Some Homotopical Aspects of de Rham Theory

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Abstract

The study of topological properties of manifolds from the point of view differential forms and the equations they satisfy is known as de Rham theory. Topological invariants of manifolds such as cohomology groups and characteristic classes can be naturally described in de Rham's language. This thesis deals with more categorical invariants of manifolds that can also be studied via differential forms. We take the point of view of representation theory, where one studies groups via their linear actions on vector spaces. In topology, the corresponding linear actions are called *infinity local systems*, and are the subject of this thesis. We describe how various aspects of de Rham theory can be categorified to the study of these representations of spaces. One new aspect that emerges is the need to replace the strict notion of associativity by a version of associativity which is more compatible with the methods of homotopy theory. This is the notion of A_∞-*structures*, which are algebraic structures where associativity only holds up to an infinite sequence of homotopies.

Keywords: de Rham cohomology, local systems, parallel transport, Riemann-Hilbert correspondence, iterated integral, representation theory, A_∞ structure, infinite groupoid, representation up to homotopy, flat connection, homotopy, holonomy, principal 2-bundle.

Resumen

El estudio de las propiedades topológicas de las variedades suaves desde el punto de vista de formas diferenciales y de las ecuaciones que dichas formas satisfacen es conocido como teoría de de-Rham. Invariantes topológicos de variedades tales como los grupos de cohomología y las clases características se pueden describir naturalmente en el lenguaje de de-Rham. Esta tesis trata con invariantes de tipo categórico que también pueden ser descritos en términos de formas diferenciales. Adoptamos el punto de vista de la teoría de representaciones, donde se estudian grupos mediante sus acciones lineales en espacios vectoriales. En topología, las correspondientes acciones lineales son llamadas *sistemas locales infinitos*, los cuales son el objeto de estudio de esta tesis. Describimos cómo varios aspectos de la teoría de de-Rham se pueden categorificar, lo que conlleva al estudio de sistemas locales. Una nueva característica que emerge en este contexto es la necesidad de reemplazar la noción de asociatividad estricta por una noción de asociatividad compatible con los métodos de teoría de homotopía. Esta nueva noción de asociatividad está codificada en las estructuras A_∞, que son estructuras algebraicas donde la asociatividad solo se cumple salvo una secuencia infinita de homotopías.

Título en español: Algunos aspectos homotópicos de la teoría de de Rham.

Palabras clave: cohomología de de Rham, sistemas locales, transporte paralelo, correspondencia de Riemann-Hilbert, integrales iteradas, teoría de representación, estructuras A_∞, grupoide infinito, representación salvo homotopía, conexión plana, homotopía, holonomía, 2-fibrado principal.

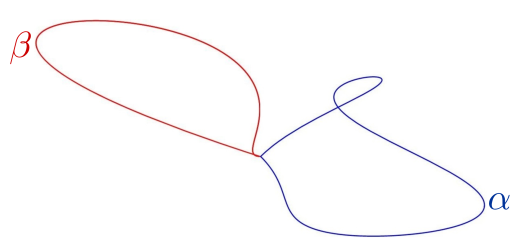
Introduction

The study of topological properties of manifolds from the point of view differential forms and the equations they satisfy is known as de Rham theory. Topological invariants of manifolds such as cohomology groups and characteristic classes can be naturally described in de Rham's language. This thesis deals with more categorical invariants of manifolds that can also be studied via differential forms. We take the point of view of representation theory, where one studies groups via their linear actions on vector spaces. In topology, the corresponding linear actions are called *infinity local systems*, and are the subject of this thesis. We describe how various aspects of de Rham theory can be categorified to the study of these representations of spaces. One new aspect that emerges is the need to replace the strict notion of associativity by a version of associativity which is more compatible with the methods of homotopy theory. This is the notion of A_∞-*structures*, which are algebraic structures where associativity only holds up to an infinite sequence of homotopies.

The theory of A_∞-structures was introduced by J. Stasheff in the 1960's, and is related to the algebraic structure that is present in the based loop space. Let X be a topological space with a base point $x_0 \in X$. The based loop space of X is the space of all continuous maps $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1) = x_0$ and is denoted ΩX. The based loop space is just the space of paths in X beginning and ending in x_0 . This space comes with an obvious binary operation $* : \Omega X \times \Omega X \to \Omega X$ defined by concatenation of paths. Explicitly, if $\alpha, \beta \in \Omega X$, then

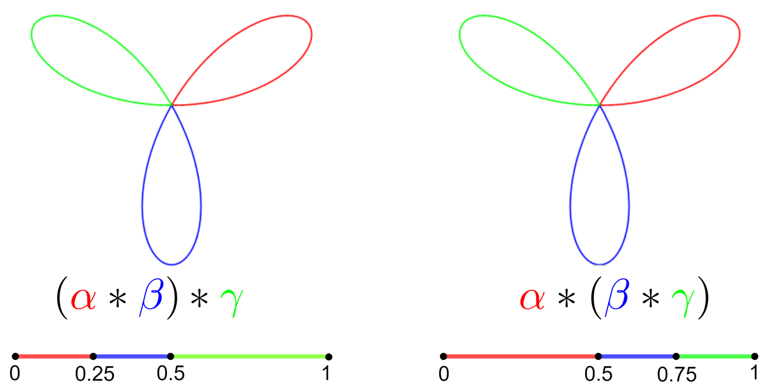
$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if} \quad t \in [0, 1/2], \\ \beta(2t-1) & \text{if} \quad t \in [1/2, 1]. \end{cases}$$

The path $\alpha * \beta$ goes over α in the first half of the interval and over β on the second half.



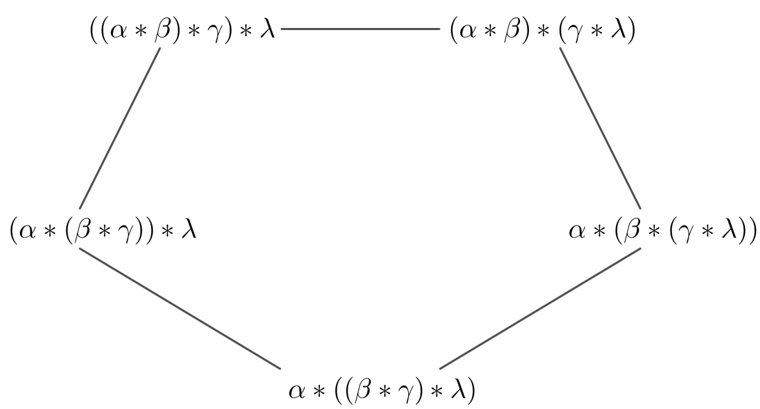
The concatenation operation is not strictly associative. The following figure shows how

$$(\alpha * \beta) * \gamma \neq \alpha * (\beta * \gamma)$$



However, there is a homotopy between the paths obtained by continuously moving the interval $[1/4, 1/2]$ towards the interval $[1/2, 3/4]$. This homotopy is just a continuous path $H : [0, 1] \to \Omega X$ joining $(\alpha * \beta) * \gamma$ to $\alpha * (\beta * \gamma)$. A homotopy exists regardless of the three paths chosen, therefore we always have a map $M_3 : [0, 1] \times (\Omega X)^3 \to \Omega X$. In this setting we denote the interval by K_3.

If we consider the concatenation of four paths, we find that there are five different ways to associate the paths and homotopies between them. If we denote the pentagon by K_4 then we have a map $M_4 : K_4 \times (\Omega X)^4 \to \Omega X$.



In general we have a sequence of polytopes $\{K_n\}_{n\geq 2}$ called the Stasheff polytopes or associahedra. The polytope K_n has dimension $n - 2$ and each vertex represents a different way of composing n paths. Notice that K_2 is a single point space. There are maps $M_n : K_n \times (\Omega X)^n \to \Omega X$ for each $n \geq 2$ providing homotopies between the ways of composing n paths and higher homotopies between them. Furthermore, the maps $\{M_n\}$ satisfy certain compatibility conditions. For more details check [25], [26]. [1]

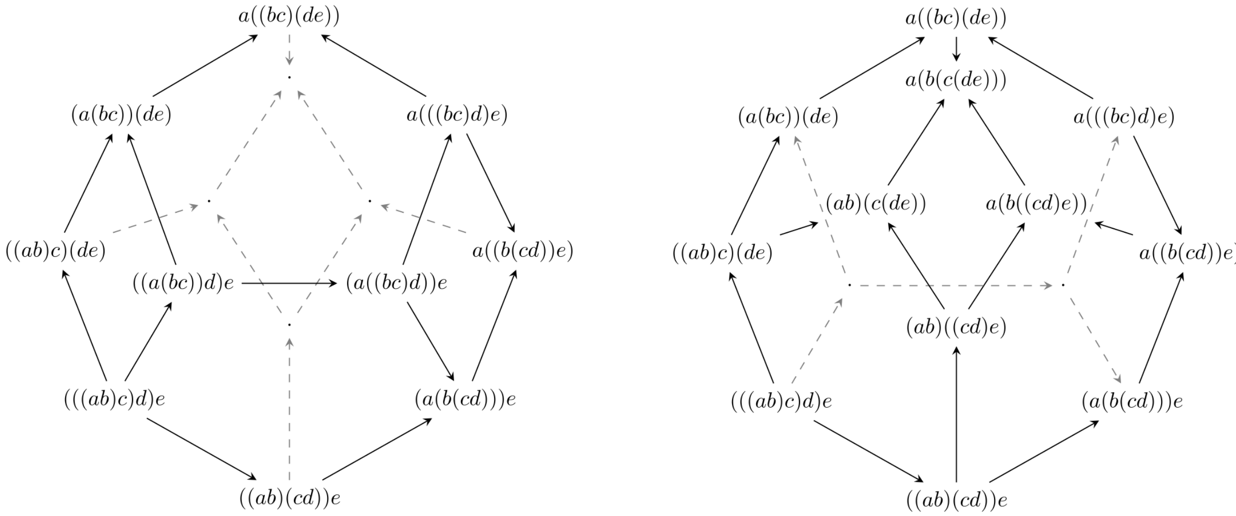


Figure 1: K_5, the associahedron of dimension 3

Based on the previous discussion one defines an A_∞-space as a topological space Y with operators $M_n : K_n \times Y^n \to Y$ for $n \geq 2$ such that certain coherence conditions are satisfied. A_∞-algebras arise as linearized versions A_∞-spaces. Let $A = C_*(Y)$ be the singular chain complex of an A_∞-space Y. There is a boundary operator $\partial : A \to A$, which we denote m_1. Furthermore, we can define operators $m_n : A^{\otimes n} \to A$ for $n \geq 2$ relying on the push-forward $(M_n)_* : C_*(K_n \times Y^n) \to A$. By Kunneth's formula we know that $C_*(K_n \times Y^n)$ is quasi-isomorphic to $C_*(K_n) \otimes A^{\otimes n}$, therefore there is a map $(M_n)_\sharp : C_*(K_n) \otimes A^{\otimes n} \to A$. We define m_n by the formula

$$m_n(\sigma_1 \otimes \cdots \otimes \sigma_n) := (M_n)_\sharp(\kappa_n \otimes \sigma_1 \otimes \cdots \otimes \sigma_n),$$

where $\kappa_n \in C_n(K_n)$ is the fundamental class of K_n. The sequence of operators $\{m_n\}$ satisfies an infinite sequence of relations derived from the corresponding compatibility conditions on $\{M_n\}$. In a few words, the topological homotopies encoded by $\{M_n\}$ turn into algebraic homotopies encoded by $\{m_n\}$. This is the origin of the theory of A_∞-algebras.

We will make use of the formalism of A_∞-structures at the categorical level. One of the classical forms of the Riemann-Hilbert correspondence states that there is a correspondence between flat connections over a manifold M and representations of the fundamental group $\pi_1(M)$. More precisely, the category of flat vector bundles over M and the category of representations of the fundamental group of M are equivalent. The main ingredient of the equivalence is the construction of holonomies along paths, which relies on solving the parallel transport differential equation. Iterated integrals are an efficient way of describing explicit solutions to the parallel transport equation. If $\gamma : [0,1] \to M$ is a path and $E \to M$ is a

[1]Figure 1.1 by Omaranto - Own workThis SVG is produced from Niles Johnson's TikZ code found at File:Associahedron K5 front.png., CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=76581752

vector bundle with fibre V and a flat connection, then the pullback of the connection via γ is determined by a form $\alpha = A(t)dt$ where $A(t) \in \mathrm{End}(V)$. A map $\varphi^\alpha : [0,1] \to \mathrm{End}(V)$ that satisfies the parallel transport equation is defined by the formula

$$\varphi^\alpha(t) = \mathrm{id}_V + \sum_{k\geq 1} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} A(s_1)\cdots A(s_k) ds_k \cdots ds_1.$$

Thus the equivalence functor $\mathsf{Loc}(M) \to \mathcal{R}\mathrm{ep}(\pi_1(M))$ may be thought of as an integration functor. This functor also has an interpretation from the point of view of Lie theory. Local systems over M are representations of the Lie algebroid TM. The Riemann-Hilbert correspondence states that representations of the Lie algebroid may be integrated to Lie groupoid representations.

There are several works describing higher versions of local systems and their properties; some references are Arias Abad-Schätz [4], Holstein [15], Malm [19], Ben-Zvi-Nadler [7], Brav-Dyckerhoff [9] and Rivera-Zeinalian [22]. We are particularly interested in the generalization of the Riemann-Hilbert correspondence that has appeared in [8], [16] and later in [2]. An ∞-local system over M is a $\mathbb{Z}$-graded vector bundle over M of finite rank together with a flat $\mathbb{Z}$-graded connection. ∞-local systems over M fit together into a category denoted $\mathsf{Loc}_\infty(M)$. On the other hand we have the fundamental ∞-groupoid of the manifold $\pi_\infty(M)$ and its representations up to homotopy $\mathcal{R}\mathrm{ep}^\infty(\pi_\infty(M))$. The fundamental ∞-groupoid is the simplicial set $\{X_\bullet\}$ where $X_k = \mathrm{Hom}_{\mathrm{smooth}}(\Delta_k, M)$ is the set of smooth k-simplices in M. A representation up to homotopy of the groupoid is a $\mathbb{Z}$-graded vector bundle over M together with a way to assign holonomies to simplices of all dimensions in a coherent manner. It is worth noting that both categories have the structure of dg-categories, which means that they are categories enriched in the category of cochain complexes. In this setting, the higher Riemann-Hilbert correpondence states that there is an A_∞-equivalence $\mathsf{Loc}_\infty(M) \to \mathcal{R}\mathrm{ep}^\infty(\pi_\infty(M))$. In other words, a flat $\mathbb{Z}$-graded connection over M provides enough data to construct holonomies along simplices of all dimensions.

The higher Riemann-Hilbert correspondence can be thought of as a categorical generalization of the de Rham theorem. $\mathsf{Loc}_\infty(M)$ and $\mathcal{R}\mathrm{ep}^\infty(\pi_\infty(M))$ are topological invariants of M and can be considered as categorical analogues of the de Rham algebra and the singular cochain algebra of M, respectively. The de Rham theorem states that both complexes are equivalent in the sense that both have isomorphic cohomologies, the equivalence is proven via an integration map $\int : \Omega^\bullet(M) \to C^\bullet(M)$ that descends to an isomorphism of algebras in cohomology. Although both complexes have the structure of differential graded algebras (dga), the integration map does not preserve the product. This should not be surprising since the de Rham complex is a commutative algebra while the singular cochain complex is noncommutative. A more complete explanation of the situation was provided by Gugenheim in [14], where he shows that a more appropriate environment for the comparison of these dga's is the category of A_∞-algebras. The main result proven by Gugenheim, known as the A_∞ de Rham theorem, states that the integration map $\int : \Omega^\bullet(M) \to C^\bullet(M)$ is merely the first component of an A_∞-morphism between dga's. Roughly speaking, an A_∞-morphism is a morphism that preserves the algebraic structure only up to homotopy, together with an infinite set of maps keeping track of the homotopies and satisfying some coherence conditions. The construction of Gugenheim's map uses Chen's theory of iterated integrals developed in [11]. Furthermore, the A_∞ de Rham theorem is instrumental in the proof of the higher Riemann-Hilbert correspondence: in [2] it is used to construct the equivalence functor $\mathsf{Loc}_\infty(M) \to \mathcal{R}\mathrm{ep}^\infty(\pi_\infty(M))$ explicitly in terms of iterated integrals.

In the categorical setting we encounter a similar situation to that of the de Rham theorem. The appropriate way of comparing the dg-categories $\mathsf{Loc}_\infty(M)$ and $\mathcal{R}\mathrm{ep}^\infty(\pi_\infty(M))$ is via an A_∞-functor, not a dg-functor. An A_∞-functor is a functor that preserves the composition only up to homotopy. Furthermore, as we will se below, the comparison of such functors is done via A_∞-natural transformations, that is to say transformations between functors in which the usual diagram is not commutative but commutative up to homotopy.

The first part of this thesis deals with some invariance properties of the category $\mathsf{Loc}_\infty(M)$. Given the higher Riemann-Hilbert correspondence and the analogy between the de Rham complex and the category of ∞-local systems, it is reasonable to explore which of the usual properties of de Rham cohomology can be generalized to the categorical setting of ∞-local systems. One of the basic properties of de Rham cohomology is that two homotopical maps $f, g : M \to N$ induce the same morphism in cohomology, i.e. $\bar{f}^* = \bar{g}^* : H(N) \to H(M)$. We prove the following categorical version of this invariance:

Theorem 0.1. Let M and N be smooth manifolds and $f, g : M \to N$ be smooth maps. If $h : [0,1] \times M \to N$ is a homotopy with $h \circ \iota_0 = f$ and $h \circ \iota_1 = g$, then there is an A_∞-natural isomorphism $\mathrm{hol} : f \Rightarrow g$ which depends on h and is given in terms of iterated integrals.

As a corollary of Theorem 2.3 one obtains an A_∞ version of the Poincaré lemma. This states that if M is contractible, then $\mathsf{Loc}_\infty(M)$ is equivalent to the category of ∞-local systems over a point, which is the category $\mathrm{DGVect}_\mathbb{R}$ of differential graded vector spaces. These results were published in [1].

The second part studies the comparison between different formalisms for higher holonomies. There are several models of 2-bundles with connections such as the ones that appear in Murray [20], Aschieri-Cantini-Jurco [5], Laurent-Gengoux-Stiénon-Xu [18], Breen-Messing [10], Wockel [29], Schommer-Pries [23]. In some of the models there are accounts of parallel transport along paths and surfaces such as in Baez-Schreiber [6] and Schreiber-Waldorf [24], in particular the relation between the connection and the parallel transport is made explicit in [28]. Not surprisingly, the parallel transport defined in the setting of principal 2-bundles is encoded in a representation of the ∞ groupoid truncated up to dimension 2, this is $\{\mathrm{Hom}_{\mathrm{smooth}}(\Delta_k, M)\}_{k\leq 2}$. This representation takes values in the structure group of the principal 2-bundle, which is a strict 2-group. This warrants a comparison between the higher holonomies provided by the higher Riemann-Hilber correspondence and the ones constructed via connections on principal 2-bundles. This comparison has already been made locally in [3], where it is shown that a truncation procedure on the $\mathbb{Z}$-graded connection yields a representation of the truncated groupoid in a strict 2-group. Here we address the problem of comparing both constructions of 2-dimensional parallel transport in the global setting. The main device for the global comparison is a two-tier construction that yields a principal 2-bundle with flat connection from an ∞-local system. At the first tier the construction is almost identical to the frame bundle construction which requires nothing more than a vector bundle, while at the second tier we rely heavily on the extra structure present in the $\mathbb{Z}$-graded vector bundle with flat $\mathbb{Z}$-graded connection. The conclusion we reach is that the following diagram is commutative

$$\begin{array}{ccc} \mathsf{Loc}_\infty^{(E,\partial)}(M) & \xrightarrow{F} & 2\text{-}\mathcal{B}un^f_{\mathcal{F}(E,\partial)}(M) \\ {\scriptstyle I}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \mathrm{Rep}_{\leq 2}(M,(E,\partial)) & \xrightarrow[H]{} & \mathrm{Rep}_{\leq 2}(M,\mathcal{F}(E,\partial)), \end{array} \tag{0.1}$$

where the first horizontal map is the two-tier frame bundle construction mapping ∞-local systems to principal 2-bundles with connection, the vertical maps are the different accounts of parallel transport, and the second horizontal map is a procedure to construct 2-group representations of the truncated groupoid from a truncated representation up to homotopy.

The structure of the thesis

Chapter 2 is concerned with the preliminaries and motivation for questions addressed in this thesis. In Section §1 we set our conventions for dealing with graded structures and provide the definitions regarding A_∞-structures. Section §2 is devoted to the study of basic properties of local systems both in the classical and the ∞ setting. Section §3 contains the main results leading to the proof of the A_∞ de Rham theorem and the higher Riemann-Hilbert correspondence. In this section we find the first instance of a major use of iterated integrals.

Chapter 3 is devoted to developing the tools necessary to prove Theorem 2.3. Once again we encounter iterated integrals, although this time is a slightly different version than the one used in the proof of the A_∞ de Rham theorem. The highlight of the chapter is Theorem 2.3 and its corollaries, which include the A_∞ version of Poincaré's lemma. This chapter is based in [1].

In the final chapter we review quickly the theory of principal 2-bundles with connections and parallel transport in principal 2-bundles. This is done following [27] and [28] closely. Later we prove the main results of the chapter: Lemma 2.3 and Theorem 2.1 establish the first horizontal map of Diagram (0.1), Theorem 3.1 provides the second horizontal arrow and finally Theorem 4.1 states the commutativity of the diagram.

Chapter 1

Preliminaries and Motivation

1 A_∞-structures

The notion of an A_∞-algebra was introduced by Stasheff in [25] and, as stated in the introduction, the category of A_∞-algebras seems to be the proper environment for the comparison of *dg*-algebras. In a few words, an A_∞-algebra is a *dg*-algebra that is associative up to homotopy; the A in the notation stands for associative. Clearly every *dg*-algebra is an example of an A_∞-algebra where the homotopy is zero. An A_∞-algebra structure includes an infinite set of operators keeping track of homotopies, and equations establishing relations between these operators; the ∞ subscript in the notation references that fact. A morphism between A_∞-algebras is a function that preserves the A_∞-structure up to homotopy, together with an infinite sequence of mappings that keep track of the homotopies, and relations between these mappings. Once again, any morphism between *dg*-algebras may be regarded as a morphism between A_∞-algebras, therefore we have that the category of *dg*-algebras is a subcategory of the category A_∞-algebras. The main aspect of this inclusion of categories is that it is not full, meaning that there are more morphisms between *dg*-algebras when regarded as A_∞- algebras. The formalism of A_∞-algebras may be extended naturally to the setting of categories relaxing everything up to homotopy and keeping track of the homotopies. So, an A_∞-category is a *dg*-category where the composition of morphisms is associative up to homotopy. Between A_∞-categories we have A_∞-functors, which preserve compositions up to homotopy. An A_∞-natural transformation is a transformation between functors where the relevant diagrams are commutative up to homotopy. As expected, *dg*-categories, *dg*-functors and *dg*-natural transformations are examples of the A_∞ versions of these notions.

We begin this section with some notation and conventions regarding graded structures. We follow with the definition of the categories of *dg*-algebras and A_∞-algebras, and provide some basic results surrounding them. The formalism of A_∞-categories will not be needed, so we will restrict our attention to A_∞-functors and A_∞-natural transformations between *dg*-categories. For a more comprehensive treatment check [17].

1.1 Graded Vector Spaces and the Koszul Sign Convention

A $\mathbb{Z}$-*graded vector space* V is a vector space decomposed as a direct sum of vector spaces over $\mathbb{Z}$:

$$V = \bigoplus_{n\in\mathbb{Z}} V^n.$$

If V is finite dimensional then only finitely many terms of the sum are non-zero and each of them must be finite dimensional. An element $v \in V$ is called *homogeneous of degree* n if $v \in V^n$, in this case we write $|v| = n$. The *suspension* of V is denoted sV and is the same vector space with a shift in degrees given by $(sV)^n = V^{n+1}$.

Let V, W be graded vector spaces. A homogeneous morphism of degree k is a linear map $f : V \to W$ that shifts the degree by $+k$, this means that for a homogeneous $v \in V$ with $|v| = n$, we have $|f(v)| = n + k$. Frequently we write $f : V^\bullet \to W^{\bullet+k}$ to state that f is homogeneous of degree k and write $|f| = k$. The identity morphism will be denoted by $1_V : V \to V$ and the subscript will be dropped whenever the space V is clear from the context. Notice that the identity is homogeneous of degree 0. The map $s : V \to sV$ that sends every element to itself is called the suspension map and has degree $|s| = -1$, its inverse has degree $|s^{-1}| = +1$.

If V_i with $i = 1, \cdots, k$ are graded vector spaces then the tensor product $V_1 \otimes \cdots \otimes V_k$ is also graded as

$$(V_1 \otimes \cdots \otimes V_k)^n = \bigoplus_{n_1+\cdots+n_k=n} V_1^{n_1} \otimes \cdots \otimes V_k^{n_k}.$$

If $v_i \in V_i$ with $i = 1, \cdots, k$ are homogeneous then $v_1 \otimes \cdots \otimes v_k$ is also homogeneous with

$$|v_1 \otimes \cdots \otimes v_k| = |v_1| + \cdots + |v_k|.$$

Suppose we have homogeneous maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$. Then we define $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ by using the Koszul sign convention:

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|v_1||f_2|} f_1(v_1) \otimes f_2(v_2)$$

for homogeneous v_1, v_2. The sign convention states that whenever a symbol passes over another symbol we introduce a sign depending on the degrees of both symbols. In our definition of $f_1 \otimes f_2$, f_2 passes over v_1 yielding the sign $(-1)^{|v_1||f_2|}$.

More generally, if $f_i : V_i \to W_i$ are homogeneous, then

$$(f_1 \otimes \cdots \otimes f_k)(v_1 \otimes \cdots \otimes v_k) = (-1)^\sigma f_1(v_1) \otimes \cdots \otimes f_k(v_k)$$

where $\sigma = |v_1||f_2| + \cdots + (|v_1 \otimes \cdots \otimes v_{k-1}|)|f_k|$. Notice that we have $|f_1 \otimes \cdots \otimes f_k| = |f_1| + \cdots + |f_k|$.

We will frequently deal with homogeneous maps of the form $f : V^{\otimes k} \to W$. If we take suspensions on all the factors we get a map $\hat{f} : (sV)^{\otimes k} \to sW$. The relation between the degrees $|f|$ and $|\hat{f}|$ is determined by the commutativity of the following diagram

$$\begin{array}{ccc} V^{\otimes k} & \xrightarrow{f} & W \\ {\scriptstyle s^{\otimes k}}\downarrow & & \uparrow{\scriptstyle s^{-1}} \\ (sV)^{\otimes k} & \xrightarrow[\hat{f}]{} & sW. \end{array}$$

Therefore we get

$$|f| = 1 + |\hat{f}| - k. \tag{1.1}$$

A *differential graded* (*dg*) vector space is a $\mathbb{Z}$-graded vector space V with a homogeneous map $d_V : V^\bullet \to V^{\bullet+1}$ satisfying the condition $d_V^2 = 0$; d_V is called a *differential*. The subscript in d_V will be dropped when V is clear from the context. A dg-vector space is also called a *cochain complex*. dg-vector spaces are the objects of a category which we denote dgVect; in this category we have

$$\mathrm{Hom}_{dg\mathrm{Vect}}(V, W) = \bigoplus_k \mathrm{Hom}^k_{dg\mathrm{Vect}}(V, W),$$

where $\mathrm{Hom}^k_{dg\mathrm{Vect}}(V, W)$ is the set of homogeneous maps of degree k.
The category dgVect is a monoidal category. Given dg-vector spaces (V, d_V) and (W, d_W), the tensor product $V \otimes W$ has a differential $d_{V\otimes W} = d_V \otimes 1_W + 1_V \otimes d_W$. To check that $d^2_{V\otimes W} = 0$ we rely on the Koszul sign convention. In general, if we have dg-vector spaces (V_i, d_{V_i}), $i = 1, \cdots, k$, then the tensor product $V_1 \otimes \cdots \otimes V_k$ has a differential

$$\sum_{i=1}^{k} 1_{V_1} \otimes \cdots 1_{V_{i-1}} \otimes d_{V_i} \otimes 1_{V_{i+1}} \otimes \cdots \otimes 1_{V_k}.$$

If (V, d_V) is a cochain complex, elements in $\mathrm{Im}(d_V)$ are called *exact* and elements in $\ker(d_V)$ are called *closed*. The condition $d_V^2 = 0$ means that for any n we have

$$\mathrm{Im}(d_V : V^{n-1} \to V^n) \subset \ker(d_V : V^n \to V^{n+1}),$$

in other words, every exact element is closed. The *cohomology* of (V, d_V) is defined as

$$H^n(V) := \frac{\ker(d_V : V^n \to V^{n+1})}{\mathrm{Im}(d_V : V^{n-1} \to V^n)}, \qquad H(V) := \bigoplus_n H^n(V).$$

If $v \in \ker(d_V)$, the class it defines in cohomology is denoted $\bar{v}$. A homogeneous morphism $f : V \to W$ of degree zero is called a *morphism of complexes* if it is compatible with the differentials, this means that $d_W \circ f = f \circ d_V$. Cochain complexes with their morphisms form a category which we denote cCh. Notice that cCh is a subcategory of dgVect with the same objects but fewer morphisms. A morphism of complexes induces a morphism $\bar{f} : H(V) \to H(W)$ by the formula $\bar{f}(\bar{v}) = \overline{f(v)}$ and this is well defined due to the compatibility condition

on f. If f induces an isomorphism in cohomology then f is called a *quasi-isomorphism*. Two morphisms of complexes $f, g : V \to W$ are *homotopic* if there is a morphism h (not necessarily of complexes) of degree $|h| = -1$ such that $f - g = d_W \circ h + h \circ d_V$. If f, g are homotopic then they define the same morphism in cohomology, this is $\bar{f} = \bar{g}$.
The set $\mathrm{Hom}_{dg\mathrm{Vect}}(V, W)$ is also a dg-vector space. The differential, which we denote $D_{V,W}$, is defined on a homogeneous morphism f by the formula

$$D_{V,W}(f) = d_W \circ f - (-1)^{|f|} f \circ d_V. \tag{1.2}$$

Let us check that $D^2_{V,W} = 0$.

$$\begin{aligned} D^2_{V,W}(f) &= D_{V,W}(d_W \circ f - (-1)^{|f|} f \circ d_V) \\ &= d_W \circ d_W \circ f - (-1)^{|f|+1} d_W \circ f \circ d_V - (-1)^{|f|} d_W \circ f \circ d_V + f \circ d_V \circ d_V \\ &= 0. \end{aligned}$$

Notice that f is a morphism of complexes if and only if $|f| = 0$ and $D_{V,W}(f) = 0$, in other words, f is closed. Two morphisms of complexes f, g are homotopic if and only if there is an h such that $D_{V,W}(h) = f - g$, in other words they define the same class in $H(\mathrm{Hom}_{dg\mathrm{Vect}}(V, W))$.

1.2 dg-Algebras and A_∞-Algebras

Definition 1.1. A dg-algebra is a dg-vector space (A, d) together with an operator $m : A \otimes A \to A$ of degree 0. The following relations must be satisfied:

- The Leibniz rule: $d(m) = m(d \otimes 1 + 1 \otimes d)$. Evaluating in $a_1 \otimes a_2 \in A \otimes A$ we get the graded Leibniz rule

  $$d(m(a_1 \otimes a_2)) = m(d(a_1) \otimes a_2) + (-1)^{|a_1|} m(a_1 \otimes d(a_2)).$$

  Notice that the appearance of the sign $(-1)^{|a_1|}$ is due to the Koszul sign convention.

- Associativity: $m(1 \otimes m) = m(m \otimes 1)$. Evaluating in $a_1 \otimes a_2 \otimes a_3$, the associativity rule may be written as

  $$m(a_1 \otimes m(a_2 \otimes a_3)) - m(m(a_1 \otimes a_2) \otimes a_3) = 0.$$

The product m defined on A induces an associative multiplication $\bar{m}$ on the cohomology $H^*(A)$: if $\bar{a}_1, \bar{a}_2 \in H^*(A)$, then $\bar{m}(\bar{a}_1 \otimes \bar{a}_2) = \overline{m(a_1 \otimes a_2)}$. The Leibniz rule guarantees that $\bar{m}$ is well defined.

Along the same lines we have the following.

Definition 1.2. Given a dg-algebra (A, d, m), we can define the bar complex of A as follows

$$B(A) := \bigoplus_{k\geq 1}(sA)^{\otimes k}.$$

The differential and multiplication define corresponding operators on the suspension of A which we will denote $\hat{d}$ and $\hat{m}$:

$$\hat{d}(sa) = sd(a) \qquad \hat{m}(sa_1, sa_2) = sm(a_1, a_2).$$

Notice that by equation (1.1) we have that $|\hat{d}| = |d| = 1$, while $|m| = 0$ and $|\hat{m}| = 1$. The fact that $|\hat{m}| = 1$ allows for the definition of a coboundary operator on $B(A)$ as follows

$$\begin{aligned} b(sa_1 \otimes \cdots \otimes sa_k) = &\sum_{i=1}^{k}(-1)^{\sum_{j=1}^{i-1}|a_j|-i+1} sa_1 \otimes \cdots \otimes sa_{i-1} \otimes \hat{d}(sa_i) \otimes sa_{i+1} \otimes \cdots \otimes sa_k \\ &+ \sum_{i=1}^{k-1}(-1)^{\sum_{j=1}^{i-1}|a_j|-i+1} sa_1 \otimes \cdots \otimes sa_{i-1} \otimes \hat{m}(sa_i, sa_{i+1}) \otimes sa_{i+2} \otimes \cdots \otimes sa_k. \end{aligned}$$

The coboundary has degree 1 and satisfies $b^2 = 0$, making $(B(A), b)$ a cochain complex.

Definition 1.3. A homogeneous morphism of dg-algebras $f : (A, d_A, m_A) \to (B, d_B, m_B)$ is a homogeneous linear map $f : A \to B$ of degree zero that is compatible with the dg-algebra structures, i.e. we have the following relations:

- $d_B f = f d_A$, i.e. f is a morphism of complexes.
- $f(m) = m(f \otimes f)$.

The category of dg-algebras and their morphisms will be denoted dgAlg.

Any morphism $f : A \to B$ induces a morphism $\bar{f} : H^*(A) \to H^*(B)$ of associative algebras given by $\bar{f}(\bar{a}) = \overline{f(a)}$.
Within dgAlg we have a tensor product operation. Given dg-algebras A and B, the tensor product $A \otimes B$ has a differential $d_{A\otimes B} = d_A \otimes 1_B + 1_A \otimes d_B$ and a product

$$m_{A\otimes B}((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = (-1)^{|a_2||b_1|} m_A(a_1 \otimes a_2) \otimes m_B(b_1 \otimes b_2).$$

$(A \otimes B, d_{A\otimes B}, m_{A\otimes B})$ is a dg-algebra. This construction may be iterated to give a dg-algebra structure to the product of multiple algebras.

Example 1.1. Let M be a smooth manifold. The de Rham algebra of M is the dg-algebra $(\Omega(M), d, \wedge)$ of differential forms with the standard de Rham differential and wedge product. The singular smooth chain complex is $C_*(M) = \bigoplus_{k\geq 0} C_k(M)$, where $C_k(M)$ is the vector space generated by smooth maps $\sigma : \Delta_k \to M$. There is a boundary operator $\partial : C_*(M) \to C_{*-1}(M)$ satisfying $\partial^2 = 0$ which makes $(C_*(M), \partial)$ into a chain complex. Dualizing we get the singular smooth cochain complex $(C^*(M), \delta)$ which can be made into a dg-algebra $(C^*(M), \delta, \cup)$ where $\cup$ denotes the cup product. The ordinary de Rham Theorem states that there is an isomorphism of algebras

$$\bar{\varphi} : (H^*(\Omega(M)), \bar{\wedge}) \to (H^*(C^*(M)), \bar{\cup})$$

induced by the map $\varphi : \Omega(M) \to C^*(M)$ given by

$$\varphi(\alpha)(\sigma) = \int_{\Delta_k} \sigma^*(\alpha), \quad \alpha \in \Omega^k(M), \sigma \in C_k(M).$$

Notice that φ is a homogeneous linear map of degree zero. Furthermore, by Stokes' Theorem, we have

$$\varphi(d\alpha)(\sigma) = \int_{\Delta_k} \sigma^*(d\alpha) = \int_{\partial\Delta_k} \sigma^*(\alpha) = \delta\varphi(\alpha)(\sigma),$$

so that φ is a morphism of cochain complexes. It is to be noted that φ fails to be a morphism of dg-algebras since it does not preserve the product. This is not unexpected since de Rham's complex is commutative (in the graded sense) and the singular cochain complex is not. However, the morphism φ induces a morphism of algebras in cohomology. The explanation of this phenomenon is that the difference between $\varphi(\alpha \wedge \beta)$ and $\varphi(\alpha) \cup \varphi(\beta)$ is a term that vanishes when we descend to cohomology. In other words, $\varphi(\alpha \wedge \beta)$ and $\varphi(\alpha) \cup \varphi(\beta)$ are homotopic.

Next we define A_∞-algebras and A_∞-morphisms.

Definition 1.4. An A_∞-algebra is a graded vector space A with a sequence of operators $m_n : A^{\otimes n} \to A$ of degree $2 - n$ for $n \geq 1$ such that the following relations are satisfied for $n \geq 1$

$$\sum_{i+j+k=n} (-1)^{i+jk} m_{n-j+1}(1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = 0.$$

For $n = 1, 2$ the relations become familiar expressions:

- For $n = 1$ we have $m_1(m_1) = 0$. Since $|m_1| = 1$, m_1 is a differential and (A, m_1) is a cochain complex.
- For $n = 2$ we have $m_1(m_2) = m_2(m_1 \otimes 1 + 1 \otimes m_1)$. This is the Leibniz rule where the multiplication is m_2.

From $n = 3$ onward we have:

- If $n = 3$ then we get

$$m_1(m_3) + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) = m_2(1 \otimes m_2 - m_2 \otimes 1).$$

  This is stating that the associator operator $m_2(1 \otimes m_2 - m_2 \otimes 1) : A^{\otimes 3} \to A$ is homotopic to zero and the homotopy is given by m_3. In other words, the product is associative up to homotopy.

- For $n > 3$ we have

$$m_1(m_n)-(-1)^{|m_n|}m_n\left(\sum_{i+1+k=n}1^{\otimes i}\otimes m_1\otimes 1^k\right)=\sum_{\substack{i+j+k=n\\1<j<n}}(-1)^{i+jk}m_{n-j+1}(1^{\otimes i}\otimes m_j\otimes 1^{\otimes k}).$$

  This is stating that $m_2,\cdots,m_{n-1}$ satisfy a relation up to a homotopy given by m_n.

Any dg-algebra (A,d,m) is an A_∞-algebra with $m_1=d$, $m_2=m$ and $m_n=0$ for $n\geq 3$.

Definition 1.5. Let (A,m_n) and (A',m'_n) be A_∞-algebras. A morphism $\varphi : A\to A'$ is a sequence of maps $\varphi_n : A^{\otimes n}\to A'$ of degree $1-n$ that satisfy the following set of relations for $n\geq 1$

$$\sum_{i+j+k=n}(-1)^{i+jk}\varphi_{n-j+1}(1^{\otimes i}\otimes m_j\otimes 1^{\otimes k})=\sum_{i_1+\cdots+i_l=n}(-1)^s m'_l(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_l})$$

where $s=\sum_{r=1}^{r=l}(l-r)(i_r-1)$.
Given morphisms $\varphi : A\to A'$ and $\psi : A'\to A''$, the composition is

$$(\psi\circ\varphi)_n=\sum_{i_1+\cdots+i_l=n}(-1)^s\psi_l(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_l}).$$

Let us explore the meaning of these relations

- For $n=1$ we get $\varphi_1(m_1)=m'_1(\varphi_1)$, this means that φ_1 is a morphism of cochain complexes.

- For $n=2$ the relation reads

$$m'_1\varphi_2+\varphi_2(m_1\otimes 1+1\otimes m_1)=\varphi_1(m_2)-m'_2(\varphi_1\otimes\varphi_1).$$

  Which means that φ_1 preserves the products up to a homotopy given by φ_2.

- For $n\geq 3$ we have a certain relation satisfied up to a homotopy given by φ_n.

A morphism φ is called a quasi isomorphism if φ_1 is a quasi isomorphism and it is called strict if $\varphi_i=0$ for $i\geq 2$. Notice that an ordinary morphism of dg-algebras of degree 0 is merely a strict A_∞-morphism. The identity morphism is the strict morphism such that $\varphi_1=\mathrm{Id}$. The category of A_∞-algebras with morphisms of degree 0 is denoted $A_\infty\mathrm{Alg}^0$. The category of A_∞-algebras with all possible morphisms, including non-homogeneous, is denoted by $A_\infty\mathrm{Alg}$, and has $dg\mathrm{Alg}$ as a non-full subcategory.
We are particularly interested in A_∞-morphisms between dg-algebras, so we write explicitly the relations in this case. If (A,d_A,m_A) and (B,d_B,m_B) are dg-algebras and $\varphi : A\to B$ is

an A_∞-morphism, then we have for $n \geq 1$

$$\sum_{i=0}^{n-1}(-1)^{n-1}\varphi_n(1^{\otimes i} \otimes d_A \otimes 1^{\otimes(n-1-i)}) + \sum_{i=0}^{n-2}(-1)^i\varphi_{n-1}(1^{\otimes i} \otimes m_A \otimes 1^{\otimes(n-2-i)})$$
$$= d_B(\varphi_n) + \sum_{i=1}^{n-1}(-1)^{i-1}m_B(\varphi_i \otimes \varphi_{n-i}). \tag{1.3}$$

Next we state some definitions and results relating Maurer-Cartan elements and A_∞-morphisms.

Definition 1.6. Let (A, d, m) be a dg-algebra. A Maurer-Cartan element of A is a homogeneous element $a \in A$ of degree one that satisfies the Maurer-Cartan equation:

$$da - a^2 = 0.$$

Remark 1.1. A Maurer-Cartan element $a \in A$ can be used to twist the algebra (A, d, m). This means that we can construct a new dg-algebra with the same elements and multiplication of A but with differential given by

$$(d - [a, -])(b) = db - [a, b] = db - ab + (-1)^{|b|}ba.$$

The Maurer-Cartan equation guarantees that $(d - [a, -])^2 = 0$.

Definition 1.7. Suppose (A, d, m) is a dg-algebra with unit. Two Maurer-Cartan elements $a, a' \in A$ are said to be gauge equivalent if the twisted complexes $(A, d - [a, -])$ and $(A, d - [a', -])$ are isomorphic via multiplication by a unit $u \in A$. In other words, the unit u defines a map $u : A \to A$ that makes the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{d-[a,-]} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow[d-[a',-]]{} & A. \end{array}$$

The commutativity of the diagram ultimately may be written as

$$a = u^{-1}a'u - u^{-1}du.$$

Under certain conditions, an A_∞-morphism $\psi : A \to B$ can be used to map elements from A to B as follows:

Definition 1.8. Let A, B be A_∞-algebras and $\psi : A \to B$ an A_∞-morphism. For an element $a \in A$ we denote by $\psi_*(a)$ the following series

$$\psi_*(a) := \sum_{n\geq 1}(-1)^{\sigma(n)}\psi_n(a^{\otimes n})$$

where $\sigma(n) = 0 + 1 + 2 + \cdots + (n-1)$.

The sign in the series is precisely the sign that appears when applying the n-th power of the suspension to $a^{\otimes n}$. Whenever the series $\psi_*(a)$ is convergent it defines an element of B; notice that this is the case when $\psi_n = 0$ for all but finitely many $n \geq 1$. The next results show some interactions between A_∞-morphisms and Maurer-Cartan elements.

Lemma 1.1. Let (A, d_A, m_A) and (B, d_B, m_B) be dg-algebras and $\psi : A \to B$ an A_∞-morphism. If $a \in A$ is a Maurer-Cartan element then $\psi_*(a) \in B$ is also Maurer-Cartan provided the series is convergent.

Proof. The proof is a simple computation:

$$\begin{aligned} d_B\psi_*(a) - \psi_*(a)\psi_*(a) &= \sum_{n\geq 1}(-1)^{\sigma(n)} d_B\psi_n(a^{\otimes n}) - \sum_{n\geq 1}\sum_{i=1}^{n-1}(1)^{\sigma(i)+\sigma(n-i)}\psi_i(a^{\otimes i})\psi_{n-i}(a^{\otimes(n-i)}) \\ &= \sum_{n\geq 1}(-1)^{\sigma(n)}\left(d_B\psi_n(a^{\otimes n}) - \sum_{i=1}^{n-1}(1)^{\sigma(i)+\sigma(n-i)-\sigma(n)}\psi_i(a^{\otimes i})\psi_{n-i}(a^{\otimes(n-i)})\right) \\ &= \sum_{n\geq 1}(-1)^{\sigma(n)}\left(d_B\psi_n(a^{\otimes n}) + \sum_{i=1}^{n-1}(1)^{i-1}m_B(\psi_i \otimes \psi_{n-i})(a^{\otimes i} \otimes a^{\otimes(n-i)})\right). \end{aligned}$$

Using the A_∞ relations (1.3) for ψ we write the previous expression as

$$\begin{aligned} &d_B\psi_*(a) - \psi_*(a)\psi_*(a) \\ &\quad = \sum_{n\geq 1}(-1)^{\sigma(n)}\left(\sum_{i=0}^{n-1}(-1)^{n-1+i}\psi_n(1^{\otimes i} \otimes d_A a \otimes 1^{\otimes(n-1-i)}) + \sum_{i=0}^{n-2}(-1)^{i}\psi_{n-1}(1^{\otimes i} \otimes a^2 \otimes 1^{\otimes(n-2-i)})\right). \end{aligned}$$

The sum can be rearranged into

$$d_B\psi_*(a) - \psi_*(a)\psi_*(a) = \sum_{n\geq 1}\sum_{i=0}^{n-1}(-1)^{\sigma(n+1)-1+i}\psi_n(1^{\otimes i} \otimes (d_A a - a^2) \otimes 1^{\otimes(n-1-i)})$$

which is zero since $d_A a - a^2 = 0$. □

Lemma 1.2. Suppose that A and B are dg-algebras with units u_A and u_B respectively and $\psi : A \to B$ is an A_∞-morphism such that $\psi_1(u_A) = u_B$. If $\psi_n(a_1 \otimes \cdots \otimes a_n)$ vanishes whenever $n \geq 2$ and at least one of the a_i is invertible, then $\psi_1 : A \to B$ sends invertible elements of A to invertible elements of B. Furthermore, if $a, a' \in A$ are gauge equivalent Maurer-Cartan elements, then $\psi_*(a)$ and $\psi_*(a')$ are gauge equivalent elements provided those series are convergent.

Proof. For the first part recall that ψ_1 preserves the product only up to a homotopy given by ψ_2, this is

$$\psi_1(a_1a_2) - \psi_1(a_1)\psi_1(a_2) = d_B\psi_2(a_1 \otimes a_2) + \psi_2(d_Aa_1 \otimes a_2 + (-1)^{|a_1|}a_1 \otimes d_Aa_2).$$

The condition that ψ_2 vanishes whenever its argument has an invertible factor means that ψ_1 preserves the product strictly when both factors are invertible. Therefore for an invertible element $u \in A$ we have

$$\psi_1(u_A) = \psi_1(uu^{-1}) = \psi_1(u)\psi_1(u^{-1}) = u_B.$$

Now suppose $a, a' \in A$ are gauge equivalent via $u \in A$, this is $ua = a'u - d_Au$. We wish to prove the following equation

$$\sum_{n\geq 1}(-1)^{\sigma(n)}\psi_1(u)\psi_n(a^{\otimes n}) = \sum_{n\geq 1}(-1)^{\sigma(n)}\psi_n(a'^{\otimes n})\psi_1(u) - \psi_1(d_Au).$$

The term on the left may be written as

$$\sum_{n\geq 1}(-1)^{\sigma(n)}m_B(\psi_1 \otimes \psi_n)(u \otimes a^{\otimes n}).$$

By the A_∞ relations (1.3) we have

$$m_B(\psi_1 \otimes \psi_n)(u \otimes a^{\otimes n}) = (-1)^n\psi_{n+1}(d_Au \otimes a^{\otimes n}) + \psi_n(ua \otimes a^{\otimes(n-1)}).$$

Therefore adding over $n \geq 1$ and using $d_Au + ua = a'u$, we get

$$\sum_{n\geq 1}(-1)^{\sigma(n)}\psi_1(u)\psi_n(a^{\otimes n}) = \sum_{n\geq 1}(-1)^{\sigma(n)}\psi_n(a'u \otimes a^{\otimes(n-1)}) - \psi_1(d_Au). \tag{1.4}$$

Using once again the A_∞ relations and $d_Au+ua = a'u$, the following equation may be proved

$$\begin{aligned}\sum_{n=1}^{k}(-1)^{\sigma(n)}\psi_n(a'^{\otimes n})\psi_1(u) + \sum_{n\geq k+1}(-1)^{\sigma(n)}\psi_n(a'^{\otimes k} \otimes a'u \otimes a^{\otimes(n-k-2)})\\ = \sum_{n=1}^{k+1}(-1)^{\sigma(n)}\psi_n(a'^{\otimes n})\psi_1(u) + \sum_{n\geq k+2}(-1)^{\sigma(n)}\psi_n(a'^{\otimes(k+1)} \otimes a'u \otimes a^{\otimes(n-k-2)}).\end{aligned}$$

Applying the previous equation inductively on the left side of (1.4) yields the desired result. □

1.3 *dg*-Categories and A_∞-Natural Transformations

Here we describe the generalization of the previous concepts to the setting of categories.

Definition 1.9. A *dg*-category $\mathcal{C}$ is a category enriched in cCh. This means that for any pair of objects $x, y \in \mathcal{C}$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a dg-vector space with differential $D_{x,y}$. Furthermore, the composition

$$\circ : \mathrm{Hom}_{\mathcal{C}}(y, z) \otimes \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

must be a morphism of complexes of degree zero. Unraveling this condition we find that if we have composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, then the Leibniz rule is satisfied

$$D_{x,z}(g \circ f) = D_{y,z}(g) \circ f + (-1)^{|g|} g \circ D_{x,y}(f).$$

Example 1.2. The category $dg\mathrm{Vect}$ is a dg-category. We have already seen that if (V, d_V) and (W, d_W) are dg-vector spaces, then the commutator (1.2) provides $\mathrm{Hom}_{dg\mathrm{Vect}}(V, W)$ with the structure of a dg-vector space. Now suppose that we have a composition

$$(U, d_U) \xrightarrow{\varphi} (V, d_V) \xrightarrow{\psi} (W, d_W).$$

Then we have

$$\begin{aligned} D_{U,W}(\psi \circ \varphi) &= d_W \circ \psi \circ \varphi - (-1)^{|\psi||\varphi|} \psi \circ \varphi \circ d_U \\ &= d_W \circ \psi \circ \varphi - (-1)^{|\psi|} \psi \circ d_V \circ \varphi + (-1)^{|\psi|} \psi \circ d_V \circ \varphi - (-1)^{|\psi||\varphi|} \psi \circ \varphi \circ d_U \\ &= (d_W \circ \psi - (-1)^{|\psi|} \psi \circ d_V) \circ \varphi - (-1)^{|\psi|} \psi \circ (d_V \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_U) \\ &= D_{V,W}(\psi) \circ \varphi - (-1)^{|\psi|} \psi \circ D_{U,V}(\varphi). \end{aligned}$$

Definition 1.10. Let $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ be a homogeneous morphism. For any other object z in $\mathcal{C}$, f induces maps

$$\begin{aligned} f^* : \mathrm{Hom}_{\mathcal{C}}(y, z) &\to \mathrm{Hom}_{\mathcal{C}}(x, z) & f_* : \mathrm{Hom}_{\mathcal{C}}(z, x) &\to \mathrm{Hom}_{\mathcal{C}}(z, y) \\ g &\mapsto f^*(g) = (-1)^{|f||g|} g \circ f & h &\mapsto f_*(h) = f \circ h. \end{aligned}$$

The maps f^* and f_* are called pullback and push forward respectively.

The following lemma is straightforward to check.

Lemma 1.3. The maps f^* and f_* are morphisms of complexes if and only if f is a closed element.

Definition 1.11. A dg-functor between dg-categories $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor such that for any pair of objects $x, y \in \mathcal{C}$, the map $\mathcal{F}_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}x, \mathcal{F}y)$ is a morphism of complexes.
The dg-functor $\mathcal{F}$ is said to be essentially surjective if for every object z of $\mathcal{D}$ there is an object x of $\mathcal{C}$ such that $\mathcal{F}x$ is isomorphic to z. $\mathcal{F}$ is said to be quasi fully faithful if for every pair of objects x, y of $\mathcal{C}$, the map $\mathcal{F}_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}x, \mathcal{F}y)$ is a quasi-isomorphism. $\mathcal{F}$ is called a quasi equivalence if it is essentially surjective and quasi fully faithful.

Definition 1.12. A dg-natural transformation ρ between the dg-functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is an assignment of a closed element $\rho x \in \mathrm{Hom}^0_{\mathcal{D}}(\mathcal{F}x, \mathcal{G}x)$ for every object x of $\mathcal{C}$ such that for every pair of objects x, y the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x,y) & \xrightarrow{\mathcal{F}_{x,y}} & \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}x, \mathcal{F}y) \\ {\scriptstyle \mathcal{G}_{x,y}}\downarrow & & \downarrow{\scriptstyle (\rho y)_*} \\ \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}x, \mathcal{G}y) & \xrightarrow[(\rho x)^*]{} & \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}x, \mathcal{G}y). \end{array}$$

The condition that ρx is closed for every x is so that the maps $(\rho x)^*$ and $(\rho y)_*$ are morphisms of complexes. When evaluated in a fixed $f : x \to y$ we get the usual commutative square of natural transformations. If ρx is always an isomorphism then ρ is called a natural isomorphism.

Any dg-algebra A may be regarded as a dg-category $\mathcal{A}$ with a single object $*$ where $Hom_{\mathcal{A}}(*, *) = A$. A morphism of dg-algebras is then a dg-functor and two morphisms are naturally isomorphic if they are conjugate.

In the previous definitions $\mathcal{C}$ is a category, which means that the composition is associative. Relaxing the associativity condition up to homotopy yields the notion of A_∞-category. We will not go into detail on A_∞-categories, however it is worth mentioning that every dg-category is an example of an A_∞-category. Similarly, A_∞-functors are functors that preserve the composition only up to homotopy and every dg-functor is an A_∞-functor. In Section 3 we will see an example of an A_∞-functor, however we will not go into the detail of its A_∞-structure. The full description of the A_∞-functor can be found in [2]. Before jumping to A_∞-natural transformations we define some maps based in the differential of the bar complex of a dg-algebra:

Definition 1.13. Let $\mathcal{C}$ be a dg-category and $x_0, \cdots, x_n$ objects in $\mathcal{C}$. We define maps

$$b_0 : \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1) \to \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1)$$

$$\begin{aligned} b_1 : & \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1) \\ & \to \bigoplus_{i=0}^{n-2} \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_i, x_{i+2}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1) \end{aligned}$$

by the formulas

$$b_0(\omega_{n-1} \otimes \cdots \otimes \omega_0) := \sum_{i=0}^{n-1} (-1)^{\sum_{j=i+1}^{n-1} |\omega_j| - n + i} \omega_{n-1} \otimes \cdots \otimes D_{x_i, x_{i+1}}(\omega_i) \otimes \cdots \otimes \omega_0,$$

$$b_1(\omega_{n-1} \otimes \cdots \otimes \omega_0) := \sum_{i=0}^{n-2} (-1)^{\sum_{j=i+1}^{n-1} |\omega_j| - n + i + 1} \omega_{n-1} \otimes \cdots \otimes \omega_{i+1} \circ \omega_i \otimes \cdots \otimes \omega_0.$$

Furthermore, we write $b = b_0 + b_1$ and make the convention that $b_1(\omega) = 0$, so that $b(\omega) = b_1(\omega)$.

Definition 1.14. Suppose $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ are *dg*-functors. An A_∞-natural transformation $\rho : \mathcal{F} \Rightarrow \mathcal{G}$ is a collection $\{\rho_n\}_{n\geq 0}$ where:

1. For every object x of $\mathcal{C}$, $\rho_0(x) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}x, \mathcal{G}x)$ is a closed morphism of degree 0.

2. For $n \geq 1$, if $x_0, \cdots, x_n$ are objects in $\mathcal{C}$, then ρ_n induces a map of degree $-n$

$$\rho_n(x_0, \cdots, x_n) : \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(x_0, x_1) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}x_0, \mathcal{G}x_n).$$

   We just write ρ_n when the objects $x_0, \cdots, x_n$ are clear from the context.

3. The following relations are satisfied for every finite sequence of homogeneous, composable morphisms $\omega_{n-1} \otimes \cdots \otimes \omega_0$ with $\omega_i \in \mathrm{Hom}_{\mathcal{C}}(x_i, x_{i+1})$:

$$\begin{aligned} &\mathcal{G}(\omega_{n-1}) \circ \rho_{n-1}(\omega_{n-2} \otimes \cdots \otimes \omega_0) - (-1)^{\sum_{i=1}^{n-1} |\omega_i| - n + 1} \rho_{n-1}(\omega_{n-1} \otimes \cdots \otimes \omega_1) \circ \mathcal{F}(\omega_0) \\ &= \rho(b(\omega_{n-1} \otimes \cdots \otimes \omega_0)) + D(\rho_n(\omega_{n-1} \otimes \cdots \otimes \omega_0)). \end{aligned}$$

   Here $\rho = \rho_n + \rho_{n-1}$.

The relation for $n = 1$ and $\omega_0 \in \mathrm{Hom}_{\mathcal{C}}(x_1, x_0)$ is

$$\mathcal{G}(\omega_0) \circ \rho_0(x_0) - (-1)^{|\omega_0|} \rho_0(x_1) \circ \mathcal{F}(\omega_0) = \rho_1(D(\omega_0)) + D\rho_1(\omega_0).$$

This means that the usual diagram of natural transformations

$$\begin{array}{ccc} \mathcal{F}(x_0) & \xrightarrow{\mathcal{F}(\omega_0)} & \mathcal{F}(x_1) \\ {\scriptstyle \rho_0(x_0)}\big\downarrow & & \big\downarrow{\scriptstyle \rho_0(x_1)} \\ \mathcal{G}(x_0) & \xrightarrow{\mathcal{G}(\omega_0)} & \mathcal{G}(x_1) \end{array}$$

is commutative up to a homotopy given by ρ_1.

Lemma 1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be *dg*-categories and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a *dg*-functor. Suppose that there is a *dg*-functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ together with A_∞-natural isomorphisms $\rho : \mathcal{G} \circ \mathcal{F} \to \mathrm{id}_{\mathcal{C}}$ and $\tau : \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_{\mathcal{D}}$. Then $\mathcal{F}$ is a quasi-equivalence.

Proof. For every object $Y \in \mathrm{Ob}\, \mathcal{D}$ we have a closed isomorphism $\tau_0(Y) \in \mathrm{Hom}^0_{\mathcal{D}}(\mathcal{F}(\mathcal{G}(Y)), Y)$, hence $\mathcal{F}$ is essentially surjective. Let us show that $\mathcal{F}$ is quasi fully faithful. By definition, if $f \in s\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a homogenous element, the A_∞-natural isomorphism $\rho : \mathcal{G} \circ \mathcal{F} \to \mathrm{id}_{\mathcal{C}}$ produces the relation

$$f \circ \rho_0(X) - \rho_0(Y) \circ \mathcal{G}(\mathcal{F}(f)) = \rho_1\left(d(f)\right) + d\left(\rho_1(f)\right).$$

This, in turn, may be written as

$$\rho_0(X)^* \circ \mathrm{id}_{\mathcal{C}} - \rho_0(Y)_* \circ (\mathcal{G} \circ \mathcal{F}) = \rho_1 \circ d + d \circ \rho_1,$$

for any pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$, where $\rho_0(X)^*$ is the pull back of $\rho_0(X)$ by X and $\rho_0(Y)_*$ is the push forward of $\rho_0(Y)$ by $\mathcal{G}(\mathcal{F}(Y))$. Hence, the morphisms of cochain complexes $\rho_0(X)^* \circ \mathrm{id}_{\mathcal{C}}$ and $\rho_0(Y)_* \circ (\mathcal{G} \circ \mathcal{F})$ are homotopic and, therefore, they induce the same morphism in cohomology. But clearly $\rho_0(X)^*$, $\rho_0(Y)_*$ and $\mathrm{id}_{\mathcal{C}}$ induce isomorphisms in cohomology, and thus so does $\mathcal{G} \circ \mathcal{F}$. It follows that $\mathcal{G} \circ \mathcal{F}$ is a quasi-isomorphism. By an entirely analogous argument, using the A_∞-natural isomorphism $\mu \colon \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_{\mathcal{D}}$, one may prove that $\mathcal{F} \circ \mathcal{G}$ is a quasi-isomorphism. The desired implication follows at once. □

2 Local Systems

In this section we define our main objects of study, ∞-local systems. We begin with the ordinary setting, a local system over a manifold is a flat connection. The classical Riemann-Hilbert correspondence states that flat connections over a manifold are in correspondence with representations of its fundamental group, giving us a topological point of view of local systems. Parallel transport is a fundamental ingredient of the Riemann-Hilbert correspondence; we will see how to find solutions to the parallel transport equation using iterated integrals. Next we consider ∞-local systems, instead of ordinary flat connections we consider flat graded connections defined on graded vector bundles. There is a Riemann-Hilbert type theorem in this setting which states that ∞-local systems are in correspondence with representations up to homotopy of the fundamental ∞-groupoid of the manifold. This higher Riemann-Hilbert correspondence relies on a notion of higher dimensional parallel transport constructed with the aid of the A_∞ version of the de Rham theorem. The A_∞ de Rham theorem states that there is an A_∞-quasi-isomorphism from the de Rham complex of a manifold to its singular cochain complex; the construction of this morphism is done using Chen's iterated integrals.
Besides the geometric and topological points of view to local systems, there are other perspectives. At the end of the section we briefly describe them and how they are related to our setting.

2.1 Ordinary Local Systems

This is a quick review of ordinary connections over a manifold, parallel transport and the classical Riemann-Hilbert correspondence. Most of the statements in this section are presented without proof, however, we will focus on the details of the parallel transport construction that will become relevant in the case of ∞-local systems. For an extensive treatment see [13].

Definition 2.1. Let M be a smooth manifold and let $\pi : E \to M$ be a vector bundle over M. A connection defined on E is a differential operator $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that satisfies the following properties:

1. ∇ is tensorial with respect to the first argument

$$\nabla_{fX}(s) = f\nabla_X(s), \quad \text{for } f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E).$$

2. ∇ satisfies the Leibniz rule with respect to the second argument

$$\nabla_X(fs) = df(X)s + f\nabla_X(s), \quad \text{for } f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E).$$

Notice that we write $\nabla_X(s)$ instead of $\nabla(X, s)$.
The curvature of ∇ is the map $R_\nabla : \mathfrak{X} \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defined by

$$R_\nabla(X, Y, s) = \nabla_X\nabla_Y(s) - \nabla_Y\nabla_X(s) - \nabla_{[X,Y]}(s).$$

∇ is called flat if its curvature is zero.

A different way of thinking about connections is as operators defined on vector valued forms. If ∇ is a connection, we may define $d_\nabla : \Omega^0(M, E) = \Gamma(E) \to \Omega^1(M, E) = \Gamma(T^*M \otimes E)$ as $d_\nabla(s)(X) = \nabla_X(s)$. Using the fact that

$$\Omega^k(M, E) \cong \Omega^k(M) \underset{C^\infty(M)}{\otimes} \Gamma(E),$$

we may extend d_∇ to an operator $d_\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$ via the Leibniz rule, i.e.

$$d_\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \otimes d_\nabla(s) \quad \text{for } \omega \in \Omega^k(M), s \in \Gamma(E).$$

If ∇ is flat, then $d_\nabla^2 = 0$, this means that a flat connection provides $\Omega(M, E)$ with the structure of a cochain complex.

Definition 2.2. An ordinary local system over M is a vector bundle together with a flat connection. We usually write a local system as a pair (E, d_∇).

Example 2.1. Let $V = \mathbb{R}^n$ be a vector space and $E = M \times V$. Forms in $\Omega^\bullet(M, E)$ are simply vectors whose entries are forms in $\Omega^\bullet(M)$, therefore we may apply the usual de Rham differential on each entry to get a connection $d : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$. Clearly $d^2 = 0$, so we have a local system over M called a trivial local system.

Suppose ∇ and ∇' are connections defined over $E \to M$ and consider the difference $\nabla - \nabla'$. For $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$ we have

$$(\nabla_X - \nabla'_X)(fs) = df(X)(s) + f\nabla_X(s) - df(X)(s) - f\nabla'_X(s) = f(\nabla_X - \nabla'_X)(s).$$

The map $(\nabla_X - \nabla'_X) : \Gamma(E) \to \Gamma(E)$ is tensorial which means that $\nabla - \nabla' \in \Omega^1(M, \mathrm{End}(E))$. In other words, if ∇ is a connection over E, any other connection is of the form $\nabla - \alpha$ where $\alpha \in \Omega^1(M, \mathrm{End}(E))$.

Example 2.2. Over a trivial vector bundle $E = M \times V$ we have the trivial connection d described in Example 2.1. Any other connection over E has the form $d - \alpha$ where $\alpha \in \Omega^1(M, \mathrm{End}(V))$, therefore we can think of α as a matrix of 1-forms. If $\omega \in \Omega(M, E)$, then the connection acts as $(d-\alpha)\omega = d\omega - \alpha \wedge \omega$ where $\alpha \wedge \omega$ is matrix multiplication using the wedge product defined on $\Omega(M)$.
For the connection $d - \alpha$ we have

$$\begin{aligned}(d-\alpha)^2(\omega) &= d^2\omega - d(\alpha \wedge \omega) - \alpha \wedge d\omega + \alpha^2 \wedge \omega \\ &= -d(\alpha) \wedge \omega + \alpha^2 \wedge \omega \\ &= (-d\alpha + \alpha^2) \wedge \omega.\end{aligned}$$

Hence $d - \alpha$ is flat if and only if $d\alpha - \alpha^2 = 0$.

The equation $d\alpha - \alpha^2 = 0$ is called the *Maurer-Cartan equation* and any form satisfying this equation is called a *Maurer-Cartan form*.

Definition 2.3. Let $f : M \to N$ be a smooth function and $E \to N$ a vector bundle with connection ∇. The pullback of ∇, denoted $f^*\nabla$, is a connection defined over the pullback bundle f^*E defined as follows: Suppose $U \subset N$ is an open set such that $E|_U \cong U \times V$, then $\nabla|_U = d - \alpha$ with $\alpha \in \Omega^1(U, \mathrm{End}(V))$. The pullback bundle trivializes over $f^{-1}(U)$, hence we can define a connection over $f^*(E)|_{f^{-1}(U)} \cong f^{-1}(U) \times V$ by $d - f^*\alpha$. These local connections can be glued together to define the pullback connection. For details on the gluing check Proposition 2.3 below.

The existence of a connection over a manifold allows us to define the parallel transport over a path on M, this is the content of the following theorem.

Theorem 2.1. Let $E \to M$ be a vector bundle with connection ∇. For every piecewise smooth path $\gamma : [0,1] \to M$ there is a linear isomorphism $\mathrm{Hol}_\gamma : E_{\gamma(1)} \to E_{\gamma(0)}$ called the parallel transport along γ. Parallel transport is compatible with composition of paths, this is $\mathrm{Hol}_{\gamma * \lambda} = \mathrm{Hol}_\lambda \circ \mathrm{Hol}_\gamma$. If the connection is flat, parallel transport is invariant under homotopies with fixed endpoints.

Proof. Consider the pullback connection $\gamma^*\nabla$ defined over γ^*E. Since $[0,1]$ is contractible, the pullback bundle is trivial, so there is an isomorphism $\psi : [0,1] \times V \to \gamma^*E$ and the connection is $\gamma^*\nabla = d - \alpha$ with $\alpha \in \Omega^1([0,1], \mathrm{End}(V))$. For each $v \in V$, the ordinary differential equation $ds - \alpha s = 0$ has a unique solution $s_v : [0,1] \to [0,1] \times V$ such that $s_v(1) = (1, v)$. We define the parallel transport by

$$\mathrm{Hol}_\gamma(e) := f \circ \psi(s_v(0))$$

where $f : \gamma^*E \to E$ is the natural map from the pullback and v is the unique $v \in V$ such that $f \circ \psi(1, v) = e$.

We will construct the solution to the equation $ds - \alpha s = 0$ using iterated integrals. Suppose $\alpha = A(t)dt$ where $A(t) \in \mathrm{End}(V)$. Define a map $\varphi^\alpha : [0,1] \to \mathrm{End}(V)$ by the formula

$$\begin{aligned}\varphi^\alpha(t) &= \mathrm{id}_V + \int_0^t A(s_1)ds_1 + \int_0^t \int_0^{s_1} A(s_1)A(s_2)ds_2 ds_1 + \cdots \\ &= \mathrm{id}_V + \sum_{k\geq 1} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} A(s_1)\cdots A(s_k)ds_k \cdots ds_1.\end{aligned}$$

Notice that the k-th term of the series is an integral over the simplex $\Delta_k(t)$ which has volume $t/k!$. Since the integrands grow as polynomials with respect to k, the series is convergent. Differentiating with respect to t yields

$$\frac{d\varphi^\alpha}{dt} = A(t)\varphi^\alpha(t). \tag{2.1}$$

The section s_v is defined by the formula $s_v(t) := (t, \varphi^\alpha(t)v)$. From (2.1) is clear that $ds_v = \alpha s_v$. The rest of the proof may be found in [13]. □

An immediate consequence of this theorem is

Corollary 2.1. Suppose $(E, d_\nabla) \to M$ is a local system and fix a point $x_0 \in M$, then there is a representation $\rho_\nabla : \pi_1(M, x_0) \to \mathrm{GL}(E_{x_0})$.

Proof. The representation is given by $\rho_\nabla([\gamma]) = \mathrm{Hol}_{\bar{\gamma}}$ where $\bar{\gamma}$ is the path γ traversed backwards. □

Let us denote the category of ordinary local systems over M by $\mathsf{Loc}(M)$ and the category of representations of $\pi_1(M, x_0)$ by $\mathcal{R}\mathrm{ep}(M, x_0)$. Then the previous result may be stated as the existence of a functor $\mathrm{Hol} : \mathsf{Loc}(M) \to \mathcal{R}\mathrm{ep}(M, x_0)$. The fact that this functor is an equivalence of categories is known as the Riemann-Hilbert correspondence (in its most basic form).

A generalization of this result is presented in the next section.

2.2 ∞-Local Systems

In this section we deal with $\mathbb{Z}$-graded connections, which are based on the idea of super connections found in [21]. Let $E \to M$ be a $\mathbb{Z}$-graded vector bundle, say $E = \bigoplus_k E^k$. A form $\omega \in \Omega^n(M, E^k)$ has form-degree n and inner-degree k. The total degree of ω is the sum of its form and inner degrees and is denoted $|\omega| = n + k$.

Definition 2.4. A $\mathbb{Z}$-graded connection over M is a graded vector bundle $E \to M$ together with a differential operator $d_E : \Omega(M, E) \to \Omega(M, E)$ that increases the total degree by one. The connection is flat if $d_E^2 = 0$.

Definition 2.5. An ∞-local system over M is a graded vector bundle together with a flat graded connection. We denote local systems as pairs (E, d_E).
The flatness condition makes $(\Omega(M, E), d_E)$ into a dg-vector space.

Let (E, d_E) and $(E', d_{E'})$ be ∞-local systems. Let $\mathrm{Hom}_{\mathrm{VB}}(E, E')$ denote the bundle of vector bundle morphisms from E to E'. Forms in $\Omega(M, \mathrm{Hom}_{\mathrm{VB}}(E, E'))$ define maps $\Omega(M, E) \to \Omega(M, E')$ as follows: take homogeneous elements

$$\alpha \otimes T \in \Omega(M) \underset{C^\infty(M)}{\otimes} \Gamma(\mathrm{Hom}_{\mathrm{VB}}(E, E')) \quad \text{and} \quad \omega \otimes s \in \Omega(M) \underset{C^\infty(M)}{\otimes} \Gamma(E),$$

then define

$$(\alpha \otimes T)(\omega \otimes s) := (-1)^{|T||\omega|} \alpha\omega \otimes T(s) \in \Omega(M) \underset{C^\infty(M)}{\otimes} \Gamma(E').$$

Definition 2.6. The category of ∞-local systems over M is denoted $\mathrm{Loc}_\infty(M)$ and is the category whose objects are ∞-local systems. If (E, d_E) and $(E', d_{E'})$ are objects, then

$$\mathrm{Hom}_{\mathrm{Loc}_\infty(M)}(E, E') = \Omega(M, \mathrm{Hom}_{\mathrm{VB}}(E, E')).$$

For a homogenous $\varphi \in \Omega(M, \mathrm{Hom}_{\mathrm{VB}}(E, E'))$ we define its differential by the already familiar commutator

$$D_{E,E'}(\varphi) = d_{E'}\varphi - (-1)^{|\varphi|}\varphi d_E.$$

Proposition 2.1. The map $D_{E,E'}(\varphi) : \Omega(M, E) \to \Omega(M, E')$ is tensorial, therefore $D_{E,E'}(\varphi) \in \Omega(M, \mathrm{Hom}_{\mathrm{VB}}(E, E'))$.

Proof. The proof is a straight forward computation:

$$\begin{aligned}
D_{E,E'}(\varphi)(f\omega) &= d_{E'}(\varphi(f\omega)) - (-1)^{|\varphi|}\varphi(d_E(f\omega)) \\
&= d_{E'}(f\varphi(\omega)) - (-1)^{|\varphi|}\varphi(df\omega + fd_e(\omega)) \\
&= df\varphi(\omega) + fd_{E'}(\varphi(\omega)) - df\varphi(\omega) - (-1)^{|\varphi|}f\varphi(d_E(\omega)) \\
&= f(d_{E'}(\varphi(\omega)) - (-1)^{|\varphi|}\varphi(d_E(\omega))) \\
&= fD_{E,E'}(\varphi)(\omega).
\end{aligned}$$

□

From Proposition 2.1 and Example 1.2 follows the statement:

Proposition 2.2. The category $\mathrm{Loc}_\infty(M)$ is a *dg*-category.

As in the ordinary case, graded connections are locally determined by forms with values in the endomorphism bundle. If $E = M \times V$ where V is a graded vector space, then any graded connection d_E defined on E has the form $d_E = d - \alpha$, where d is the trivial connection from Example 2.1 and $\alpha \in \Omega(M, \mathrm{End}(V))$ with total degree $|\alpha| = 1$. Notice that in this case α is a sum of differential forms

$$\alpha = \sum_{i \geq 0} \alpha_i, \quad \alpha_i \in \Omega^i(M, \mathrm{End}^{1-i}(V)).$$

In particular $d-\alpha_1$ is an ordinary connection over E. As in Example 2.2, a graded connection is flat if and only if the Maurer-Cartan equation is satisfied $d\alpha-\alpha^2=0$. This equation splits into a family of equations when we regard the homogeneous components with respect to form degree:

- In form degree 0 we have $\alpha_0^2=0$. Since $\alpha_0\in\Gamma(\mathrm{End}^1(V))$, we get that α_0 provides the fibres of E with the structure of dg-vector spaces. For this reason we often write $\delta=\alpha_0$ and regard (E,δ) as a complex of vector bundles.
- In form degree 1 we have $d\alpha_0=\alpha_0\wedge\alpha_1+\alpha_1\wedge\alpha_0$. This means that α_0 is d-closed up to a homotopy provided by α_1.
- In form degree 2 we have $d\alpha_1-\alpha_1^2=\alpha_0\wedge\alpha_2+\alpha_2\wedge\alpha_0$. This means that the ordinary connection $d-\alpha_1$ is not strictly flat but flat up to a homotopy provided by α_2.
- In general we have

$$d\alpha_k-\sum_{i=1}^{k}\alpha_i\wedge\alpha_{k+1-i}=\alpha_0\wedge\alpha_{k+1}+\alpha_{k+1}\wedge\alpha_0,$$

  so we get that a relation is satisfied up to a homotopy given by α_{k+1}.

The following lemma gives a formula for the differential of a morphism between trivial ∞-local systems:

Lemma 2.1. Let $(E,D)=(M\times V,d-\alpha)$ and $(E',D')=(M\times W,d-\beta)$ be trivial ∞-local systems. If $\varphi:(E,D)\to(E',D')$ is a homogenous morphism, then its differential is

$$D_{E,E'}(\varphi)=d\varphi-\alpha'\varphi+(-1)^{|\varphi|}\varphi\alpha. \tag{2.2}$$

Proof. The proof is straight forward

$$\begin{aligned}
D_{E,E'}(\varphi)&=d_{E'}\varphi-(-1)^{|\varphi|}\varphi d_E\\
&=(d-\alpha')\varphi-(-1)^{|\varphi|}\varphi(d-\alpha)\\
&=d\circ\varphi-\alpha'\varphi-(-1)^{|\varphi|}\varphi\circ d+(-1)^{|\varphi|}\varphi\alpha\\
&=d\varphi+(-1)^{|\varphi|}\varphi\circ d-\alpha'\varphi-(-1)^{|\varphi|}\varphi\circ d+(-1)^{|\varphi|}\varphi\alpha\\
&=d\varphi-\alpha'\varphi+(-1)^{|\varphi|}\varphi\alpha.
\end{aligned}$$

□

Now we are interested in seeing how the defining form of a connection changes under automorphisms of the bundle. Suppose we have the graded connection $(M\times V,d-\alpha)$ and an automorphism $\varphi\in\Gamma(M\times\mathrm{GL}(V))$ of degree 0. The map $\varphi(d-\alpha)\varphi^{-1}:\Omega(M,V)\to\Omega(M,V)$

is again a graded connection, therefore there is another form β such that $\varphi(d-\alpha)\varphi^{-1} = d-\beta$. In other words, there is a form β that makes the following diagram commutative:

$$\begin{array}{ccc} \Omega(M,V) & \xrightarrow{d-\alpha} & \Omega(M,V) \\ \varphi\downarrow & & \downarrow\varphi \\ \Omega(M,V) & \xrightarrow[d-\beta]{} & \Omega(M,V). \end{array}$$

Unraveling the commutativity condition we find that α and β are related by the equation $\alpha = \varphi^{-1}\beta\varphi - \varphi^{-1}d\varphi$. Since the equation was derived from the commutativity of the diagram, it is clear that the relation established is an equivalence relation, hence the following definition:

Definition 2.7. Let $\alpha, \beta \in \Omega(M,V)$ have total degree 1. We say that α and β are gauge equivalent if there is an automorphism $\varphi \in \Gamma(M \times \mathrm{GL}(V))$ of degree 0 such that $\alpha = \varphi^{-1}\beta\varphi - \varphi^{-1}d\varphi$.

If $(d-\alpha)^2 = 0$ and $\varphi(d-\alpha)\varphi^{-1} = d-\beta$, then we have that $(d-\beta)^2 = 0$, which means that the gauge equivalence relation may be restricted to a relation between Maurer-Cartan forms. In this setting we get that α is gauge equivalent to β if and only if the complexes $(\Omega(M,V), d-\alpha)$ and $(\Omega(M,V), d-\beta)$ are isomorphic. With the gauge equivalence relation at hand we are ready to construct pullbacks of local systems.

Proposition 2.3. Let $f : M \to N$ be a smooth map between manifolds. Then f induces a dg-functor of degree 0 $f^* : \mathrm{Loc}_\infty(N) \to \mathrm{Loc}_\infty(M)$ called the pullback functor.

Proof. Let (E, d_E) be an ∞-local system over N. Let $\{U_i\}_i$ be a trivializing cover of N and suppose $\varphi_i : E|_{U_i} \to U_i \times V$ are trivializations. For each i there is a Maurer-Cartan form $\alpha^i \in \Omega(U_i, \mathrm{End}(V))$ such that $\varphi_i(d_E|_{U_i})\varphi_i^{-1} = d - \alpha^i$. If $U_i \cap U_j$ is nonempty, then the forms $\alpha^i|_{U_i\cap U_j}$ and $\alpha^j|_{U_i\cap U_j}$ are gauge equivalent due to the commutativity of the following diagram

$$\begin{array}{ccc} \Omega(U_i\cap U_j,V) & \xrightarrow{d-\alpha^i} & \Omega(U_i\cap U_j,V) \\ \varphi_i^{-1}\downarrow & & \downarrow\varphi_i^{-1} \\ \Omega(U_i\cap U_j,E) & \xrightarrow{d_E} & \Omega(U_i\cap U_j,E) \\ \varphi_j\downarrow & & \downarrow\varphi_j \\ \Omega(U_i\cap U_j,V) & \xrightarrow{d-\alpha^j} & \Omega(U_i\cap U_j,V). \end{array}$$

Now consider the pullback bundle $f^*E \to M$. We have a trivializing cover $\{f^{-1}(U_i)\}_i$ of M with trivializations $\hat{\varphi}_i : f^*E|_{f^{-1}(U_i)} \to f^{-1}(U_i) \times V$ induced by the φ_i. Notice that the forms $f^*\alpha^i \in \Omega(f^{-1}(U_i), \mathrm{End}(V))$ are also Maurer-Cartan forms, therefore we have flat connections $d - f^*\alpha^i$ defined on $\Omega(f^{-1}(U_i), V)$. Furthermore, whenever the set $f^{-1}(U_i) \cap f^{-1}(U_j)$ is nonempty, the forms $f^*\alpha^i|_{f^{-1}(U_i)}$ and $f^*\alpha^j|_{f^{-1}(U_j)}$ are gauge equivalent. Now we can define a flat graded connection locally over every trivializing open $d^i_{f^*E} : \Omega(f^{-1}(U_i), f^*E) \to$

$\Omega(f^{-1}(U_i), f^*E)$ by the formula $d^i_{f^*E} = \hat{\varphi}_i^{-1}(d - f^*\alpha^i)\hat{\varphi}_i$. The gauge equivalence condition implies that the following diagram is commutative

$$\begin{array}{ccc} \Omega(f^{-1}(U_i)\cap f^{-1}(U_j), f^*E) & \xrightarrow{d^i_{f^*E}} & \Omega(f^{-1}(U_i)\cap f^{-1}(U_j), f^*E) \\ {\scriptstyle \hat{\varphi}_j^{-1}\hat{\varphi}_i}\downarrow & & \downarrow{\scriptstyle \hat{\varphi}_j^{-1}\hat{\varphi}_i} \\ \Omega(f^{-1}(U_i)\cap f^{-1}(U_j), f^*E) & \xrightarrow{d^j_{f^*E}} & \Omega(f^{-1}(U_i)\cap f^{-1}(U_j), f^*E). \end{array}$$

This allows the definition of a global graded flat connection d_{f^*E} such that $d_{f^*E}|_{f^{-1}(U_i)} = d^i_{f^*E}$. We will write $f^*d_E = d_{f^*E}$.

We have defined the pullback functor on objects. On morphisms we have the usual pullback of differential forms

$$f^* : \Omega(N, \mathrm{Hom}_{BV}(E, E')) \to \Omega(M, \mathrm{Hom}_{BV}(f^*E, f^*E')).$$

Notice that our definition of f^* guarantees that the following diagram is commutative

$$\begin{array}{ccc} \Omega(N, E) & \xrightarrow{d_E} & \Omega(N, E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega(M, f^*E) & \xrightarrow{f^*d_E} & \Omega(M, f^*E). \end{array}$$

In turn this implies that $D_{f^*E, f^*E'} f^* = f^* D_{E,E'}$, making f^* a dg-functor. □

2.3 The ∞-Groupoid and Representations up to Homotopy

In this section we state some generalities on simplicial sets. This will be relevant when we define the infinity groupoid of a manifold and its representations up to homotopy. Furthermore, the particular realization of the simplices we present here will be convenient for interpreting integrals as iterated integrals.

Definition 2.8. For every $k \geq 0$ we denote $[k] = \{0, \cdots, k\}$. The simplex category, denoted Δ, is the category whose objects are the sets $[k]$ for $k \geq 0$. A morphism $f : [k] \to [l]$ is a function that preserves the order, i.e. if $m, n \in [k]$ with $m \leq n$, then $f(m) \leq f(n)$. Among the morphisms there are some particularly important ones:

- The co-face morphisms are maps $\tilde{p}^k_i : [k] \to [k+1]$, with $i \in [k+1]$. The map $\tilde{p}^k_i$ is the only injective, order preserving map whose image does not contain the element i.
- The co-degeneracy morphisms are maps $\tilde{s}^k_i : [k] \to [k-1]$, with $i \in [k-1]$. The map $\tilde{s}^k_i$ is the only surjective, order preserving map such that $\tilde{s}^k_i(i) = \tilde{s}^k_i(i+1)$.
- The $k+1$ possible ways to include $[0]$ in $[k]$ which we denote by $\tilde{v}^k_i : [0] \to [k]$ with $i \in [k]$.

The simplex category has a geometrical realisation which will be convenient for our purposes, however it is to be noted that the relevance of the simplex category lies in its combinatorial properties, not the geometric properties that may arise from the geometric description.

Let k be a non-negative integer, the k-dimensional simplex, denoted Δ_k, is the set

$$\Delta_k := \{(t_1, \cdots, t_k) \in \mathbb{R}^k \bigm| 1 \geq t_1 \geq \cdots \geq t_k \geq 0\}.$$

The 0-simplex Δ_0 is merely the single point space $\{0\}$. The k-simplex may be included in the $k+1$-simplex as a face in $k+2$ different ways. This inclusions are denoted $\hat{p}_i^k : \Delta_k \to \Delta_{k+1}$ for $i \in [k+1]$ and are defined by the formula

$$\hat{p}_i^k(t_1, \cdots, t_k) = \begin{cases} (1, t_1, \cdots, t_k) & \text{if} \quad i = 0, \\ (t_1, \cdots, t_{i-1}, t_i, t_i, t_{i+1}, \cdots, t_k) & \text{if} \quad 1 \leq i \leq k, \\ (t_1, \cdots, t_k, 0) & \text{if} \quad i = k+1. \end{cases}$$

The components of the boundary of Δ_{k+1} will be denoted $\partial_i \Delta_{k+1}$ where

$$\begin{aligned} \partial_0 \Delta_{k+1} &= \hat{p}_0^k(\Delta_k) = \{(1, t_1, \cdots, t_k) \quad \bigm| \quad (t_1, \cdots, t_k) \in \Delta_k\}, \\ \partial_i \Delta_{k+1} &= \hat{p}_i^k(\Delta_k) = \{(t_1, \cdots, t_{i-1}, t_i, t_i, t_{i+1}, \cdots, t_k) \quad \bigm| \quad (t_1, \cdots, t_k) \in \Delta_k\}, \\ \partial_{k+1} \Delta_{k+1} &= \hat{p}_{k+1}^k(\Delta_k) = \{(t_1, \cdots, t_k, 0) \quad \bigm| \quad (t_1, \cdots, t_k) \in \Delta_k\}. \end{aligned}$$

On the other hand we have k projection maps $\hat{s}_i^k : \Delta_k \to \Delta_{k-1}$ for $i \in [k-1]$ that collapse the k-simplex onto one of its faces. The projections are defined by

$$\hat{s}_i^k(t_1, \cdots, t_k) = (t_1, \cdots, t_i, t_{i+2}, \cdots, t_k).$$

The $k+1$ different ways of including the 0-simplex as a vertex in the k simplex will be denoted $\hat{v}_i^k : \Delta_0 \to \Delta_k$ for $i \in [k]$. These inclusions may be obtained as compositions of the $\hat{p}_i^k$ as follows

$$\hat{v}_i^k = p_k^{k-1} \cdots p_{i+1}^i p_0^{i-1} \cdots p_0^0.$$

Put simply, $\hat{v}_i^k(0) = (1, \cdots, 1, 0, \cdots, 0)$ where the tuple has i ones and $k-i$ zeroes.

The correspondence between the combinatorial and geometric descriptions of the simplex category is straightforward: the set $[k]$ maps to the simplex Δ_k. If $\tilde{f} : [k] \to [n]$ is an order-preserving map, then we have a corresponding function $\hat{f} : \Delta_k \to \Delta_n$ mapping the i-th vertex of Δ_k into the $\tilde{f}(i)$-th vertex of Δ_n. This assignment maps $\tilde{p}_i^k$, $\tilde{s}_i^k$ and $\tilde{v}_i^k$ into $\hat{p}_i^k$, $\hat{s}_i^k$ and $\hat{v}_i^k$ respectively.

Definition 2.9. A simplicial set is a contraviariant functor $X : \Delta \to \mathsf{Set}$ from the simplex category to the category of sets. We usually denote the set $X(\Delta_k)$ by X_k and the whole simplicial set by $X_\bullet$. The co-face and co-degeneracy maps turn into face and degeneracy maps respectively

$$X(\hat{p}_i^k) = p_i^k : X_{k+1} \to X_k, \quad X(\hat{s}_i^k) = s_i^k : X_{k-1} \to X_k.$$

The vertex maps turn into constant functions $v_i^k : X_k \to X_0$. When the dimension of the simplex is clear from the context we will simply write p_i, s_i and v_i for the face, degeneracy and vertex maps.

The face and degeneracy maps satisfy the following relations called the *simplicial identities*:

1. $p_i^{k-1} p_j^k = p_{j-1}^{k-1} p_i^k$ if $i < j$.
2. $s_i^{k+1} s_j^k = s_{j+1}^{k+1} s_i^k$ if $i \le j$.
3. $p_i^{k-1} s_j^k = s_{j-1}^{k-1} p_i^{k-2}$ if $i < j$.
4. $p_i^{k-1} s_j^k = \mathrm{Id}$ if $i = j$ or $i = j + 1$.
5. $p_i^{k-1} s_j^k = s_j^{k-1} p_{i-1}^{k-2}$ if $i > j + 1$.

Example 2.3. Let M be a smooth manifold. We define a simplicial set $X_\bullet$ by $X_k = \mathrm{Hom}_{\mathrm{Smooth}}(\Delta_k, M)$, the set of smooth maps from Δ_k to M. The whole simplicial set is called the infinity groupoid of M and is denoted $\pi_\infty(M)$.

We aim to define representations up to homotopy of simplicial sets, for this purpose we need the following:

Definition 2.10. Let A be an algebra and $X_\bullet$ a simplicial set. A cochain of degree k with values in A is a map $F : X_k \to A$.

The algebra structure in A allows us to provide the set of cochains with some extra structure. First we need some notation. There is a map $P_k^r : X_k \to X_r$ called the r-dimensional frontal face map. For an element $x \in X_k$ we define

$$P_k^r(x) = p_{r-1}^r \cdots p_{k-1}^{k-2} p_k^{k-1}(x).$$

Similarly, the r-dimensional back face $Q_k^r : X_k \to X_r$ is defined by

$$Q_k^r(x) = p_0^r \cdots p_0^{k-1}(x).$$

We use the front and back maps to define the cup product of cochains with values in A. Let F and G be cochains with degrees k and l respectively. The cup product, denoted $F \cup G$, is a cochain of degree $k + l$ defined by the formula

$$F \cup G(x) = F(P_{k+l}^k(x)) G(Q_{k+l}^l(x)), \quad x \in X_{k+l}.$$

Remark 2.1. In the case of the infinity groupoid, the back and frontal face maps are the pullbacks of the following maps respectively:

$$U_i : \Delta_i \to \Delta_n, \quad (t_1 \cdots, t_i) \mapsto (1, \cdots, 1, t_1 \cdots, t_i),$$
$$V_i : \Delta_i \to \Delta_n, \quad (t_1 \cdots, t_i) \mapsto (t_1 \cdots, t_i, 0, \cdots, 0).$$

Definition 2.11. Let $X_\bullet$ be a simplicial set. A representation up to homotopy of $X_\bullet$ is comprised of

- A $\mathbb{Z}$-graded vector space $E_x = \bigoplus_k E_x^k$ for each 0-simplex $x \in X_0$.
- A sequence of cochains $\{F_k\}_{k\geq 0}$ such that F_k is a cochain of degree k with $F_k(x) \in \mathrm{Hom}^{1-k}(E_{v_k(x)}, E_{v_0(x)})$ for $x \in X_k$.

For each $k \geq 0$ we require the following relation to be satisfied:

$$\sum_{j=1}^{k-1}(-1)^j F_{k-1}(p_j(x)) - \sum_{j=0}^{k}(-1)^j (F_j \cup F_{k-j})(x) = 0. \tag{2.3}$$

The cup product used in relation (2.3) is defined with compositions. Explicitly we have $(F_j \cup F_{k-j})(x) = F_j(P_k^j(x)) \circ F_{k-j}(Q_k^{k-j}(x))$,

$$E_{v_k(x)} \xrightarrow{F_{k-j}(Q_k^{k-j}(x))} E_{v_j x} \xrightarrow{F_j(P_k^j(x))} E_{v_0(x)}.$$

As usual we will explore the meaning of the relations for low values of k.

- For $k = 0$ we have a point $x \in X_0$ and a linear map $F_0(x) : E_x \to E_x$ of degree 1 with the property $F_0(x) \circ F_0(x) = 0$. Therefore $(E_x, F_0(x))$ is a cochain complex.
- For $k = 1$ we have that a 1-simplex γ is a path in M from x_0 to x_1. $F_1(\gamma)$ is a morphism $\mathrm{Hom}^0(E_{x_1}, E_{x_0})$ such that

  $$F_0(x_0) \circ F_1(\gamma) = F_1(\gamma) \circ F_0(x_1).$$

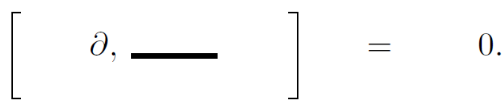
  In other words, $F_1(\gamma)$ is a morphism of cochain complexes.
- For $k = 2$ consider a 2-simplex σ with vertices x_0, x_1, x_2 and edges $\gamma_{i,j}$ connecting vertex x_i to x_j. The relation reads

  $$F_1(\gamma_{0,1}) \circ F_1(\gamma_{1,2}) - F_1(\gamma_{0,2}) = F_0(x_0) \circ F_2(\sigma) + F_2(\sigma) \circ F_1(x_2).$$

  This means that the cochain complex morphisms going from E_{x_2} to E_{x_0} defined by applying F_1 to the edges of σ are homotopic via a homotopy given by $F_2(\sigma)$.

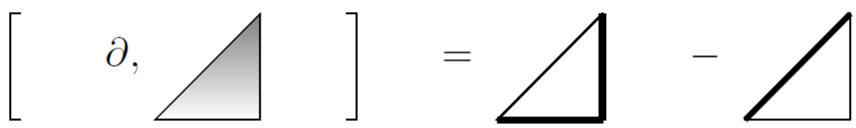
When the simplicial set is the infinity groupoid of a manifold, a representation up to homotopy is the assignment of holonomies for simplices of all dimensions in a coherent manner. In this case we have the following pictorial representation of the first three relations. A shaded face of a simplex represents the holonomy assigned to it:

- For a path we have .



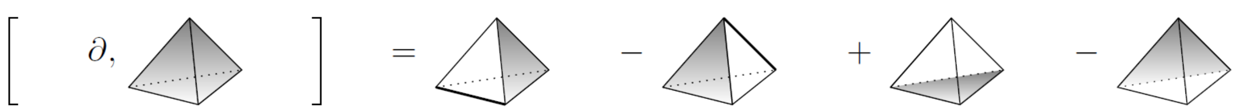
  This relation states that the holonomy assigned to a path is a morphism between the cochain complexes lying over the endpoints of the path.

- For the triangle we get ,



  which means that the holonomy assigned to a 2-simplex is an (algebraic) homotopy between the cochain maps assigned to its edges.

- For the tetrahedron the relation is .



Definition 2.12. Let $X_\bullet$ be a simplicial set and $(E,F_\bullet),(E',F'_\bullet)$ representations up to homotopy of $X_\bullet$. A morphism of representations of degree n is a sequence $\varphi=\{\varphi_k\}_{k\geq 0}$ with φ_k a k-cochain such that for a k-simplex $\sigma\in X_k$, we have

$$\varphi_k(\sigma)\in \mathrm{Hom}^{n-k}(E_{v_k(\sigma)},E'_{v_0(\sigma)}).$$

The identity morphism is the sequence $\varphi_0=\mathrm{Id}$ and $\varphi_k=0$ for $k\geq 1$.
If $\varphi:(E,F)\to(E',F')$ and $\varphi':(E',F')\to(E'',F'')$ are morphisms, then the composition is the sequence $\{(\varphi'\circ\varphi)_k\}_{k\geq 0}$ where

$$(\varphi'\circ\varphi)_k:=\sum_{i+j=k}(-1)^{jn}(\varphi'_j\cup\varphi_i).$$

The differential of φ is the sequence $\{D(\varphi)_k\}_{k\geq 0}$ where

$$D(\varphi)_k(\sigma) := \sum_{i+j=k}(-1)^{jn}F'_j \cup \varphi_i(\sigma) + \sum_{i+j=k}(-1)^{n+j+1}\varphi_j F_i(\sigma) + \sum_{i=1}^{k-1}(-1)^{j+n}\varphi_{k-1}(p_i(\sigma)).$$

Proposition 2.4. Representations up to homotopy of $X_\bullet$ form a dg-category with the morphisms, composition and differential given in Definition 2.12.

The category of representations up to homotopy of $X_\bullet$ will be denoted $\mathcal{R}\mathrm{ep}^\infty(X_\bullet)$. In particular, if the simplicial set is $\pi_\infty(M)$, we will denote the category by $\mathcal{R}\mathrm{ep}^\infty(M)$.

3 The A_∞ de Rham Theorem and Higher Parallel Transport

The main goal of this chapter is to prove a theorem of Riemann-Hilbert type relating ∞-local systems to representations of the infinity groupoid of a manifold that first appeared in [8] and [16]. The theorem will be a consequence of the A_∞ de Rham theorem which was first proved by Gugenheim in [14]). Here we follow the presentation given in [2]. The A_∞ theorem states that there is an A_∞- quasi-isomorphism $\psi : (\Omega^\bullet(M), d, \wedge) \to (C^\bullet(M), \delta, \cup)$ from the de Rham algebra of a manifold to its smooth singular cochain algebra. The zeroth component of ψ is precisely the morphism used in the proof of the classical de Rham theorem. The morphism ψ is constructed as a composition of two maps $\psi = \mathrm{S} \circ \mathrm{C}$ where $\mathrm{C} : B(\Omega^\bullet(M)) \to \Omega(PM)$ is the iterated integral map going from the bar complex of $\Omega(M)$ to the de Rham algebra of the path space PM. The second map $\mathrm{S} : \Omega(PM) \to C^\bullet(M)$ is called Igusa's map and is defined in terms of a family of maps relating the cellular structures of the cubes $\{I^k\}_{k\geq 0}$ and the simplices $\{\Delta_k\}_{k\geq 0}$.

3.1 Iterated Integrals: A Scalar Version

Here we develop a version of one of the main tools used throughout this thesis, Chen's iterated integrals. Iterated integrals were formalized by Chen in [11]. In this text we use iterated integrals for two major purposes:

1. To construct an A_∞-quasi-isomorphism between the de Rham algebra and the algebra of smooth cochains of a smooth manifold, this is known as the A_∞ de Rham theorem. In turn this will allow for the definition of higher dimensional parallel transport.
2. To construct an A_∞-natural transformation between some dg-functors. This will lead to an A_∞ version of the Poincaré lemma.

Each purpose requires a different version of iterated integrals. Although the particularities of the constructions differ, both versions have mostly the same properties. For the A_∞ de Rham theorem we use a scalar version of iterated integrals. For the A_∞ Poincaré lemma, a vector valued version is needed and will be constructed in the next chapter.

The path space PM is the space of smooth maps from the interval to M, this is $C^\infty(I, M)$. Let X be a smooth manifold. We say that $f : X \to PM$ is smooth if the evaluation map $f^{ev} : X \times I \to M$, $f^{ev}(x,t) = f(x)(t)$ is smooth. The idea to define differential forms on PM is that it is enough to know all the possible pullbacks of a differential form $\alpha \in \Omega(PM)$ to finite dimensional manifolds X via smooth maps $f : X \to PM$. So a differential form $\alpha \in \Omega(PM)$ is a choice of a differential form $f^*(\alpha) \in \Omega(X)$ for every pair (X, f) where X is finite dimensional and $f : X \to PM$ is smooth. The choice must be natural in the following sense; suppose (Y, g) is another pair, and there is a smooth map $h : X \to Y$ such that $g \circ h = f$, i.e., the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & PM \\ {\scriptstyle h}\downarrow & & \| \\ Y & \xrightarrow[g]{} & PM, \end{array} \tag{3.1}$$

then we must have $h^*(g^*(\alpha)) = f^*(\alpha)$.

Differential forms on PM may be rigorously defined as follows:

- Let $C(-, PM)$ be the category whose objects are pairs (X, f) where X is a finite dimensional smooth manifold and $f : X \to PM$ is smooth. A morphism $h : (X, f) \to (Y, g)$ is a smooth map $h : X \to Y$ that makes diagram 3.1 commutative.
- The functor $\bar{\mathbb{R}} : C(-, PM) \to \mathrm{Vect}$ is the constant functor assigning $\mathbb{R}$ to every object (X, f) and $\mathrm{Id}_{\mathbb{R}}$ to every morphism.
- The functor $\Omega C(-, PM) \to \mathrm{Vect}$ assigns $\Omega(X)$ to an object (X, f) and h^* to a morphism $h : (X, f) \to (Y, g)$.
- A differential form $\alpha \in \Omega(PM)$ is a natural transformation $\alpha : \bar{\mathbb{R}} \to \Omega$.

The dg algebra structure on $\Omega(PM)$ is defined by declaring

$$f^*(d\alpha) = df^*\alpha, \quad \text{and} \quad f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta), \quad \text{for } \alpha, \beta \in \Omega(PM).$$

The scalar version of the iterated integral map is a map going from the bar complex of the de Rham algebra of M to the de Rham algebra of the path space.

$$\mathrm{C} : B(\Omega(M)) \to \Omega(PM).$$

In order to define a form $\mathrm{C}(sa_1 \otimes \cdots \otimes sa_n)$ in $\Omega(PM)$ fix a pair (X, f) with X a smooth manifold and $f : X \to PM$ a smooth function. Using f^{ev} we can define maps $f_{(n)} : \Delta_n \times X \to$

M^n by the formula

$$f_{(n)}(t_1, \cdots, t_n, x) := (f^{ev}(t_1, x), \cdots, f^{ev}(t_n, x)),$$

which in turn define pullbacks $f_{(n)}^* : \Omega(M^n) \to \Omega(\Delta_n \times X)$. We also denote by $p_i : M^n \to M$ the projection onto the i-th component and define the map $P : (s\Omega(M))^{\otimes n} \to \Omega(M^n)$ by the formula

$$P(s\alpha_1 \otimes \cdots \otimes s\alpha_n) := (-1)^{\sum_{i=1}^n |\alpha_i|(n-i)} p_1^*\alpha_1 \wedge \cdots \wedge p_n^*\alpha_n.$$

The sign in the formula is precisely the sign obtained from the Koszul convention applied to $s^{\otimes n}$:

$$(s^{\otimes n})(\alpha_1 \otimes \cdots \otimes \alpha_n) = (-1)^{\sum_{i=1}^n |\alpha_i|(n-i)} s\alpha_1 \otimes \cdots \otimes s\alpha_n.$$

The last step is a push forward map $\pi_* : \Omega(\Delta_n \times X) \to \Omega(X)$; it is defined locally by

$$f(t_1, \cdots, t_n, x) dt_1 \cdots dt_n dX \mapsto \left(\int_{\Delta_n} f(t_1, \cdots, t_n, x) dt_1 \cdots dt_n \right) dX. \tag{3.2}$$

Thus we have the composition

$$(s\Omega(M))^{\otimes n} \xrightarrow{P} \Omega(M^n) \xrightarrow{f_{(n)}^*} \Omega(\Delta_n \times X) \xrightarrow{\pi_*} \Omega(X),$$

which allows us to define the iterated integral map $\mathrm{C} : B(\Omega(M)) \to \Omega(PM)$: for $s\alpha_1 \otimes \cdots \otimes s\alpha_n \in (s\Omega(M))^{\otimes n}$ and a pair (X, f), we define C by the formula

$$\begin{aligned} f^*\mathrm{C}(s\alpha_1 \otimes \cdots \otimes s\alpha_n) &= \pi_*(f_{(n)}^*(P(s\alpha_1 \otimes \cdots \otimes s\alpha_n))) \\ &= (-1)^{\sum_{i=1}^n |\alpha_i|(n-i)} \pi_*(f_{(n)}^*(p_1^*(\alpha_1) \wedge \cdots \wedge p_n^*(\alpha_n)). \end{aligned}$$

Lemma 3.1. The push forward $\pi_* : \Omega(\Delta_n \times X) \to \Omega(X)$ is a morphism of left $\Omega(X)$-modules of degree $-n$. This is, for every $\omega \in \Omega(X)$ and $\alpha \in \Omega(\Delta_n \times X)$ we have

$$\pi_*(\pi^*(\omega) \wedge \alpha) = (-1)^{|\omega|n} \omega \wedge \pi_*(\alpha).$$

Furthermore, let $\partial\pi$ be the composition

$$\partial\Delta_n \times X \xrightarrow{\iota \times \mathrm{Id}} \Delta_n \times X \xrightarrow{\pi} X,$$

where $\partial\Delta_n$ is the boundary of the simplex. Then the following formula holds:

$$\pi_* \circ d - (-1)^n d \circ \pi_* = (\partial\pi)_* \circ (\iota \times \mathrm{Id})^*. \tag{3.3}$$

The push forward along $\partial\pi$ is understood as the sum over the push forwards of the components of the boundary $\partial\Delta_n$.

Proof. First we check that the push forward is a morphism of $\Omega(X)$-modules, notice that it is enough to check this locally. Suppose $(x_1, \cdots, x_m)$ are coordinates for X and write

$$\omega = f dx_{i_1} \cdots dx_{i_k} \in \Omega^k(X),$$
$$\alpha = g dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l} \in \Omega^{n+l}(\Delta_n \times X).$$

Then we have

$$\begin{aligned} \int_{\Delta_n} \pi^*(\omega) \wedge \alpha &= \int_{\Delta_n} f dx_{i_1} \cdots dx_{i_k} g dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{kn} f \left(\int_{\Delta_n} g dt_1 \cdots dt_n \right) dx_{i_1} \cdots dx_{i_k} dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{kn} f dx_{i_1} \cdots dx_{i_k} \left(\int_{\Delta_n} g dt_1 \cdots dt_n \right) dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{kn} \omega \wedge \int_{\Delta_n} \alpha. \end{aligned}$$

The fact that π_* has degree $-n$ is also clear from the previous computation. For the second statement first we consider a form $\alpha = g dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l}$. Notice that restricting this form to the boundary of Δ_n nullifies it, therefore in this case we merely need to check that $\pi_* \circ d = (-1)^n d \circ \pi_*$ which is straight forward:

$$d \int_{\Delta_n} g dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l} = \sum_{i=1}^{m} \int_{\Delta_n} \frac{\partial g}{\partial x_i} dt_1 \cdots dt_n dx_i dx_{j_1} \cdots dx_{j_l} = (-1)^n \pi_*(d\alpha).$$

Next we consider forms of the type $\alpha = g dt_1 \cdots \widehat{dt_k} \cdots dt_n dx_{j_1} \cdots dx_{j_l}$, where the hat denotes that dt_k is not a factor of α. For this form the term $d \circ \pi_*$ vanishes, leaving us to check that $(-1)^{n+1} \pi_* \circ d = (\partial \pi)_* \circ (\iota \times \mathrm{Id})^*$. Notice that when we restrict α to the boundary of Δ_n the only terms that remain correspond to the components $t_{k-1} = t_k$ and $t_k = t_{k+1}$ of the boundary. In the cases $k = 1$ and $k = n$ we have the components $t_1 = 1, t_1 = t_2$ and $t_{n-1} = t_n, t_n = 0$ respectively. With this the computation is straight forward

$$\begin{aligned} &(-1)^{n+1} \int_{\Delta_n} \frac{\partial g}{\partial t_k} dt_k dt_1 \cdots \widehat{dt_k} \cdots dt_n dx_{j_1} \cdots dx_{j_l} = (-1)^{n+k} \int_{\Delta_n} \frac{\partial g}{\partial t_k} dt_1 \cdots dt_n dx_{j_1} \cdots dx_{j_l} \\ &= (-1)^{k-1} \int_{\partial_{k-1}\Delta_n} g dt_1 \cdots \widehat{dt_k} \cdots dt_n dx_{j_1} \cdots dx_{j_l} + (-1)^k \int_{\partial_k \Delta_n} g dt_1 \cdots \widehat{dt_k} \cdots dt_n dx_{j_1} \cdots dx_{j_l}. \end{aligned}$$

For any other homogeneous form, equation (3.3) is trivial. □

Remark 3.1. Chen's map has the property that the image of an element $s\alpha_1 \otimes \cdots \otimes s\alpha_n$ for which one of the factors is a function is zero. If α_i is a function, the vector field $\frac{\partial}{\partial t_i}$ annihilates the form $f_{(n)}^*(p_1^*(\alpha_1) \wedge \cdots \wedge p_n^*(\alpha_n))$, thus the integral over Δ_n vanishes.

Lemma 3.2. Chen's map is natural, i.e. if $h : M \to N$ is a smooth map, then the following diagram is commutative:

$$\begin{array}{ccc} B(\Omega(N)) & \xrightarrow{\mathrm{C}} & \Omega(PN) \\ {\scriptstyle Bh^*}\downarrow & & \downarrow{\scriptstyle (Ph)^*} \\ B(\Omega(M)) & \xrightarrow[\mathrm{C}]{} & \Omega(PM). \end{array}$$

The map Bh^* is the map induced on the bar complex by the pullback $h^* : \Omega(N) \to \Omega(M)$ and $Ph : PM \to PN$ is the push forward of paths along h.

Proof. Consider the diagram

$$\begin{array}{ccccc} (s\Omega(N))^{\otimes n} & \xrightarrow{P} & \Omega(N^n) & \xrightarrow{(Ph\circ f)^*_{(n)}} & \Omega(\Delta_n \times X) \\ {\scriptstyle Bh^*}\downarrow & & {\scriptstyle (h^n)^*}\downarrow & & {\scriptstyle \mathrm{Id}}\downarrow \\ (s\Omega(M))^{\otimes n} & \xrightarrow[P]{} & \Omega(M^n) & \xrightarrow[f^*_{(n)}]{} & \Omega(\Delta_n \times X). \end{array}$$

The left square is clearly commutative. The second square is commutative since

$$(Ph \circ f)_{(n)} = (h \times \cdots \times h) \circ f_{(n)},$$

The naturality of C follows. □

Proposition 3.1. For any element $s\alpha_1 \otimes \cdots \otimes s\alpha_n$ of $B(\Omega(M))$, the following identity holds:

$$\begin{aligned} d(\mathrm{C}(s\alpha_1 \otimes \cdots \otimes s\alpha_n)) = &\mathrm{C}(D(s\alpha_1 \otimes \cdots \otimes s\alpha_n)) + \\ &\mathrm{ev}_1^*(\alpha_1) \wedge \mathrm{C}(s\alpha_2 \otimes \cdots \otimes s\alpha_n) \\ &- (-1)^{|\alpha_1|+\cdots+|\alpha_{n-1}|}\mathrm{C}(s\alpha_1 \otimes \cdots \otimes s\alpha_{n-1}) \wedge \mathrm{ev}_0^*(\alpha_n). \end{aligned}$$

In the previous equation, D is the differential defined on the bar complex; $\mathrm{ev}_0, \mathrm{ev}_1 : PM \to M$ are the evaluation maps in $t = 0$ and $t = 1$ respectively.

Proof. It suffices to establish the formula for the pullback of $\mathrm{C}(s\alpha_1 \otimes \cdots \otimes s\alpha_n)$ by any smooth map $f : X \to PM$. Using Lemma 3.1 we see that $f^* d\mathrm{C}(s\alpha_1 \otimes \cdots \otimes s\alpha_n)$ is equal to

$$\begin{aligned} (-1)^{\sum_{i=1}^n |\alpha_i|(n-i)}&((-1)^n(\pi_* d(f^*_{(n)}(p_1^*\alpha_1 \wedge \cdots \wedge p_n^*\alpha_n))) \\ &+ (-1)^{n+1}((\partial\pi)_*(\iota \times \mathrm{Id})^* f^*_{(n)}(p_1^*\alpha_1 \wedge \cdots \wedge p_n^*\alpha_n))). \end{aligned}$$

The first term gives

$$\sum_{i=1}^{n} (-1)^{|\alpha_1|+\cdots+|\alpha_{i-1}|} f^*\mathrm{C}(s\alpha_1 \otimes \cdots \otimes s(-d\alpha_i) \otimes \cdots \otimes s\alpha_n).$$

The second term yields

$$\sum_{i=1}^{n-1}(-1)^{|\alpha_1|+\cdots+|\alpha_i|} f^*\mathrm{C}(s\alpha_1\otimes\cdots\otimes s(\alpha_i\wedge\alpha_{i+1})\otimes\cdots\otimes s\alpha_n)$$
$$+ (ev_1\circ f)^*\alpha_1 f^*\mathrm{C}(s\alpha_2\otimes\cdots\otimes s\alpha_n)$$
$$- (-1)^{|\alpha_1|+\cdots+|\alpha_{n-1}|} f^*\mathrm{C}(s\alpha_1\otimes\cdots\otimes s\alpha_{n-1})\wedge(ev_0\circ f)^*\alpha_n.$$

□

Definition 3.1. The set

$$C_+^\infty(I,\partial I) := \{\phi : I\to I \mid \phi \text{ is smooth, monotone and } \phi(0)=0, \phi(1)=1\}$$

is a monoid under composition and acts on $C^\infty(X,PM)$ via reparametrizations:

$$(\phi\bullet f)^{ev}(t,x) := f^{ev}(\phi(t),x).$$

A differential form $\omega\in\Omega(PM)$ is reparametrization invariant if for any smooth map $f : X\to PM$ and any $\phi\in C_+^\infty(I,\partial I)$ the equation

$$f^*\omega = (\phi\bullet f)^*\omega$$

holds. The subcomplex of invariant differential forms will be denoted by $\Omega_{inv}(PM)$.

Lemma 3.3. The image of Chen's map is contained in the subcomplex $\Omega_{inv}(PM)$ of reparametrization invariant differential forms of PM.

Proof. Pick an arbitrary $f : X\to PM$ and $\phi\in C_+^\infty(I,\partial I)$. We want to check that

$$f^*\mathrm{C}(s\alpha_1\otimes\cdots\otimes s\alpha_n) = (\phi\bullet f)^*\mathrm{C}(s\alpha_1\otimes\cdots\otimes s\alpha_n)$$

holds. Using the equality

$$(\phi\bullet f)_{(n)} = f_{(n)}\circ(\phi\times\cdots\times\phi\times\mathrm{Id}) : \Delta_n\times X\to M^n,$$

we see that it suffices to prove the equation

$$\pi_* = \pi_*\circ(\phi\times\cdots\times\phi\times\mathrm{Id})^* : \Omega(\Delta_n\times X)\to\Omega(X).$$

The equation follows from the substitution rule for integrals. □

3.2 Igusa's Map

Next we construct Igusa's map, this is

$$\mathrm{S} : \Omega(PM)\to sC(M).$$

The definition relies on a sequence of maps going from cubes to the path spaces of simplices:

Definition 3.2. For each $k \geq 0$ we define a map

$$\Theta_{(k)} : I^{k-1} \to P\Delta_k$$

as a composition of the following maps:

- The map $\lambda_k : I^{k-1} \to PI^k$ which is defined as follows: take a point $(x_1, \cdots, x_{k-1}) \in I^{k-1}$ and consider the sequence of points

$$\begin{aligned} P_0 &= (x_1, \cdots, x_{k-2}, x_{k-1}, 1), \\ P_1 &= (x_1, \cdots, x_{k-2}, x_{k-1}, 0), \\ P_2 &= (x_1, \cdots, x_{k-2}, 0, 0), \\ &\vdots \\ P_k &= (0, \cdots, 0). \end{aligned}$$

  The path $\lambda_k(x_1, \cdots, x_{k-2}, x_{k-1})$ is the piecewise linear path

$$[P_0, P_1] * \cdots * [P_{k-1}, P_k]$$

  which can be made to be smooth when composed with an appropriate reparametrization. The reparametrization chosen will be irrelevant due to the reparametrization invariance property of the iterated integral map stated in Lemma 3.3.

- The map $P\pi_k : PI^k \to P\Delta_k$ where $\pi_k : I^k \to \Delta_k$ is such that

$$\pi_k(x_1, \cdots, x_k) := (t_1, \cdots, t_k) \text{ where } t_i = \max\{x_i, \cdots, x_k\}.$$

Then we have $\Theta_{(k)} = P\pi_k \circ \lambda_k$. We also make the convention that $\Theta_{(0)}$ is the trivial map from a point to a point. The adjoint map of $\Theta_{(k)}$ will be denoted $\Theta_k : I^k \to \Delta_k$.

The degree of the adjoint map Θ_k is $(-1)^k$, this is

$$\int_{I^k} \Theta_k^* \alpha = (-1)^k \int_{\Delta_k} \alpha, \qquad \alpha \in \Omega(\Delta_k).$$

Furthermore, notice that the image of $\Theta_{(k)}$ lies in the subset of $P\Delta_k$ comprised of paths beginning at the last vertex v_k and ending at the zeroth vertex v_0 of Δ_k.

The sequence of maps Θ_k relates the family of cubes I^k with the family of simplices Δ_k. We already know that the simplices satisfy some combinatorial properties given in terms of face and degeneracy maps. We can also define face maps for the family of cubes:

Definition 3.3. For $k \geq 1$ we define the maps $\partial_i^{\pm} : I^k \to I^{k+1}$ for $i = 1, \cdots, k+1$ as follows:

$$\partial_i^-(x_1, \cdots, x_k) = (x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_k),$$
$$\partial_i^+(x_1, \cdots, x_k) = (x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_k).$$

In the case $k = 0$ we have that $I^0 = *$ is the single point space and $\partial^{\pm} : * \to I$ are the constant functions $\partial^- = 0$ and $\partial^+ = 1$.

The following lemma states some compatibility conditions between the face maps of the cubes and the face maps of the simplices:

Lemma 3.4. The sequence $\Theta_{(k)}$ satisfies the properties:

1. For $1 \leq i \leq k-1$ the following diagram is commutative

$$\begin{array}{ccc} I^{k-2} & \xrightarrow{\partial_i^-} & I^{k-1} \\ {\scriptstyle \Theta_{(k-1)}} \downarrow & & \downarrow {\scriptstyle \Theta_{(k)}} \\ P\Delta_{k-1} & \xrightarrow[\overline{P\partial_i}]{} & P\Delta_k . \end{array}$$

The map $\overline{P\partial_i}$ is the composition

$$P\Delta_{k-1} \xrightarrow{\hat{R}_i} P\Delta_{k-1} \xrightarrow{P\partial_i} P\Delta_k$$

where $\delta_i : \Delta_{k-1} \to \Delta_k$ is the i-th face map and $\hat{R}_i$ is induced by the following reparametrization:

$$R_i(t) := \begin{cases} \frac{kt}{k-1} & \text{for} \quad 0 \leq t \leq \frac{k-i-1}{k}, \\ \frac{k-i-1}{k-1} & \text{for} \quad \frac{k-i-1}{k} \leq t \leq \frac{k-i}{k}, \\ \frac{kt-1}{k-1} & \text{for} \quad \frac{k-i}{k} \leq t \leq 1. \end{cases}$$

2. For $1 \leq i \leq k-1$ the following diagram is commutative

$$\begin{array}{ccccc} I^{k-2} & \xrightarrow{\partial_i^+} & I^{k-1} & \xrightarrow{\Theta_{(k)}} & P\Delta_k \\ \cong \downarrow & & & & \uparrow {\scriptstyle \mu_i} \\ I^{i-1} \times I^{k-i-1} & & \xrightarrow[\Theta_{(i)} \times \Theta_{(k-i)}]{} & & P\Delta_i \times P\Delta_{k-i}. \end{array}$$

The map μ_i is the concatenation

$$\mu_i(\alpha, \beta)(t); = \begin{cases} U_{k-i}\left(\beta\left(\frac{kt}{k-i}\right)\right) & \text{for} \quad 0 \leq t \leq \frac{k-i}{k}, \\ V_i\left(\alpha\left(\frac{k}{i}(t - \frac{k-i}{k})\right)\right) & \text{for} \quad \frac{k-i}{k} \leq t \leq 1. \end{cases}$$

where U_i and V_i are the maps defined in Remark 2.1.

The following lemma explains an interaction between the concatenation maps μ_i from Lemma 3.4 and the iterated integral map C.

Lemma 3.5. Let $\omega_1, \cdots, \omega_n$ be differential forms on Δ_k and let $f : X \to P\Delta_i$ and $g : Y \to \Delta_{k-i}$ be smooth functions. Then

$$\int_{X\times Y} (f\times g)^*(\mu_i)^* C(s\omega_1\otimes\cdots\otimes s\omega_n)$$
$$= \sum_{j=0}^{n} \left(\int_X f^*C(sV_i^*\omega_1\otimes\cdots\otimes sV_i^*\omega_l)\right)\times\left(\int_Y g^*C(sU_{k-i}^*\omega_{l+1}\otimes\cdots\otimes sU_{k-i}^*\omega_n)\right).$$

Here we are extending C to the augmented bar complex $\mathbb{R}\oplus B(\Omega(\Delta_k))$ by setting $C(1) := 1$.

The proofs of Lemmas 3.4 and 3.5 can be found in [2].

Definition 3.4. Igusa's map $\mathrm{S} : \Omega(PM) \to sC(M)$ is defined as follows: let $\alpha \in \Omega(PM)$, then $\mathrm{S}(\alpha) := s\varsigma(\alpha)$ is the suspension of the cochain $\varsigma(\alpha)$ defined on a k-simplex $\sigma : \Delta_k \to M$ as

$$\varsigma(\alpha)(\sigma) = \int_{I^{k-1}} \Theta_{(k)}^* P\sigma^*\alpha.$$

The purpose of Lemmas 3.4 and 3.5 is to prove the following formula:

Proposition 3.2. Let $\omega_1, \cdots, \omega_n$ be differential forms on M, then the following equation holds:

$$\mathrm{S}(d\mathrm{C}(s\omega_1\otimes\cdots\otimes s\omega_n)) = \hat{\delta}(\mathrm{S}(\mathrm{C}(s\omega_1\otimes\cdots\otimes s\omega_n))) +$$
$$\sum_{i=1}^{n-1} \mathrm{S}(\mathrm{C}(s\omega_1\otimes\cdots\otimes s\omega_i)) \mathbin{\hat{\cup}} \mathrm{S}(\mathrm{C}(s\omega_{i+1}\otimes\cdots\otimes s\omega_n)).$$

Recall that the hat in the notation means we are considering the differential and the product defined on the suspension.

Proof. Take $\alpha = \mathrm{C}(s\omega_1\otimes\cdots\otimes s\omega_n) \in \Omega(PM)$ and a k-simplex $\sigma : \Delta_k \to M$. Then we have

$$\mathrm{S}(d\alpha)(\sigma) = \int_{I^{k-1}} d\Theta_{(k)}^*(P\sigma)^*\alpha = \int_{\partial I^{k-1}} \iota^*\Theta_{(k)}^*(P\sigma)^*\alpha,$$

where ι is the inclusion of ∂I^{k-1} into I^{k-1}. The right hand side of the previous equation may be written in terms of the inclusions $\partial_i^{\pm} : I^{k-2} \to I^{k-1}$:

$$\int_{\partial I^{k-1}} \iota^*\Theta_{(k)}^*(P\sigma)^*\alpha = \sum_{i=1}^{k-1}(-1)^i\int_{I^{k-2}}(\partial_i^-)^*\Theta_{(k)}^*(P\sigma)^*\alpha - \sum_{i=1}^{k-1}(-1)^i\int_{I^{k-2}}(\partial_i^+)^*\Theta_{(k)}^*(P\sigma)^*\alpha.$$

For the terms in the first summation we use part 1 of Lemma 3.4 and the naturality of C. We get the equation

$$\int_{I^{k-2}} (\partial_i^-)^* \Theta^*_{(k)} (P\sigma)^* \alpha = \int_{I^{k-2}} \Theta^*_{(k-1)} (P\partial_i^* \sigma)^* \alpha = \mathrm{S}(\alpha)(\partial_i^* \sigma).$$

Adding from $i = 1$ to $k - 1$ yields

$$\sum_{i=1}^{k-1} (-1)^i \int_{I^{k-2}} (\partial_i^-)^* \Theta^*_{(k)} (P\sigma)^* \alpha = \hat{\delta}(\mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n)))(\sigma) \\ - \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n))(\partial_0^* \sigma) - (-1)^k \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n))(\partial_k^* \sigma).$$

For the terms in the second summation we use part 2 of Lemma 3.4 together with Lemma 3.5 to write

$$\int_{I^{k-2}} (\partial_i^+)^* \Theta^*_{(k)} (P\sigma)^* \alpha = \sum_{j=0}^{n} \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_j))(V_i^* \sigma) \mathrm{S}(\mathrm{C}(s\omega_{j+1} \otimes \cdots \otimes s\omega_n))(U_{k-j}^* \sigma).$$

Adding these expressions over the possible values of i yields

$$\sum_{i=1}^{k-1} (-1)^i \int_{I^{k-2}} (\partial_i^+)^* \Theta^*_{(k)} (P\sigma)^* \alpha = \sum_{i=1}^{n-1} -\mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_i)) \,\hat{\cup}\, \mathrm{S}(\mathrm{C}(s\omega_{i+1} \otimes \cdots \otimes s\omega_n))(\sigma) \\ - \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n))(\partial_0^* \sigma) - (-1)^k \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n))(\partial_k^* \sigma).$$

Subtracting both summations yields the equation. □

3.3 The A_∞ de Rham Theorem

In this section we define the A_∞-morphism going from the de Rham algebra of a manifold to its smooth singular cochain algebra. Recall from Definition 1.5 that an A_∞-morphism between dg-algebras A and B is a sequence of maps $\psi_k : A^{\otimes k} \to B$ satisfying a set of relations. In this situation the sequence will be obtained as the composition of the Chen iterated integral map C and Igusa's map S for $k > 1$. For $k = 1$ the definition is slightly different. First let us compute $\mathrm{S} \circ \mathrm{C}(s\alpha)(\sigma)$ for $\alpha \in \Omega(M)$, $|\alpha| \neq 0$ and $\sigma : \Delta_k \to M$:

$$\mathrm{S}(\mathrm{C}(s\alpha))(\sigma) = \int_{I^{k-1}} \Theta^*_{(k)} P\sigma^* \mathrm{C}(s\alpha) = \int_{I^{k-1}} \Theta^*_{(k)} \mathrm{C}(s\sigma^* \alpha) \\ = \int_{I^k} \Theta^*_k (\sigma^* \alpha) = (-1)^k \int_{\Delta_k} \sigma^* \alpha.$$

Notice that the previous equation ceases to be true when $|\alpha| = 0$ since $C(s\alpha) = 0$ while the integral $\int_{\Delta_k} \sigma^* \alpha$ may differ from zero in case $k = 0$. Based on this observation we define the A_∞-morphism.

Definition 3.5. Let M be a smooth manifold. We define a family of maps $\psi = \{\psi_n\}$ with

$$\psi_n : (s\Omega(M))^{\otimes n} \to sC(M)$$

as follows:

- For $n = 1$, take an arbitrary simplex $\sigma : \Delta_k \to M$ and define

$$\psi_1(s\alpha)(\sigma) := (-1)^k \int_{\Delta_k} \sigma^*\alpha.$$

  Notice that $\psi_1(s\alpha) = \mathrm{S} \circ \mathrm{C}(s\alpha)$ whenever $|\alpha| > 0$.

- If $n > 1$ then we define

$$\psi_n(s\alpha_1 \otimes \cdots \otimes s\alpha_n) = \mathrm{S} \circ \mathrm{C}(s\alpha_1 \otimes \cdots \otimes s\alpha_n).$$

Remark 3.2. Notice that if α is a form of degree zero, then $\psi_1(s\alpha) = s\alpha$. Furthermore, from Remark 3.1 we get that for $n > 1$, $\psi_n(s\alpha_1 \otimes \cdots \otimes s\alpha_n)$ vanishes whenever one of the factors α_i is a form of degree zero. In other words, the sequence ψ satisfies the hypothesis of Lemma 1.2.

Theorem 3.1 (Gugenheim)**.** The sequence $\psi := \{\psi_n\}$ is an A_∞-quasi-isomorphism from $(\Omega(M), -d, \wedge)$ to $(C(M), \delta, \cup)$. The construction is natural with respect to pullbacks along smooth maps.

Proof. From the formulas given in Propositions 3.1 and 3.2, we have the following

$$\begin{aligned} \mathrm{S} \circ \mathrm{C}(D(s\omega_1 \otimes \cdots \otimes s\omega_n)) =& \hat{\delta}(\mathrm{S} \circ \mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n)) \\ &+ \sum_{i=1}^{n-1} (\mathrm{S} \circ \mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_i)) \hat{\cup} (\mathrm{S} \circ \mathrm{C}(s\omega_{i+1} \otimes \cdots \otimes s\omega_n)) \\ &- \mathrm{S}(\mathrm{ev}_1^*(\omega_1) \wedge \mathrm{C}(s\omega_2 \otimes \cdots \otimes s\omega_n)) \\ &+ (-1)^{\sum_{j=1}^{n-1} |\omega_j|} \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_{n-1}) \wedge \mathrm{ev}_0^*(\omega_n)). \end{aligned} \tag{3.4}$$

For $n = 1$ and $n = 2$, the previous equation turns into

$$\psi_1(\hat{d}s\omega_1) = \hat{\delta}\psi_1(s\omega_1) \quad \text{and}$$

$$\psi_1(s\omega_1 \hat{\wedge} s\omega_2) - \psi_1(s\omega_1) \hat{\cup} \psi_1(s\omega_2) = \hat{\delta}\psi_2(s\omega_1 \otimes \omega_2) - \psi_2(\hat{d}(s\omega_1) \otimes s\omega_2 + (-1)^{|s\omega_1|} s\omega_1 \otimes \hat{d}(s\omega_2))$$

respectively, which are the first two relations required of an A_∞-morphism.

For $n \geq 3$ we recall that $\psi_n(s\omega_1 \otimes \cdots \otimes s\omega_n) = \mathrm{S} \circ \mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n)$ when $n \geq 2$ or $n = 1$

and $|\omega_1| > 0$. Thus equation (3.4) turns into

$$\begin{aligned}
\mathrm{S} \circ \mathrm{C}(D(s\omega_1 \otimes \cdots \otimes s\omega_n)) =& \hat{\delta}(\psi_n(s\omega_1 \otimes \cdots \otimes s\omega_n)) \\
&+ \sum_{i=2}^{n-2} (\psi_i(s\omega_1 \otimes \cdots \otimes s\omega_i)) \hat{\cup} (\psi_{n-i}(s\omega_{i+1} \otimes \cdots \otimes s\omega_n)) \\
&- \mathrm{S}(\mathrm{ev}_1^*(\omega_1) \wedge \mathrm{C}(s\omega_2 \otimes \cdots \otimes s\omega_n)) \\
&+ (\mathrm{S} \circ \mathrm{C}(s\omega_1)) \hat{\cup} (\psi_{n-1}(s\omega_2 \otimes \cdots \otimes s\omega_n)) \\
&+ (-1)^{\sum_{j=1}^{n-1} |\omega_j|} \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_{n-1}) \wedge \mathrm{ev}_0^*(\omega_n)) \\
&+ (\psi_{n-1}(s\omega_1 \otimes \cdots \otimes s\omega_{n-1})) \hat{\cup} (\mathrm{S} \circ \mathrm{C}(s\omega_n)).
\end{aligned}$$

In the previous expression, the third line vanishes whenever $|\omega_1| > 0$ and the fourth line vanishes when $|\omega_1| = 0$. Similarly, the fifth line vanishes whenever $|\omega_n| > 0$ and the sixth line vanishes when $|\omega_n| = 0$. Computing directly the third and fifth lines (when they are not null) yields

$$\begin{aligned}
-\mathrm{S}(\mathrm{ev}_1^*(\omega_1) \wedge \mathrm{C}(s\omega_2 \otimes \cdots \otimes s\omega_n)) &= (\psi_1(s\omega_1)) \hat{\cup} (\psi_{n-1}(s\omega_2 \otimes \cdots \otimes s\omega_n)) \\
(-1)^{\sum_{j=1}^{n-1} |\omega_j|} \mathrm{S}(\mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_{n-1}) \wedge \mathrm{ev}_0^*(\omega_n)) &= (\psi_{n-1}(s\omega_1 \otimes \cdots \otimes s\omega_{n-1})) \hat{\cup} (\psi_1(s\omega_n)).
\end{aligned}$$

In any case, equation 3.4 turns into

$$\begin{aligned}
\mathrm{S} \circ \mathrm{C}(D(s\omega_1 \otimes \cdots \otimes s\omega_n)) =& \hat{\delta}(\mathrm{S} \circ \mathrm{C}(s\omega_1 \otimes \cdots \otimes s\omega_n)) \\
&+ \sum_{i=1}^{n-1} (\psi_i(s\omega_1 \otimes \cdots \otimes s\omega_i)) \hat{\cup} (\psi_{n-i}(s\omega_{i+1} \otimes \cdots \otimes s\omega_n)),
\end{aligned}$$

proving the A_∞ relations for $n \geq 3$.

The naturality of ψ is a direct consequence of the naturality of both S and C. Furthermore, since ψ_1 is exactly the map used in the proof of the classical de Rham theorem, it is immediate that ψ is a quasi-isomorphism. □

Remark 3.3. To get an A_∞-morphism from $(\Omega(M), d, \wedge) \to (C(M), \delta, \cup)$ we simply compose the sequence ψ with the isomorphism $(\Omega(M), d, \wedge) \to (\Omega(M), -d, \wedge)$ given by $\alpha \mapsto (-1)^{|\alpha|}\alpha$. From this point forward, ψ will denote this composition and not merely the morphism defined in 3.5.

3.4 Higher Parallel Transport

Here we show how Gugenheim's A_∞-morphism may be used to construct representations up to homotopy from ∞-local systems. Here is a quick summary of the procedure:

- We begin with local system over M whose vector bundle is trivial, i.e., $E = M \times V$ for some graded vector space V. The flat graded connection defined over E is determined by a Maurer-Cartan element $\alpha \in \Omega(M) \otimes \mathrm{End}(V)$.

- The A_∞-morphism $\psi : \Omega(M) \to C(M)$ may be extended to

$$\psi^V : \Omega(M) \otimes \mathrm{End}(V) \to C(M) \otimes \mathrm{End}(V).$$

This extension is still an A_∞-morphism.

- Mapping α under ψ^V as in Definition 1.8 will yield a Maurer-Cartan element $\psi^V(\alpha) \in C(M) \otimes \mathrm{End}(V)$ according to Lemma 1.1.

- Evaluating $\psi^V(\alpha)$ on a simplex $\sigma : \Delta_k \to M$ yields an endomorphism of V which can be interpreted as a map going from the fibre over the last vertex of σ to the fibre over its zeroth vertex. The Maurer-Cartan equation for $\psi^V(\alpha)$ unravels into the relations required to define a representation up to homotopy.

- Finally, according to Lemma 1.2, ψ^V preserves gauge equivalences, which allows for the definition of a global representation up to homotopy.

The following results fill the details of the procedure:

Lemma 3.6. Let $\varphi : (A, d_A, m_A) \to (B, d_B, m_B)$ be an A_∞-morphism between dg-algebras and let V be a graded vector space. Then both $A \otimes \mathrm{End}(V)$ and $B \otimes \mathrm{End}(V)$ have dg-algebra structures and φ may be extended to an A_∞-morphism

$$\varphi^V : A \otimes \mathrm{End}(V) \to B \otimes \mathrm{End}(V).$$

Proof. For the dg-algebra structure we consider $\mathrm{End}(V)$ as a dg-algebra with differential equal to zero. Then the tensor product of dg-algebras is again a dg-algebra as stated right after Definition 1.3. Now we extend φ by the formula

$$\varphi_n^V((a_1 \otimes T_1) \otimes \cdots \otimes (a_n \otimes T_n)) = (-1)^s \varphi_n(a_1 \otimes \cdots \otimes a_n) \otimes T_1 \cdots T_n,$$

where $s = |T_1||a_2| + (|T_1| + |T_2|)|a_3| + \cdots + (|T_1| + \cdots + |T_{n-1}|)|a_n|$. The A_∞ relations in this setting take the form

$$(-1)^s \left(\sum_{i=0}^{n-1} (-1)^{n-1} \varphi_n(1^{\otimes i} \otimes d_A \otimes 1^{\otimes(n-1-i)}) + \sum_{i=0}^{n-2} (-1)^i \varphi_{n-1}(1^{\otimes i} \otimes m_A \otimes 1^{\otimes(n-2-i)}) \right) \otimes (T_1 \cdots T_n)$$
$$= (-1)^s \left(d_B(\varphi_n) + \sum_{i=1}^{n-1} (-1)^{i-1} m_B(\varphi_i \otimes \varphi_{n-i}) \right) \otimes (T_1 \cdots T_n).$$

□

Remark 3.4. According to Remark 3.2, ψ satisfies the hypothesis of Lemma 1.2, hence the same is true for ψ^V.

Lemma 3.7. An ∞-local system with trivial vector bundle $(E, d_E) = (M \times V, d - \alpha)$ induces a representation up to homotopy of the ∞-groupoid $\pi_\infty(M)$.

Proof. We wish to define a sequence of cochains $\{F_k\}$ such that $F_k(\sigma) \in \mathrm{Hom}^{1-k}(E_{v_k(\sigma)}, E_{v_0(\sigma)})$ for a k-simplex $\sigma : \Delta_k \to M$. Let $\beta := \psi^V(\alpha) \in C^\bullet(M) \otimes \mathrm{End}(V)$. We know that β is a Maurer-Cartan form, therefore it is homogeneous of degree $|\beta| = 1$ and satisfies the Maurer-Cartan equation. Since

$$(C^\bullet(M) \otimes \mathrm{End}(V))^1 = \bigoplus_{i \geq 0} C^i(M) \otimes \mathrm{End}^{1-i}(V),$$

the form β may be written as a sum $\beta = \sum_{i \geq 0} \beta_i$ where $\beta_i \in C^i(M) \otimes \mathrm{End}^{1-i}(V)$. Let $\mathbf{1}$ denote the cochain that assigns the identity to each 1-simplex and consider the element $\beta - \mathbf{1} \in C^\bullet(M) \otimes \mathrm{End}(V)$, this is

$$\beta - \mathbf{1} = \beta_0 + (\beta_1 - \mathbf{1}) + \beta_2 + \beta_3 \cdots$$

Since both β and $\mathbf{1}$ are Maurer-Cartan forms, we have the following equation

$$\delta(\beta - \mathbf{1}) - (\beta - \mathbf{1}) \cup (\beta - \mathbf{1}) - \mathbf{1} \cup \beta - \beta \cup \mathbf{1} = \delta\beta - \beta \cup \beta + \delta\mathbf{1} - \mathbf{1} \cup \mathbf{1} = 0.$$

If we consider the k-th homogeneous component of the previous equation we get

$$\sum_{j=0}^{k} \beta_{k-1}(p_j) - \sum_{j=0}^{k} (-1)^j (\beta_j \cup \beta_{k-j}) - \mathbf{1} \cup \beta_{k-1} - \beta_{k-1} \cup \mathbf{1} = 0.$$

The first and last term on the first summation cancel out with $\mathbf{1} \cup \beta_{k-1} + \beta_{k-1} \cup \mathbf{1}$, leaving us with

$$\sum_{j=1}^{k-1} \beta_{k-1}(p_j) - \sum_{j=0}^{k} (-1)^j (\beta_j \cup \beta_{k-j}) = 0,$$

which is precisely the relation required of a representation up to homotopy.

Finally we define the cochains of the representation. For a k-simplex σ we define $F_k(\sigma) : \{v_k(\sigma)\} \times V \to \{v_0(\sigma)\} \times V$ as

$$F_k(\sigma)(v_k(\sigma), u) = (v_0(\sigma), (\beta - \mathbf{1})_k(\sigma)(u)), \qquad u \in V.$$

To simplify the notation we will write $F_k(\sigma) = (\beta - \mathbf{1})_k(\sigma)$, where the morphism is understood to go from the fiber over $v_k(\sigma)$ to the fiber over $v_0(\sigma)$. □

Theorem 3.2. An ∞-local system over M induces a representation up to homotopy of the ∞-groupoid $\pi_\infty(M)$.

Proof. Let (E, d_E) be an A_∞-local system over M whose fibers are isomorphic to the graded vector space V. We will define the sequence of cochains $\{F_k\}$ that determine the representation. Let $\sigma : \Delta_k \to M$ be a k-simplex. The vector bundle $\sigma^* E$ is trivial, therefore there is an isomorphism $\varphi : \sigma^*(E, d_E) \to (\Delta_k \times V, d - \alpha)$ and we have the following diagram:

$$\begin{array}{ccccc} \Delta_k \times V & \xleftarrow{\varphi} & \sigma^* E & \xrightarrow{\hat{\sigma}} & E \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_k & \xleftarrow[\mathrm{id}]{} & \Delta_k & \xrightarrow[\sigma]{} & M. \end{array}$$

By Lemma 3.7, $(\Delta_k \times V, d - \alpha)$ induces a representation up to homotopy of $\pi_\infty(\Delta_k)$ which we denote $\{F^\alpha\}$. Evaluation of F_k^α at the fundamental cochain $\mathrm{id} : \Delta_k \to \Delta_k$ yields a map $F_k^\alpha(\mathrm{id}) : \{v_k\} \times V \to \{v_0\} \times V$. Let us denote by φ_{v_0}, $\hat{\sigma}_{v_0}$, φ_{v_k} and $\hat{\sigma}_{v_k}$ the restrictions of φ and $\hat{\sigma}$ to the fibers of $\sigma^* E$ over the zeroth and k-th vertices of Δ_k respectively. Then we define

$$F_k(\sigma) := \hat{\sigma}_{v_0} \varphi_{v_0}^{-1} F_k^\alpha(\mathrm{id}) \varphi_{v_k} \hat{\sigma}_{v_k}^{-1} : E_{\sigma(v_k)} \to E_{\sigma(v_0)}.$$

Let us check that the definition does not depend on the choice of trivialisation for $\sigma^* E$: suppose $\varphi' : \sigma^*(E, d_E) \to (\Delta_k \times V, d - \alpha')$ is another trivialisation, then α and α' are gauge equivalent via the element $\varphi' \varphi^{-1} \in \Omega^0(\Delta_k, \mathrm{End}(V))$. Let $\beta := \psi^V(\alpha)$ and $\beta' := \psi^V(\alpha')$, then by Lemma 1.2 we have the gauge equivalence of β and β' via $\varphi' \varphi^{-1}$, this is

$$\beta = \varphi \varphi'^{-1} \cup \beta' \cup \varphi' \varphi^{-1} - \varphi \varphi'^{-1} \cup \delta(\varphi' \varphi^{-1}).$$

For $k \neq 1$, the gauge equivalence relation is merely a conjugation by $\varphi' \varphi^{-1}$, therefore we have

$$\varphi_{v_0}^{-1} F_k^\alpha(\mathrm{id}) \varphi_{v_k} = \varphi_{v_0}^{-1} \left(\varphi_{v_0} \varphi'^{-1}_{v_0} F_k^{\alpha'}(\mathrm{id}) \varphi'_{v_k} \varphi_{v_k}^{-1} \right) \varphi_{v_k} = \varphi'^{-1}_{v_0} F_k^{\alpha'}(\mathrm{id}) \varphi'_{v_k}.$$

For $k = 1$ we use the gauge equivalence equation and compute $\varphi \varphi'^{-1} \cup \delta(\varphi' \varphi^{-1})(\mathrm{id})$ to get

$$\begin{aligned} \varphi_{v_0}^{-1} F_1^\alpha(\mathrm{id}) \varphi_{v_1} &= \varphi_{v_0}^{-1} (\beta_1)(\mathrm{id}) \varphi_{v_1} - \varphi_{v_0}^{-1} \mathbf{1}(\mathrm{id}) \varphi_{v_1} \\ &= \varphi_{v_0}^{-1} \varphi_{v_0} \varphi'^{-1}_{v_0} (\beta'_1)(\mathrm{id}) \varphi'_{v_1} \varphi_{v_1}^{-1} \varphi_{v_1} - \varphi_{v_0}^{-1} \varphi_{v_0} \varphi'^{-1}_{v_0} \left(\varphi'_{v_1} \varphi_{v_1}^{-1} - \varphi'_{v_1} \varphi_{v_1}^{-1} \right) \varphi_{v_1} - \varphi_{v_0}^{-1} \varphi_{v_1} \\ &= \varphi'^{-1}_{v_0} (\beta'_1)(\mathrm{id}) \varphi'_{v_1} - \varphi'^{-1}_{v_0} \varphi'_{v_1} + \varphi_{v_0}^{-1} \varphi_{v_1} - \varphi_{v_0}^{-1} \varphi_{v_1} \\ &= \varphi'^{-1}_{v_0} (\beta'_1 - \mathbf{1})(\mathrm{id}) \varphi'_{v_1} \\ &= \varphi'^{-1}_{v_0} F_1^{\alpha'}(\mathrm{id}) \varphi'_{v_1}. \end{aligned}$$

The fact that the family of cochains $\{F_k\}$satisfies the relations of a representation up to homotopy stems from the fact that each $\{F_k^\alpha\}$ does. $\square$

We have just defined a map $\int : \mathsf{Loc}_\infty(M) \to \mathcal{R}\mathrm{ep}^\infty(M)$. This map can be extended to an A_∞-functor between the dg-categories that is actually an equivalence of categories. For details on this construction we refer the reader to [2].

Chapter 2

An A_∞ Version of the Poincaré Lemma

In this chapter we set out to prove some homotopical properties of the category $\mathsf{Loc}_\infty(M)$. Given the dg-structure of the category and bearing in mind the A_∞ de Rham theorem, any sort of property of $\mathsf{Loc}_\infty(M)$ will be proven with the proper considerations of the A_∞-structure. Once again iterated integrals prove to be instrumental for our purposes, thus we devote the first section to the development of the version better suited to prove our results. Once we have established the basic properties of iterated integrals, we move towards the proof of the main theorem of the chapter, Theorem 2.3. The theorem states that homotopic maps $f, g : M \to N$ induce functors $f^*, g^* : \mathsf{Loc}_\infty(N) \to \mathsf{Loc}_\infty(M)$ that are A_∞-naturally isomorphic. In the process we show some examples highlighting the importance of the A_∞-structure. The third section contains the corollaries of our main result, including the A_∞ version of Poincaré's lemma. The final section of the chapter contains a long computation necessary to prove the relations required of the A_∞-natural transformation.

1 Iterated Integrals: A Vector Valued Version

For the present chapter we will adopt a slightly different notation for simplices. We will denote by $\Delta_k(t)$ the k-simplex of width t for $t \in [0,1]$, this is

$$\Delta_k(t) = \{(s_1, \cdots, s_k) \in \mathbb{R} \quad \big| \quad t \geq s_1 \geq \cdots \geq s_k \geq 0\}.$$

The components of the boundary of $\Delta_k(t)$ will be denoted $\partial_i \Delta_k(t)$ where

$$\begin{aligned}
\partial_0 \Delta_k(t) &= \{(s_1, \cdots, s_k) \in \Delta_k(t) \quad \big| \quad s_1 = t\}, \\
\partial_i \Delta_k(t) &= \{(s_1, \cdots, s_k) \in \Delta_k(t) \quad \big| \quad s_i = s_{i+1}\}, \quad i = 1, \cdots, k-1, \\
\partial_k \Delta_k(t) &= \{(s_1, \cdots, s_k) \in \Delta_k(t) \quad \big| \quad s_k = 0.\}
\end{aligned}$$

For any smooth manifold M we denote by $\pi_{k,i,t} : \Delta_k(t) \times M \to [0,t] \times M$ the projection onto the i-th component of the simplex. We will omit both k and t from the notation when the dimension and width of the simplex are clear from the context.

Let V be a vector space and consider the trivial vector bundles $\Delta_k \times M \times V$ and $M \times V$. The push forward $\int_{\Delta_k} : \Omega(\Delta_k \times M) \to \Omega(M)$ defined in (3.2) may be extended to forms with values in V

$$\int_{\Delta_k} \otimes \mathrm{Id} : \Omega(\Delta_k \times M) \otimes V \to \Omega(M) \otimes V.$$

We will abuse the notation and just write $\int_{\Delta_k}$ for the map $\int_{\Delta_k} \otimes \mathrm{Id}$. Now consider the case where V is a graded vector space and we take forms with values in $\mathrm{End}(V)$. The vector space $\Omega(M, \mathrm{End}(V))$ may be provided with a dg-algebra structure with the product

$$(\alpha \otimes T) \wedge (\beta \otimes S) = (-1)^{|\beta||T|}(\alpha \wedge \beta) \otimes (T \circ S) \quad \text{for} \quad \alpha, \beta \in \Omega(M) \text{ and } T, S \in \mathrm{End}(V).$$

In a similar way, the action $\Omega(M) \circlearrowleft \Omega(\Delta_k \times M)$ may be extended to an action $\Omega(M, \mathrm{End}(V)) \circlearrowleft \Omega(\Delta_k \times M, \mathrm{End}(V))$. The first part of Lemma 3.1 still holds in this context:

Lemma 1.1. The map $\int_{\Delta_k} : \Omega(\Delta_k \times M, \mathrm{End}(V)) \to \Omega(M, \mathrm{End}(V))$ is a morphism of $\Omega(M, \mathrm{End}(V))$-modules of degree $-k$.

Proof. Take $\alpha \in \Omega(M)$, $\beta \in \Omega(\Delta_k \times M)$ and $T, S \in \mathrm{End}(V)$. Then

$$\begin{aligned}
\int_{\Delta_k} (\alpha \otimes T) \wedge (\beta \otimes S) =& (-1)^{|\beta||T|} \left(\int_{\Delta_k} \alpha \wedge \beta \right) \otimes (T \circ S) \\
=& (-1)^{|\beta||T| - |\alpha| k} \left(\alpha \wedge \int_{\Delta_k} \beta \right) \otimes (T \circ S) \\
& (-1)^{|\beta||T| - |\alpha| k + |T|(|\beta| - k)} (\alpha \otimes T) \wedge \left(\left(\int_{\Delta_k} \beta \right) \otimes S \right) \\
& (-1)^{-(|\alpha| + |T|)k} (\alpha \otimes T) \wedge \left(\left(\int_{\Delta_k} \beta \right) \otimes S \right).
\end{aligned}$$

Notice that the sign obtained is given by the Kozsul convention when considering the total degree of $\alpha \otimes T$, not just the form degree. □

For each $t \in [0,1]$ and a sequence of possibly non-homogeneous forms $\{\theta_i\}_{i \geq 1} \in \Omega([0,1] \times M, \mathrm{End}(V))$ consider the series

$$\varphi(t) = \mathrm{id} + \sum_{k \geq 1} \int_0^t \pi_1^*(\theta_1) \int_0^{s_1} \pi_2^*(\theta_2) \cdots \int_0^{s_{k-1}} \pi_k^*(\theta_k). \tag{1.1}$$

If the forms are homogeneous, then the previous iterated integrals can be written as integrals over the simplices as follows:

$$\varphi(t) = \mathrm{id} + \sum_{k\geq 1}(-1)^{(k-1)|\theta_1|+(k-2)|\theta_2|+\cdots+|\theta_{k-1}|}\int_{\Delta_k(t)}\pi_1^*(\theta_1)\cdots\pi_k^*(\theta_k). \tag{1.2}$$

Notice that the sign $(-1)^{(k-1)|\theta_1|+(k-2)|\theta_2|+\cdots+|\theta_{k-1}|}$ is precisely the sign obtained when commuting forms with integrals. The sum above is not finite and is not necessarily convergent, however it will converge whenever the sequence of forms is uniformly bounded. Suppose each form can be written locally as $\theta_i = f_i(t,x)dtdX_i$. We say that the sequence $\{\theta_i\}_{i\geq 1}$ is uniformly bounded if there is a function $B : M \to \mathbb{R}$ such that $|f_i(t,x)| \leq B(x)$ for every $i \geq 1$ and every $t \in [0,1]$. Notice in particular that constant sequences and sequences with finitely many different forms are uniformly bounded.

Lemma 1.2. Suppose the sequence $\{\theta_i\}_{i\geq 1} \in \Omega([0,1]\times M, \mathrm{End}(V))$ is uniformly bounded. Then the series (1.2) is convergent.

Proof. The volume of the k-simplex of width t is $t^k/k!$. If $B(x)$ is a bound for the sequence then we have

$$\left|\int_{\Delta_k(t)} f_1(s_1,x)\cdots f_k(s_k,x)ds_k\cdots ds_1\right| \leq \frac{t^k}{k!}B^k(x).$$

The convergence of the series follows. □

We are particularly interested in the case of constant sequences.

Definition 1.1. Suppose $\theta_i = \alpha$ for every i. Then (1.2) can be written as

$$\varphi(t) = \mathrm{id} + \sum_{k\geq 1}(-1)^{\sigma(k)|\alpha|}\int_{\Delta_k(t)}\pi_1^*(\alpha)\cdots\pi_k^*(\alpha)$$

where $\sigma(k) = 0+1+\cdots+(k-1)$. Since the sequence is uniformly bounded, the series is convergent and defines a form in $\Omega(M,\mathrm{End}(V))$ which we will denote $\varphi^\alpha(t)$. In fact we have a smooth map $\varphi^\alpha : [0,1] \to \Omega(M,\mathrm{End}(V))$.

Next we prove some facts about the forms φ^α. We denote by $\iota_t : M \to [0,1]\times M$ the inclusion at level t, this is, $\iota_t(x) = (t,x)$. Also, let $i_{\frac{\partial}{\partial t}}$ denote the contraction with the vector field $\frac{\partial}{\partial t}$. The first result we prove is that $\varphi^\alpha(t)$ defines a solution for the parallel transport differential equation.

Proposition 1.1. $\varphi^\alpha(t)$ satisfies the following differential equation.

$$\begin{cases} \frac{d\varphi^\alpha}{dt} = \iota_t^* i_{\frac{\partial}{\partial t}}\alpha \wedge \varphi^\alpha(t), \\ \varphi^\alpha(0) = \mathrm{id}. \end{cases} \tag{1.3}$$

Proof. Clearly $\varphi^\alpha(0) = \mathrm{id}$. Let us compute the derivative.

$$\begin{aligned}
\frac{d\varphi^\alpha}{dt} &= \frac{d}{dt}\int_0^t \pi_1^*(\alpha) + \sum_{k\geq 2}\frac{d}{dt}\int_0^t \pi_1^*(\alpha)\int_0^{s_1}\pi_2^*(\alpha)\cdots\int_0^{s_{k-1}}\pi_k^*(\alpha) \\
&= \iota_t^* i_{\frac{\partial}{\partial t}}\alpha + \iota_t^* i_{\frac{\partial}{\partial t}}\alpha \sum_{k\geq 2}\int_0^t \pi_2^*(\alpha)\int_0^{s_2}\pi_3^*(\alpha)\cdots\int_0^{s_{k-1}}\pi_k^*(\alpha) \\
&= \iota_t^* i_{\frac{\partial}{\partial t}}\alpha\left(\mathrm{id} + \sum_{k\geq 2}\int_0^t \pi_2^*(\alpha)\int_0^{s_2}\pi_3^*(\alpha)\cdots\int_0^{s_{k-1}}\pi_k^*(\alpha)\right) \\
&= \iota_t^* i_{\frac{\partial}{\partial t}}\alpha\varphi^\alpha(t).
\end{aligned}$$

□

Next we prove the gauge invariance property of iterated integrals.

Proposition 1.2. Let $\alpha \in \Omega([0,1]\times M, \mathrm{End}(V))$ and $\beta \in \Omega([0,1]\times M, \mathrm{End}(W))$. If $\psi \in \Omega^0([0,1]\times M, \mathrm{Hom}(V,W))$ is invertible such that $\alpha = \psi^{-1}\beta\psi - \psi^{-1}d\psi$, then the following equation holds:

$$\varphi^\alpha(t) = (\iota_t^*\psi)^{-1}\varphi^\beta(t)(\iota_0^*\psi). \tag{1.4}$$

Proof. The strategy of the proof is to show that the right side of equation (1.4) satisfies the differential equation (1.3). By Proposition 1.1, φ^α also satisfies the equation making both sides of (1.4) equal.

Let us compute the derivative of $(\iota_t^*\psi)^{-1}\varphi^\beta(t)(\iota_0^*\psi)$ with respect to t:

$$\begin{aligned}
\frac{d}{dt}(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi) &= \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + \iota_t^*\psi^{-1}\frac{d\varphi^\beta}{dt}\iota_0^*\psi \\
&= \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + \iota_t^*\psi^{-1}\iota_t^* i_{\frac{\partial}{\partial t}}\beta\varphi^\beta(t)\iota_0^*\psi \\
&= \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + (\iota_t^*\psi^{-1}\iota_t^* i_{\frac{\partial}{\partial t}}\beta\iota_t^*\psi)(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi).
\end{aligned} \tag{1.5}$$

Since $\alpha = \psi^{-1}\beta\psi - \psi^{-1}d\psi$, we also have

$$\iota_t^* i_{\frac{\partial}{\partial t}}\alpha = \iota_t^*\psi^{-1}\iota_t^* i_{\frac{\partial}{\partial t}}\beta\iota_t^*\psi - \iota_t^*\psi^{-1}\frac{d}{dt}(\iota_t^*\psi).$$

Replacing in (1.5) we get

$$\frac{d}{dt}(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi) = \frac{d}{dt}(\iota_t^*\psi^{-1})\varphi^\beta(t)\iota_0^*\psi + \iota_t^* i_{\frac{\partial}{\partial t}}\alpha\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi + \iota_t^*\psi^{-1}\frac{d}{dt}(\iota_t^*\psi)\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi.$$

By the product rule we know that $d(\iota_t^*\psi^{-1})/dt = -\iota_t^*\psi^{-1}\big(d(\iota_t^*\psi)/dt\big)\iota_t^*\psi^{-1}$, hence the first and last term on the right cancel out leaving us with

$$\frac{d}{dt}(\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi) = i_{\frac{\partial}{\partial t}}\alpha(t)\iota_t^*\psi^{-1}\varphi^\beta(t)\iota_0^*\psi,$$

completing the proof. □

The following computation will be relevant in the next section.

Proposition 1.3. Suppose $\theta_1, \cdots, \theta_k \in \Omega([0,1]\times M, \mathrm{End}(V))$ are homogeneous forms, then

$$\begin{aligned} d\int_{\Delta_k(t)} &\pi_1^*\theta_1\wedge\cdots\wedge\pi_k^*\theta_k \\ &= \sum_{i=1}^{k}(-1)^{k+\sum_{j=1}^{i-1}|\theta_j|}\int_{\Delta_k(t)}\pi_1^*\theta_1\wedge\cdots\wedge\pi_{i-1}^*\theta_{i-1}\wedge\pi_i^*d\theta_i\wedge\pi_{i+1}^*\theta_{i+1}\wedge\cdots\wedge\pi_k^*\theta_k \\ &\quad + \sum_{i=1}^{k-1}(-1)^i\int_{\Delta_{k-1}(t)}\pi_1^*\theta_1\wedge\cdots\wedge\pi_{i-1}^*\theta_{i-1}\wedge\pi_i^*(\theta_i\wedge\theta_{i+1})\wedge\pi_{i+1}^*\theta_{i+2}\wedge\cdots\wedge\pi_{k-1}^*\theta_k \\ &\quad + (-1)^k\left(\int_{\Delta_{k-1}(t)}\pi_1^*\theta_1\wedge\cdots\wedge\pi_{k-1}^*\theta_{k-1}\right)\wedge\iota_0^*\theta_k \\ &\quad + (-1)^{(k-1)|\theta_1|}\iota_t^*\theta_1\wedge\left(\int_{\Delta_{k-1}(t)}\pi_1^*\theta_2\wedge\cdots\wedge\pi_{k-1}^*\theta_k\right). \end{aligned}$$

Proof. For $k=1$ the equation is

$$d\int_0^t\theta_1 = \iota_t^*\theta_1 - \iota_0^*\theta_1 - \int_0^t d\theta_1,$$

which is straightforward to check. For the general case we write

$$\int_{\Delta_k(t)}\pi_1^*\theta_1\wedge\cdots\wedge\pi_k^*\theta_k = (-1)^{(k-1)|\theta_1|}\int_0^t\iota_s^*\,\mathrm{i}_{\frac{\partial}{\partial s}}\,\theta_1\wedge\left(\int_{\Delta_{k-1}(s)}\pi_1^*\theta_2\wedge\cdots\wedge\pi_{k-1}^*\theta_k\right)ds,$$

and proceed by induction on k. Using case $k=1$ we get

$$\begin{aligned} d\int_{\Delta_k(t)}\pi_1^*\theta_1\wedge\cdots\wedge\pi_k^*\theta_k =& (-1)^{(k-1)|\theta_1|}d\left(\int_0^t\iota_s^*\,\mathrm{i}_{\frac{\partial}{\partial s}}\,\theta_1\wedge\left(\int_{\Delta_{k-1}(s)}\pi_1^*\theta_2\wedge\cdots\wedge\pi_{k-1}^*\theta_k\right)ds\right) \\ =& (-1)^{(k-1)|\theta_1|}\iota_t^*\left(\iota_s^*\,\mathrm{i}_{\frac{\partial}{\partial s}}\,\theta_1\wedge\left(\int_{\Delta_{k-1}(s)}\pi_1^*\theta_2\wedge\cdots\wedge\pi_{k-1}^*\theta_k\right)ds\right) \\ &- (-1)^{(k-1)|\theta_1|}\iota_0^*\left(\iota_s^*\,\mathrm{i}_{\frac{\partial}{\partial s}}\,\theta_1\wedge\left(\int_{\Delta_{k-1}(s)}\pi_1^*\theta_2\wedge\cdots\wedge\pi_{k-1}^*\theta_k\right)ds\right) \\ &- (-1)^{(k-1)|\theta_1|}\int_0^t d\left(\iota_s^*\,\mathrm{i}_{\frac{\partial}{\partial s}}\,\theta_1\wedge\left(\int_{\Delta_{k-1}(s)}\pi_1^*\theta_2\wedge\cdots\wedge\pi_{k-1}^*\theta_k\right)ds\right). \end{aligned}$$

The line corresponding to ι_0^* vanishes leaving us with

$$
\begin{aligned}
d\int_{\Delta_k(t)} \pi_1^*\theta_1 \wedge \cdots \wedge \pi_k^*\theta_k =& (-1)^{(k-1)|\theta_1|}\iota_t^*\theta_1 \wedge \left(\int_{\Delta_{k-1}(t)} \pi_1^*\theta_2 \wedge \cdots \wedge \pi_{k-1}^*\theta_k\right) \\
&- (-1)^{(k-1)|\theta_1|}\int_0^t d\left(\iota_s^* \, \mathrm{i}_{\frac{\partial}{\partial s}}\, \theta_1\right) \wedge \left(\int_{\Delta_{k-1}(s)} \pi_1^*\theta_2 \wedge \cdots \wedge \pi_{k-1}^*\theta_k\right) ds \\
&- (-1)^{k|\theta_1|-1}\int_0^t \iota_s^* \, \mathrm{i}_{\frac{\partial}{\partial s}}\, \theta_1 \wedge \left(d\int_{\Delta_{k-1}(s)} \pi_1^*\theta_2 \wedge \cdots \wedge \pi_{k-1}^*\theta_k\right) ds.
\end{aligned}
$$

The second term can be rewritten as

$$
\begin{aligned}
&-(-1)^{(k-1)|\theta_1|}\int_0^t d\left(\iota_s^* \, \mathrm{i}_{\frac{\partial}{\partial s}}\, \theta_1\right) \wedge \left(\int_{\Delta_{k-1}(s)} \pi_1^*\theta_2 \wedge \cdots \wedge \pi_{k-1}^*\theta_k\right) ds \\
&= (-1)^k \int_{\Delta_k(t)} \pi_1^* d\theta_1 \wedge \pi_2^*\theta_2 \wedge \cdots \wedge \pi_k^*\theta_k.
\end{aligned}
$$

Using the formula for the case $k-1$ in the last term we get

$$
\begin{aligned}
&(-1)^{k|\theta_1|}\int_0^t \iota_s^* \, \mathrm{i}_{\frac{\partial}{\partial s}}\, \theta_1 \wedge \left(d\int_{\Delta_{k-1}(s)} \pi_1^*\theta_2 \wedge \cdots \wedge \pi_{k-1}^*\theta_k\right) ds \\
=& \sum_{i=2}^{k} (-1)^{k+\sum_{j=1}^{i-1}|\theta_j|} \int_{\Delta_k(t)} \pi_1^*\theta_1 \wedge \cdots \wedge \pi_{i-1}^*\theta_{i-1} \wedge \pi_i^* d\theta_i \wedge \pi_{i+1}^*\theta_{i+1} \wedge \cdots \wedge \pi_k^*\theta_k \\
&+ \sum_{i=2}^{k-1} (-1)^i \int_{\Delta_{k-1}(t)} \pi_1^*\theta_1 \wedge \cdots \wedge \pi_{i-1}^*\theta_{i-1} \wedge \pi_i^*(\theta_i \wedge \theta_{i+1}) \wedge \pi_{i+1}^*\theta_{i+2} \wedge \cdots \wedge \pi_{k-1}^*\theta_k \\
&+ (-1)^k \left(\int_{\Delta_{k-1}(t)} \pi_1^*\theta_1 \wedge \cdots \wedge \pi_{k-1}^*\theta_{k-1}\right) \wedge \iota_0^*\theta_k \\
&- \int_{\Delta_{k-1}(s)} \pi_1^*(\theta_1 \wedge \theta_2) \wedge \pi_2^*\theta_3 \wedge \cdots \wedge \pi_{k-1}^*\theta_k.
\end{aligned}
$$

Combining the terms we arrive at the desired formula. □

As a direct consequence of Proposition 1.3, we can prove the following formula:

Corollary 1.1. If α is a homogeneous element of $\Omega([0,1]\times M, \mathrm{End}(V))$, we have

$$\begin{aligned} d\varphi^\alpha(t) = &\sum_{k=1}^{\infty}\sum_{i=1}^{k}(-1)^{k+(i-1+\sigma(k))|\alpha|}\int_{\Delta_k(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{i-1}^*\alpha\wedge\pi_i^*d\alpha\wedge\pi_{i+1}^*\alpha\wedge\cdots\wedge\pi_k^*\alpha \\ &+\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}(-1)^{i+\sigma(k)|\alpha|}\int_{\Delta_{k-1}(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{i-1}^*\alpha\wedge\pi_i^*(\alpha\wedge\alpha)\wedge\pi_{i+1}^*\alpha\wedge\cdots\wedge\pi_k^*\alpha \\ &+\sum_{k=1}^{\infty}(-1)^{k+\sigma(k)|\alpha|}\left(\int_{\Delta_{k-1}(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{k-1}^*\alpha\right)\wedge\iota_0^*\alpha \\ &+\sum_{k=1}^{\infty}(-1)^{(k-1+\sigma(k))|\alpha|}\iota_t^*\alpha\wedge\left(\int_{\Delta_{k-1}(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{k-1}^*\alpha\right). \end{aligned}$$

Proposition 1.4. If α is a Maurer-Cartan element, then

$$d\varphi^\alpha(t) = \iota_t^*\alpha\wedge\varphi^\alpha(t) - \varphi^\alpha(t)\wedge\iota_0^*\alpha.$$

Proof. First notice that $|\alpha| = 1$ since it is Maurer-Cartan, thus the signs on the formula from Corollary 1.1 simplify to

$$(-1)^{k+i-1+\sigma(k)},\quad (-1)^{i+\sigma(k)},\quad (-1)^{k+\sigma(k)},\quad \text{and}\quad (-1)^{k-1+\sigma(k)}$$

respectively. Next, notice that $\sigma(k) = \sigma(k-1)+k-1$, from which we obtain

$$(-1)^{k-1+\sigma(k)} = (-1)^{\sigma(k-1)} \quad \text{and} \quad (-1)^{k+\sigma(k)} = (-1)^{\sigma(k-1)+1}.$$

Therefore,

$$\begin{aligned} d\varphi^\alpha(t) = &\sum_{k=1}^{\infty}\sum_{i=1}^{k}(-1)^{i+\sigma(k-1)}\int_{\Delta_k(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{i-1}^*\alpha\wedge\pi_i^*d\alpha\wedge\pi_{i+1}^*\alpha\wedge\cdots\wedge\pi_k^*\alpha \\ &-\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}(-1)^{i+\sigma(k-1)}\int_{\Delta_{k-1}(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{i-1}^*\alpha\wedge\pi_i^*(\alpha\wedge\alpha)\wedge\pi_{i+1}^*\alpha\wedge\cdots\wedge\pi_k^*\alpha \\ &-\sum_{k=1}^{\infty}(-1)^{\sigma(k-1)}\left(\int_{\Delta_{k-1}(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{k-1}^*\alpha\right)\wedge\iota_0^*\alpha \\ &+\sum_{k=1}^{\infty}(-1)^{\sigma(k-1)}\iota_t^*\alpha\wedge\left(\int_{\Delta_{k-1}(t)}\pi_1^*\alpha\wedge\cdots\wedge\pi_{k-1}^*\alpha\right). \end{aligned}$$

Furthermore, α satisfies the Maurer-Cartan equation, so that $d\alpha-\alpha\wedge\alpha = 0$. Thus the first and the second term on the right hand side of this equality cancel out, leaving us with

$$\begin{aligned} d\varphi^\alpha(t) &= \iota_1^*\alpha \wedge \left(\sum_{k=1}^{\infty}(-1)^{\sigma(k-1)}\int_{\Delta_{k-1}(t)} \pi_1^*\alpha \wedge \cdots \wedge \pi_{k-1}^*\alpha\right) \\ &\quad - \left(\sum_{k=1}^{\infty}(-1)^{\sigma(k-1)}\int_{\Delta_{k-1}(t)} \pi_1^*\alpha \wedge \cdots \wedge \pi_{k-1}^*\alpha\right) \wedge \iota_0^*\alpha \\ &= \iota_t^*\alpha \wedge \varphi^\alpha(t) - \varphi^\alpha(t) \wedge \iota_0^*\alpha, \end{aligned}$$

as desired. □

2 The A_∞-Natural Transformation and Homotopy Invariance

Having developed our main tool of the chapter we are ready to prove some theorems regarding ∞-local systems.

Lemma 2.1. Let $E = [0,1] \times M \times V$ be a trivial graded vector bundle and $\alpha \in \Omega([0,1] \times M, \mathrm{End}(V))$ a Maurer-Cartan form so that $D = d - \alpha$ is a flat graded connection. The form $\varphi^\alpha(t) \in \Omega(M, \mathrm{End}(V))$ defines a zero degree, closed isomorphism of ∞-local systems $\varphi^\alpha(t) : \iota_0^*(E, D) \to \iota_t^*(E, D)$.

Proof. It is clear that $\varphi^\alpha(t)$ defines a morphism of ∞-local systems. Let us check that this morphism is closed, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \Omega(M,V) & \xrightarrow{d-\iota_0^*\alpha} & \Omega(M,V) \\ {\scriptstyle \varphi^\alpha(t)}\downarrow & & \downarrow{\scriptstyle \varphi^\alpha(t)} \\ \Omega(M,V) & \xrightarrow[d-\iota_t^*\alpha]{} & \Omega(M,V). \end{array}$$

We need to prove that $\varphi^\alpha(t)(d-\iota_0^*\alpha 0) = (d-\iota_t^*\alpha 0)\varphi^\alpha(t)$. The previous equation is equivalent to $d\varphi^\alpha(t) = \iota_t^*\alpha\varphi^\alpha(t) - \varphi^\alpha(t)\iota_0^*\alpha$, which was proven in Proposition 1.4.
To see that $\varphi^\alpha(t)$ is an isomorphism we write $\alpha = \sum_i \alpha_i$ where $\alpha_i \in \Omega^i([0,1] \times M, \mathrm{End}(V))$. Recall that α_1 is an ordinary connection over E and, according to the proof of Theorem 2.1, the iterated integral $\varphi^{\alpha_1}(t)$ defines a parallel transport which is invertible. Now write $\varphi^\alpha(t) = \varphi_0^\alpha(t) + \eta$ where $\varphi_0^\alpha(t)$ is the component of form degree zero and η is the sum of the rest of the components. Notice that $\varphi_0^\alpha(t) = \varphi^{\alpha_1}(t)$ and η is a nilpotent form, thus $\varphi^\alpha(t)$ is the sum of an invertible element and a nilpotent element, which means it is invertible. □

With the interpretation of the iterated integral as an isomorphism of ∞-local systems, the gauge invariance property given in Proposition 1.2 may be viewed as the commutativity of a diagram:

Lemma 2.2. Suppose $(E_1, D_1) = ([0,1]\times M\times V, d-\alpha)$ and $(E_2, D_2) = ([0,1]\times M\times W, d-\beta)$ are ∞-local systems and $\psi \in \Omega([0,1]\times M, \mathrm{Hom}(V,W))$ is a closed isomorphism. Then the following diagram is commutative

$$\begin{array}{ccc} \iota_0^*(E_1, D_1) & \xrightarrow{\iota_0^*\psi} & \iota_0^*(E_2, D_2) \\ {\scriptstyle \varphi^\alpha(t)}\downarrow & & \downarrow{\scriptstyle \varphi^\beta(t)} \\ \iota_t^*(E_1, D_1) & \xrightarrow{\iota_t^*\psi} & \iota_t^*(E_2, D_2). \end{array}$$

Now we aim to generalize Lemma 2.1 to non-trivial ∞-local systems. As expected, we rely on the gauge invariance property.

Theorem 2.1. Let (E, D) be an ∞-local system over $[0,1]\times M$. Then for each $t \in [0,1]$ there is a closed isomorphism $\varphi : \iota_0^*(E,D) \to \iota_t^*(E,D)$.

Proof. Take an open covering $\{U_i\}$ of M where each U_i is contractible so that $(E_i, D_i) := (E,D)|_{[0,1]\times U_i}$ is a trivial vector bundle. Let $\varphi_i : \iota_0^*(E_i, D_i) \to \iota_t^*(E_i, D_i)$ be the closed isomorphism from Lemma 2.1. If $\psi_{ij} : (E_i, D_i)|_{[0,1]\times(U_i\cap U_j)} \to (E_j, D_j)|_{[0,1]\times(U_i\cap U_j)}$ is the transition function, then by Lemma 2.2 we have the commutativity of the following diagram:

$$\begin{array}{ccc} \iota_0^*(E_i, D_i)|_{[0,1]\times(U_i\cap U_j)} & \xrightarrow{\iota_0^*\psi_{ij}} & \iota_0^*(E_j, D_j)|_{[0,1]\times(U_i\cap U_j)} \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ \iota_t^*(E_i, D_i)|_{[0,1]\times(U_i\cap U_j)} & \xrightarrow{\iota_t^*\psi_{ij}} & \iota_t^*(E_j, D_j)|_{[0,1]\times(U_i\cap U_j)}. \end{array}$$

Thus the morphisms φ_i can be glued together into a closed isomorphism $\varphi : \iota_0^*(E,D) \to \iota_t^*(E,D)$. □

It is worth noting that the isomorphism we have just defined is not a natural transformation. Here is an example:

Example 2.1. Consider the local systems

$$(E_0, d_{E_0}) = ([0,1]\times M\times V_0, d-\alpha_0) \quad \text{and} \quad (E_1, d_{E_1}) = ([0,1]\times M\times V_1, d-\alpha_1),$$

where $V_0 = V_1 = \mathbb{R}$, $\alpha_0 = dt$ and $\alpha_1 = tdt$ for $t \in [0,1]$. Let us compute the iterated integral maps for α_0 and α_1:

$$\begin{aligned}\varphi^{\alpha_0}(1) &= 1 + \sum_{k\geq 1} \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k \\ &= 1 + \sum_{k\geq 1} \frac{1}{k!} = e. \\ \varphi^{\alpha_1}(1) &= 1 + \sum_{k\geq 1} \int_0^1 s_1 ds_1 \int_0^{s_1} s_2 ds_2 \cdots \int_0^{s_{k-1}} s_k ds_k \\ &= 1 + \sum_{k\geq 1} \frac{1}{2 * 4 * \cdots * 2k} \\ &= 1 + \sum_{k\geq 1} \frac{(1/2)^k}{k!} = e^{1/2}.\end{aligned}$$

Therefore the morphisms $\varphi^{\alpha_0} : \iota_0^*(E_0, d_{E_0}) \to \iota_1^*(E_0, d_{E_0})$ and $\varphi^{\alpha_1} : \iota_0^*(E_1, d_{E_1}) \to \iota_1^*(E_1, d_{E_1})$ are multiplication by e and $e^{1/2}$ respectively. Now consider the morphism $\omega : (E_0, d_{E_0}) \to (E_1, d_{E_1})$ given by the constant form $\omega = 1 \in \Omega([0,1] \times M, \mathrm{Hom}(V_0, V_1)) \cong \Omega([0,1] \times M)$. It is clear that the following diagram is not commutative

$$\begin{array}{ccc} \iota_0^*(E_0, d_{E_0}) & \xrightarrow{\iota_0^*\omega = 1} & \iota_0^*(E_1, d_{E_1}) \\ {\scriptstyle \varphi^{\alpha_0} = e}\downarrow & & \downarrow{\scriptstyle \varphi^{\alpha_1} = e^{1/2}} \\ \iota_1^*(E_0, d_{E_0}) & \xrightarrow[\iota_1^*\omega = 1]{} & \iota_1^*(E_1, d_{E_1}). \end{array}$$

The correct statement is that the isomorphism φ of Theorem 2.1 is actually the first component of an A_∞-natural isomorphism between the dg-functors ι_0^* and ι_t^*.

Theorem 2.2. For every $t \in [0,1]$ there is an A_∞-natural isomorphism $\rho : \iota_0^* \Rightarrow \iota_t^*$.

Proof. As in the proof of Theorem 2.1, we will construct the transformation locally and later show that the maps constructed are gauge invariant.

Let (E_i, D_i) be local systems over $M \times I$ for $i = 0, \cdots, n$. Assume each (E_i, D_i) is trivialized, hence $E_i = M \times I \times V_i$ and $D_i = d - \alpha_i$ with $\alpha_i \in \Omega(M \times I, \mathrm{End}(V_i))$ of total degree 1 satisfying $d\alpha_i - \alpha_i^2 = 0$. Let us define

$$\rho_n^\alpha(t) : \mathrm{Hom}(E_{n-1}, E_n) \otimes \cdots \otimes \mathrm{Hom}(E_0, E_1) \to \mathrm{Hom}(i_0^*(E_0), i_t^*(E_n))$$

as follows. Take homogeneous elements $\omega_i \in \mathrm{Hom}(E_i, E_{i+1})$, $i = 0, \cdots, n-1$, this is, $\omega_i \in \Omega(M \times I, \mathrm{Hom}(V_i, V_{i+1}))$. If we make $V = \bigoplus_i V_i$, then the forms $\{\alpha_i\}$ and $\{\omega_i\}$ may be seen as elements of $\Omega(M \times I, \mathrm{End}(V))$. Let $\eta := \sum_i \alpha_i + \sum_i \omega_i$ and define $\tilde{\rho}_n^\alpha(t)(\omega_0, \cdots, \omega_{n-1}) := \varphi^\eta(t) \in \Omega(M, \mathrm{End}(V))$. We are interested in the $(0, n)$ component of this form which goes from V_0 to V_n, therefore we define

$$\rho_n^\alpha(\omega_{n-1}, \cdots, \omega_0) := (\tilde{\rho}_n^\alpha(\omega_{n-1}, \cdots, \omega_0))_{0,n} \in \Omega(M, \mathrm{Hom}(V_0, V_n)).$$

A general term of ρ_n^α is an integral

$$\int_{\Delta_k(t)} \lambda_n \pi^*_{i_{n-1}}(\omega_{n-1})\lambda_{n-1} \cdots \lambda_1 \pi^*_{i_0}(\omega_0)\lambda_0 \tag{2.1}$$

where $\lambda_j = \pi^*_{i_{j-1}+p_j}(\alpha_j) \cdots \pi^*_{i_{j-1}+1}(\alpha_j)$. In words, the integrands contain pullbacks of all the $\{\omega_i\}_{i=0}^{n-1}$ as factors, ordered with descending indices from left to right. Furthermore, there may be products of pullbacks of α_i between ω_i and ω_{i-1}. Notice that since the amount of factors in the integrand must equal k and all of the ω_i appear in the product, the integrals relevant to us are over $\Delta_k(t)$ with $k \geq n$. Also notice that since the $\{\alpha_i\}$ have total degree 1, the total degree of the forms yielded by the integrals is $\sum_{i=0}^{n-1} |\omega_i| - n$, which makes ρ_n^α a map of degree $-n$. Notice that ρ_0^α is just the isomorphism φ given in Theorem 2.1, which we know is a closed isomorphism.

Next we prove that this construction does not depend on the trivializations chosen for each E_i. Suppose that (E_i, D_i) is trivialized both as $([0,1] \times M \times V_i, d - \alpha_i)$ and $([0,1] \times M \times W_i, d - \beta_i)$, and for each i let $\psi_i \in \Omega^0([0,1] \times M, \mathrm{Hom}(V_i, W_i))$ be an isomorphism between trivializations. Take $\omega_i \in \Omega([0,1] \times M, \mathrm{Hom}(V_i, V_{i+1}))$ and define $\mu_i \in \Omega([0,1] \times M, \mathrm{Hom}(W_i, W_{i+1}))$ by the formula $\omega_i = \psi_{i+1}^{-1}\mu_i\psi_i$. We want to prove the following equation

$$\rho_n^\alpha(\omega_{n-1}, \cdots, \omega_0) = (i_t^*\psi_n)^{-1}\rho_n^\beta(\mu_{n-1}, \cdots, \mu_0)(i_0^*\psi_0). \tag{2.2}$$

Consider the direct sums $V = \bigoplus V_i$ and $W = \bigoplus W_i$. Then we have an isomorphism $\psi = \bigoplus \psi_i : V \to W$. Furthermore, if we call $\eta = \sum \alpha_i + \sum \omega_i$ and $\theta = \sum \beta_i + \sum \mu_i$, then the forms are gauge equivalent, i.e., $\eta = \psi^{-1}\theta\psi - \psi^{-1}d\psi$. By Proposition 1.2, for every $t \in [0,1]$ the following equation holds

$$\varphi^\eta(t) = (i_t^*\psi)^{-1}\varphi^\theta(t)(i_0^*\psi)$$

In particular this implies that the $(0,n)$ components are equal. That is precisely equation (2.2).

Finally we check that the A_∞ relations are satisfied. It is enough to check the relations locally, hence we need to verify that

$$\begin{aligned}\iota_t^*\omega_{n-1} \wedge \rho_{n-1}^\alpha(\omega_{n-2} \otimes \cdots \otimes \omega_0) - (-1)^{\sum_{j=1}^{n-1}|\omega_j| - n+1}\rho_{n-1}^\alpha(\omega_{n-1} \otimes \cdots \otimes \omega_1) \wedge \iota_0^*\omega_0 \\ = \rho^\alpha(b(\omega_{n-1} \otimes \cdots \otimes \omega_0)) + D\rho_n^\alpha(\omega_{n-1} \otimes \cdots \otimes \omega_0),\end{aligned}$$

Using equation (2.2) we can write the differential D in the previous relation as follows:

$$
\begin{aligned}
&D\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)\\
&= d\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0) - i_t^*(\alpha_n)\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0) + (-1)^{|\omega_{n-1}|+\cdots+|\omega_0|-n}\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)i_0^*(\alpha_0).
\end{aligned}
$$

Therefore the equation we need to verify is

$$
\begin{aligned}
&d\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0)\\
&\quad = \iota_t^*\omega_{n-1}\wedge\rho_{n-1}^\alpha(\omega_{n-2}\otimes\cdots\otimes\omega_0) - (-1)^{\sum_{j=1}^{n-1}|\omega_j|-n+1}\rho_{n-1}^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_1)\wedge\iota_0^*\omega_0\\
&\qquad + \iota_t^*\alpha_n\wedge\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0) - (-1)^{\sum_{j=0}^{n-1}|\omega_j|-n}\rho_n^\alpha(\omega_{n-1}\otimes\cdots\otimes\omega_0)\wedge\iota_0^*\alpha_0\\
&\qquad - \rho^\alpha(b(\omega_{n-1}\otimes\cdots\otimes\omega_0)),
\end{aligned}
$$

This equation will be proved at the end of the chapter since it requires quite a bit of work. □

Example 2.2. Recall from Example 2.1 the local systems

$$(E_0, d_{E_0}) = ([0,1]\times M\times V_0, d-\alpha_0) \quad \text{and} \quad (E_1, d_{E_1}) = ([0,1]\times M\times V_1, d-\alpha_1).$$

We have seen that for the morphism $\omega = 1 : (E_0, d_{E_0}) \to (E_1, d_{E_1})$ we have

$$\iota_1^*\omega\circ\varphi^{\alpha_0} - \varphi^{\alpha_1}\circ\iota_0^*\omega = e - e^{1/2} \neq 0,$$

which means that the iterated integral map is not a natural transformation between the functors ι_0^* and ι_1^*. According to Theorem 2.2, the correct statement is that the transformation is natural up to a homotopy given by ρ_1, this is

$$\iota_1^*\omega\circ\varphi^{\alpha_0} - \varphi^{\alpha_1}\circ\iota_0^*\omega = \rho_1(D(\omega)) + D\rho_1(\omega).$$

Let us compute the right side of this equation and verify that it is equal to $e - e^{1/2}$. First notice that since $\omega = 1$ is a constant we have $\rho_1(\omega) = 0$, thus it is enough to compute $\rho_1(D(\omega))$. Next we have $D(\omega)$ which is the commutator of ω with the differentials $d-\alpha_0$ and $d-\alpha_1$:

$$D(\omega) = d\circ\omega - \alpha_1\omega - \omega\circ d + \omega\alpha_0 = \alpha_0 - \alpha_1 = dt - tdt.$$

Now according to expression (2.1), a general term of $\rho_1(dt - tdt)$ is an integral

$$\int_{\Delta_{k+l+1}} \pi_1^*(\alpha_1)\cdots\pi_k^*(\alpha_1)\pi_{k+1}^*(dt-tdt)\pi_{k+2}^*(\alpha_0)\cdots\pi_{k+l+1}^*(\alpha_0),$$

where $k, l \geq 0$. Writing as an iterated integral we get

$$\int_0^1 s_1 ds_1 \int_0^{s_1} s_2 ds_2 \cdots \int_0^{s_{k-1}} s_k ds_k \int_0^{s_k} (ds_{k+1} - s_{k+1}ds_{k+1}) \int_0^{s_{k+1}} ds_{k+2}\cdots\int_0^{s_{k+l}} ds_{k+l+1}.$$

A straight forward computation shows that the previous integral is equal to

$$\frac{(l+2)(l+4)\cdots(l+2k+2)-(l+1)(l+3)\cdots(l+2k+1)}{(l+2k+2)!},$$

hence

$$\rho_1(D(\omega)) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(l+2)(l+4)\cdots(l+2k+2)-(l+1)(l+3)\cdots(l+2k+1)}{(l+2k+2)!}.$$

Adding the first 225 terms of the series yields the number

$$1.0695605577589137880067937658168375492095947265625,$$

which coincides with $e - e^{1/2}$ up to the 14th decimal space.

As a corollary of Theorem 2.2 we get our main result of this section:

Theorem 2.3. Let M and N be smooth manifolds and $f, g : M \to N$ be smooth maps. If $h : [0,1] \times M \to N$ is a homotopy with $h \circ \iota_0 = f$ and $h \circ \iota_1 = g$, then there is an A_∞-natural isomorphism $\mathrm{hol} : f \Rightarrow g$ which depends on h and is given in terms of iterated integrals.

Proof. Consider the sequence

$$\mathsf{Loc}_\infty(N) \xrightarrow{h^*} \mathsf{Loc}_\infty([0,1]\times M) \underset{\iota_0^*}{\overset{\iota_1^*}{\rightrightarrows}} \mathsf{Loc}_\infty(M).$$

From Theorem 2.2 we have an A_∞-natural transformation $\rho : \iota_0^* \Rightarrow \iota_1^*$ given in terms of iterated integrals. Composing the transformation with the functor h^* we get an A_∞-natural transformation

$$\mathrm{hol} := \rho \circ h^* : f^* = \iota_0^* \circ h^* \Rightarrow g^* = \iota_1^* \circ h^*.$$

□

3 An A_∞ Version of Poincare's Lemma

From Theorem 2.3 we get several corollaries, including a categorified version of Poincare's lemma.

Corollary 3.1. If $f : M \to N$ is a homotopy equivalence, then $f^* : \mathsf{Loc}_\infty(N) \to \mathsf{Loc}_\infty(M)$ is a quasi-equivalence of dg-categories.

Proof. Suppose that $g : N \to M$ is a homotopy inverse of f, then $g \circ f \simeq \mathrm{id}_M$ and $f \circ g \simeq \mathrm{id}_N$. By Theorem 2.3 there are A_∞-natural isomorphisms $(g\circ f)^* \Rightarrow \mathrm{id}_{\mathsf{Loc}_\infty(M)}$ and $(f\circ g)^* \Rightarrow \mathrm{id}_{\mathsf{Loc}_\infty(N)}$. The result then follows from Lemma 1.4. □

Applying this corollary to a contractible space we get Poincare's lemma:

Corollary 3.2 (Categorified Poincaré lemma)**.** If M is contractible, then $\mathsf{Loc}_\infty(M)$ is quasi-equivalent to $\mathsf{DGVect}_{\mathbb{R}}$.

Proof. Since M is contractible, it has the same homotopy type as a point. Thus, according to Corollary 3.1, there is a quasi-equivalence between $\mathsf{Loc}_\infty(M)$ and $\mathsf{Loc}_\infty(\{*\})$ which is clearly the category $\mathsf{DGVect}_{\mathbb{R}}$. □

Poincare's lemma gives a simplified local description of ∞-local systems

Corollary 3.3. For an arbitrary manifold M, any ∞-local system (E, D) is locally isomorphic to a trivial ∞-local system, that is, every point has an open neighbourhood U in which $(E|_U, D|_U)$ is isomorphic to an ∞-local system of the form $(U \times V, d)$.

Proof. Any point in M has a contractible neighbourhood. □

4 Computation of $d\rho_n^\alpha(\omega_{n-1} \otimes \cdots \otimes \omega_0)$

Here we have the computation of $d\rho_n(\omega_{n-1}, \cdots, \omega_0)$.

Take a general term of $\rho_n(\omega_{n-1}, \cdots, \omega_0)$, this is an integral

$$\int_0^t \pi_1^*(\lambda_1) \int_0^{s_1} \pi_2^*(\lambda_2) \cdots \int_0^{s_{k-1}} \pi_k^*(\lambda_k) = (-1)^\chi \int_{\Delta_k} \pi_1^*(\lambda_1) \cdots \pi_k^*(\lambda_k)$$

where $\chi = (k-1)|\lambda_1| + \cdots + |\lambda_{k-1}|$ and the sequence $\lambda_1, \cdots, \lambda_k$ contains all of the ω_i with descending index and possibly some copies of α_i between ω_i and ω_{i-1}. Notice that for this to happen we must have $k \geq n$. We compute d of the integral:

$$(-1)^\chi d \int_{\Delta_k} \pi_1^*(\lambda_1) \cdots \pi_k^*(\lambda_k) = (-1)^\chi \int_{\partial \Delta_k} \pi_1^*(\lambda_1) \cdots \pi_k^*(\lambda_k) + (-1)^{\chi+k} \int_{\Delta_k} d\big(\pi_1^*(\lambda_1) \cdots \pi_k^*(\lambda_k)\big).$$

The integral over the border components is:

$$\begin{aligned}
(-1)^\chi \int_{\partial \Delta_k} \pi_1^*(\lambda_1) \cdots \pi_k^*(\lambda_k) =& \sum_{j=0}^{k} (-1)^{\chi+j} \int_{\partial_j \Delta_k} \pi_1^*(\lambda_1) \cdots \pi_k^*(\lambda_k) \\
=& (-1)^\chi \int_{\Delta_{k-1}} i_t^*(\lambda_1) \pi_1^*(\lambda_2) \cdots \pi_{k-1}^*(\lambda_k) \\
&+ \sum_{j=1}^{k-1} (-1)^{\chi+j} \int_{\Delta_{k-1}} \pi_1^*(\lambda_1) \cdots \pi_j^*(\lambda_j \lambda_{j+1}) \cdots \pi_{k-1}^*(\lambda_k) \\
&+ (-1)^{\chi+k} \int_{\Delta_{k-1}} \pi_1^*(\lambda_1) \cdots \pi_{k-1}^*(\lambda_{k-1}) i_0^*(\lambda_k).
\end{aligned}$$

Rearranging into iterated integrals we get

$$\begin{aligned}(-1)^{\chi}\int_{\partial\Delta_k}\pi_1^*(\lambda_1)\cdots\pi_k^*(\lambda_k) =& i_t^*(\lambda_1)\int_0^t \pi_1^*(\lambda_2)\cdots\int_0^{s_{k-2}}\pi_{k-1}^*(\lambda_k)\\ &+\sum_{j=1}^{k-1}(-1)^{|\lambda_1|+\cdots+|\lambda_j|+j}\int_0^t\pi_1^*(\lambda_1)\cdots\int_0^{s_{j-1}}\pi_j^*(\lambda_j\lambda_{j+1})\cdots\int_0^{s_{k-2}}\pi_{k-1}^*(\lambda_{k-1})\\ &+(-1)^{|\lambda_1|+\cdots+|\lambda_{k-1}|+k}\int_0^t\pi_1^*(\lambda_1)\int_0^{s_1}\pi_2^*(\lambda_2)\cdots\int_0^{s_{k-2}}\pi_{k-1}^*(\lambda_{k-1})i_0^*(\lambda_k).\end{aligned}\tag{4.1}$$

For the other term we use Leibniz's rule and rearrange as an iterated integral

$$\begin{aligned}(-1)^{\chi+k}\int_{\Delta_k}d\big(\pi_1^*(\lambda_1)\cdots\pi_k^*(\lambda_k)\big) &= \sum_{j=1}^{k}(-1)^{\chi+|\lambda_1|+\cdots+|\lambda_{j-1}|+k}\int_{\Delta_k}\pi_1^*(\lambda_1)\cdots\pi_j^*(d\lambda_j)\cdots\pi_k^*(\lambda_k)\\ &=\sum_{j=1}^{k}(-1)^{|\lambda_1|+\cdots+|\lambda_{j-1}|-j}\int_0^t\pi_1^*(\lambda_1)\cdots\int_0^{s_{j-1}}\pi_j^*(d\lambda_j)\cdots\int_0^{s_{k-1}}\pi_k^*(\lambda_k).\end{aligned}\tag{4.2}$$

We recognize different kinds of terms within $d\rho_n^\alpha$:

- Terms containing $i_t^*\lambda_1$ and $i_0^*\lambda_k$. They arise from the first and last lines of (4.1). We consider the possibilities for λ_1 and λ_k

  ⋆ $\lambda_1=\omega_{n-1}$ and $\lambda_k=\omega_0$. In this case the sign involved is

  $$(-1)^{|\lambda_1|+\cdots+|\lambda_{k-1}|+k}=(-1)^{|\omega_{n-1}|+\cdots+|\omega_1|-n}.$$

  Then the terms are

  $$\begin{aligned}&i_t^*(\omega_{n-1})\int_0^t\pi_1^*(\lambda_2)\cdots\int_0^{s_{k-2}}\pi_{k-1}^*(\lambda_k)\\ &+(-1)^{|\omega_{n-1}|+\cdots+|\omega_1|-n}\int_0^t\pi_1^*(\lambda_1)\int_0^{s_1}\pi_2^*(\lambda_2)\cdots\int_0^{s_{k-2}}\pi_{k-1}^*(\lambda_{k-1})i_0^*(\omega_0).\end{aligned}$$

  Adding over the simplexes of all dimensions and the admissible values of the λ_j we get

  $$i_t^*(\omega_{n-1})\rho_{n-1}^\alpha(\omega_{n-2},\cdots,\omega_0)+(-1)^{|\omega_{n-1}|+\cdots+|\omega_1|-n}\rho_{n-1}^\alpha(\omega_{n-1},\cdots,\omega_1)i_0^*(\omega_0).\tag{4.3}$$

  ⋆ $\lambda_1=\alpha_n$ and $\lambda_k=\alpha_0$. The sign is

  $$(-1)^{|\lambda_1|+\cdots+|\lambda_{k-1}|+k}=(-1)^{|\omega_{n-1}|+\cdots+|\omega_0|-n+1}.$$

Once again adding over the pertinent terms we get

$$i_t^*(\alpha_n)\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0) - (-1)^{|\omega_{n-1}|+\cdots+|\omega_0|-n}\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)i_0^*(\alpha_0). \tag{4.4}$$

- Terms containing $d\lambda_j$. These terms arise from (4.2). There are two possibilities for λ_j:

  ⋆ $\lambda_j = \omega_i$. The sign is

$$(-1)^{|\lambda_1|+\cdots+|\lambda_{j-1}|-j} = (-1)^{|\omega_{n-1}|+\cdots+|\omega_{i+1}|-n+i}.$$

  Adding over we get

$$(-1)^{|\omega_{n-1}|+\cdots+|\omega_{i+1}|-n+i}\rho_n^\alpha(\omega_{n-1},\cdots,d\omega_i,\cdots,\omega_0). \tag{4.5}$$

  ⋆ $\lambda_j = \alpha_i$. The sign is

$$(-1)^{|\lambda_1|+\cdots+|\lambda_{j-1}|-j} = (-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i-1}.$$

  By the Maurer-Cartan condition the integral may be written as

$$(-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i-1}\int_0^t \pi_1^*(\lambda_1)\cdots\int_0^{s_{j-1}} \pi_j^*(\alpha_i^2)\cdots\int_0^{s_{k-1}} \pi_k^*(\lambda_k). \tag{4.6}$$

- Terms containing products $\lambda_j\lambda_{j+1}$. These are obtained from the second line of (4.1). Again we consider all possibilities for λ_j and λ_{j+1}

  ⋆ $\lambda_j = \lambda_{j+1} = \alpha_i$. The sign is

$$(-1)^{|\lambda_1|+\cdots+|\lambda_j|+j} = (-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i}.$$

  The integral

$$(-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i}\int_0^t \pi_1^*(\lambda_1)\cdots\int_0^{s_{j-1}} \pi_j^*(\alpha_i^2)\cdots\int_0^{s_{k-2}} \pi_{k-1}^*(\lambda_{k-1}). \tag{4.7}$$

  ⋆ $\lambda_j = \alpha_{i+1}$ and $\lambda_{j+1} = \omega_i$. The sign is

$$(-1)^{|\lambda_1|+\cdots+|\lambda_j|+j} = (-1)^{|\omega_{n-1}|+\cdots+|\omega_{i+1}|-n+i-1}.$$

  Adding all the relevant terms we get

$$(-1)^{|\omega_{n-1}|+\cdots+|\omega_{i+1}|-n+i-1}\rho_n^\alpha(\omega_{n-1},\cdots,\alpha_{i+1}\omega_i,\cdots,\omega_0). \tag{4.8}$$

  ⋆ $\lambda_j = \omega_i$ and $\lambda_{j+1} = \alpha_i$. The sign is

$$(-1)^{|\lambda_1|+\cdots+|\lambda_j|+j} = (-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i}.$$

Adding all the relevant terms we get

$$(-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i}\rho_n^\alpha(\omega_{n-1},\cdots,\omega_i\alpha_i,\cdots,\omega_0). \tag{4.9}$$

⋆ $\lambda_j=\omega_i$ and $\lambda_{j+1}=\omega_{i-1}$. The sign is

$$(-1)^{|\lambda_1|+\cdots+|\lambda_j|+j}=(-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i}.$$

Adding all the relevant terms we get

$$(-1)^{|\omega_{n-1}|+\cdots+|\omega_i|-n+i}\rho_n^\alpha(\omega_{n-1},\cdots,\omega_i\omega_{i-1},\cdots,\omega_0). \tag{4.10}$$

Finally we give a succinct expression. Notice that terms given in equations (4.6) and (4.7) cancel out, which means that

$$d\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)=(4.3)+(4.4)+(4.5)+(4.8)+(4.9)+(4.10).$$

Terms in (4.5)+(4.8)+(4.9)+(4.10) come together conveniently

$$(4.5)+(4.8)+(4.9)+(4.10)=-\rho_n^\alpha(b(\omega_{n-1},\cdots,\omega_0)).$$

Which means that

$$\begin{aligned}
&d\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)\\
=&i_t^*(\omega_{n-1})\rho_{n-1}^\alpha(\omega_{n-2},\cdots,\omega_0)+(-1)^{|\omega_{n-1}|+\cdots+|\omega_1|-n}\rho_{n-1}^\alpha(\omega_{n-1},\cdots,\omega_1)i_0^*(\omega_0)\\
&+i_t^*(\alpha_n)\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)-(-1)^{|\omega_{n-1}|+\cdots+|\omega_0|-n}\rho_n^\alpha(\omega_{n-1},\cdots,\omega_0)i_0^*(\alpha_0)\\
&-\rho_n^\alpha(b(\omega_{n-1},\cdots,\omega_0)).
\end{aligned} \tag{4.11}$$

Chapter 3

2-Dimensional Holonomies

In Chapter 1 we discussed a Riemann-Hilbert type theorem that states an equivalence between the category of ∞-local systems and the category of representations up to homotopy of the ∞-groupoid of a manifold. The correspondence is given by a holonomy functor Hol which assigns a representation up to homotopy of the ∞-groupoid to every flat, $\mathbb{Z}$-graded connection. In other words, the structure of a flat, $\mathbb{Z}$-graded connection over a manifold allows for the definition of higher parallel transport, i.e., parallel transport along simplices of all dimensions higher than 1. On the other hand there is another formalism of 2-dimensional parallel transport that is defined in principal 2-bundles found in [28]. The goal of this chapter is to compare the 2-dimensional parallel transport constructed in Chapter 1 following [2] to the one found in [28]. We will see that the 2-dimensional holonomies from Chapter 1 only encompass the simpler cases of parallel transport in principal 2-bundles. However, it is to be noted that the holonomies of Chapter 1 are defined for all higher dimensions.

1 Flat Connections on Principal 2-Bundles

In this section we give a quick review of principal 2-bundles and connections over them. The exposition given here follows closely the work of Waldorf in [27] and [28].

Definition 1.1. A Lie Crossed Module Γ is a tuple $\Gamma = (H, G, \tau, \alpha)$, where H, G are Lie groups, $\tau : H \to G$ is a Lie group homomorphism and $\alpha : G \times H \to H$ is an action such that $\tau(\alpha(g, h)) = g\tau(h)g^{-1}$ and $\alpha(\tau(h), h') = hh'h^{-1}$, for all $g \in G$ and $h, h' \in H$. A strict Lie 2-group is a small 2-category with a single object in which the set of 1-morphisms and the set of 2-morphisms are Lie groups and all structural maps (source, target, identity and composition) are Lie group homomorphisms.

Throughout this chapter, all the Lie 2-groups we consider are strict Lie 2-groups, so even if we drop the word "strict" it should be understood that we are referring to a strict Lie 2-group.

Lemma 1.1. There is a one to one correspondence between Lie crossed modules and strict 2-groups.

Proof. Given a Lie crossed module $\Gamma = (H, G, \tau, \alpha)$, we can define in a natural way a Lie 2-group (which we also denote as Γ) where the 1-morphisms are the elements of the group G and the 2-morphisms are the elements of the semidirect product $H \ltimes G$. The source and target maps are $s(h, g) = g$ and $t(h, g) = \tau(h)g$. $\square$

Remark 1.1. In view of the previous lemma, we will make no distinction between a Lie crossed module and a strict Lie 2-group.

Definition 1.2. A differential Lie crossed module γ is a tuple $\gamma = (\mathfrak{h}, \mathfrak{g}, t, a)$ where $\mathfrak{h}$ and $\mathfrak{g}$ are Lie algebras, $t : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism and $a : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ is an action by derivations such that $a(t(Y_1, Y_2)) = [Y_1, Y_2]$ and $ta[X, Y] = [X, t(Y)]$ for $X \in \mathfrak{g}$ and $Y_1, Y_2, Y \in \mathfrak{h}$.

Clearly, differentiating a Lie crossed module yields a differential Lie crossed module.

Definition 1.3. A Lie groupoid $\mathcal{X}$ is a groupoid in which the set of objects and the set of morphisms are manifolds. The source (s) and target (t) maps are submersions and all the structural maps are smooth.

A right action of a Lie groupoid $\mathcal{X}$ on a smooth manifold M is an anchor map $\alpha : M \to \mathrm{Obj}(\mathcal{X})$ together with an action map $\circ : M\, {}_\alpha\!\times_t \mathrm{Mor}(\mathcal{X}) \to M$ that satisfy the following conditions

$$(x \circ f) \circ g = x \circ (f \circ g), \quad x \circ \mathrm{Id}_{\alpha(x)} = x, \quad \alpha(x \circ g) = s(g), \quad x \in M,\ f, g \in \mathrm{Mor}(\mathcal{X}).$$

A left action is merely a right action of the opposite Lie groupoid.

A principal $\mathcal{X}$-bundle over a manifold M is another manifold P with a surjective submersion $\pi : P \to M$ and a right action of $\mathcal{X}$ on P that preserves the projection and such that the map $P\, {}_\alpha\!\times_t \mathrm{Mor}(\mathcal{X}) \to P \times_M P$ given by $(p, f) \mapsto (p, p \circ f)$ is a diffeomorphism.

Given Lie groupoids $\mathcal{X}$, $\mathcal{Y}$; an anafunctor $F : \mathcal{X} \to \mathcal{Y}$ is a smooth manifold F called the total space, with a left action of $\mathcal{X}$ and a right action of $\mathcal{Y}$ such that the actions are commutative and the left anchor $\alpha_l : F \to \mathrm{Obj}(\mathcal{X})$ with the right action of $\mathcal{Y}$ is a principal $\mathcal{Y}$-bundle.

Given anafunctors $F, F' : \mathcal{X} \to \mathcal{Y}$, a transformation $f : F \Rightarrow F'$ is a smooth function $f : F \to F'$ that is $\mathcal{X}$-equivariant, $\mathcal{Y}$-equivariant and preserves the anchor maps.

Remark 1.2. Any smooth functor $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ may be used to define an anafunctor $F : \mathcal{X} \to \mathcal{Y}$ with total space $F := \mathrm{Obj}(\mathcal{X})\, {}_\mathcal{F}\!\times_t \mathrm{Mor}(\mathcal{Y})$, anchors $\alpha_l(x, f) := x$ and $\alpha_r(x, f) := s(f)$ and actions $g \circ (x, f) := (t(g), \mathcal{F}g \circ f)$ and $(x, f) \circ g := (x, f \circ g)$. Thus any smooth functor may be considered an anafunctor. Similarly, a smooth transformation $\mathcal{T} : \mathcal{F} \Rightarrow \mathcal{F}'$ induces a transformation between the respective anafunctors given by $f_\mathcal{T}(x, g) := (x, \mathcal{T}(x) \circ g)$.

Definition 1.4. A *right action* of a Lie 2-group Γ on a groupoid $\mathcal{X}$ is a functor $R : \mathcal{X} \times \Gamma \to \mathcal{X}$ such that $R(p, 1) = p$ and $R(\rho, \mathrm{id}_1) = \rho$, for all $p \in \mathrm{Obj}(\mathcal{X})$ and $\rho \in \mathrm{Mor}(\mathcal{X})$. Furthermore, if

$m : \Gamma\times\Gamma \to \Gamma$ is the multiplication functor, then the following diagram must be commutative:

$$\begin{array}{ccc} \mathcal{X}\times\Gamma\times\Gamma & \xrightarrow{\mathrm{id}_{\mathcal{X}}\times m} & \mathcal{X}\times\Gamma \\ {\scriptstyle R\times\mathrm{id}_\Gamma}\downarrow & & \downarrow{\scriptstyle R} \\ \mathcal{X}\times\Gamma & \xrightarrow[R]{} & \mathcal{X}. \end{array}$$

If $\mathcal{X}$ and $\mathcal{Y}$ are groupoids with smooth right Γ-actions, a Γ-equivariant anafunctor $F : \mathcal{X} \to \mathcal{Y}$ is an anafunctor with a smooth action $\rho : F \times \mathrm{Mor}(\Gamma) \to F$ that preserves the anchors, i.e. makes the following diagrams commutative

$$\begin{array}{ccc} F\times\mathrm{Mor}(\Gamma) & \xrightarrow{\rho} & F \\ {\scriptstyle \alpha_l\times t}\downarrow & & \downarrow{\scriptstyle \alpha_l} \\ \mathrm{Obj}(\mathcal{X})\times\mathrm{Obj}(\Gamma) & \xrightarrow[R]{} & \mathrm{Obj}(\mathcal{X}), \end{array} \qquad \begin{array}{ccc} F\times\mathrm{Mor}(\Gamma) & \xrightarrow{\rho} & F \\ {\scriptstyle \alpha_r\times s}\downarrow & & \downarrow{\scriptstyle \alpha_r} \\ \mathrm{Obj}(\mathcal{Y})\times\mathrm{Obj}(\Gamma) & \xrightarrow[R]{} & \mathrm{Obj}(\mathcal{Y}). \end{array}$$

Furthermore, ρ must be compatible with the actions of $\mathcal{X}$ and $\mathcal{Y}$, which means that

$$\rho(\chi\circ f\circ\eta, \gamma_l\circ\gamma\circ\gamma_r) = R(\chi,\gamma_l)\circ\rho(f,\gamma)\circ R(\eta,\gamma_r),$$

for $\chi \in \mathrm{Mor}(\mathcal{X})$, $\eta \in \mathrm{Mor}(\mathcal{Y})$, $f \in F$ and $\gamma_l, \gamma, \gamma_r \in \mathrm{Mor}(\Gamma)$.

If $F, F' : \mathcal{X} \to \mathcal{Y}$ are Γ-equivariant anafunctors, then a transformation $f : F \Rightarrow F'$ is called Γ-equivariant if the smooth map $f : F \to F'$ is $\mathrm{Mor}(\Gamma)$-equivariant.

Given a smooth manifold M we denote by M_{dis} the trivial groupoid where the objects are the points of M and the only morphisms are identities.

Definition 1.5. A *Principal* Γ*-2-Bundle* over M is a groupoid $\mathcal{P}$ together with a right action $R : \mathcal{P}\times\Gamma \to \mathcal{P}$ and a functor $\pi : \mathcal{P} \to M_{dis}$ that is a surjective submersion at the level of objects. The action must preserve the functor π, and the functor

$$(\mathrm{pr}_1, R) : \mathcal{P}\times\Gamma \to \mathcal{P}\times_M\mathcal{P}$$

must be a weak equivalence.

Principal Γ-2-bundles over a fixed manifold M can be given the structure of a 2-category and, as such, can be classified by non-abelian cohomology. We will focus on an aspect of this classification which is how a principal Γ-2-bundle can be constructed from local data given by cocycles.

Definition 1.6. Let $\Gamma = \{H, G, \tau, \alpha\}$ be a Lie crossed module and $\{U_i\}_{i\in I}$ an open covering of a manifold M. A Γ-cocycle on the covering is comprised of the following data:

1) For each pair $i, j \in I$ a smooth map $g_{ij} : U_i \cap U_j \to G$.

2) for each triple $i, j, k \in I$ a smooth map $a_{ijk} : U_i \cap U_j \cap U_k \to H$.

The cocycle conditions for the maps are the following:

$$g_{ik} = (\tau \circ a_{ijk}) g_{ij} g_{jk}, \qquad \text{on } U_i \cap U_j \cap U_k,$$
$$a_{ijl} a_{jkl} = a_{ikl} \alpha(g_{kl}, a_{ijk}), \qquad \text{on } U_i \cap U_j \cap U_k \cap U_l.$$

We denote a Γ-cocycle by (g, a).

Lemma 1.2. Given a Γ-cocycle defined on an open covering of M, we can construct a principal Γ-2-bundle over M.

Proof. Let (g, a) be a Γ-cocycle defined on the open covering $\{U_i\}_{i \in I}$. We define a Lie groupoid $\mathcal{P}$ as follows

$$\mathrm{Obj}(\mathcal{P}) := \bigsqcup_{i \in I} U_i \times G, \quad \mathrm{Mor}(\mathcal{P}) := \bigsqcup_{i,j \in I} (U_i \cap U_j) \times H \times G.$$

Source and target maps are

$$s(i, j, x, h, g) := (j, x, g), \quad t(i, j, x, h, g) := (i, x, g_{ij}(x)^{-1} \tau(h) g).$$

The composition of morphisms is given by

$$(i, j, x, h_2, g_2) \circ (j, k, x, h_1, g_1) := (i, k, x, a_{ijk}(x) \alpha(g_{jk}(x), h_2) h_1, g_1).$$

The projection $\pi : \mathcal{P} \to M_{dis}$ is $\pi(i, x, g) := x$. The action is defined by

$$R((i, x, g), g') := (i, x, gg'), \quad R((i, j, x, h, g), (h', g')) := (i, j, x, h\alpha(g, h'), gg').$$

It is straight forward to check that $(\mathrm{pr}_1, R) : \mathcal{P} \times \Gamma \to \mathcal{P} \times_M \mathcal{P}$ is a weak equivalence. □

As in the classical setting, a connection on a principal Γ-2-bundle is defined in terms of a differential form. If Γ is a Lie 2-group with differential Lie crossed module γ, then for any Lie groupoid $\mathcal{P}$ there is a differential graded Lie algebra $\Omega(\mathcal{P}, \gamma)$. The details of the definition and properties of $\Omega(\mathcal{P}, \gamma)$ can be found in [27]. We will simply state that a differential form $\psi \in \Omega^k(\mathcal{P}, \gamma)$ has three components $\psi = (\psi^a, \psi^b, \psi^c)$ with

$$\psi^a \in \Omega^k(\mathrm{Obj}(\mathcal{P}), \mathfrak{g}), \quad \psi^a \in \Omega^k(\mathrm{Mor}(\mathcal{P}), \mathfrak{h}), \quad \psi^a \in \Omega^{k+1}(\mathrm{Obj}(\mathcal{P}), \mathfrak{h}).$$

Example 1.1. Given a Lie 2-group $\Gamma = (H, G, \tau, \alpha)$ with differential Lie crossed module $\gamma = (\mathfrak{h}, \mathfrak{g}, t, a)$, there is a canonical Maurer-Cartan form $\Theta \in \Omega^1(\Gamma, \gamma)$. The form has two non-trivial components $\Theta^a \in \Omega^1(G, \mathfrak{g})$ $\Theta^b \in \Omega^1(H \times G, \mathfrak{h})$ given by

$$\Theta^a := \theta^G \quad \text{and} \quad \Theta^b := (\alpha_{\mathrm{pr}_G^{-1}})_*(\mathrm{pr}_H^* \theta^H),$$

where θ^G and θ^H are the Maurer-Cartan forms of G and H respectively. The form Θ satisfies the Maurer-Cartan equation.

Definition 1.7. Let $F : \mathcal{P} \to \Gamma$ be a smooth functor. Write $F_0 : \mathrm{Obj}(\mathcal{P}) \to G$ for the map at the level of objects, $F_G : \mathrm{Mor}(\mathcal{P}) \to G$ and $F_H : \mathrm{Mor}(\mathcal{P}) \to H$ for the projections onto G and H respectively of the map at the level of morphisms. There is a linear map $\mathrm{Ad}_F : \Omega^k(\mathcal{P}, \gamma) \to \Omega^k(\mathcal{P}, \gamma)$ called the adjoint action defined as follows:

$$\begin{aligned}
\mathrm{Ad}_F(\psi)^a &:= \mathrm{Ad}_{F_0}(\psi^a),\\
\mathrm{Ad}_F(\psi)^b &:= \mathrm{Ad}_{F_H}((\alpha_{F_G})_*(\psi^b) + (\tilde{\alpha}_{F_H^{-1}})_*(\mathrm{Ad}_{F_G}(s^*\psi^a)),\\
\mathrm{Ad}_F(\psi)^c &:= (\alpha_{F_0})_*(\psi^c).
\end{aligned}$$

Definition 1.8. If $\mathcal{P}$ is a principal Γ-2-bundle, then a connection on $\mathcal{P}$ is a 1-form $\Omega \in \Omega^1(\mathcal{P}, \gamma)$ such that the following equation holds over $\mathcal{P} \times \Gamma$:

$$R^*\Omega = \mathrm{Ad}^{-1}_{\mathrm{pr}_\Gamma}(\mathrm{pr}^*_{\mathcal{P}}\Omega) + \mathrm{pr}^*_\Gamma\Theta.$$

A connection is a triple of ordinary differential forms $\Omega = (\Omega^a, \Omega^b, \Omega^c)$ with $\Omega^a \in \Omega^1(\mathcal{P}_0, \mathfrak{g})$, $\Omega^b \in \Omega^1(\mathcal{P}_1, \mathfrak{h})$ and $\Omega^c \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$ such that

$$\begin{aligned}
R^*\Omega^a &= \mathrm{Ad}_g^{-1}(p^*\Omega^a) + g^*\Theta && \text{over } \mathrm{Obj}(\mathcal{P}) \times \mathrm{Obj}(\Gamma),\\
R^*\Omega^b &= (\alpha_{g^{-1}})_*(\mathrm{Ad}_h^{-1}(p^*\Omega^b) + (\tilde{\alpha}_h)_*(p^*s^*\Omega^a) + h^*\Theta) && \text{over } \mathrm{Mor}(\mathcal{P}) \times \mathrm{Mor}(\Gamma),\\
R^*\Omega^c &= (\alpha_{g^{-1}})_*(p^*\Omega^c) && \text{over } \mathrm{Obj}(\mathcal{P}) \times \mathrm{Obj}(\Gamma).
\end{aligned}$$

Here p, g and h are the projections to either $\mathrm{Obj}(\mathcal{P})$ or $\mathrm{Mor}(\mathcal{P})$, G and H, respectively.

The curvature of a connection is $\mathrm{curv}(\Omega) = D\Omega + \frac{1}{2}[\Omega \wedge \Omega] \in \Omega^2(\mathcal{P}, \gamma)$. The connection is called flat if $\mathrm{curv}(\Omega) = 0$.

Principal Γ-2-bundles over M with flat connection fit into a 2-category which we denote 2-$\mathcal{B}un^f_\Gamma(M)$. The details of the bicategory structure can be found in [27]. Furthermore, if we fix a principal Γ-2-bundle $\mathcal{P}$, the category of flat connections defined over $\mathcal{P}$ will be denoted 2-$\mathcal{B}un^f_{\mathcal{P}}(M)$. The classification of principal Γ-2-bundles over M by non-abelian cohomology may be refined to classify bundles with flat connection. First we define a differential Γ-cocycle, which is a Γ-cocycle with the local data required to build a connection.

Definition 1.9. Let $\Gamma = (H, G, \tau, \alpha)$ be a Lie crossed module and $\{U_i\}_{i \in I}$ an open covering of a manifold M. A differential Γ-cocycle on the covering is comprised of the following data:

1) A Γ-cocycle (g, a).

2) On every open set U_i a couple of forms (A_i, B_i) where $A_i \in \Omega^1(U_i, \mathfrak{g})$ and $B_i \in \Omega^2(U_i, \mathfrak{h})$.

3) For each pair $i, j \in I$, a form $\varphi_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{h})$.

The forms must satisfy the following conditions:

$$A_j + \tau_*(\varphi_{ij}) = \mathrm{Ad}_{g_{ij}}(A_i) - g_{ij}^*\bar{\theta} \qquad \text{on } U_i \cap U_j, \tag{1.1}$$

$$(\alpha_{g_{ij}})_*(B_i) = B_j + d\varphi_{ij} + \frac{1}{2}[\varphi_{ij} \wedge \varphi_{ij}] + \alpha_*(A_j \wedge \varphi_{ij}) \quad \text{on } U_i \cap U_j, \tag{1.2}$$

$$\varphi_{jk} + (\alpha_{g_{jk}})_*(\varphi_{ij}) - a_{ijk}^*\theta = \mathrm{Ad}^{-1}_{a_{ijk}}(\varphi_{ik}) + (\tilde{\alpha}_{a_{ijk}})_*(A_k) \qquad \text{on } U_i \cap U_j \cap U_k. \tag{1.3}$$

A pair (A_i, B_i) satisfying the conditions stated above is called a Γ-connection. We denote differential Γ-cocycles by (g, a, A, B, φ) A differential Γ-cocycle has two curvatures associated to it. The "3-curvature" is

$$\mathrm{curv}(A_i, B_i) := dB_i + \alpha_*(A_i)(B_i) \in \Omega^3(U_i, \mathfrak{h}). \tag{1.4}$$

The "fake curvature" is

$$\mathrm{fcurv}(A_i, B_i) := dA_i + \frac{1}{2}[A_i, A_i] - \tau_*(B_i) \in \Omega^2(U_i, \mathfrak{g}). \tag{1.5}$$

A Γ-cocycle is called flat if both curvatures vanish.

A flat differential Γ-cocycle encodes the necessary local data to construct a principal Γ-2-bundle with flat connection, that is the content of the following lemma:

Lemma 1.3. Given a differential Γ-cocycle defined on an open covering of M, we can construct a principal Γ-2-bundle with a flat connection over M.

Proof. Let $\mathcal{P}$ be the bundle obtained from Lemma 1.2. We define a connection on $\mathcal{P}$ as follows:

$$\begin{aligned}
\Omega^a|_{U_i \times G} &:= \mathrm{Ad}^{-1}_{\mathrm{pr}_G}(A_i) + \mathrm{pr}_G^*\theta \\
\Omega^b|_{(U_i \cap U_j) \times H \times G} &:= (\alpha_{\mathrm{pr}_G^{-1}})_*(\mathrm{Ad}^{-1}_{\mathrm{pr}_H}(\varphi_{ij}) + (\tilde{\alpha}_{\mathrm{pr}_H})_*(A_j) + \mathrm{pr}_H^*\theta) \\
\Omega^c|_{U_i \times G} &:= -(\alpha_{\mathrm{pr}_G^{-1}})_*(B_i).
\end{aligned}$$

The forms defined above meet the criteria required of a flat connection on $\mathcal{P}$. □

2 From Local Systems to Principal 2-Bundles

Here we present a construction that assigns to each ∞-local system over M, a principal 2-bundle with a flat connection. At the level of objects the construction is merely the frame bundle construction, which requires only the vector bundle structure of the ∞-local system. To perform the construction at the level of morphisms we will rely on the cochain complex structure of the fibers provided by the flat $\mathbb{Z}$-graded connection. Its worth noting that this construction yields a simple class of principal 2-bundles.

The first step toward the frame 2-bundle construction is to obtain a Lie 2-group out of the fibres of an ∞-local system. For that purpose we rely on the following lemmas due to Faria-Picken [12].

Lemma 2.1. Let (V, ∂) be a cochain complex. There is a Lie crossed module denoted $\Gamma(V, \partial) = (H, G, \tau, \alpha)$ where:

1. $G = GL^0(V)$, the group of automorphisms of complexes of (V, ∂).
2. $H = GL^{-1}(V)$, where
$$GL^{-1}(V) = \frac{\mathrm{End}^{-1}(V)'}{[\partial, \mathrm{End}^{-2}(V)]}.$$
Here $\mathrm{End}^{-1}(V)'$ is the space of degree -1 endomorphisms of V such that $[\partial, h] + \mathrm{id}$ is invertible. The group structure is given by
$$h * h' := h + h' + h[\partial, h'].$$
3. $\tau : H \to G$ is given by $\tau(h) := [\partial, h] + \mathrm{id}$.
4. $\alpha : G \to \mathrm{Aut}(H)$ is such that $\alpha_g(h) = ghg^{-1}$.

Lemma 2.2. Let (V, ∂) be a cochain complex. The differential Lie crossed module associated to $\Gamma(V, \partial)$ is $\gamma(V, \partial) = (\mathfrak{h}, \mathfrak{g}, \tau_*, \alpha_*)$ where:

1. $\mathfrak{g} = \mathrm{gl}^0(V)$ is the vector space of degree zero cochain maps $V \to V$, endowed with the commutator bracket.
2. $\mathfrak{h} = \mathrm{gl}^{-1}(V)$, with
$$\mathrm{gl}^{-1}(V) = \frac{\mathrm{End}^{-1}(V)}{[\partial, \mathrm{End}^{-2}(V)]}.$$
This quotient is endowed with the Lie bracket
$$[T, S] = S\partial T - T\partial S + ST\partial - TS\partial;$$
and $[\partial, \mathrm{End}^{-2}(V)]$ is the ideal generated by elements of the form $\partial h - h\partial$.
3. $\tau_* : \mathfrak{h} \to \mathfrak{g}$ is given by $\tau_*(S) := \tau S + S\tau$.
4. The action α_* is such that $\alpha_*(R)(S) = RS - SR$.

The proof of Lemmas 2.1 and 2.2 may be found in [12] and [3].

Remark 2.1. When we interpret the crossed module $\Gamma(V, \partial)$ as a Lie 2-group, we have that a pair $(g, h) : g \to g'$ is a morphism if $g' = \tau(h)g$. Unraveling this equation leads to $g' - g = [\partial, hg]$, which means that hg is a homotopy from g' to g.

Definition 2.1. A cochain complex bundle is a graded vector bundle $p : E \to M$ with a morphism of bundles $\partial : E^\bullet \to E^{\bullet+1}$ such that $\partial^2 = 0$. The local trivializations are required to be isomorphisms of complexes of vector bundles on each fibre, i.e., there is a complex (V, ∂) such that if E is trivial over a certain open $U \subset M$, then the trivialization map $\varphi_U : p^{-1}(U) \to U \times V$ restricts to an isomorphism of complexes $\varphi_x : E_x \to (V, \partial)$.

Lemma 2.3. Given a cochain complex bundle (E, ∂), there is an associated principal 2-bundle $\mathcal{F}(E, \partial)$ which we call the frame 2-bundle.

Proof. Consider a bundle of complexes $(E, \partial) \to M$ and a point $x_0 \in M$. We let $(V, \partial) = (E_{x_0}, \partial_{x_0})$ and $\Gamma = \Gamma(V, \partial) = (H, G, \tau, \alpha)$. We define a groupoid $\mathcal{F}(E, \partial)$ and an action $\mathcal{F}(E, \partial) \times \Gamma \to \mathcal{F}(E, \partial)$ as follows:

1. For every $x \in M$, let

$$\mathrm{Obj}(\mathcal{F}(E, \partial))_x := \{\rho : V \to E_x \mid \rho \text{ is an isomorphism of complexes of degree } 0\}.$$

   We set $\mathrm{Obj}(\mathcal{F}(E, \partial)) = \bigsqcup_{x \in M} \mathrm{Obj}(\mathcal{F}(E, \partial))_x$. The projection $\pi : \mathrm{Obj}(\mathcal{F}(E, \partial)) \to M$ is such that $\pi^{-1}(x) = \mathrm{Obj}(\mathcal{F}(E, \partial))_x$.

2. Let $\rho, \rho' \in \mathrm{Obj}(\mathcal{F}(E, \partial))_x$. Define the set

$$\mathrm{Hom}_{\mathcal{F}(E,\partial)}(\rho, \rho') := \{\xi : V \to E_x \mid \partial_x \xi + \xi \partial = \rho - \rho'\} / \sim$$

   where $\xi \sim \xi'$ if they are homotopic. So morphisms between ρ and ρ' are homotopy classes of homotopies between them. The fact that being homotopic is an equivalence relation provides the identity morphisms (reflexiveness), every homotopy is invertible (symmetry) and a way to compose homotopies (transitiveness). The source and target maps are denoted s and t respectively. Given a homotopy $\xi \in \mathrm{Hom}_{\mathcal{F}(E,\partial)}(\rho, \rho')$ for $\rho, \rho' \in \mathrm{Obj}(\mathcal{F}(E, \partial))_x$, we define $\pi(\xi) = \mathrm{id}_x$.

Next we define an action $R : \mathcal{F}(E, \partial) \times \Gamma \to \mathcal{F}(E, \partial)$. On objects it is defined by composition: $R : \mathrm{Obj}(\mathcal{F}(E, \partial)) \times G \to \mathrm{Obj}(\mathcal{F}(E, \partial))$ is such that $(\rho, g) \mapsto R(\rho, g) = \rho g$. At this level the action is clearly free and transitive along the fibres. Suppose $\xi : \rho \to \rho'$, then we have that $(\xi, (h, g)) : (\rho, g) \to (\rho', \tau(h) g)$ is a morphism in $\mathcal{F}(E, \partial) \times \Gamma$. We define $R : \mathrm{Mor}(\mathcal{F}(E, \partial)) \times (H \ltimes G) \to \mathrm{Mor}(\mathcal{F}(E, \partial))$ as

$$(\xi, (h, g)) \mapsto R(\xi, (h, g)) = \xi \tau(h) g - s(\xi) h g.$$

Let us check that $R(\xi, (h, g))$ is indeed a morphism $\rho g \to \rho' \tau(h) g$:

$$\begin{aligned} [\partial, R(\xi, (h, g))] =& [\partial, \xi \tau(h) g - \rho h g] \\ =& [\partial, \xi] \tau(h) g - \rho [\partial, h] g \\ =& \rho g - \rho' \tau(h) g. \end{aligned}$$

Let us check that the action just defined makes $\mathcal{F}(E,\partial)$ a principal Γ-2-bundle, i.e. the functor $(\mathrm{pr}, R) : \mathcal{F}(E,\partial)\times\Gamma \to \mathcal{F}(E,\partial)\times_M \mathcal{F}(E,\partial)$ is a weak equivalence. Since the action is transitive along the fibres on objects, the functor (pr, R) is surjective. Next we check that the functor is fully faithful: consider objects (ρ, g) and (ρ', g') in $\mathcal{F}(E,\partial)\times\Gamma$. In order to have morphisms $(\rho, g)\to(\rho', g')$ there must be at least one $h \in H$ such that $\tau(h)g = g'$. If this is the case we have a function

$$(\mathrm{pr}, R) : \mathrm{Hom}_{\mathcal{F}(E,\partial)\times\Gamma}((\rho, g),(\rho', g')) \to \mathrm{Hom}_{\mathcal{F}(E,\partial)\times_M\mathcal{F}(E,\partial)}((\rho, \rho g),(\rho', \rho' g')).$$

The function is injective: take morphisms $(\xi,(h, g))$ and $(\xi',(h', g))$, and suppose they map to the same morphism, this is $(\xi, \xi\tau(h)g - s(\xi)hg) = (\xi', \xi'\tau(h')g - s(\xi')h'g)$. Clearly we must have $\xi = \xi'$. The equation from the second component may be rewritten as

$$\xi(\tau(h) - \tau(h')) = s(\xi)(h - h').$$

Next we notice that, since $\tau(h)g = \tau(h')g$ we have $\tau(h) = \tau(h')$, therefore we conclude that $h = h'$ and the function is injective.
The function is surjective: take a pair of homotopies $\xi : \rho \to \rho'$ and $\sigma : \rho g \to \rho' g'$. Define $h = \rho^{-1}\xi g' g^{-1} - \rho^{-1}\sigma g^{-1}$. We have

$$\begin{aligned}\tau(h)g =&[\partial, \rho^{-1}\xi g' - \rho^{-1}\sigma] + g\\ =&\rho^{-1}[\partial, \xi]g' - \rho^{-1}[\partial, \sigma] + g\\ =&\rho^{-1}(\rho - \rho')g' - \rho^{-1}(\rho g - \rho' g') + g = g'.\end{aligned}$$

Hence $(\xi,(h, g)) \in \mathrm{Hom}_{\mathcal{F}(E,\partial)\times\Gamma}((\rho, g),(\rho', g'))$. Furthermore, we have

$$R(\xi,(h, g)) = \xi\tau(h)g - \rho hg = \xi g' - \xi g' + \sigma = \sigma,$$

proving surjectivity. □

Remark 2.2. As E is a cochain complex bundle, the usual cocycles g_{ij} defined from local trivialisations of the bundle take values in the group $G = GL^0(V)$. Setting $a_{ijk} = 0 \in H = GL^{-1}(V)$ is easy to check that we have a $\Gamma(V,\partial)$-cocycle (g, a). The principal 2-bundle built from this cocycle using Lemma 1.2 is precisely the frame 2-bundle $\mathcal{F}(E,\partial)$, making clear the statement that the frame 2-bundle construction yields only a simple class of principal 2-bundles.

Now we aim to build a flat connection on the frame 2-bundle from an ∞-local system. First we see that the flat $\mathbb{Z}$-graded connection of a local system provides the vector bundle with a cochain complex bundle structure:

Proposition 2.1. Let (E, D) be a local system over a connected n-manifold M with projection $\pi : E \to M$. Then E is a cochain complex bundle.

Proof. For an arbitrary $x_0 \in M$, choose a neighbourhood $U \subset M$ over which E is trivial and a coordinate chart $\psi : U \to \mathbb{R}^n$ that maps x_0 to the origin. Over U the connection takes

the form $D = d + \partial + \cdots$, where $\partial \in \Omega^0(U, \mathrm{End}^1(E|_U))$. Furthermore, since the connection is flat we have that $\partial^2 = 0$, which means that the fibers have a chain complex structure with differential ∂. Now for any other $x \in U$ let γ_x be a path in U connecting x to x_0 such that $\psi\gamma_x$ is a radial path connecting $\psi(x)$ to the origin. Parallel transport along these paths provides isomorphisms of chain complexes $T_{\gamma_x} : E_x \to E_{x_0}$. A local trivialisation as a cochain complex bundle over U is defined by $\varphi : \pi^{-1}(U) \to U \times E_{x_0}$, $\varphi(p) = (\pi(p), T_{\gamma_{\pi(p)}}(p))$.
The fact that M is connected guarantees that all the fibers over M are isomorphic as chain complexes. □

From Proposition 2.1 and Lemma 2.3 we get that every ∞-local system has an associated principal 2-bundle.

Theorem 2.1. Let (E, D) be an ∞-local system over a connected smooth manifold M. Then there is a flat connection Ω_D defined on the frame 2-bundle $\mathcal{F}(E, \partial)$.

Proof. First let us fix a point $x_0 \in M$ and call $(V, \partial) = (E_{x_0}, \partial_{x_0})$, the fiber over x_0 with its cochain complex structure. We will define a differential $\Gamma(V, \partial)$-cocycle from the $\mathbb{Z}$-graded connection D. Let $\{U_i\}_{i \in I}$ be an open covering of M such that the vector bundle E trivialises as a cochain complex bundle over each U_i with fiber (V, ∂). Let (g, a) be $\Gamma(V, \partial)$-cocycle defined in Remark 2.2. For differential components of the cocycle consider the local form of D over U_i which is $D = d + \sum_{k \geq 0} \omega_i^k$ where $\beta_i^0 = \partial \in \Omega^0(U_i, \mathrm{End}^1(V))$ and $\beta_i^k \in \Omega^k(U_i, \mathrm{End}^{1-k}(V))$. Let $\mathrm{pr} : \mathrm{End}^{-1}(V) \to \mathrm{End}^{-1}(V)/[\partial, \mathrm{End}^{-2}(V)]$ be the natural projection to the quotient, then we define

$$A_i = \omega_i^1 \in \Omega^1(U_i, \mathrm{End}^0(V)), \quad B_i = -\mathrm{pr} \circ \omega_i^2 \in \Omega^2(U_i, \mathfrak{h}).$$

The Maurer-Cartan equation for $\omega_i = \sum_{k \geq 0} \omega_i^k$ has the following implications:

- The term of degree one of the equation is $\partial\omega_i^1 = \omega_i^1\partial$, so ω_i^1 actually takes values in the algebra of cochain maps and $A_i = \omega_i^1 \in \Omega^1(U_i, \mathfrak{g})$.
- The term of degree two is $\partial\omega_i^2 + \omega_i^2\partial = d\omega_i^1 + \omega_i^1 \wedge \omega_i^1$. Projecting to the quotient and rewriting appropriately yields $\tau_*(B_i) = dA_i + [A_i, A_i]/2$, which means that the fake curvature of the $\Gamma(V, \partial)$-cocycle vanishes.
- The term of degree three is $d\omega_i^2 + \omega_i^1\omega_i^2 - \omega_i^2\omega_i^1 = \partial\omega_i^3 - \omega_i^3\partial$. Projecting to the quotient and rewriting we get $dB_i + \alpha_*(A_i)(B_i) = 0$, thus the 3-curvature also vanishes.

The final component of the cocycle is the form $\varphi_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{h})$, this one we take to be zero $\varphi_{ij} = 0$. Now suppose that $U_i \cap U_j$ is non-empty, then the forms ω_i and ω_j are gauge equivalent via the cocycle g_{ij}, this is $\omega_i = g_{ij}\omega_j g_{ji} + g_{ij} dg_{ji}$. After projecting onto the quotient, the terms of degree one and two of the gauge equivalence equation may be written respectively as

$$A_i = g_{ij} A_j g_{ji} + g_{ij} dg_{ji}, \quad \text{and} \quad B_i = g_{ij} B_j g_{ji}.$$

The previous equations are the conditions (1.1) and (1.2) required of a differential cocycle. Condition (1.3) is satisfied trivially. The proof is completed by applying Lemma 1.3. □

Remark 2.3. Fix a cochain complex bundle (E, ∂). Let $\mathsf{Loc}_\infty^{(E,\partial)}(M)$ denote the category of ∞-local systems over M such that their cochain complex bundle is isomorphic to (E, ∂). We have defined a map

$$F : \mathsf{Loc}_\infty^{(E,\partial)}(M) \to 2\text{-}\mathcal{B}un^f_{\mathcal{F}(E,\partial)}(M)$$
$$(E, D) \mapsto (\mathcal{F}(E, \partial), \Omega_D).$$

We will only be concerned with this map at the level of objects and will disregard the rest of the functorial structure.

3 Representations of the Fundamental 2-Groupoid

We summarize the 2-dimensional parallel transport constructed in Section 3.4 for ∞-local systems and the construction of parallel transport in principal 2-bundles given in [28]. Both notions of parallel transport are presented as representations of the fundamental 2-groupoid of a manifold.

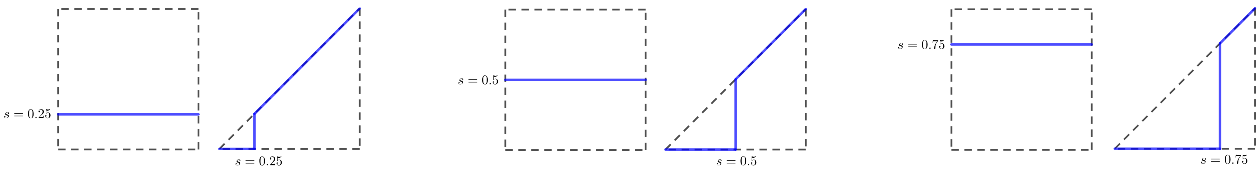
Definition 3.1. The fundamental 2-groupoid of a manifold M is the truncation up to dimension 2 of the ∞-groupoid defined in Example 2.3 and is denoted $\pi_{\leq 2}(M)$. In other words, the fundamental 2-groupoid is the simplicial set $X_k = \mathrm{Hom}_{\mathrm{PS}}(\Delta_k, M)$ for $k \leq 2$ where PS stands for piecewise smooth. The bicategory structure is as follows:

- The objects of $\pi_{\leq 2}(M)$ are the points of M, so $\mathrm{Obj}(\pi_{\leq 2}(M)) = \mathrm{Hom}_{\mathrm{PS}}(\Delta_0, M)$.
- A 1-morphism between the points $x, y \in M$ is a path connecting x to y, so $1 - \mathrm{Mor}(\pi_{\leq 2}(M)) = \mathrm{Hom}_{\mathrm{PS}}(\Delta_1, M)$. The composition of morphisms is the usual composition of paths.
- The 2-morphisms of $\pi_{\leq 2}(M)$ are fixed ends homotopies between paths modulo higher homotopies. Let us denote the vertices of Δ_2 by v_0, v_1 and v_2 and the edges by $[v_0, v_1]$, $[v_1, v_2]$ and $[v_0, v_2]$. Then a map $\sigma : \Delta_2 \to M$ is a fixed ends homotopy between the paths $\sigma([v_0, v_2])$ and $\sigma([v_0, v_1] * [v_1, v_2])$. Thus $2 - \mathrm{Mor}(\pi_{\leq 2}(M)) = \mathrm{Hom}_{\mathrm{PS}}(\Delta_2, M)$ with the usual vertical and horizontal compositions of homotopies.

Remark 3.1. To elaborate on how a 2-simplex $\sigma : \Delta_2 \to M$ may be considered a homotopy, recall the map $\Theta_2 : I^2 \to \Delta_2$ which is the adjoint map of $\Theta_{(2)}$ given in Definition 3.2 of Chapter 1. This map sends horizontal segments on I^2 into piecewise linear paths on Δ_2.

Therefore, for a 2-simplex $\sigma : \Delta_2 \to M$, the composition $\sigma \circ \Theta_2 : I^2 \to M$ is a homotopy with fixed ends between the paths $\sigma([v_0, v_2])$ and $\sigma([v_0, v_1] * [v_1, v_2])$.

Definition 3.2. Let (E, ∂) be a cochain complex bundle over M. The linear 2-groupoid of (E, ∂) is the 2-category $\mathcal{GL}(E, \partial)$ such that



- The objects are the fibers (E_x, ∂_x) for $x \in M$.
- The 1-morphisms are cochain complex morphisms $(E_x, \partial_x) \to (E_y, \partial_y)$ with the usual composition.
- The 2-morphisms are (algebraic) homotopies between morphisms of complexes, with the usual horizontal and vertical composition of homotopies.

A linear representation of $\pi_{\leq 2}(M)$ on (E, ∂) is a 2-functor $\mathcal{R} : \pi_{\leq 2}(M) \to \mathcal{GL}(E, \partial)$. Linear representations of $\pi_{\leq 2}(M)$ on (E, ∂) will be denoted $\mathrm{Rep}_{\leq 2}(M, (E, \partial))$.

Definition 3.3. Let Γ be a Lie 2-group and $\mathcal{P}$ a principal Γ-2-bundle over M. The Γ-torsor category of $\mathcal{P}$ is the 2-category $\Gamma\text{-Tor}(\mathcal{P})$ such that

- The objects of are the fibers $\mathcal{P}_x$ for $x \in M$. These are called Γ-torsors.
- The 1-morphisms are Γ-equivariant anafunctors $\mathcal{P}_x \Rightarrow \mathcal{P}_y$.
- The 2-morphisms are Γ-equivariant transformations between anafunctors.

A 2-group representation of $\pi_{\leq 2}(M)$ on $\mathcal{P}$ is a 2-functor $\mathcal{S} : \pi_{\leq 2}(M) \to \Gamma\text{-Tor}(\mathcal{P})$. 2-group representations of $\pi_{\leq 2}(M)$ on $\mathcal{P}$ will be denoted $\mathrm{Rep}_{\leq 2}(M, \mathcal{P})$.

The following theorem relates linear representations to 2-group representations.

Theorem 3.1. Let (E, ∂) be a cochain complex bundle over M. Then there is a map $H : \mathrm{Rep}_{\leq 2}(M, (E, \partial)) \to \mathrm{Rep}_{\leq 2}(M, \mathcal{F}(E, \partial))$.

Proof. Suppose $\mathcal{R} : \pi_{\leq 2}(M) \to \mathcal{GL}(E, \partial)$ is a linear representation, we will define the functor

$$H\mathcal{R} : \pi_{\leq 2}(M) \to \Gamma\text{-Tor}(\mathcal{F}(E, \partial)).$$

For every point $x \in M$, the fiber $\mathcal{F}(E, \partial)_x$ is a Γ-torsor, so we set $H\mathcal{R}(E_x) = \mathcal{F}(E, \partial)_x$.

If $\rho : x \to y$ is a path connecting x to y, then $\mathcal{R}(\rho) : E_x \to E_y$ is a morphism of complexes. The push forward $\mathcal{R}(\rho)_* : \mathcal{F}(E, \partial)_x \to \mathcal{F}(E, \partial)_y$ given by composition to the left with $\mathcal{R}(\rho)$ is a Γ-equivariant functor, hence it induces a Γ-equivariant anafunctor and we define $H\mathcal{R}(\rho) = \mathcal{R}(\rho)_*$. It can be checked easily that the functor preserves composition of paths.

Finally take paths $\rho, \rho' : x \to y$ and let $\xi : \rho \to \rho'$ be a homotopy between them. Then $\mathcal{R}(\xi) : \mathcal{R}(\rho) \to \mathcal{R}(\rho')$ is an (algebraic) homotopy. The homotopy $\mathcal{R}(\xi)$ induces a natural transformation between the functors $\mathcal{R}(\rho)_*, \mathcal{R}(\rho')_* : \mathcal{F}(E,\partial)_x \to \mathcal{F}(E,\partial)_y$ as follows: let $\alpha : V \to E_x$ be an object in $\mathcal{F}(E,\partial)_x$, then $\mathcal{R}(\xi)_*(\alpha) = \mathcal{R}(\xi)\circ\alpha : \mathcal{R}(\rho)_*(\alpha) \to \mathcal{R}(\rho')_*(\alpha)$ is a homotopy. Let us check that the push forward $\mathcal{R}(\xi)_*$ is indeed a natural transformation: consider a morphism $\mathcal{A} : \alpha \to \beta$ in $\mathcal{F}(E,\partial)_x$, then we have the following diagram

$$\begin{array}{ccc} \mathcal{R}(\rho)_*(\alpha) & \xrightarrow{\mathcal{R}(\rho)_*(\mathcal{A})} & \mathcal{R}(\rho)_*(\beta) \\ {\scriptstyle \mathcal{R}(\xi)_*(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathcal{R}(\xi)_*(\beta)} \\ \mathcal{R}(\rho')_*(\alpha) & \xrightarrow[\mathcal{R}(\rho')_*(\mathcal{A})]{} & \mathcal{R}(\rho')_*(\beta). \end{array}$$

A simple computation shows that

$$[\mathcal{R}(\xi)_*(\mathcal{A}),\partial] = (\mathcal{R}(\xi)\circ\alpha + \mathcal{R}(\rho')\circ\mathcal{A}) - (\mathcal{R}(\rho)\circ\mathcal{A} + \mathcal{R}(\xi)\circ\beta),$$

which shows that the diagram is commutative up to homotopy. Once again, since the natural transformation is defined by composition to the left, we can see that it is Γ-equivariant. We define $H\mathcal{R}(\xi) = \mathcal{R}(\xi)_*$. The fact that $H\mathcal{R}$ preserves the horizontal and vertical composition of 2-morphisms derives from the same fact for $\mathcal{R}$ and the properties of the push forward. □

3.1 Linear Representations From Local Systems

As seen in Chapter 1, a representation up to homotopy of $\pi_\infty(M)$ may be constructed from an ∞-local system over M. Restricting such a representation to the fundamental 2-groupoid we get a linear representation of $\pi_{\leq 2}(M)$. We will write explicitly the local formulae for this representation.

Suppose we have a trivial vector bundle $E = M \times V$ with V a $\mathbb{Z}$-graded vector space, and a flat, $\mathbb{Z}$-graded connection $d - \alpha$ where $\alpha \in \Omega(M, \mathrm{End}(V))$ is a Maurer-Cartan element. We know that $\alpha = \sum_i \alpha_i$ where $\alpha_i \in \Omega^i(M, \mathrm{End}^{1-i}(V))$. As shown in Section 3.4 of Chapter 1, applying the iterated integral map to α yields a Maurer-Cartan element $\beta \in C^\bullet(M)\otimes\mathrm{End}(V)$ with $\beta = \sum \beta_i$ and $\beta_i \in C^i(M)\otimes\mathrm{End}^{1-i}(V)$. We are interested in terms β_1 and β_2 of this form. For β_1 we have

$$\beta_1(\gamma) = \mathrm{id}_V + \sum_{n\geq 1}(-1)^{(n-1)n/2}\int_{\Delta_n}\pi_1^*\gamma^*\alpha_1\cdots\pi_n^*\gamma^*\alpha_1,$$

where $\gamma : \Delta_1 \to M$ is a path and $\pi_j : \Delta_k \to \Delta_1$ is the projection onto the j-th component. This term turns out to be the usual parallel transport with respect to α_1 along the inverse of the path γ. For the formula of β_2 we recall the map $\Theta_2 : I^2 \to \Delta_2$ of Remark 3.1. Next we consider the map $(\Theta_2)_{(n)} : \Delta_n \times I \to (\Delta_2)^n$ given by the formula

$$(\Theta_2)_{(n)}(t_1,\cdots,t_n,s) = (\Theta_2(t_1,s),\cdots,\Theta_2(t_n,s)).$$

If we denote by $p_i : (\Delta_2)^n \to \Delta_2$ the projection onto the i-th component and $\omega_i = \sigma^*\alpha_i$, then we have

$$\beta_2(\sigma) = \sum_{m,n\geq 0} (-1)^{m+n+1} \int_{\Delta_{m+n+1}\times I} (\Theta_2)^*_{(m+n+1)} \left(p_1^*(\omega_1)\cdots p_m^*(\omega_1)p_{m+1}^*(\omega_2)p_{m+2}^*(\omega_1)\cdots p_{m+n+1}^*(\omega_1)\right),$$

for a simplex $\sigma : \Delta_2 \to M$. Both $\beta_1(\gamma)$ and $\beta_2(\sigma)$ must be interpreted as maps going from the last vertex to the first vertex of γ and σ respectively. The relations of a representation up to homotopy regarding β_1 and β_2 are merely stating that we have a 2-functor $\pi_{\leq 2}(M) \to \mathcal{GL}(E, \partial)$.

Remark 3.2. We have just described a map $I : \mathsf{Loc}_\infty^{(E,\partial)}(M) \to \mathrm{Rep}_{\leq 2}(M, (E, \partial))$ for any fixed cochain complex bundle (E, ∂).

3.2 2-Group Representations From Principal 2-Bundles

Similarly to the case of ∞-local systems and linear representations, a principal Γ-2-bundle with flat connection can be used to construct a 2-group representation of the fundamental 2-groupoid. We summarize the construction which appeared in [28]. We begin defining the 2-group representation locally.

From Lemma 1.3 we know that a principal Γ-2-bundle with flat connection is locally determined by a differential Γ-cocycle. So let us consider a 2 group $\Gamma = (H, G, \tau, \alpha)$ with differential Lie crossed module $\gamma = (\mathfrak{h}, \mathfrak{g}, \tau_*, \alpha_*)$, and the principal Γ-2-bundle $\mathcal{P} = M_{dis} \times \Gamma$. Let (A, B) be a flat Γ-connection over M, this is $A \in \Omega^1(M, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{h})$ are forms such that the curvatures (1.4) and (1.5) vanish. A representation of $\pi_{\leq 2}(M)$ in Γ is defined as follows

- Let $\gamma : \Delta_1 \to M$ be a piecewise smooth path. Let $g_\gamma : \Delta_1 \to G$ be the path that satisfies the initial condition $g_\gamma(0) = \mathrm{id}_G$ and the differential equation

$$\frac{dg_\gamma(t)}{dt} = -\left(R_{g_\gamma(t)}\right)_* A\left(\frac{d\gamma}{dt}\right) \tag{3.1}$$

  where $(R_g)_*$ is the differential at $\mathrm{id}_G \in G$ of the map given by right multiplication with g.
  We define the smooth functor $\mathrm{Hol}_\gamma : \mathcal{P}_{\gamma(0)} \Rightarrow \mathcal{P}_{\gamma(1)}$ at the level of objects by $\mathrm{Hol}_\gamma(\gamma(0), g) = (\gamma(1), g_\gamma(1)g)$, which is clearly a G-equivariant map. If $(\mathrm{id}_{\gamma(0)}, h, g) : (\gamma(0), g) \to (\gamma(0), \tau(h)g)$ is a morphism, we define $\mathrm{Hol}_\gamma(\mathrm{id}_{\gamma(0)}, h, g) = (\gamma(1), \alpha(g_\gamma(1), h), g_\gamma(1)g)$. This is indeed a morphism $g_\gamma(1)g \to g_\gamma(1)\tau(h)g$ since

$$\tau(\alpha(g_\gamma(1), h))g_\gamma(1)g = g_\gamma(1)\tau(h)g_\gamma(1)^{-1}g_\gamma(1)g = g_\gamma(1)\tau(h)g.$$

  Thus Hol_γ is a Γ-equivariant functor.

- Let $\sigma : \Delta_2 \to M$ be a 2-simplex and $\Sigma := \sigma \circ \Theta_2 : I^2 \to M$. Denote by γ_0 and γ_1 the paths corresponding to $\sigma([v_0, v_1] * [v_1, v_2])$ and $\sigma([v_0, v_2])$ respectively. We write $\Sigma_s(t) = \Sigma(s, t)$, note that $\Sigma_0 = \gamma_0$ and $\Sigma_1 = \gamma_1$. The holonomy Hol_σ is a Γ-equivariant transformation $\mathrm{Hol}_\sigma : \mathrm{Hol}_{\gamma_0} \Rightarrow \mathrm{Hol}_{\gamma_1}$ defined as follows: let $h_\sigma : [0,1] \to H$ be the path that satisfies the initial condition $h_\sigma(0) = \mathrm{id}_H$ and the differential equation

$$\frac{dh(s)}{ds} = \left(L_{h(s)}\right)_* \left(\int_0^1 \alpha \left(g_{\Sigma_s(t)^{-1}}\right)_* \left(B\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right)\right) dt \right), \tag{3.2}$$

where $(L_h)_*$ is the differential at $\mathrm{id}_H \in H$ of the map given by left multiplication with h. The transformation is defined by $\mathrm{Hol}_\sigma(\sigma(v_0), g) = (\sigma(v_2), h_\sigma(1), g_{\gamma_0} g) : (\sigma(v_2), g_{\gamma_0} g) \to (\sigma(v_2), g_{\gamma_1} g)$.

Now for the global representation we consider a principal Γ-2-bundle $\mathcal{P}$ over M with flat connection Ω. For each path $\gamma : [0,1] \to M$ and each 2-simplex $\sigma : \gamma \Rightarrow \gamma'$ in M, we define an anafunctor $\mathrm{Hol}_\gamma : \mathcal{P}_{\gamma(0)} \to \mathcal{P}_{\gamma(1)}$ and a Γ-equivariant transformation $\mathrm{Hol}_\sigma : \mathrm{Hol}_\gamma \Rightarrow \mathrm{Hol}_{\gamma'}$ respectively.

For paths $\gamma : [0,1] \to M$ the idea is to define the set $\mathrm{Hol}_\gamma(\lambda)$ with respect to a fixed subdivision λ of $[0,1]$, then a smooth manifold Hol_γ is constructed by taking a direct limit over the possible subdivisions λ.

Definition 3.4. Let $\pi : \mathcal{P} \to M_{\mathrm{dis}}$ be a principal Γ-bundle with a connection $\Omega = (\Omega^a, \Omega^b, \Omega^c)$.

- A path $\beta : [a, b] \to \mathrm{Obj}(\mathcal{P})$ is *horizontal*, if $\Omega^a(\beta'(t)) = 0$ for all $t \in [a, b]$.
- A path $\rho : [a, b] \to \mathrm{Mor}(\mathcal{P})$ is *horizontal*, if $\Omega^b(\rho'(t)) = 0$ for all $t \in [a, b]$.

Definition 3.5. For $0 < n \in \mathbb{N}$, let $T_n := \{(t_i)_{i=0}^n \mid 0 = t_0 < t_1 < \cdots < t_n = 1\}$ be the set of possible n-fold subdivisions of the interval $[0,1]$. For $\lambda \in T_n$ we define the set

$$\begin{aligned} \mathrm{Hol}_\gamma(\lambda) := \{(\{\rho_i\}_{i=0}^n, \{\gamma_i\}_{i=1}^n) \mid \rho_i \in \mathrm{Mor}(\mathcal{P}),\ \gamma_i : [t_{i-1}, t_i] \to \mathrm{Obj}(\mathcal{P}) \text{ are horizontal paths,} \\ \pi \circ \gamma_i = \gamma|_{[t_{i-1}, t_i]},\ t(\rho_i) = \gamma_{i+1}(t_i) \text{ and } s(\rho_i) = \gamma_i(t_i)\}/\sim . \end{aligned}$$

We think about the elements of $\mathrm{Hol}_\gamma(\lambda)$ as formal compositions of paths in $\mathrm{Obj}(\mathcal{P})$ and morphisms in $\mathrm{Mor}(\mathcal{P})$, using the notation $\xi = \rho_0 * \gamma_1 * \rho_1 * \cdots * \gamma_n * \rho_n$ for a representative ξ of an element in $\mathrm{Hol}_\gamma(\lambda)$. The relation $\sim$ is the relation generated by $\{\sim_j\}_{1 \le j \le n}$ where

$$\rho_0 * \gamma_1 * \cdots * \gamma_n * \rho_n \sim_j \rho'_0 * \gamma'_1 * \cdots * \gamma'_n * \rho'_n$$

if there exist a horizontal path $\tilde{\rho} : [t_{j-1}, t_j] \to \mathrm{Mor}(\mathcal{P})$ such that:

- $\gamma_j = s(\tilde{\rho})$, $\gamma'_j = t(\tilde{\rho})$ and $\gamma'_i = \gamma_i$ for all $1 \le i \le n$, $i \ne j$,
- $\rho'_{j-1} = \tilde{\rho}(t_{j-1}) \circ \rho_{j-1}$, $\rho'_j = \rho_j \circ \tilde{\rho}(t_j)^{-1}$ and $\rho'_i = \rho_i$ for all $0 \le i \le n, i \ne j, j-1$.

The anchor maps are given by:

$$\alpha_l : \mathrm{Hol}_\alpha(\lambda) \to \mathcal{P}_{\gamma(0)} \;:\; \rho_0 * \gamma_1 * \cdots * \gamma_n * \rho_n \mapsto s(\rho_0),$$

$$\alpha_r : \mathrm{Hol}_\alpha(\lambda) \to \mathcal{P}_{\gamma(1)} \;:\; \rho_0 * \gamma_1 * \cdots * \gamma_n * \rho_n \mapsto t(\rho_n).$$

The left $\mathcal{P}_{\gamma(0)}$-action $\mathrm{Mor}(\mathcal{P}_{\gamma(0)})\ {}_s\times_{\alpha_l} \mathrm{Hol}_\gamma(\lambda) \to \mathrm{Hol}_\gamma(\lambda)$ and right $\mathcal{P}_{\gamma(1)}$-action $\mathrm{Hol}_\gamma(\lambda)\ {}_{\alpha_r}\times_t \mathrm{Mor}(\mathcal{P}_{\gamma(1)}) \to \mathrm{Hol}_\gamma(\lambda)$ on the set $\mathrm{Hol}_\gamma(\lambda)$ are defined by

$$\rho \circ (\rho_0 * \gamma_1 * \cdots * \gamma_n * \rho_n) := (\rho_0 \circ \rho^{-1}) * \gamma_1 * \cdots * \gamma_n * \rho_n,$$

$$(\rho_0 * \gamma_1 * \cdots * \gamma_n * \rho_n) \circ \rho := \rho_0 * \gamma_1 * \cdots * \gamma_n * (\rho^{-1} \circ \rho_n).$$

The $\mathrm{Mor}(\Gamma)$-action $\mathrm{Hol}_\gamma(\lambda) \times \mathrm{Mor}(\Gamma) \to \mathrm{Hol}_\gamma(\lambda)$ is given by

$$(\rho_0 * \gamma_1 * \cdots * \gamma_n * \rho_n) \cdot (h, g) := R(\rho_0, (h^{-1}, \tau(h)g)) * R(\gamma_1, g) * \cdots * R(\gamma_n, g) * R(\rho_n, g).$$

Finally, to define the smooth manifold Hol_γ consider the set $T := \bigsqcup_{n\in\mathbb{N}} T_n$ which is directed by inclusion, i.e. $\lambda \leq \lambda'$ if $\lambda \subset \lambda'$. If $\lambda \leq \lambda'$ then we have a map $f_{\lambda,\lambda'} : \mathrm{Hol}_\gamma(\lambda) \to \mathrm{Hol}_\gamma(\lambda')$ defined by adding identities $\rho_i = \mathrm{id}$ and splitting γ_i in two parts, at all points of λ' that are not in λ. The anafunctor Hol_γ is the direct limit of the direct system of sets $\{\mathrm{Hol}_\gamma(\lambda)\}_{\lambda\in T}$. The anchor maps and the actions defined on each $\mathrm{Hol}_\gamma(\lambda)$ descend to Hol_γ.

Proposition 3.1. The smooth manifold Hol_γ together with the anchor maps α_l and α_r, the actions $\mathrm{Mor}(\mathcal{P}_{\gamma(0)})\ {}_s\times_{\alpha_l} \mathrm{Hol}_\gamma(\lambda) \to \mathrm{Hol}_\gamma(\lambda)$, $\mathrm{Hol}_\gamma(\lambda)\ {}_{\alpha_r}\times_t\ \mathrm{Mor}(\mathcal{P}_{\gamma(1)}) \to \mathrm{Hol}_\gamma(\lambda)$ and $\mathrm{Hol}_\gamma(\lambda) \times \mathrm{Mor}(\Gamma) \to \mathrm{Hol}_\gamma(\lambda)$, define a Γ-equivariant anafuntor $\mathrm{Hol}_\gamma : \mathcal{P}_{\gamma(0)} \to \mathcal{P}_{\gamma(1)}$.

Before proceeding to the construction of the 2-dimensional holonomies we need the following definition:

Definition 3.6. A smooth bigon in M is a smooth map $\Sigma : I^2 \to M$ such that $\Sigma(s, 0) = x$ and $\Sigma(s, 1) = y$ for all $s \in I$. In other words, a bigon is a fixed-ends homotopy between the paths $\gamma(t) = \Sigma(0, t)$ and $\gamma'(t) = \Sigma(1, t)$. In this case we write $\Sigma : \gamma \Rightarrow \gamma'$.

A bigon $\Sigma : \gamma \Rightarrow \gamma'$ is called small, if there exist $n \in \mathbb{N}$, $\lambda \in T_n$ and sections $\sigma_i : U_i \to \mathrm{Obj}(\mathcal{P})$ defined on open sets U_i such that

$$\Sigma(\{(s,t) \mid t_{i-1} \leq t \leq t_i, 0 \leq s \leq 1\}) \subset U_i.$$

To define the holonomy along a 2-simplex $\sigma : \Delta_2 \to M$ we first consider the bigon $\Sigma = \sigma \circ \Theta_2 : \gamma \to \gamma'$. The idea is to subdivide Σ in small bigons Σ_i and then define a Γ-equivariant transformation $\varphi^{\mathrm{small}}_{\Sigma_i} : \mathrm{Hol}_{\gamma_{i-1}} \to \mathrm{Hol}_{\gamma_i}$ between the parallel transports along γ_{i-1} and γ_i, for each small bigon Σ_i in the subdivision of Σ. Finally the Γ-equivariant transformation for Σ is defined as the composition $\varphi_\Sigma(s) := \varphi^{\mathrm{small}}_{\Sigma_n} \circ \cdots \circ \varphi^{\mathrm{small}}_{\Sigma_1}$.

Definition 3.7. Let $\Sigma : \gamma \Rightarrow \gamma'$ be a bigon and $\xi \in \mathrm{Hol}_\gamma$. A *horizontal lift* of Σ with source

ξ is a tuple $(n, \lambda, \{\Phi_i\}_{i=1}^n, \{\rho_i\}_{i=1}^n, \{g_i\}_{i=1}^n)$ consisting of $n \in \mathbb{N}$, a subdivision $\lambda \in T_n$ and smooth maps

- $\Phi_i : [0,1] \times [t_{i-1}, t_i] \to \mathrm{Obj}(\mathcal{P})$,
- $\rho_i : [0,1] \to \mathrm{Mor}(\mathcal{P})$ with ρ_0 and ρ_n constant,
- $g_i : [0,1] \to G$ with $g_i(0) = 1$,

such that the following conditions are satisfied:

1. Φ_i is a lift of Σ, i.e., $\pi \circ \Phi_i = \Sigma|_{[0,1]\times[t_{i-1},t_i]}$ for all $1 \le i \le n$.
2. $t(\rho_i(s)) = \Phi_{i+1}(s, t_i)$ for all $0 \le i < n$ and $s(\rho_i(s)) = R(\Phi_i(s,t_i), g_i(s))$ for all $1 \le i \le n$.
3. The paths $\gamma_i'(t) := \Phi_i(1,t)$, $\nu_i(s) := \Phi_i(s, t_{i-1})$ and ρ_i are horizontal for all $1 \le i \le n$.
4. $\xi = \rho_n * \gamma_n * \cdots * \gamma_1 * \rho_0$ with $\gamma_i(t) := \Phi_i(0,t)$ and $\rho_i := \rho_i(0)$.

Lemma 3.1. For every small bigon $\Sigma : \gamma \Rightarrow \gamma'$ and every $\xi \in \mathrm{Hol}_\gamma$ there exists a horizontal lift with source ξ.

Finally, for an arbitrary bigon $\Sigma : \gamma \Rightarrow \gamma'$ there exists a subdivision $s \in T_n$ such that the pieces $\Sigma_i(s,t) := \Sigma((s_i - s_{i-1})s + s_{i-1}, t)$ are small. Then we define

$$\varphi_\Sigma(s) := \varphi_{\Sigma_n}^{\text{small}} \circ \cdots \circ \varphi_{\Sigma_1}^{\text{small}}.$$

The map $\varphi_\Sigma(s)$ is independent of the choice of s.

Proposition 3.2. The map $\varphi_\Sigma : \mathrm{Hol}_\gamma \to \mathrm{Hol}_{\gamma'}$ is a Γ-equivariant transformation.

Proposition 3.3. Let $\mathcal{P}$ be a principal Γ-2-bundle with flat connection Ω. Then the assignments $x \mapsto \mathcal{P}_x$, $[\gamma] \mapsto \mathrm{Hol}_{[\gamma]}$, and $[\Sigma] \mapsto \varphi_{[\Sigma]}$ form a 2-functor

$$\mathrm{tra}_{\mathcal{P}} : \pi_{\le 2}(M) \to \Gamma\text{-}\mathrm{Tor}(\mathcal{P}).$$

The equivalence between the local and global definitions of holonomies may be found in Section 5 of [28]. The main observation is that the global construction of holonomies relies on lifting paths and surfaces horizontally, which is done locally by solving the differential equations (3.1) and (3.2).

Remark 3.3. The parallel transport we have just described is a map $T : 2\text{-}\mathcal{B}un^f_{\mathcal{P}}(M) \to \mathrm{Rep}_{\le 2}(M, \mathcal{P})$ for any fixed Γ-2-bundle $\mathcal{P}$.

4 Comparison of 2-Dimensional Holonomies

We have seen how 2-dimensional holonomies can be constructed in two different settings: for ∞-local systems and for principal 2-bundles with flat connection. In the case of ∞-local systems, the holonomies may be used to define a linear representation of the fundamental 2-groupoid. For principal 2-bundles with flat connection, we use holonomies to define a 2-group representation. Furthermore, we have a way to construct principal 2-bundles with flat connection from ∞-local systems and a way to construct 2-group representations from linear representations. All the constructions are regarded as maps as stated in Remarks 2.3, 3.2, 3.3 and Theorem 3.1. Putting this maps together we get the square

$$\begin{array}{ccc} \mathsf{Loc}_\infty^{(E,\partial)}(M) & \xrightarrow{F} & 2\text{-}\mathcal{B}un^f_{\mathcal{F}(E,\partial)}(M) \\ {\scriptstyle I}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \mathrm{Rep}_{\leq 2}(M,(E,\partial)) & \xrightarrow[H]{} & \mathrm{Rep}_{\leq 2}(M,\mathcal{F}(E,\partial)) \end{array} \tag{4.1}$$

which leads us to our final theorem:

Theorem 4.1. Diagram (4.1) is commutative.

Since holonomies are compatible under composition, it is enough to prove the commutativity of the diagram locally. The local commutativity has already been proven in [3], here we review the main ideas of the proof.

Consider a local system $(M \times V, d-\omega)$ where $\omega = \partial + \omega_1 + \omega_2 + \cdots$. We will write $\Gamma(V,\partial) = \{H, G, \tau, \alpha\}$. The associated principal 2-bundle $\mathcal{F}(E,\partial)$ has a connection determined by the pair (A, B) where

$$A = \omega_1, \qquad B = -\mathrm{pr} \circ \omega_2,$$

according to the proof of Theorem 2.1. The paths $g_\gamma : \Delta_1 \to G$ and $h_\sigma : \Delta_1 \to H$ that solve the differential equations (3.1) and (3.2) respectively are used to define a 2-group representation as follows:

- For a path $\gamma : \Delta_1 \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$, a functor $\mathrm{Hol}_\gamma : \mathcal{F}(E,\partial)_x \to \mathcal{F}(E,\partial)_y$ is defined by multiplication to the left with the element $g_\gamma(1)$.

- For a 2-simplex $\sigma : \Delta_2 \to M$ such that its 0th vertex is x and its last vertex is y, multiplication to the left with the element $h_\sigma(1)$ yields a natural transformation.

On the other hand, the iterated integral map provides forms $\beta_1 \in C^1(M) \otimes \mathrm{End}^0(V)$ and $\beta_2 \in C^2(M) \otimes \mathrm{End}^{-1}(V)$. A linear representation is defined as follows:

- For a path $\gamma : \Delta_1 \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$, the map $\mathrm{hol}_\gamma : \{x\} \times V \to \{y\} \times V$ is defined by $(x, v) \mapsto (y, \beta_1(\bar{\gamma})(v))$.

- For a 2-simplex $\sigma : \Delta_2 \to M$ such that its 0th vertex is x and its last vertex is y, the map $\mathrm{hol}_\sigma : \{x\} \times V \to \{y\} \times V$ is defined by $(x, v) \mapsto (y, \beta_2(\bar{\sigma})(v))$.

Applying Theorem 3.1 to this representation we get the 2-group representation:

- The functor $(\mathrm{hol}_\gamma)_* : \mathcal{F}(E,\partial)_x \to \mathcal{F}(E,\partial)_y$ is the push forward under the map hol_γ.
- The push forward $(\mathrm{hol}_\sigma)_*$ is a natural transformation.

To prove that both 2-group representations coincide, it is enough to construct solutions to Equations (3.1) and (3.2) using the forms β_1 and β_2:

- For Equation (3.1), let γ_s be the restriction of γ to the subinterval $\Delta_1(s)$ of Δ_1. We know that β_1 actually takes values in the group G, therefore we may define $g_\gamma(s) = \beta_1(\bar{\gamma}_s)$ for $s \in [0,1]$. Note that $g_\gamma(1) = \beta_1(\bar{\gamma})$.
- For Eequation (3.2), let σ_s be the restriction of σ to $\Delta_2(s) \subset \Delta_2$. We define $h_\sigma(s) = \mathrm{pr}(\beta_2(\bar{\sigma}_s))$, where pr is the projection onto the quotient $\mathrm{End}^{-1}(V)'/[\partial, \mathrm{End}^{-2}(V)] = H$. Also note that $h_\sigma(1) = \beta_2(\bar{\sigma})$.

Equation (3.1) is the usual parallel transport equation and $\beta_1(\bar{\gamma}_s)$ is the usual holonomy along the inverse of the path γ_s. Thus it is clear that both representations coincide for paths. The following results provide the proof that the representations coincide for 2-simplices.

Lemma 4.1. If $h : [0,1] \to H$ is a solution to Equation (3.2), then

$$h(1) = \left(\int_0^1 \int_0^1 \alpha(g_{\Sigma_s}(t)^{-1})_* \left(B\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) \right) \mathrm{Hol}_{\Sigma_s}^{-1} dt ds \right) \mathrm{Hol}_{\Sigma_1}.$$

Proof. Equation (3.2) may be rewritten as

$$\begin{aligned}
\frac{dh(s)}{ds} &= \left(L_{h(s)}\right)_* \left(\int_0^1 \alpha\left(g_{\Sigma_s(t)^{-1}}\right)_* \left(B\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) \right) dt \right) \\
&= \int_0^1 \alpha\left(g_{\Sigma_s(t)^{-1}}\right)_* \left(B\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) \right) dt \\
&\quad + h(s) \left(\int_0^1 \mathrm{Ad}_{g_{\Sigma_s}(t)^{-1}} \tau_*(B) \left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) dt \right) \\
&= \int_0^1 \alpha\left(g_{\Sigma_s(t)^{-1}}\right)_* \left(B\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) \right) dt + h(s)\mathrm{Hol}_{\Sigma_s}^{-1} \frac{d\mathrm{Hol}_{\Sigma_s}}{ds}.
\end{aligned}$$

Therefore we get the following equation

$$\frac{d\left(h(s)\mathrm{Hol}_{\Sigma_s}^{-1}\right)}{ds} = \left(\int_0^1 \alpha\left(g_{\Sigma_s(t)^{-1}}\right)_* \left(B\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) \right) dt \right) \mathrm{Hol}_{\Sigma_s}^{-1}.$$

Integrating the previous equation leads to the desired result. □

Lemma 4.2. Define

$$a_s(t) := A_{\Sigma(s,t)}\left(\frac{\partial \Sigma}{\partial t} \right) \quad \text{and} \quad b_s(t) := B_{\Sigma(s,t)}\left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right).$$

Then the integral

$$\int_0^1 \int_0^1 \alpha(g_{\Sigma_s}(t)^{-1})_* \left(B \left(\frac{\partial}{\partial t}\Sigma_s(t), \frac{\partial}{\partial s}\Sigma_s(t) \right) \right) \mathrm{Hol}_{\Sigma_s}^{-1} dt ds \tag{4.2}$$

is equal the iterated integral

$$\sum_{m,n\geq 0} \int_{\Delta_{m+n+1}\times I} a_s(1-t_1)\cdots a_s(1-t_m) b_s(1-t_{m+1}) a_s(1-t_{m+2})\cdots a_s(1-t_{m+n+1}) dt_1 \cdots dt_{m+n+1} ds.$$

Proof. A solution to Equation (3.1) in terms of iterated integrals is

$$g_\gamma(t) = \mathrm{id} + \sum_{n\geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} a(t_1)\cdots a(t_n) dt_1 \cdots dt_n.$$

Similarly we have

$$g_\gamma(t)^{-1} = \mathrm{id} + \sum_{n\geq 1} (-1)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} a(t_1)\cdots a(t_n) dt_1 \cdots dt_n.$$

and

$$g_{\bar{\gamma}}(t) = \mathrm{id} + \sum_{n\geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} a(1-t_1)\cdots a(1-t_n) dt_1 \cdots dt_n.$$

replacing the previous expressions in (4.2) yields the result. □

Proposition 4.1. The integral

$$\sum_{m,n\geq 0} \int_{\Delta_{m+n+1}\times I} a_s(1-t_1)\cdots a_s(1-t_m) b_s(1-t_{m+1}) a_s(1-t_{m+2})\cdots a_s(1-t_{m+n+1}) dt_1 \cdots dt_{m+n+1} ds.$$

can be written as

$$\sum_{m,n\geq 0} (-1)^{m+n} \int_{\Delta_{m+n+1}\times I} \mu_{(m+n+1)}^* (p_1^*\Sigma^*\omega_1 \cdots p_m^*\Sigma^*\omega_1 p_{m+1}^*\Sigma^*(-\mathrm{pr}(\omega_2)) p_{m+2}^*\Sigma^*\omega_1 \cdots p_{m+n+1}^*\Sigma^*\omega_1), \tag{4.3}$$

where $\mu_{(k)} : \Delta_k \times I \to (I^2)^{\times k}$ is defined by

$$\mu_{(k)}(t_1, \cdots, t_k, s) = ((1-t_1, s), \cdots, (1-t_k, s)).$$

Replacing $\Sigma = \sigma \circ \Theta_2$ in Equation (4.3) we get

$$\sum_{m,n\geq 0} (-1)^{m+n+1} \int_{\Delta_{m+n+1}\times I} (\Theta_2)_{(m+n+1)}^* \left(p_1^*(\omega_1)\cdots p_m^*(\omega_1) p_{m+1}^*(\omega_2) p_{m+2}^*(\omega_1)\cdots p_{m+n+1}^*(\omega_1) \right),$$

which is $\beta_2(\sigma)$.

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