\alpha} \in \text{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) If f is an element as in Remark 6(v), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

PROPOSITION 1.16 ([GL11], THEOREM 7). *If A is a polynomial extension of a ring R with respect to the set of indeterminates $\{x_1, \ldots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(1) *For each $x^{\alpha} \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_{\alpha} := \sigma^{\alpha}(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$ such that*

$$x^{\alpha} r = r_{\alpha} x^{\alpha} + p_{\alpha,r}, \tag{1.20}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, so is r_{α}.

(2) *For each $x^{\alpha}, x^{\beta} \in \text{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R \setminus \{0\}$ and $p_{\alpha,\beta} \in A$ such that*

$$x^{\alpha} x^{\beta} = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{1.21}$$

where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

In $\text{Mon}(A)$, we define the total order

$$x^{\alpha} \succeq x^{\beta} \Longleftrightarrow \begin{cases} x^{\alpha} = x^{\beta} \\ \text{or} \\ x^{\alpha} \neq x^{\beta} \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^{\alpha} \neq x^{\beta}, |\alpha| = |\beta| \text{ but there exists } i \text{ with } \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases} \tag{1.22}$$

If $x^{\alpha} \succeq x^{\beta}$ but $x^{\alpha} \neq x^{\beta}$, we write $x^{\alpha} \succ x^{\beta}$. Every element $f \in A \setminus \{0\}$ can be represented in a unique way as $f = r_1 x^{\alpha_1} + \cdots + r_t x^{\alpha_t}$, with $r_i \in R \setminus \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that x^{α_1} is the *leading monomial* of f and we write $\text{lm}(f) := x^{\alpha_1}$; r_1 is the *leading coefficient* of f, $\text{lc}(f) := r_1$; and $r_1 x^{\alpha_1}$ is the *leading term* of f, $\text{lt}(f) := r_1 x^{\alpha_1}$. It is clear that $\succeq$ is an *monomial order* in the sense of [GL11], i.e., the following conditions hold

(i) For every $x^{\alpha}, x^{\beta}, x^{\gamma}, x^{\lambda} \in \text{Mon}(A)$

$$x^{\alpha} \succeq x^{\beta} \Longrightarrow \text{lm}(x^{\gamma} x^{\alpha} x^{\lambda}) \succeq \text{lm}(x^{\gamma} x^{\beta} x^{\lambda});$$

(ii) $x^{\alpha} \succeq 1$, for every $x^{\alpha} \in \text{Mon}(A)$; and

(iii) $\succeq$ is degree compatible, i.e., $|\alpha| \geq |\beta| \Longrightarrow x^{\alpha} \succeq x^{\beta}$.

EXAMPLE 1.5. (i) Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a PBW extension over R. A is a positively SG ring with graduation $A = \bigoplus_{n \in \mathbb{N}} A_n$, where

$$A_n := {}_R\langle x^{\alpha} \in \text{Mon}(A) \mid \deg(x^{\alpha}) = n\rangle,$$

i.e., A_n are the set of homogeneous polynomials of degree n.

REMARK 7. (i) From Definition 1.12 (iv), it is clear that skew PBW extensions are more general than iterated skew polynomial rings. For example, universal enveloping algebras of finite dimensional Lie algebras and some 3-dimensional skew polynomial algebras in the sense of Bell and Smith [BS90] cannot be expressed as iterated skew polynomial rings but are skew PBW extensions. Quasi-commutative skew PBW extensions are isomorphic to iterated Ore extensions of endomorphism type [LR14, Theorem 2.3].

(ii) Skew PBW extensions of endomorphism type over a ring are more general than iterated Ore extensions of endomorphism type of the same ring. Let us illustrate the situation with two and three indeterminates.

For the iterated Ore extension of endomorphism type $R[x;\sigma_x][y;\sigma_y]$, if $r \in R$ then we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. Now, if we have $\sigma(R)\langle x,y\rangle$ a skew PBW extension of endomorphism type over R, then for any $r \in R$, Definition 1.12 establishes that $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and r_2 belong to R.

If we have the iterated Ore extension $R[x;\sigma_x][y;\sigma_y][z;\sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. For the skew PBW extension of endomorphism type $\sigma(R)\langle x,y,z\rangle$, $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r_0' + r_1'x + r_2'y + r_3'z$, and $zy = d_{2,3}yz + r_0'' + r_1''x + r_2''y + r_3''z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r_0', r_0'', r_1, r_1', r_1'', r_2, r_2', r_2'', r_3$, r_3', r_3'' of R. As the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

(iii) Ambiskew polynomial rings (Section 1.2.1) are skew PBW extensions over B, that is, $R(B,\sigma,c,p) \cong \sigma(B)\langle y,x\rangle$.

(iv) PBW extensions introduced by Bell and Goodearl [BG88] (Section 1.2.3) are particular examples of skew PBW extensions. More exactly, the first objects satisfy the relation $x_i r = r x_i + \delta_i(r)$, for every $i = 1,\dots,n$, and each $r \in R$, and the elements d_{ij} in Definition 1.12 (iv) are equal to the identity of R. As examples of PBW extensions, we mention the following: the enveloping algebra of a finite-dimensional Lie algebra; any differential operator ring $R[\theta_1,\dots,\theta_1;\delta_1,\dots,\delta_n]$ formed from commuting derivations $\delta_1,\dots,\delta_n$; differential operators introduced by Rinehart; twisted or smash product differential operator rings, and others (for more details, see [BG88, p. 27]).

(v) 3-dimensional skew polynomial algebras and bi-quadratic algebras on three generators with PBW bases (Sections 1.2.4 and 1.2.5, respectively) are skew PBW extensions.

(vi) The algebra $U_q'(\mathfrak{so}_3)$ is a skew PBW extension over $\mathbb{k}$, i.e., $U_q'(\mathfrak{so}_3) \cong \sigma(\mathbb{k})\langle I_1, I_2, I_3\rangle$ [FGL+20, Example 1.3.3].

(vii) Using techniques such as those presented in [FGL+20, Theorem 1.3.1], it can be shown that AW(3) is a skew PBW extension of endomorphism type, that is, AW(3) $\cong \sigma(\mathbb{R})\langle K_0, K_1, K_2\rangle$.

PROPOSITION 1.17 ([LR14, PROPOSITION 4.1]). *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension over R. If R is a domain, then A is a domain.*

From Proposition 1.17 it follows that if R is a domain and $f, g \in A \setminus \{0\}$, then $\deg(fg) = \deg(f) + \deg(g)$. Hence, if A is a skew PBW extension over a domain R, then the units of A are exactly the units of R [FGL+20, Corollary 3.2.2.].

PROPOSITION 1.18 ([FGL+20, PROPOSITION 3.2.3.]). *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a ring R. If R is a domain, then $J(A) = 0$.*

From Definition 1.12 it follows that skew PBW extensions are not $\mathbb{N}$-graded rings. With this in mind, Proposition 1.19 allows to define a subfamily of these extensions, the *graded skew PBW extensions* (Definition 1.15) introduced by Suárez in his PhD Thesis [Sua17] (see also [Su7]). Before presenting its definition, we recall the following facts:

- If $R = \bigoplus_{p\in\mathbb{N}} R_p$ and $S = \bigoplus_{p\in\mathbb{N}} S_p$ are $\mathbb{N}$-graded rings, then a map $\varphi : R \to S$ is called *graded* if $\varphi(R_p) \subseteq S_p$, for each $p \in \mathbb{N}$. For $m \in \mathbb{N}$, $R(m) := \bigoplus_{p\in\mathbb{N}} R(m)_p$, where $R(m)_p := R_{p+m}$.
- Suppose that $\sigma : R \to R$ is a graded algebra automorphism and $\delta : R(-1) \to R$ is a graded σ-derivation (i.e., a degree +1 graded σ-derivation δ of R). Let $B := R[x;\sigma,\delta]$ be the associated *graded Ore extension* of R, that is, $B = \bigoplus_{p\geq 0} Rx^p$ as an R-module, and for $r \in R$, $xr = \sigma(r)x + \delta(r)$. If we consider x to have degree 1 in B, then under this grading B is a connected graded algebra generated in degree 1 (for more details, see [CS08, Pha12]).

PROPOSITION 1.19 ([Su7, PROPOSITION 2.7(II)]). *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If the following conditions hold:*

(1) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation, for all $1 \leq i \leq n$, and*

(2) *$x_j x_i - d_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in Definition 1.12 (iv) and $d_{i,j} \in R_0$,*

then A is an $\mathbb{N}$-graded algebra with graduation given by $A = \bigoplus_{p\geq 0} A_p$, where for $p \geq 0$, A_p is the $\Bbbk$-space generated by the set

$$\left\{ r_t x^\alpha \mid t + |\alpha| = p,\ r_t \in R_t \text{ and } x^\alpha \in \mathrm{Mon}(A) \right\}.$$

DEFINITION 1.15 ([Su7, DEFINITION 2.6]). Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If A satisfies both conditions established in Proposition 1.19, then we say that A is a *graded skew PBW extension over R*.

Some properties of graded skew PBW extensions can be found in [SAR21, SCR21, SRS23, SLR17]. Next, we recall some examples of these objects.

EXAMPLE 1.6. The Jordan plane, homogenized enveloping algebras (Section 1.2.8), and some classes of diffusion algebras (Section 1.2.6, [Su7, Examples 2.9]) are graded skew PBW extensions. If we assume the condition of PBW basis, then graded Clifford algebras defined by Le Bruyn [Bru95] are also examples of graded skew PBW extensions. Let us see the details.

Let $\Bbbk$ be an algebraically closed field such that $\mathrm{char}(\Bbbk) \neq 2$ and let $M_1, \ldots, M_n \in \mathbb{M}_n(\Bbbk)$ be symmetric matrices of order $n \times n$ with entries in $\Bbbk$. A *graded Clifford algebra* $\mathscr{A}$ is a $\Bbbk$-algebra on

degree-one generators $x_1,\ldots,x_n$ and on degree-two generators $y_1,\ldots,y_n$ with defining relations given by

(i) $x_ix_j + x_jx_i = \sum_{k=1}^{n} (M_k)_{ij}y_k$ for all $i,j = 1,\ldots,n$;

(ii) y_k central for all $k = 1,\ldots,n$.

Note that the commutative polynomial ring $R = \mathbb{k}[y_1,\ldots,y_n]$ is an $\mathbb{N}$-graded algebra where $R_0 = \mathbb{k}, R_1 = \{0\}$, $y_1,\ldots,y_n \in R_2$, and $R_i = \{0\}$, for $i \geq 3$. If we suppose that the set $\{x_1^{a_1}\cdots x_n^{a_n} \mid a_i \in \mathbb{N}, i = 1,\ldots,n\}$ is a left PBW R-basis for $\mathscr{A}$, then the graded Clifford algebra $\mathscr{A}$ is a graded skew PBW extension over the connected algebra R, that is, $\mathscr{A} \cong \sigma(R)\langle x_1,\ldots,x_n\rangle$. Indeed, from the relations (i) and (ii) above, it is clear that $\sigma_i = \mathrm{id}_R$, $\delta_i = 0$, $d_{i,j} = -1 \in R_0$, for $1 \leq i,j \leq n$, and $\sum_{k=1}^{n}(M_k)_{ij}y_k \in R_2$, where $d_{i,j}$ is given as in Definition 1.12 (iv). In this way, $\mathscr{A}$ is a bijective skew PBW extension that satisfies both conditions of Proposition 1.19.

CHAPTER 2

MORPHISMS AND CV-POLYNOMIALS OF ITERATED ORE EXTENSIONS AND DOUBLE ORE EXTENSIONS

In this chapter, we formulate several characterizations of morphisms and cv-polynomials of iterated Ore extensions and double extensions. The results in this chapter contribute to the study of morphisms and pseudo-linear transformations of noncommutative rings of polynomial type.

With this aim, in Section 2.1 we recall the key results of Rimmer's paper [Rim78] on automorphisms of the Ore extensions of automorphism type $R[x;\sigma]$ which fix R elementwise. Next, Section 2.2 contains the more important results formulated by Ferrero and Kishimoto [FK80] (see also Kikumasa [Kik90]) on automorphisms of skew polynomial rings of derivation type. Section 2.3 contains some definitions and preliminaries on homomorphisms and cv-polynomials of Ore extensions of mixed type in one indeterminate presented by Lam and Leroy [LL92]. Surprisingly, Lam and Leroy in their article did not consider the works of Rimmer, Ferrero, Kishimoto and Kikumasa, so according to the search we carried out in the literature, this is the first place where the four treatments are presented in order to generalize some of their results to the more general context of the iterated Ore extensions.

The original results of this chapter begin in Section 2.4. There, we investigate the behavior of morphisms and cv-polynomials of iterated Ore extensions. More exactly, Sections 2.4.1, 2.4.2 and 2.4.3 present the characterization of morphisms and cv-polynomials of two-step, three-step and n-step iterated Ore extensions, respectively. We organize information in Tables 2.1, 2.2 and 2.3 with the aim of presenting different possibilities along with their conditions. It is important to say that we have found some mistakes in particular cases of [LL92], which we illustrate in Examples 2.6 and 2.13, and present their corresponding versions. Our important results are formulated in Theorems 2.19, 2.20, 2.21, 2.22, 2.23, 2.24, 2.25, 2.26 for length two iterated Ore extensions, Theorems 2.27, 2.28, 2.29 for length three iterated Ore extensions, and Theorems 2.30, 2.31, 2.33 for length n iterated Ore extensions.

Having in mind the theory of homomorphisms and cv-polynomials of Ore extensions presented in Section 2.4, and the importance of double extensions in noncommutative algebra, in

Section 2.5 we present a first approach toward a theory of homomorphisms and cv-polynomials between these objects. More exactly, since double extensions are defined by using matrix notation, in Section 2.5.1 we introduce the notion of *dcv-matrix* that encodes pairs of cv-polynomials (Definition 2.10). We consider different examples that illustrate dcv-matrices (Examples 2.18, 2.20, 2.21, and Tables 2.4 and 2.5), and show that the computations are non-trivial. Under certain conditions, Theorem 2.34 characterizes dcv-matrices, while Theorem 2.35 presents a characterization of isomorphisms between double extensions. Next, Section 2.5.2 contains examples and a characterization of dcv-matrices for trimmed double extensions with their corresponding Nakayama automorphism (Theorem 2.36). Then, in Section 2.5.3, we investigate when a homomorphism of double extensions is a homomorphism of two-step iterated extensions (Theorem 2.37) in an analogous way to the comparison between double extensions and two-step iterated Ore extensions presented by Zhang and Zhang and Carvalho et al. (Section 1.2.2).

Finally, Section 2.6 presents some ideas for a future work.

2.1 ORE EXTENSIONS OF AUTOMORPHISM TYPE IN ONE INDETERMINATE

When R is a commutative ring and σ is the identity automorphism of R, Gilmer [Gil68] determined all the automorphisms of $R[x;\sigma] = R[x]$ which fix R elementwise. Parmenter [Par] extended this result to the case where R is commutative and σ is any automorphism of R (the cases where σ is the identity on R and R is any ring has been considered by Coleman and Enochs [CE71]). These three results are corollaries of Rimmer's paper [Rim78, Theorem 1]. More exactly, for a unital ring (not necessarily commutative) R and σ any automorphism of R, Rimmer [Rim78] determined all the automorphisms of the skew polynomial ring of automorphism type $R[x;\sigma]$ which fix R elementwise. As an application of his results, he investigated isomorphisms between different skew polynomial rings wich preserve an underlying ring isomorphism [Rim78, Theorem 3].

Let us recall some of the most important facts established by Rimmer in his paper.

PROPOSITION 2.1. (1) [Rim78, Lemma 1] *If r_1 is a central unit of R and if $r_0 \in R$ is such that $r_0 r = \sigma(r) r_0$ for all $r \in R$, then $x \mapsto r_0 + r_1 x$ induces an automorphism of $R[x;\sigma]$ which fixes all elements of R.*

(2) [Rim78, Lemma 2] *Suppose $x \mapsto r_0 + r_1 x + \cdots + r_n x^n$ induces a homomorphism ψ of $R[x;\sigma]$ which fixes each element of R. Then ψ is surjective if and only if*

- *$\sigma(r) r_0 = r_0 r$, for all $r \in R$,*
- *r_1 is a central unit,*
- *$x \mapsto x + r_1^{-1} r_2 x^2 + \cdots + r_1^{-1} r_n x^n$ induces a surjective homomorphism, μ say, of $R[x;\sigma]$ which fixes every element of R.*

  In this case there exists an automorphism θ of $R[x;\sigma]$, which also fixes R elementwise, induced by $x \mapsto r_0 + r_1 x$, such that $\psi = \theta\mu$. Thus ψ is an automorphism precisely when μ is an automorphism.

PROPOSITION 2.2 ([RIM78, THEOREM 1]). *Let R be any ring with identity and let σ be any automorphism of R. Then $x \mapsto \sum_{i=0}^{n} r_i x^i$ induces an automorphism of $R[x;\sigma]$ which fixes each element of R if and only if*

(1) $\sigma(r)r_i = r_i\sigma^i(r)$, *for all* $r \in R$ *and all* $i \in \{0,1,\ldots,n\}$,

(2) r_1 *is a central unit in* R, *and*

(3) r_i *is nilpotent for all* $i \in \{2,\ldots,n\}$.

Further, these three conditions hold if and only if $x \mapsto \sum_{i=0}^{n} r_i x^i$ induces a surjective homomorphism of $R[x;\sigma]$ which fixes each element of R.

Parmenter [Par] noted that there is not always a homomorphism of $R[x;\sigma]$ which fixes each element of R and maps x to $\sum_{i=0}^{n} r_i x^i$. As we can check, there is at most one such homomorphism, and this exists precisely when $\left(\sum r_i x^i\right) r = \sigma(r)\left(\sum r_i x^i\right)$, for all $r \in R$. Equivalently,

$$\text{for all } r \in R, \text{ and for all } i \in \{0,\ldots,n\},\ \sigma(r)r_i = r_i\sigma(r). \tag{2.1}$$

Of course, this is precisely condition (1) of Proposition 2.2. In particular, $\sigma(r)r_1 = r_1\sigma(r)$, for all $r \in R$, which means that r_1 is a central element of R.

PROPOSITION 2.3 ([RIM78, THEOREM 2]). *The map $x \mapsto x + r_2x^2 + \cdots r_nx^n$ induces a surjective homomorphism ϕ of $R[x;\sigma]$ which fixes R elementwise precisely when it induces an automorphism, and this occurs if and only if*

(1)' $\sigma(r)r_i = r_i\sigma^i(r)$, *for all* $r \in R$ *and all* $i \in \{2,\ldots,n\}$, *and*

(2)' r_i *is nilpotent for all* $i \in \{2,\ldots,n\}$.

Rimmer [Rim78, Section 3] dealt with isomorphisms between skew polynomial rings whose underlying rings are isomorphic. As we will see, for an element $f(x) \in R[x;\sigma]$, Proposition 2.4 establishes conditions for which the R-linear map $R[x;\sigma] \to R[x;\sigma]$ defined by $x^k \mapsto f(x)^k$ is an R-ring automorphism.

PROPOSITION 2.4 ([RIM78, THEOREM 3]). *Let R and R' be any rings with an isomorphism $\phi : R \to R'$, and let σ and σ' be automorphisms of R and R', respectively. Then the map $x \mapsto b_0 + b_1x + \cdots + b_nx^n$ extends ϕ to an isomorphism $\phi : R[x;\sigma] \to R'[x;\sigma']$ if and only if*

(1) $(\sigma \circ \phi)(r)b_i = b_i(\phi \circ \sigma')^i(r)$, *for all* $r \in R$ *and all* $0 \le i \le n$,

(2) b_1 *is a unit, and*

(3) b_i *is nilpotent for all* $2 \le i \le n$.

Further these three conditions occur precisely when ϕ is a surjective homomorphism.

If v is a unit in R, Rimmer denoted by i_v the inner automorphism of R given by $i_v(r) = vrv^{-1}$, for all $r \in R$.

PROPOSITION 2.5 ([RIM78, LEMMA 4]). *Let R and R' be any rings with automorphisms σ and σ', respectively, and let $\phi : R \to R'$ be any isomorphism. If the map $x \mapsto b_0 + b_1 x + \cdots + b_n x^n$ extends ϕ to a surjective homomorphism between $R[x;\sigma]$ and $R[x;\sigma']$, then b_1 is a unit and $\phi^{-1}\sigma\phi = \sigma' i_{b_1}$.*

PROPOSITION 2.6 ([RIM78, LEMMA 5]). *Let $\phi : R \to R'$ be an isomorphism, and let σ and σ' automorphisms of R and R', respectively. Further, let v be a unit of R' such that $\phi^{-1}\sigma\phi = \sigma' i_v$. Then the map $\theta : R[x;\sigma] \to R'[x;\sigma']$ determined by $x \mapsto vx$ and $r \mapsto \phi(r)$ for all $r \in R$ is an isomorphism.*

2.2 ORE EXTENSIONS OF DERIVATION TYPE IN ONE INDETERMINATE

Ferrero and Kishimoto [FK80] studied automorphisms of skew polynomial rings of derivation type. They discussed conditions on $f(x) \in R[x;\delta]$ for the R-linear map $R[x;\delta] \to R[x;\delta]$ defined by $x^k \mapsto f(x)^k$ to be a R-ring automorphism. Kikumasa [Kik90] characterized automorphisms of a certain skew polynomial ring of derivation type.

Throughout this section, N will mean the union of all nilpotent ideals of R.

PROPOSITION 2.7 ([FK80, LEMMA 2]). *The R-linear map $\phi : R[x;\delta] \to R[x;\delta]$ defined by $x^k \mapsto (r_0 + r_1 x)^k$ is an R-ring automorphism if and only if r_1 is a central unit and $r_0 r - r r_0 = \delta(r)(r_1 - 1)$ for all $r \in R$.*

In what follows, Ferrero and Kishimoto [FK80, p. 23] assumed that $\delta(N) \subset N$ and that the R-linear map $\phi : R[x;\delta] \to R[x;\delta]$ defined by $x^k \mapsto \left(\sum_{i=0}^{n} b_i x^i\right)^k$, $n \geq 2$, is an R-ring automorphism. Notice that if $b_n \neq 0$, then $\phi^{-1}(x) = \sum_{j=0}^{m} c_j x^j$ with $c_m \neq 0$ and $m \geq 2$ (Proposition 2.7).

The following results are the most important presented by Ferrero and Kishimoto.

PROPOSITION 2.8. (1) [FK80, Theorem 1] (1) *b_1 is a unit.* (2) *b_i are nilpotent for $i \geq 2$, and therefore $\{b_i \mid i \geq 2\} \subset N$.*

(2) [FK80, Theorem 2] *If b_i are nilpotent for $i \geq 2$, then ϕ is a monomorphism.*

(3) [FK80, Theorem 3] *If N_0 is nilpotent, then ϕ is an R-ring automorphism.*

(4) [FK80, Corollary 2] *Assume that one of the following conditions is fulfilled:*

(1') *R is Noetherian.*

(2') *There exists a monic polynomial $f(x)$ with coefficients in R such that $f(\delta)(b_i) = 0$ for $i \geq 2$.*

If b_i are nilpotent for $i \geq 2$, then ϕ is an R-ring automorphism.

2.3 ORE EXTENSIONS OF MIXED TYPE IN ONE INDETERMINATE

We recall some definitions and results about morphisms and cv-polynomials of Ore extensions in one indeterminate presented by Lam and Leroy [LL92] (see also [Ler95, LLO08]).

According to [LL92, p. 86], let $S' = D[x';\sigma',\delta']$ and $S = D[x;\sigma,\delta]$ be Ore extensions over the same division ring D. Consider any D-homomorphism $\phi : S' \to S$. Writing $p(x) := \phi(x') \in S$, we have $\phi(x'a) = \phi(x')\phi(a) = p(x)a$ for any $a \in D$. On the other hand, $\phi(x'a) = \phi(\sigma'(a)x' + \delta'(a)) = \sigma'(a)p(x) + \delta'(a)$. Therefore, necessarily

$$p(x)a = \sigma'(a)p(x) + \delta'(a), \quad \text{for any } a \in D. \tag{2.2}$$

This fact indicates how the homomorphisms between Ore extensions should be defined and characterize them in a polynomial sense.

DEFINITION 2.1 ([LL92, P. 86]). Let $S' = D[x';\sigma',\delta']$ and $S = D[x;\sigma,\delta]$. Consider $p(x) \in S$ satisfying (2.2). We can define $\phi : S' \to S$ by

$$\phi\left(\sum_{i=0}^{n} a_i x'^i\right) = \sum_{i=0}^{n} a_i p(x)^i.$$

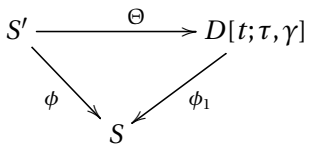
This D-homomorphism ϕ is unique from S' to S sending x' to $p(x)$.

As we can see, the polynomial satisfying condition (2.2) plays an important role in describing the homomorphisms between Ore extensions. This fact motivates the following definition.

DEFINITION 2.2 ([LL92, DEFINITION 2.4]). Let $S' = D[x';\sigma',\delta']$ and $S = D[x;\sigma,\delta]$. A polynomial $p(x) \in S$, satisfying that $p(x)a = \sigma'(a)p(x) + \delta'(a)$ is called *change-of-variable polynomial* (or *cv-polynomial*) with respect to (σ',δ'). We shall say that $p(x) \in S$ is a *cv-polynomial* if it is a cv-polynomial with respect to some pair (σ',δ').

If $S' = D[x';\sigma',\delta']$, $S = D[x;\sigma,\delta]$ and $\phi : S' \to S$ is a D-homomorphism, then ϕ is injective if and only if the associated cv-polynomial $p(x) = \phi(x')$ has degree ≥ 1, and ϕ is surjective (whence bijective) if and only if $p(x)$ has degree $= 1$ [LL92, p. 86].

REMARK 8 ([LL92, P. 86]). We should remark that our requirement that $\phi : S' \to S$ be a homomorphism over D is not strictly necessary. In general, we can deal with homomorphisms $\phi : S' \to S$ with the property that $\theta := \phi|_D$ is an automorphism of D. In this case, one can check as above that $p(x) := \phi(x')$ is a cv-polynomial with respect to (τ,γ), where $\tau = \theta\sigma'\theta^{-1}$ and $\gamma = \theta\delta'\theta^{-1}$. In fact, ϕ can be factored as $\phi_1 \circ \Theta$ where $\Theta : S' \to D[t;\tau,\gamma]$ is defined by $\Theta\left(\sum a_i x'^i\right) = \sum \theta(a_i)t^i$ and $\phi_1 : D[t;\tau,\gamma] \to S$ is defined by $\phi_1\left(\sum b_i t^i\right) = \sum b_i p(x)^i$.



The latter map ϕ_1 here is a D-homomorphism. Since the homomorphism Θ can be handled

separately, it is sufficient to work with the D-homomorphism ϕ_1. Therefore, we shall focus our attention on D-homomorphisms. *All homomorphisms between Ore extension considered below will be assumed to be D-homomorphisms.*

EXAMPLE 2.1 ([LL92, P. 86, 87]). (i) For $D[x']$ and $D[x]$, the cv-polynomials in $D[x]$ with respect to the quasi–derivation $(\mathrm{id}_D, 0)$ are exactly the polynomials in the center of S.

(ii) Recall that for an element $u \in R^*$, σ_u denotes the *inner automorphism* of R associated with u, defined by $\sigma_u(r) = uru^{-1}$, for all $r \in R$ (this is Lam and Leroy's notation in contrast to Rimmer's notation above). If σ is an endomorphism of R, the *inner order* of σ denoted by $o(\sigma)$, is defined to be the smallest positive integer n such that σ^n is an inner automorphism; if no such integer n exists, we take $o(\sigma)$ to be ∞. In particular, if σ is an endomorphism which is not an automorphism, we have by definition $o(\sigma) = \infty$. On the other hand, for any element $a \in R$, the rule $\delta_a(r) = ar - ra$ defines a derivation δ_a on R, called the *inner derivation by a*. Any derivation on R which is not inner derivation is called an *outer derivation*. For an endomorphism σ of R, we write $\delta_{a,\sigma} := ar - \sigma(r)a$ (see [LL92, GJ04] for more details), and consider in the natural way the notions of *σ-inner* and *σ-outer* derivation.

- if δ is an inner derivation of R by c, say $\delta = \delta_{c,\sigma}$, then $R[x;\sigma,\delta] = R[x-c;\sigma]$;
- if σ is an inner automorphism of R, say $\sigma = \sigma_u$ with $u \in R^*$, then $R[x;\sigma,\delta] = R[u^{-1}x;u^{-1}\delta]$.

As we can see, we have an homomorphism between Ore extensions that turns out to be an isomorphism. More exactly, in the first case we have $\phi : R[x';\sigma] \to R[x;\sigma,\delta]$ defined by $\phi\left(\sum a_i x'^i\right) = \sum a_i (x-c)^i$, while in the second one, $\psi : R[x';u^{-1}\delta] \to R[x;\sigma,\delta]$ is defined by $\psi\left(\sum a_i x'^i\right) = \sum a_i (u^{-1}x)^i$. Of course, the change of variable in both cases is determined by a (σ,δ)-polynomial, $x-c$ and $u^{-1}x$, respectively, which are the images of the indeterminate x'.

(iii) The constant polynomial $p(x) = c \in D \subset D[x;\sigma,\delta]$ is a cv-polynomial with respect to (σ',δ') if and only if $ca = \sigma'(a)c + \delta'(a)$ for all $a \in D$, or equivalently, $\delta' = \delta_{c,\sigma'}$. In this way, any constant polynomial in $D[x;\sigma,\delta]$ is a cv-polynomial. Notice that here σ' is arbitrary, while δ' is uniquely determined by σ' and c.

(iv) For any linear polynomial $p(x) = ux + c$ with $u \in D^*$, let us see that

$$(ux+c)a = \sigma'(a)(ux+c) + \delta'(a), \quad \text{where} \quad \sigma' = \sigma_u \circ \sigma \quad \text{and} \quad \delta' = u\delta + \delta_{c,\sigma'}.$$

Since

$$(ux+c)a = uxa + ca = u(\sigma(a)x + \delta(a)) + ca = u\sigma(a)x + u\delta(a) + ca,$$

and,

$$\begin{aligned}\sigma'(a)(ux+c)+\delta'(a) &= (\sigma_u\circ\sigma)(a)(ux+c)+(u\delta+\delta_{c,\sigma'})(a)\\ &= \sigma_u(\sigma(a))(ux+c)+u\delta(a)+ca-\sigma'(a)c\\ &= u\sigma(a)u^{-1}(ux+c)+u\delta(a)+ca-\sigma_u(\sigma(a))c\\ &= u\sigma(a)u^{-1}(ux+c)+u\delta(a)+ca-u\sigma(a)u^{-1}c\\ &= u\sigma(a)x+u\delta(a)+ca,\end{aligned}$$

it follows that $(ux+c)a=\sigma'(a)(ux+c)+\delta'(a)$, whence $p(x)=ux+c$ is a cv-polynomial with respect to the quasi-derivation (σ',δ') on D.

(v) Let $S=D[x;\sigma,\delta]$ and $p(x)=x^2+c\in S$, we have

$$p(x)a=\sigma^2(a)p(x)+[\delta(\sigma(a))+\sigma(\delta(a))]x+(\delta^2+\delta_{c,\sigma^2})(a),$$

for any $a\in D$. Therefore, if $p(x)$ is a cv-polynomial, we must have $\sigma\delta=-\delta\sigma$; conversely, if $\sigma\delta=-\delta\sigma$, then $p(x)$ is a cv-polynomial with respect to $(\sigma^2,\delta^2+\delta_{c,\sigma^2})$.

(vi) Let $S'=D[x';\sigma',\delta']$, $S''=D[x'';\sigma'',\delta'']$ and $S=D[x;\sigma,\delta]$ be Ore extensions. Consider cv-polynomials $p(x),q(x)\in S$ with respect to (σ',δ') and (σ'',δ''), respectively. Then $p(x)+q(x)$ is also a cv-polynomial with respect to $(\sigma',\delta'+\delta'')$.

For $S=R[x;\sigma,\delta]$, a polynomial $f(x)\in S$ is said to be *right invariant* if $f(x)S\subseteq Sf(x)$, and *right semi-invariant* if $f(x)R\subseteq Rf(x)$. If no confusion arises, we suppress the adjective "right" and simply say invariant and semi-invariant polynomials.

EXAMPLE 2.2 ([LL92, P. 87]). Let $S=D[x;\sigma,\delta]$ and $p(x)=\sum_{i=0}^{n}b_ix^i$ be a semi-invariant polynomial of degree n. Then for any $a\in D$, $p(x)a=cp(x)$ for some c, and a comparison of the (left) coefficient of x^n gives $c=b_n\sigma^n(a)b_n^{-1}$, so $p(x)$ is a cv-polynomial with respect to $(\sigma_{b_n}\circ\sigma^n,\delta=0)$. Conversely, any cv-polynomial with respect to any $(\sigma',0)$ is a semi-invariant polynomial, since if $\delta=0$, then $p(x)a=\sigma'(a)p(x)$ and therefore $p(x)D\subseteq Dp(x)$.

For a ring R and a quasi-derivation (σ,δ) on R, δ is said to be *algebraic* if there exists a nonzero polynomial $g(x)\in S=R[x;\sigma,\delta]$ such that $g(\delta)=0$. Here, the evaluation of a polynomial $g(x)=\sum_{i=0}^{n}a_ix^i\in S$ at δ is defined to be the operator $g(\delta)=\sum_{i=0}^{n}a_i\delta^i$ on S [LL92, p. 87].

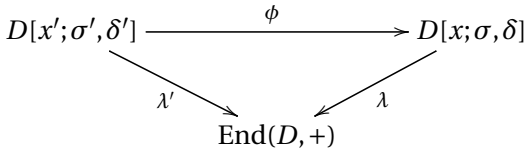
Some results characterizing the cv-polynomials are presented below.

PROPOSITION 2.9 ([LL92, THEOREM 2.12]). *Let $S'=D[x';\sigma',\delta']$ and $S=D[x;\sigma,\delta]$, $p(x)=\sum_{i=0}^{n}b_ix^i\in S$ be of degree $n\geq 0$.*

(1) *If $p(x)$ is a cv-polynomial with respect to the quasi-derivation (σ',δ'), then $\delta'=(p-b_0)(\delta)+\delta_{b_0,\sigma'}$, and if $n\geq 1$, then $\sigma'=\sigma_{b_n}\circ\sigma^n$.*

(2) *If δ is not an algebraic derivation, then $p(x)$ is a cv-polynomial with respect to the quasi-derivation (σ',δ') if and only if $\delta'=(p-b_0)(\delta)+\delta_{b_0,\sigma'}$.*

PROPOSITION 2.10 ([LL92, COROLLARY 2.14]). *We say that an injective homomorphism $\phi : D[x';\sigma',\delta'] \to D[x;\sigma,\delta]$ is homogeneous if the associated cv-polynomial $p(x) := \phi(x')$ has zero constant term. In this case, δ' and δ are related by the equation $\delta' = p(\delta)$. If we consider the homomorphisms $\lambda : D[x;\sigma,\delta] \to \mathrm{End}(D,+)$ defined by $\lambda(x) = \delta$ and $\lambda' : D[x';\sigma',\delta'] \to \mathrm{End}(D,+)$ given by $\lambda'(x') = \delta'$, then we have the commutative diagram*



Taking into account Proposition 2.10, next it will be shown that the injective homomorphisms between Ore extensions are characterized from the homogeneous cv-polynomials.

PROPOSITION 2.11 ([LL92, COROLLARY 2.15]). *If there is an injection ϕ from $S' = D[x';\sigma',\delta']$ to $S = D[x;\sigma,\delta]$ (defined by a cv-polynomial $p(x) = \phi(x')$), then there is an Ore extension $S'' = D[x'';\sigma'',\delta'']$ with an isomorphism $\psi : S'' \to S'$ such that S'' admits a homogeneous injction ϕ_1 into S satisfying $\phi = \phi_1 \circ \psi^{-1}$. If $p(x)$ happens to have a root $c \in D$, then there is also an Ore extension $\overline{S} = D[\overline{x};\sigma,\overline{\delta}]$ with an isomorphism $\tau : \overline{S} \to S$ such that S' has a homogeneous injection ϕ_2 into $\overline{S}$ satisfying $\phi = \tau \circ \phi_2$.*

From [LL92, Corollary 2.19], we know that if $\phi : D[x',\sigma',\delta'] \to D[x,\sigma,\delta]$ is an injective homomorphism with $\phi(x') = p(x)$, then $g(x') = \sum_{i=0}^{n} b_i x'^i$ is a cv-polynomial in $D[x',\sigma',\delta']$ if and only if its image $\phi(g) = \sum_{i=0}^{n} b_i p(x)^i$ is a cv-polynomial in $D[x,\sigma,\delta]$.

For the Ore extension $S = R[x;\sigma,\delta]$ and any element $a \in R$, $x^i a = \sum_{j=0}^{i} f_j^i(a) x^i$, where $f_j^i \in \mathrm{End}(D,+)$ is the sum of all possible products with j factors of σ and $i-j$ factors of δ. For example $f_j^j = \sigma^j$, $f_0^i = \delta^i$ and $f_{j-1}^j = \sigma^{j-1}\delta + \sigma^{j-2}\delta\sigma + \cdots + \delta\sigma^{j-1}$. If we assume that σ and δ commute, then $f_j^i = \binom{i}{j}\sigma^j\delta^{i-j}$. Let $M_n(a)$ denote the $n \times n$ matrix whose (i,j)-entry is $f_j^i(a)$ with $1 \le i,j \le n$, where, of course, $f_j^i(a)$ is taken to be zero if $i < j$, that is, $M_n(a)$ is a lower triangular matrix [LL88a, Section 2].

The next result characterizes cv-polynomials by using an eigenvector interpretation.

PROPOSITION 2.12 ([LL92, THEOREM 2.16]). *Let $p(x) = \sum_{i=0}^{n} b_i x^i \in S = D[x;\sigma,\delta]$ be a polynomial of degree $n \ge 0$. Then the following assertions are equivalent:*

(1) *$p(x)$ is a cv-polynomial.*

(2) *For any $a \in D$, and $j = 1,2,\ldots,n$, we have $\sum_{i=j}^{n} b_i f_i^j(a) = b_n \sigma^n(a) b_n^{-1} b_j$.*

(3) *$(b_1,\ldots,b_n)$ is a left eigenvector for each matrix $M_n(a)$, where $a \in D$.*

(4) *$p(x)D \subseteq Dp(x) + D$.*

If these conditions hold, then $up(x) + c$ is also a cv-polynomial, for any $u,c \in D$.

At the end of Section 2.4.1 (Examples 2.12, 2.13, Theorem 2.26, and Example 2.14) we say a few words about the extension of Proposition 2.12.

2.3.0.1 CHANGE OF VARIABLE AND SEMI-INVARIANT POLYNOMIALS

As we saw above, the cv-polynomials are a generalization of semi-invariant polynomials. In this section is deepened into the relationship between these polynomials.

PROPOSITION 2.13 ([LL92, PROPOSITION 3.1]). *Let $f(x) \in S$ be a monic semi-invariant polynomial of degree $m \geq 1$. Let $p(x) \in S = R[x;\sigma,\delta]$ with $p(x) = q(x)f(x) + r(x)$ where $\deg(r(x)) < m$. Then:*

(1) *$p(x)$ is a cv-polynomial with respect to (σ',δ') if and only if $r(x)$ is a cv-polynomial with respect to (σ',δ') and $q(x)\sigma^m(a) = \sigma'(a)q(x)$ for all $a \in D$.*

(2) *If $p(x)$ is a left multiple of $f(x)$, then $p(x)$ is a cv-polynomial if and only if it is semi-invariant.*

Now, assume that $p(x)$ is a cv-polynomial and that σ is an automorphism. Then

(3) *$q(x)$ is semi-invariant.*

(4) *If $\deg(p(x)) \geq m$ and $r(x) \notin D$, then $o(\sigma) < \infty$ and $\deg(p(x)) \cong \deg(r(x)) (\mathrm{mod}\ o(\sigma))$.*

PROPOSITION 2.14 ([LL88B, PROPOSITION 2.9]). *Assume σ is an automorphism, and let $f(x)$ be a monic non-constant right semi-invariant polynomial in S of the least degree (if it exists). Then any right semi-invariant polynomial $p(x)$ lies in $D[f(x)] = \left\{ \sum_{i\geq 0} c_i f(x)^i \mid c_i \in D \right\}$ (the subring of S generated by D and $f(x)$). In particular, $\deg(p(x))$ must be a multiple of $\deg(f(x))$.*

PROPOSITION 2.15 ([LL92, THEOREM 3.4]). *Assume that σ is an automorphism. Let $f(x)$ be a monic semi-invariant polynomial of minimal degree $m \geq 1$, and let $p(x)$ be any cv-polynomial, say of degree $n \geq 1$. Then:*

(1) *$p(x)$ can be represented in the form $\sum_{i\geq 1} c_i f(x)^i + r(x)$, where $c_i \in D$ and $r(x)$ is a cv-polynomial of degree $< m$. In particular, if $n \geq m$, then $m|n$.*

(2) *If $o(\sigma) = \infty$ and $n \geq m$, then $p(x)$ is the sum of a semi-invariant polynomial and a constant.*

(3) *If $n \geq m$, then $n|\deg(q(x))$ for any cv-polynomial $q(x)$ of degree $\geq n$. In particular, if $n \leq m$, then $n|m$.*

COROLLARY 2.16 ([LL92, COROLLARY 3.5]). *Assume that σ is an automorphism and that δ is not σ-inner. Let $f(x)$ be a monic semi-invariant polynomial of minimal degree $m \geq 1$, and let $p(x)$ be a monic cv-polynomial of minimal degree $n \geq 2$ (if both exist). Then $n|\deg(P(x))$ for any non-linear cv-polynomial $P(x)$. In particular, we have $n|m$ (so, for instance, if m is a prime, then we can conclude that $n = m$).*

The following result shows another connection between cv-polynomials and semi-invariant polynomials based on the formal differentiation process under some conditions such as commutativity between endomorphism σ and σ-derivation δ.

PROPOSITION 2.17 ([LL92, THEOREM 3.8]). *Assume that σ is an automorphism, and that $\sigma\delta = \delta\sigma$. Let $p(x) = \sum_{i=0}^{n} b_i x^i$ be any cv-polynomial in S. Then the formal derivative $p_1(x) := \sum_{i=1}^{n} i\, b_i\, t^{i-1}$ is semi-invariant. More generally,*

$$p_j(x) := \sum_{i=0}^{n-j} \binom{i+j}{j} b_{i+j} x^i \quad (1 \leq j \leq n)$$

are all semi-invariant polynomials.

If the hypothesis $\sigma\delta = \delta\sigma$ does not hold in Proposition 2.17, then the conclusion does not follow. In Example 2.1 (v), one can observe this fact, being $p(x)$ a cv-polynomial, but its formal derivatives would not be a semi-invariant polynomial.

From [LL92, Corollary 3.11] we know that if σ is an automorphism, $\sigma\delta = \delta\sigma$, and $p(x) \in S$ is a cv-polynomial of degree $n \geq 2$, such that $R[x;\sigma,\delta]$ has no non-constant semi-invariant polynomial of degree $< n$, then $\operatorname{char}(D) = p > 0$ and $p(x)$ has the form $\sum_{i=0}^{n} c_i x^i + c$.

2.4 ITERATED ORE EXTENSIONS

Motivated by the study of homomorphisms and cv-polynomials presented in Sections 2.1, 2.2, and 2.3, in this section we investigate both notions for iterations of these extensions.

2.4.1 TWO-STEP ITERATED ORE EXTENSIONS

We start with Definition 2.3 which is a direct extension of [LL92, Definition 2.4]. As we will see, the ring of coefficients $R[x_1;\sigma_1,\delta_1]$ is the same for both extensions, and the cv-polynomial only acts on the second indeterminate. As expected, in this situation we get similar results to those obtained by Lam and Leroy [LL92].

DEFINITION 2.3. Let $S' = R[x_1;\sigma_1,\delta_1][x_2';\sigma_2',\delta_2']$ and $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$ be two iterated Ore extensions. We say that $p(x_2)$ is a *cv-polynomial for two-step iterated Ore extensions* with respect to the quasi-derivation (σ_2',δ_2') on $R[x_1;\sigma_1,\delta_1]$ if the homomorphism $\phi : S' \to S$ given by $\phi(x_1) = x_1$ and $\phi(x_2') := p(x_2) \in S$, satisfies

$$\phi(x_2'x_1) = \sigma_2'(x_1)p(x_2) + \delta_2'(x_1) \quad \text{and} \quad \phi(x_2'x_1) = p(x_2)x_1,$$

i.e.,

$$p(x_2)x_1 = \sigma_2'(x_1)p(x_2) + \delta_2'(x_1), \tag{2.3}$$

and

$$x_1 r = \sigma_1(r)x_1 + \delta_1(r), \quad \text{and} \quad p(x_2)r = \sigma_2(r)p(x_2) + \delta_2(r).$$

Conversely, if a polynomial $p(x_2) \in S$ satisfies (2.3) for a quasi-derivation (σ_2',δ_2') on $R[x_1;\sigma_1,\delta_1]$, then we define the homomorphism $\phi : S' \to S$ by $\phi(x_2') = p(x_2)$.

Let us see some easy examples that illustrate Definition 2.3.

EXAMPLE 2.3. Let $S' = R[x_1;\sigma_1,\delta_1][x_2';\sigma_2',\delta_2']$ and $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$ as in Definition 2.3.

(i) If $p(x_2) := c \in R$, then $p(x_2)x_1 = cx_1$ and $p(x_2)r = cr = \sigma_2'(r)c + \delta_2'(r)$, whence if we take $\delta_2' := \delta_{2(c,\sigma_2')}(r) = cr - \sigma_2'(r)c$, it follows that $p(x_2)$ is a cv-polynomial.

(ii) Let $p(x_2) = ax_2 + b \in S$. Then

$$p(x_2)x_1 = (ax_2+b)x_1 = ax_2x_1 + bx_1 = a\sigma_2(x_1)x_2 + a\delta_2(x_1) + bx_1,$$

and

$$\sigma_2'(x_1)(ax_2+b) + \delta_2'(x_1) = \sigma_2'(x_1)(ax_2) + \sigma_2'(x_1)b + \delta_2'(x_1).$$

If $\sigma_2'(x_1) = \sigma'_{2(a,\sigma_2)}(x_1) := (\sigma_a \circ \sigma_2)(x_1) = a\sigma_2(x_1)a^{-1}$ and $\delta_2' = a\delta_2 + \delta_{2(b,\sigma_2')}$, an easy computation shows that $p(x_2)$ is a cv-polynomial.

(iii) Let $p(x_2) = x_2^2 + c \in S$. Suppose that $\sigma_2' = \sigma_2^2$, $\delta_2' = \delta_2^2 + \delta_{2(c,\sigma_2')}$, and $\sigma_2\delta_2 = -\delta_2\sigma_2$. Since

$$\begin{aligned} p(x_2)x_1 = (x_2^2+c)x_1 &= x_2^2x_1 + cx_1 = \sigma_2^2(x_1)x_2^2 + \delta_2\sigma_2(x_1)x_2 + \sigma_2\delta_2(x_1)x_2 + \delta_2^2(x_1) + cx_1 \\ &= \sigma_2^2(x_1)x_2^2 + \delta_2^2(x_1) + cx_1, \end{aligned}$$

and

$$\begin{aligned} \sigma_2'(x_1)p(x_2) + \delta_2'(x_1) &= \sigma_2'(x_1)(x_2^2+c) + \delta_2'(x_1) = \sigma_2'(x_1)x_2^2 + \sigma_2'(x_1)c + \delta_2'(x_1) \\ &= \sigma_2^2(x_1)x_2^2 + \sigma_2^2(x_1)c + \delta_2^2(x_1) + cx_1 - \sigma_2^2(x_1)c \\ &= \sigma_2^2(x_1)x_2^2 + \delta_2^2(x_1) + cx_1, \end{aligned}$$

then $\phi(x_2'x_1) = p(x_2)x_1 = \sigma_2'(x_1)p(x_2) + \delta_2'(x_1)$, that is, $p(x)$ is a cv-polynomial with respect to the quasi-derivation $\left(\sigma_2^2, \delta_2^2 + \delta_{2(c,\sigma_2')}\right)$ on $R[x_1;\sigma_1,\delta_1]$.

Now, we introduce the following definition of cv-polynomial more general than Definition 2.3.

DEFINITION 2.4. Let $S' = R[x_1';\sigma_1',\delta_1'][x_2';\sigma_2',\delta_2']$ and $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$. Consider the homomorphism $\phi : S' \to S$ given by $\phi(x_1') := p_1(x_1) = p_1 \in R[x_1;\sigma_1,\delta_1]$ and $\phi(x_2') := p_2(x_1,x_2) = p_2 \in S$. We say that the polynomials p_i's ($i = 1,2$) are the *cv-polynomials with respect to the quasi-derivations* (σ_i',δ_i')'s, $i = 1,2$, on R and $R[x_1';\sigma_1',\delta_1']$, respectively, if the relations

$$\begin{aligned} \phi(x_2'x_1') &= \sigma_2'(\phi(x_1'))p_2 + \delta_2'(\phi(x_1')) = p_2p_1, \\ \phi(x_1'r) &= \sigma_1'(r)p_1 + \delta_1'(r) = p_1r, \quad \text{for all } r \in R \\ \phi(x_2'r) &= \sigma_2'(r)p_2 + \delta_2'(r) = p_2r, \quad \text{for all } r \in R, \end{aligned}$$

are satisfied. Equivalently,

$$p_2p_1 = \sigma_2'(p_1)p_2 + \delta_2'(p_1), \tag{2.4}$$
$$p_1r = \sigma_1'(r)p_1 + \delta_1'(r), \tag{2.5}$$
$$p_2r = \sigma_2'(r)p_2 + \delta_2'(r), \tag{2.6}$$

Conversely, if two polynomials $p_1 \in R[x_1;\sigma_1,\delta_1]$ and $p_2 \in S$ satisfy conditions (2.4) - (2.6), where (σ_1',δ_1') and (σ_2',δ_2') are quasi-derivations on R and $R[x_1';\sigma_1',\delta_1']$, respectively, then we define the homomorphism $\phi : S' \to S$ by $\phi(x_i') = p_i$ $(i = 1,2)$.

Note that $p_1 \in R[x_1;\sigma_1,\delta_1]$, but $\sigma_2',\delta_2' : R[x_1';\sigma_1',\delta_1'] \to R[x_1';\sigma_1',\delta_1']$ For this reason, it is necessary that σ_2' and δ_2', are defined in terms of σ_2 and δ_2, as shown in Examples 2.5, 2.7 and 2.8.

The next example shows that constant polynomials are cv-polynomials (c.f. Example 2.3 (i)).

EXAMPLE 2.4. Consider the Ore extensions S' and S as in Definition 2.4. Let $p_1 := c$ and $p_2 := d$ be elements of R. With the aim of showing that conditions (2.4) - (2.6) hold, take the inner derivation $\delta_2'(c) := \delta_{2(d,\sigma_2')}(c) = dc - \sigma_2'(c)d$. Then

$$\phi(x_2'x_1') = p_2p_1 = dc \quad \text{and} \quad \sigma_2'(c)d + \delta_2'(c) = \sigma_2'(c)d + dc - \sigma_2'(c)d = dc.$$

In addition, for any $r \in R$ and the inner derivation given by $\delta_1' = \delta_{1(c,\sigma_1')}$, it is clear that $\phi(x_1'r) = p_1r = cr$. Also, note that $\sigma_1'(r)c + \delta_1'(r) = \sigma_1'(r)c + cr - \sigma_1'(r)c = cr$. In the same way, $\phi(x_2'r) = p_2r = \sigma_2'(r)p_2 + \delta_2'(r) = dr$. These facts show that any pair of constant polynomials in S correspond to a morphism in the sense of cv-polynomials. Finally, note that the values of the endomorphisms σ_i''s are arbitrary, while the corresponding δ_i''s are determined uniquely by the values of σ_i''s and the constant values associated with the polynomials p_i's.

EXAMPLE 2.5. Throughout this example, we fix $p_2 = c \in R$ as a constant polynomial, and consider some possibilities for the polynomial p_1 in the indeterminate x_1. This case is similar to that one considered by Lam and Leroy [LL92]. Table 2.1 contains conditions to guarantee that we get homomorphisms and cv-polynomials in the sense of Definition 2.4.

$p_2 = c$		
cv-polynomial	(σ_i',δ_i')	**Conditions**
$p_1 = d$	$\delta_1' = \delta_{1(d,\sigma_1')}$, $\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	
$p_1 = a_1x_1 + a_0$	$\begin{cases}\sigma_1' = a_1\sigma_1a_1^{-1},\\ \delta_1' = a_1\delta_1 + \delta_{1(a_0,\sigma_1')},\end{cases}$ $\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	
$p_1 = a_2x_1^2 + a_0$	$\begin{cases}\sigma_1' = a_2\sigma_1^2a_2^{-1},\\ \delta_1' = a_2\delta_1^2 + \delta_{1(a_0,\sigma_1')},\end{cases}$ $\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	$\sigma_1\delta_1 = -\delta_1\sigma_1$
$p_1 = a_nx_1^n + a_0$	$\begin{cases}\sigma_1' = a_n\sigma_1^na_n^{-1},\\ \delta_1' = a_n\delta_1^n + \delta_{1(a_0,\sigma_1')},\end{cases}$ $\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	$\sigma_1\delta_1 = -\delta_1\sigma_1$

$p_1 = f(x_1)x_1 + a_0$, $f(x_1) = a_2x_1 + a_1$	$\delta_1' = a_2\delta_1^2 + a_1\delta_1 + \delta_{1(a_0,\sigma_1')}$,	$\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	$\sigma_1\delta_1 = -\delta_1\sigma_1$, $f(x_1)\sigma_1(-) = \sigma_1'(-)f(x_1)$
$p_1 = f(x_1)x_1 + b$, $f(x_1) = \sum_{i=0}^n a_i x_1^i$	$\delta_1' = \sum_{i=1}^n a_i\delta_1^i + \delta_{1(b,\sigma_1')}$,	$\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	$\sigma_1\delta_1 = -\delta_1\sigma_1$, $f(x_1)\sigma_1(-) = \sigma_1'(-)f(x_1)$
$p_1 = \sum_{i=0}^n a_i x_1^i$	$\delta_1' = \sum_{i=1}^n a_i\delta_1^i + \delta_{1(a_0,\sigma_1')}$,	$\begin{cases}\sigma_2' = \sigma_{2c},\\ \delta_2' = \delta_{2(c,\sigma_2')}\end{cases}$	$\sigma_1\delta_1 = -\delta_1\sigma_1$, $a_i\sigma_1^i(-) = \sigma_1'(-)a_i$

Table 2.1: cv-polynomials and quasi-derivations

Recall that a σ-derivation δ of a ring R is said to be *algebraic* if there exists a non-zero polynomial $g(x) \in R[x;\sigma,\delta]$ such that $g(\delta) = 0$ (here, the evaluation of a polynomial $g(x) = \sum_{i=0}^n a_i x^i \in R$ at δ is defined to be the operator $g(\delta) = \sum_{i=0}^n a_i\delta^i$ on $R[x;\sigma,\delta]$) [LL92, p. 87].

The following result is one of the most important formulated by Lam and Leroy (we consider their notation to inner automorphisms).

PROPOSITION 2.18 ([LL92, THEOREM 2.12]). *Let $p(x) = \sum_{i=0}^n r_i x^i \in D[x;\sigma,\delta]$ be of degree $n \geq 0$, and let (σ',δ') be any quasi-derivation on D.*

(1) *If $p(x)$ is a cv-polynomial with respect to (σ',δ'), then $\delta' = (p - b_0)(\delta) + \delta_{b_0,\sigma'}$, and if $n \geq 1$, then $\sigma' = I_{b_n} \circ \sigma^n$, where I_{b_n} denotes the inner automorphism $I_{b_n}(r) := \sigma_{b_n}(r) = b_n r b_n^{-1}$.*

(2) *If δ is not an algebraic quasi-derivation, then $p(x)$ is a cv-polynomial with respect to (σ',δ') if and only if $\delta' = (p - b_0)(\delta) + \delta_{b_0,\sigma'}$.*

Having in mind Table 2.1, the following examples motivate the formulation of our Theorem 2.19 that extends Proposition 2.18.

EXAMPLE 2.6. (i) Consider the Ore extensions $S' = D[x';\sigma',\delta']$ and $S = D[x;\sigma,\delta]$ and the polynomial $p(x) = a_2x^2 + a_1x + a_0 \in S$ and $r \in D$. Suppose that p is a cv-polynomial with respect to the quasi-derivation (σ',δ') on D. Then $p(x)r = \sigma_1'(r)p(x) + \delta'(r)$ for any $r \in D$, whence

$$\begin{aligned} pr &= (a_2x^2 + a_1x + a_0)r = a_2x^2r + a_1xr + a_0r \\ &= a_2\sigma^2(r)x^2 + a_2\delta\sigma(r)x + a_2\sigma\delta(r)x + a_2\delta^2(r) + a_1\sigma(r)x + a_1\delta(r) + a_0r. \\ &= a_2\sigma^2(r)x^2 + [a_2\delta\sigma(r) + a_2\sigma\delta(r) + a_1\sigma(r)]x + a_2\delta^2(r) + a_1\delta(r) + a_0r. \end{aligned} \tag{2.7}$$

On the other hand,

$$\sigma'(r)p + \delta'(r) = \sigma'(r)a_2x^2 + \sigma'(r)a_1x + \sigma'(r)a_0 + \delta'(r). \tag{2.8}$$

By Proposition 2.18, $\delta' = (p - a_0)(\delta) + \delta_{(a_0,\sigma')}$. Besides, as p has degree 2, $\sigma' = \sigma_{a_2} \circ \sigma^2$. By

replacing this last expression in (2.8), we get

$$\begin{aligned}\sigma'(r)p+\delta'(r) &= (\sigma_{a_2}\circ\sigma^2)a_2x^2+(\sigma_{a_2}\circ\sigma^2)a_1x+(\sigma_{a_2}\circ\sigma^2)a_0\\ &\quad+(p-a_0)(\delta)(r)+\delta_{(a_0,\sigma')}(r)\\ &= a_2\sigma^2(r)a_2^{-1}a_2x^2+a_2\sigma^2(r)a_2^{-1}a_1x+a_2\sigma^2(r)a_2^{-1}a_0\\ &\quad+a_2\delta^2(r)+a_1\delta(r)+a_0r-a_2\sigma^2(r)a_2^{-1}a_0\\ &= a_2\sigma^2(r)x^2+a_2\sigma^2(r)a_2^{-1}a_1x+a_2\delta^2(r)+a_1\delta(r)+a_0r. \end{aligned}\tag{2.9}$$

Nevertheless, it is clear that expressions (2.7) and (2.9) are not necessarily the same.

(ii) Consider the polynomial $p=a_2x^2+a_1x+a_0$ belonging to the Jordan plane $\mathscr{J}(\Bbbk)\cong\Bbbk[y][x;\sigma,\delta]$ (Section 1.2.1). Suposse that p is a cv-polynomial with respect to the quasi-derivation (σ',δ') on $\Bbbk[y]$. Then $py=\sigma'(y)p+\delta'(y)$ for $y\in\Bbbk[y]$, whence

$$\begin{aligned}py &= (a_2x^2+a_1x+a_0)y\\ &= a_2x^2y+a_1xy+a_0y\\ &= a_2\sigma^2(y)x^2+a_2\delta\sigma(y)x+a_2\sigma\delta(y)x+a_2\delta^2(y)+a_1\sigma(y)x+a_1\delta(y)+a_0y.\\ &= a_2yx^2-a_2y^2x-a_2y^2x+a_2y^4+a_1qyx+a_1yx-a_1y^2+a_0y. \end{aligned}\tag{2.10}$$

Since

$$\sigma'(y)p+\delta'(y)=\sigma'(y)a_2x^2+\sigma'(y)a_1x+\sigma'(y)a_0+\delta'(y).\tag{2.11}$$

By Proposition 2.18, we know that $\delta'=(p-a_0)(\delta)+\delta_{(a_0,\sigma')}$. Besides, as p has degree 2, then $\sigma'=\sigma_{a_2}\circ\sigma^2$. By replacing this last expression in (2.8), we get

$$\begin{aligned}\sigma'(y)p+\delta'(y) &= (\sigma_{a_2}\circ\sigma^2)a_2x^2+(\sigma_{a_2}\circ\sigma^2)a_1x+(\sigma_{a_2}\circ\sigma^2)a_0+(p-a_0)(\delta)(y)+\delta_{(a_0,\sigma')}(y)\\ &= a_2\sigma^2(y)a_2^{-1}a_2x^2+a_2\sigma^2(y)a_2^{-1}a_1x+a_2\sigma^2(y)a_2^{-1}a_0\\ &\quad+a_2\delta^2(y)+a_1\delta(y)+a_0r-a_2\sigma^2(y)a_2^{-1}a_0\\ &= a_2\sigma^2(y)x^2+a_2\sigma^2(y)a_2^{-1}a_1x+a_2\delta^2(y)+a_1\delta(y)+a_0y\\ &= a_2yx^2+a_1yx+a_2y^4-a_1y^2+a_0y. \end{aligned}\tag{2.12}$$

However, expressions (2.10) and (2.12) do not coincide.

(iii) Let $p=a_2x^2+a_1x+a_0$ be an element belonging to the *q-meromorphic Weyl algebra* $\mathscr{J}_q(\Bbbk)\cong\Bbbk[y][x;\sigma,\delta]$ (Section 1.2.1). Suppose that p is a cv-polynomial with respect to the quasi-derivation (σ',δ') on $\Bbbk[y]$. Then $py=\sigma'(y)p+\delta'(y)$ for $y\in\Bbbk[y]$, and

$$\begin{aligned}py &= (a_2x^2+a_1x+a_0)y\\ &= a_2x^2y+a_1xy+a_0y\\ &= a_2\sigma^2(y)x^2+a_2\delta\sigma(y)x+a_2\sigma\delta(y)x+a_2\delta^2(y)+a_1\sigma(y)x+a_1\delta(y)+a_0y.\\ &= a_2q^2yx^2-a_2qy^2x-a_2qy^2x+a_2y^4-a_1y^2+a_0y. \end{aligned}\tag{2.13}$$

By using that

$$\sigma'(y)p+\delta'(y)=\sigma'(y)a_2x^2+\sigma'(y)a_1x+\sigma'(y)a_0+\delta'(y), \tag{2.14}$$

Proposition 2.18 asserts that $\delta'=(p-a_0)(\delta)+\delta_{(a_0,\sigma')}$. Besides, as p has degree 2, $\sigma'=\sigma_{a_2}\circ\sigma^2$, and by replacing this last expression in (2.14),

$$\begin{aligned}\sigma'(y)p+\delta'(y)&=(\sigma_{a_2}\circ\sigma^2)a_2x^2+(\sigma_{a_2}\circ\sigma^2)a_1x+(\sigma_{a_2}\circ\sigma^2)a_0\\&\quad+(p-a_0)(\delta)(y)+\delta_{(a_0,\sigma')}(y)\\&=a_2\sigma^2(y)a_2^{-1}a_2x^2+a_2\sigma^2(y)a_2^{-1}a_1x+a_2\sigma^2(y)a_2^{-1}a_0\\&\quad+a_2\delta^2(y)+a_1\delta(y)+a_0r-a_2\sigma^2(y)a_2^{-1}a_0\\&=a_2\sigma^2(y)x^2+a_2\sigma^2(y)a_2^{-1}a_1x+a_2\delta^2(y)+a_1\delta(y)+a_0y\\&=a_2q^2yx^2+a_1q^2y+a_2y^4-a_1y^2+a_0y.\end{aligned} \tag{2.15}$$

Nevertheless, it is clear that expressions (2.13) and (2.15) are different.

THEOREM 2.19. *Let $p_1=\sum_{i=0}^n a_ix_1^i\in S=R[x_1;\sigma_1,\delta_1]$ be of degree $n\geq 0$ and let (σ_1',δ_1') be any quasi-derivation on R. If p_1 is a cv-polynomial with respect to (σ_1',δ_1'), then $\delta_1'=(p_1-a_0)(\delta_1)+\delta_{(a_0,\sigma_1')}$ and if $n\geq 1$, then $\sigma_1\delta_1=-\delta_1\sigma_1$ and $a_i\sigma_1^i=\sigma_1'a_i$ for every $1\leq i\leq n$.*

Proof. The first part of the assertion is precisely the content of Proposition 2.18. Now, if $n\geq 2$, since p_1 is a cv-polynomial, then for every $r\in R$, $p_1r=\sigma_1'(r)p_1+\delta_1'(r)$, i.e., $p_1r-\sigma_1'(r)p_1=\delta_1'(r)$, and hence

$$\begin{aligned}\sum_{i=0}^n a_ix_1^ir-\sigma_1'(r)\sum_{i=0}^n a_ix_1^i&=[a_n\sigma_1^n(r)-\sigma_1'(r)a_n]x_1^n+[a_{n-1}\sigma_1^{n-1}(r)-\sigma_1'(r)a_{n-1}]x_1^{n-1}\\&\quad+\cdots+[a_1\sigma_1(r)-\sigma_1'(r)a_1]x_1-\sigma_1'(r)a_0+p_{\sigma_1,\delta_1}+\sum_{i=1}^n a_i\delta_1^i(r)+a_0r=\delta_1'(r),\end{aligned} \tag{2.16}$$

where p_{σ_1,δ_1} are the possible combinations between σ_1 and δ_1. Since (2.16) is equal to $\delta_1'(r)=(p_1-a_0)(\delta_1)(r)+\delta_{(a_0,\sigma_1')}(r)=\sum_{i=1}^n a_i\delta_1^i(r)+a_0r-\sigma_1'(r)a_0$, it follows that

$$\left[a_n\sigma_1^n(r)-\sigma_1'(r)\right]x_1^n+\left[a_{n-1}\sigma_1^{n-1}(r)-\sigma_1'(r)\right]x_1^{n-1}+\ldots+\left[a_1\sigma_1(r)-\sigma_1'(r)a_1\right]x_1+p_{\sigma_1,\delta_1}=0,$$

whence $a_i\sigma_1^i(r)=\sigma_1'(r)a_i$ for $1\leq i\leq n$ and $p_{\sigma_1,\delta_1}=0$. Therefore, $\sigma_1\delta_1=-\delta_1\sigma_1$ as desired. □

REMARK 9. Rimmer [Rim78] also imposed the condition $a_i\sigma_1^i(r)=\sigma_1(r)a_i$ in his study of homomorphisms between Ore extensions of endomorphism type (that fix the ring of coefficients).

EXAMPLE 2.7. In Table 2.2, we consider $p_1=c\in R$ and p_2 a polynomial of degree one with respect to the indeterminate x_2. As we see, if $p_2=ax_1x_2+b$ then σ_2' is determined according to one of the following possibilities: either mixed compositions cancel each other or the condition $(a_1x_1)\sigma_2(-)=\sigma_2'(-)(a_1x_1)$ holds.

$p_1 = c$ **and** p_2 **with degree one in the indeterminate** x_2		
cv-polynomials	(σ'_i, δ'_i)	**Conditions**
$p_2 = ax_2 + b$	$\delta'_1 = \delta_{1(c,\sigma'_1)}$ $\begin{cases}\sigma'_2 = a\sigma_2 a^{-1},\\ \delta'_2 = a\delta_2 + \delta_{2(b,\sigma'_2)}\end{cases}$	
$p_2 = ax_1x_2 + b$	$\delta'_1 = \delta_{1(c,\sigma'_1)}$ $\begin{cases}\sigma'_2 = a\sigma_1\sigma_2 a^{-1},\\ \delta'_2 = ax_1\delta_2 + \delta_{2(b,\sigma'_2)}\end{cases}$	$\delta_1\sigma_2 = 0$
$p_2 = ax_1x_2 + b$	$\delta'_1 = \delta_{1(c,\sigma'_1)}$ $\delta'_2 = ax_1\delta_2 + \delta_{2(b,\sigma'_2)}$	$(a_1x_1)\sigma_2(-) = \sigma'_2(-)(a_1x_1)$
$p_2 = f(x_1)x_2 + b,$ $f(x_1) = a_1x_1 + a_0$	$\delta'_1 = \delta_{1(c,\sigma'_1)}$ $\begin{cases}\delta'_2 = a_1x_1\delta_2 + a_0\delta_2 + \delta_{2(b,\sigma'_2)},\\ = f(x_1)\delta_2 + \delta_{2(b,\sigma'_2)}\end{cases}$	$f(x_1)\sigma_2(-) = \sigma'_2(-)f(x_1)$
$p_2 = f(x_1)x_2 + b,$ $f(x_1) = a_2x_1^2 + a_1x_1 + a_0$	$\delta'_1 = \delta_{1(c,\sigma'_1)}$ $\delta'_2 = f(x_1)\delta_2 + \delta_{2(b,\sigma'_2)}$	$f(x_1)\sigma_2(-) = \sigma'_2(-)f(x_1)$
$p_2 = f(x_1)x_2 + g(x_1),$ $f(x_1) = \sum_{i=0}^{n} a_i x_1^i,$ $g(x_1) = \sum_{j=0}^{m} b_j x_1^j,$	$\delta'_1 = \delta_{1(c,\sigma'_1)},$ $\delta'_2 = f(x_1)\delta_2 + \delta_{2(g(x_1),\sigma'_2)}$	$f(x_1)\sigma_2(-) = \sigma'_2(-)f(x_1)$

Table 2.2: cv-polynomials and quasi-derivations

EXAMPLE 2.8. In Table 2.3, p_1 is a degree one polynomial with respect to x_1, and p_2 is a polynomial in two indeterminates x_1 and x_2. From Example 2.7 we know that σ_i' and δ_i' are determined by the choice of p_i.

The results obtained in the previous examples illustrate Theorem 2.20. It is natural to say that a polynomial $p(x_1,x_2) \in S = D[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$ is *right invariant* if $p(x_1,x_2)R[x_1;\sigma_1,\delta_1] \subseteq R[x_1;\sigma_1,\delta_1]p(x_1,x_2)$, and *right semi-invariant* if $p(x_1,x_2)R \subseteq Rp(x_1,x_2)$.

THEOREM 2.20. *Consider x_2^n a monomial right invariant of degree $n \geq 1$. Let $p_2(x_1,x_2)$ as $p_2(x_1,x_2) = f(x_1)x_2^n + g(x_1,x_2)$, where the degree of $g(x_1,x_2)$ with respect to x_2 is less than or equal to n. Then:*

(1) *p_2 is a cv-polynomial with respect to the quasi-derivation (σ_2',δ_2') if and only if $g(x_1,x_2)$ is a cv-polynomial with respect to (σ_2',δ_2') and $f(x_1)\sigma_2^n(-) = \sigma_2'(-)f(x_1)$.*

(2) *If p_2 is a left multiple of $f(x_1)$, then p_2 is a cv-polynomial if and only if is invariant.*

Proof. (1) Suppose that p_2 is a cv-polynomial with respect to the quasi-derivation (σ_2',δ_2'). Then

$$\begin{aligned} p_2p_1 &= \sigma_2'(p_1)p_2 + \delta_2'(p_1) = \sigma_2'(p_1)[f(x_1)x_2^n + g(x_1,x_2)] + \delta_2'(p_1) \\ &= \sigma_2'(p_1)f(x_1)x_2^n + [\sigma_2'(p_1)g(x_1,x_2) + \delta_2'(p_1)], \quad \text{and} \end{aligned} \tag{2.17}$$

$$p_2p_1 = \left[f(x_1)x_2^n + g(x_1,x_2)\right]p_1 = f(x_1)x_2^np_1 + g(x_1,x_2)p_1 = f(x_1)\sigma_2^n(p_1)x_2^n + g(x_1,x_2)p_1, \tag{2.18}$$

since x_2^n is a monomial invariant. Both expressions (2.44) and (2.45) show that $f(x_1)\sigma_2^n(-) = \sigma_2'(-)f(x_1)$ and

$$g(x_1,x_2)p_1 = \sigma_2'(p_1)g(x_1,x_2) + \delta_2'(p_1). \tag{2.19}$$

Conversely, if $g(x_1,x_2)$ is a cv-polynomial with respect to the quasi-derivation (σ_2',δ_2'), then $g(x_1,x_2)p_1 = \sigma_2'(p_1)g(x_1,x_2) + \delta_2'(p_1)$, and since $f(x_1)\sigma_2^n(-) = \sigma_2'(-)f(x_1)$,

$$\begin{aligned} p_2p_1 &= f(x_1)x_2^np_1 + g(x_1,x_2)p_1 = f(x_1)\sigma_2^n(p_1)x_2^n + g(x_1,x_2)p_1 \\ &= \sigma_2'(p_1)f(x_1)x_2^n + \sigma_2'(p_1)g(x_1,x_2) + \delta_2'(p_1) \\ &= \sigma_2'(p_1)[f(x_1)x_2^n + g(x_1,x_2)] + \delta_2'(p_1) = \sigma_2'(p_1)p_2 + \delta_2'(p_1). \end{aligned}$$

(2) If p_2 is a cv-polynomial with respect to (σ_2',δ_2'), then $p_2p_1 = \sigma_2'(p_1)p_2 + \delta_2'(p_1)$, and having in mind that p_2 is a left multiple of $f(x_1)$, $g(x_1,x_2) = 0$. In this way, $\delta_2'(p_1) = 0$ due to expression (2.46). Besides, since $f(x_1)\sigma_2^n(-) = \sigma_2'(-)f(x_1)$, then $p_2p_1 = f(x_1)\sigma_2^n(p_1)x_2^n$, which shows that p_2 is invariant. Now, assume that p_2 is invariant. By assumption $g(x_1,x_2) = 0$, whence $p_2p_1 = f(x_1)x_2^np_1 = f(x_1)\sigma_2^n(p_1)x_2^n$. If we suppose that there exist σ_2' such that $f(x_1)\sigma_2^n(p_1) = \sigma_2'(p_1)f(x_1)$, then p_2 is a cv-polynomial with respect to the quasi-derivation $(\sigma_2',\delta_2' = 0)$.

□

p_1 **with degree one in** x_1 **and** p_2 **arbitrary**		
cv-polynomials	(σ'_i, δ'_i)	**Conditions**
$p_1 = a_1x_1 + a_0$, $p_2 = b_1x_2 + b_0$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\begin{cases}\sigma'_2 = b_1\sigma_2b_1^{-1},\\ \delta'_2 = b_1\delta_2 + \delta_{2(b_0,\sigma'_2)}\end{cases}$	
$p_1 = a_1x_1 + a_0$, $p_2 = b_1x_1x_2 + b_0$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\begin{cases}\sigma'_2 = b_1\sigma_1\sigma_2b_1^{-1}\\ \delta'_2 = b_1x_1\delta_2 + \delta_{2(b_0,\sigma'_2)}\end{cases}$	$\delta_1\sigma_2 = 0$
$p_1 = a_1x_1 + a_0$, $p_2 = f(x_1)x_2 + c$, $f(x_1) = b_1x_1 + b_0$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\begin{cases}\delta'_2 = b_1x_1\delta_2 + b_0\delta_2 + \delta_{2(c,\sigma'_2)}\\ \quad = f(x_1)\delta_2 + \delta_{2(c,\sigma'_2)}\end{cases}$	$f(x_1)\sigma_2(-) = \sigma'_2(-)f(x_1)$
$p_1 = a_1x_1 + a_0$, $p_2 = f(x_1)x_2 + g(x_1)$, $f(x_1) = \sum_{i=0}^{n} a_ix_1^i$, $g(x_1) = \sum_{j=0}^{m} b_jx_1^j$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\delta'_2 = f(x_1)\delta_2 + \delta_{2(g(x_1),\sigma'_2)}$	$f(x_1)\sigma_2(-) = \sigma'_2(-)f(x_1)$
$p_1 = a_1x_1 + a_0$, $p_2 = b_2x_2^2 + b_0$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\begin{cases}\sigma'_2 = b_2\sigma_2^2b_2^{-1},\\ \delta'_2 = b_2\delta_2^2 + \delta_{2(b_0,\sigma'_2)}\end{cases}$	$\sigma_2\delta_2 = -\delta_2\sigma_2$ $b_2\sigma_2^2(-) = \sigma'_2(-)b_2$
$p_1 = a_1x_1 + a_0$, $p_2 = b_1x_1x_2^2 + b_0$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\delta'_2 = b_1x_1\delta_2^2 + \delta_{2(b_0,\sigma'_2)}$	$\sigma_2\delta_2 = -\delta_2\sigma_2$ $b_1x_1\sigma_2^2(-) = \sigma_2^{'}(-)b_1x_1$
$p_1 = a_1x_1 + a_0$, $p_2 = f(x_1)x_2^n + g(x_1)$ $f(x_1) = \sum_{i=0}^{n} a_ix_1^i$ $g(x_1) = \sum_{j=0}^{m} b_jx_1^j$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\delta'_2 = f(x_1)\delta_2^n + \delta_{2(g(x_1),\sigma'_2)}$	$\sigma_2\delta_2 = -\delta_2\sigma_2$ $f(x_1)\sigma_2^n(-) = \sigma_2^{'}(-)f(x_1)$
$p_1 = a_1x_1 + a_0$, $p_2 = f(x_2)x_2 + g(x_1)$ $f(x_2) = \sum_{i=0}^{n} a_ix_2^i$ $g(x_1) = \sum_{j=0}^{m} b_jx_1^j$	$\begin{cases}\sigma'_1 = a_1\sigma_1a_1^{-1},\\ \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}\end{cases}$ $\delta'_2 = \sum_{i=0}^{n} a_i\delta_2^i + \delta_{2(g(x_1),\sigma'_2)}$	$\sigma_2\delta_2 = -\delta_2\sigma_2$ $f(x_2)\sigma_2(-) = \sigma_2^{'}(-)f(x_2)$
$p_1 = a_1x_1 + a_0$, $p_2 = \sum_{j=0}^{m}\sum_{i=0}^{n} b_{ij}x_1^ix_2^j$	$\sigma'_1 = a_1\sigma_1a_1^{-1}, \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_1)}, \quad \delta'_2 = \sum b_{ij}x_1^i\delta_2^i + \delta_{2(b_{i0}x_1^i,\sigma'_2)}$	$\sigma_2\delta_2 = -\delta_2\sigma_2$ $b_{ij}x_1^i\sigma_2^j(-) = \sigma_2^{'}(-)b_{ij}x_1^i$

Table 2.3: cv-polynomials and quasi-derivations

Let R be a ring and (σ_2,δ_2) a quasi-derivation on $R[x_1;\sigma_1,\delta_1]$. We say that δ_2 is *algebraic* if there exists a non-zero polynomial g in the two-step iterated Ore extension $S=R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$ such that $g(x_1,\delta_2)=0$. The evaluation of a polynomial $g(x_1,x_2)=\sum_{i=0}^{n}q_i(x_1)x_2^i$ at δ_2 is defined to be the operator $g(x_1,\delta_2)=\sum_{i=0}^{n}q_i(x_1)\delta_2^i$ on S.

The following result extends Theorem 2.19 for iterated Ore extensions.

THEOREM 2.21. *Let $S'=R[x_1';\sigma_1',\delta_1'][x_2';\sigma_2',\delta_2']$ and $S=R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$. Consider $p_1(x_1)\in R[x_1;\sigma_1,\delta_1]$, $p_2(x_1,x_2)\in S$ given by $p_1(x_1)=\sum_{i=0}^{k}a_ix_1^i$ and $p_2(x_1,x_2)=\sum_{i=0}^{n}\sum_{j=0}^{m}b_{ij}x_1^ix_2^j$.*

(1) *If p_2 is a cv-polynomial with respect to the quasi-derivation (σ_2',δ_2') then $\delta_2'=(p_2-B_{00})(\delta_2)+\delta_{2(B_{00},\sigma_2')}$, where $B_{00}=\sum_{i=0}^{n}b_{i0}x_1^ix_2^0$, and if also $m\geq 1$, then $b_{ij}x_1^i\sigma_2^j(-)=\sigma_2'(-)b_{ij}x_1^i$ and $\sigma_2\delta_2=-\delta_2\sigma_2$.*

(2) *If δ_i is not an algebraic derivation for $1\leq i\leq 2$, then p_i is a cv-polynomial $1\leq i\leq 2$ with respect to the quasi-derivation (σ_i',δ_i') if and only if $\delta_1'=(p_1-a_0)\delta_1+\delta_{1(a_0,\sigma_1')}$ and $\delta_2'=(p_2-B_{00})(\delta_2)+\delta_{2(B_{00},\sigma_2')}$, where $B_{00}=\sum_{i=0}^{n}b_{i0}x_1^ix_2^0$.*

Proof. (1) Consider the homomorphism $\lambda:S\to \mathrm{End}(R[x_1;\sigma_1,\delta_1],+)$, given by $\lambda(r)=$ left multiplication by r on R, $\lambda(x_2)=\delta_2$, and $\lambda(x_1)=\lambda_{x_1}$ with $\lambda_{x_1}(r)=r$, for every $r\in R$, and $\lambda_{x_1}(x_1)=x_1$. Suppose that p_2 is a cv-polynomial with respect to (σ_2',δ_2'). Then

$$\begin{aligned} p_2p_1=\phi(x_2'x_1')&=\sigma_2'(\phi(x_1'))p_2+\delta_2'(\phi(x_1')) \\ p_2r=\phi(x_2'r)&=\sigma_2'(r)p_2+\delta_2'(r). \end{aligned} \tag{2.20}$$

If we evaluate the expression (2.20) with respect to the homomorphism λ, we get

$$p_2(x_1,\delta_2)p_1(x_1)=\sigma_2'(p_1(x_1))p_2(x_1,\delta_2)+\delta_2'(p_1(x_1)). \tag{2.21}$$

Now, by evaluating $\delta_2(1)$, expression (2.21) guarantees that

$$p_2(x_1,\delta_2)p_1(x_1)=\sigma_2'(p_1(x_1))\sum_{i=0}^{n}b_{i0}x_1^ix_2^0+\delta_2'(p_1(x_1)),$$

since $\delta_2^j(1)=0$ for $j\geq 1$. Therefore, by taking $B_{00}=\sum_{i=0}^{n}b_{i0}x_1^ix_2^0$, it follows that

$$\begin{aligned} \delta_2'(p_1(x_1))&=p_2(x_1,\delta_2)p_1(x_1)-\sigma_2'(p_1(x_1))B_{00} \\ &=p_2(x_1,\delta_2)p_1(x_1)-B_{00}p_1(x_1)+B_{00}p_1(x_1)-\sigma_2'(p_1(x_1))B_{00} \\ &=(p_2(x_1,\delta_2)-B_{00})p_1(x_1)+\delta_{2(B_{00},\sigma_2')}(p_1(x_1)) \\ &=(p_2(x_1,x_2)-B_{00})(\delta_2)(p_1(x_1))+\delta_{2(B_{00},\sigma_2')}(p_1(x_1)). \end{aligned}$$

and thus $\delta_2'=(p_2-B_{00})(\delta_2)+\delta_{2(B_{00},\sigma_2')}$.

For the next part, let $m \geq 1$. Since p_2 is a cv-polynomial, $p_2p_1 - \sigma_2'(p_1)p_2 = \delta_2'(p_1)$, and so

$$\begin{aligned}&\left(\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}x_1^i x_2^j\right)\left(\sum_{i=0}^{k} a_i x_1^i\right) - \sigma_2'\left(\sum_{i=0}^{k} a_i x_1^i\right)\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}x_1^i x_2^j \\ &= [b_{nm}x_1^n\sigma_2^m(a_k x_1^k) - \sigma_2'(a_k x_1^k)b_{nm}x_1^n]x_2^m + \cdots - \sigma_2'(p_1)B_{00} \\ &\quad + p_{\sigma_2,\delta_2} + \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}x_1^i\delta_2^j(p_1) + B_{00}p_1 = \delta_2'(p_1),\end{aligned} \tag{2.22}$$

where p_{σ_2,δ_2} are the possible combinations between σ_2 and δ_2. Since (2.22) is equal to

$$\delta_2'(p_1) = (p_2 - B_{00})(\delta_2)(p_1) + \delta_{2(B_{00},\sigma_2')}(p_1) = \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}x_1^i\delta_2^j(p_1) + B_{00}p_1 - \sigma_1'(p_1)B_{00},$$

necessarily the equality

$$[b_{nm}x_1^n\sigma_2^m(a_k x_1^k) - \sigma_2'(a_k x_1^k)b_{nm}x_1^n]x_2^m + \cdots + p_{\sigma_2,\delta_2} = 0.$$

must be satisfied. Therefore, $b_{ij}x_1^i\sigma_2^j(a_i x_1^i) = \sigma_2'(a_i x_1^i)b_{ij}x_1^i$ for $1 \leq i \leq n$, $1 \leq j \leq m$, $p_{\sigma_2,\delta_2} = 0$, and so $\sigma_2\delta_2 = -\delta_2\sigma_2$, as desired.

(2) The first implication has been proven in Theorem 2.19(1). Let us suppose that δ_2 is not an algebraic derivation, and consider $q(x_1,x_2) = p_2(x_1,x_2) - B_{00}$. Then $\delta_2' = q(x_1,\delta_2) + \delta_{2(B_{00},\sigma_2')}$, and so $q(x_1,\delta_2) = \delta_2' - \delta_{2(B_{00},\sigma_2')}$ is a σ_2'-derivation, which implies that for any $r \in R$,

$$q(x_1,\delta_2)(p_1(x_1)r) = \sigma_2'(p_1(x_1))q(x_1,\delta_2)(r) + q(x_1,\delta_2)(p_1(x_1))r.$$

In this way,

$$q(x_1,\delta_2)\lambda(p_1(x_1)) = \lambda(\sigma_2'(p_1(x_1)))q(x_1,\delta_2) + \lambda(q(x_1,\delta_2)(p_1(x_1))),$$

which holds in the image of λ. Since δ_2 is not algebraic, λ is injective, and so there exists λ^{-1}, and

$$q(x_1,x_2)(p_1(x_1)) = \sigma_2'(p_1(x_1))q(x_1,x_2) + q(x_1,\delta_2)(p_1(x_1)).$$

By replacing $q(x_1,x_2)$ and $q(x_1,\delta_2)$, we get

$$\begin{aligned}(p_2(x_1,x_2) - B_{00})p_1(x_1) &= \sigma_2'(p_1(x_1))(p_2(x_1,x_2) - B_{00}) + (\delta_2' - \delta_{2(B_{00},\sigma_2')})(p_1(x_1)) \\ &= \sigma_2'(p_1(x_1))p_2(x_1,x_2) - \sigma_2'(p_1(x_1))B_{00} \\ &\quad + \delta_2'(p_1(x_1)) - \delta_{2(B_{00},\sigma_2')}(p_1(x_1)) \\ &= \sigma_2'(p_1(x_1))p_2(x_1,x_2) - \sigma_2'(p_1(x_1))B_{00} \\ &\quad + \delta_2'(p_1(x_1)) - B_{00}p_1(x_1) + \sigma_2'(p_1(x_1))B_{00} \\ &= \sigma_2'(p_1(x_1))p_2(x_1,x_2) + \delta_2'(p_1(x_1)) - B_{00}p_1(x_1),\end{aligned}$$

and cancelling $B_{00}p_1(x_1)$, it follows that $p_2p_1 = \sigma_2'(p_1)p_2 + \delta_2'(p_1)$.

Following the same ideas of part (2) of the proof above, but in this case with $q(x_1) = p_1(x_1) - a_0$, $\delta_1' = p_1(\delta_1) + \delta_{1(a_0,\sigma_1')}$, and evaluating with the homomorphism $\alpha : R[x_1;\sigma_1,\delta_1] \to \mathrm{End}(R,+)$, $\alpha(x_1) = \delta_1$ and $\lambda(r)$ defined as the left multiplication by r on R for each $r \in R$, we can assert that $p_1 r = \sigma_1'(p_1)r + \delta_1'(r)$. These facts show that p_i $(i = 1,2)$ is a cv-polynomial with respect to the quasi-derivation (σ_i', δ_i') $(i = 1,2)$.

□

EXAMPLE 2.9. Let $\mathscr{J}' = \Bbbk[y'][x';\sigma_2',\delta_2']$ and $\mathscr{J} = \Bbbk[y][x;\sigma_2,\delta_2]$ be two Jordan's planes (Section 1.2.1). Consider the polynomials $p_1 = a_1 y + a_0$ and $p_2 = b_1 yx + b_0$. By Theorem 2.21, we know that $\delta_2' = (p_2 - B_{00})(\delta_2) + \delta_{2(B_{00},\sigma')}$. Let $\sigma_2' = a_1 \sigma a_1^{-1}$ and $\sigma_2(y) = y$ (recall that $\delta_2(y) = -y^2$). Since

$$\begin{aligned} p_2 p_1 &= (b_1 yx + b_0)(a_1 y + a_0) \\ &= a_1 b_1 yxy + a_0 b_1 yx + a_1 b_0 y + a_0 b_0 \\ &= a_1 b_1 y(\sigma_2(y)x + \delta_2(y)) + a_0 b_1 yx + a_1 b_0 y + a_0 b_0 \\ &= a_1 b_1 y^2 x - a_1 b_1 y^3 + a_0 b_1 yx + a_1 b_0 y + a_0 b_0, \end{aligned}$$

and

$$\begin{aligned} \sigma_2'(p_1)p_2 + \delta_2'(p_1) &= \sigma_2'(a_1 y + a_0)(b_1 yx + b_0) + \delta_2'(a_1 y + a_0) \\ &= a_1 b_1 y^2 x + a_1 b_0 y + a_0 b_1 yx + a_0 b_0 + \delta_2'(a_1 y) + \delta_2'(a_0) \\ &= a_1 b_1 y^2 x + a_1 b_0 y + a_0 b_1 yx + a_0 b_0 + \sigma_2'(a_1)\delta_2'(y) + \delta_2'(a_1)y \\ &= a_1 b_1 y^2 x + a_1 b_0 y + a_0 b_1 yx + a_0 b_0 + a_1[b_1 y\delta_2(y) + b_0\sigma_2'(y) - yb_0] \\ &= a_1 b_1 y^2 x + a_1 b_0 y + a_0 b_1 yx + a_0 b_0 - a_1 b_1 y^3, \end{aligned}$$

it follows that the polynomials p_i's, $i = 1,2$, are cv-polynomials with respect to the quasi-derivation (σ_i', δ_i'), $i = 1,2$. Note that the conditions for σ_1' and δ_1' are satisfied.

Lam and Leroy [LL92, p. 86] asserted that a D-homomorphism $\phi : D[x';\sigma',\delta'] \to D[x;\sigma,\delta]$ is *injective* if and only if the associated cv-polynomial $p(x) = \phi(x')$ has degree greater or equal 1, and ϕ is *surjective* (respectively, *bijective*) if and only if $p(x)$ has degree one. Theorem 2.22 extends this result to two-step iterated Ore extensions and it is one of the most important results of the thesis.

THEOREM 2.22. *If ϕ is a homomorphism of two-step iterated Ore extensions as in Definition 2.4, then ϕ is an isomorphism if and only if the associated cv-polynomials p_1 and p_2 have degree one with respect to the indeterminates x_1 and x_2, respectively.*

Proof. Let ϕ be an isomorphism of two-step iterated Ore extensions. By Definition 2.4, $\phi(x_1') = p_1 \in R[x_1;\sigma_1,\delta_1]$ and $\phi(x_2') := p_2 \in S$.

With the aim of showing that the cv-polynomials p_1 and p_2 necessarily have a degree greater or equal to one, consider the following cases:

(1) $p_1 = c$ and $p_2 = c^{-1}dc$, with $c, d \in R$. Since we can find elements $f, g \in S'$ such that $f = dx_1'$ and $g = cx_2'$. Then $\phi(f) = dc = \phi(g)$, but since $f \neq g$, it contradicts the injectivity of ϕ.

(2) $p_1 = c \in R$, and $p_2 \in S$ with degree greater than zero. Let $f, g \in S'$ such that $f = dx_1'$ and $g = dc$. It follows that $\phi(f) = dc = \phi(g)$, which contradicts again the injectivity of ϕ.

(3) $p_1 \in R[x_1;\sigma_1,\delta_1]$ of degree greater than zero and $p_2 = d \in R$. Consider elements $f, g \in S'$ such that $f = d$ and $g = x_2'$. Again, $\phi(f) = d = \phi(g)$, which contradicts again the injectivity of ϕ.

Assume that the cv-polynomials p_1 and p_2 have degree $n, k > 1$, respectively. We write

$$p_1 = \sum_{i=0}^{n} a_i x_1^i \text{ and } p_2 = \sum_{j=0}^{k} q_j(x_1) x_2^j,$$

where $a_i \in R$, $q_j(x_1) \in R[x_1;\sigma_1,\delta_1]$ for $0 \leq i \leq n$ and $0 \leq j \leq k$, and $a_n, q_k(x_1)$ are non-zero elements. Since ϕ is surjective, for the indeterminate $x_1 \in S$ there exists a polynomial $g \in S'$ such that

$$x_1 = \phi(g) = \phi\left(\sum_{i,j=0}^{w} c_{ij} x_1'^i x_2'^j\right) = \sum_{i,j=0}^{w} c_{ij}\phi(x_1'^i x_2'^j) = \sum_{i,j=0}^{w} c_{ij} p_1^i p_2^j.$$

In this way, $w = 1$, the degree of p_1 is one, and the degree of p_2 is zero, which gives us a contradiction. Since the cv-polynomials have degree greater than one, it follows that there is no polynomial $g \in S'$ with $\phi(g) = x_1$. Similarly, if we take the indeterminate $x_2 \in S$, we get that $w = 1$, p_1 must have degree zero or one, and p_2 must have degree one, which is again a contradiction. Therefore, both p_1 and p_2 must have degree exactly one.

Now, let $\phi : S' \to S$ be the homomorphism defined as $\phi(x_1') = p_1 = ax_1 + b$ and $\phi(x_2') = p_2 = f(x_1)x_2 + g(x_1)$, with $a, b \in R$ and $f(x_1), g(x_1) \in R[x_1;\sigma_1,\delta_1]$. First of all, we show that both polynomials are cv-polynomials. With this aim, consider the following equalities:

$$\begin{aligned}\phi(x_2'x_1') = p_2p_1 &= (f(x_1)x_2 + g(x_1))(ax_1 + b) \\ &= f(x_1)x_2ax_1 + f(x_1)x_2b + g(x_1)ax_1 + g(x_1)b \\ &= f(x_1)[\sigma_2(ax_1)x_2 + \delta_2(ax_1)] + f(x_1)x_2b + g(x_1)ax_1 + g(x_1)b \\ &= f(x_1)\sigma_2(ax_1)x_2 + f(x_1)\delta_2(ax_1) + f(x_1)x_2b + g(x_1)ax_1 + g(x_1)b,\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\sigma_2'(\phi(x_1'))p_2 + \delta_2'(\phi(x_1')) &= \sigma_2'(ax_1 + b)(f(x_1)x_2 + g(x_1)) + \delta_2'(ax_1 + b) \\ &= \sigma_2'(ax_1)f(x_1)x_2 + \sigma_2'(ax_1)g(x_1) + \sigma_2'(b)f(x_1)x_2 \\ &\quad + \sigma_2'(b)g(x_1) + \delta_2'(ax_1) + \delta_2'(b).\end{aligned} \tag{2.24}$$

Having in mind that (2.23) and (2.24) must coincide, let $\sigma_2' := (\sigma_{2f(x_1)} \circ \sigma_2)$, that is, $(\sigma_{2f(x_1)} \circ \sigma_2)(-) = f(x_1)\sigma_2(-)(f(x_1))^{-1}$, so $f(x_1)$ is a unit, say $f(x_1) = \gamma \in R^*$. Since $\delta_2' = (\gamma\delta_2 + \delta_{2(g(x_1),\sigma_2')})$, it follows that $p_1 = ax + b$ and $p_2 = \gamma x_2 + g(x_1)$, whence

$$\begin{aligned}p_2p_1 &= \gamma\sigma_2(ax_1)x_2 + \gamma\delta_2(ax_1) + \gamma x_2b + g(x_1)ax_1 + g(x_1)b \\ &= \gamma\sigma_2(ax_1)x_2 + \gamma\delta_2(ax_1) + \gamma\sigma_2(b)x_2 + \gamma\delta_2(b) + g(x_1)ax_1 + g(x_1)b,\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\sigma_2'(\phi(x_1'))p_2+\delta_2'(\phi(x_1')) &= \sigma_2'(ax_1)\gamma x_2+\sigma_2'(ax_1)g(x_1)+\sigma_2'(b)\gamma x_2\\ &\quad+\sigma_2'(b)g(x_1)+\delta_2'(ax_1)+\delta_2'(b)\\ &= \gamma\sigma_2(ax_1)\gamma^{-1}\gamma x_2+\gamma\sigma_2(ax_1)\gamma^{-1}g(x_1)+\gamma\sigma_2(b)\gamma^{-1}\gamma x_2\\ &\quad+\gamma\sigma_2(b)\gamma^{-1}g(x_1)+\gamma\delta_2(ax_1)+g(x_1)ax_1\\ &\quad-\gamma\sigma_2(ax_1)\gamma^{-1}g(x_1)+\gamma\delta_2(b)+g(x_1)b-\gamma\sigma_2(b)\gamma^{-1}g(x_1)\\ &= \gamma\sigma_2(ax_1)x_2+\gamma\sigma_2(b)x_2+\gamma\delta_2(ax_1)+g(x_1)ax_1\\ &\quad+\gamma\delta_2(b)+g(x_1)b.\end{aligned}\tag{2.26}$$

If we compare (2.25) and (2.26), then

$$p_2p_1=\sigma_2'(\phi(x_1'))p_2+\delta_2'(\phi(x_1'))=\sigma_2'(p_1)p_2+\delta_2'(p_1),$$

as desired.

Now, since

$$\begin{aligned}p_1r=(ax+b)r=axr+br=a\sigma_1(r)x+a\delta_1(r)+br,\quad\text{and}\\ \sigma_1'(r)p_1+\delta_1'(r)=\sigma_1'(r)ax+\sigma_1'(r)b+\delta_1'(r),\end{aligned}$$

then $\sigma_1'=a\sigma_1a^{-1}$ and $\delta_1'=a\delta_1+\delta_{1(b,\sigma_1')}$. In this way,

$$\sigma_1'(r)p_1+\delta_1'(r)=a\sigma_1(r)a^{-1}ax+a\sigma_1(r)a^{-1}b+a\delta_1(r)+br-a\sigma_1(r)a^{-1}b,\tag{2.27}$$

or equivalently,

$$p_1r=a\sigma_1(r)x+a\delta_1(r)+br=\sigma_1'(r)p_1+\delta_1'(r).$$

All these facts guarantee that the polynomials $p_1=ax_1+b$ and $p_2=\gamma x_2+g(x_1)$ are cv-polynomials with respect to the quasi-derivations $(\sigma_1',\delta_1')=(a\sigma_1a^{-1},a\delta_1+\delta_{1(b,\sigma_1')})$ and $(\sigma_2',\delta_2')=(\gamma\sigma_2\gamma^{-1},\gamma\delta_2+\delta_{2(g(x_1),\sigma_2')})$, respectively.

Finally, let us verify that ϕ is an isomorphism. We will show that there exists a homomorphism $\psi:S\to S'$ such that $\psi\circ\phi=\mathrm{id}_{S'}$ and $\phi\circ\psi=\mathrm{id}_S$. Consider $x_1'=\psi(\phi(x_1'))$, and the equalities

$$x_1'=\psi(\phi(x_1'))=\psi(p_1)=\psi(ax_1+b)=\psi(ax_1)+\psi(b).$$

From the last expression, it follows that $\psi(x_1)=a^{-1}x_1'-a^{-1}\psi(b)$.

In the same way, let us consider $x_2'=\psi(\phi(x_2'))$. Since

$$\begin{aligned}x_2'=\psi(\phi(x_2'))=\psi(p_2)=\psi(\gamma x_2+g(x_1))=\psi(\gamma x_2)+\psi(g(x_1))\\ =\gamma\psi(x_2)+g(\psi(x_1))=\gamma\psi(x_2)+g(a^{-1}x_1'-a^{-1}\psi(b)),\end{aligned}$$

we get $\psi(x_2)=\gamma^{-1}x_2'-\gamma^{-1}g(a^{-1}x_1'-a^{-1}\psi(b))$. These facts show that $\psi\circ\phi=\mathrm{id}_{S'}$, where

$p_1 = ax_1 + b$ and $p_2 = \gamma x_2 + g(x_1)$. In the same way, it can be shown that $\phi \circ \psi = \mathrm{id}_S$, whence the homomorphism defined by $\phi(x_1') = p_1 = ax_1 + b$ and $\phi(x_2') = p_2 = \gamma x_2 + g(x_1)$ is an isomorphism. □

Next, we characterize the automorphisms corresponding with cv-polynomials for the Jordan plane $\mathscr{J}(\Bbbk)$ (Theorem 2.23) and the q-meromorphic Weyl algebra (also known as the q-skew Jordan's plane) $\mathscr{J}_q(\Bbbk)$ (Theorem 2.24).

THEOREM 2.23. *If ϕ is an automorphism of the two-step iterated Ore extension $\mathscr{J}$ (Section 1.2.1), then the correspoding cv-polynomials are given by $\phi(y) = p_1 = \gamma y$ and $\phi(x) = p_2 = x + g(y)$ with respect to (σ_i, δ_i), $i = 1, 2$, for some $\gamma \in \Bbbk^*$ and $g(y) \in \Bbbk[y]$.*

Proof. Consider ϕ an automorphism of $\mathscr{J}$. Then ϕ is an isomorphism of two-step iterated Ore extensions, so Theorem 2.22 establishes that its associated cv-polynomials have degree 1. Let $\phi(y) := p_1 = \gamma y + \beta$ with $\gamma, \beta \in \Bbbk^*$ and $\phi(x) := p_2 = f(y)x + g(y)$ where $f(y), g(y) \in \Bbbk[y]$. The idea is to see that these polynomials satisfy conditions (2.4) - (2.6) with respect to the quasi-derivations (σ_i, δ_i), $i = 1, 2$, respectively.

Note that the equality $p_2 p_1 = \sigma_2(p_1)p_2 + \delta_2(p_1)$ holds if the expressions

$$\begin{aligned} p_2 p_1 &= (f(y)x + g(y))(\gamma y + \beta) \\ &= f(y)x\gamma y + f(y)x\beta + g(y)\gamma y + g(y)\beta \\ &= \gamma f(y)xy + \beta f(y)x + \gamma g(y)y + \beta g(y) \\ &= \gamma f(y)(yx - y^2) + \beta f(y)x + \gamma g(y)y + \beta g(y) \\ &= \gamma f(y)yx - \gamma f(y)y^2 + \beta f(y)x + \gamma g(y)y + \beta g(y), \end{aligned}$$

and

$$\begin{aligned} \sigma_2(p_1)p_2 + \delta_2(p_1) &= \sigma_2(\gamma y + \beta)(f(y)x + g(y)) + \delta_2(\gamma y + \beta) \\ &= (\sigma_2(\gamma y) + \sigma_2(\beta))(f(y)x + g(y)) + \delta_2(\gamma y) + \delta_2(\beta) \\ &= \gamma y f(y)x + \gamma y g(y) + \beta f(y)x + \beta g(y) + \sigma_2(\gamma)\delta_2(y) + \delta_2(\gamma)y + \delta_2(\beta) \\ &= \gamma f(y)yx + \gamma g(y)y + \beta f(y)x + \beta g(y) - \gamma y^2 \end{aligned}$$

coincide, i.e., $\gamma f(y) = \gamma$, whence $f(y) \in \Bbbk^*$ and necessarily $f(y) = 1$. Thus, $p_2 = x + g(y)$ and $p_1 = \gamma y + \beta$. It is straightforward to see that $p_1 r = \sigma_1(r)p_1 + \delta_1(r)$ and $p_2 r = \sigma_1(r)p_2 + \delta_2(r)$ for any $r \in \Bbbk$.

Now, since ϕ is an automorphism, there exists ϕ^{-1} such that $y = \phi^{-1}(\phi(y))$, that is,

$$y = \phi^{-1}(\phi(y)) = \phi^{-1}(p_1) = \phi^{-1}(\gamma y + \beta) = \phi^{-1}(\gamma y) + \phi^{-1}(\beta).$$

From the last expression, it follows that $\phi^{-1}(\beta) = 0$, so $\beta = 0$. In addition, $\phi^{-1}(y) = \gamma^{-1} y$.

In the same way, let us consider $x = \phi^{-1}(\phi(x))$. Then

$$x = \phi^{-1}(\phi(x)) = \phi^{-1}(p_2) = \phi^{-1}(x + g(y)) = \phi^{-1}(x) + \phi^{-1}(g(y)) = \phi^{-1}(x) + g(\phi^{-1}(y))$$
$$= \phi^{-1}(x) + g(\gamma^{-1}y),$$

so $\phi^{-1}(x) = x - g(\gamma^{-1}y)$. We conclude that the homomorphism defined by $\phi(y) = p_1 = \gamma y$ and $\phi(x) = p_2 = x + g(y)$, with $g(y) \in \Bbbk[y]$, is an automorphism of Jordan plane $\mathscr{J}$. □

EXAMPLE 2.10. Shirikov [Shi05, Proposition 3.1] characterized the automorphisms of the Jordan's plane $\mathscr{J}(\Bbbk)$ defined over a field $\Bbbk$ of characteristic zero (he does not consider $\mathscr{J}(\Bbbk)$ as an iterated Ore extension). He proved that these automorphisms are given by $\phi(x) = \gamma x + g(y)$ and $\phi(y) = \gamma y$, for some $\gamma \in \Bbbk^*$ and $g(y) \in \Bbbk[y]$. If we compare with Theorem 2.23, there is a little difference concerning the coefficient of the indeterminate x. This is because our treatment concerns the definition of homomorphism through the notion of cv-polynomial. Next, we present an example of an automorphism of $\mathscr{J}(\Bbbk)$ that does not correspond to a pair of cv-polynomials.

Consider the automorphism ϕ of $\mathscr{J}(\Bbbk)$ given by the polynomials $\phi(x) = p_2 = 2x + y^2$ and $\phi(y) = p_1 = 2y$. Since

$$p_2p_1 = \sigma_2(p_1)p_2 + \delta_2(p_1) = (2x + y^2)(2y) = 4xy + 2y^3$$
$$= 4(\sigma_2(y)x + \delta_2(y)) + 2y^3 = 4yx - 4y^2 + 2y^3,$$
$$\sigma_2(2y)(2x + y^2) + \delta_2(2y) = 2y(2x + y^2) - 2y^2 = 4yx + 2y^3 - 2y^2,$$

it follows that $p_2p_1 \neq \sigma_2(2y)(2x + y^2) + \delta_2(2y)$, that is, condition (2.4) does not hold.

THEOREM 2.24. *If ϕ is an automorphism of the q-*meromorphic Weyl algebra $\mathscr{J}_q(\Bbbk)$ *(Section 1.2.1), then the corresponding cv-polynomials are given by $\phi(y) = p_1 = \gamma y$ and $\phi(x) = p_2 = x$ with respect to (σ_i, δ_i) $(i = 1, 2)$, respectively, for some element $\gamma \in \Bbbk^*$.*

Proof. Let $\phi : \mathscr{J}_q(\Bbbk) \cong \Bbbk[y][x; \sigma_2, \delta_2] \to \mathscr{J}_q(\Bbbk) \cong \Bbbk[y][x; \sigma_2, \delta_2]$ be an automorphism of $\mathscr{J}_q(\Bbbk)$. Then ϕ acts as an isomorphism on the iterated Ore extension, and by Theorem 2.22, we have that the associated cv-polynomials have a degree of 1. We define $\phi(y) := p_1 = \gamma y + \beta$, where γ and β are non-zero elements in $\Bbbk$, and $\phi(x) := p_2 = f(y)x + g(y)$, such that $f(y)$ and $g(y)$ are elements of $\Bbbk[y]$. Let us see that the given polynomials satisfy the conditions (2.4) - (2.6):

$$p_2p_1 = (f(y)x + g(y))(\gamma y + \beta)$$
$$= f(y)x\gamma y + f(y)x\beta + g(y)\gamma y + g(y)\beta$$
$$= \gamma f(y)xy + \beta f(y)x + \gamma g(y)y + \beta g(y)$$
$$= \gamma f(y)(qyx - y^2) + \beta f(y)x + \gamma g(y)y + \beta g(y)$$
$$= \gamma q f(y)yx - \gamma f(y)y^2 + \beta f(y)x + \gamma g(y)y + \beta g(y),$$
$$\sigma_2(p_1)p_2 + \delta_2(p_1) = \sigma_2(\gamma y + \beta)(f(y)x + g(y)) + \delta_2(\gamma y + \beta)$$
$$= (\sigma_2(\gamma y) + \sigma_2(\beta))(f(y)x + g(y)) + \delta_2(\gamma y) + \delta_2(\beta)$$

$$= \gamma q y f(y)x + \gamma q y g(y) + \beta f(y)x + \beta g(y) + \sigma_2(\gamma)\delta_2(y) + \delta_2(\gamma)y + \delta_2(\beta)$$
$$= \gamma q f(y) yx + \gamma q g(y) y + \beta f(y)x + \beta g(y) - \gamma y^2.$$

It is necessary to ensure that the equality $\gamma f(y) = \gamma$ holds, that is, $f(y)$ is a unit, and so $f(y) = 1$. In addition, we must satisfy $\gamma g(y)y = \gamma q g(y)y$. Given that $q \neq 1$, then $g(y) = 0$. In this way, $p_2 = x$ and $p_1 = \gamma y + \beta$. It is straightforward to see that for any $r \in \Bbbk$, $p_1 r = \sigma_1(r)p_1 + \delta_1(r)$ and $p_2 r = \sigma_1(r)p_2 + \delta_2(r)$ hold. Hence, conditions (2.4), (2.5) and (2.6) are satisfied.

Since ϕ is an automorphism, there exists ϕ^{-1} such that $y = \phi^{-1}(\phi(y))$. Then

$$y = \phi^{-1}(\phi(y)) = \phi^{-1}(\phi(p_1)) = \phi^{-1}(\gamma y + \beta) = \phi^{-1}(\gamma y) + \phi^{-1}(\beta).$$

From the last expression, $\phi^{-1}(\beta) = 0$, and so $\beta = 0$. Also, we get $\phi^{-1}(y) = \gamma^{-1}y$.

Similarly, if we consider $x = \phi^{-1}(\phi(x))$, then $x = \phi^{-1}(\phi(x)) = \phi^{-1}(p_2) = \phi^{-1}(x)$, and so $\phi^{-1}(x) = x$, i.e., we conclude that the homomorphism described by $\phi(y) = p_1 = \gamma y$ and $\phi(x) = p_2 = x$ is an automorphism of the q-skew Jordan plane $\mathscr{J}_q$. □

DEFINITION 2.5. Consider $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$. The σ_2-derivation δ_2 is *quasi-algebraic* if satisfies one of the following conditions:

(i) There exist constants $q_i(x_1) \in R[x_1;\sigma_1,\delta_1]$ with $q_n(x_1) = 1$, such that $\sum_{i=1}^{n} q_i(x_1)\delta_2^i$ is an σ_2^n-inner derivation.

(ii) There exist constants $q_i(x_1) \in R[x_1;\sigma_1,\delta_1]$, $1 \leq i \leq n$ not all zero, such that $\sum_{i=1}^{n} q_i(x_1)\delta_2^i$ is an σ_2'-inner derivation for some endomorphism σ_2'.

THEOREM 2.25. *Let $S' = R[x_1';\sigma_1',\delta_1'][x_2';\sigma_2',\delta_2']$ and $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$. Assume that $\sigma_i\delta_i = -\delta_i\sigma_i$ for $i = 1,2$. Suppose that one of the following two conditions*

(1) *p_2 can be written as $p_2 = q(x_2) + f(x_1)$ and σ_i is an automorphism with $o(\sigma_i) = \infty$, or*

(2) *δ_i is not quasi-algebraic,*

is true for both values of i. If $\phi : S' \to S$ and $\psi : S \to S'$ are injective ring homomorphisms, then both are isomorphisms.

Proof. Assume that one of the homomorphisms ϕ or ψ is not an isomorphism. Then $\phi\circ\psi : S \to S$ is not an isomorphism, and by Theorem 2.22, the associated cv-polynomials p_1 and p_2 have degree ≥ 2. First, we denote $p_2 = \sum_{j=0}^{m} b_j x_2^j + f(x_1)$ with $m \geq 2$. Suppose that condition (1) holds. By Theorem 2.21(1), $b_j\sigma_2^j(-) = \sigma_2(-)b_j$, and since σ_2 is an automorphism, $\sigma_2^{j-1} = \sigma_{2(b_j^{-1})}$, for every $1 \leq j \leq m$, so $o(\sigma_2) \leq j - 1 \leq m - 1 < \infty$, a contradiction. In an analogous way, for $p_1 = \sum_{i=0}^{n} a_i x_1^i$ with $n \geq 2$, $a_i\sigma_1^i = \sigma_1 a_i$, and this implies that $\sigma_1^{i-1} = \sigma_{1(a_i^{-1})}$, for every $1 \leq i \leq n$, thus $o(\sigma_1) \leq i - 1 \leq n - 1 < \infty$.

Suppose that δ_2 is not quasi-algebraic. Then δ_2 is not algebraic and by Theorem 2.21(2), $\delta_2 = (p_2 - B_{00})(\delta_2) + \delta_{2(B_{00},\sigma_2)}$, where $B_{00} = \sum_{i=0}^{n} b_{i0} x_1^i x_2^0$. Since p_2 has degree $m \geq 2$,

$$\delta_2 - \sum_{i=0}^{n}\sum_{j=1}^{m} b_{ij} x_1^i \delta_2^j = \delta_2 - \sum_{j=1}^{m} q_j(x_1)^i \delta_2^j = \delta_{2(B_{00},\sigma_2)},$$

whence δ_2 is a σ_2'-inner derivation with $\sigma_2' = \sigma_2^m$, i.e, δ_2 is quasi-algebraic, a contradiction. Following the same reasoning, Theorem 2.21(2) implies that $\delta_1 = (p_1 - a_0)\delta_1 + \delta_{1(a_0,\sigma_1')}$, which shows that δ_1 is quasi-algebraic, and we get a contradiction. □

REMARK 10. It is worth asking whether the theorem is still valid if we change the hypothesis (2) so that δ_2 is not algebraic. This question was studied by Lam and Leroy [LL92, Remark 5.7], where they showed that with a quasi-algebraic but not algebraic derivation two injective homomorphisms can be constructed, but these do not lead to an isomorphism. For two-step iterated Ore extensions, if we fix the ring $R[x_1]$ for both Ore extensions, then we obtain Lam aand Leroy's result.

EXAMPLE 2.11. For $\mathbb{k}$ a field of characteristic zero and an element $f(x) \in \mathbb{k}[x]$, Alev and Dumas [AD97] considered the algebra Λ defined as the Ore extension

$$\Lambda := \Lambda(f(x)) := \mathbb{k}[x]\left[y; \delta := f(x)\frac{d}{dx}\right] = \mathbb{k}\langle x, y \mid yx - xy = f(x)\rangle.$$

Note that if $f(x) = 0$, then $\Lambda(0) = \mathbb{k}[x,y]$, and if $f(x) = 1$, then $\Lambda(1)$ is the first Weyl algebra. They investigated the group of automorphisms of Λ, and showed that for elements $f(x)$, $g(x) \in \mathbb{k}[x]$, $\Lambda(f(x)) \cong \Lambda(g(x))$ if and only if $g(x) = \lambda f(\alpha x + \beta)$ for some elements $\lambda, \alpha \in \mathbb{k}^*$ and $\beta \in \mathbb{k}$ [AD97, Proposition 3.6(i)]. As one can check after some computations, for some particular values of λ, α and β, the theory of cv-polynomials developed above illustrates Alev and Dumas's result.

To finish this section, we present some remarks about eigenvector interpretations for the coefficient vector of a cv-polynomial formulated in Proposition 2.12. The following example illustrates this result.

EXAMPLE 2.12. [LL92, Example (2.10)] The polynomial $p(x) = x^2 + b_0 \in S = D[x;\sigma,\delta]$ is a cv-polynomial with respect to the quasi-derivation $(\sigma^2, \delta^2 + \delta_{(c,\sigma^2)})$ on D, provided that $\sigma\delta = -\delta\sigma$. By Proposition 2.12(2), for any $a \in D$ and $j = 1, 2, \ldots, n$, we get $\sum_{i=j}^{n} b_i f_j^i(a) = b_n \sigma^n(a) b_n^{-1} b_j$. In particular, for $j = 1$,

$$\sum_{i=1}^{2} b_i f_1^i(a) = b_1 f_1^1(a) + b_2 f_1^2(a) = 0\sigma(a) + 1\sigma\delta(a) + 1\delta\sigma(a) = 0,$$

and for $j = 2$, $\sum_{i=2}^{2} b_i f_2^i(a) = b_2 f_2^2(a) = \sigma^2(a)$. Then, for $j = 1$, $\sum_{i=1}^{2} b_i f_j^i(a) = 0 = b_2\sigma^2(a)b_2^{-1}0$, and for $j = 2$, $\sum_{i=2}^{2} b_i f_2^i(a) = \sigma^2(a) = b_2\sigma^2(a)b_2^{-1}b_2$ as Proposition 2.12(2) asserts.

On the other hand, following Proposition 2.12(3), $[b_1 \;\; b_2]$ is a left eigenvector for each $a \in D$

of the matrix $M_2(a)$ with eigenvalue $b_2\sigma^2(a)b_2^{-1}$, whence

$$\begin{aligned}
[b_1 \;\; b_2]\begin{bmatrix} f_1^1(a) & f_2^1(a) \\ f_1^2(a) & f_2^2(a) \end{bmatrix} &= b_2\sigma^2(a)b_2^{-1}[b_1 \;\; b_2] \\
[0 \;\; 1]\begin{bmatrix} \sigma(a) & 0 \\ \sigma\delta(a)+\delta\sigma(a) & \sigma^2(a) \end{bmatrix} &= \sigma^2(a)[0 \;\; 1] \\
[\sigma\delta(a)+\delta\sigma(a) \;\; \sigma^2(a)] &= [0 \;\; \sigma^2(a)] \\
[0 \;\; \sigma^2(a)] &= [0 \;\; \sigma^2(a)].
\end{aligned}$$

Note that without the condition that $\sigma\delta(a) = -\delta\sigma(a)$, Proposition 2.12 would be false. Examples 2.13 illustrates this situation.

EXAMPLE 2.13. (i) Consider $p(x) = b_2x^2 + b_1x + b_0 \in S = D[x;\sigma,\delta]$ (Example (2.5)). This polynomial is a cv-polynomial with respect to the quasi-derivation $(\sigma', \delta' = b_2\delta^2 + b_1\delta + \delta_{(b_0,\sigma')})$ whenever $\sigma\delta = -\delta\sigma$ and $b_i\sigma^i(a) = \sigma'(a)b_i$. Now, for any $a \in D$ and $j = 1,2,\ldots,n$, by Proposition 2.12(2), we get $\sum_{i=j}^{n} b_i f_j^i(a) = b_n\sigma^n(a)b_n^{-1}b_j$. For $j = 1$,

$$\sum_{i=1}^{2} b_i f_1^i(a) = b_1 f_1^1(a) + b_2 f_1^2(a) = b_1\sigma(a) + b_2\sigma\delta(a) + b_2\delta\sigma(a) = b_1\sigma(a).$$

while for $j = 2$, $\sum_{i=2}^{2} b_i f_2^i(a) = b_2 f_2^2(a) = b_2\sigma^2(a)$. In this way, for $j = 1$, we obtain $\sum_{i=1}^{2} b_i f_1^i(a) = b_1\sigma(a) \neq b_2\sigma^2(a)b_2^{-1}b_1$, and for $j = 2$, $\sum_{i=2}^{2} b_i f_2^i(a) = b_2\sigma^2(a) = b_2\sigma^2(a)b_2^{-1}b_2$.

According to Proposition 2.12(3), $[b_1 \;\; b_2]$ is a left eigenvector for each $a \in D$ of the matrix $M_2(a)$ with eigenvalue $b_2\sigma^2(a)b_2^{-1}$, which implies that

$$\begin{aligned}
[b_1 \;\; b_2]\begin{bmatrix} f_1^1(a) & f_2^1(a) \\ f_1^2(a) & f_2^2(a) \end{bmatrix} &= b_2\sigma^2(a)b_2^{-1}[b_1 \;\; b_2] \\
[b_1 \;\; b_2]\begin{bmatrix} \sigma(a) & 0 \\ \sigma\delta(a)+\delta\sigma(a) & \sigma^2(a) \end{bmatrix} &= b_2\sigma^2(a)b_2^{-1}[b_1 \;\; b_2] \\
[b_1\sigma(a) + b_2\sigma\delta(a) + b_2\delta\sigma(a) \;\; b_2\sigma^2(a)] &= [b_2\sigma^2(a)b_2^{-1}b_1 \;\; b_2\sigma^2(a)] \\
[b_1\sigma(a) \;\; b_2\sigma^2(a)] &= [b_2\sigma^2(a)b_2^{-1}b_1 \;\; b_2\sigma^2(a)].
\end{aligned}$$

However, it is clear that this equality is not necessarily true.

(ii) Consider the polynomial $p = b_2x^2 + b_1x + b_0 \in \mathscr{J}_q(\Bbbk) \cong \Bbbk[y][x;\sigma,\delta]$. Suppose that p is a cv-polynomial with respect to the quasi-derivation $(\sigma', \delta' = b_2\delta^2 + b_1\delta + \delta_{(b_0,\sigma')})$, whenever $\sigma\delta = -\delta\sigma$ and $b_i\sigma^i(y) = \sigma'(y)b_i$. For $y \in \Bbbk[y]$, and $j = 1,2,\ldots,n$, by Proposition 2.12(2), $\sum_{i=j}^{n} b_i f_j^i(y) = b_n\sigma^n(y)b_n^{-1}b_j$, so for $j = 1$,

$$\sum_{i=1}^{2} b_i f_1^i(y) = b_1 f_1^1(y) + b_2 f_1^2(y) = b_1\sigma(y) + b_2\sigma\delta(y) + b_2\delta\sigma(y) = b_1\sigma(y) = b_1qy$$

and for $j=2$, $\sum_{i=2}^{2} b_i f_2^i(y) = b_2 f_2^2(y) = b_2\sigma^2(y) = b_2 q^2 y$. In this way, for $j=1$,

$$\sum_{i=1}^{2} b_i f_1^i(y) = b_1\sigma(y) = b_1 q y \neq b_2\sigma^2(y)b_2^{-1}b_1 = b_1 q^2 y$$

and for $j=2$, $\sum_{i=2}^{2} b_i f_2^i(y) = b_2\sigma^2(y) = b_2 q^2 y = b_2\sigma^2(y)b_2^{-1}b_2$.

Now, Proposition 2.12(3) asserts that $[b_1 \;\; b_2]$ is a left eigenvector for $y \in \Bbbk[y]$ of the matrix $M_2(y)$ with eigenvalue $b_2\sigma^2(y)b_2^{-1}$, that is,

$$[b_1 \;\; b_2]\begin{bmatrix} f_1^1(y) & f_2^1(y) \\ f_1^2(y) & f_2^2(y) \end{bmatrix} = b_2\sigma^2(y)b_2^{-1}[b_1 \;\; b_2]$$

$$[b_1 \;\; b_2]\begin{bmatrix} \sigma(y) & 0 \\ \sigma\delta(y)+\delta\sigma(y) & \sigma^2(y) \end{bmatrix} = b_2\sigma^2(y)b_2^{-1}[b_1 \;\; b_2]$$

$$[b_1\sigma(y) + b_2\sigma\delta(y) + b_2\delta\sigma(y) \;\; b_2\sigma^2(y)] = [b_2\sigma^2(y)b_2^{-1}b_1 \;\; b_2\sigma^2(y)]$$

$$[b_1\sigma(y) \;\; b_2\sigma^2(y)] \neq [b_2\sigma^2(y)b_2^{-1}b_1 \;\; b_2\sigma^2(y)]$$

$$[b_1 q y \;\; b_2 q^2 y] \neq [b_1 q^2 y \;\; b_2 q^2 y].$$

Our version of [LL92, Theorem 2.16] taking into account the previous observation and removing the hypothesis that the Ore extension is over a division ring is the content of the following result.

THEOREM 2.26. *Let $p_1(x) = \sum_{i=0}^{n} a_i x_1^i \in S = R[x_1;\sigma_1,\delta_1]$ be a polynomial of degree $n \geq 0$. Then the following assertions are equivalent:*

(1) *$p_1(x_1)$ is a cv-polynomial.*

(2) *For any $r \in R$, and $j = 1,2,\ldots,n$, we have $\sum_{i=j}^{n} a_i f_j^i(r) = a_j\sigma_1^j(r)$ whenever $\sigma_1\delta_1 = -\delta_1\sigma_1$.*

(3) *$[a_1,\ldots,0],\ldots,[0,\ldots,a_i,\ldots,0]$ are left eigenvectors for each matrix $M_n(r)$, where $r \in R$.*

(4) *$p_1(x_1)R \subseteq Rp_1(x_1) + R$.*

If these conditions hold, then $up_1(x_1) + c$ is also a cv-polynomial, for any $u, c \in R$.

Proof. If $p_1(x_1)$ has degree $n = 0$, the conditions follow immediately.

- (1) $\Rightarrow$ (2) Let $p_1(x)$ be a cv-polynomial with respect to (σ_1',δ_1'). For any $r \in R$,

$$p_1(x_1)r = \sum_{i=0}^{n} a_i x_1^i r = \sum_{i=0}^{n} a_i \sum_{j=0}^{i} f_j^i(r)x_1^j = \sum_{j=0}^{n}\left(\sum_{i=j}^{n} a_i f_j^i(r)\right)x_1^j. \tag{2.28}$$

Now, by Theorem 2.19,

$$p_1(x_1)r = \sigma_1'(r)p_1(x_1) + \delta_1' = a_j\sigma_1^j a_j^{-1}\sum_{i=0}^{n} a_i x_1^i + \delta_1'(r).$$

These two expressions show that identity is satisfied for each $j = 1,2,\ldots,n$, whenever that $\sigma_1\delta_1 = -\delta_1\sigma_1$.

- (2) $\Rightarrow$ (3) The matrix $M_n(r)$ that is composed by $(i,j)-$entry of the $f_j^i(r)$ with $1 \leq i,j \leq n$, so the equation in Part (2) can be expressed in the form

$$\begin{aligned}
[a_1,\ldots,0]M_n(r) &= a_1\sigma_1 a_1^{-1}[a_1,\ldots,0] \\
&\vdots \\
[0,\ldots,a_i,\ldots,0]M_n(r) &= a_i\sigma_1^i a_i^{-1}[0,\ldots,a_i,\ldots,0],
\end{aligned}$$

for each $r \in R$ and $0 \leq i \leq n$, and $[0,\ldots,a_i,\ldots,0]$ is a left eigenvector for $M_n(r)$ with eigenvalue $a_i\sigma_1^i a_i^{-1}$.

- (3) $\Rightarrow$ (4) For every $r \in R$,

$$[0,\ldots,a_i,\ldots,0]M_n(r) = \alpha(r)[0,\ldots,a_i,\ldots,0],$$

for some eigenvalue $\alpha(r) \in R$. Thus, for $1 \leq j \leq n$, we obtain $\sum_{i=j}^{n} a_i f_j^i(r) = (\alpha(r))a_j$. Using this fact in the expression (2.28),

$$p_1(x_1)r \in \sum_{j=1}^{n} (\alpha(r))a_j x_1^j + R \subseteq Rp_1(x_1) + R, \quad \text{for any } r \in R.$$

- (4) $\Rightarrow$ (1) For any $r \in R$, we have determined constants $r_1, r_2 \in R$ such that $p_1(x_1)r = r_1p_1(x_1) + r_2$. By defining $\sigma_1'(r) = r_1$ and $\delta_1'(r) = r_2$, it is easy to see that (σ_1', δ_1') is a quasi-derivation on R, whence $p_1(x_1)$ is a cv-polynomial with respect to (σ_1', δ_1').

- To prove that the last statement of the theorem is true, consider Part (4). We have $p_1(x_1)R \subseteq Rp_1(x_1) + R$, and so

$$(up_1(x_1) + c)R \subseteq u(Rp_1(x_1) + R) + cR \subseteq R(up_1(x_1) + c) + R,$$

since this is also a cv-polynomial.

□

EXAMPLE 2.14. Consider again Example 2.13. For any $r \in R$ and $j = 1,2,\ldots,n$, Theorem 2.26(2)

asserts that $\sum_{i=j}^{n} a_i f_j^i(r) = a_j \sigma^j(a)$, and hence

$$\sum_{i=1}^{2} a_i f_1^i(r) = a_1 f_1^1(r) + a_2 f_1^2(r) = a_1\sigma(r) + a_2\sigma\delta(r) + a_2\delta\sigma(r) = a_1\sigma(a), \quad j = 1,$$
$$\sum_{i=2}^{2} a_i f_2^i(r) = a_2 f_2^2(r) = a_2\sigma^2(r), \quad j = 2,$$

while $\sum_{i=1}^{2} a_i f_1^i(r) = b_1\sigma(r) = b_1\sigma(r)$, and $\sum_{i=2}^{2} a_i f_2^i(r) = a_2\sigma^2(r) = a_2\sigma^2(r)$.

By Theorem 2.26(3), $[a_1 \ \ 0]$ and $[0 \ \ a_2]$ are left eigenvectors for each $r \in R$ of the matrix $M_2(r)$ with eigenvalues $a_1\sigma(r)a_1^{-1}$ and $a_2\sigma(r)^2a_2^{-1}$, respectively, so that

$$[a_1 \ \ 0]\begin{bmatrix} f_1^1(r) & f_2^1(r) \\ f_1^2(r) & f_2^2(r) \end{bmatrix} = a_1\sigma(r)a_1^{-1}[a_1 \ \ 0]$$
$$[a_1 \ \ 0]\begin{bmatrix} \sigma(r) & 0 \\ \sigma\delta(r)+\delta\sigma(r) & \sigma^2(r) \end{bmatrix} = a_1\sigma(r)a_1^{-1}[a_1 \ \ 0]$$
$$[a_1\sigma(r) \ \ 0] = [a_1\sigma(r)a_1^{-1}a_1 \ \ 0]$$
$$[a_1\sigma(r) \ \ 0] = [a_1\sigma(r) \ \ 0],$$
$$[0 \ \ a_2]\begin{bmatrix} f_1^1(r) & f_2^1(r) \\ f_1^2(r) & f_2^2(r) \end{bmatrix} = a_2\sigma(r)^2a_2^{-1}[0 \ \ a_2]$$
$$[0 \ \ a_2]\begin{bmatrix} \sigma(r) & 0 \\ \sigma\delta(r)+\delta\sigma(r) & \sigma^2(r) \end{bmatrix} = a_2\sigma(r)^2a_2^{-1}[0 \ \ a_2]$$
$$[a_2\sigma\delta(r)+\delta\sigma(r) \ \ a_2\sigma^2(r)] = [0 \ \ a_2\sigma(r)^2a_2^{-1}a_2]$$
$$[0 \ \ a_2\sigma^2(r)] = [0 \ \ a_2\sigma^2(r)],$$

which shows that Theorem 2.26(4) holds.

REMARK 11. Once more again, in a personal communication, Professor Lopes told us that in [BLO15] he studied the isomorphisms and automorphisms of Ore extensions of derivation type over a polynomial ring in one indeterminate. The results presented above can be illustrated with those corresponding in his paper.

2.4.2 THREE-STEP ITERATED ORE EXTENSIONS

In this section, we consider the question on homomorphisms and cv-polynomials of three-step iterated Ore extensions. We follow the ideas presented in Section 2.4.1.

DEFINITION 2.6. Consider two iterated Ore extensions with three indeterminates given by $S' = R[x_1';\sigma_1',\delta_1'][x_2';\sigma_2',\delta_2'][x_3';\sigma_3',\delta_3']$ and $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2][x_3;\sigma_3,\delta_3]$. The homomorphism $\phi : S' \to S$ defined as $\phi(x_1') := p_1(x_1) = p_1 \in R[x_1;\sigma_1,\delta_1]$, $\phi(x_2') := p_2(x_1,x_2) = p_2 \in$

$R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$ and $\phi(x_3') := p_3(x_1,x_2,x_3) = p_3 \in S$, satisfies

$$\begin{aligned}
\phi(x_3'x_2') &= \sigma_3'(\phi(x_2'))p_3 + \delta_3'(\phi(x_2')),\\
\phi(x_3'x_1') &= \sigma_3'(\phi(x_1'))p_3 + \delta_3'(\phi(x_1')),\\
\phi(x_2'x_1') &= \sigma_2'(\phi(x_1'))p_2 + \delta_2'(\phi(x_1')),\\
\phi(x_1'r) &= \sigma_1'(r)p_1 + \delta_1'(r),\\
\phi(x_2'r) &= \sigma_2'(r)p_2 + \delta_2'(r),\\
\phi(x_3'r) &= \sigma_3'(r)p_3 + \delta_3'(r),
\end{aligned}$$

for $r \in R$, or equivalently,

$$\begin{aligned}
p_3p_2 &= \sigma_3'(p_2)p_3 + \delta_3'(p_2),\\
p_3p_1 &= \sigma_3'(p_1)p_3 + \delta_3'(p_1),\\
p_2p_1 &= \sigma_2'(p_1)p_2 + \delta_2'(p_1),\\
p_1r &= \sigma_1'(r)p_1 + \delta_1'(r),\\
p_2r &= \sigma_2'(r)p_2 + \delta_2'(r),\\
p_3r &= \sigma_3'(r)p_3 + \delta_3'(r).
\end{aligned}$$

The polynomials p_i's are the *cv-polynomials for iterated Ore extensions with respect to the quasi-derivations* (σ_i',δ_i') for $i = 1,2,3$, respectively.

Conversely, if a polynomial p_i for $i = 1,2,3$ satisfies the above conditions for quasi-derivations in the sense above, then we can define $\phi : S' \to S$ by $\phi(x_i') = p_i$ with $i = 1,2,3$.

EXAMPLE 2.15. Let $\phi : S' \to S$ be the homomorphism as in Definition 2.6. We define

$$\begin{aligned}
\phi(x_1') &= p_1(x_1) = a_1x_1 + a_0,\\
\phi(x_2') &= p_2(x_1,x_2) = b_{11}x_1x_2 + b_{00},\\
\phi(x_3') &= p_3(x_1,x_2,x_3) = c_{111}x_1x_2x_3 + c_{000}.
\end{aligned}$$

Let us see that these are cv-polynomials with respect to the quasi-derivations (σ_i',δ_i'), $i = 1,2,3$. From Example 2.8, with $p_2 = f(x_1)x_2 + b_{00}$ where $f(x_1) = b_{11}x_1$, we know that $\sigma_1' = a_1\sigma_1a_1^{-1}$, $\delta_1' = a_1\delta_1 + \delta_{1(a_0,\sigma_1')}$, $\delta_2' = f(x_1)\delta_2 + \delta_{2(b_{00},\sigma_2')}$ and $f(x_1)\sigma_2(-) = \sigma_2'(-)f(x_1)$.

Let us show that $p_3p_1 = \sigma_3'(p_1)p_3 + \delta_3'(p_1)$. Consider $p_3 = q(x_1,x_2)x_3 + c_{000}$ with $q(x_1,x_2) = c_{111}x_1x_2$. Then

$$\begin{aligned}
p_3p_1 &= (c_{111}x_1x_2x_3 + c_{000})(a_1x_1 + a_0) = c_{111}x_1x_2x_3a_1x_1 + c_{111}x_1x_2x_3a_0 + c_{000}a_1x_1 + c_{000}a_0\\
&= c_{111}x_1x_2\sigma_3(a_1x_1)x_3 + c_{111}x_1x_2\delta_3(a_1x_1) + c_{111}x_1x_2\sigma_3(a_0)x_3\\
&\quad + c_{111}x_1x_2\delta_3(a_0) + c_{000}a_1x_1 + c_{000}a_0\\
&= q(x_1,x_2)\sigma_3(a_1x_1)x_3 + q(x_1,x_2)\delta_3(a_1x_1) + q(x_1,x_2)\sigma_3(a_0)x_3\\
&\quad + q(x_1,x_2)\delta_3(a_0) + c_{000}a_1x_1 + c_{000}a_0. \qquad (2.29)
\end{aligned}$$

Notice that

$$\begin{aligned}\sigma_3'(p_1)p_3+\delta_3'(p_1) &= \sigma_3'(a_1x_1+a_0)(c_{111}x_1x_2x_3+c_{000})+\delta_3'(a_1x_1+a_0)\\ &= \sigma_3'(a_1x_1)c_{111}x_1x_2x_3+\sigma_3'(a_1x_1)c_{000}+\sigma_3'(a_0)c_{111}x_1x_2x_3\\ &\quad+\sigma_3'(a_0)c_{000}+\delta_3'(a_1x_1)+\delta_3'(a_0)\\ &= \sigma_3'(a_1x_1)q(x_1,x_2)x_3+\sigma_3'(a_1x_1)c_{000}+\sigma_3'(a_0)q(x_1,x_2)x_3\\ &\quad+\sigma_3'(a_0)c_{000}+\delta_3'(a_1x_1)+\delta_3'(a_0). \qquad (2.30)\end{aligned}$$

With the aim of that conditions (2.29) and (2.30) coincide, let $\delta_3' := q(x_1,x_2)\delta_3+\delta_{3(c_{000},\sigma_3')}$ and assume that the condition $q(x_1,x_2)\sigma_3(-)=\sigma_3'(-)q(x_1,x_2)$ holds. For the equality $p_3p_2=\sigma_3'(p_2)p_3+\delta_3'(p_2)$, we get

$$\begin{aligned}p_3p_2 &= (c_{111}x_1x_2x_3+c_{000})(b_{11}x_1x_2+b_{00})\\ &= c_{111}x_1x_2x_3b_{11}x_1x_2+c_{111}x_1x_2x_3b_{00}+c_{000}b_{11}x_1x_2+c_{000}b_{00}\\ &= c_{111}x_1x_2\sigma_3(b_{11}x_1x_2)x_3+c_{111}x_1x_2\delta_3(b_{11}x_1x_2)+c_{111}x_1x_2\sigma_3(b_{00})x_3\\ &\quad+c_{111}x_1x_2\delta_3(b_{00})+c_{000}b_{11}x_1x_2+c_{000}b_{00}\\ &= q(x_1,x_2)\sigma_3(b_{11}x_1x_2)x_3+q(x_1,x_2)\delta_3(b_{11}x_1x_2)+q(x_1,x_2)\sigma_3(b_{00})x_3\\ &\quad+q(x_1,x_2)\delta_3(b_{00})+c_{000}b_{11}x_1x_2+c_{000}b_{00},\end{aligned}$$

and

$$\begin{aligned}\sigma_3'(p_2)p_3+\delta_3'(p_2) &= \sigma_3'(b_{11}x_1x_2+b_{00})(c_{111}x_1x_2x_3+c_{000})+\delta_3'(b_{11}x_1x_2+b_{00})\\ &= \sigma_3'(b_{11}x_1x_2)c_{111}x_1x_2x_3+\sigma_3'(b_{11}x_1x_2)c_{000}\\ &\quad+\sigma_3'(b_{00})c_{111}x_1x_2x_3+\sigma_3'(b_{00})c_{000}+\delta_3'(b_{11}x_1x_2)+\delta_3'(b_{00})\\ &= \sigma_3'(b_{11}x_1x_2)q(x_1,x_2)x_3+\sigma_3'(b_{11}x_1x_2)c_{000}\\ &\quad+\sigma_3'(b_{00})q(x_1,x_2)x_3+\sigma_3'(b_{00})c_{000}+\delta_3'(b_{11}x_1x_2)+\delta_3'(b_{00}).\end{aligned}$$

It is straightforward to see that by using δ_3' and the condition assumed above, we get the desired equality.

As in the case of two-step iterated Ore extensions, a polynomial $p(x_1,x_2,x_3)$ belonging to the iterated Ore extension $S=R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2][x_3;\sigma_3,\delta_3]$ is said to be *right invariant* if

$$p(x_1,x_2,x_3)R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]\subseteq R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]p(x_1,x_2,x_3),$$

and *right semi-invariant* if $p(x_1,x_2,x_3)R\subseteq Rp(x_1,x_2,x_3)$.

The following result is the natural extension of Theorem 2.20.

THEOREM 2.27. *Let x_3^n be a a monomial invariant of degree $n\geq 1$. Let $p_3(x_1,x_2,x_3)$ with $p_3(x_1,x_2,x_3)=f(x_1,x_2)x_3^n+g(x_1,x_2,x_3)$ where the degree of $g(x_1,x_2,x_3)$ is less than or equal to n. Then:*

(1) *p_3 is a cv-polynomial with respect to (σ_3',δ_3') if and only if $g(x_1,x_2,x_3)$ is a cv-polynomial with respect to (σ_3',δ_3') and $f(x_1,x_2)\sigma_3^n(-)=\sigma_3'(-)f(x_1,x_2)$.*

(2) *If p_3 is a left multiple of $f(x_1,x_2)$, then p_3 is a cv-polynomial if and only if is invariant.*

Proof. The ideas are completely analogous to the proof of Theorem 2.20.

(1) First, assume that p_3 is a cv-polynomial with respect to the quasi-derivation (σ_3',δ_3'). Then $p_3p_2=\sigma_3'(p_2)p_3+\delta_3'(p_2)$, and

$$\begin{aligned} p_3p_2 &= \sigma_3'(p_2)[f(x_1,x_2)x_3^n+g(x_1,x_2,x_3)]+\delta_3'(p_2)\\ &= \sigma_3'(p_2)f(x_1,x_2)x_3^n+[\sigma_3'(p_2)g(x_1,x_2,x_3)+\delta_3'(p_2)] \end{aligned} \tag{2.31}$$

Now,

$$\begin{aligned} p_3p_2 &= [f(x_1,x_2)x_3^n+g(x_1,x_2,x_3)]p_2 = f(x_1,x_2)x_3^np_2+g(x_1,x_2,x_3)p_2\\ &= f(x_1,x_2)\sigma_3^n(p_2)x_3^n+g(x_1,x_2,x_3)p_2, \end{aligned} \tag{2.32}$$

since $q(x_3)$ is a monomial invariant. From (2.31) and (2.32) it follows that $f(x_1,x_2)\sigma_3^n(-)=\sigma_3'(-)f(x_1,x_2)$, and

$$g(x_1,x_2,x_3)p_2=\sigma_3'(p_2)g(x_1,x_2,x_3)+\delta_3'(p_2). \tag{2.33}$$

Conversely, if $g(x_1,x_2,x_3)$ is a cv-polynomial with respect to (σ_3',δ_3'), then $g(x_1,x_2,x_3)p_2=\sigma_3'(p_2)g(x_1,x_2,x_3)+\delta_3'(p_2)$, and having in mind that $f(x_1,x_2)\sigma_3^n(-)=\sigma_3'(-)f(x_1,x_2)$,

$$\begin{aligned} p_3p_2 &= f(x_1,x_2)x_3^np_2+g(x_1,x_2,x_3)p_2 = f(x_1,x_2)\sigma_3^n(p_2)x_3^n+g(x_1,x_2,x_3)p_2\\ &= \sigma_3'(p_2)f(x_1,x_2)x_3^n+\sigma_3'(p_2)g(x_1,x_2,x_3)+\delta_3'(p_2)\\ &= \sigma_3'(p_2)[f(x_1,x_2)x_3^n+g(x_1,x_2,x_3)]+\delta_3'(p_2)=\sigma_3'(p_2)p_3+\delta_3'(p_2). \end{aligned}$$

(2) If p_3 is a cv-polynomial with respect to the quasi-derivation (σ_3',δ_3'), then $p_3p_2=\sigma_3'(p_2)p_3+\delta_3'(p_2)$. Since p_3 is a left multiple of $f(x_1,x_2)$, we get $g(x_1,x_2,x_3)=0$. Due to the equation (2.33), $\delta_3'(p_2)=0$, and besides $f(x_1,x_2)\sigma_3^n(-)=\sigma_3'(-)f(x_1,x_2)$, whence

$$p_3p_2=\sigma_3'(p_2)f(x_1,x_2)x_3^n=f(x_1,x_2)\sigma_3^n(p_2)x_3^n,$$

which shows that p_3 is invariant.

Now, assume that p_3 is invariant. By assumption, $g(x_1,x_2,x_3)=0$, and so $p_3p_2=f(x_1,x_2)x_3^np_2=f(x_1,x_2)\sigma_3^n(p_2)x_3^n$. If there exist σ_3' such that $f(x_1,x_2)\sigma_3^n(p_2)=\sigma_3'(p_2)f(x_1,x_2)$, then p_3 is a cv-polynomial with respect to $(\sigma_3',\delta_3'=0)$.

□

As above, if (σ_3,δ_3) is a quasi-derivation on a ring R, δ_3 is said to be *algebraic* if there exists a non-zero polynomial $g\in S=R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2][x_3;\sigma_3,\delta_3]$ such that $g(x_1,x_2,\delta_3)=0$. The evaluation of $g(x_1,x_2,x_3)=\sum_{i=0}^n q_i(x_1,x_2)x_3^i$ at δ_3 is defined to be the operator $g(x_1,x_2,\delta_3)=\sum_{i=0}^n q_i(x_1,x_2)\delta_3^i$ on S.

THEOREM 2.28. *Let* $S' = R[x_1';\sigma_1',\delta_1'][x_2';\sigma_2',\delta_2'][x_3';\sigma_3',\delta_3']$ *and* $S = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2][x_3;\sigma_3,\delta_3]$. *Consider* $p_1,p_2,p_3 \in S$ *given by* $p_1(x_1) = \sum_{i=0}^{k} a_i x_1^i$, $p_2(x_1,x_2) = \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij} x_1^i x_2^j$ *and* $p_3(x_1,x_2,x_3) = \sum_{i=0}^{t}\sum_{j=0}^{s}\sum_{l=0}^{w} c_{ijl} x_1^i x_2^j x_3^l$.

(1) *If* p_3 *is a cv-polynomial with respect to the quasi-derivation* (σ_3',δ_3'), *then* $\delta_3' = (p_3 - C_0)(\delta_3) + \delta_{3(C_0,\sigma_3')}$, *where* $C_0 = \sum_{i=0}^{t}\sum_{j=0}^{s} c_{ij0} x_1^i x_2^j x_3^0$, *and if also* $w \geq 1$, *then we obtain the equalities* $c_{ijl} x_1^i x_2^j \sigma_3^l(-) = \sigma_3'(-) c_{ijl} x_1^i x_2^j$ *and* $\sigma_3\delta_3 = -\delta_3\sigma_3$.

(2) *If* δ_3 *is not an algebraic derivation, then* p_3 *is a cv-polynomial with respect to* (σ_3',δ_3') *if and only if* $\delta_3' = (p_3 - C_0)(\delta_3) + \delta_{3(C_0,\sigma_3')}$, *where the element* C_0 *is given by* $\sum_{i=0}^{t}\sum_{j=0}^{s} c_{ij0} x_1^i x_2^j x_3^0$.

Proof. (1) Consider the homomorphism $\lambda : S \to \mathrm{End}(R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2],+)$ defined by $\lambda(x_3) = \delta_3$ and for any $r \in R$, $\lambda(r)$ denotes the left multiplication by r on R. The maps $\lambda(x_1) = \lambda_{x_1}$ and $\lambda(x_2) = \lambda_{x_2}$ are defined in the same way that the proof of Theorem 2.21. Assume that p_3 is a cv-polynomial with respect to (σ_3',δ_3'). Then

$$\begin{aligned} p_3 p_1 &= \phi(x_3'x_1') = \sigma_3'(p_1)p_3 + \delta_3'(p_1), \quad \text{and} \\ p_3 p_2 &= \phi(x_3'x_2') = \sigma_3'(p_2)p_3 + \delta_3'(p_2). \end{aligned} \tag{2.34}$$

By evaluating the equation (2.34) with respect to the homomorphism λ, we get

$$p_3(x_1,x_2,\delta_3)p_1(x_1) = \sigma_3'(p_1(x_1))p_3(x_1,x_2,\delta_3) + \delta_3'(p_1(x_1)). \tag{2.35}$$

Since $\delta_3^j(1) = 0$, for $j \geq 1$,

$$p_3(x_1,x_2,\delta_3)p_1(x_1) = \sigma_3'(p_1(x_1))\sum_{i=0}^{t}\sum_{j=0}^{s} c_{ij0} x_1^i x_2^j x_3^0 + \delta_3'(p_1(x_1)).$$

If $C_0 = \sum_{i=0}^{t}\sum_{j=0}^{s} c_{ij0} x_1^i x_2^j x_3^0$, then

$$\begin{aligned} \delta_3'(p_1(x_1)) &= p_3(x_1,x_2,\delta_3)p_1(x_1) - \sigma_3'(p_1(x_1))C_0 \\ &= p_3(x_1,x_2,\delta_3)p_1(x_1) - C_0p_1(x_1) + C_0p_1(x_1) - \sigma_3'(p_1(x_1))C_0 \\ &= (p_3(x_1,x_2,\delta_3) - C_0)p_1(x_1) + \delta_{3(C_0,\sigma_3')}(p_1(x_1)) \\ &= (p_2(x_1,x_2,x_3) - C_0)(\delta_3)(p_1(x_1)) + \delta_{3(C_0,\sigma_3')}(p_1(x_1)), \end{aligned}$$

whence $\delta_3' = (p_3 - C_0)(\delta_3) + \delta_{3(C_0,\sigma_3')}$.

For the next part, consider $w \geq 1$. Since p_3 is a cv-polynomial, $p_3p_1 - \sigma_3'(p_1)p_3 = \delta_3'(p_1)$,

and so

$$\begin{aligned}
&\left(\sum_{i=0}^{t}\sum_{j=0}^{s}\sum_{l=0}^{w} c_{ijl}x_1^i x_2^j x_3^l\right)\left(\sum_{i=0}^{k} a_i x_1^i\right) - \sigma_3'\left(\sum_{i=0}^{k} a_i x_1^i\right)\sum_{i=0}^{t}\sum_{j=0}^{s}\sum_{l=0}^{w} c_{ijl}x_1^i x_2^j x_3^l \\
&= [c_{tsw}x_1^t x_2^s \sigma_3^w(a_k x_1^k) - \sigma_3'(a_k x_1^k)c_{tsw}x_1^t x_2^s]x_3^w + \cdots - \sigma_3'(p_1)C_0 \\
&\quad + p_{\sigma_3,\delta_3} + \sum_{i=0}^{t}\sum_{j=0}^{t}\sum_{l=0}^{w} c_{ijl}x_1^i x_2^j \delta_3^l(p_1) + C_0 p_1 = \delta_3'(p_1),
\end{aligned} \tag{2.36}$$

where p_{σ_3,δ_3} are the possible combinations between σ_3 and δ_3. Since (2.36) is equal to

$$\delta_3'(p_1) = (p_3 - C_0)(\delta_3)(p_1) + \delta_{3(C_0,\sigma_3')}(p_1) = \sum_{i=0}^{t}\sum_{j=0}^{s}\sum_{l=0}^{w} c_{ijl}x_1^i x_2^j \delta_3^l(p_1) + C_0 p_1 - \sigma_3'(p_1)C_0,$$

we obtain

$$[c_{tsw}x_1^t x_2^s \sigma_3^w(a_k x_1^k) - \sigma_3'(a_k x_1^k)c_{tsw}x_1^t x_2^s]x_3^w + \cdots + p_{\sigma_3,\delta_3} = 0,$$

and hence, $c_{ijl}x_1^i x_2^j \sigma_3^l(-) = \sigma_3'(-)c_{ijl}x_1^i x_2^j$ for $1 \le i \le t$, $1 \le j \le s$, $1 \le l \le w$, and $p_{\sigma_3,\delta_3} = 0$, and so $\sigma_3\delta_3 = -\delta_3\sigma_3$ as desired.

(2) Let us assume that δ_3 is not an algebraic derivation and let $q(x_1,x_2,x_3) = p_2(x_1,x_2,x_3) - C_0$. Then $\delta_3' = q(x_1,x_2,\delta_3) + \delta_{3(C_0,\sigma_3')}$, and $q(x_1,x_2,\delta_3) = \delta_3' - \delta_{3(C_0,\sigma_3')}$ is a σ_3'-derivation, so for any $r \in R$,

$$q(x_1,x_2,\delta_3)(p_1(x_1)r) = \sigma_3'(p_1(x_1))q(x_1,x_2,\delta_3)(r) + q(x_1,x_2,\delta_3)(p_1(x_1))r.$$

If we evaluate by the homomorphism λ, then

$$q(x_1,x_2,\delta_3)\lambda(p_1(x_1)) = \lambda(\sigma_3'(p_1(x_1)))q(x_1,x_2,\delta_3) + \lambda(q(x_1,x_2,\delta_3)(p_1(x_1)))$$

which holds in the image of λ. Since δ_3 is not algebraic, λ is injective and there exists λ^{-1}, whence

$$q(x_1,x_2,x_3)(p_1(x_1)) = \sigma_3'(p_1(x_1))q(x_1,x_2,x_3) + q(x_1,x_2,\delta_3)(p_1(x_1)).$$

By replacing $q(x_1,x_2,x_3)$ and $q(x_1,x_3,\delta_3)$,

$$\begin{aligned}
(p_3(x_1,x_2,x_3) - C_0)p_1(x_1) &= \sigma_3'(p_1(x_1))(p_3(x_1,x_2,x_3) - C_0) + (\delta_3' - \delta_{3(C_0,\sigma_3')})(p_1(x_1)) \\
&= \sigma_3'(p_1(x_1))p_3(x_1,x_2,x_3) - \sigma_3'(p_1(x_1))C_0 \\
&\quad + \delta_3'(p_1(x_1)) - \delta_{3(C_0,\sigma_3')}(p_1(x_1)) \\
&= \sigma_3'(p_1(x_1))p_3(x_1,x_2,x_3) - \sigma_3'(p_1(x_1))C_0 \\
&\quad + \delta_3'(p_1(x_1)) - C_0 p_1(x_1) + \sigma_3'(p_1(x_1))C_0 \\
&= \sigma_3'(p_1(x_1))p_3(x_1,x_2,x_3) + \delta_3'(p_1(x_1)) - C_0 p_1(x_1),
\end{aligned}$$

whence $C_0 p_1(x_1)$, and $p_3 p_1 = \sigma_3'(p_1)p_3 + \delta_3'(p_1)$ as desired.

□

THEOREM 2.29. *Let ϕ be a homomorphism of the iterated Ore extension as in Definition 2.6. ϕ is an isomorphism if and only if the associated cv-polynomials have degree one with respect to x_1, x_2 and x_3, respectively.*

Proof. It follows by using an inductive argument to the presented in the proof of Theorem 2.22.

□

2.4.3 n-STEP ITERATED ORE EXTENSIONS

Having in mind the ideas developed in Sections 2.4.1 and 2.4.2, in this short section we present a general framework of homomorphisms between iterated Ore extensions of n-step iterated Ore extensions.

DEFINITION 2.7. Let $S' = R[x_1';\sigma_1',\delta_1']\cdots[x_n';\sigma_n',\delta_n']$ and $S = R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$. Let the homomorphism ϕ defined as $\phi(x_i') := p_i$, where $p_i = p(x_1, x_2, \ldots, x_i)$, with $1 \leq i \leq n$, and such that p_i satisfies for $1 \leq j < i \leq n$ and $r \in R$,

$$\phi(x_i'x_j') = \sigma_i'(\phi(x_j'))p_i + \delta_i'(\phi(x_j')) \quad \text{and} \quad \phi(x_i'r) = \sigma_i'(r)p_i + \delta_i'(r),$$

or equivalently,

$$p_ip_j = \sigma_i'(p_j)p_i + \delta_i'(p_j) \quad \text{and} \quad p_ir = \sigma_i'(r)p_i + \delta_i'(r).$$

We say that the polynomials p_i are the *cv-polynomials for iterated Ore extensions with respect to* the quasi-derivations (σ_i', δ_i') for $1 \leq i \leq n$.

Conversely, if a polynomial $p_i \in S$, where p_i depends on the indeterminates $x_1, x_2, \ldots, x_i$, with $1 \leq i \leq n$, satisfies the above conditions, where (σ_i', δ_i') are the corresponding quasi-derivations, then we can define $\phi : S' \to S$ by $\phi(x_i') = p_i$.

EXAMPLE 2.16. Consider the single parameter quantum matrix algebra $O_p(M_n(\Bbbk))$ described in Section 1.2.1. Then

$$O_p(M_n(\Bbbk)) \cong \Bbbk[x_{11};\sigma_{11},\delta_{11}]\cdots[x_{1n};\sigma_{1n},\delta_{1n}]\cdots[x_{n1};\sigma_{n1},\delta_{n1}]\cdots[x_{nn};\sigma_{nn},\delta_{nn}], \tag{2.37}$$

with σ_{ij} and δ_{ij}, $1 \leq i, j, l, m \leq n$, subject to the relations

$$\sigma_{ij}(x_{lm}) = \begin{cases} px_{lm}, & i > l, j = m, \\ px_{lm}, & i = l, j > m, \\ x_{lm}, & i > l, j < m, \\ x_{lm}, & i > l, j > m. \end{cases}$$

and $\delta_{ij}(x_{lm}) = (p - p^{-1})x_{im}x_{lj}$ for $i > l, j > m$, otherwise, zero.

Now, consider $S' = O'_p(M_n(\Bbbk))$ and $S = O_p(M_n(\Bbbk))$ which generating basis $\{x'_{ij}\}$ and $\{x_{ij}\}$, respectively. The idea is to show that if we defined the homomorphism $\phi : S' \to S$ as $\phi(x'_{ij}) := p_{ij} = b_{ji}x_{ji}$, then the polynomials p_{ij} for $1 \leq i, j \leq n$ are cv-polynomials. For $1 \leq i, j, l, m \leq n$, we have

$$\begin{aligned}\phi(x'_{ij}x'_{lm}) = p_{ij}p_{lm} = \sigma'_{ij}(\phi(x'_{lm}))p_{ij} + \delta'_{ij}(\phi(x'_{lm})), \quad \text{and}\\ \phi(x'_{ij}r) = p_{ij}r = \sigma'_{ij}(r)p_{ij} + \delta'_{ij}(r).\end{aligned}$$

Suppose that $\sigma'_{ij} = \sigma_{ji}$ and $\delta'_{ij} = \delta_{ji}$. Since

$$p_{ij}p_{lm} = b_{ji}x_{ji}b_{ml}x_{ml} = \sigma_{ji}(b_{ml}x_{ml})b_{ji}x_{ji} + \delta_{ji}(b_{ml}x_{ml}) = b_{ji}b_{ml}\sigma_{ji}(x_{ml})x_{ji} + b_{ml}\delta_{ji}(x_{ml}),$$

and

$$\begin{aligned}\sigma'_{ij}(p_{lm})p_{ij} + \delta'_{ij}(p_{lm}) &= \sigma_{ji}(b_{ml}x_{ml})b_{ji}x_{ji} + \delta_{ji}(b_{ml}x_{ml})\\ &= b_{ji}b_{ml}\sigma_{ji}(x_{ml})x_{ji} + b_{ml}\delta_{ji}(x_{ml}),\end{aligned}$$

it follows that $\phi(x'_{ij}r) = p_{ij}r = \sigma'_{ij}(r)p_{ij} + \delta'_{ij}(r)$ as desired (recall that σ_{ij} is the identity and δ_{ij} is zero for elements $r \in \Bbbk$ for every $1 \leq i, j \leq n$). Therefore, the polynomials $p_{ij} = b_{ji}x_{ji}$ are cv-polynomials with respect to the quasi-derivation $(\sigma'_{ij}, \delta'_{ij})$, and ϕ is a homomorphism of iterated Ore extensions.

As in the previous sections, for S an iterated Ore extensions with n indeterminates and $p_i \in S$, where p_i is a polynomial in the indeterminates x_1, x_2 up to x_i, for $1 \leq i \leq n$, we say that p_i is *right invariant* if $p_i R[x_1;\sigma_1,\delta_1]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}] \subseteq R[x_1;\sigma_1,\delta_1]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}]p_i$.

THEOREM 2.30. *Fix i and consider x_i^n a monomial invariant of degree $n \geq 1$. Let $p_i \in S$ as $p_i = f(x_1,\ldots,x_{i-1})x_i^n + g(x_1,\ldots,x_i)$ with the degree of g less than or equal to n. Then:*

(1) *p_i is a cv-polynomial with respect to (σ'_i, δ'_i) if and only if $g(x_1,\ldots,x_i)$ is a cv-polynomial with respect to (σ'_i, δ'_i) and $f(x_1,\ldots,x_{i-1})\sigma_i^n(-) = \sigma'_i(-)f(x_1,\ldots,x_{i-1})$.*

(2) *If p_i is a left multiple of $f(x_1,\ldots,x_{i-1})$, then p_i is a cv-polynomial if and only if is invariant.*

Proof. (1) If p_i is cv-polynomial with respect to the quasi-derivation (σ'_i, δ'_i), then $p_ip_j = \sigma'_i(p_j)p_i + \delta'_i(p_j)$, and hence

$$\begin{aligned}p_ip_j &= \sigma'_i(p_j)[f(x_1,\ldots,x_{i-1})x_i^n + g(x_1,\ldots,x_i)] + \delta'_i(p_j)\\ &= \sigma'_i(p_j)f(x_1,\ldots,x_{i-1})x_i^n + [\sigma'_i(p_j)g(x_1,\ldots,x_i)] + \delta'_i(p_j). \qquad (2.38)\end{aligned}$$

Also, note that

$$\begin{aligned}p_ip_j &= [f(x_1,\ldots,x_{i-1})x_i^n + g(x_1,\ldots,x_i)]p_j = f(x_1,\ldots,x_{i-1})x_i^np_j + g(x_1,\ldots,x_i)p_j\\ &= f(x_1,\ldots,x_{i-1})\sigma_i^n(p_j)x_i^n + g(x_1,\ldots,x_i)p_j, \qquad (2.39)\end{aligned}$$

where the last expression is obtained due to x_i^n is a monomial invariant of degree n. By (2.38) and (2.39), $f(x_1,\dots,x_{i-1})\sigma_i^n = \sigma_i' f(x_1,\dots,x_{i-1})$, and

$$g(x_1,\dots,x_i)p_j = \sigma_i'(p_j)g(x_1,\dots,x_i) + \delta_i'(p_j), \tag{2.40}$$

which shows that $g(x_1,\dots,x_i)$ is a cv-polynomial with respect to (σ_i',δ_i').

Conversely, if $g(x_1,\dots,x_i)$ is a cv-polynomial with respect to the quasi-derivation (σ_i',δ_i'), then

$$g(x_1,\dots,x_i)p_j = \sigma_i'(p_j)g(x_1,\dots,x_i) + \delta_i'(p_j),$$

and by using that $f(x_1,\dots,x_{i-1})\sigma_i^n(-) = \sigma_i'(-)f(x_1,\dots,x_{i-1})$,

$$\begin{aligned} p_ip_j &= f(x_1,\dots,x_{i-1})x_i^n p_j + g(x_1,\dots,x_i)p_j = f(x_1,\dots,x_{i-1})\sigma_i^n(p_j)x_i^n + g(x_1,\dots,x_i)p_j \\ &= \sigma_i'(p_j)f(x_1,\dots,x_{i-1})x_i^n + \sigma_i'(p_j)g(x_1,\dots,x_i) + \delta_i'(p_j) \\ &= \sigma_i'(p_j)[f(x_1,\dots,x_{i-1})x_i^n + g(x_1,\dots,x_i)] + \delta_i'(p_j) = \sigma_i'(p_j)p_i + \delta_i'(p_j). \end{aligned}$$

(2) Suppose that p_i is a cv-polynomial with respect to the quasi-derivation (σ_i',δ_i'). Then $p_ip_j = \sigma_i'(p_j)p_i + \delta_i'(p_j)$, and since p_i is a left multiple of $f(x_1,\dots,x_{i-1})$, it follows that $g(x_1,\dots,x_i) = 0$ and $\delta_i'(p_j) = 0$ due to (2.40). From (1),

$$f(x_1,\dots,x_{i-1})\sigma_i^n(-) = \sigma_i'(-)f(x_1,\dots,x_{i-1}),$$

and

$$p_ip_j = \sigma_i'(p_j)f(x_1,\dots,x_{i-1})x_i^n = f(x_1,\dots,x_{i-1})\sigma_i^n(p_j)x_i^n,$$

which shows that p_i is invariant.

Finally, assume that p_i es invariant for each i. Then $g(x_1,\dots,x_i) = 0$, and

$$p_ip_j = f(x_1,\dots,x_{i-1})x_i^n p_j = f(x_1,\dots,x_{i-1})\sigma_i^n(p_j)x_i^n.$$

If σ_i' satisfies $f(x_1,\dots,x_{i-1})\sigma_i^n(-) = \sigma_i'(-)f(x_1,\dots,x_{i-1})$, we get that p_i is a cv-polynomial.

□

For $S = R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$, δ_i is said to be *algebraic* if there exists a non-zero polynomial $g \in S$ such that $g(x_1,x_2,\dots,\delta_i,\dots,x_n) = 0$. The evaluation of a polynomial $g(x_1,\dots,x_n) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ at δ_i is defined to be $g(x_1,x_2,\dots,\delta_i,\dots,x_n) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x_1^{\alpha_1}x_2^{\alpha_2}\cdots\delta_i^{\alpha_i}\cdots x_n^{\alpha_n}$ on S.

The following theorem shows that if each of the cv-polynomials has a degree greater than or equal to one, then these can be characterized with a suitable quasi-derivation (σ_i',δ_i'). This theorem extends Proposition 2.18.

THEOREM 2.31. *Consider iterated Ore extensions S' and S with n indeterminates. Let $p_i \in S$ given by $p_i = \sum_{\alpha\in\mathbb{N}^i} a_\alpha x^\alpha$ for $1 \le i \le n$.*

(1) *If p_i is a cv-polynomial with respect to the quasi-derivation (σ_i',δ_i'), then $\delta_i' = (p_i - D_0)(\delta_i) +$*

$\delta_{i(D_0,\sigma_i')}$, *with* $D_0 = \sum_{\alpha\in\mathbb{N}^{i-1}} a_\alpha x^\alpha$ *and* $\alpha_i = 0$. *If the degree of* x_i *is greater than or equal to* 1, *denote it as* β, *then* $\sigma_i\delta_i = -\delta_i\sigma_i$ *and* $a_{\alpha_1\cdots\alpha_{i-1}}x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}\sigma_i^{\alpha_i}(-) = \sigma_i'(-)a_{\alpha_1\cdots\alpha_{i-1}}x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}$.

(2) *If* δ_i *is not an algebraic derivation, then* p_i *is a cv-polynomials with respect to* (σ_i', δ_i') *if and only if* $\delta_i' = (p_i - D_0)(\delta_i) + \delta_{i(D_0,\sigma_i')}$, *where* $D_0 = \sum_{\alpha\in\mathbb{N}^i} a_\alpha x^\alpha$ *with* $\alpha_i = 0$.

Proof. It follows by considering an inductive reasoning on the proofs presented in Theorems 2.21 and 2.28.

□

From the discussion above we have immediately the following result.

PROPOSITION 2.32. *Let* $S' = R[x_1';\sigma_1',\delta_1']\cdots[x_n';\sigma_n',\delta_n']$ *and* $S = R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$. *The homomorphism* $\phi : S' \to S$ *defined by* $\phi(x_i') = p(x_i) = x_i$ *for each* $i = 1,\ldots,n$, *is an isomorphism.*

EXAMPLE 2.17. Consider $O_p(M_2(\Bbbk)) \cong \Bbbk[x_{11}'][x_{12}';\sigma_{12}'][x_{21}';\sigma_{21}'][x_{22}';\sigma_{22}',\delta_{22}']$ and $O_q(M_2(\Bbbk)) \cong \Bbbk[x_{11}][x_{12};\sigma_{12}][x_{21};\sigma_{21}][x_{22};\sigma_{22},\delta_{22}]$ the single parameter quantum matrix algebras generated by $\{x_{ij}'\}$ and $\{x_{ij}\}$ respectively with $1 \le i,j \le 2$. Gaddis [Gad14, Proposition 3.1] showed that $O_p(M_2(\Bbbk))$ and $O_q(M_2(\Bbbk))$ are isomorphic if and only if $p = q^{\pm 1}$. By using the results obtained in this section, it can be seen that the homomorphism $\phi : O_p(M_2(\Bbbk)) \to O_q(M_2(\Bbbk))$ given by $\phi(x_{ij}') := p_{ij} = x_{ij}$ is an isomorphism in the sense of cv-polynomials formulated in Definition 2.7.

THEOREM 2.33. *Let* ϕ *be a homomorphism of the iterated Ore extensions as in Definition 2.7.* ϕ *is an isomorphism if and only if the associated cv-polynomials* p_i *for* $1 \le i \le n$ *have degree one with respect to* x_i.

Proof. To prove the first implication, it suffices to follow the ideas of Theorem 2.22 and Theorem 2.29, considering each p_i for $1 \le i \le n$ as a constant polynomial or of degree greater than one and thus having a contradiction.

Conversely, consider the homomorphism $\phi : S' \to S$ given by $\phi(x_i') = p_i = a_i x_i + g(x_1,\ldots,x_{i-1})$, where each of the polynomials p_i have degree one with respect to x_i. Following the same ideas of Theorem 2.29, we have that each p_i satisfies that $p_i p_j = \sigma_i'(p_j)p_i + \delta_i'(p_j)$ for $j < i$. Likewise, to verify that ϕ is an isomorphism, consider the homomorphism $\psi : S \to S'$ such that $\psi(x_i) = a_i^{-1}x_i' - a_i^{-1}g(\psi(x_1),\ldots,\psi(x_{i-1}))$. With this homomorphism it is straightforward to verify that $\psi\circ\phi = \mathrm{id}_{S'}$ and $\phi\circ\psi = \mathrm{id}_S$, checking that ϕ is an isomorphism. □

REMARK 12. The results obtained in this section can be illustrated with the class of *q-skew polynomial rings* which have been investigated by some authors (e.g., Goodearl and Letzter [GL94] and Haynal [Hay08]). Also, Lopes's papers [CL09, LR22] on the automorphisms and isomorphisms of a large class of iterated Ore extensions which can be viewed as ambiskew polynomial rings (in the second case, of endomorphism type), are useful to exemplify the theory developed in this section.

2.5 DOUBLE ORE EXTENSIONS

In this section, we present a first approach toward a theory of homomorphisms and cv-polynomials between right double extensions. Consider the notation in Definition 1.8.

2.5.1 DEFINITIONS AND PRELIMINARIES

DEFINITION 2.8. Let $B' = R_{P'}[y_1', y_2'; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of R. Consider the map $\phi : B' \to B$ given by

$$\phi\left(\sum_{i,j} a_{i,j} y_1'^{n(i)} y_2'^{m(j)}\right) = \sum_{i,j} a_{i,j} q_1^{n(i)} q_2^{m(j)}, \quad \text{with } q_1, q_2 \in B.$$

If we want ϕ to be an R-homomorphism, it must satisfies

$$\phi(y_2' y_1') = \phi(y_2')\phi(y_1') = q_2 q_1,$$

and given the relation (1.2), necessarily

$$\begin{aligned}\phi(y_2' y_1') &= \phi(p_{12}' y_1' y_2' + p_{11}' y_1'^2 + \tau_1' y_1' + \tau_2' y_2' + \tau_0') \\ &= p_{12}' q_1 q_2 + p_{11}' q_1^2 + \tau_1' q_1 + \tau_2' q_2 + \tau_0'.\end{aligned}$$

In this way, the relation

$$q_2 q_1 = p_{12}' q_1 q_2 + p_{11}' q_1^2 + \tau_1' q_1 + \tau_2' q_2 + \tau_0' \tag{2.41}$$

must be satisfied. These facts motivate the following definition. Let $B^{\oplus 2} := \begin{bmatrix} B \\ B \end{bmatrix}$.

DEFINITION 2.9. Let $B' = R_{P'}[y_1', y_2'; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of R. We define the map $\varphi : B'^{\oplus 2} \to B^{\oplus 2}$, a homomorphism in terms of ϕ (Definition 2.8), as

$$\varphi \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} := \begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

such that condition

$$\varphi\left(\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} r\right) = \varphi \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \varphi(r) = \begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} r = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r, \quad r \in R,$$

holds.

On the other hand, since

$$\begin{aligned}\varphi\left(\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} r\right) &= \varphi\left(\sigma'(r)\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \delta'(r)\right) \\ &= \sigma'(r)\begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} + \delta'(r) \\ &= \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r),\end{aligned}$$

it is necessary that for any $r \in R$,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r). \tag{2.42}$$

Equivalently,

$$q_1 r = \sigma'_{11}(r)q_1 + \sigma'_{12}(r)q_2 + \delta'_1(r) \quad \text{and} \quad q_2 r = \sigma'_{21}(r)q_1 + \sigma'_{22}(r)q_2 + \delta'_2(r),$$

which allows us to say that the polynomials q_i's $(i = 1,2)$ are the *cv-polynomials with respect to* (σ', δ').

Due to the similarities between Ore extensions and double extensions, the terminology in the following definition is motivated by the theory of cv-polynomials between Ore extensions presented in Section 2.4.

DEFINITION 2.10. Let $B' = R_{P'}[y_1', y_2'; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of R. A matrix

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in B^{\oplus 2} \quad \text{satisfying} \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r),$$

with q_1 and q_2 subject to the relation (2.41), is called a *double-change-of-variable matrix* (or *dcv-matrix*) with respect to (σ', δ').

EXAMPLE 2.18. Let $B' = R_{P'}[y_1', y_2'; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of R. Consider $\varphi : B'^{\oplus 2} \to B^{\oplus 2}$ given by

$$\varphi\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} := \begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_2 y_1^2 + a_0 \\ c \end{bmatrix},$$

with $a_2 \in R^*$ and $a_0, c \in R$. Let us find the conditions that guarantee that polynomials q_1 and q_2 satisfy relation (2.41). Since

$$\phi(y_2' y_1') = q_2 q_1 = c(a_2 y_1^2 + a_0) = c a_2 y_1^2 + c a_0,$$

and

$$\phi(p'_{12}y'_1y'_2 + p'_{11}y'^2_1 + \tau'_1y'_1 + \tau'_2y'_2 + \tau'_0) = p'_{12}(a_2y_1^2 + a_0)c + p'_{11}(a_2y_1^2 + a_0)^2 + \tau'_1(a_2y_1^2 + a_0) + \tau'_2c + \tau'_0,$$

let $p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \tau'_1 = c$. Then $\phi(y'_2y'_1) = \phi(p'_{12}y'_1y'_2 + p'_{11}y'^2_1 + \tau'_1y'_1 + \tau'_2y'_2 + \tau'_0)$ as desired.

Let us see that condition (2.42) holds. Consider

$$\sigma'(r) = \begin{bmatrix} \sigma'_{11}(r) = a_2\sigma_{11}^2(r)a_2^{-1} & 0 \\ 0 & \sigma'_{22}(r) \end{bmatrix},$$

and

$$\delta'(r) = \begin{bmatrix} \delta'_1(r) = (a_2\delta_1^2 + \delta_{1(a_0,\sigma'_{11})})(r) \\ \delta'_2(r) = (\delta_{2(c,\sigma'_{22})})(r) \end{bmatrix}.$$

Then

$$\begin{aligned}
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} a_2y_1^2 + a_0 \\ c \end{bmatrix} r = \begin{bmatrix} a_2y_1^2r + a_0r \\ cr \end{bmatrix} \\
&= \begin{bmatrix} a_2y_1^2r \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix} \\
&= \begin{bmatrix} a_2y_1[\sigma_{11}(r)y_1 + \sigma_{12}(r)y_2 + \delta_1(r)] \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix} \\
&= \begin{bmatrix} a_2y_1\sigma_{11}(r)y_1 + a_2y_1\sigma_{12}(r)y_2 + a_2y_1\delta_1(r) \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix} \\
&= \begin{bmatrix} a_2[\sigma_{11}^2(r)y_1^2 + \sigma_{12}(\sigma_{11}(r))y_2y_1 + \delta_1(\sigma_{11}(r))y_1] \\ +a_2[\sigma_{11}(\sigma_{12}(r))y_1y_2 + \sigma_{12}^2(r)y_2^2 + \delta_1(\sigma_{12}(r))y_2] \\ +a_2[\sigma_{11}(\delta_1(r))y_1 + \sigma_{12}(\delta_1(r))y_2 + \delta_1^2(r)] \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix}.
\end{aligned}$$

Besides, if $\sigma_{12} = \sigma_{21} = 0$ and $\sigma_{11}\delta_1 = -\delta_1\sigma_{11}$,

$$\begin{aligned}
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} a_2\sigma_{11}^2(r)y_1^2 + a_2\delta_1(\sigma_{11}(r))y_1 \\ +a_2\sigma_{11}(\delta_1(r))y_1 + \delta_1^2(r) \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix} \\
&= \begin{bmatrix} a_2\sigma_{11}^2(r)y_1^2 + \delta_1^2(r) \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r) &= \sigma'(r)\begin{bmatrix} a_2y_1^2 + a_0 \\ c \end{bmatrix} + \delta'(r) \\
&= \begin{bmatrix} \sigma'_{11}(r) & \sigma'_{12}(r) \\ \sigma'_{21}(r) & \sigma'_{22}(r) \end{bmatrix}\begin{bmatrix} a_2y_1^2 + a_0 \\ c \end{bmatrix} + \begin{bmatrix} \delta'_1(r) \\ \delta'_2(r) \end{bmatrix} \\
&= \begin{bmatrix} a_2\sigma_{11}^2(r)a_2^{-1} & 0 \\ 0 & \sigma'_{22}(r) \end{bmatrix}\begin{bmatrix} a_2y_1^2 + a_0 \\ c \end{bmatrix} \\
&\quad + \begin{bmatrix} (a_2\delta_1^2 + \delta_{1(a_0,\sigma'_{11})})(r) \\ (\delta_{2(c,\sigma'_{22})})(r) \end{bmatrix} \\
&= \begin{bmatrix} a_2\sigma_{11}^2(r)a_2^{-1}a_2y_1^2 + a_2\sigma_{11}^2(r)a_2^{-1}a_0 \\ \sigma'_{22}(r)c \end{bmatrix} \\
&\quad + \begin{bmatrix} a_2\delta_1^2(r) + a_0r - a_2\sigma_{11}^2(r)a_2^{-1}a_0 \\ cr - \sigma'_{22}(r)c \end{bmatrix} \\
&= \begin{bmatrix} a_2\sigma_{11}^2(r)y_1^2 + \delta_1^2(r) \\ 0 \end{bmatrix} + \begin{bmatrix} a_0r \\ cr \end{bmatrix},
\end{aligned}$$

we get

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r), \quad \text{i.e.} \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_2y_1^2 + a_0 \\ c \end{bmatrix}$$

is a dcv-matrix with respect to (σ', δ').

EXAMPLE 2.19. Table 2.4 presents some possibilities of polynomials by considering q_2 as a constant. We formulate conditions to obtain dcv-matrices.

EXAMPLE 2.20. Let $B' = R_{P'}[y'_1, y'_2; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of R. Let $\varphi : B'^{\oplus 2} \to B^{\oplus 2}$ given by

$$\begin{aligned}
\varphi\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} &:= \begin{bmatrix} \phi(y'_1) \\ \phi(y'_2) \end{bmatrix} \\
&= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\
&= \begin{bmatrix} ay_1 + c \\ by_2 + c \end{bmatrix},
\end{aligned}$$

with $a, b \in R^*$, $c \in R$, and a, b belonging to the center of R. Consider the conditions $p'_{12} = 0$, $p'_{11} = ba^{-1}$, $p_{12} = 1, \sigma_{11} = \sigma_{21}$, $\sigma_{12} = \sigma_{22}$, $\delta_1 = \delta_2$, $\tau'_1 = c - bca^{-1}, \tau'_2 = 0, \tau'_0 = 0$.

Note that

$$\begin{aligned}
\phi(y_2'y_1') &= q_2q_1 \\
&= (by_2+c)(ay_1+c) \\
&= by_2ay_1 + by_2c + cay_1 + c^2 \\
&= b[\sigma_{21}(a)y_1 + \sigma_{22}(a)y_2 + \delta_2(a)]y_1 \\
&\quad + b[\sigma_{21}(c)y_1 + \sigma_{22}(c)y_2 + \delta_2(c)] + cay_1 + c^2 \\
&= b\sigma_{21}(a)y_1^2 + b\sigma_{22}(a)y_2y_1 + b\delta_2(a)y_1 + b\sigma_{21}(c)y_1 \\
&\quad + b\sigma_{22}(c)y_2 + b\delta_2(c) + cay_1 + c^2 \\
&= b\sigma_{21}(a)y_1^2 + b\sigma_{22}(a)[p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0] \\
&\quad + b\delta_2(a)y_1 + b\sigma_{21}(c)y_1 + b\sigma_{22}(c)y_2 + b\delta_2(c) + cay_1 + c^2, \\
&= p_{12}b\sigma_{22}(a)y_1y_2 + [b\sigma_{21}(a) + p_{11}b\sigma_{22}(a)]y_1^2 \\
&\quad + [\tau_1b\sigma_{22}(a) + b\delta_2(a) + b\sigma_{21}(c) + ca]y_1 \\
&\quad + [\tau_2b\sigma_{22}(a) + b\sigma_{22}(c)]y_2 + [\tau_0b\sigma_{22}(a) + b\delta_2(c) + c^2],
\end{aligned}$$

and by the above conditions,

$$\begin{aligned}
\phi(y_2'y_1') = q_2q_1 &= b\sigma_{12}(a)y_1y_2 + [b\sigma_{11}(a) + p_{11}b\sigma_{12}(a)]y_1^2 \\
&\quad + [\tau_1b\sigma_{12}(a) + b\delta_1(a) + b\sigma_{11}(c) + ca]y_1 \\
&\quad + [\tau_2b\sigma_{12}(a) + b\sigma_{12}(c)]y_2 + [\tau_0b\sigma_{12}(a) + b\delta_1(c) + c^2].
\end{aligned}$$

$q_2 = c$		
cv-polynomial	(σ', δ')	**Conditions**
$q_1 = d$	$\begin{bmatrix} \sigma'_{11} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = \delta_{1(d,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = cdc^{-1}d^{-1}, \\ p'_{11} = \tau'_1 = \tau'_2 = \tau'_0 = 0 \end{cases}$
$q_1 = a_1 y_1 + a_0$	$\begin{bmatrix} \sigma'_{11} = a_1\sigma_{11}a_1^{-1} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = a_1\delta_1 + \delta_{1(a_0,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \\ \sigma_{12} = 0 \end{cases}$
$q_1 = a_2 y_1^2 + a_0$	$\begin{bmatrix} \sigma'_{11} = a_2\sigma_{11}^2a_2^{-1} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = a_2\delta_1^2 + \delta_{1(a_0,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \\ \sigma_{11}\delta_1 = -\delta_1\sigma_{11} \end{cases}$
$q_1 = a_n y_1^n + a_0$	$\begin{bmatrix} \sigma'_{11} = a_n\sigma_{11}^n a_n^{-1} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = a_n\delta_1^n + \delta_{1(a_0,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \\ \sigma_{11}\delta_1 = -\delta_1\sigma_{11} \end{cases}$
$q_1 = f(y_1)y_1 + a_0,$ $f(y_1) = a_2 y_1 + a_1$	$\begin{bmatrix} \sigma'_{11} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = a_2\delta_1^2 + a_1\delta_1 + \delta_{1(a_0,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \\ \sigma_{11}\delta_1 = -\delta_1\sigma_{11} \\ f(y_1)\sigma_{11}(-) = \sigma'_{11}(-)f(y_1) \end{cases}$
$q_1 = f(y_1)y_1 + b,$ $f(y_1) = \sum_{i=0}^n a_i y_1^i$	$\begin{bmatrix} \sigma'_{11} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = \sum_{i=1}^n a_i\delta_1^i + \delta_{1(b,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \\ \sigma_{11}\delta_1 = -\delta_1\sigma_{11} \\ f(y_1)\sigma_{11}(-) = \sigma'_{11}(-)f(y_1) \end{cases}$
$q_1 = \sum_{i=0}^n a_i y_1^i$	$\begin{bmatrix} \sigma'_{11} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = \sum_{i=1}^n a_i\delta_1^i + \delta_{1(a_0,\sigma'_{11})} \\ \delta'_2 = \delta_{2(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \\ \sigma_{11}\delta_1 = -\delta_1\sigma_{11} \\ a_i\sigma_{11}^i(-) = \sigma'_{11}(-)a_i \end{cases}$
$q_1 = a_1 y_1 y_2 + a_0$	$\begin{bmatrix} \sigma'_{11} = a_1\sigma_{11}\sigma_{22}a_1^{-1} & \sigma'_{12}=0 \\ \sigma'_{21}=0 & \sigma'_{22} \end{bmatrix}, \begin{bmatrix} \delta'_1 = \delta_{1(a_0,\sigma'_{11})} \\ \delta'_2 = \delta_1\delta_2 + \delta_{(c,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = p'_{11} = \tau'_2 = \tau'_0 = 0, \\ \tau'_1 = c \end{cases}$

Table 2.4: dcv-matrices

On the other hand,

$$\begin{aligned}
&\phi(p'_{12}y'_1y'_2 + p'_{11}y_1'^2 + \tau'_1y'_1 + \tau'_2y'_2 + \tau'_0)\\
&= p'_{12}q_1q_2 + p'_{11}q_1^2 + \tau'_1q_1 + \tau'_2q_2 + \tau'_0\\
&= p'_{12}(ay_1 + c)(by_2 + c) + p'_{11}(ay_1 + c)^2 + \tau'_1(ay_1 + c) + \tau'_2(by_2 + c) + \tau'_0\\
&= p'_{12}ay_1by_2 + p'_{12}ay_1c + p'_{12}cby_2 + p'_{12}c^2 + p'_{11}ay_1ay_1 + p'_{11}ay_1c\\
&\quad + p'_{11}cay_1 + p'_{11}c^2 + \tau'_1ay_1 + \tau'_1c + \tau'_2by_2 + \tau'_2c + \tau'_0\\
&= p'_{12}a\sigma_{11}(b)y_1y_2 + p'_{12}a\sigma_{12}(b)y_2^2 + p'_{12}a\delta_1(b)y_2 + p'_{12}a\sigma_{11}(c)y_1\\
&\quad + p'_{12}a\sigma_{12}(c)y_2 + p'_{12}a\delta_1(c) + p'_{12}cby_2 + p'_{12}c^2 + p'_{11}a\sigma_{11}(a)y_1^2\\
&\quad + p'_{11}a\sigma_{12}(a)[p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0] + p'_{11}a\delta_1(a)y_1\\
&\quad + p'_{11}a\sigma_{11}(c)y_1 + p'_{11}a\sigma_{12}(c)y_2 + p'_{11}a\delta_1(c) + p'_{11}cay_1 + p'_{11}c^2\\
&\quad + \tau'_1ay_1 + \tau'_1c + \tau'_2by_2 + \tau'_2c + \tau'_0\\
&= [p'_{12}a\sigma_{11}(b) + p'_{11}a\sigma_{12}(a)]y_1y_2 + [p'_{11}a\sigma_{11}(a) + p'_{11}p_{11}a\sigma_{12}(a)]y_1^2\\
&\quad + [p'_{12}a\sigma_{11}(c) + p'_{11}a\sigma_{12}(a)\tau_1 + p'_{11}a\delta_1(a) + p'_{11}a\sigma_{11}(c) + p'_{11}ca + \tau'_1a]y_1\\
&\quad + [p'_{12}a\delta_1(b) + p'_{12}a\sigma_{12}(c) + p'_{12}cb + p'_{11}a\sigma_{12}(a)\tau_2 + p'_{11}a\sigma_{12}(c) + \tau'_2b]y_2\\
&\quad + p'_{12}a\sigma_{12}(b)y_2^2 + [p'_{12}a\delta_1(c) + p'_{12}c^2 + p'_{11}a\sigma_{12}(a)\tau_0 + p'_{11}a\delta_1(c)\\
&\quad + p'_{11}c^2 + \tau'_1c + \tau'_2c + \tau'_0].
\end{aligned}$$

Again, by the conditions described above,

$$\begin{aligned}
&\phi(p'_{12}y'_1y'_2 + p'_{11}y_1'^2 + \tau'_1y'_1 + \tau'_2y'_2 + \tau'_0)\\
&= [ba^{-1}a\sigma_{12}(a)]y_1y_2 + [ba^{-1}a\sigma_{11}(a) + ba^{-1}p_{11}a\sigma_{12}(a)]y_1^2\\
&\quad + [ba^{-1}a\sigma_{12}(a)\tau_1 + ba^{-1}a\delta_1(a) + ba^{-1}a\sigma_{11}(c) + ba^{-1}ca + (c - bca^{-1})a]y_1\\
&\quad + [ba^{-1}a\sigma_{12}(a)\tau_2 + ba^{-1}a\sigma_{12}(c)]y_2\\
&\quad + [ba^{-1}a\sigma_{12}(a)\tau_0 + ba^{-1}a\delta_1(c) + ba^{-1}c^2 + (c - bca^{-1})c]\\
&= b\sigma_{12}(a)y_1y_2 + [b\sigma_{11}(a) + p_{11}b\sigma_{12}(a)]y_1^2 + [\tau_1b\sigma_{12}(a) + b\delta_1(a) + b\sigma_{11}(c) + ca]y_1\\
&\quad + [\tau_2b\sigma_{12}(a) + b\sigma_{12}(c)]y_2 + [\tau_0b\sigma_{12}(a) + ba\delta_1(c) + c^2].
\end{aligned}$$

These equalities show that $\phi(y'_2y'_1) = \phi(p'_{12}y'_1y'_2 + p'_{11}y_1'^2 + \tau'_1y'_1 + \tau'_2y'_2 + \tau'_0)$, i.e., the polynomials q_1 and q_2 satisfy (2.41).

With the aim of showing that condition (2.42) holds, consider

$$\sigma'(r) = \begin{bmatrix} \sigma_a(\sigma_{11}(r)) & \sigma_b(\sigma_{12}(r)) \\ \sigma_a(\sigma_{21}(r)) & \sigma_b(\sigma_{22}(r)) \end{bmatrix} \text{ and } \delta'(r) = \begin{bmatrix} (a\delta_1 + \delta_{c,\sigma'_1})(r) \\ (b\delta_2 + \delta_{c,\sigma'_2})(r) \end{bmatrix},$$

where

$$\delta_{c,\sigma'_i}(r) = \begin{cases} cr - [a\sigma_{11}(r)a^{-1}c + b\sigma_{12}(r)b^{-1}c], & \text{if } i = 1 \\ cr - [a\sigma_{21}(r)a^{-1}c + b\sigma_{22}(r)b^{-1}c], & \text{if } i = 2. \end{cases}$$

Since

$$\begin{aligned}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} q_1 r \\ q_2 r \end{bmatrix} = \begin{bmatrix} (ay_1+c)r \\ (by_2+c)r \end{bmatrix} = \begin{bmatrix} ay_1 r \\ by_2 r \end{bmatrix} + \begin{bmatrix} cr \\ cr \end{bmatrix} \\ &= \begin{bmatrix} a[\sigma_{11}(r)y_1+\sigma_{12}(r)y_2+\delta_1(r)] \\ b[\sigma_{21}(r)y_1+\sigma_{22}(r)y_2+\delta_2(r)] \end{bmatrix} + \begin{bmatrix} cr \\ cr \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{bmatrix} \begin{bmatrix} ay_1 \\ by_2 \end{bmatrix} + \begin{bmatrix} a\delta_1(r) \\ b\delta_2(r) \end{bmatrix} + \begin{bmatrix} cr \\ cr \end{bmatrix},\end{aligned}$$

and,

$$\begin{aligned}\sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r) &= \sigma'(r)\begin{bmatrix} ay_1+c \\ by_2+c \end{bmatrix} + \delta'(r) = \sigma'(r)\begin{bmatrix} ay_1 \\ by_2 \end{bmatrix} + \sigma'(r)\begin{bmatrix} c \\ c \end{bmatrix} + \delta'(r) \\ &= \begin{bmatrix} \sigma_a(\sigma_{11}(r)) & \sigma_b(\sigma_{12}(r)) \\ \sigma_a(\sigma_{21}(r)) & \sigma_b(\sigma_{22}(r)) \end{bmatrix} \begin{bmatrix} ay_1 \\ by_2 \end{bmatrix} + \begin{bmatrix} \sigma_a(\sigma_{11}(r)) & \sigma_b(\sigma_{12}(r)) \\ \sigma_a(\sigma_{21}(r)) & \sigma_b(\sigma_{22}(r)) \end{bmatrix} \begin{bmatrix} c \\ c \end{bmatrix} \\ &\quad + \begin{bmatrix} (a\delta_1 + \delta_{c,\sigma_1'})(r) \\ (b\delta_2 + \delta_{c,\sigma_2'})(r) \end{bmatrix} \\ &= \begin{bmatrix} a\sigma_{11}(r)a^{-1}ay_1 + b\sigma_{12}(r)b^{-1}by_2 \\ a\sigma_{21}(r)a^{-1}ay_1 + b\sigma_{22}(r)b^{-1}by_2 \end{bmatrix} + \begin{bmatrix} a\sigma_{11}(r)a^{-1}c + b\sigma_{12}(r)b^{-1}c \\ a\sigma_{21}(r)a^{-1}c + b\sigma_{22}(r)b^{-1}c \end{bmatrix} \\ &\quad + \begin{bmatrix} a\delta_1(r) + cr - [a\sigma_{11}(r)a^{-1}c + b\sigma_{12}(r)b^{-1}c] \\ b\delta_2(r) + cr - [a\sigma_{21}(r)a^{-1}c + b\sigma_{22}(r)b^{-1}c] \end{bmatrix} \\ &= \begin{bmatrix} a\sigma_{11}(r)y_1 + b\sigma_{12}(r)y_2 \\ a\sigma_{21}(r)y_1 + b\sigma_{22}(r)y_2 \end{bmatrix} + \begin{bmatrix} a\delta_1(r) + cr \\ b\delta_2(r) + cr \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{bmatrix} \begin{bmatrix} ay_1 \\ by_2 \end{bmatrix} + \begin{bmatrix} a\delta_1(r) \\ b\delta_2(r) \end{bmatrix} + \begin{bmatrix} cr \\ cr \end{bmatrix},\end{aligned}$$

we obtain $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r)$, and hence $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} ay_1+c \\ by_2+c \end{bmatrix}$ is a dcv-matrix with respect to (σ', δ').

EXAMPLE 2.21. Let $B' = R_{P'}[y_1', y_2'; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of R. Let $\varphi : B'^{\oplus 2} \to B^{\oplus 2}$ given by

$$\varphi \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} := \begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_1 y_1 + a_0 \\ b_1 y_2 + b_0 \end{bmatrix},$$

with $a_1, b_1 \in R^*$ and $a_0, b_0 \in R$. Consider the conditions,

$$\begin{aligned} &p_{12}' = 0; \quad p_{11}' = b_1 a_1^{-1}, \quad \tau_1' = b_0 - b_1 a_1^{-1} a_0; \quad \tau_2' = 0; \quad \tau_0' = b_0 a_0 - b_1 a_1^{-1} a_0^2 \\ &\sigma_{11} = \sigma_{21}; \quad \sigma_{12} = \sigma_{22}; \quad \delta_1 = \delta_2. \end{aligned}$$

Note that

$$
\begin{aligned}
&\phi(p'_{12}y'_1y'_2 + p'_{11}y_1^{'2} + \tau'_1y'_1 + \tau'_2y'_2 + \tau'_0)\\
&= p'_{12}q_1q_2 + p'_{11}q_1^2 + \tau'_1q_1 + \tau'_2q_2 + \tau'_0\\
&= p'_{12}(a_1y_1 + a_0)(b_1y_2 + b_0) + p'_{11}(a_1y_1 + a_0)^2 + \tau'_1(a_1y_1 + a_0) + \tau'_2(b_1y_2 + b_0) + \tau'_0\\
&= p'_{12}[a_1y_1b_1y_2 + a_1y_1b_0 + a_0b_1y_2 + a_0b_0] + p'_{11}[a_1y_1a_1y_1 + a_1y_1a_0 + a_0a_1y_1 + a_0^2]\\
&\quad + \tau'_1a_1y_1 + \tau'_1a_0 + \tau'_2b_1y_2 + \tau'_2b_0 + \tau'_0\\
&= p'_{12}a_1[\sigma_{11}(b_1)y_1 + \sigma_{12}(b_1)y_2 + \delta_1(b_1)]y_2 + p'_{12}a_1[\sigma_{11}(b_0)y_1 + \sigma_{12}(b_0)y_2 + \delta_1(b_0)]\\
&\quad + p'_{12}a_0b_1y_2 + p'_{12}a_0b_0 + p'_{11}a_1[\sigma_{11}(a_1)y_1 + \sigma_{12}(a_1)y_2 + \delta_1(a_1)]y_1\\
&\quad + p'_{11}a_1[\sigma_{11}(a_0)y_1 + \sigma_{12}(a_0)y_2 + \delta_1(a_0)] + p'_{11}a_0a_1y_1 + p'_{11}a_0^2\\
&\quad + \tau'_1a_1y_1 + \tau'_1a_0 + \tau'_2b_1y_2 + \tau'_2b_0 + \tau'_0\\
&= p'_{12}a_1\sigma_{11}(b_1)y_1y_2 + p'_{12}a_1\sigma_{12}(b_1)y_2^2 + p'_{12}a_1\delta_1(b_1)y_2 + p'_{12}a_1\sigma_{11}(b_0)y_1\\
&\quad + p'_{12}a_1\sigma_{12}(b_0)y_2 + p'_{12}a_1\delta_1(b_0) + p'_{12}a_0b_1y_2 + p'_{12}a_0b_0 + p'_{11}a_1\sigma_{11}(a_1)y_1^2\\
&\quad + p'_{11}a_1\sigma_{12}(a_1)y_2y_1 + p'_{11}a_1\delta_1(a_1)y_1 + p'_{11}a_1\sigma_{11}(a_0)y_1 + p'_{11}a_1\sigma_{12}(a_0)y_2\\
&\quad + p'_{11}a_1\delta_1(a_0) + p'_{11}a_0a_1y_1 + p'_{11}a_0^2 + \tau'_1a_1y_1 + \tau'_1a_0 + \tau'_2b_1y_2 + \tau'_2b_0 + \tau'_0\\
&= p'_{12}a_1\sigma_{11}(b_1)y_1y_2 + p'_{12}a_1\sigma_{12}(b_1)y_2^2 + p'_{12}a_1\delta_1(b_1)y_2\\
&\quad + p'_{12}a_1\sigma_{11}(b_0)y_1 + p'_{12}a_1\sigma_{12}(b_0)y_2 + p'_{12}a_1\delta_1(b_0)\\
&\quad + p'_{12}a_0b_1y_2 + p'_{12}a_0b_0 + p'_{11}a_1\sigma_{11}(a_1)y_1^2\\
&\quad + p'_{11}a_1\sigma_{12}(a_1)p_{12}y_1y_2 + p'_{11}a_1\sigma_{12}(a_1)p_{11}y_1^2 + p'_{11}a_1\sigma_{12}(a_1)\tau_1y_1\\
&\quad + p'_{11}a_1\sigma_{12}(a_1)\tau_2y_2 + p'_{11}a_1\sigma_{12}(a_1)\tau_0 + p'_{11}a_1\delta_1(a_1)y_1\\
&\quad + p'_{11}a_1\sigma_{11}(a_0)y_1 + p'_{11}a_1\sigma_{12}(a_0)y_2 + p'_{11}a_1\delta_1(a_0) + p'_{11}a_0a_1y_1 + p'_{11}a_0^2\\
&\quad + \tau'_1a_1y_1 + \tau'_1a_0 + \tau'_2b_1y_2 + \tau'_2b_0 + \tau'_0\\
&= [p'_{12}a_1\sigma_{11}(b_1) + p'_{11}a_1\sigma_{12}(a_1)p_{12}]y_1y_2 + [p'_{11}a_1\sigma_{11}(a_1) + p'_{11}a_1\sigma_{12}(a_1)p_{11}]y_1^2\\
&\quad + [p'_{12}a_1\sigma_{12}(b_1)]y_2^2\\
&\quad + [p'_{12}a_1\sigma_{11}(b_0) + p'_{11}a_1\sigma_{12}(a_1)\tau_1 + p'_{11}a_1\delta_1(a_1) + p'_{11}a_1\sigma_{11}(a_0) + p'_{11}a_0a_1 + \tau'_1a_1]y_1\\
&\quad + [p'_{12}a_1\delta_1(b_1) + p'_{12}a_1\sigma_{12}(b_0) + p'_{12}a_0b_1 + p'_{11}a_1\sigma_{12}(a_1)\tau_2 + p'_{11}a_1\sigma_{12}(a_0) + \tau'_2b_1]y_2\\
&\quad + [p'_{12}a_1\delta_1(b_0) + p'_{12}a_0b_0 + p'_{11}a_1\sigma_{12}(a_1)\tau_0 + p'_{11}a_1\delta_1(a_0) + p'_{11}a_0^2 + \tau'_1a_0\tau'_2b_0 + \tau'_0].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\phi(y'_2y'_1) = q_2q_1 &= (b_1y_2 + b_0)(a_1y_1 + a_0) = b_1y_2a_1y_1 + b_1y_2a_0 + b_0a_1y_1 + b_0a_0\\
&= b_1[\sigma_{21}(a_1)y_1 + \sigma_{22}(a_1)y_2 + \delta_2(a_1)]y_1\\
&\quad + b_1[\sigma_{21}(a_0)y_1 + \sigma_{22}(a_0)y_2 + \delta_2(a_0)] + b_0a_1y_1 + b_0a_0\\
&= b_1\sigma_{21}(a_1)y_1^2 + b_1\sigma_{22}(a_1)y_2y_1 + b_1\delta_2(a_1)y_1\\
&\quad + b_1\sigma_{21}(a_0)y_1 + b_1\sigma_{22}(a_0)y_2 + b_1\delta_2(a_0) + b_0a_1y_1 + b_0a_0\\
&= b_1\sigma_{21}(a_1)y_1^2 + b_1\sigma_{22}(a_1)[p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0]\\
&\quad + b_1\delta_2(a_1)y_1 + b_1\sigma_{21}(a_0)y_1 + b_1\sigma_{22}(a_0)y_2 + b_1\delta_2(a_0) + b_0a_1y_1 + b_0a_0,
\end{aligned}
$$

or equivalently,

$$\begin{aligned}
q_2q_1 &= b_1\sigma_{21}(a_1)y_1^2 + b_1\sigma_{22}(a_1)p_{12}y_1y_2 + b_1\sigma_{22}(a_1)p_{11}y_1^2 + b_1\sigma_{22}(a_1)\tau_1y_1 \\
&\quad + b_1\sigma_{22}(a_1)\tau_2y_2 + b_1\sigma_{22}(a_1)\tau_0 + b_1\delta_2(a_1)y_1 + b_1\sigma_{21}(a_0)y_1 \\
&\quad + b_1\sigma_{22}(a_0)y_2 + b_1\delta_2(a_0) + b_0a_1y_1 + b_0a_0 \\
&= [b_1\sigma_{22}(a_1)p_{12}]y_1y_2 + [b_1\sigma_{21}(a_1) + b_1\sigma_{22}(a_1)p_{11}]y_1^2 \\
&\quad + [b_1\sigma_{22}(a_1)\tau_1 + b_1\delta_2(a_1) + b_1\sigma_{21}(a_0) + b_0a_1]y_1 \\
&\quad + [b_1\sigma_{22}(a_1)\tau_2 + b_1\sigma_{22}(a_0)]y_2 + [b_1\sigma_{22}(a_1)\tau_0 + b_1\delta_2(a_0) + b_0a_0].
\end{aligned}$$

Using the conditions described above,

$$\begin{aligned}
\phi(y_2'y_1') &= [b_1\sigma_{22}(a_1)p_{12}]y_1y_2 + [b_1\sigma_{21}(a_1) + b_1\sigma_{22}(a_1)p_{11}]y_1^2 \\
&\quad + [b_1\sigma_{22}(a_1)\tau_1 + b_1\delta_2(a_1) + b_1\sigma_{21}(a_0) + b_0a_1]y_1 \\
&\quad + [b_1\sigma_{22}(a_1)\tau_2 + b_1\sigma_{22}(a_0)]y_2 + [b_1\sigma_{22}(a_1)\tau_0 + b_1\delta_2(a_0) + b_0a_0] \\
&= [b_1\sigma_{12}(a_1)p_{12}]y_1y_2 + [b_1\sigma_{21}(a_1) + b_1\sigma_{12}(a_1)p_{11}]y_1^2 \\
&\quad + [b_1\sigma_{12}(a_1)\tau_1 + b_1\delta_1(a_1) + b_1\sigma_{21}(a_0) + b_0a_1]y_1 \\
&\quad + [b_1\sigma_{12}(a_1)\tau_2 + b_1\sigma_{12}(a_0)]y_2 + [b_1\sigma_{12}(a_1)\tau_0 + b_1\delta_1(a_0) + b_0a_0],
\end{aligned}$$

while

$$\begin{aligned}
&\phi(p_{12}'y_1'y_2' + p_{11}'y_1'^2 + \tau_1'y_1' + \tau_2'y_2' + \tau_0') \\
&= [b_1\sigma_{12}(a_1)p_{12}]y_1y_2 + [b_1\sigma_{21}(a_1) + b_1\sigma_{12}(a_1)p_{11}]y_1^2 \\
&\quad + [b_1\sigma_{12}(a_1)\tau_1 + b_1\delta_1(a_1) + b_1\sigma_{21}(a_0) + b_0a_1]y_1 \\
&\quad + [b_1\sigma_{12}(a_1)\tau_2 + b_1\sigma_{12}(a_0) + \tau_2'b_1]y_2 + [b_1\sigma_{12}(a_1)\tau_0 + b_1\delta_1(a_0) + b_0a_0].
\end{aligned}$$

These facts guarantee that $\phi(y_2'y_1') = \phi(p_{12}'y_1'y_2' + p_{11}'y_1'^2 + \tau_1'y_1' + \tau_2'y_2' + \tau_0')$, i.e., the polynomials q_1 and q_2 satisfy relation (2.41).

With the aim of showing that condition (2.42) holds, consider

$$\sigma'(r) = \begin{bmatrix} \sigma_{a_1}(\sigma_{11}(r)) & \sigma_{b_1}(\sigma_{12}(r)) \\ \sigma_{a_1}(\sigma_{21}(r)) & \sigma_{b_1}(\sigma_{22}(r)) \end{bmatrix},$$

$$\delta'(r) = \begin{bmatrix} (a_1\delta_1 + \delta_{a_0,\sigma_1'})(r) \\ (b_1\delta_2 + \delta_{b_0,\sigma_2'})(r) \end{bmatrix},$$

where

$$\begin{cases} \delta_{a_0,\sigma_1'}(r) = a_0r - [a_1\sigma_{11}(r)a_1^{-1}a_0 + b_1\sigma_{12}(r)b_1^{-1}b_0], \\ \delta_{b_0,\sigma_2'}(r) = b_0r - [a_1\sigma_{21}(r)a_1^{-1}a_0 + b_1\sigma_{22}(r)b_1^{-1}b_0]. \end{cases}$$

Since

$$\begin{aligned}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} q_1 r \\ q_2 r \end{bmatrix} = \begin{bmatrix} (a_1 y_1 + a_0) r \\ (b_1 y_2 + b_0) r \end{bmatrix} \\ &= \begin{bmatrix} a_1 y_1 r \\ b_1 y_2 r \end{bmatrix} + \begin{bmatrix} a_0 r \\ b_0 r \end{bmatrix} \\ &= \begin{bmatrix} a_1[\sigma_{11}(r) y_1 + \sigma_{12}(r) y_2 + \delta_1(r)] \\ b_1[\sigma_{21}(r) y_1 + \sigma_{22}(r) y_2 + \delta_2(r)] \end{bmatrix} + \begin{bmatrix} a_0 r \\ b_0 r \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{bmatrix} \begin{bmatrix} a_1 y_1 \\ b_1 y_2 \end{bmatrix} + \begin{bmatrix} a_1 \delta_1(r) \\ b_1 \delta_2(r) \end{bmatrix} + \begin{bmatrix} a_0 r \\ b_0 r \end{bmatrix},\end{aligned}$$

and,

$$\begin{aligned}\sigma'(r) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r) &= \sigma'(r) \begin{bmatrix} a_1 y_1 + a_0 \\ b_1 y_2 + b_0 \end{bmatrix} + \delta'(r) \\ &= \sigma'(r) \begin{bmatrix} a_1 y_1 \\ b_1 y_2 \end{bmatrix} + \sigma'(r) \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \delta'(r) \\ &= \begin{bmatrix} \sigma_{a_1}(\sigma_{11}(r)) & \sigma_{b_1}(\sigma_{12}(r)) \\ \sigma_{a_1}(\sigma_{21}(r)) & \sigma_{b_1}(\sigma_{22}(r)) \end{bmatrix} \begin{bmatrix} a_1 y_1 \\ b_1 y_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} \sigma_{a_1}(\sigma_{11}(r)) & \sigma_{b_1}(\sigma_{12}(r)) \\ \sigma_{a_1}(\sigma_{21}(r)) & \sigma_{b_1}(\sigma_{22}(r)) \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \begin{bmatrix} (a_1 \delta_1 + \delta_{a_0, \sigma_1'})(r) \\ (b_1 \delta_2 + \delta_{b_0, \sigma_2'})(r) \end{bmatrix} \\ &= \begin{bmatrix} a_1 \sigma_{11}(r) a_1^{-1} a_1 y_1 + b_1 \sigma_{12}(r) b_1^{-1} b_1 y_2 \\ a_1 \sigma_{21}(r) a_1^{-1} a_1 y_1 + b_1 \sigma_{22}(r) b_1^{-1} b_1 y_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} a_1 \sigma_{11}(r) a_1^{-1} a_0 + b_1 \sigma_{12}(r) b_1^{-1} b_0 \\ a_1 \sigma_{21}(r) a_1^{-1} a_0 + b_1 \sigma_{22}(r) b_1^{-1} b_0 \end{bmatrix} \\ &\quad + \begin{bmatrix} a_1 \delta_1(r) + a_0 r - [a_1 \sigma_{11}(r) a_1^{-1} a_0 + b_1 \sigma_{12}(r) b_1^{-1} b_0] \\ b_1 \delta_2(r) + b_0 r - [a_1 \sigma_{21}(r) a_1^{-1} a_0 + b_1 \sigma_{22}(r) b_1^{-1} b_0] \end{bmatrix} \\ &= \begin{bmatrix} a_1 \sigma_{11}(r) y_1 + b_1 \sigma_{12}(r) y_2 \\ a_1 \sigma_{21}(r) y_1 + b_1 \sigma_{22}(r) y_2 \end{bmatrix} + \begin{bmatrix} a_1 \delta_1(r) + a_0 r \\ b_1 \delta_2(r) + b_0 r \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{bmatrix} \begin{bmatrix} a_1 y_1 \\ b_1 y_2 \end{bmatrix} + \begin{bmatrix} a_1 \delta_1(r) \\ b_1 \delta_2(r) \end{bmatrix} + \begin{bmatrix} a_0 r \\ b_0 r \end{bmatrix},\end{aligned}$$

we obtain $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r)$ and $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_1 y_1 + a_0 \\ b_1 y_2 + b_0 \end{bmatrix}$ is a dcv-matrix with respect to (σ', δ').

EXAMPLE 2.22. In Table 2.5, we present necessary conditions to obtain polynomials and dcv-matrices depending of q_1 and q_2.

$q_2 = b_1 y_2 + b_0$		
cv-polynomial	(σ', δ')	**Conditions**
$q_1 = d$	$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} = 0 \\ \sigma'_{21} = 0 & \sigma'_{22} = \mathrm{id} \end{bmatrix}, \quad \begin{bmatrix} \delta'_1 = \delta_{1(d,\sigma'_{11})} \\ \delta'_2 = b_1\delta_2 + \delta_{2(b_0,\sigma'_{22})} \end{bmatrix}$	$\begin{cases} p'_{12} = 1, \\ p'_{11} = \tau'_1 = \tau'_2 = 0 \\ \sigma_{21} = 0 \\ \sigma_{22} = \mathrm{id} \\ \tau'_0 = b_1\delta_2(d) \\ d \text{ commutes} \end{cases}$
$q_1 = a_1 y_1 + b_0$	$\begin{bmatrix} \sigma'_{11} = \sigma_{a_1}(\sigma_{11}(r)) & \sigma'_{12} = \sigma_{b_1}(\sigma_{12}(r)) \\ \sigma'_{21} = \sigma_{a_1}(\sigma_{21}(r)) & \sigma'_{22} = \sigma_{b_1}(\sigma_{22}(r)) \end{bmatrix}, \quad \begin{bmatrix} \delta'_1 = a_1\delta_1 + \\ cr - [a_1\sigma_{11}(r)a_1^{-1}b_0 + b_1\sigma_{12}(r)b_1^{-1}b_0] \\ \delta'_2 = b_1\delta_2 + \\ cr - [a_1\sigma_{21}(r)a_1^{-1}b_0 + b_1\sigma_{22}(r)b_1^{-1}b_0] \end{bmatrix}$	$\begin{cases} p'_{12} = \tau'_2 = \tau'_0 = 0, \\ p'_{11} = b_1a_1^{-1} \\ p_{12} = 1 \\ \tau'_1 = b_0 - b_1b_0a_1^{-1} \\ \sigma_{11} = \sigma_{21} \\ \sigma_{12} = \sigma_{22} \\ \delta_1 = \delta_2 \\ a_1, b_1 \text{ commute in } R \end{cases}$
$q_1 = a_1 y_1 + a_0$	$\begin{bmatrix} \sigma'_{11} = \sigma_{a_1}(\sigma_{11}(r)) & \sigma'_{12} = \sigma_{b_1}(\sigma_{12}(r)) \\ \sigma'_{21} = \sigma_{a_1}(\sigma_{21}(r)) & \sigma'_{22} = \sigma_{b_1}(\sigma_{22}(r)) \end{bmatrix}, \quad \begin{bmatrix} \delta'_1 = a_1\delta_1 + \\ a_0r - [a_1\sigma_{11}(r)a_1^{-1}a_0 + b_1\sigma_{12}(r)b_1^{-1}b_0] \\ \delta'_2 = b_1\delta_2 + \\ b_0r - [a_1\sigma_{21}(r)a_1^{-1}a_0 + b_1\sigma_{22}(r)b_1^{-1}b_0] \end{bmatrix}$	$\begin{cases} p'_{12} = \tau'_2 = 0, \\ p'_{11} = b_1a_1^{-1} \\ \tau'_1 = b_0 - b_1a_1^{-1}a_0 \\ \tau'_0 = b_0a_0 - b_1a_1^{-1}a_0^2 \\ \sigma_{11} = \sigma_{21} \\ \sigma_{12} = \sigma_{22} \\ \delta_1 = \delta_2 \end{cases}$

Table 2.5: dcv-matrices

Motivated by the notion of semi-invariant polynomial introduced by Lam and Leroy [LL92, p. 84], we say that a monomial $y_i^n \in B$ is *right double semi-invariant* if $y_i^n R \subseteq R y_1^n + R y_2^n$ for $i = 1,2$.

THEOREM 2.34. *Consider y_i^n a monomial right double semi-invariant of degree $n \geq 1$. Let*

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} f(y_1,y_2)y_1^n + g_1(y_1,y_2) \\ f(y_1,y_2)y_2^n + g_2(y_1,y_2) \end{bmatrix},$$

where for $i = 1,2$ the degree of the polynomial $g_i(y_1,y_2)$ is less than or equal to n and it is satisfied that $q_2 q_1 = p'_{12} q_1 q_2 + p'_{11} q_1^2 + \tau'_1 q_1 + \tau'_2 q_2 + \tau'_0$. Then $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is a dcv-matrix with respect to (σ', δ') if and only if $\begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix}$ is a dcv-matrix with respect to (σ', δ') and $f(y_1,y_1)\sigma^n(-) = \sigma'(-)f(y_1,y_2)$.

Proof. First, suppose that $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is a dcv-matrix with respect to (σ', δ'). Then

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r) = \sigma'(r) \begin{bmatrix} f y_1^n + g_1(y_1,y_2) \\ f y_2^n + g_2(y_1,y_2) \end{bmatrix} + \delta'(r),$$

or what is the same,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r) \begin{bmatrix} f y_1^n \\ f y_2^n \end{bmatrix} + \sigma'(r) \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} + \delta'(r) \tag{2.43}$$

$$= \begin{bmatrix} \sigma'_{11}(r)f & \sigma'_{12}(r)f \\ \sigma'_{21}(r)f & \sigma'_{22}(r)f \end{bmatrix} \begin{bmatrix} y_1^n \\ y_2^n \end{bmatrix} + \sigma'(r) \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} + \delta'(r), \tag{2.44}$$

and

$$\begin{aligned} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} f y_1^n + g_1(y_1,y_2) \\ f y_2^n + g_2(y_1,y_2) \end{bmatrix} r = \begin{bmatrix} f y_1^n r + g_1(y_1,y_2) r \\ f y_2^n r + g_2(y_1,y_2) r \end{bmatrix} \\ &= \begin{bmatrix} f\sigma_{11}^n(r) y_1^n + f\sigma_{12}^n(r) y_2^n \\ f\sigma_{21}^n(r) y_1^n + f\sigma_{22}^n(r) y_2^n \end{bmatrix} + \begin{bmatrix} g_1(y_1,y_2) r \\ g_2(y_1,y_2) r \end{bmatrix} \\ &= \begin{bmatrix} f\sigma_{11}^n(r) & f\sigma_{12}^n(r) \\ f\sigma_{21}^n(r) & f\sigma_{22}^n(r) \end{bmatrix} \begin{bmatrix} y_1^n \\ y_2^n \end{bmatrix} + \begin{bmatrix} g_1(y_1,y_2) r \\ g_2(y_1,y_2) r \end{bmatrix}, \end{aligned} \tag{2.45}$$

due to that y_i^n is a monomial right double semi-invariant for $i = 1,2$. Both expressions (2.44) and (2.45) show that $f(y_1,y_2)\sigma^n(-) = \sigma'(-)f(y_1,y_2)$ and

$$\begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} r = \sigma'(r) \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} + \delta'(r). \tag{2.46}$$

Conversely, if $\begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix}$ is a dcv-matrix with respect to (σ',δ'), then

$$\begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} r = \sigma'(r) \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} + \delta'(r),$$

and since $f(y_1,y_2)\sigma^n(-) = \sigma'(-)f(y_1,y_2)$, we get

$$\begin{aligned}
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} f y_1^n r + g_1(y_1,y_2)r \\ f y_2^n r + g_2(y_1,y_2)r \end{bmatrix} \\
&= \begin{bmatrix} f\sigma_{11}^n(r)y_1^n + f\sigma_{12}^n(r)y_2^n \\ f\sigma_{21}^n(r)y_1^n + f\sigma_{22}^n(r)y_2^n \end{bmatrix} + \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} \\
&= \begin{bmatrix} f\sigma_{11}^n(r) & f\sigma_{12}^n(r) \\ f\sigma_{21}^n(r) & f\sigma_{22}^n(r) \end{bmatrix} \begin{bmatrix} y_1^n \\ y_2^n \end{bmatrix} + \begin{bmatrix} g_1(y_1,y_2)r \\ g_2(y_1,y_2)r \end{bmatrix} \\
&= \begin{bmatrix} \sigma'_{11}(r)f & \sigma'_{12}(r)f \\ \sigma'_{21}(r)f & \sigma'_{22}(r)f \end{bmatrix} \begin{bmatrix} y_1^n \\ y_2^n \end{bmatrix} + \sigma'(r) \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} + \delta'(r) \\
&= \sigma'(r) \begin{bmatrix} f y_1^n \\ f y_2^n \end{bmatrix} + \sigma'(r) \begin{bmatrix} g_1(y_1,y_2) \\ g_2(y_1,y_2) \end{bmatrix} + \delta'(r) = \sigma'(r) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r).
\end{aligned}$$

□

Theorem 2.22 characterizes isomorphisms between double extensions.

THEOREM 2.35. *Let $\varphi : B'^{\oplus 2} \to B^{\oplus 2}$ be a homomorphism of right double extensions as in Definition 2.9. If φ is an isomorphism, then the polynomials q_1 and q_2 of the dcv-matrix have degree one with respect to the indeterminates y_1 and y_2, respectively.*

Proof. Let φ be an isomorphism of right double extensions. Since φ depends of the map ϕ, it is clear that for φ to be an isomorphism it is necessary that ϕ be an isomorphism. By Definition 2.9,

$$\varphi \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} := \begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},$$

so we consider the following cases:

(1) $q_1 = c$ and $q_2 = c^{-1}dc$, with $c,d \in R$. We can find elements $f,g \in B'$ such that $f = dy_1'$ and $g = cy_2'$, whence

$$\varphi \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \phi(f) \\ \phi(g) \end{bmatrix} = \begin{bmatrix} dq_1 \\ cq_2 \end{bmatrix} = \begin{bmatrix} dc \\ dc \end{bmatrix},$$

but since $f \neq g$, it contradicts the injectivity of ϕ.

(2) $q_1 = c \in R$, and $q_2 \in B$ with degree greater than zero. Let $f,g \in B'$ such that $f = dy_1'$ and

$g = dc$. Then

$$\varphi \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} \phi(f) \\ \phi(g) \end{bmatrix} = \begin{bmatrix} dq_1 \\ dc \end{bmatrix} = \begin{bmatrix} dc \\ dc \end{bmatrix},$$

which contradicts again the injectivity of ϕ.

(3) $q_1 \in B$ of degree greater than zero and $q_2 = d \in R$. Consider elements $f, g \in B'$ such that $f = d$ and $g = y_2'$. Then

$$\varphi \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} \phi(f) \\ \phi(g) \end{bmatrix} = \begin{bmatrix} d \\ q_2 \end{bmatrix} = \begin{bmatrix} d \\ d \end{bmatrix},$$

a contradiction.

Suppose that the polynomials q_1 and q_2 have degree $n, k > 1$, respectively. Since φ is surjective, for the indeterminates $y_1, y_2 \in B$ there exist polynomials $f, g \in B'$ such that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \varphi \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \phi(f) \\ \phi(g) \end{bmatrix} = \begin{bmatrix} \phi\left(\sum_{i,j} a_{i,j} y_1'^{n(i)} y_2'^{m(j)}\right) \\ \phi\left(\sum_{i,j} b_{i,j} y_1'^{k(i)} y_2'^{l(j)}\right) \end{bmatrix} = \begin{bmatrix} \sum_{i,j} a_{i,j} q_1^{n(i)} q_2^{m(j)} \\ \sum_{i,j} b_{i,j} q_1^{k(i)} q_2^{l(j)} \end{bmatrix},$$

whence necessarily the degree of q_1 is one and the degree of q_2 is zero. However, since the polynomials have degree greater than one it follows that there are no polynomials $f, g \in B'$ with $\varphi \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, which gives us a contradiction. We conclude that both p_1 and p_2 must have degree exactly one. □

2.5.2 HOMOMORPHISMS BETWEEN TRIMMED DOUBLE ORE EXTENSIONS

One of the particular cases of the double extensions is presented by Zhang and Zhang [ZZ08, Convention 1.6.(c)] as a *trimmed double extension*, for which δ is the zero map and $\tau = \{0, 0, 0\}$. We use the short notation $B = R_p[y_1, y_2; \sigma]$ to denote this subclass of extensions.

Next, we explore the homomorphisms of trimmed double extension and compare our results with those corresponding by Zhu et al. [ZvOZ17] where they computed the Nakayama automorphism of a trimmed double extension.

THEOREM 2.36. *Consider two trimmed double extensions $B' = R_{P'}[y_1', y_2'; \sigma']$ and $B = R_P[y_1, y_2; \sigma]$, where*

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_i y_1^i \\ \sum_{j=1}^{m} b_i y_2^j \end{bmatrix},$$

and it is satisfied that $\phi(y_2' y_1') = \phi(p_{12}' y_1' y_2' + p_{11}' y_1'^2)$, i.e, $q_2 q_1 = p_{12}' q_1 q_2 + p_{11}' q_1^2$. Then $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is a dcv-matrix if and only if the degree of q_1 and q_2 is the same $(n = m)$, $a_i = b_i$ for $1 \leq i \leq n$, $\sigma_{ll}\sigma_{lk}(-) = 0$, $\sigma_{lk}\sigma_{ll}(-) = 0$, and $a_i \sigma_{pq}^i(-) = \sigma_{pq}'(-) a_i$ for $l \neq k$ and $1 \leq p, q, l, k \leq 2$.

Proof. Suppose that $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is dcv-matrix. By Definition 2.10, $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, which implies that

$$\begin{aligned}
\begin{bmatrix} q_1 r \\ q_2 r \end{bmatrix} &= \begin{bmatrix} \sum_{i=1}^{n} a_i y_1^i r \\ \sum_{j=1}^{m} b_i y_2^j r \end{bmatrix} \\
&= \begin{bmatrix} a_n \sigma_{11}^n(r) y_1^n + a_n \sigma_{12}^n(r) y_2^n + a_{n-1} \sigma_{11}^{n-1}(r) y_1^{n-1} \\ +a_{n-1} \sigma_{12}^{n-1}(r) y_2^{n-1} + \cdots + a_1 \sigma_{11}(r) y_1 + a_1 \sigma_{12}(r) y_2 \\ + \sum_{l,k=0}^{n} p_{\sigma_{11}\sigma_{12}} y_1^l y_2^k \\ b_m \sigma_{21}^m(r) y_1^m + b_m \sigma_{22}^m(r) y_2^m + b_{m-1} \sigma_{21}^{m-1}(r) y_1^{m-1} \\ +b_{m-1} \sigma_{22}^{m-1}(r) y_2^{m-1} + \cdots + b_1 \sigma_{21}(r) y_1 + b_1 \sigma_{22}(r) y_2 \\ + \sum_{l,k=0}^{m} p_{\sigma_{22}\sigma_{21}} y_1^l y_2^k \end{bmatrix},
\end{aligned} \tag{2.47}$$

where $p_{\sigma_{11},\sigma_{12}}$ are the possible combinations between σ_{11} and σ_{12} and the same sense for $p_{\sigma_{22},\sigma_{21}}$. On the other hand,

$$\begin{aligned}
\sigma'(r) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} &= \begin{bmatrix} \sigma'_{11}(r) & \sigma'_{12}(r) \\ \sigma'_{21}(r) & \sigma'_{22}(r) \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n} a_i y_1^i \\ \sum_{j=1}^{m} b_i y_2^m \end{bmatrix} \\
&= \begin{bmatrix} \sigma'_{11}(r) \sum_{i=1}^{n} a_i y_1^i + \sigma'_{12}(r) \sum_{j=1}^{m} b_i y_2^j \\ \sigma'_{21}(r) \sum_{i=1}^{n} a_i y_1^i + \sigma'_{22}(r) \sum_{j=1}^{m} b_i y_2^j \end{bmatrix} \\
&= \begin{bmatrix} \sigma'_{11}(r) a_n y_1^n + \sigma'_{11}(r) a_{n-1} y^{n-1} + \cdots + \sigma'_{11}(r) a_1 y_1 + \sigma'_{12}(r) b_m y_2^m \\ +\sigma'_{12}(r) b_{m-1} y_2^{m-1} + \cdots + \sigma'_{12}(r) b_1 y_2 \\ \sigma'_{21}(r) a_n y_1^n + \sigma'_{21}(r) a_{n-1} y^{n-1} + \cdots + \sigma'_{21}(r) a_1 y_1 + \sigma'_{22}(r) b_m y_2^m \\ +\sigma'_{22}(r) b_{m-1} y_2^{m-1} + \cdots + \sigma'_{22}(r) b_1 y_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma'_{11}(r) a_n y_1^n + \sigma'_{12}(r) b_m y_2^m + \sigma'_{11}(r) a_{n-1} y^{n-1} \\ +\sigma'_{12}(r) b_{m-1} y_2^{m-1} + \cdots + \sigma'_{11}(r) a_1 y_1 + \sigma'_{12}(r) b_1 y_2 \\ \sigma'_{21}(r) a_n y_1^n + \sigma'_{22}(r) b_m y_2^m + \sigma'_{21}(r) a_{n-1} y^{n-1} \\ +\sigma'_{22}(r) b_{m-1} y_2^{m-1} + \cdots + \sigma'_{21}(r) a_1 y_1 + \sigma'_{22}(r) b_1 y_2 \end{bmatrix}.
\end{aligned} \tag{2.48}$$

If we compare (2.47) and (2.48), then necessarily

$$\sum_{l,k=0}^{n} p_{\sigma_{11},\sigma_{12}} y_1^l y_2^k = \sum_{l,k=0}^{m} p_{\sigma_{22},\sigma_{21}} y_1^l y_2^k = 0,$$

i.e, the possible combinations between σ_{ll} and σ_{lk}, $1 \leq l, k \leq 2$, $l \neq k$ are zero. In addition, $n = m$, $a_i = b_i$ for $1 \leq i \leq n$ and $a_i \sigma_{pq}^i(-) = \sigma'_{pq}(-) a_i$ for $1 \leq p, q \leq 2$, as desired.

It is straightforward to prove the other implication. □

EXAMPLE 2.23. (1) Consider the double extension [ZZ09, Subcase 4.3.1]. The algebra $\mathbb{H}$ is generated by x_1, x_2, y_1, y_2 subject to the relations

$$\begin{aligned} x_2x_1 &= x_1x_2 + x_1^2, & y_2y_1 &= -y_1y_2, \\ y_1x_1 &= x_1y_2, & y_1x_2 &= fx_1y_2 + x_2y_2, \\ y_2x_1 &= x_1y_1, & y_2x_2 &= fx_1y_1 + x_2y_1, \end{aligned}$$

where $f \neq 0$. This is the trimmed double extension $\mathbb{H} := R_P[y_1, y_2; \sigma]$ with $R = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - x_1x_2 - x_1^2\rangle$. Let $\varphi : \mathbb{H}^{\oplus 2} \to \mathbb{H}^{\oplus 2}$ defined by

$$\varphi \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \lambda y_1 \\ \lambda y_2 \end{bmatrix}, \quad \lambda \in \Bbbk.$$

Since

$$\phi(y_2y_1) = q_2q_1 = \lambda^2 y_2y_1 = -\lambda^2 y_1y_2 = p_{12}q_1q_2 + p_{11}y_1 = \phi(p_{12}y_1y_2 + p_{11}y_1^2),$$

by Theorem 2.36, $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is a dcv-matrix. Note also that

$$\begin{bmatrix} \lambda y_1x_1 \\ \lambda y_2x_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1y_2 \\ \lambda x_1y_1 \end{bmatrix} = \begin{bmatrix} 0 & x_1 \\ x_1 & 0 \end{bmatrix} \begin{bmatrix} \lambda y_1 \\ \lambda y_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11}(x_1) & \sigma_{12}(x_1) \\ \sigma_{21}(x_1) & \sigma_{22}(x_1) \end{bmatrix} \begin{bmatrix} \lambda y_1 \\ \lambda y_2 \end{bmatrix},$$

and a simple calculation shows that the properties of the dcv-matrix are satisfied for every $a \in R_P$. By Theorem 2.22, we get an isomorphism.

(2) Zhu et al. [ZvOZ17, Remark 3.13.(1)] found the Nakayama automorphism of algebra $\mathbb{H}$. Our results agree with yours by taking $\lambda := -h^2$.

(3) Zhu et al. [ZvOZ17, Remark 3.13.(2)] considered the trimmed double extension presented in [ZZ09, Subcase 4.3.3], that is, $\mathbb{N} := R_P[y_1, y_2; \sigma]$ with $R_P = \Bbbk\{x_1, x_2\}/\langle x_2x_2 + x_1x_2\rangle$, $P = (-1, 0)$, and σ given by the matrix

$$\begin{bmatrix} 0 & 0 & -g & f \\ 0 & 0 & f & -g \\ g & f & 0 & 0 \\ f & g & 0 & 0 \end{bmatrix},$$

with $f, g \in \Bbbk$ and $f^2 \neq g^2$. They found the Nakayama automorphism of $\mathbb{N}$ which is defined as $y_1 = (g^2 - f^2)y_1$ and $y_2 = (g^2 - f^2)y_2$. In this way, if we consider

$$\varphi \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} (g^2 - f^2)y_1 \\ (g^2 - f^2)y_2 \end{bmatrix},$$

then we get

$$\begin{aligned}\phi(y_2y_1) &= ((g^2-f^2)y_2)((g^2-f^2)y_1) = (g^2-f^2)^2y_2y_1\\ &= -(g^2-f^2)^2y_1y_2 = p_{12}q_1q_2 + p_{11}y_1 = \phi(p_{12}y_1y_2 + p_{11}y_1^2),\end{aligned}$$

and by Theorem 2.36 it follows that it is a dcv-matrix, and of course, it is an isomorphism by Theorem 2.22.

2.5.3 TWO-STEP ITERATED ORE EXTENSIONS VS. DOUBLE ORE EXTENSIONS

We investigate when a homomorphism of double extensions is a homomorphism of two-step iterated extensions in the sense of cv-polynomials defined in Section 2.4.1. Before, consider the following example.

EXAMPLE 2.24. From [ZZ09, Subcase 4.2.3 (ii) (iii)], we know that the algebra $\mathbb{D}$ is generated by the indeterminates x_1, x_2, y_1, y_2 subject to the relations

$$\begin{aligned} x_2x_1 &= -x_1x_2, & y_2y_1 &= py_1y_2,\\ y_1x_1 &= -px_1y_1, & y_1x_2 &= -p^2x_2y_1 + x_1y_2,\\ y_2x_1 &= px_1y_2, & y_2x_2 &= x_1y_1 + x_2y_2,\end{aligned}$$

where $p \in \{-1,1\}$. $\mathbb{D}$ can be expressed $R_P[y_1,y_2;\sigma,\delta,\tau]$, with $R = \Bbbk[x_1][x_2;\alpha]$ and $\alpha(x_1) = -x_1$. On the other hand, the algebra $\mathbb{E}$ is generated by the indeterminates x_1, x_2, y_1, y_2 satisfying the relations

$$\begin{aligned} x_2x_1 &= -x_1x_2, & y_2y_1 &= py_1y_2,\\ y_1x_1 &= x_1y_2 + x_2y_2, & y_1x_2 &= x_1y_2 - x_2y_2,\\ y_2x_1 &= -x_1y_1 + x_2y_1, & y_2x_2 &= x_1y_1 + x_2y_1,\end{aligned}$$

with $p^2 = -1$. It can be seen that $\mathbb{E}$ is the double extension $R_P[y_1,y_2;\sigma,\delta,\tau]$.

Consider the homomorphism

$$\begin{aligned}\varphi : \mathbb{D}^{\oplus 2} &\to \mathbb{E}^{\oplus 2}\\ \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} &\mapsto \begin{bmatrix} \phi(y_1') \\ \phi(y_2') \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1, x_2 \in R.\end{aligned}$$

If we suppose $p'_{12} = -1$, since in the algebra $\mathbb{D}$ we have $p'_{11} = \tau'_1 = \tau'_2 = 0 = \tau'_0 = 0$, then let us show that q_1 and q_2 satisfy (2.41). Then $\phi(y_2'y_1') = q_2q_1 = x_2x_1 = -x_1x_2$, and

$$\begin{aligned}&\phi(p'_{12}y_1'y_2' + p'_{11}y_1'^2 + \tau'_1y_1' + \tau'_2y_2' + \tau'_0) = p'_{12}q_1q_2 + p'_{11}q_1^2 + \tau'_1q_1 + \tau'_2q_2 + \tau'_0\\ &= p'_{12}x_1x_2 + p'_{11}x_1^2 + \tau'_1x_1 + \tau'_2x_2 + \tau'_0 = -x_1x_2,\end{aligned}$$

whence $\phi(y_2'y_1') = \phi(p'_{12}y_1'y_2' + p'_{11}y_1'^2 + \tau'_1y_1' + \tau'_2y_2' + \tau'_0)$ as desired.

With respect to the condition (2.42), let

$$\sigma'(r) = \begin{bmatrix} \alpha(r) & 0 \\ 0 & \alpha(r) \end{bmatrix} \text{ and } \delta'(r) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for every $r \in \Bbbk[x_1]$. Note that, for $r \in \Bbbk$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \begin{bmatrix} q_1 r \\ q_2 r \end{bmatrix} = \begin{bmatrix} x_1 r \\ x_2 r \end{bmatrix},$$

and

$$\begin{aligned} \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r) &= \sigma'(r)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \delta'(r) \\ &= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r x_1 \\ r x_2 \end{bmatrix} = \begin{bmatrix} x_1 r \\ x_2 r \end{bmatrix}, \end{aligned}$$

which shows that $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \sigma'(r)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \delta'(r)$, and so it follows that $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a dcv-matrix with respect to (σ', δ').

From Propositions 1.8 and 1.9 we know when a double extension can be expressed as a two-step iterated Ore extension. In an analogous way, Theorem 2.37 establishes when a homomorphism of double extensions in the sense of dcv-matrix can be expressed as a homomorphism of two-step iterated Ore extensions in the sense of the cv-polynomials defined in Section 2.4.

THEOREM 2.37. *Let $B' = R_{P'}[y_1', y_2'; \sigma', \delta', \tau']$ and $B = R_P[y_1, y_2; \sigma, \delta, \tau]$ be right double extensions of the $\Bbbk$-algebra R. Consider the homomorphism φ between $B'^{\oplus 2}$ and $B^{\oplus 2}$ as in Definition 2.9. If the conditions*

$$\begin{gathered} p'_{12}\phi(y_1') = \sigma'_2(\phi(y_1')), \quad p'_{11} = 0, \quad \tau'_1\phi(y_1') = \delta'_2(\phi(y_1')), \quad \tau'_2 = \tau'_0 = 0, \\ \sigma'_{11}(r) = \sigma'_1(r), \quad \sigma'_{22}(r) = \sigma'_2(r), \quad \sigma'_{12}(-) = \sigma'_{21}(-) = 0, \end{gathered}$$

are satisfied, then φ can be presented as a homomorphism between iterated Ore extension.

Proof. Consider the relation (2.41). If $p'_{12}\phi(y_1') = \sigma'_2(\phi(y_1'))$, $p'_{11} = 0$, $\tau'_1\phi(y_1') = \delta'_2(\phi(y_1'))$, and $\tau'_2 = \tau'_0 = 0$, then

$$\phi(y_2'y_1') = q_2 q_1 = \sigma'_2(\phi(y_1'))q_2 + \delta'_2(\phi(y_1')) = \sigma'_2(q_1)q_2 + \delta'_2(q_1),$$

which is precisely the condition in the definition of cv-polynomials. On the other hand,

condition (2.42) implies

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \begin{bmatrix} \sigma'_{11}(r) & \sigma'_{12}(r) \\ \sigma'_{21}(r) & \sigma'_{22}(r) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \delta'_1(r) \\ \delta'_2(r) \end{bmatrix},$$
$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r = \begin{bmatrix} \sigma'_1(r) & 0 \\ 0 & \sigma'_2(r) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \delta'_1(r) \\ \delta'_2(r) \end{bmatrix},$$

or equivalently,

$$q_1 r = \sigma'_1(r) q_1 + \delta'_1(r), \quad \text{and} \quad q_2 r = \sigma'_2(r) q_2 + \delta'_2(r),$$

which are precisely the other two conditions in Definition 2.4. □

EXAMPLE 2.25. Consider Example 2.24. According to the homomorphism $\varphi : \mathbb{D}^{\oplus 2} \to \mathbb{E}^{\oplus 2}$ that maps the generators of the double extension to $x_1, x_2 \in R$ with $R = \Bbbk[x_1][x_2; \alpha]$, $p'_{12} = -1, p'_{11} = 0, \tau'_1 = \tau'_2 = 0 = \tau'_0 = 0$, and $\sigma_{11} = \sigma_{22} = \alpha, \sigma_{12} = \sigma_{21} = 0, \delta_1 = \delta_2 = 0$, the conditions of Theorem 2.37 are satisfied.

REMARK 13. Theorem 2.37 can be reformulated by using Proposition 1.8(iii). More exactly, in this case σ'_{21} need not be necessarily zero and $\delta'_2(r) = \delta'_2(r) - \sigma'_{21}(r) q_1$ for every $r \in R$, whence

$$\begin{aligned} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} r &= \begin{bmatrix} \sigma'_{11}(r) & \sigma'_{12}(r) \\ \sigma'_{21}(r) & \sigma'_{22}(r) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \delta'_1(r) \\ \delta'_2(r) \end{bmatrix} \\ &= \begin{bmatrix} \sigma'_1(r) & 0 \\ \sigma'_{21}(r) & \sigma'_2(r) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \delta'_1(r) \\ \delta'_2(r) - \sigma'_{21}(r) q_1 \end{bmatrix}, \end{aligned}$$

which are conditions in Definition 2.4.

EXAMPLE 2.26. The examples shown in Table 2.4 correspond to homomorphisms of iterated extensions. This is due that q_2 is a constant and σ'_{12} and σ'_{21} are zero.

2.6 FUTURE WORK

Research on morphisms between graded rings is extensive and can be approached from different points of view. Below we present some of them that are of interest to us.

First, it is a pending task to investigate the version of Theorem 2.26 for two-step, three-step, and n-step iterated Ore extensions. Also, the question on cv-polynomials and homomorphisms by considering the nilpotent elements of the coefficient ring R of the iterated Ore extension, such as in the works of Rimmer, Ferrero and Kishimoto, and Kikumasa, it is also of our interest.

Lam and Leroy's paper [LL92] motivated several articles about morphisms of noncommutative rings and matrix applications. For instance, these authors [LL94, LL88b] formulated a general Hilbert 90 Theorem for division rings in the setting of Ore extensions extending all known forms of this theorem (until this moment), while Leroy [Ler95, Ler12] studied relations between evaluation of Ore polynomials and pseudo-linear transformations. It is interesting

to investigate all these notions for different families of semi-graded rings such as iterated Ore extensions and double extensions.

Of course, it is a pending task to exemplify dcv-matrices consisting of polynomials of degree two.

On the other hand, Gómez and Suárez [GS20] presented necessary and sufficient conditions for a graded trimmed double extension to be a graded skew PBW extension (Definition 1.15), and in fact, they proved Artin-Schelter and skew Calabi-Yau properties for graded skew PBW extensions. Having in mind theory established in this chapter, a question arises whether it is possible to develop the theory of homomorphisms and cv-polynomials for graded skew PBW extensions, and if so, compare it with the results obtained here and those corresponding in [CLM11, LL92, SAR21, SCR21, LRS17, SR19].

Last but not least, Lü et al. [JLZ15] proved that the universal enveloping algebra of a Poisson-Ore extension is a length two iterated Ore extension of the original universal enveloping algebra, and then showed that the Poisson enveloping algebra of a double Poisson-Ore extension is an iterated double extension [LOWY18]. Related with this, Lou et al. [LOW20] gave a definition of Poisson double extension - which may be considered as an analogue of double extension -, and showed that algebras in a class of double extensions are deformation quantizations of Poisson double extensions. These works together Zambrano's paper [Zam20] where he gave a description of Poisson brackets on some families of skew PBW extensions, motivate us to ask ourselves about the relations between all these algebras through the notion of homomorphism and cv-polynomial developed in this chapter.

CHAPTER 3

MAPS BETWEEN SEMI-GRADED RINGS

In this chapter, we study morphisms between schematic semi-graded rings. As we said in the Introduction, our interest in this topic is due to Professor Jason Gaddis who asked us if we can use geometric tools to study families of semi-graded rings. We would like to express our sincere thanks to Professor Gaddis for introducing us to this project.

In Section 3.1 we start by considering the key notions of noncommutative algebraic geometry for $\mathbb{N}$-graded case following Rosenberg [Ros95] and Van den Bergh [VdB01]. Then, we recall Smith's result [Smi03, Theorem 3.2], [Smi16, Theorem 1.2] on maps between $\mathbb{N}$-graded rings.

Section 3.2 contains definitions and preliminaries on schematic semi-graded rings defined by Chacón and Reyes [Cha22, CR23]. For the completeness of the thesis, in Section 3.2.1 we recall the notion of schematic algebra introduced by Van Oystaeyen and Willaert [vOW95] in the setting of connected $\mathbb{N}$-graded rings, with some of its key results. Section 3.2.2 presents definitions and preliminaries on schematic semi-graded rings.

Having in mind Smith's papers [Smi03, Smi16] on maps between noncommutative projective spaces of the form $\mathrm{Proj}_{nc}A$, and the notion of schematic semi-graded ring, in Section 3.3 we consider maps in the Smith's sense but now in the more general setting of noncommutative projective spaces over semi-graded rings defined by Lezama and Latorre [LL17]. Our more important result, Theorem 3.21, extends Smith's key result [Smi03, Theorem 3.2] from the category of schematic $\mathbb{N}$-graded rings to the category of schematic semi-graded rings. This result contributes to the research on noncommutative geometry for semi-graded rings developed in the literature (e.g. [FGL$^+$20, HR20, Lez20, Lez21, LL17, LG19, RS22, SRS23]).

Section 3.4 presents some ideas for a future work.

3.1 MAPS BETWEEN GRADED RINGS

Let us recall some definitions and preliminaries of noncommutative projective geometry with the key results formulated by Smith [Smi03] on maps between $\mathbb{N}$-graded rings.

Following Rosenberg [Ros95, Ros98] and Van den Bergh's [VdB01] ideas of the noncommu-

tative projective geometry, Smith [Smi03, Smi16] takes a *Grothendieck category* as the basic non-commutative geometric object. More exactly, we think of a Grothendieck category $\mathsf{Mod}X$ (the idea for the notation $X = \mathsf{Mod}X$ is Van den Bergh's) as "the quasi-coherent sheaves on an imaginary non-commutative space X". The standard commutative example is the category $\mathsf{Qcoh}X$ of quasi-coherent sheaves on a quasi-separated, quasi-compact scheme X. The two non-commutative models are $\mathsf{Mod}R$, the category of right modules over a ring, and $\mathsf{Proj}A$, the non-commutative projective spaces having a (not necessarily commutative) graded ring A as homogeneous coordinate ring, which were defined by Verevkin [Ver92b, Ver92a] and Artin and Zhang [AZ94] (for more details, see [Smi02, Definition 2.2]). A map $g : Y \to X$ between two spaces is an adjoint pair of functors (g^*, g_*) with $g_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ and g^* left adjoint to g_*. If g_* is faithful and has a right adjoint, then g is called affine. An immediate example is a ring homomorphism $\varphi : R \to S$ that induces an affine map $g : Y \to X$ between the affine spaces defined by $\mathsf{Mod}Y := \mathsf{Mod}S$ and $\mathsf{Mod}X := \mathsf{Mod}R$.

Following Smith [Smi03, Section 2], for $\Bbbk$ a field, an $\mathbb{N}$-graded $\Bbbk$-algebra $A = A_0 \bigoplus A_1 \bigoplus \cdots$ is locally finite if $\dim_{\Bbbk} A_i < \infty$ for all i. The augmentation ideal $\mathfrak{m}$ of A is $A_1 \bigoplus A_2 \bigoplus \cdots$. Note that if A_0 is finite dimensional and A is right Noetherian, then it follows that $\dim_{\Bbbk} A_i < \infty$ for all i because $A_{\geq i}/A_{\geq i+1}$ is a Noetherian $A/\mathfrak{m}$-module. We write $\mathsf{GrMod}A$ for the category of $\mathbb{Z}$-graded right A-modules, and define

$$\mathsf{Tails}A = \mathsf{GrMod}A/\mathsf{Fdim}A,$$

where $\mathsf{Fdim}A$ is the full subcategory consisting of direct limits of finite dimensional A-modules. Equivalently, $\mathsf{Fdim}A$ consists of those modules in which every element is annihilated by a suitably large power of $\mathfrak{m}$. The symbol π means the quotient functor $\mathsf{GrMod}A \to \mathsf{Tails}A$ and ω denotes its right adjoint.

The noncommutative projective space X with homogeneous coordinate ring A is defined by $\mathsf{Mod}X = \mathsf{Tails}A$. We write $\mathsf{Proj}_{nc}\ A := (\mathsf{Mod}X, \mathscr{O}_X)$, where $\mathscr{O}_X$ is the image of A in $\mathsf{Mod}X$. Hence, $\mathsf{Proj}_{nc}\ A$ is an enriched quasi-scheme in the sense of [VdB01]. For Y another noncommutative projective space with homogeneous coordinate ring B, a map $f : \mathsf{Proj}_{nc}\ B \to \mathsf{Proj}_{nc}\ A$ is a map $f : Y \to X$ such that $f^*\mathscr{O}_X \cong \mathscr{O}_Y$. If A is a commutative $\mathbb{N}$-graded $\Bbbk$-algebra, we write $\mathsf{Proj}A$ for the classical projective scheme. We view a quasi-separated, quasi-compact scheme X as a noncommutative space by associating to it the enriched space $(\mathsf{Qcoh}X, \mathscr{O}_X)$. As it can be seen, the assignation $X \mapsto (\mathsf{Qcoh}X, \mathscr{O}_X)$ is a faithful functor.

In his papers, Smith considered the question on maps between noncommutative projective spaces. He called a map $g : Y \to X$ a closed immersion if it is affine and the essential image of $\mathsf{Mod}Y$ in $\mathsf{Mod}X$ under g^* is closed under submodules and quotients. A closed subspace Z of a space X is a full subcategory $\mathsf{Mod}Z$ of $\mathsf{Mod}X$ that is closed under submodules and quotient modules in $\mathsf{Mod}X$ and such that the inclusion functor $i_* : \mathsf{Mod}Z \to \mathsf{Mod}X$ has both a left adjoint i^* and a right adjoint $i^!$. Hence, two spaces are isomorphic if their module categories are equivalent. In this way, a map $Y \to X$ is a closed immersion if and only if it is an isomorphism from Y to a closed subspace X.

For J a graded ideal in a not necessarily commutative and locally finite $\mathbb{N}$-graded $\Bbbk$-algebra A, Smith [Smi03, Theorem 3.2] showed that a surjective homomorphism $A \to A/J$ of graded rings

induces a closed immersion $i : \mathrm{Proj}_{nc} A/J \to \mathrm{Proj}_{nc} A$ between the non-commutative projective spaces with homogeneous coordinate rings A and A/J.

3.2 SCHEMATIC SEMI-GRADED RINGS

It is well-known that Serre [Ser55] proved a theorem that describes the coherent sheaves on a projective scheme in terms of graded modules. Following Artin and Zhang [AZ94, Section 1], a commutative graded $\Bbbk$-algebra is associated to a projective scheme $\mathrm{Proj} A$, and the geometry of this scheme can be described in terms of the quotient category $\mathsf{qgr}\, A = \mathsf{gr}\, A/\mathsf{tors}$, where $\mathsf{gr}\, A$ denotes the category of graded modules and tors denotes its subcategory of torsion modules. For A a finitely generated commutative graded $\Bbbk$-algebra and X its associated projective scheme, if $\mathsf{coh}\, X$ denotes the category of coherent sheaves on X and $\mathscr{O}_X(n)$ is the nth power of the twisting sheaf on X [Har77, p. 117], then we have a functor $\Gamma_* : \mathsf{coh}\, X \to \mathsf{qgr}\ A$ given by

$$\Gamma_*(\mathscr{F}) = \bigoplus_{d=-\infty}^{\infty} \mathrm{H}^0(X, \mathscr{F} \otimes \mathscr{O}_X(d)),$$

and *Serre's theorem* asserts that if A is generated over $\Bbbk$ by elements of degree one, then Γ_* defines an equivalence of categories $\mathsf{coh}\, X \to \mathsf{qgr}\ A$ [Ser55, Section 59, Proposition. 7.8, p. 252], [GD61, 3.3.5] and [Har77, Proposition. II. 5.15].

Artin and Zhang [AZ94] extended Serre's theorem. For A an $\mathbb{N}$-graded algebra over a commutative Noetherian ring, they defined the associated projective scheme to be the pair $\mathrm{Proj}\ A = (\mathsf{qgr}\ A, \mathscr{A})$, where $\mathsf{qgr}\ A$ is the quotient category above and $\mathscr{A}$ is the object determined by the right module A_A, and considered the autoequivalence s of $\mathsf{qgr}\ A$ defined by the shift of degrees in $\mathsf{gr}\ A$. The object $\mathscr{A}$ plays the role of the *structure sheaf* of $\mathrm{Proj}\ A$ and s the role of the *polarization* defined by the projective embedding (this definition is the same as is given by Verevkin [Ver92a, Ver92b]). Since Serre's theorem does not hold for all commutative graded algebras, i.e., the functor defined by Γ_* need not be an equivalence, Artin and Zhang's definition of $\mathrm{Proj}\ A$ is compatible with the classical definition for commutative graded rings only under some additional hypotheses, such as that A is generated in degree one. In the literature, the noncommutative version of Serre's theorem is known as *Serre-Artin-Zhang-Verevkin theorem* [AZ94, Ver92a, Ver92b].

Van Oystaeyen and Willaert [vOW95] studied above $\mathrm{Proj}\ A$ by developing a kind of scheme theory similar to the commutative theory (note that Manin [Man91] commented the failure of attempts to obtain a noncommutative scheme theory à la Grothendieck [GD61] for quantized algebras). They noticed that this theory is possible only if the connected and $\mathbb{N}$-graded algebra considered contains "enough" Ore sets. Algebras satisfying this condition are called *schematic*, and some examples are homogenizations of almost commutative algebras, Rees rings of universal enveloping algebras of Lie algebras, and three-dimensional Sklyanin algebras. They constructed a *generalized Grothendieck topology* for the free monoid on all Ore sets of a schematic algebra R, and defined a *noncommutative site* (see [vOW96b] for more details) as a category with coverings on which sheaves can be defined, and formulated the Serre's theorem. As a consequence of their treatment, an equivalence between the category of all coherent sheaves

and the category Proj A was obtained in the sense of noncommutative algebraic geometry introduced by Artin [Art92]. In a series of papers [vO00, vOW96b, vOW96a, vOW97, Wil98], Van Oystaeyen and Willaert investigated several properties of schematic $\mathbb{N}$-graded rings.

Having in mind Van Oystaeyen and Willaert's ideas [vOW95] about a scheme theory for noncommutative $\mathbb{N}$-graded rings, and the interest on noncommutative algebraic geometry of semi-graded rings, Chacón in his PhD Thesis [Cha22] (see also [CR24, CR23]), presented a noncommutative scheme theory for semi-graded rings following the ideas presented in [vOW95], and investigated the *schematicness* of these objects in a more general context than $\mathbb{N}$-graded rings. With this theory, he generalized the results established in [vOW95] for $\mathbb{N}$-graded algebras to the semi-graded setting (as a matter of fact, he does not impose the condition of connectedness of the algebra appearing in [vOW95]), and present another approach to the Serre-Artin-Zhang-Verevkin theorem for semi-graded rings ([CR23, Examples 5.27 and 5.28] show that his theory and the corresponding presented by Lezama [Lez21, LL17] are independent).

3.2.1 GRADED SETTING

We start by recalling the theory of *schematic rings* introduced by Van Oystaeyen and Willaert [vOW95] to $\mathbb{N}$-graded rings.

In the noncommutative setting, for a *noncommutative positively graded Noetherian* $\Bbbk$-algebra $R = \Bbbk \oplus R_1 \oplus R_2 \oplus \cdots$ with $R = \Bbbk[R_1]$ (notice that R is connected, that is, $R_0 = \Bbbk$), Van Oystaeyen and Willaert [vOW95] presented his interpretation of Serre's theorem for algebras with enough Ore sets called *schematic algebras*. With the aim of presenting the key ideas developed by them, we consider some notions of torsion theory that we will use freely throughout the chapter. For more details, we refer to Goldman [Gol69], Stenstrom [Ste75] or Van Oystaeyen [vO78].

DEFINITION 3.1 ([vOW95, SECTION 2]). Let $\mathscr{L}$ be a set of left ideals of an arbitrary ring R. $\mathscr{L}$ is said to be *a filter* if it satisfies the following conditions:

T_1: If $I \in \mathscr{L}$ and $I \subseteq J$, then $J \in \mathscr{L}$.

T_2: If $I, J \in \mathscr{L}$, then $I \cap J \in \mathscr{L}$.

T_3: If $I \in \mathscr{L}$ and $a \in R$, then $(I : a) := \{r \in R \mid ra \in I\} \in \mathscr{L}$.

The functor $\kappa : R-\text{Mod} \to R-\text{Mod}$ defined by

$$\kappa(M) = \{m \in M \mid \text{ there exists } I \in \mathscr{L} \text{ with } Im = 0\},$$

is a *left exact preradical*, that is, a left exact subfunctor of the identity functor on the category $R-\text{Mod}$. A module M satisfying $\kappa(M) = M$ is called a κ*-torsion module*, and if $\kappa(M) = 0$, then M is said to be a κ*-torsion-free module*. It is straightforward to see that the family of torsion modules are closed under quotient objects and coproducts, while the torsion-free modules are closed under subobjects and products.

The filter $\mathscr{L}$ is called *idempotent* (also called a *Gabriel topology*) when it satisfies the following condition:

T_4: If $I \lhd_l R$ and there exists $J \in \mathscr{L}$ such that for all $a \in J$ the relation $(I : a) \in \mathscr{L}$ holds, then $I \in \mathscr{L}$.

Condition T_4 implies that $\mathscr{L}$ is closed under products and that the functor κ is a radical, that is, $\kappa(M/\kappa(M)) = 0$, for all $M \in R-\text{Mod}$.

PROPOSITION 3.1 ([CHA22, PROPOSITION 3.2]; [CR23, PROPOSITION 2.2]). *If R is a left Noetherian ring and $J_1 \supseteq J_2 \supseteq \cdots$ is a descending chain of two-sided ideals of R, then the set*

$$\mathscr{A} = \{I \lhd_l R \mid \text{ there exist elements } n, m \in \mathbb{N} \text{ with } (J_m)^n \subseteq I\},$$

is an idempotent filter.

Consider a ring R, $\mathscr{L}$ an idempotent filter of left ideals of R and its associated radical κ. For an R-module M, we need the quotient module $Q_\kappa(M)$ of M (Definition 3.3).

DEFINITION 3.2 ([CHA22, DEFINITION 3.3]; [CR23, DEFINITION 2.3]). Let $M \in R-\text{Mod}$. Consider the family Ω_M of pairs (I, f) with $I \in \mathscr{L}$ and $f : I \to M$ an R-homomorphism. We define the relation $\sim$ on Ω_M as $(I_1, f_1) \sim (I_2, f_2)$ if and only if there exists an element $J \in \mathscr{L}$ such that $J \subseteq I_1 \cap I_2$ and $f|_J = g|_J$.

It is straightforward to see that $\sim$ is an equivalence relation. The equivalence class of the element (I, f) is denoted as $[I, f]$, and the set of equivalence classes will be written as $M_{\mathscr{L}}$. For two elements $[I, f], [J, g] \in M_{\mathscr{L}}$, we define their sum as $[I, f] + [J, g] = [I \cap J, f + g]$. It is easy to see that this sum is well defined and that $(M_{\mathscr{L}}, +)$ is an Abelian group. It is also easy to see that if $I, J \in \mathscr{L}$ and $f \in \text{Hom}(I, R)$, then $f^{-1}(J) \in \mathscr{L}$. In this way, when we take elements $[I, f] \in R_{\mathscr{L}}$ and $[J, g] \in M_{\mathscr{L}}$, we can define the product of these elements as $[I, f] \cdot [J, g] = [f^{-1}(J), g \circ f]$. Notice that this product is well defined, and so $R_{\mathscr{L}}$ is actually a ring with identity $[R, \text{id}_R]$. Thus, $M_{\mathscr{L}}$ is a left $R_{\mathscr{L}}$-module.

Let $m \in M$. We define the application $\beta(m) : R \to M$ given by $\beta(m)(r) = rm$. It is well-known that $\beta : M \to \text{Hom}(R, M)$ is an isomorphism of R-modules. If we consider $\varphi_M : M \to M_{\mathscr{L}}$ defined by $\varphi_M(m) = [R, \beta(m)]$, then it follows that φ_R is a ring homomorphism, and so we can consider $R_{\mathscr{L}}$ and $M_{\mathscr{L}}$ as R-modules with the action given by $r[I, f] := [R, \beta(r)][I, f]$. Note that φ_M is actually a homomorphism of R-modules. Since $\text{Ker}(\varphi_M) = \kappa(M)$, the fundamental isomorphism theorem guarantees that $\varphi_M(M) \cong M/\kappa(M)$. In this way, if $\kappa(M) = 0$ then we can embed M into $M_{\mathscr{L}}$. For an element $\xi \in M_{\mathscr{L}}$ given by $\xi = [I, f]$, and an element $a \in I$, notice that $a\xi = [R, \beta(f(a))] = \varphi_M(f(a))$, which shows that $I\xi \subseteq \varphi_M(M)$, that is, $\text{Coker}(\varphi_M)$ is a κ-torsion module.

DEFINITION 3.3 ([STE75]; [VOW95, SECTION 2]; [CR23, DEFINITION 2.4]). The *quotient module of M* with respect to κ is defined as $Q_\kappa(M) = (M/\kappa(M))_{\mathscr{L}}$. Since $\mathscr{L}$ is idempotent, it follows that $\kappa(M/\kappa(M)) = 0$. Hence, we can embed $M/\kappa(M)$ into $Q_\kappa(M)$.

Equivalently, the *quotient module* of M respect to κ is given by

$$Q_\kappa(M) = \varinjlim_{I \in \mathscr{L}} \mathrm{Hom}_R(I, M/\kappa(M)).$$

$Q_\kappa(M)$ turns out to be a module over the ring $Q_\kappa(R)$.

Following [Ste75], recall that an R-module E is κ-*injective* if for every R-module M and each submodule N such that $\kappa(M/N) = M/N$, every R-homomorphism $N \to E$ can be extended to an R-homomorphism $M \to E$. We say that E is κ-*closed* (also known as *faithfully* κ-*injective*) if the extension of the homomorphism is unique. It is straightforward to see that E is κ-closed if and only if E is κ-injective and κ-torsion-free. By using these notions, we can characterize $Q_\kappa(M)$ in the following way: $Q_\kappa(M)$ is the unique κ-closed module containing $N = M/\kappa(M)$ such that $Q_\kappa(M)/N$ is κ-torsion.

EXAMPLE 3.1 ([vOW95, P. 111]). (i) Consider S a left Ore set in an arbitrary ring R. The set

$$\mathscr{L}(S) = \{I \lhd_l R \mid I \cap S \neq \emptyset\}$$

is an idempotent filter. If κ_S denotes its corresponding radical and $Q_S(M)$ is the module of quotients of M, then it is straightforward to see that $Q_S(M)$ is isomorphic to $S^{-1}M$, i.e., the classical Ore localization of M at S.

(ii) If $R = \bigoplus_{k\geq 0} R_k$ is a positively graded Noetherian ring and R_+ denotes the two-sided ideal $\bigoplus_{k\geq 1} R_k$, by Proposition 3.1, the set

$$\mathscr{L}(\kappa_+) = \{I \lhd_l R \mid \text{there exists } n \in \mathbb{N} \text{ with } (R_+)^n \subseteq I\}$$

is an idempotent filter. The corresponding radical is denoted by κ_+.

From the treatment above, having in mind that the filter $\mathscr{L}(\kappa_+)$ is idempotent, Van Oystaeyen and Willaert [vOW95] formed the *quotient category* (R,κ_+)-gr, that is, the full subcategory of $Q_{\kappa_+}(R)$-gr consisting of modules of the form $Q_{\kappa_+}(M)$ for some graded R-module M. Notice that (R,κ_+)-gr is equivalent to the full subcategory of R-gr consisting of the κ_+-closed modules and define Proj R as the Noetherian objects in (R,κ_+)-gr. Since they wanted to describe the objects of Proj R by means of objects of usual module categories in the same way as for commutative algebras, they need modules determined by Ore localizations. This is the content of the following definition.

DEFINITION 3.4 ([vOW95, DEFINITION 1]). The noncommutative positively graded Noetherian $\Bbbk$-algebra $R = \Bbbk \oplus R_1 \oplus R_2 \oplus \cdots$ with $R = \Bbbk[R_1]$ is *schematic* if there is a finite set I of homogeneous left Ore sets of R such that for every $S \in I$, $S \cap R_+ \neq \emptyset$, and such that one of the following equivalent properties is satisfied:

(i) for each $(r_S)_{S\in I} \in \prod_{S\in I} S$, there exists $m \in \mathbb{N}$ such that $(R_+)^m \subseteq \sum_{S\in I} Rr_S$,

(ii) $\bigcap_{S\in I} \mathscr{L}(S) = \mathscr{L}(\kappa_+)$,

(iii) $\bigcap_{S\in I} \kappa_S(M) = \kappa_+(M)$, for all $M \in R-\mathrm{Mod}$,

(iv) $\bigwedge_{S\in I} \kappa_S = \kappa_+$ where $\bigwedge$ denotes the infimum of torsion theories.

Van Oystaeyen and Willaert [vOW95, vOW96b] constructed the *noncommutative site*, a category with coverings on which sheaves can be defined, and formulated the Serre's theorem. Examples 3.2 and 3.3 contain remarkable examples of schematic algebras.

EXAMPLE 3.2. Recall that if R is a *positively* filtered $\Bbbk$-algebra by the family $(F_nR)_{n\geq 0}$ (i.e., $F_0R = \Bbbk$), $\sigma : R \to G(R)$ is the principal symbol map, and $\widehat{R}$ is the Rees-ring of R, it is well-known that $G(R)$ and $\widehat{R}$ are positively graded and there is a canonical central element X in $\widehat{R}$ of degree 1 such that $\widehat{R}/\langle X\rangle \cong G(R)$. If $\widehat{R}$ is Noetherian, this is equivalent to $G(R)$ being Noetherian or the filtration of R being Zariskian. Notice that if S is a multiplicatively set in R such that $\sigma(S)$ is a multiplicative set in $G(R)$, then $\widehat{R} = \{sX^{\deg\sigma(S)} \mid s \in S\}$ is a multiplicative set (consisting of homogeneous elements) in $\widehat{R}$. For more details about graded rings, Zariskian filtrations and Rees rings, see Li and Van Oystaeyen's book [LvO96].

For R positively filtered by $(F_nR)_{n\geq 0}$, if $G(R)$ is schematic then $\widehat{R}$ is schematic [vOW97, Theorem 1]. In this way, since for an almost commutative ring R there exists a filtration on R such that $G(R)$ is commutative, it follows that its Rees-ring is schematic. For example, the algebra R generated by three elements x, y and z of degree 1 with relations $xy - yx = z^2$, $xz - zx = 0$, and $yz - zy = 0$, is schematic since it is the Rees-ring of the first Weyl algebra $A_1(\Bbbk)$ with respect to the Bernstein-filtration (this algebra is known as the *homogenized Weyl algebra*).

Van Oystaeyen and Willaert [vOW97, p. 199] said that "it is probably not true that the class of schematic algebras is closed under iterated Ore extensions since Ore sets in a ring R need not be Ore in an Ore extension $R[x;\sigma,\delta]$". Nevertheless, the following proposition shows that under suitable conditions, these extensions are schematic.

PROPOSITION 3.2 ([vOW97, THEOREM 3]). *Given a positively graded ring R which is generated by R_1 and which is schematic by means of Ore sets S_i, given σ a graded automorphism of R and δ a σ-derivation of degree 1, then for all $s_i \in \prod S_i$ and for all $m \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that*

$$(R[x;\sigma,\delta]_+)^p \subseteq M := \sum_i R[x;\sigma,\delta]s_i + R[x;\sigma,\delta]x^m,$$

where $R[x;\sigma,\delta]$ denotes the Ore extension considered with graduation $(R[x;\sigma,\delta])_n = \bigoplus_{k=0}^{n} R_k x^{n-k}$.

Proposition 3.2 is one of the results that Van Oystaeyen and Willaert [vOW97] used to show that the algebras in Example 3.3 are schematic.

EXAMPLE 3.3 ([vOW97, EXAMPLES 2-5]). (i) The *coordinate ring of quantum* 2×2*-matrices* $\mathcal{O}_q(M_2(\mathbb{C}))$ with $q \in \mathbb{C}$ is generated by elements a, b, c and d subject to the relations

$$\begin{aligned} ba &= q^{-2}ab, & ca &= q^{-2}ac, & bc &= cb,\\ db &= q^{-2}bd, & dc &= q^{-2}cd, & ad - da &= (q^2 - q^{-2})bc. \end{aligned}$$

(ii) *Quantum Weyl algebras* $A_n^{\overline{q},\Lambda}$ defined by Alev and Dumas [AD94] are given by an $n\times n$ matrix $\Lambda = (\lambda_{ij})$ with $\lambda_{ij} \in \Bbbk^*$ and a row vector $\overline{q} = (q_1, \ldots, q_n)$, where $q_i \neq 0$ for every i, the algebra is generated by elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to relations $(i < j)$ given

by

$$x_i x_j = \mu_{ij} x_j x_i, \qquad x_i y_j = \lambda_{ji} y_j x_i, \qquad y_j y_i = \lambda_{ji} y_i y_j,$$
$$x_j y_i = \mu_{ij} y_i x_j \qquad x_j y_j = 1 + q_j y_j x_j + \sum_{i<j} (q_i - 1) y_i x_i,$$

where $\mu_{ij} = \lambda_{ij} q_i$.

(iii) *Three dimensional Sklyanin algebras $A_{\Bbbk}$ over a field $\Bbbk$* according to Artin et al. [ATdB90] are graded $\Bbbk$-algebras generated by three homogeneous elements x, y and z of degree 1 satisfying the relations

$$axy + byx + cz^2 = 0, \quad ayz + bzy + cx^2 = 0, \quad \text{and} \quad azx + bxz + cy^2 = 0,$$

where $a, b, c \in \Bbbk$.

(iv) Color Lie super algebras defined by Rittenberg and Wyler [RW78].

REMARK 14. Of course, there are examples of non-schematic algebras. If we take the graded algebra $\Bbbk\{x, y\}/\langle yx - xy - x^2\rangle$ and suppose that $\operatorname{char}(\Bbbk) = 0$, then its subalgebra generated by y and xy is not left schematic [vOW97, p. 203].

3.2.2 SEMI-GRADED SETTING

Next, we present the theory of schematic semi-graded rings developed by Chacón and Reyes [Cha22, CR23] on non-$\mathbb{N}$-graded and non-connected algebras.

With the aim of defining *good* Ore sets (Definition 3.5), for R an SG ring and an element $n \in \mathbb{Z}$, we consider the following sets:

$$R'_n = \{r \in R_n \mid \text{ for all } m \in \mathbb{Z}, \text{ and for all } h \in R_m, rh \in R_{n+m}\},$$
$$R''_n = \{r \in R'_n \mid \text{ for all } m \in \mathbb{Z}, \text{ and for all } h \in R_m, hr \in R_{n+m}\},$$
$$R' = \bigcup_{n\in\mathbb{Z}} R'_n,$$
$$R'' = \bigcup_{n\in\mathbb{Z}} R''_n.$$

DEFINITION 3.5 ([CHA22, DEFINITION 3.23]; [CR23, DEFINITION 3.8]). Let R be an SG ring and consider a left Ore set S of R. We say that S is *good* if the following conditions hold:

(i) $S \subseteq R''$, and,

(ii) if $s \in S$ and $r \in R'$, then there exist elements $u \in R'$ and $v \in S$ such that $us = vr$.

From Definition 3.5 it follows that for any elements $s_1, \ldots, s_k \in S$, there exist $r_1, \ldots, r_k \in R'$ such that $r_i s_i = r_j s_j \in S$, for every i, j.

Let R be an SG ring and M an SG R-module. We say that M is *localizable semi-graded* (LSG) if for every element $(n, m) \in \mathbb{Z}^2$, the inclusion $R'_n M_m \subseteq M_{n+m}$ holds [Cha22, Definition 3.24].

PROPOSITION 3.3 ([CHA22, PROPOSITION 3.25]; [CR23, PROPOSITION 3.10]). *Let R be an* SG *ring, S a good left Ore set, and M an* LSG *R-module. Then $S^{-1}M$ is an* LSG *R-module with semi-graduation given by*

$$(S^{-1}M)_n = \left\{ \frac{f}{s} \mid f \in \bigcup_{k\in\mathbb{Z}} M_k,\ \deg(f) - \deg(s) = n \right\}.$$

The next result shows that the localization of an SG ring by considering a good Ore set is again an SG ring.

PROPOSITION 3.4 ([CHA22, PROPOSITION 3.26]; [CR23, PROPOSITION 3.11]). *Let R be an* SG *ring and S a good left Ore set. Then $S^{-1}R$ is an* SG *ring with semigraduation given by*

$$(S^{-1}R)_n = \left\{ \frac{f}{s} \mid f \in \bigcup_{k\in\mathbb{Z}} R_k,\ \deg(f) - \deg(s) = n \right\}.$$

PROPOSITION 3.5 ([CHA22, PROPOSITION 3.27]). *Let R be an* SG *ring, S a good left Ore set and M an* LSG *R-module. Then $S^{-1}M$ is an* SG *$S^{-1}R$-module.*

Let LSG$-R$ be the full subcategory of SGR$-R$ whose objects are the LSG R-modules. This subcategory is closed for subobjects, quotients and coproducts, so it is Abelian (see [CR24, Section 3.2] for more details).

Following Van Oystaeyen and Willaert's ideas developed in [vOW95], in this section we define the notion of *schematicness* in the setting of semi-graded rings. For a positively SG ring R, we define $R_+ = \bigoplus_{k\geq 1} R_k$ and we say that a left Ore set S is *non-trivial* if $S \cap R_+ \neq \emptyset$.

We start with the following observation:

REMARK 15. If R is a positively SG left Noetherian ring, then Proposition 3.1 shows that

$$\mathscr{L}(\kappa_+) = \{I \lhd_l R \mid \text{there exist } n, m \in \mathbb{N} \text{ with } (R_{\geq m})^n \subseteq I\}$$

is an idempotent filter. The corresponding left exact radical is denoted by κ_+ and $Q_{\kappa_+}(M)$ is the module of quotients of M.

DEFINITION 3.6 ([CHA22, DEFINITION 3.29]; [CR23, DEFINITION 4.1]). Let R be a positively SG left Noetherian ring. R is called (*left*) *schematic* if there is a finite set I of non-trivial good left Ore sets of R such that for each $(x_S)_{S\in I} \in \prod_{S\in I} S$, there exist $t, m \in \mathbb{N}$ such that $(R_{\geq t})^m \subseteq \sum_{S\in I} Rx_s$.

The following result illustrates some characterizations of being schematic (c.f. Definition 3.4).

PROPOSITION 3.6 ([CHA22, PROPOSITION 3.30]; [CR23, PROPOSITION 4.2]). *Let R be a positively* SG *left Noetherian algebra and $S_1, \ldots, S_n$ a finite set of non-trivial good left Ore sets of R. The following conditions are equivalent:*

(1) *For each $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} S_i$, there exist elements $t, m \in \mathbb{N}$ such that $(R_{\geq t})^m \subseteq \sum_{S\in I} Rx_s$.*

(2) *Let $I \lhd_l R$. If I has no trivial intersection with every S_i, then I contains a power of $R_{\geq t}$ for*

some $t \in \mathbb{N}$.

(3) $\bigcap_{i=1}^{n} \mathscr{L}(S_i) = \mathscr{L}(\kappa_+)$.

If R is schematic by considering the *good* left Ore sets S_i, then $\bigcap_{i=1}^{n} \kappa_{S_i}(M) = \kappa_+(M)$, for every R-module M. If M is an $\mathsf{LSG}-$module, then for each $i = 1, \ldots, n$ we have that $\kappa_{S_i}(M)$ is an SG submodule, and so $\kappa_+(M)$ is also an SG submodule. These facts imply that $M/\kappa_+(M)$ is an SG R-module, and so a submodule of $Q_{\kappa_+}(M)$. The idea is to show that $Q_{\kappa_+}(M)$ is semi-graded. For the remainder of the section, $\mathscr{L} := \mathscr{L}(\kappa_+)$.

Let us start by taking an LSG R-module M such that $\kappa_+(M) = 0$. It is clear that $Q_{\kappa_+}(M) = M_{\mathscr{L}}$ and $\varphi_M(M) \cong M$. Thus, $\varphi_M(M)$ is a submodule of $M_{\mathscr{L}}$ which is an SG R-module where $\varphi_M(m)$ is homogeneous of degree k if and only if m is homogeneous of degree k. If we want $M_{\mathscr{L}}$ to be an LSG R-module, it must be satisfied that if ξ is homogeneous of degree k, then for every $s \in R'$, the element $s\xi \in (\varphi_M(M))_{\deg(s)+k}$. Now, since there exists $I \in \mathscr{L}$ with $I\xi \subseteq \varphi_M(M)$, the following definition makes sense.

Definition 3.7 ([Cha22, Definition 3.31]; [CR23, Definition 4.3]). Let $\xi \in M_{\mathscr{L}}$. We say that the element ξ is *homogeneous of degree* k if there exists $I \in \mathscr{L}$ such that $I\xi \subseteq \varphi_M(M)$, and for every element $s \in I \cap R'$, $s\xi \in (\varphi_M(M))_{\deg(s)+k}$.

Notice that if the condition above is satisfied for I, then it also holds for every $J \subseteq I$. Lemma 3.7 shows that this condition is true for ideals containing I.

Remark 16. Since the good Ore sets S_i are non-trivial, there exist elements $s_i' \in S_i \cap R_+$ for $i = 1, \ldots, n$, whence $\alpha_i = \deg(s_i') > 0$. If we define $m := \operatorname{lcm}\{\alpha_i\}_{1 \leq i \leq n}$ and $s_i'' := (s_i')^{m/\alpha_i}$, then we obtain $s_i'' \in S_i \cap R_+$, and all of them have the same degree. Now, if we consider an element $I \in \mathscr{L}$, there exist $t, n \in \mathbb{Z}$ such that $(R_{\geq t})^n \subseteq I$. Thus, $s_i = (s_i'')^{tn} \in I \cap S_i$, which implies that $\sum_{i=1}^{n} Rs_i \subseteq I$. In this way, for each $I \in \mathscr{L}$ there exist elements $s_i \in S_i$, all with the same positive degree, satisfying the relation $\sum_{i=1}^{n} Rs_i \subseteq I$.

Lemma 3.7 ([CR23, Lemma 3.3]; [CR23, Lemma 4.5]). *Let $I, J \in \mathscr{L}$ be ideals such that $I \subseteq J$ and $I\xi, J\xi \subseteq \varphi_M(M)$. If for every $s \in I \cap R'$ the element $s\xi \in \big(\varphi_M(M)\big)_{\deg(s)+k}$, then the same property holds for each $s \in J \cap R'$.*

From Lemma 3.7, it is sufficient to guarantee the property by considering any ideal I such that $I\xi \subseteq \varphi_M(M)$. Our purpose is to give a more simple method to verify that the element ξ is homogeneous. Let $\xi = [I, f]$. Since $I\xi \subseteq \varphi_M(M)$, the element ξ is homogeneous of degree k if and only if for each $s \in I \cap R'$ the element $s\xi = [R, \beta(f(s))] = \varphi_M(f(s)) \in (\varphi_M(M))_{\deg(s)+k}$, or equivalently, for all $s \in I \cap R'$, the element $f(s) \in M_{\deg(s)+k}$. For a morphism $f : I \to M$, we will say that f is *homogeneous of degree* k if for each $s \in I \cap R'$, the element $f(s)$ is homogeneous of degree $\deg(s) + k$. Hence, $[I, f]$ is homogeneous of degree k (in $M_{\mathscr{L}}$) if and only if f is homogeneous of degree k. Let $(M_{\mathscr{L}})_k$ be the family of homogeneous elements of degree k. It is clear that $(M_{\mathscr{L}})_k$ is a subgroup and $\varphi_M(M_k) \subseteq (M_{\mathscr{L}})_k$.

We will say that the morphism $f : I \to M$ is *strongly homogeneous of degree* k if for every homogeneous element $s \in I$, the element $f(s)$ is homogeneous of degree $\deg(s) + k$. It is clear that in the setting of graded rings, the notions of homogeneous morphism and strongly homogeneous

morphism coincide. On the other hand, $[I, f]$ it will be called *strongly homogeneous of degree* k if some of its representative elements is strongly homogeneous of degree k. Let $\overline{(M_{\mathscr{L}})_k}$ be the family of strongly homogeneous elements of degree k. It is straightforward to see that $\overline{R_{\mathscr{L}}} = \bigoplus \overline{(R_{\mathscr{L}})_k}$ is a graded ring and $\overline{M_{\mathscr{L}}} = \bigoplus \overline{(M_{\mathscr{L}})_k}$ is an $\overline{R_{\mathscr{L}}}$-graded module. Note also that if $s \in R''$ then $\varphi_R(s) \in \overline{R_{\mathscr{L}}}$; in particular $\overline{R_{\mathscr{L}}}$, is an extension of the graded ring $\varphi_R(\bigoplus R''_k)$. As it is clear, $\overline{R_{\mathscr{L}}}$ is an R-submodule of $R_{\mathscr{L}}$ if and only if R is graded. This last remark shows that in the setting of non-graded rings is not appropriate to consider strongly homogeneous morphisms [Cha22, Remark 3.34].

By [Cha22, Proposition 3.35], the sum $\sum \overline{(M_{\mathscr{L}})_k}$ is direct. Let $[I, f] \in M_{\mathscr{L}}$ with $I = \sum_{i=1}^{n} Rs_i$, for some elements $s_i \in S_i \cap R''_k$. Since there are finitely s_i's, we can consider that the homogeneous decompositions of the elements $f(s_i)$ have the same length, say $f(s_i) = \sum_{t=\alpha}^{\beta} (f(s_i))_{t+k}$, where $(f(s_i))_j$ is the j-th homogeneous component of $f(s_i)$. By taking $f_t(s_i) = (f(s_i))_{t+k}$, we have $f(s_i) = \sum_{t=\alpha}^{\beta} f_t(s_i)$. For elements $t = \alpha, \ldots, \beta$, we define the maps $f_t : I \to M$ in the natural way as $f_t(\sum a_i s_i) = \sum a_i f_t(s_i)$. From [Cha22, Propositions 3.36 and 3.37] we know that these maps are homogeneous $R-$homomorphisms of degree t.

PROPOSITION 3.8 ([CHA22, PROPOSITION 3.38]; [CR23, PROPOSITION 4.10]). *If M is an* LSG *R-module with $\kappa_+(M) = 0$, then $Q_{\kappa_+}(M) = M_{\mathscr{L}}$ is an* LSG *R-module with semigraduation given by*

$$M_{\mathscr{L}} = \bigoplus_k \overline{(M_{\mathscr{L}})_k}.$$

PROPOSITION 3.9 ([CHA22, THEOREM 3.39]; [CR23, THEOREM 4.11]). *If M is an* LSG *$R-$module, then $Q_{\kappa_+}(M)$ is an* LSG *$R-$module.*

Different families of schematic semi-graded rings can be found in [Cha22, Section 3.3] or [CR24, Section 5].

3.3 MAPS BETWEEN SCHEMATIC SEMI-GRADED RINGS

By considering the ideas formulated in Section 1.1, let SGR be the *category of semi-graded rings* whose objects are the semi-graded rings and whose morphisms are the homogeneous ring homomorphisms. If we fix an SG ring R, $\mathsf{SGR} - R$ denotes the *category of semi-graded modules over R* whose morphisms are the homogeneous $R-$homomorphisms. It is straightforward to see that $\mathsf{SGR} - R$ is preadditive, and that its zero object is the trivial module. Let $f : M \to N$ be a morphism in $\mathsf{SGR} - R$. Since $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are semi-graded submodules, it follows that $N/\mathrm{Im}(f)$ is a semi-graded module. This fact guarantees that the category $\mathsf{SGR} - R$ has kernels and cokernels. If f is a monomorphism of $\mathsf{SGR} - R$, then f is the kernel of the canonical homomorphism $j : N \to N/\mathrm{Im}(f)$. If f is an epimorphism, then f is the cokernel of the inclusion $i : \mathrm{Ker}(f) \to M$. In this way, the category $\mathsf{SGR} - R$ is normal and conormal.

If $\{M_i\}_{i \in I}$ is a family of objects of $\mathsf{SGR} - R$, then their direct sum $\bigoplus_{i \in I} M_i$ is a semi-graded ring with semi-graduation given by

$$\left(\bigoplus_{i \in I} M_i\right)_p := \bigoplus_{i \in I} (M_i)_p, \ p \in \mathbb{Z}.$$

It is easy to see that this object with the natural inclusions coincides with the coproduct of the familiy of objects $\{M_i\}_{i\in I}$ in $\mathsf{SGR}-R$. Therefore, $\mathsf{SGR}-R$ is an Abelian category.

This section contains the original results of the chapter. We start with the following fact on morphisms in the category $\mathsf{SGR}-R$.

PROPOSITION 3.10. *Let $f : N \to M$ be a morphism of $\mathsf{SGR}-R$ and $X \subseteq N$. Then $f(\langle X\rangle^{\mathsf{SG}}) = \langle f(X)\rangle^{\mathsf{SG}}$.*

Proof. Let $y \in \langle f(X)\rangle^{\mathsf{SG}}$. Then $y \in M'$ for every SG-submodule M' of M with $f(X) \subseteq M'$. Since $\langle X\rangle^{\mathsf{SG}}$ is an SG-submodule of N and $X \subseteq \langle X\rangle^{\mathsf{SG}}$, then $f(\langle X\rangle^{\mathsf{SG}})$ is an SG-submodule of M such that $f(X) \subseteq f(\langle X\rangle^{\mathsf{SG}})$, whence $y \in f(\langle X\rangle^{\mathsf{SG}})$. This shows that $\langle f(X)\rangle^{\mathsf{SG}} \subseteq f(\langle X\rangle^{\mathsf{SG}})$.

On the other hand, if $y \in f(\langle X\rangle^{\mathsf{SG}})$ then there exists $x \in \langle X\rangle^{\mathsf{SG}}$ with $f(x) = y$. Let M' be an SG-submodule of M such that $M' \supseteq f(X)$. Then $f^{-1}(M')$ is an SG-submodule of N with $X \subseteq f^{-1}(M')$, which implies that $x \in f^{-1}(M')$, and hence $y = f(x) \in M'$ and $y \in \langle f(X)\rangle^{\mathsf{SG}}$. Thus, $f(\langle X\rangle^{\mathsf{SG}}) \subseteq \langle f(X)\rangle^{\mathsf{SG}}$. □

Having in mind Definition 3.1, we can say that for R an SG-ring and M an R-module, an element $m \in M$ is a *torsion element* if there exist $n, t \geq 0$ such that $R^n_{\geq t} m = 0$. The set of torsion elements of M is denoted by $T(M)$. M is called a *torsion module* if $T(M) = M$, and it is said to be *torsion-free* if $T(M) = 0$. From [LL17, Remark 5.4] we know that $T(M)$ is an R-submodule of M.

If R is schematic by considering the *good* left Ore sets S_i as in Definition 3.6, and we define

$$\kappa_{S_i}(M) = \{m \in M \mid \exists s \in S_i, sm = 0\},$$

for each i, then $\bigcap_{i=1}^{n} \kappa_{S_i}(M) = T(M)$ for every SG R-module M. From the ideas presented in Section 3.2.2, if M is an LSG R-module, then for all $i = 1, \ldots, n$, $\kappa_{S_i}(M)$ is an SG submodule of M, and so $T(M)$ is also an SG submodule of M. These facts imply that $M/T(M)$ is an SG R-module.

Let $\mathsf{TOR}-R$ be the full subcategory of $\mathsf{SGR}-R$ consisting of the torsion modules of R. Following the ideas presented by Lezama and Latorre [LL17, Theorem 5.5], it is easy to see that $\mathsf{TOR}-R$ is a Serre subcategory of $\mathsf{SGR}-R$.

For R and S two SG rings, if $f : R \to S$ is a homomorphism of SG rings and M is an SG S-module, by defining $r\,m := f(r)m$, for elements $r \in R_a$ and $m \in M_b$, we get $f(r) \in S_a$, and so $r\,m = f(r)m \in \bigoplus_{c\leq a+b} M_c$. This guarantees that M is also an SG R-module with the semi-graduation given as S-module. Under these conditions, it is clear that if $g : M \to N$ is a homomorphism of SG S-modules, then g is also a homomorphism of SG R-modules. In this way, we obtain the functor

$$f_* : \mathsf{SGR}-\mathsf{S} \to \mathsf{SGR}-\mathsf{R}, \tag{3.1}$$

$$M \mapsto f_*(M) = M \tag{3.2}$$

$$g \mapsto f_*(g) = g \tag{3.3}$$

where $f_*(M)$ is the same module M considered as an R-module. In particular, if J is an SG

ideal of R (that is, a two-sided ideal that is an SG submodule of R), then R/J is an SG ring and the canonical morphism $f : R \to R/J$ is homogeneous. This fact guarantees the existence of the functor

$$f_* : \mathsf{SGR} - R/J \to \mathsf{SGR} - R. \tag{3.4}$$

As in the case of Smith [Smi03, Theorem 3.2], [Smi16, Theorem 1.2], we are interested in the existence of induced closed immersions of projective spaces, so our aim in this section is to define a pair of adjoint functors to f_*.

DEFINITION 3.8. Consider M an SG R-module. We define $f^!(M)$ as the largest SG R-submodule of M that is annihilated by J.

We have the following immediate result on $f^!(M)$.

PROPOSITION 3.11. *$f^!(M)$ is an SG R/J-module by defining $\overline{r}\, m = r\, m$, for every $\overline{r} \in R/J$ and each $m \in f^!(M)$.*

Proof. Let $r, s \in R$ with $\overline{r} = \overline{s}$. Then $r - s \in J$, and since $m \in f^!(M)$, m is annihilated by J, which implies that $0 = (r-s)m = rm - sm$, i.e., $rm = sm$. It is clear that the product defined satisfies the definition of module. Finally, for elements $\overline{r} \in (R/J)_a$ and $m \in M_b$, without loss of generality we can take $r \in R_a$, and get $\overline{r}\, m = r\, m \in \bigoplus_{c \leq a+b} M_c$, whence the assertion follows. □

PROPOSITION 3.12. *Let $g : M \to N$ be a morphism of SG R-modules. Then $g(f^!(M)) \subseteq f^!(N)$.*

Proof. Consider $m \in f^!(M)$ and $j \in J$. Since m is annihilated by J, $jm = 0$, and so $jg(m) = g(jm) = g(0) = 0$. This shows that $g(m)$ is annihilated by J, and so $g(m) \in f^!(N)$. □

Propositions 3.11 and 3.12 allow us to define the functor

$$f^! : \mathsf{SGR} - R \to \mathsf{SGR} - R/J \tag{3.5}$$

$$M \mapsto f^!(M) \tag{3.6}$$

$$g \mapsto f^!(g) = g|_{f^!(M)}, \tag{3.7}$$

that sends every morphism to its restriction. If no confusion arises, we write $f^!(g) = g$.

PROPOSITION 3.13. *$(f_*, f^!)$ is an adjoint pair.*

Proof. Consider M and N be objects belonging to $\mathsf{SGR} - R/J$ and $\mathsf{SGR} - R$, respectively. If $g : f_*(M) \to N$ is a morphism of $\mathsf{SGR} - R$, then it is clear that $\mathrm{Im}(g) \subseteq f^!(N)$, and hence we get the isomorphism $\mu_{MN} : \mathrm{Hom}(M, f^!(N)) \to \mathrm{Hom}(f_*(M), N)$ given by $\mu_{MN}(g) = g$. This guarantees the required natural isomorphism. □

With the aim of defining the other adjoint pair, recall that in the graded setting, to every R-module M is assigned the R/J-module M/JM. However, since in the case of semi-graded rings JM is not necessarily an SG submodule of M (for instance, take $R = \Bbbk[x]$, $J = Rx$, and $M = A_1(\Bbbk)$, the first Weyl algebra), then we cannot assert that M/JM is an object in $\mathsf{SGR} - R/J$. This fact

motivates us to consider $f^*(M) := M/\langle JM\rangle^{\mathsf{SG}}$. Precisely, since $\langle JM\rangle^{\mathsf{SG}}$ is an SG submodule of M, then $f^*(M)$ is also an SG R-module, and it is easy to see that $f^*(M)$ is an SG R/J-module by defining $\overline{a}\cdot\overline{m} = \overline{am}$, and considering the semi-graduation of $f^*(M)$ as an SG R-module. In this way, if $\alpha : N \to M$ is a morphism of $\mathsf{SGR}-R$ modules, then it is possible to define

$$\begin{aligned} \alpha^* : f^*(N) &\to f^*(M) \\ \overline{m} &\mapsto \overline{\alpha(m)}. \end{aligned}$$

More exactly,

PROPOSITION 3.14. *α^* is a morphism of the category $\mathsf{SGR}-R/J$.*

Proof. Since $\alpha(JN) \subseteq JM$, then $\alpha(\langle JN\rangle^{\mathsf{SG}}) = \langle \alpha(JN)\rangle^{\mathsf{SG}} \subseteq \langle JM\rangle^{\mathsf{SG}}$, which shows that α^* is well-defined. It is immediate to see that α^* is a homogeneous R/J-homomorphism. □

From the discussion above, we assert the existence of the functor

$$f^* : \mathsf{SGR}-R \to \mathsf{SGR}-R/J, \tag{3.8}$$

such that the following proposition holds:

PROPOSITION 3.15. *(f^*, f_*) is an adjoint pair.*

Proof. Consider M an object in $\mathsf{SGR}-R$ and N an object in $\mathsf{SGR}-R/J$. Let $h : M \to f_*(N)$ a morphism in $\mathsf{SGR}-R$. If $m \in M$ and $j \in J$, then $h(jm) = jh(m) = \overline{j}h(m) = 0$, which implies $JM \subseteq \mathrm{Ker}(h)$, and hence $\langle JM\rangle^{\mathsf{SG}} \subseteq \mathrm{Ker}(h)$. Then the map $\lambda_{MN}(h) : f^*(M) \to N$ given by $\lambda_{MN}(h)(\overline{m}) = h(m)$ is well-defined, and in fact, it is clear that it is a morphism in $\mathsf{SGR}-R/J$.

On the other hand, if $g : f^*(M) \to N$ is a morphism in $\mathsf{SGR}-R/J$, it is straightforward to see that the map $\lambda'_{MN}(g) : M \to f_*(N)$ defined by $\lambda'_{MN}(g)(m) = g(\overline{m})$ is a morphism in $\mathsf{SGR}-R$.

Finally, note that the maps

$$\begin{aligned} &\lambda_{MN} : \mathrm{Hom}(M, f_*(N)) \to \mathrm{Hom}(f^*(M), N), \text{ and} \\ &\lambda'_{MN} : \mathrm{Hom}(f^*(M), N) \to \mathrm{Hom}(M, f_*(N)) \end{aligned}$$

are inverses of each other, and correspond to the natural isomorphisms that guarantee that (f^*, f_*) is an adjoint pair. □

DEFINITION 3.9. Let $\mathscr{A}$ be a subcategory of $\mathsf{SGR}-R$ (or $\mathsf{Mod}-R$). We say that an object E of $\mathscr{A}$ is *T-injective* if the following condition holds: for an object M and N a subobject of M (both in $\mathscr{A}$) with M/N being of T-torsion, then every morphism from N to E can be extended to M. If this extension is unique, then E is said to be T-closed.

As it occurs in the graded case, E is T-closed if and only if E is T-torsion-free and T-injective. Note that this notion of being T-closed is equivalent to the corresponding introduced by Gabriel [Gab62].

If M is an R-module, there exists a T-closed module $Q(M)$ in $\mathsf{Mod}-R$ which is called the *module of quotients* of M satisfying the following property: there exists an injective homomorphism $\Phi_M : M/T(M) \to Q(M)$ such that $\mathrm{Coker}(\Phi_M)$ is torsion. For more details on the construction of $Q(M)$, see Goldman [Gol69, Section 3], Stenstrom [Ste75, Chapter IX] or Van Oystaeyen [vO78].

If M is an object in $\mathsf{LSG}-R$, then $Q(M)$ is also an object in $\mathsf{LSG}-R$ [CR24, Theorem 4.11]. In this case, the semi-graduation of $Q(M)$ is defined as follows: since $M' = \mathrm{Im}(\Phi_M)$ is isomorphic to $M/T(M)$, then M' has a natural semi-graduation. Now, for $\xi \in Q(M)$, we say that ξ is *homogeneous of degree* k if and only if there exist elements $n, t \in \mathbb{N}$ such that $R^n_{\geq t}\xi \subseteq M'$, and for all $s \in R^n_{\geq t} \cap R'$, the element $s\xi$ is homogeneous of degree $k + \deg(s)$. By considering this semi-graduation, it is straightforward to see that Φ_M is a morphism in $\mathsf{LSG}-R$.

PROPOSITION 3.16. *If M is an $\mathsf{LSG}-R$ module, then $Q(M)$ is T-closed in $\mathsf{LSG}-R$.*

Proof. We know that $Q(M)$ is T-torsion-free and T-injective in $\mathsf{Mod}-R$. Let $N \in \mathsf{LSG}-R$, N' an SG submodule of N such that N/N' is T-torsion, and $f : N' \to Q(M)$ an homogeneous R-homomorphism. Since $Q(M)$ is T-injective in $\mathsf{Mod}-R$, there exists an R-homomorphism $g : N \to Q(M)$ that extends f, we only have to show that g is also homogeneous.

Consider $m \in N_k$. By using that N/N' is T-torsion, there exist elements $n_1, t_1 \in \mathbb{N}$ with $R^{n_1}_{\geq t_1} m \subseteq N'$. On the other hand, let M' be the isomorphic image of $M/T(M)$ in $Q(M)$. Since $Q(M)/M'$ is T-torsion, there exist $t_2, n_2 \in \mathbb{N}$ with $R^{n_2}_{\geq t_2} g(m) \subseteq M'$. If $n := \max\{n_1, n_2\}$ and $t := \max\{t_1, t_2\}$, then $R^n_{\geq t} g(m) \subseteq M'$, and for every $s \in R^n_{\geq t} \cap R'$, we get that sm is homogeneous and $sm \in N'$ (recall that N is LSG). Hence, $sg(m) = g(sm) = f(sm) \in (M')_{\deg(s)+k}$ due to that f is homogeneous. By considering the semi-graduation of $Q(M)$, it follows that $g(m) \in (Q(M))_k$, whence g is homogeneous. □

Since in general we do not know if the canonical functor $\pi : \mathsf{SGR}-R \to \mathsf{SGR}-R/\mathsf{TOR}-R$ has a right adjoint, from now on we consider a schematic ring R and the full subcategory $\mathsf{LSG}-R$.

Since $\mathsf{LSG}-R$ is a subcategory of $\mathsf{SGR}-R$ closed for subobjects and quotients, then $\tau := \mathsf{TOR}-R \cap \mathsf{LSG}-R$ is a Serre subcategory of $\mathsf{LSG}-R$. As a matter of fact, due to [Gab62, Proposition 4, Section 17], the canonical functor $\pi : \mathsf{LSG}-R \to \mathsf{LSG}-R/\tau$ has a right adjoint, say ω. On the other hand, by [Gab62, Corollary 4, Section 17; Proposition 13(a), Section 17] the category $\mathsf{LSG}-R/\tau$ is equivalent to the full subcategory $\mathscr{C}-R$ of $\mathsf{LSG}-R$ consisting of the T-closed modules. Note that we can define the functor

$$\begin{aligned} Q : \mathsf{LSG}-R &\to \mathscr{C}-R \\ M &\mapsto Q(M), \end{aligned}$$

and in fact, $Q = \omega\pi$. Without loss of generality, from now on we assume that $Q(M)$ is an extension of $M/T(M)$ and Φ_M is the inclusion map.

Having in mind that our objects of interest are schematic rings, we need to guarantee that if J is a SG ideal of a schematic ring R then R/J is also schematic. We say that J is a *compatible* ideal of R if $R'/J = (R/J)'$. The following result shows the importance of this condition.

PROPOSITION 3.17. *If R is a schematic ring and J is a compatible* SG *ideal of R, then R/J is also schematic.*

Proof. It is easy to see that if S is a good Ore set of R, then S/J is also a multiplicative set that satisfies Ore's condition. Now, by using that R is left Noetherian, then R/J also is, and it is well-known that S/J is an Ore set of R/J. Since J is compatible, we get S/J is a good Ore set.

On the other hand, for every $n \in \mathbb{N}$, it can be seen that $R_{\geq n}/J = (R/J)_{\geq n}$, which guarantees that for $S_1, \ldots, S_k$ good Ore sets of R satisfying the schematic condition, then $S_1/J, \ldots, S_k/J$ are good Ore sets of R/J satisfying this condition. □

Notice that if M is an $\mathsf{LSG}-R/J$ module, then $f_*(M)$ is an $\mathsf{LSG}-R$ module. If J is a compatible ideal of R, then for each $\mathsf{LSG}-R$ module M, we get that $f^*(M)$ and $f^!(M)$ are $\mathsf{LSG}-R/J$ modules.

PROPOSITION 3.18. *If E is an object in $\mathscr{C}-R$, then $f^!(E)$ is an object in $\mathscr{C}-R/J$.*

Proof. Consider M an object in $\mathsf{LSG}-R/J$, N a subobject of M with M/N of torsion, and $g: N \to f^!(E)$ a morphism in $\mathsf{LSG}-R/J$. Let $j: f^!(E) \to E$ be the inclusion. By considering the morphism $j \circ g: f_*(N) \to E$ and using that E is closed, there exists a homogeneous R-homomorphism $h: f_*(M) \to E$ that extends to $j \circ g$. Besides, it is clear that $f^!(h): M \to f^!(E)$ extends to g.

Finally, due to $f^!(E)$ is a submodule of E, and E is torsion-free, then $f^!(E)$ is torsion-free in $\mathsf{LSG}-R$, and so it is torsion-free in $\mathsf{LSG}-R/J$. □

We define the morphisms

$$i_*: \mathscr{C}-R/J \to \mathscr{C}-R, \quad i^!: \mathscr{C}-R \to \mathscr{C}-R/J, \quad \text{and} \quad i^*: \mathscr{C}-R \to \mathscr{C}-R/J,$$

as the restriction maps of Qf_*, $f^!$, and $Q'f^*$, respectively, where Q is the functor assigning the module of quotients in the category $\mathsf{LSG}-R$ and Q' is the same functor in $\mathsf{LSG}-R/J$.

LEMMA 3.19. *$(i_*, i^!)$ is an adjoint pair and i_* is fully faithful.*

Proof. 1. $(i_*, i^!)$ is an adjoint pair: Let M be an object in $\mathscr{C}-R/J$ and N an object in $\mathscr{C}-R$. If $g: M \to f^!(N)$ is a homogenous R/J-homomorphism and $j: f^!(N) \to N$ is the inclusion map, then $j \circ g: M \to N$ is a homogeneous R-homomorphism. Since N is T-closed, there exists a unique map $\overline{g}: i_*(M) = Q(M) \to N$ extending $j \circ g$. This fact allows to obtain the map $\gamma_{MN}: \mathrm{Hom}(M, f^!(N)) \to \mathrm{Hom}(i_*(M), N)$ defined by $\gamma_{MN}(g) = \overline{g}$. It is clear that γ_{MN} is a monomorphism.

On the other hand, note that $M \subseteq f^!(Q(M))$ and $f^!(Q(M))/M$ is a torsion module, whence $f^!(Q(M))$ is the module of quotients of M (in $\mathsf{LSG}-R/J$), and by using that M is T-closed, we get $f^!(Q(M)) = M$. In this way, if $h: i_*(M) \to N$ is a homogeneous R-homomorphism, then $h = \gamma_{MN}(f^!(h))$, and so γ_{MN} is a bijective map. It is straightforward to see that these maps induce the natural isomorphism that show that $(i_*, i^!)$ is an adjoint pair.

2. i_* is fully faithful: From the definition of f_* we know that it is fully faithful. It is easy to show that when restricted to free torsion modules, the functor Q is also fully faithful. From these facts we get that i_* is fully faithful

□

With the aim of proving the other adjunction, we require that J satisfies the following additional $(*)$: for any elements $n, t \in \mathbb{N}$, when $x \in J \cap R^n_{\geq t}$ there exist $n_x, t_x \in \mathbb{N}$ such that $R^{n_x}_{\geq t_x} x \subseteq JR^n_{\geq t}$. Let us just mention that this condition holds trivially in commutative cases. The importance of condition $(*)$ is shown in the following lemma which is key in the proof of Theorem 3.21.

LEMMA 3.20. *If J satisfies condition $(*)$ and E is an object in $\mathscr{C} - R/J$, then $f_*(E)$ is an object in $\mathscr{C} - R$.*

Proof. Recall that the construction of the module of quotients in $\mathsf{LSG} - R$ is the same as in the category $\mathsf{Mod} - R$, so it follows that a semi-graded R-module M is closed in $\mathsf{LSG} - R$ if and only if is closed in $\mathsf{Mod} - R$. In this way, we only have to show that E is closed in $\mathsf{Mod} - R$.

Consider elements $n, t \in \mathbb{N}$, $I = R^n_{\geq t}$ and $g : I \to E$ an R-homomorphism. We define the maps

$$\begin{aligned} g^* &: I/JI \to E, \quad \overline{x} \mapsto g(x) \\ j &: I/JI \to R/J, \quad \overline{x} \mapsto x. \end{aligned}$$

Since E is an R/J-module, g^* is well-defined. It is clear that g and j are R/J-homomorphisms. Note that $\mathrm{Coker}(j)$ is torsion, and by using that J satisfies $(*)$, we get that $\mathrm{Ker}(j)$ is torsion. As E is closed in $\mathsf{Mod} - R/J$ there exist an R/J-homomorphism $h : R/J \to E$ with $h \circ j = g^*$. If we take the canonical map $\theta : R \to R/J$, then the R-homomorphism $h \circ \theta : R \to E$ extends g, and by [Gol69, Proposition 3.2] E is T-injective in $\mathsf{Mod} - R$. Besides, since E is T-torsion-free in $\mathsf{Mod} - R$ (due to that E is closed in $\mathsf{Mod} - R/J$), it follows that E is T-closed in $\mathsf{Mod} - R$. □

Finally, we get the most important result of the chapter that extends partially Smith's result [Smi03, Theorem 3.2], [Smi16, Theorem 1.2].

THEOREM 3.21. *Let R be a schematic SG ring and J a compatible SG ideal of R satisfying $(*)$. Then the map $i : \mathscr{C} - R/J \to \mathscr{C} - R$ given by the functors $(i^*, i_*, i^!)$ is a closed immersion.*

Proof. By Lemma 3.19, we only have to show that (i^*, i_*) is an adjoint pair and that the essential image of i_* is closed for subobjects and quotients.

Let M be an object in $\mathscr{C} - R$ and N an object in $\mathscr{C} - R/J$. Due to Lemma 3.20, we get $(i_* = f_*)$, and hence the morphism $\upsilon_{MN} : \mathrm{Hom}(M, i_*(N)) \to \mathrm{Hom}(i^*(M), N)$ defined by $\upsilon_{MN} = Q' \circ \lambda_{MN}$, where λ_{MN} is the map defined in 3.15, induces the required natural isomorphism.

By last, note that in the category $\mathscr{C} - R$ the subobjects of an object M are the closed SG-submodules of M, and if N is a subobject of M then the quotient is given by $Q(M/N)$. Besides, since $\mathrm{Im}(i_*)$ is the subcategory consisting of the SG-R-modules M such that $f^!(M) = M$, and

in this case we have $Q(M/N) = Q'(M/N)$, it follows that $\mathrm{Im}(i_*)$ is closed under quotients and submodules. □

We get immediately Smith's result in the schematic graded case formulated by Van Oystaeyen and Willaert [vOW95].

COROLLARY 3.22 ([SMI03, THEOREM 3.2]; [SMI16, THEOREM 1.2]). *Let J be a graded ideal in an $\mathbb{N}$-graded schematic $\Bbbk$-algebra A satisfying $(*)$. Then the homomorphism $A \to A/J$ induces a closed immersion $i : \mathrm{Proj}_{nc} A/J \to \mathrm{Proj}_{nc} A$.*

.

Proof. Since A is $\mathbb{N}$-graded, then the category $\mathsf{GrMod} - A$ coincides with the category $\mathsf{LSG} - A$. Finally, it is clear that in the graded case, the graded ideals are SG compatible. □

3.4 FUTURE WORK

For R a schematic SG ring and J a compatible SG ideal of R that satisfies $(*)$, we have showed that the canonical map $R \to R/J$ induces a categorical map $\mathscr{C} - R/J \to \mathscr{C} - R$. Since in the $\mathbb{N}$-graded setting the categories $\mathsf{LSG} - R$ and $\mathsf{gr}R$ coincide, it follows that the category $\mathscr{C} - R$ is equivalent to the category $\mathrm{Proj}_{nc}(R)$ defined by Smith, Theorem 3.21 presents a partial generalization of Smith's result [Smi03, Theorem 3.2]; [Smi16, Theorem 1.2] since he did not assume the condition of schematicness on the $\mathbb{N}$-graded ring. As we saw above, in the semi-graded case we need this condition to guarantee the existence of the module of quotients in (at least) the category $\mathsf{LSG} - R$. An immediate task is to investigate if the category $\mathsf{SGR} - R$, or an appropriate subcategory of this, has modules of quotients. Thinking about it, a possible way to solve this problem is due to Gabriel [Gab62], which consists in to prove that $\mathsf{SGR} - R$ has enough injectives (in fact, this would allow to formulate of notion of *cohomology* in the semi-graded setting). Precisely, it is important to note that our methodology here is different from that considered by Smith, since he used the well-known fact that the category of modules over an $\mathbb{N}$-graded ring is a Grothendieck category, while in the semi-graded context we do not know if $\mathsf{SGR} - R$ is also Grothendieck; we only know that $\mathsf{SGR} - R$ is $\mathtt{Ab5}$ [Cha22, Section 1.5]. Of course, if we found a positive answer then automatically we guarantee the existence of enough injectives in $\mathsf{SGR} - R$. This will be our line of research in the near future.

We are also interested in Smith's papers [Smi01, Smi02]. In the first, he called a non-commutative space X *integral* if there is an indecomposable injective X-module $\mathscr{E}_X$ such that its endomorphism ring is a division ring and every X-module is a subquotient of a direct sum of copies of $\mathscr{E}_X$. A Noetherian scheme is integral in this sense if and only if it is integral in the usual sense. Smith proved that several classes of non-commutative spaces over $\mathbb{N}$-graded rings are integral. On the other hand, in [Smi02] he investigated the concepts of closed points, closed subspaces, open subspaces, weakly closed and weakly open subspaces, and effective divisors, on a non-commutative space over an $\mathbb{N}$-graded ring. As expected, the characterization of all these notions in the setting of (schematic) semi-graded rings is a very interesting problem.

As a possible topic of research on morphisms of semi-graded rings, we have the X-inner automorphisms. Montgomery [Mon81] computed the X-inner automorphisms of filtered algebras R whose associated graded ring $G(R)$ is a commutative domain. She used this result to classify the X-inner automorphisms of enveloping algebras $U(L)$ of Lie algebras L, as well as the R-stabilizing X-inner automorphisms of skew polynomial rings of derivation type $R[x;\delta]$. The results appearing in [Mon81, Mon83] have been generalized by several authors. For instance, Osterburg and Passman [OP90] computed the R-stabilizing X-inner automorphisms of smash products $R\#U(L)$. Leroy and Matczuk [LM92] found the R-stabilizing X-inner automorphisms of skew polynomial rings of mixed type $R[x;\sigma,\delta]$ with σ an automorphism of R. For the Ore extension $R[x;\sigma,\delta]$ over a prime ring R with σ an automorphism of R, Leroy and Matczuk [LM92] determined the center, the extended centroid, and its X-inner automorphisms. Related with this, Bergen and Wilson [BW99] examined the X-inner automorphisms of *semi-commutative algebras* (a semi-commutative algebra is, by definition, a filtered algebra A whose associated graded ring $G(A)$ is the coordinate ring of quantum space $\mathcal{O}_q(\Bbbk^n)$) defined by McConnell [McC90] and Musson [Mus93] which include several quantum algebras such as quantum symplectic space, quantum Euclidean space, quantum matrices, q-analogs of the Heisenberg algebra, and the quantum Weyl algebra are examples of semi-commutative algebras. We consider a natural task to study this kind of morphisms for semi-graded rings.

Last but not least, Bell and Zhang proved that for A and B two connected graded algebras finitely generated in degree one, if A is isomorphic to B as ungraded algebras, then they are also isomorphic to each other as graded algebras [BZ17, Theorem 1]. Equivalently, suppose that an algebra A has two graded algebra decompositions $A = \bigoplus_{i=0}^{\infty} A_i = \bigoplus_{i=0}^{\infty} B_i$, such that

(1) $A_0 = B_0 = \Bbbk$,

(2) A is generated by A_1 (resp. by B_1), and

(3) either A_1 or B_1 is finite dimensional over $\Bbbk$.

Then there is an algebra automorphism $\phi : A \to A$ such that $\phi(A_i) = B_i$, for all i [BZ17, Corollary 2]. The natural question concerns if for two connected semi-graded algebras the result holds.

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