The figure 113 shows an example of a DB with a repulsive zone next to an attracting sliding zone. Their common border (closed curve in fig.) has all point of double- tangency type (or of one of the other groups of points).

A trajectory from the centre toward the outside of the DB has the following sequence of points: .

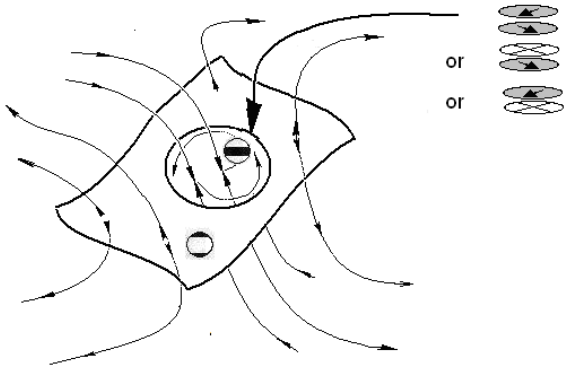


Figure 113. DB with dynamics of the type and .

Figure 114 shows an example of a DB with crossing zone from the region Z_1 to Z_2 next to an attracting sliding zone. Their common border (closed curve in fig.) has points of single tangency type (or of one of the other groups of points).

If we consider a curve of points in the DB starting inside the border and ending outside the border we get the following sequence of symbols: .

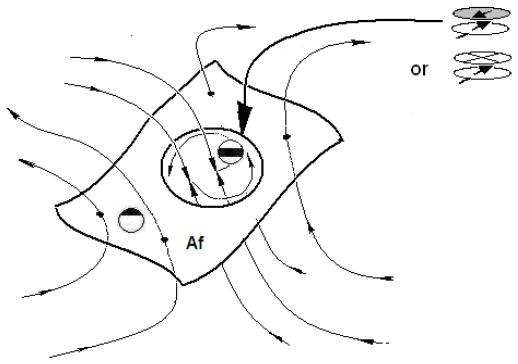


Figure 114. DB with dynamics of the type and .

Figure 115 shows an example of a DB with a crossing zone from the region Z_2 to Z_1 next to an attracting sliding zone. Their common border (closed curve in fig.) has all points of single tangency type (or of one of the other groups of points).

If we consider a curve of points in the DB starting inside the border and ending outside the border we get the following sequence of symbols: .

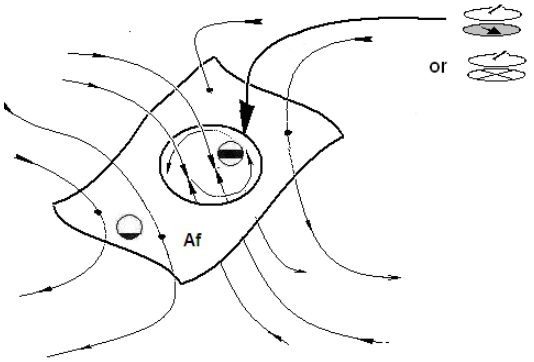


Figure 115. DB with dynamics of the type and .

Figure 116 shows an example of a DB with a crossing zone from the region Z_1 to Z_2 next to a repulsive zone. Their common border (closed curve in fig.) has all points of the single tangency type (or of one of the other groups of points).

If we consider a curve of points in the DB starting inside the border and ending outside the border we get the following sequence of symbols: .

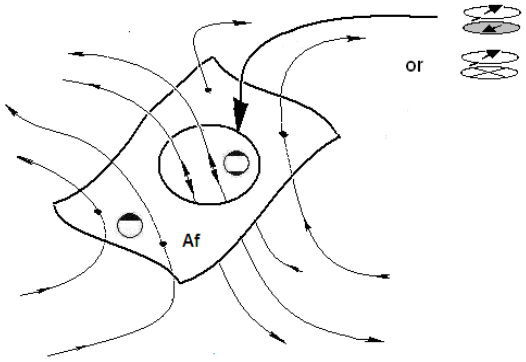
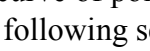


Figure 116. DB with dynamics of the type and .

Figure 117 shows an example of a DB with a crossing zone from the region Z_2 to Z_1 next to a repulsive zone. Their common border (closed curve in fig.) has all points of the single tangency type (or of one of the other groups of points).

If we consider a curve of points in the DB starting inside the border and ending outside the border we get the following sequence of symbols:  .

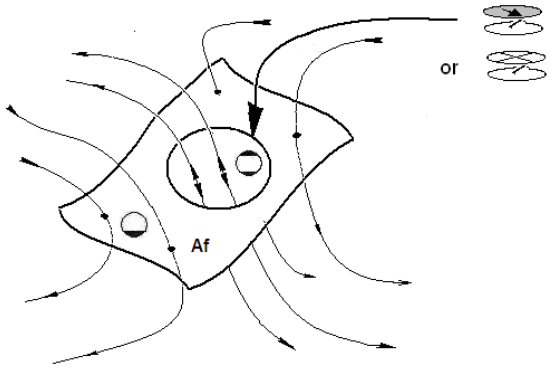


Figure 117. DB with dynamics of the type and .

Thus far, there are four individual types of dynamics and eight classes of DBs with two types of dynamics. Any change from one to another after a change in the value of a parameter is considered a bifurcation.

If we take each case and we add the additional groups presented previously, we realise that there are too many possibilities to be described with detail. Instead, we will depict the classification procedure.

Let
1. Ψ_A be an area in the DB of a system with only one class of dynamics in the DB.
2. Ψ_f and Ψ_g be the areas in the DB of a system with two types of dynamics in the DB.

With these types of areas, there is only one bifurcation of regions.

$$\Psi_A \rightarrow \Psi g \cup \Psi f \qquad (4.153)$$

or (reversing the parameter change)

$$\Psi g \cup \Psi f \rightarrow \Psi_A \qquad (4.154)$$

By combining this bifurcation of regions with the possibilities listed earlier, we can generate all the types of bifurcations taking into account the dynamics of the zones in the DB. See table 4.4.

Table 4.4. Possibilities of change from one dynamics into two dynamics and vice versa.

No.	Structure initial/final	Structure Final/initial
1	Ψ_A	Ψ_g + Ψ_g
2	Ψ_A	Ψ_g + Ψ_g
3	Ψ_A	Ψ_g + Ψ_g
4	Ψ_A	Ψ_g + Ψ_g
5	Ψ_A	Ψ_g + Ψ_g
6	Ψ_A	Ψ_g + Ψ_g
7	Ψ_A	Ψ_g + Ψ_g
8	Ψ_A	Ψ_g + Ψ_g
9	Ψ_A	Ψ_g + Ψ_g
10	Ψ_A	Ψ_g + Ψ_g
11	Ψ_A	Ψ_g + Ψ_g
12	Ψ_A	Ψ_g + Ψ_g

Identification of the type of dynamics in zones and their borders in the DB

The type of dynamics, the shape, and the size of a sector or an area in the DB are identified by using the area detector of the SPT method.

The area detector moves across all points of the DB in an iterative way. At each iteration one of variables of the DB is increased or decreased. Each iteration identifies the type of point from the catalogue developed in this work. This process yields a numeric and graphic map of the dynamics on the discontinuity surface. This procedure is used with a big stepsize in order to decrease the computational time although this can lead to a missing identification process. Thus, in this case, the code sequences developed in the first section of this chapter are used to ensure that, on the common border of two identified zones, the dynamics corresponds to one in the database.

Another option (see sector 4.6) is to use the continuation or tracking of singular and special points. The tracking method uses the knowledge of the codes for identifying the sequence of codes found in the neighbourhood of two types of dynamics. If this method is

started at a point which belongs to the common border of two zones, it only moves to a new position if it finds the same type of point. This method based on continuation only analyses a small fraction of points in the DB and therefore does not use so much computational time.

Chapter 5

The Singular points method in the identification of global bifurcations

Before starting the development of methods for the detection of bifurcations of cycles, which are periodic orbits including sliding segments, the characteristics that exhibit a sliding cycle are defined. Next we start an outline about the different definitions of bifurcation which are found in the literature regarding sliding systems. The objective is to compare the qualitative definitions with the capacity of the SPT method in order to propose equivalences between two sliding systems. Then, a group of cycles are taken and coded as sequences of elements in different regions of the system and the DBs as it was done in the previous chapter, using the SPT main ideas. With the cycle types that where mostly found in applications, we define their codes, and a bifurcation analysis method is started. Basically, the values of the parameter are changed and the cycles codes are analyzed as they are computed. The sequences of cycles presented become the standard code for each bifurcation and they can be used for tracking the dynamics of the system. Finally, bifurcations which are well-known in the literature are used for the validation of the proposed method (see also (86) (87) (88).

5.1 **Sliding bifurcations in limit cycles**

5.1.1 Characteristics of a sliding cycle

In smooth systems, the definition of a periodic cycle or orbit is related to a set of one type of element; a curve of points which belong to a region where the vector field is defined. Therefore, it only demands that the trajectory evolves by the same point $\boldsymbol{x}_0$ in the curve after a fixed period of time τ.

$$\Phi(\boldsymbol{x}_0, t_0) = \Phi(\boldsymbol{x}_0, (t_0 + \tau)) \tag{5.1}$$

In nonsmooth systems, the definition can be more complex. Consider a dynamical system where the whole state space has been divided into a set of sub-regions Z_j where the dynamics are represented by its own set of equations

$$\dot{x} = \{\mathbf{f}_1(\boldsymbol{x}, \boldsymbol{\alpha}), \mathbf{f}_2(\boldsymbol{x}, \boldsymbol{\alpha}), \mathbf{f}_3(\boldsymbol{x}, \boldsymbol{\alpha}), \ldots, \mathbf{f}_m(\boldsymbol{x}, \boldsymbol{\alpha})\} \quad (5.2)$$

with

$\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_{3,\ldots,} \mathbf{f}_m)$
$\boldsymbol{x} = (x_1, x_2, x_{3,\ldots,} x_n) \ \forall x \in Z_j$ with $j = (1, 2, \ldots, m)$
and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ the vector of parameters.

Each region Z_j with dynamics represented by the vector field $\mathbf{f}_j$ can contribute with a segment (or set of segments) of orbit Φ_j to the cycle. In systems where several sub-regions interact, a periodic orbit may or may not visit all the regions. Also it can visit the same region one or more times in a way that depends only on the evolution of the dynamics. Figure 118 illustrates cycles with different number of elements: in case (a), there is only one element; in case (b), we have four elements: one segment in each region and two different points in the DB. In case (c) we have also four elements. One segment is in region Z_1 and three elements are in the DB: one sliding segment and two special points. Finally, in case (d) we have eight elements with two visits to the DB.

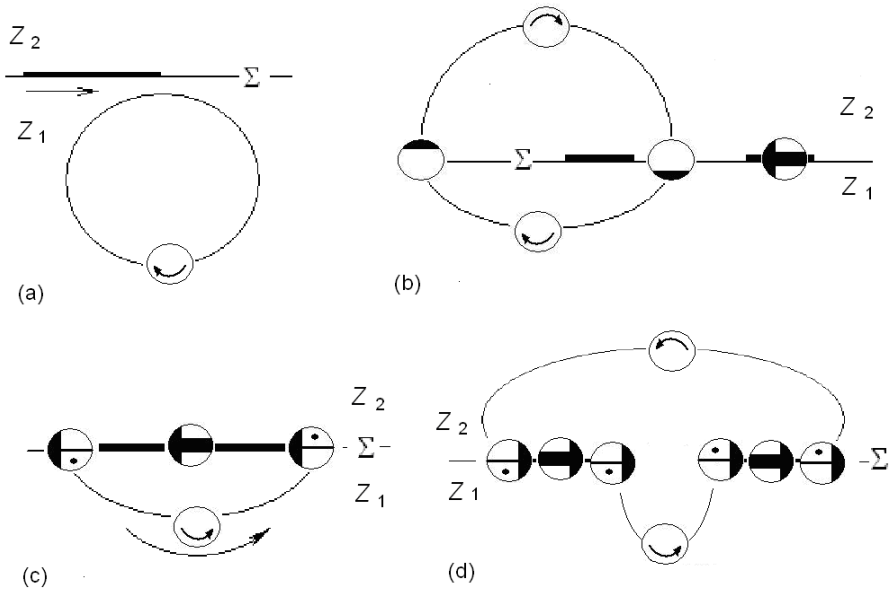


Figure 118. Examples of cycles with different number of elements.

In the case of the Filippov systems, the discontinuity surface defined by H is divided into two types of hyper-surfces (H-surf). In the first one, Ψs, we have sliding (attracting or escaping) dynamics and in the second one, Ψc, we have crossing dynamics.

If an orbit arrives to a crossing zone, it contributes with a point. It is also possible that an orbit gets to the zone defined by H in which there is sliding and in this way, generates a

segment of the orbit Φ_s. Later, the orbit reaches the limit of the sliding zone in a singular or special point and leaves the sliding zone. The last part of the evolution contributes with two elements: a sliding segment, and a singular or special point $\Omega_{sin_or_sp}$.

The sliding vector field $G_{1,2}$ in the discontinuity surface $\Sigma_{1,2}$ is defined by Filippov's method (57) or the equivalent Utkin's method (56). In the following, we take Filippov's solution

$$G_{1,2} = \lambda\, \mathbf{f}_1(\boldsymbol{x}) + (1-\lambda)\, \mathbf{f}_2(\boldsymbol{x})$$

where (5.6)

$$\lambda = \langle\, ,\ \mathbf{f}_2(\boldsymbol{x}) \rangle\ /\ \langle\ H_x(\boldsymbol{x}), \mathbf{f}_2(\boldsymbol{x}) - \mathbf{f}_1(\boldsymbol{x}) \rangle \quad H_x(\boldsymbol{x}) \tag{5.7}$$

As defined in (3) the set of crossing points Ωc is

$$\Omega c_{1,2} = \{\, \boldsymbol{x} \in \Sigma_{1,2} \text{ such that } \sigma(\boldsymbol{x}) > 0 \,\} \tag{5.8}$$

with

$$\sigma(x) = \{\langle\ H_x(\boldsymbol{x})\ ,\ \mathbf{f}_1(\boldsymbol{x})\rangle\ \langle\ H_x(\boldsymbol{x})\ ,\ \mathbf{f}_2(\boldsymbol{x})\rangle\,\} \tag{5.9}$$

For the crossing points we do not need the use of the sliding vector field. We only need the equations of the vector fields $\mathbf{f}_1$ and $\mathbf{f}_2$ since once the orbit evolving (with $\mathbf{f}_1$) in the Z_1 region arrives to the DB, the integration process switches to the vector field $\mathbf{f}_2$ and the orbit continues in the region Z_2.

Thus it is possible to describe the characteristics of a sliding cycle through the view of the method of sequences following the SPT method. A sliding cycle is then

1. A sequence of ordered elements (segments and points represented by symbols). Each sub-region Z_j generates a type of orbit Φ_j and each discontinuity surface Σ can generate four types of elements. Namely sliding segments (Φs), singular points ($\Omega_{singular}$), special points ($\Omega_{special}$) and crossing points (Ω_c).

2. The time taken by each sequence remains constant. Moreover, if we assume that a cycle are conformed by three segment, two normal Φ_1 and Φ_2 from the regions Z_1 and Z_2 and the other Φ_S from the DB between Z_1 and Z_2; and Ω_a, Ω_b and Ω_c are points belonging to a cycle; and Ω_a is a point of Φ_1, Ω_b a point of Φ_2 and Ω_c a point of Φ_S; and Γ is the time that takes the application of the evolution operator (Ó) to get again the same point.

   $$\Omega_a = \acute{O}\,(\Omega_a, \Gamma) \tag{5.10}$$

Then, it will be also true for any point belonging to any segment of the cycle

$$\Omega_b = Ø(\Omega_b, \Gamma) \quad (5.11)$$

$$\Omega_c = Ø(\Omega_c, \Gamma) \quad (5.12)$$

Remark: The number and types of elements required to complete a cycle depend on the complexity of the system and the definition does not exclude that at any time during the evolution of the system, the cycle is composed by only one type of segment.

5.1.2 Different definitions of bifurcation of sliding cycles

The study of periodic orbit bifurcations in nonsmooth systems of Filippov type is divided into standard, and non-standard bifurcations which are also called sliding bifurcations. The standard bifurcations have been widely studied and referenced (66), (65), (45). Each one of these authors has defined bifurcation in similar ways, but thus far, there is no unified concept (46); this leads to difficulties in the comparison of different contributions. Guckenheimer and Holmes (66) define bifurcation as a value of the parameter α when the system is not structurally stable. R. Seydel (65) defines bifurcation as a parameter value where the number of fixed points or period solutions changes. Y. Kuznetzov (45), in 1995, defined bifurcation as the appearance of a topologically non-equivalent phase portrait under a variation of parameters.

For non-standard bifurcations, which appear in non-smooth systems, there are also several definitions such as those by Kuznetzov *et al.* (4), Di Bernardo *et al.* (62), and Guardia *et al.* (58), which have been carefully studied. Thus we will take an operative qualitative definition of bifurcation for global sliding so-called bifurcations founded in the literature, such as with the intention of checking if the algorithms of the SPT method are able to detect those dynamics. Basically we will proceed in a similar way we did with local sliding bifurcations.

Next we outline the procedure of the SPT method when it is analyzing a cycle. We do so for performing a comparison with the definitions of bifurcation referenced before and decide if the method SPT could lose or add the detection of some type of bifurcation.

The procedure is as follows:

1. Select a point in the cycle where the analysis will be started.
2. Select the parameter range for analysis.

3. Set the size of the increment in the change of the parameter.
4. Run the integration process to collect all the different class of segments and points belonging to cycle.
5. Increment the value of parameter and repeat the fourth step until the value of the parameter have the value of the limit of the previously specified range.
6. Compare the changes in the sequences obtained in the fourth step with the information in the database of the global sliding bifurcations (which are shown in the following sections).

For each value of the parameter, the result is a sequence of coded data that shows the quantity, the class and the order of the points in the DB.

Thus, the equivalence between two cycles defined operatively is:

1. The quantity of components or elements in the sequence is the same. (the points and the segments are considered as elements)
2. The class of each element in the sequence is the same.
3. The order in the sequence of the components in the DB is the same (in fact, the order is implicitly taken into account in the definition of a sequence).

On the contrary, two systems are defined non-equivalent when any of the previous conditions fails.

Before we continue with the explanation of the abovementioned method, we will check the definitions of global sliding bifurcation presented in (3), (62) and (58) with our previously presented concept of non-equivalent systems.

Doing a comparison between the previous items of the definition of local sliding bifurcation and the rules in the procedure of the SPT method, we see that the main characteristic to check the conditions is the ability of the SPT method to detect regular and special point such as pseudo-equilibria, tangent, anticollinear, singular, and so on.

This task is efficiently done by the method. Also it is able to detect global sliding bifurcations in orbits as they are defined in (62). The first definition presented in (58), based in the concept of Σ-equivalence is perhaps one of the most strict but it is the one that better fits with the way that the method SPT work. Regarding this definition, additionally to the points previously mentioned, it is necessary to detect the arrival and departing points, which is also done by the SPT method.

We note in the following some details regarding the SPT method:

1. A cycle that crosses but does not visit the sliding region, compared with a cycle which crosses once but then hits just a point of the sliding region (grazing) seems to be very similar, but they are not equivalent.

$$^{S}C=\{(\Phi_1), (\Omega c_{(1,2)}), (\Phi_2), (\Omega c_{(2,1)})\} <\neq> \{(\Phi_1), (\Omega c_{(1,2)}), (\Phi_2), (\Omega s\ (+))\} \quad (5.13)$$

2. The quantity of elements of the sliding cycle (^{S}C) should also remain constant. Let Φ_1 and Φ_2 be the flow of a orbit in the region Z_1 and Z_2, respectively, $\Omega c_{(1,2)}$ and $\Omega c_{(1,2)}$ be the points in the DB when the orbit crosses from the Z_1 to the Z_2 regions and vice versa, $\Phi s_{(1,2)}$ be a segment of the DB with sliding dynamics, and $\Omega spe_{(2,1)}$ be a special point.

$$^{S}C=\{(\Phi_1), (\Omega c_{(1,2)}), (\Phi_2), (\Omega c_{(2,1)})\} <\neq> \{(\Phi_1), (\Omega c_{(1,2)}), (\Phi_2), \Phi s_{(1,2),}\ (\Omega_{spe(2,1)})\} \quad (5.14)$$

   Both dynamics may look similar in a special case but a careful study in the phase space shows a dramatically change that it is effectively registered in the sequence of elements presented in (5.14).

3. Finally, the order of each element in the sequence indexed from the initial point should not be changed. And also the direction of the evolution (preservation of time orientation) must be same.

$$^{S}C=\{(\Phi_1), (\Omega c_{(1,2)}), (\Phi_2), (\Omega c_{(2,1)})\} <\neq> \{(\Phi_2), (\Omega c_{(2,1)}), (\Phi_1), (\Omega c_{(1,2)}))\} \quad (5.15)$$

   The first cycle evolves in clockwise and the second in counter-clockwise direction.

In order to illustrate the conditions presented, assume a periodic ^{S}C cycle in a system with three vector fields in the regions Z_1, Z_2, and Z_3 separated by the DB $\Sigma_{1,2}$, $\Sigma_{1,3}$, and $\Sigma_{3,2}$. The cycles are composed by regular segments Φ_1, Φ_2, and Φ_3 and by sliding segments $\Phi s_{(1,2)}^{\leftarrow}$, $\Phi s_{(1,2)}^{\rightarrow}$, $\Phi s_{(1,3)}^{\leftarrow}$, $\Phi s_{(1,3)}^{\rightarrow}$, $\Phi s_{(3,2)}^{\leftarrow}$, and $\Phi s_{(3,2)}^{\rightarrow}$ in the DB. Assume that the orbit of the system is as it is shown in figure 119(a) and, after a variation in one parameter, it shows the following cycles (figure 119 (b), (c), (d), (e), and (f)).

a) We have a cycle $^{S}C_a = \{(\Phi_1),(\Omega c_{(1,2)}), (\Phi_2), (\Phi s_{(1,2)}), (\Omega_{singular\ (2,1)})\}$.

b) First condition fails; some elements belong to another region, different from the initial one. Then its classification varies. $^{S}C_b = \{(\Phi_1),(\Omega c_{(1,2)}), (\Phi_2), (\Phi s_{(1,3)}),(\Omega_{spe\ (3,1)})\}$.

c) First and third conditions fail; the cycle evolves in counter clockwise and thus the direction of time is not preserved.
   $^{S}C_a = \{(\Phi s_{(1,2)}), (\Omega_{singular\ (2,1)}) ,(\Phi_2) ,(\Omega c_{(2,1)}), (\Phi_1)\}$.

d) All conditions fail; the cycle is spread across three regions instead of two regions.

$^{S}C_d = \{(\Phi_2), (\Omega c_{(2,1)}), (\Phi_1)\ (\Omega c_{(1,2)}), (\Phi_2), (\Phi c_{(2,3)}), (\Phi_3), (\Phi s_{(3,2)}), (\Phi_{sin(3,2)})\}$.

e) Third condition fails; the order of the elements varies with relation to the initial conditions. $^{S}C_e = \{(\Phi_1), (\Phi s_{(1,2)}),(\Omega_{spe(2,1)}), (\Phi_2), (\Omega c_{(1,2)})\}$

f) Second condition fails; the number of elements varies with relation to the initial conditions. Also, the first condition fails.
$^{S}C_f = \{(\Phi_1),(\Omega c_{(1,2)}), (\Phi_{2a}), (\Phi s_{(1,2)a}),(\Omega_{spe(2,1)a}), (\Phi_{2b}), (\Phi s_{(1,2)b}),(\Omega_{spe(2,1)b})\}$

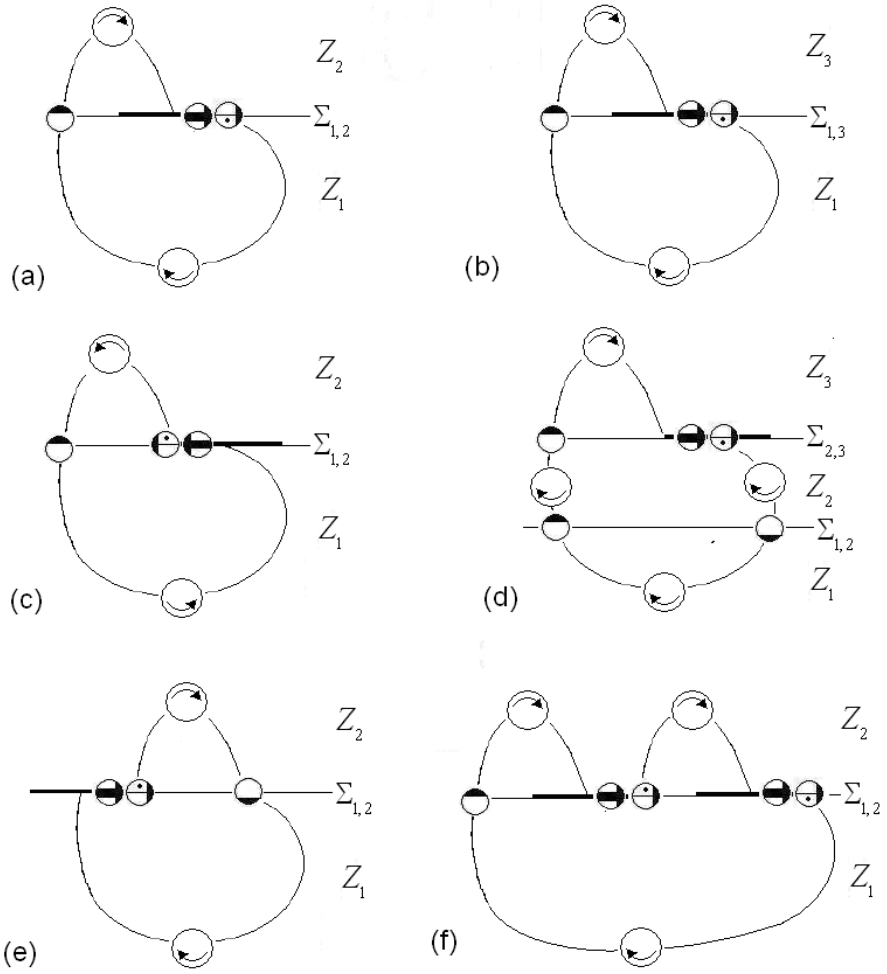


Figure 119. Phase portrait of cycles without equivalence among them.

It is worth to note that the condition for equivalence for sliding cycles must be also fulfilled in the DB. In order to emphasize this point we show in the following a second group of cycles presenting faults against these conditions. Note that within the DB of nonsmooth systems dimension three or higher, smooth bifurcations can also be found. Figure 120 illustrates four cycles. The system has two regions and a DB, which has

attracting sliding and crossing dynamics in both directions. In the four cases the dynamics evolving in the regions Z_1 and Z_2 are not modified but the dynamics in the DB effectively has been modified to make comparisons. In case (a), there is no cycle, there are several segments including sliding and evolution in both regions. In case (b), a cycle is born which has a part of it in each region and two sliding segments. The number of sliding segment changes from 4 to 2. In case (c), the cycle now has two visits to region Z_2, and as a result, the number of sliding segments changes from 2 to 3. In case (d), which is equivalent to case (b), we observe a cycle with two sliding segments but with a pattern of non-hyperbolic stability that needs to be analysed in relation to case (b) by using the basic definition of topological equivalence for smooth systems.

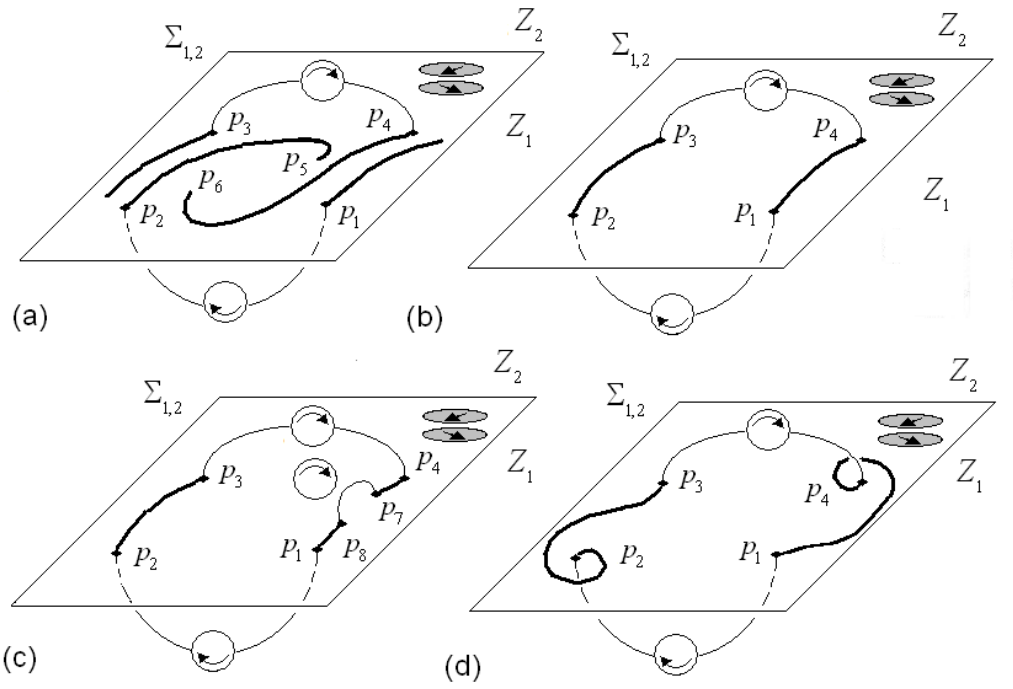


Figure 120. Cycles without SPT equivalence in the DB.

5.2 Symbols

From here, we will begin the description of standard or well-known cycles and other news cycles generated by synthesis. Because of the high number of elements forming a cycle and the length of the sequences of the elements, it is necessary to introduce various symbols to clearly describe these cycles. Next, we will list some of these symbols.

A complete orbit that only belongs to a single region is symbolized as $\Phi\Phi_1$ or $\Phi\Phi_2$. A sub-index 1 or 2 is assigned to indicate the regions to which the orbit belongs. The other alternatives of sub-indices are *i* and *j*.

In two- and three-dimensional systems, the displacement over the DB from the left side of the x-axis toward the right is defined as a positive sliding movement. The crossing point of one orbit through the boundary surface between two regions is symbolized as crossing from 1 to 2 by $\Omega c_{(1,2)}$ or Ωc ($\uparrow$) and crossing from 2 to 1 by $\Omega c_{(2,1)}$ or Ωc ($\downarrow$).

The points Ω_{SS}, Ω_{SU}, Ω_{SSV}, Ω_{SUV}, Ω_{STV}, Ω_{SVT}, Ω_{STT}, Ω_{SVV}, Ω_{ACS}, Ω_{ACU}, Ω_{SVP}, and Ω_{UVP} are symbols for singular and special points and will be used in the next section with the same definition as that in section 4.2.

The arrival of a regular segment to an intermediate point belonging to a sliding sector is symbolized as $\Omega s(\sim)$. The singular points at which an attractive sliding segment finished are symbolized as $\Omega_{SS(+)}$, $\Omega_{STV(+)}$, $\Omega_{SVT(+)}$, and $\Omega_{STT(+)}$.

The symbol ↵ means that it is an equation of a periodic orbit, and therefore, its sequence is repeated continuously. The symbol ¤ means that the orbit has arrived at a point where the dynamics stop. The symbols ‘ and ’ indicate the CW and CCW directions in the evolution of system.

The sequence of elements forming a cycle is symbolized as ${}^{S}C_{xx}$. A sequence of cycles that informs the presence of a bifurcation is symbolized as ${}^{S}\beta_{xx}$.

Detection of sliding bifurcation in limit cycles

Before continuing with the section on the synthesis of cycles and the coding of the well-known bifurcations, we consider important to briefly present the operating principle of the SPT method when it is automated to detect bifurcations. After that, additional information on the bifurcation in sliding cycles supplement the comprehension of the principle.

The main objective of an automatic detector of bifurcations is to realize the task of detection without the supervision of a person or an operator who in the absence of a detector, continuously looks for the evolution of the trajectory and knows when a bifurcation has occurred. The second objective is to track the bifurcation by continuation. In order to achieve these objectives, the proposed method requires a considerable amount of information: the data of the integration process, the theory presented in the previous chapter, the codes of the sequences of elements that conform each cycle, and the codes of the sequences of the cycles. The entire task has been divided into four steps. The first one is to fill the matrix that maintains the information of the sequence of elements that are appearing in the integration process. The second step is the comparison between the matrix of elements and the matrix containing a group of specific patterns of cycles. This has the

target of identifying what type of cycle is presented. The third step is the comparison of a sequence of cycles and for that a third matrix that stores a group of patterns is used. In that, each pattern represents a specific sequence of cycles that correspond to a bifurcation. Finally, in the continuation process, once a bifurcation is detected, the parameter values are changed infinitesimally with the objective of not moving away from the values of the bifurcation. Instead, the objective is to recover the bifurcation and in this way obtain a continuation of the phenomena.

The process is initialized with a trajectory, which is presented as a group of points coming from an integration process.

$$\boldsymbol{\Phi} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots\ldots, \boldsymbol{x}_n) \tag{5.16}$$

$$\boldsymbol{X}_n = \boldsymbol{x}_{n-1} + \int_0^{dt} \mathbf{f}\,(\boldsymbol{x}_{n-1}, dt) \tag{5.17}$$

Matrix of elements

In the first step, as shown in figure 121, after each iteration of the numerical integration, one point is added to the graphic scheme. The new point includes the values of the states, the amount of the time since the simulation started and the data of the vector fields involve in the dynamics. This method adds another data such as the type of point (crossing, attractive sliding, repulsive, isolated or singular from the list of points 36 points) that is met during each integration step.

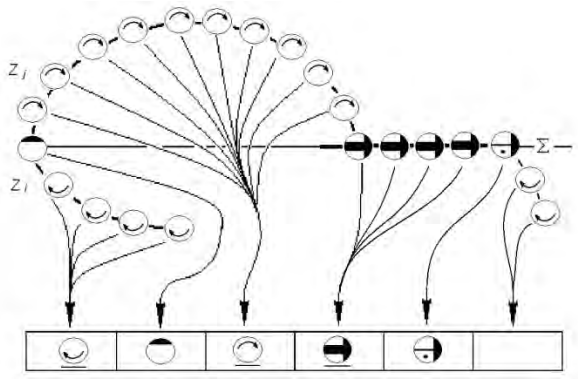


Figure 121. Filling sequence of the matrix with elements appearing in the simulation.

Before a point is recorded in a cell of the matrix of elements, it is compared with the point that was met immediately before. The data of a repeated point are not included in the matrix. Instead, the repetition of a point turns the point that was met immediately before into a piece of curve of the same type. This procedure is carried out with the objective of avoiding a situation in which the matrix is filled or saturated with the the same type data (which typically occurs at the beginning of the process). Figure 5.5 shows a schematic representation of the the matrix filling process. Note that a piece of curve is recognized in the figures by a symbol Φ (or a figure of point with an underline) and the points with a symbol Ω (or a figure).

For example, consider figure 5.5, and assume that the filling process has just collected the third element. In the third position of the matrix there is a point of the type . Now, if the following point in the integration process is of the same type, the type ., is not recorded in the 4th position of the matrix; however, if instead of a point , we had a point of the type , this point would be recorded in the 4th position of the matrix.

It is very important that the time and location of any nonsmooth event is solved accurately (11). To ensure that, if during the simulation, the trajectory hits the DB, an additional function is used. This function computes the trajectory in backward time after the hit and restarts the integration process with a smaller step time that guarantees (up until a prescribed precision) that the hitting point really falls on the DB. This procedure avoids accuracy problems (described in chapter 3) and, most important, it avoids losing the dynamics inside the DB. Once a point is detected in the DB another function tests if the point is crossing or sliding (or impacting). In the second case the integration is stopped and it is continued with the Filippov vector field while the dynamics remains sliding (or the DB of impact and its dynamics is activated).
In order to detect if a periodic limit cycle appears, the process checks if a part of the computed sequence in the matrix of elements is repeated. For illustrating this, assume for example that we have a sequence of a grazing cycle ΦΦi→Ωs (+), Then, after some time nΓ, the matrix of elements would contain a cycle with the sequence ΦΦi→Ωs (+)→ΦΦi→Ωs (+)→ΦΦi→Ωs (+)…, which is not fully correct. In order to avoid this situation, a function which performs the evaluation of when the element of the matrix in the position *n* is equal to the element in the position *(n/k)+1* runs in parallel with the integrator. *k* is a number which counts the elements of a specific cycle.

While picking elements for the matrix, other situations can be found which are not allowed by the SPT method. For example, it is impossible to accept the sequence Φi→Φj because it implies a change of region. In the change, a crossing point must be found, and an admissible sequence would be Φi→Ωij→Φj. Thus, a function that keeps only admissible sequences runs parallel to the integrator.

Comparison between matrix of elements and matrix of cycles

The second step regards the comparison between two matrices: the matrix of cycles and the matrix of elements. The first one stores sequences of elements for cycles which have been identified in literature. The second one stores the sequences of elements that are computed during the integration process. The integrator assumes that a cycle is completed when a sequence of elements is continuously repeated and the reference point of the time lapsed until the same element is repeated becomes constant. Then, as a cycle is detected, the SPT method increase or decrease a parameter value. The decision is taken according to a point of the limit cycle which lies in the DB and the selection of the point is based on a study of the diagram of the local dynamic generated by the SPT method, which is detailed in chapter 4. In some sense, this point should be the most characteristic point, i.e., it should be the point that it is suspected that will appear in all the cycles that will form the sequence of cycles of a bifurcation.

For example, in the case of the system illustrated in figure 5.5 which evolves in the CW direction and has a sliding piece on the right of the DB, the best moment to change the value of the parameter is when the trajectory is close to the singular point of the right of the sliding piece.

If the integrator is programmed to look for a cycle, the procedure is slightly different. In this case, the integrator takes the number of elements that the cycle which is being considered has and reserves the same quantity of cells for storing the same elements during the integration process. Then, as illustrated infigure 5.6, as a new element appears, a comparison is carried out until all the elements of the stored cycle are identical to the elements that are picked up from the integration. Assume k is the index of the pattern matrix. For each new element υ, which could be a unitary element Ω or a group of points Φ entering into the collection matrix, the integration is stopped and a condition is checked for:

$$\upsilon_j = \upsilon_k \tag{5.18}$$

If both elements match, the counter is incremented.

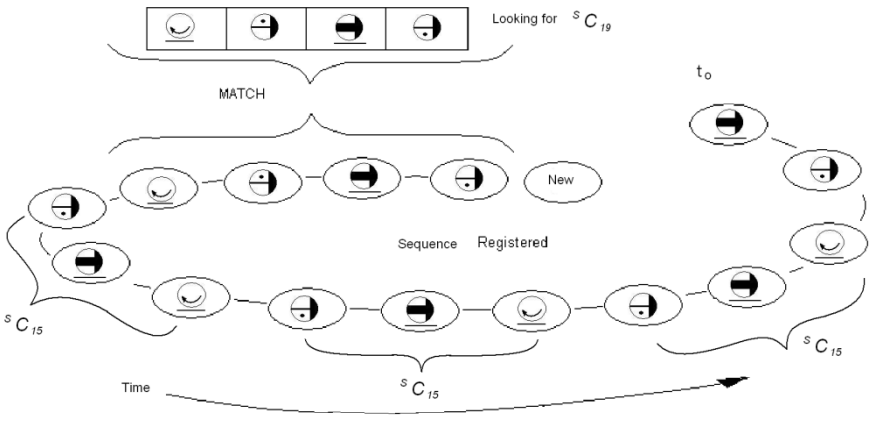


Figure 122. Matching operation.

Comparison between the matrix of sequence of cycles and the bifurcations pattern matrix

The proposed method can be used in two cases. The first one, when it is not known much about the phenomenon and it is wanted to detect which sequences of cycles are present as a parameter is varied within a range. The second one, when we are looking for a specific sequence that corresponds to a known bifurcation.

In the first case, the integrator uses a matrix that stores information regarding the cycles that are found during the time that the method is active. Fig.5.7 illustrates the procedure. Each time the integrator detects if a sequence is being repeated, it stores the information of the cycle, and changes the parameter value in order to continue with the next identification. If a sequence cycle is repeated although the parameter value is varied, the information is not recorded.

The second case is more complex because a double comparison must be performed: one which is similar to the described in the previous case, and the other one, which regards to the comparison of the sequence of cycles to a specific sequence, until a full matching is detected. In this second case, we need to guarantee that there is no equivalence between two consecutive cycles and, therefore, the matrix of the sequences of cycles must have stored various sequences that were previously theoretically studied and that correspond to well-known bifurcations.

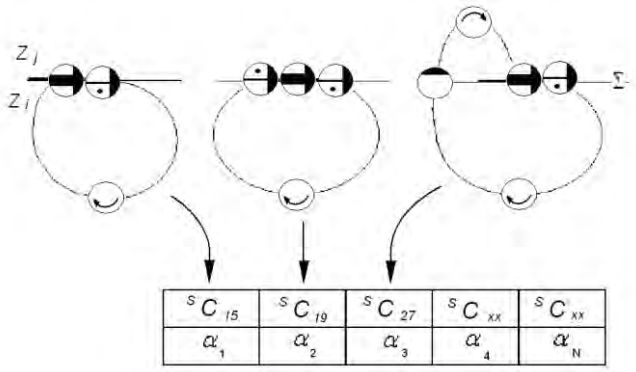


Figure 123. Sequence of tracking a cycle bifurcation.

Cycles continuation

Cycle continuation for systems with two or more parameters is still possible using the cycle sequences. Methods in the spirit of predictor-corrector shown in (81) and (65) can be used. In this case the function which predicts is the sequence of elements of the cycle that generates the bifurcation, and the sequences of the other corresponding two cycles are used for the correction. Once the parameter values that generate the bifurcation cycle are computed, the continuation can be initialized.

In the prediction step, a tangent or secant scheme can be tried, taking the last computed data of a cycle ${}^{S}C_5$. Here, in order to show the use of the sequences of cycles in the method, we consider the case where the bifurcation curve has always a positive slope and has no branches.

With the aim of deciding towards which cycle the systems evolve as the parameters are increased or decreased, a sensibility analysis is performed. For example, the bifurcation corresponding to ${}^{S}C_S$ has a sequence of cycles $({}^{S}C_3)\,[{}^{S}C_4]({}^{S}C_5)$. As it is shown in Fig. 4.72, both parameters are alternatively varied. Assume that a direct proportional sensibility exists for parameter α_1. This implies that a small increment in the parameter value tends to change the cycle into ${}^{S}C_5$ and a small decrement tends to change the cycle into ${}^{S}C_3$. Changing α_1, the cycle ${}^{S}C_4$ is obtained. Then, the second parameter α_2 is decreased (in this case the initial point has a high value). After the change in parameter α_2, the cycle ${}^{S}C_4$ changes to ${}^{S}C_3$ or to ${}^{S}C_5$. In the first case, the continuation algorithm increases α_1 until the cycle type ${}^{S}C_5$ is found again. In the second case, the algorithm acts conversely. The process is continuously iterated until the prescribed final value of parameter α_2 is reached.

5.4 Synthesis of cycles

In this section, the method of synthesis is defined or standardized for a large set of cycles that will serve in the method, in the task of identifying the sequences of elements forming a cycle. These standardized cycles in turn become a part of the sequences of cycles that form the bifurcations. In this group of cycles, the cycles that belong to the well-known bifurcations are also included. Therefore, instead of synthesis, the cycles are broken into their constitutive elements, defined, and classified according to the functions and characteristics of these elements.

Although the explanation and the cycles defined here are simple, the method is oriented and presents its best advantages in complex high-dimensional and multi-DB systems as is observed in the example analyzed in 158] (see Appendix).

In the synthesis process, all basic elements are used to making combinations with the objective of obtaining all possible cycles. After a cycle is completed, it is tested to verify that the sequence of elements is not contrary to the rules related to the sliding or impact dynamics. Initially, some cycles will not have a physical system associated with them as has happened before with equations that have waited decades until some scientific system was found to be described with them. Complementarily, some cycles will not have a mathematical representation, and they will be retired from the classification.

From the operational point of view, at the beginning, the synthesis will seem useless in comparison with the generation of cycles using the practical knowledge because the latter looks easier, but as the complexity of the cycles increases, it is impossible to manage the problem without systematizing it. Recently, Dercole (82) has proposed a method based on previous knowledge, named Puzzle, which operates by assuming how a cycle could evolve to generate other cycles. Further, he has built a catalogue of the possible asymptotic behaviours of a second-order Filippov system. To obtain a wide database, the ideal procedure would be to have the collector of cycles and sequences of cycles installed several years ago in a dynamics simulator of some nonsmooth systems. Nevertheless, because of this deficiency and calculating that as the order of the systems is increased, the visualization of the cycles is arduous, the synthesis described above is an alternative method for obtaining cycles.

To recollect, we distinguished as elements the segments of the orbit and the special points on the DB. The segments were classified as orbits evolving in a certain vector field that is recognized by a sub-index. The points belonging to the DB were classified as crossing, attractive sliding, special and singular according to the classification given in section 4.2. The sliding points form segments with the same name.

Without a loss of generality, and to be synthetic, some rules should be taken: figures that do not have an in-depth explanation will be drawn as two-dimensional figures. The right side of the DB is always taken as the starting point. The synthesis begins by using the least number of elements, and then, more elements are added.

Note: In this section, the figures present as vector fields Z_i and Z_j with sub-indices i = 1 and j = 2.

Cycles confirmed by one, two, and three elements

Following the procedure, we should use the minimum number of elements.

Cycles confirmed by one element

An orbit evolving in one of the vector fields without making contact with the limit boundary. (See figure 124).

Isolated orbits in Z_1 or Z_2

$$^{S}C_1 \Rightarrow \{`(\Phi\Phi_1)\hookleftarrow\} \quad (5.19)$$

$$^{S}C_2 \Rightarrow \{`(\Phi\Phi_{2)}\hookleftarrow\} \quad (5.20)$$

The other two possibilities are presented when the system evolves in the CCW direction.

$$^{S}C_3 \Rightarrow \{`(\Phi\Phi_1)\hookleftarrow\} \quad (5.21)$$

$$^{S}C_4 \Rightarrow \{`(\Phi\Phi_2)\hookleftarrow\} \quad (5.22)$$

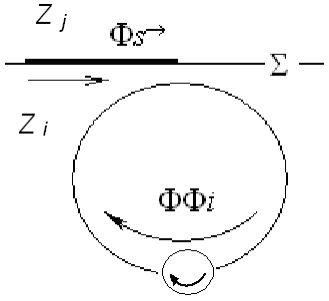


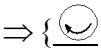
Figure 124. Sequence of a limit cycle type C_1.

Tangent cycles

The following two cases use two elements. In these cases, the orbit only touches a point of the sliding segment.

There are orbits tangents in Z_1 or in Z_2 (see figures 125 and 126).

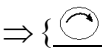
$^{S}C_5 \Rightarrow \{`(\Phi_1)\ (\Omega_{STV\rightarrow}{}^{X})\hookleftarrow\}$
or

$^{S}C_5 \Rightarrow \{$  $\}$ (5.23)

and

$^{S}C_6 \Rightarrow \{$ ‘(Φ_2) $(\Omega_{SVT}{}^{\leftarrow}{}_X)$ $\}$

or

$^{S}C_6 \Rightarrow \{$  $\}$ (5.24)

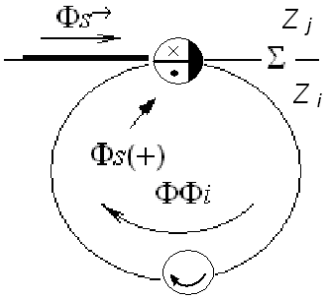
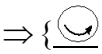


Figure 125. Sequence of a limit cycle type $^{S}C_5$.

The other two possibilities are presented when the system evolves in the CCW direction.

$^{S}C_7 \Rightarrow \{$ ‘(Φ_1) $(\Omega_{STV\leftarrow}{}^{X})$ $\}$

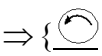
or

$^{S}C_7 \Rightarrow \{$  $\}$ (5.25)

and

$^{S}C_8 \Rightarrow \{$ ‘(Φ_2) $(\Omega_{SVT}{}^{\rightarrow}{}_X)$ $\}$

or

$^{S}C_8 \Rightarrow \{$  $\}$ (5.26)

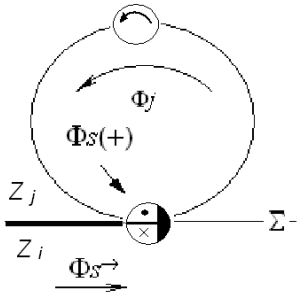


Figure 126. Sequence of a limit cycle type $^{S}C_8$.

Cycles with two regular segments and two points without touching sliding segments

This cycle is assembled with two segments, one in each vector field and two crossing points. The cycles do not touch the sliding segment; therefore, these segments can be in any position of the DB with the exception of the point at which the orbit crosses.

$^{S}C_{9} \Rightarrow \{`(\Phi_1)\ (\Omega c_{1,2})\ (\Phi_j)\ (\Omega c_{2,1})\ \hookleftarrow\}$

or

$^{S}C_{9} \Rightarrow \{$ $\hookleftarrow\}$ (5.27)

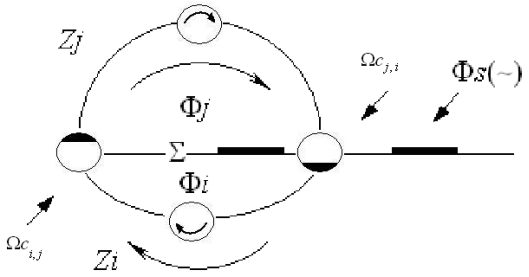


Figure 127. Sequence of a limit cycle type $^{S}C_{9.}$

Another possibility is presented when the system evolves in the CCW direction.

$^{S}C_{10} \Rightarrow \{`(\Phi_1)\ (\Omega c_{1,2})\ (\Phi_2)\ (\Omega c_{2,1})\ \hookleftarrow\}$

or

$^{S}C_{10} \Rightarrow \{$ $\hookleftarrow\}$ (5.28)

Cycles with two regular segments and two points, one of them touches the sliding segments

Orbits with right sliding touch and $\Phi s^{\rightarrow}$.

$^{S}C_{11} \Rightarrow \{`(\Phi_i)\ (\Omega c_{1.2})\ (\Phi_2)\ (\Omega_{SS}{}^{\rightarrow}{}_{T})\ \hookleftarrow\}$

or

$^{S}C_{11} \Rightarrow \{$ $\hookleftarrow\}$ (5.29)

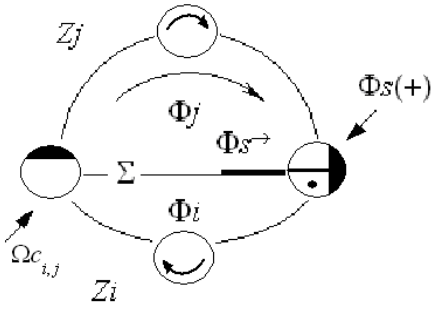


Figure 128. Sequence of a limit cycle type ${}^{S}C_{11.}$

Another possibility is presented when the system evolves in the CCW direction.

${}^{S}C_{12} \Rightarrow \{`(\Phi_2)\ (\Omega_{SS}{}^{\leftarrow}{}_{T})\ (\Phi_1)\ (\Omega c_{1.2})\ \hookleftarrow\}$

or

${}^{S}C_{12} \Rightarrow \{$ $\}$ (5.30)

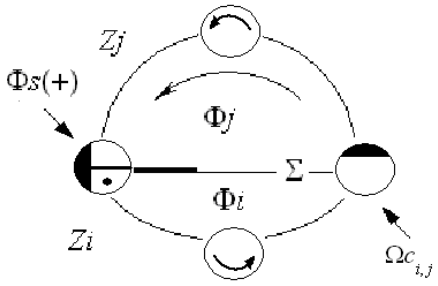


Figure 5.13: Sequence of a limit cycle type ${}^{S}C_{12}$

Orbits with left sliding touch and $\Phi s^{\leftarrow}$.

${}^{S}C_{13} \Rightarrow \{`(\Phi_1)\ (\Omega_{SS\leftarrow}{}^{T})\ (\Phi_2)\ (\Omega c_{2,1})\ \hookleftarrow\}$

or

${}^{S}C_{13} \Rightarrow \{$ $\}$ (5.31)

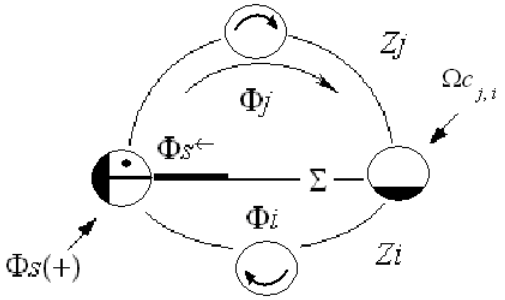


Figure 129. Sequence of a limit cycle type $^{S}C_{13.}$

Another possibility is presented when the system evolves in the CCW direction.

$^{S}C_{14} \Rightarrow \{`(\Phi_1)\ (\Omega_{SS\leftarrow}{}^{T})\ (\Phi_2)\ (\Omega c_{2,1})\ \hookleftarrow\}$
or
$^{S}C_{14} \Rightarrow \{$ $\hookleftarrow\}$ (5.32)

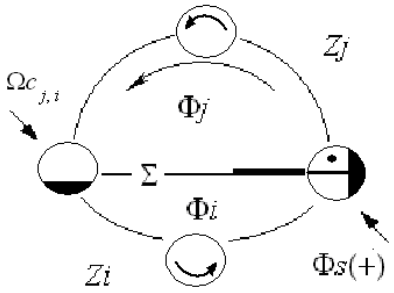


Figure 130. Sequence of a limit cycle type $^{S}C_{14.}$

Cycles with one regular and one sliding segments and two points belonging to sliding segments.

Incomplete sliding with half cycle and $\Phi s^{\rightarrow}$.

In this case, the sliding trajectory is incomplete; it does not travel over all sliding segment, and the regular segment intercepts the sliding segment in an intermediate point (see the following figures).

Cycle in Z_I or Z_2.

$^{S}C_{15} \Rightarrow \{`(\Phi_1)\ (\Phi_{SS\rightarrow})\ (\Omega_{SS}{}^{\rightarrow}{}_{T})\ \hookleftarrow\}$
or
$^{S}C_{15} \Rightarrow \{$ $\hookleftarrow\}$ (5.33)

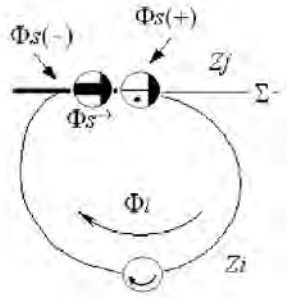


Figure 131. Sequence of a sliding limit cycle type ${}^{S}C_{15.}$

Another possibility is presented when the system evolves in the CCW direction.

${}^{S}C_{16} \Rightarrow \{ (\Phi_{SS\leftarrow}) (\Omega_{SS}{}^{\leftarrow}{}_{T}) (\Phi_1) \}$

or

${}^{S}C\ 16 \Rightarrow \{$ $\}$ (5.34)

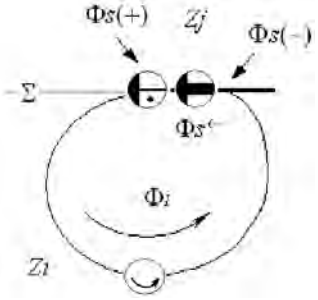


Figure 132. Sequence of a sliding limit cycle type ${}^{S}C_{16.}$

Cycle in Z_J or Z_2.

${}^{S}C_{17} \Rightarrow \{ (\Phi_2) (\Phi_{SS\rightarrow}) (\Omega_{SS\rightarrow}{}^{T}) \}$

or

${}^{S}C_{17} \Rightarrow \{$ $\}$ (5.35)

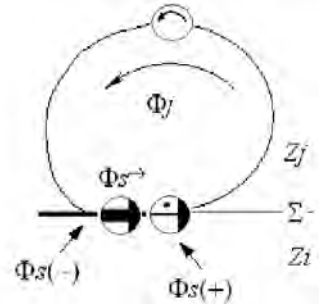


Figure 133. Sequence of a sliding limit cycle type ${}^{S}C_{17}$.

Another possibility is presented when the system evolves in the CCW direction.

${}^{S}C_{18} \Rightarrow \{‘ (\Phi_{SS\leftarrow}) (\Omega_{SS\leftarrow}{}^{T}) (\Phi_2) \hookleftarrow\}$
or
${}^{S}C_{18} \Rightarrow \{$ $\hookleftarrow\}$ (5.36)

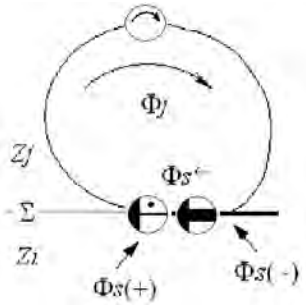


Figure 134. Sequence of a sliding limit cycle type ${}^{S}C_{18}$.

Complete sliding with half cycle and $\Phi s^{\rightarrow}$.

In these cycles, both points are tangentially singular.

Cycle in Z_I or Z_1.

${}^{S}C_{19} \Rightarrow \{‘ (\Phi_1) (\Omega_{SS\rightarrow}{}^{T}) (\Phi_{SS\rightarrow}) (\Omega_{SS}{}^{\leftarrow}{}_{T}) \hookleftarrow\}$
or
${}^{S}C_{19} \Rightarrow \{$ $\hookleftarrow\}$ (5.37)

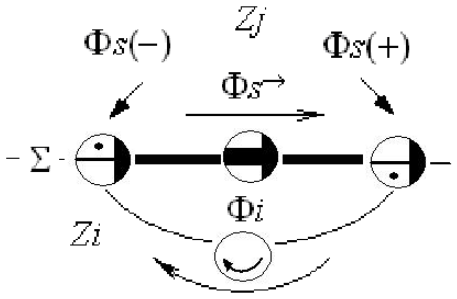


Figure 135. Sequence of a sliding limit cycle type $^{S}C_{19.}$

Another possibility is presented when the system evolves in the CCW direction.

$^{S}C_{20} \Rightarrow \{$‘ $(\Phi_{SS\rightarrow})\ (\Omega_{SS}{}^{\leftarrow}{}_{T})\ (\Phi_{1})\ (\Omega_{SS\leftarrow}{}^{T})$ ↵$\}$

or

$^{S}C_{20} \Rightarrow \{$ ↵$\}$ (5.38)

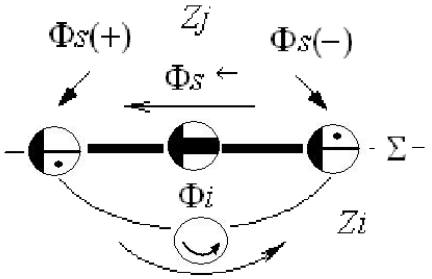


Figure 136. Sequence of a sliding limit cycle type $^{S}C_{20.}$

Complete sliding with half cycle in Z_J or Z_2.

$^{S}C_{21} \Rightarrow \{$‘ $(\Phi_{SS\leftarrow})\ (\Omega_{SS\leftarrow}{}^{T})\ (\Phi_{2})\ (\Omega_{SS}{}^{\leftarrow}{}_{T})$ ↵$\}$

or

$^{S}C_{21} \Rightarrow \{$ ↵$\}$ (5.39)

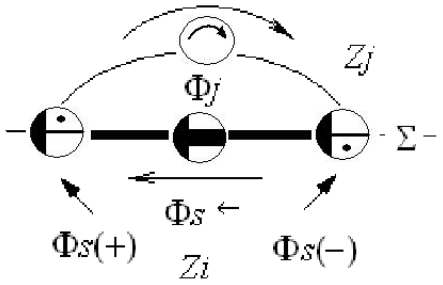


Figure 137. Sequence of a sliding limit cycle type ${}^{S}C_{21}$.

Another possibility is presented when the system evolves in the CCW direction.

${}^{S}C_{22} \Rightarrow \{$‘ (Φ_2) $(\Omega_{SS}{}^{\rightarrow}{}_{T})$ $(\Phi_{SS\rightarrow})$ $(\Omega_{SS\rightarrow}{}^{T})$ ↵$\}$

or

${}^{S}C_{22} \Rightarrow \{$ ↵$\}$ (5.40)

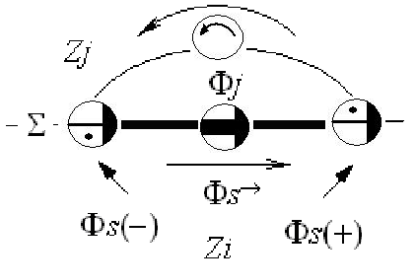


Figure 138. Sequence of a sliding limit cycle type ${}^{S}C_{22}$.

Cycles with two regular and one sliding segments and three points

It is defined as a mixed orbit, has more than two components, and includes regular and sliding segments (see figure 139): Mixed low orbit)

Mixed cycle right complete sliding segment with $\Phi s^{\rightarrow}$

${}^{S}C_{23} \Rightarrow \{$‘ (Φ_1) $(\Omega c_{1,2})$ (Φ_2) $(\Phi_{SS\rightarrow})$ $(\Omega_{SS}{}^{\rightarrow}{}_{T})$ ↵$\}$

or

${}^{S}C_{23} \Rightarrow \{$ ↵$\}$ (5.41)

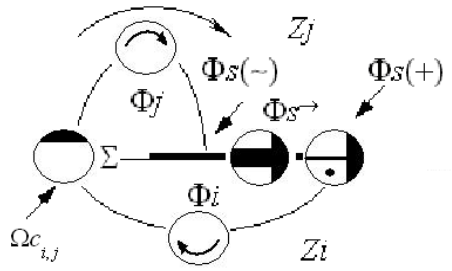


Figure 139. Sequence of a sliding limit cycle type ${}^{S}C_{23.}$

Another possibility is presented when the system evolves in the CCW direction.

${}^{S}C_{24} \Rightarrow \{$‘ (Φ_2) $(\Phi_{SS\leftarrow})$ $(\Omega_{SS}{}^{\leftarrow}{}_{T})$ (Φ_1) $(\Omega c_{2,1})$ ↵$\}$

or

${}^{S}C_{24} \Rightarrow \{$ ↵$\}$ (5.42)

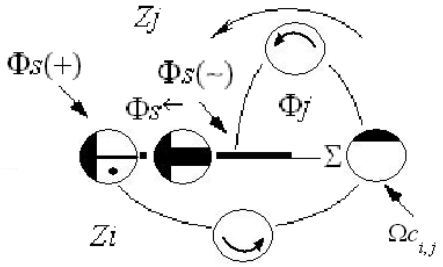


Figure 140. Sequence of a sliding limit cycle type ${}^{S}C_{24.}$

Mixed cycle right complete sliding segment with $\Phi s^{\leftarrow}$

${}^{S}C_{25} \Rightarrow \{$‘ (Φ_1) $(\Phi_{SS\leftarrow})$ $(\Omega_{SS\leftarrow}{}^{T})$ (Φ_2) $(\Omega c_{2,1})$ ↵$\}$

or

${}^{S}C_{25} \Rightarrow \{$ ↵$\}$ (5.43)

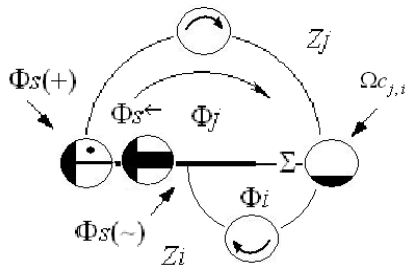


Figure 141. Sequence of a sliding limit cycle type ${}^{S}C_{25.}$

Another possibility is presented when the system evolves in the CCW direction.

${}^{S}C_{26} \Rightarrow \{$ ‘ $(\Phi_2)\ (\Omega c_{2,1})\ (\Phi_1)\ (\Phi_{SS\rightarrow})\ (\Omega_{SS\rightarrow}{}^{T})$ ↵ $\}$

or

${}^{S}C_{26} \Rightarrow \{$ ↵ $\}$ (5.44)

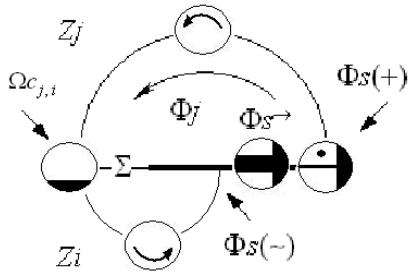


Figure 142. Sequence of a sliding limit cycle type ${}^{S}C_{26.}$

The four last points have the sliding segment complete; there are other four possibilities with the same dynamics but with an incomplete sliding segment.

Mixed cycle right incomplete sliding segment with $\Phi s^{\rightarrow}$

${}^{S}C_{27} \Rightarrow \{$ ‘ $(\Phi_1)\ (\Phi_{SS\rightarrow})\ (\Omega_{SS\rightarrow}{}^{T})\ (\Phi_2)\ (\Omega c_{2,1})$ ↵ $\}$

or

${}^{S}C_{27} \Rightarrow \{$ ↵ $\}$ (5.45)

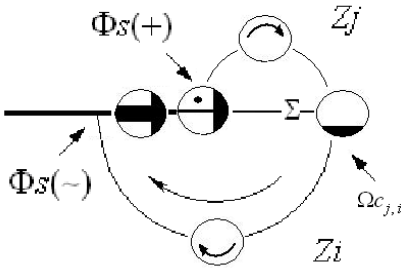


Figure 143. Sequence of a sliding limit cycle type $^{S}C_{27.}$

$^{S}C_{28} \Rightarrow \{$ ‘ $(\Phi_2)\ (\Phi_{SS\rightarrow})(\Omega_{SS}{}^{\rightarrow}{}_{T})\ (\Phi_1)(\Omega c_{1,2})$ ↵ $\}$
or
$^{S}C_{28} \Rightarrow \{$ ↵ $\}$ (5.46)

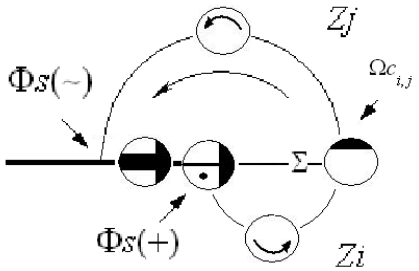


Figure 144. Sequence of a sliding limit cycle type $^{S}C_{28.}$

Mixed cycle right incomplete sliding segment with $\Phi s^{\leftarrow}$

$^{S}C_{29} \Rightarrow \{$ ‘ $(\Phi_{SS\leftarrow})\ (\Omega_{SS\leftarrow}{}^{T})\ (\Phi_2)\ (\Omega c_{2,1})\ (\Phi_1)$ ↵ $\}$
or
$^{S}C_{29} \Rightarrow \{$ ↵ $\}$ (5.47)

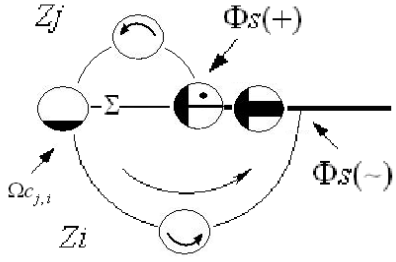


Figure 145. Sequence of a sliding limit cycle type $^{S}C_{29.}$

$^{S}C_{30} \Rightarrow \{`(\Phi_{SS\leftarrow})(\Omega_{SS}{}^{\leftarrow}{}_{T})(\Phi_{1})(\Omega c_{1,2})(\Phi_{2})\ \hookleftarrow\}$

or

$^{S}C_{30} \Rightarrow \{$ $\}$ (5.48)

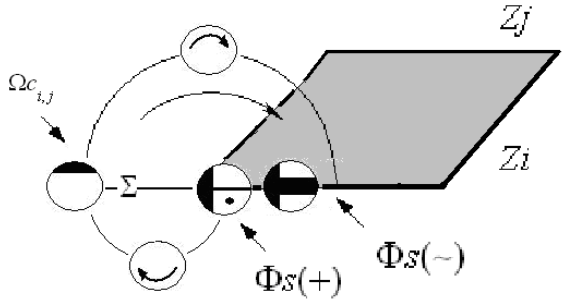


Figure 146. Sequence of a sliding limit cycle type $^{S}C_{30.}$

Cycles with two sliding segment

Some cycles suffer from a transformation that generates the partition of the sliding segments to two segments. Between the new sliding segments, the dynamics could evolve in one or two regions.

All the cycles with two sliding segments could have 6, 7, or 8 elements. Here, we will present the figures of cycles with 8 elements but also we will present a, b, and c versions of each equation for the cycles with 7 and 6 elements.

When both segments have a positive direction, the evolution is in CW direction; this presents two main possibilities.

$^{S}C_{31} \Rightarrow \{`(\Phi_{1,a})(\Omega_{SSa\rightarrow}{}^{T})(\Phi_{SSa\rightarrow})(\Omega_{SS}{}^{\rightarrow}{}_{T})`(\Phi_{1,b})`(\Omega_{SSb\rightarrow}{}^{T})(\Phi_{SSb\rightarrow})(\Omega_{SS}{}^{\rightarrow}{}_{T})\hookleftarrow\}$

or

$^{S}C_{31} \Rightarrow \{$ $\}$ (5.49)

$^{S}C_{31a} \Rightarrow \{$ $\}$

$^{S}C_{31b} \Rightarrow \{$ $\}$

$^{S}C_{31c} \Rightarrow \{$ $\}$

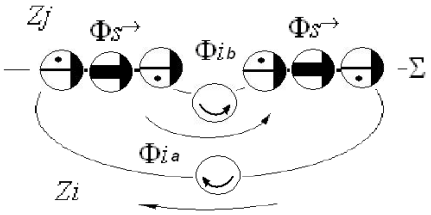


Figure 147. Sequence of a sliding limit cycle type ${}^{S}C_{31.}$

An examination of the previous equation reveals that it has a code that repeats and confuses the identification. In order to avoid this, an additional condition in relation to the period is added.

For the case in which intermediate orbit evolves in the Z_J vector field:

${}^{S}C_{32} \Rightarrow \{`(\Omega_{SS}{}^{\leftarrow}{}_{T})(\Phi_{SSa\leftarrow})(\Omega_{SSa\leftarrow}{}^{T})`(\Phi_{2,a})`(\Omega_{SS}{}^{\leftarrow}{}_{T})(\Phi_{SSa\leftarrow})(\Omega_{SSa\leftarrow}{}^{T})\ (\Phi_{2,b})\leftarrow\}$
or
${}^{S}C_{32} \Rightarrow \{$ $\}$ (5.50)
${}^{S}C_{32a} \Rightarrow \{$ $\}$
${}^{S}C_{32b} \Rightarrow \{$ $\}$
${}^{S}C_{32c} \Rightarrow \{$ $\}$

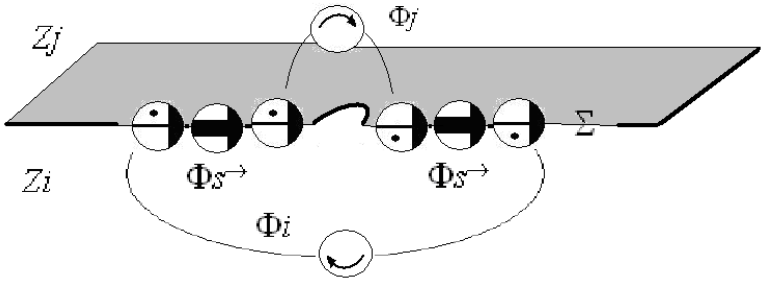


Figure 148. Sequence of a sliding limit cycle type ${}^{S}C_{32.}$

The other two possibilities are presented when the system evolves in the CCW direction.

${}^{S}C_{33} \Rightarrow \{`(\Omega_{SS\leftarrow}{}^{T})(\Phi_{SSa\leftarrow})(\Omega_{SSa}{}^{\leftarrow}{}_{T})`(\Phi_{1,a})`\ (\Omega_{SS\leftarrow}{}^{T})(\Phi_{SSa\leftarrow})(\Omega_{SSa}{}^{\leftarrow}{}_{T})\ (\Phi_{1,b})\leftarrow\}$
or
${}^{S}C_{33} \Rightarrow \{$ $\}$ (5.51)
${}^{S}C_{33a} \Rightarrow \{$ $\}$
${}^{S}C_{33b} \Rightarrow \{$ $\}$
${}^{S}C_{33c} \Rightarrow \{$ $\}$

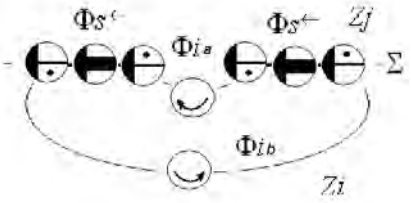
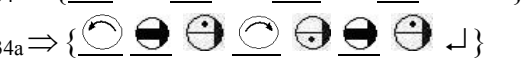


Figure 149. Sequence of a sliding limit cycle type ${}^{S}C_{33.}$

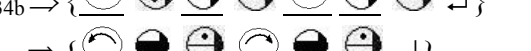
${}^{S}C_{34} \Rightarrow \{`(\Phi_{2,a})(\Omega_{SSa}{}^{\rightarrow}{}_{T})(\Phi_{SSa\rightarrow})(\Omega_{SS\rightarrow}{}^{T})`(\Phi_{2,b})`(\Omega_{SSa}{}^{\rightarrow}{}_{T})(\Phi_{SSa\rightarrow})(\Omega_{SS\rightarrow}{}^{T})\,\hookleftarrow\}$

or

${}^{S}C_{34} \Rightarrow \{$ $\hookleftarrow\}$ (5.52)

${}^{S}C_{34a} \Rightarrow \{$  $\hookleftarrow\}$

${}^{S}C_{34b} \Rightarrow \{$ $\hookleftarrow\}$

${}^{S}C_{34c} \Rightarrow \{$  $\hookleftarrow\}$

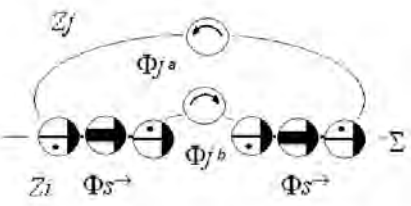


Figure 150. Sequence of a sliding limit cycle type ${}^{S}C_{34.}$

The other four possibilities are presented when the biggest oscillation evolves in the upper vector field.

${}^{S}C_{35} \Rightarrow \{`(\Phi_{1})(\Omega_{SS\rightarrow}{}^{T})(\Phi_{SS\rightarrow})(\Omega_{SS}\rightarrow{}^{T})(\Phi_{2})(\Omega_{SS}{}^{\rightarrow}{}_{T})(\Phi_{SS\rightarrow})\,(\Omega_{SS}{}^{\rightarrow}{}_{T})\hookleftarrow\}$

or

${}^{S}C_{35} \Rightarrow \{$ $\hookleftarrow\}$ (5.53)

${}^{S}C_{35a} \Rightarrow \{$ $\hookleftarrow\}$

${}^{S}C_{35b} \Rightarrow \{$ $\hookleftarrow\}$

${}^{S}C_{35c} \Rightarrow \{$ $\hookleftarrow\}$

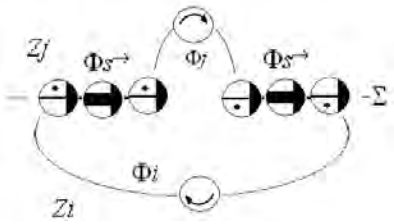


Figure 151. Sequence of a sliding limit cycle type ${}^{S}C_{35.}$

${}^{S}C_{36} \Rightarrow \{$‘ $(\Omega_{SS\leftarrow T})(\Phi_{SS\leftarrow})\ (\Omega_{SS\leftarrow T})(\Phi_1)(\Omega_{SS\leftarrow}{}^{T})(\Phi_{SS\leftarrow})(\Omega_{SS\leftarrow}{}^{T})(\Phi_2)$ ↵$\}$
or
${}^{S}C_{36} \Rightarrow \{$ ↵$\}$ (5.54)
${}^{S}C_{36a} \Rightarrow \{$ ↵$\}$
${}^{S}C_{36b} \Rightarrow \{$ ↵$\}$
${}^{S}C_{36c} \Rightarrow \{$ ↵$\}$

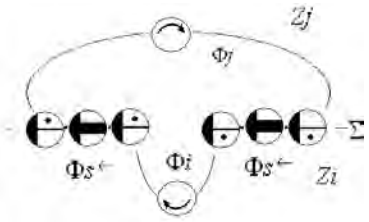


Figure 152. Sequence of a sliding limit cycle type ${}^{S}C_{36.}$

The other two possibilities are presented when the system evolves in the CCW direction.

${}^{S}C_{37} \Rightarrow \{$‘ $(\Omega_{SS\leftarrow}{}^{T})(\Phi_{SS\leftarrow})(\Omega_{SS\leftarrow}{}^{T})(\Phi_2)(\Omega_{SS\leftarrow T})(\Phi_{SS\leftarrow})\ (\Omega_{SS\leftarrow T})(\Phi_1)$ ↵$\}$
or
${}^{S}C_{37} \Rightarrow \{$ ↵$\}$ (5.55)
${}^{S}C_{37a} \Rightarrow \{$ ↵$\}$
${}^{S}C_{37b} \Rightarrow \{$ ↵$\}$
${}^{S}C_{37c} \Rightarrow \{$ ↵$\}$

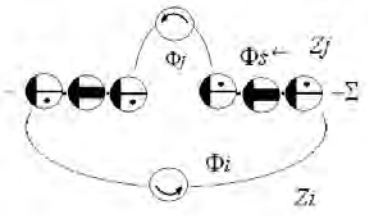


Figure 153. Sequence of a sliding limit cycle type ${}^{S}C_{37.}$

$^{S}C_{38} \Rightarrow \{`(\Phi_2)(\Omega_{SS}{}^{\rightarrow}{}_{T})(\Phi_{SS\rightarrow})\ (\Omega_{SS}{}^{\rightarrow}{}_{T})(\Phi_1)(\Omega_{SS\rightarrow}{}^{T})(\Phi_{SS\rightarrow})(\Omega_{SS}{\rightarrow}^{T})\ \hookleftarrow\}$

or

$^{S}C_{38} \Rightarrow \{$ $\hookleftarrow\}$ (5.56)

$^{S}C_{38a} \Rightarrow \{$ $\hookleftarrow\}$

$^{S}C_{38b} \Rightarrow \{$ $\hookleftarrow\}$

$^{S}C_{38c} \Rightarrow \{$ $\hookleftarrow\}$

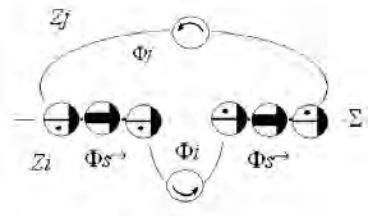


Figure 154. Sequence of a sliding limit cycle type $^{S}C_{38.}$

Pseudo cycles or dead cycles

These orbits are not periodic and as such, they should not be included, but in many cases, they are the result of a perturbation in a parameter. Therefore, to be able to obtain a complete sequence of the cycles of a bifurcation with the dynamics that begin as a periodic orbit and then stop, it is necessary to include these cycles.

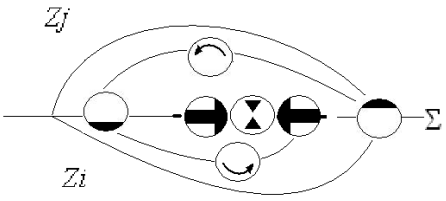


Figure 155. Incomplete cycle with an attractive segment inside the cycle.

Only two cases are presented. These dynamics are characteristic of cycles with a sliding segment having both directions (see figures 155 and 156). Some cycles leave the DB, but after evolving for a while, they return to a point of the sliding segment with a singularity that stops the dynamics of the system (pseudo saddle point).

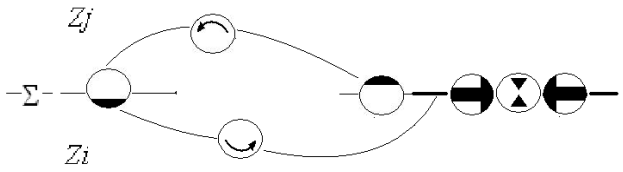


Figure 156. Incomplete cycle with an attractive segment outside the cycle.

The two main classes are cycles with an interior attractive segment of double sliding and cycles with an attractive segment of double sliding outside the cycle.

$^{S}C_{39} \Rightarrow \{ \circlearrowleft (\Omega c_{1,2})\ (\Phi_2)\ (\Omega c_{2,1})\ (\Phi_1)\ (\Omega_{SPP})\ (\Phi_{SS\leftarrow}) \hookleftarrow \}$

or

$^{S}C_{39} \Rightarrow \{$ $\}$ (5.57)

and

$^{S}C_{40} \Rightarrow \{ \circlearrowleft (\Omega c_{1,2})\ (\Phi_2)\ (\Omega c_{2,1})\ (\Phi_1)\ (\Omega_{SPP})\ (\Phi_{SS\rightarrow}) \hookleftarrow \}$

or

$^{S}C_{40} \Rightarrow \{$ $\}$ (5.58)

Before finish, we will just introduce the kind of cycles that can be found when a system combines sliding dynamics with impact dynamics. In next figure is show an orbit evolving in the region Z_j when it impacts a barrier. The bounce speed are not in the same region, is the region Z_i. Here the evolution moves to the DB, in a section in which sliding is possible. The orbit moves through the sliding segment and at the end abandons the sliding segment toward the region Z_j. In this way the cycle is complete.

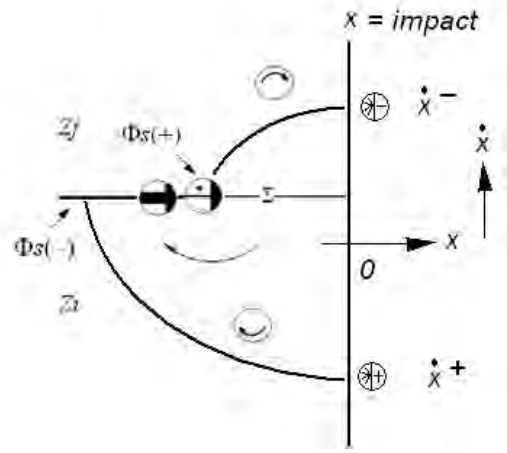


Figure 157. Impact cycle with an attractive sliding segment.

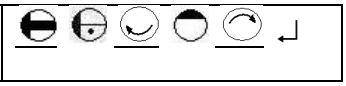
The code of this cycle is:

$^{S}C_{41} \Rightarrow \{`\ (\Phi_2)\ (\boldsymbol{X}^{\bullet -})\ (\boldsymbol{X}^{\bullet +})\ (\Phi_1)\ (\Phi_{SS\rightarrow})\ (\Omega_{SS\rightarrow}{}^{T})\ \hookleftarrow\}$
or
$^{S}C_{41} \Rightarrow \{$ $\}$

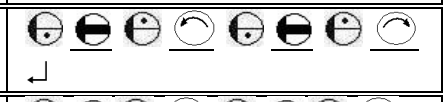
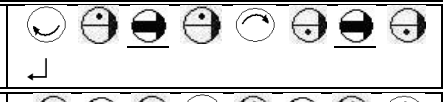
Summarizing this section, it has been syntetized a group of cycles, then we can define the first 41 forms of cycles, but as the process continued other cycles containing other types of singular and special points, more DBs, and higher dimensions can be added. Thus, the database of cycles (see table 5.1) will be bigger. Meanwhile, we will use these cycles in the next section to analyse standard forms of global bifurcations.

Table 5.1. Database of sliding cycles

Cycle	Text symbol	Graphic symbol
C_1	{ ‘($\Phi\Phi_1$) ↵}	
C_2	{ ‘($\Phi\Phi_{2)}$ ↵}	
C_3	{ '($\Phi\Phi_1$) ↵}	
C_4	{ '($\Phi\Phi_2$) ↵}	
${}^{S}C_5$	{ ‘(Φ_1) ($\Omega_{STV\rightarrow}{}^{X}$) ↵}	
${}^{S}C_6$	{ ‘(Φ_2) (${}^{\leftarrow}{}_{X}\Omega_{SVT}$) ↵}	
${}^{S}C_7$	{ ' (Φ_1) ($\Omega_{STV\leftarrow}{}^{X}$) ↵}	
${}^{S}C_8$	{ ' (Φ_2) (${}^{\rightarrow}{}_{X}\Omega_{SVT}$) ↵}	
${}^{S}C_9$	{ ‘(Φ_1) ($\Omega c_{1,2}$) (Φ_j) ($\Omega c_{2,1}$) ↵}	
${}^{S}C_{10}$	{ '(Φ_1) ($\Omega c_{1,2}$) (Φ_2) ($\Omega c_{2,1}$) ↵}	
${}^{S}C_{11}$	{ ‘(Φ_i) ($\Omega c_{1.2}$) (Φ_2) (${}^{\rightarrow}{}_{T}\Omega_{SS}$) ↵}	
${}^{S}C_{12}$	{ '(Φ_2) (${}^{\leftarrow}{}_{T}\Omega_{SS}$) ($\Phi_1$) ($\Omega c_{1.2}$) ↵}	
${}^{S}C_{13}$	{ ‘(Φ_1) ($\Omega_{SS\leftarrow}{}^{T}$) ($\Phi_2$) ($\Omega c_{2,1}$) ↵}	
${}^{S}C_{14}$	{ '(Φ_1) ($\Omega_{SS\leftarrow}{}^{T}$) ($\Phi_2$) ($\Omega c_{2,1}$) ↵}	
${}^{S}C_{15}$	{ ‘(Φ_1) ($\Phi_{SS\rightarrow}$) (${}^{\rightarrow}{}_{T}\Omega_{SS}$) ↵}	
${}^{S}C_{16}$	{ '($\Phi_{SS\leftarrow}$) (${}^{\leftarrow}{}_{T}\Omega_{SS}$) ($\Phi_1$) ↵}	
${}^{S}C_{17}$	{ '(Φ_2) ($\Phi_{SS\rightarrow}$) ($\Omega_{SS\rightarrow}{}^{T}$) ↵}	
${}^{S}C_{18}$	{ ‘($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow}{}^{T}$) ($\Phi_2$) ↵}	
${}^{S}C_{19}$	{ ‘(Φ_1) ($\Omega_{SS\rightarrow}{}^{T}$) ($\Phi_{SS\rightarrow}$) (${}^{\leftarrow}{}_{T}\Omega_{SS}$) ↵}	
${}^{S}C_{20}$	{ '($\Phi_{SS\rightarrow}$) (${}^{\leftarrow}{}_{T}\Omega_{SS}$) ($\Phi_1$) ($\Omega_{SS\leftarrow}{}^{T}$) ↵}	
${}^{S}C_{21}$	{ ‘($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow}{}^{T}$) ($\Phi_2$) (${}^{\leftarrow}{}_{T}\Omega_{SS}$) ↵}	
${}^{S}C_{22}$	{ '(Φ_2) (${}^{\rightarrow}{}_{T}\Omega_{SS}$) ($\Phi_{SS\rightarrow}$) ($\Omega_{SS\rightarrow}{}^{T}$) ↵}	
${}^{S}C_{23}$	{ ‘(Φ_1) ($\Omega c_{1,2}$) (Φ_2) ($\Phi_{SS\rightarrow}$) (${}^{\rightarrow}{}_{T}\Omega_{SS}$) ↵}	
${}^{S}C_{24}$	{ '(Φ_2) ($\Phi_{SS\leftarrow}$) (${}^{\leftarrow}{}_{T}\Omega_{SS}$) ($\Phi_1$) ($\Omega c_{2,1}$) ↵ }	
${}^{S}C_{25}$	{‘ (Φ_1) ($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow}{}^{T}$) ($\Phi_2$) ($\Omega c_{2,1}$) ↵}	
${}^{S}C_{26}$	{ '(Φ_2) ($\Omega c_{2,1}$) (Φ_1) ($\Phi_{SS\rightarrow}$) ($\Omega_{SS\rightarrow}{}^{T}$) ↵ }	
${}^{S}C_{27}$	{ ‘(Φ_1) ($\Phi_{SS\rightarrow}$) ($\Omega_{SS\rightarrow}{}^{T}$) ($\Phi_2$) ($\Omega c_{2,1}$) ↵}	
${}^{S}C_{28}$	{ '(Φ_2) ($\Phi_{SS\rightarrow}$)($\Omega_{SS}{}^{\rightarrow}{}_{T}$) ($\Phi_1$)($\Omega c_{1,2}$) ↵}	
${}^{S}C_{29}$	{ '($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow}{}^{T}$) ($\Phi_2$) ($\Omega c_{2,1}$) ($\Phi_1$) ↵ }	

$^{S}C_{30}$	{ ‘($\Phi_{SS\leftarrow}$) ($\Omega_{SS}{}^{\leftarrow}{}_{T}$) ($\Phi_1$) ($\Omega c_{1,2}$) ($\Phi_2$) ↵}	

Continuation of table 3

Cycle	Text symbol	Graphic symbol
$^{S}C_{31}$	{‘($\Phi_{1,a}$)($\Omega_{SSa\rightarrow}{}^{T}$)($\Phi_{SSa\rightarrow}$)($\Omega_{SS}{}^{\rightarrow}{}_{T}$)‘($\Phi_{1,b}$)‘($\Omega_{SSb\rightarrow}{}^{T}$) ($\Phi_{SSb\rightarrow}$) ($\Omega_{SS}{}^{\rightarrow}{}_{T}$) ↵}	
$^{S}C_{32}$	{ ‘($\Omega_{SS}{}^{\leftarrow}{}_{T}$)($\Phi_{SSa\leftarrow}$)($\Omega_{SSa\leftarrow}{}^{T}$)‘($\Phi_{2,a}$)‘($\Omega_{SS}{}^{\leftarrow}{}_{T}$)($\Phi_{SSa\leftarrow}$) ($\Omega_{SSa\leftarrow}{}^{T}$) ($\Phi_{2,b}$) ↵}	
$^{S}C_{33}$	{ ‘($\Omega_{SS\leftarrow}{}^{T}$)($\Phi_{SSa\leftarrow}$)($\Omega_{SSa}{}^{\leftarrow}{}_{T}$)‘($\Phi_{1,a}$)‘ ($\Omega_{SS\leftarrow}{}^{T}$)($\Phi_{SSa\leftarrow}$)($\Omega_{SSa}{}^{\leftarrow}{}_{T}$) ($\Phi_{1,b}$) ↵}	
$^{S}C_{34}$	{ ‘($\Phi_{2,a}$)($\Omega_{SSa}{}^{\rightarrow}{}_{T}$)($\Phi_{SSa\rightarrow}$)($\Omega_{SS\rightarrow}{}^{T}$)‘($\Phi_{2,b}$)‘($\Omega_{SSa}{}^{\rightarrow}{}_{T}$) ($\Phi_{SSa\rightarrow}$)($\Omega_{SS\rightarrow}{}^{T}$) ↵}	
$^{S}C_{35}$	{ ‘(Φ_1)($\Omega_{SS\rightarrow}{}^{T}$)($\Phi_{SS\rightarrow}$)($\Omega_{SS}{\rightarrow}^{T}$)($\Phi_2$)($\Omega_{SS}{}^{\rightarrow}{}_{T}$)($\Phi_{SS\rightarrow}$) ($\Omega_{SS}{}^{\rightarrow}{}_{T}$) ↵}	
$^{S}C_{36}$	{ ‘($\Omega_{SS\leftarrow T}$)($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow T}$)(Φ_1)($\Omega_{SS\leftarrow}{}^{T}$)($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow}{}^{T}$)($\Phi_2$) ↵}	
$^{S}C_{37}$	{ ‘ ($\Omega_{SS\leftarrow}{}^{T}$)($\Phi_{SS\leftarrow}$)($\Omega_{SS\leftarrow}{}^{T}$)($\Phi_2$)($\Omega_{SS\leftarrow T}$)($\Phi_{SS\leftarrow}$) ($\Omega_{SS\leftarrow T}$)($\Phi_1$) ↵}	
$^{S}C_{38}$	{ ‘(Φ_2)($\Omega_{SS}{}^{\rightarrow}{}_{T}$)($\Phi_{SS\rightarrow}$) ($\Omega_{SS}{}^{\rightarrow}{}_{T}$)($\Phi_1$)($\Omega_{SS\rightarrow}{}^{T}$)($\Phi_{SS\rightarrow}$) ($\Omega_{SS}{\rightarrow}^{T}$) ↵}	
$^{S}C_{39}$	{ ‘($\Omega c_{1,2}$) (Φ_2) ($\Omega c_{2,1}$) (Φ_1) (Ω_{SPP}) ($\Phi_{SS\leftarrow}$) ↵}	
$^{S}C_{40}$	{‘($\Omega c_{1,2}$) (Φ_2) ($\Omega c_{2,1}$) (Φ_1) (Ω_{SPP}) ($\Phi_{SS\rightarrow}$) ↵}	
$^{S}C_{41}$	$^{S}C_{41} \Rightarrow$ {‘ (Φ_2) ($\boldsymbol{X}^{\bullet-}$) ($\boldsymbol{X}^{\bullet+}$) ($\Phi_1$) ($\Phi_{SS\rightarrow}$) ($\Omega_{SS\rightarrow}{}^{T}$) ↵}	

5.5 Methodical Process in the Face of a Change of Parameter

In this section, we take the abovementioned orbits and study the changes in the equivalence after the variation of parameter values.

The result could be a growing cycle, a change in direction of some sliding segment, the displacement of a vector field, the displacement of the DB, the destruction of a cycle, the appearance of a singular point in the discontinuity boundary, the transformation of a slide segment, and vice versa.

An additional alternative is a method used in (82) for carrying out bifurcation analysis. This so-called Puzzle is based on the modification of a cycle by supposing reasonable solutions.

Each one of these results can generate a change in the chain of elements that constitute a cycle, thereby generating a bifurcation.

Next, these results are analyzed in some of the 39 cycles formalized in the previous section. In this section, in order to describe the bifurcation, we will use the symbols ${}^{S}\beta_{xx}$ for naming a specific sequence or cycles of a bifurcation. The central cycle is the one that requires a narrow range of parameter values for its existence and appears contained in the [ ] symbol.

$${}^{S}\beta_{xx} \Rightarrow \{({}^{S}C_x)\ [\ {}^{S}C_y]\ ({}^{S}C_z)\} \qquad (5.59)$$

Isolated cycle growing

In the face of a change in the parameter value, a change in size takes place. If the size of the cycle is incremented, then a contact between the orbit and the DB is produced. When the contact is with a point of a sliding segment, the dynamics are strongly affected. If the cycle is continuously growing, another change takes place in the sequence of cycles. The orbit changes the number of segments that compose it from one to two. The initial and final segments are of different types; this confirms that a bifurcation has been generated (see the following figure). The sequence of cycles is

$${}^{S}\beta_1 \Rightarrow \{({}^{S}C_1)\ [\ {}^{S}C_5]\ ({}^{S}C_{15})\} \qquad (5.59)$$

From here, the bifurcations will be presented in terms of cycles but the content corresponds to the information presented in the following section.

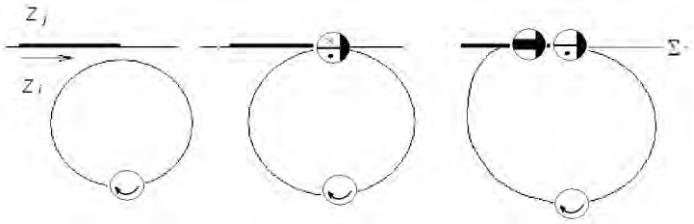


Figure 158. Sequence of change of the bifurcation ${}^{S}\beta_1$.

Carrying out a similar procedure with the other three isolated cycles leads to the following bifurcations:

$${}^{S}\beta_2 \Rightarrow \{({}^{S}C_2)\ [\ {}^{S}C_6]\ ({}^{S}C_{17})\} \qquad (5.60)$$

$${}^{S}\beta_3 \Rightarrow \{({}^{S}C_3)\ [\ {}^{S}C_7]\ ({}^{S}C_{16})\} \tag{5.61}$$

$${}^{S}\beta_4 \Rightarrow \{({}^{S}C_4)\ [\ {}^{S}C_8]\ ({}^{S}C_{18})\} \tag{5.62}$$

Half incomplete sliding cycles growing

If cycle ${}^{S}C_{15}$ grows continuously, it generates a change to complete the sliding half cycle and then, to transform it to a mixed cycle (see the following figure).

The number of segments is increased from two to three. The sequence of cycles is

$${}^{S}\beta_5 \Rightarrow \{({}^{S}C_{15})\ [\ {}^{S}C_{19}]\ ({}^{S}C_{27})\} \tag{5.63}$$

Carrying out a similar procedure with the other three half incomplete growing sliding cycles leads to the following bifurcations:

$${}^{S}\beta_6 \Rightarrow \{({}^{S}C_{17})\ [\ {}^{S}C_{21}]\ ({}^{S}C_{36})\} \tag{5.64}$$

$${}^{S}\beta_7 \Rightarrow \{({}^{S}C_{16})\ [\ {}^{S}C_{20}]\ ({}^{S}C_{38})\} \tag{5.65}$$

$${}^{S}\beta_8 \Rightarrow \{({}^{S}C_{18})\ [\ {}^{S}C_{22}]\ ({}^{S}C_{29})\} \tag{5.66}$$

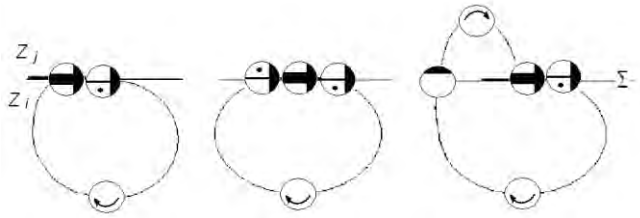


Figure 159. Sequence of change of the bifurcation ${}^{S}\beta_5$.

Mixed sliding cycles growing

If a cycle of the ${}^{S}C_{29}$ type grows continuously, it generates a change, begin to touching the right side of the sliding segment and after, changes to a normal cycle or to a non hyperbolic dynamics (see figure 160).

The composition of the cycle has changed; from three segments, it is down to two. The segments before and after are of different types. The sequence of cycles is

$${}^{S}\beta_9 \Rightarrow \{({}^{S}C_{27})\ [\ {}^{S}C_{11}]\ ({}^{S}C_9)\} \tag{5.67}$$

Carrying out a similar procedure with the other three mixed growing sliding cycles leads to the following bifurcations:

$${}^{S}\beta_{10} \Rightarrow \{({}^{S}C_{36})\ [\ {}^{S}C_{13}]\ ({}^{S}C_{9})\} \tag{5.68}$$

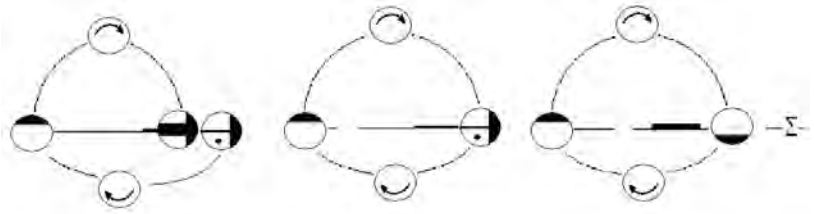


Figure 160. Sequence of the change of the bifurcation ${}^{S}\beta_{10.}$

$${}^{S}\beta_{11} \Rightarrow \{({}^{S}C_{38})\ [\ {}^{S}C_{14}]\ ({}^{S}C_{10})\} \tag{5.69}$$

$${}^{S}\beta_{12} \Rightarrow \{({}^{S}C_{29})\ [\ {}^{S}C_{12}]\ ({}^{S}C_{10})\} \tag{5.70}$$

Sliding segment with contrary direction

Another possibility is given when the sliding segment has an opposite direction (see the following figure).

$$\beta_{13} = \left\lfloor \begin{array}{c} \Phi\Phi_{i}\hookleftarrow \\ \Phi\Phi_{i} \mapsto \Phi_{S}(+)\hookleftarrow \\ \Phi_{i} \mapsto C_{Cij} \mapsto \Phi_{j} \mapsto \Phi_{S}(\square) \mapsto \Phi_{S}^{\leftarrow} \mapsto \Phi_{S}(+)\hookleftarrow \end{array} \right\rfloor \tag{5.71}$$

It has two conditions:

Below the sliding segment should co-exist lines having two different directions. Some lines should point toward the discontinuity limit, and the others away from it. The vectors next to the Φs(+) point should point away from the point. After evaluating the orbit changes as a bifurcation, we think that it is necessary to find a differential equation with dynamics that match with the sequence described before. See figure 161.

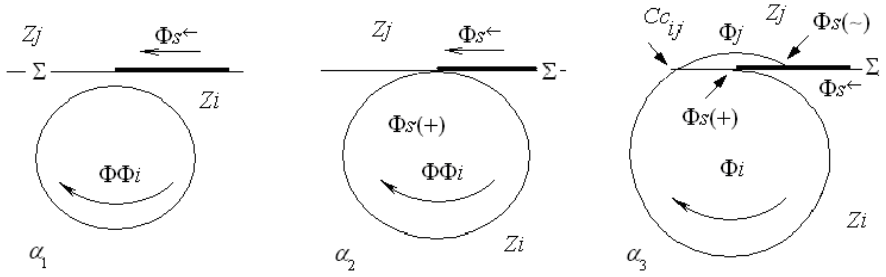


Figure 161. Change due to sliding segment in contrary direction.

One maching is: $\mathbf{f}_a=[f_1=y, f_2=-x+x^2+xy-1]$, $\mathbf{f}_b=[f_1=y, f_2=-10x+1,3y-2]$. In this way, we show how this method is able to introduce new sequences that correspond to sliding bifurcations with a mathematical representation but without a real phenomen

Summarizing, as database of these section we have the table 5.2 which is used as source of information by the method SPT to detect what class of bifurcation is presenting at the moment that in the process of gathering a terna of cycles is registered.

Table 4. Database of global bifurcations

Type of bifurcation	Terna of Cycles
${}^{S}\beta_1$	$\{({}^{S}C_1)\ [\ {}^{S}C_5]\ ({}^{S}C_{15})\}$
${}^{S}\beta_2$	$\{({}^{S}C_2)\ [\ {}^{S}C_6]\ ({}^{S}C_{17})\}$
${}^{S}\beta_3$	$\{({}^{S}C_3)\ [\ {}^{S}C_7]\ ({}^{S}C_{16})\}$
${}^{S}\beta_4$	$\{({}^{S}C_4)\ [\ {}^{S}C_8]\ ({}^{S}C_{18})\}$
${}^{S}\beta_5$	$\{({}^{S}C_{15})\ [\ {}^{S}C_{19}]\ ({}^{S}C_{27})\}$
${}^{S}\beta_6$	$\{({}^{S}C_{17})\ [\ {}^{S}C_{21}]\ ({}^{S}C_{36})\}$
${}^{S}\beta_7$	$\{({}^{S}C_{16})\ [\ {}^{S}C_{20}]\ ({}^{S}C_{38})\}$
${}^{S}\beta_8$	$\{({}^{S}C_{18})\ [\ {}^{S}C_{22}]\ ({}^{S}C_{29})\}$
${}^{S}\beta_9$	$\{({}^{S}C_{27})\ [\ {}^{S}C_{11}]\ ({}^{S}C_{9})\}$
${}^{S}\beta_{10}$	$\{({}^{S}C_{36})\ [\ {}^{S}C_{13}]\ ({}^{S}C_{9})\}$
${}^{S}\beta_{11}$	$\{({}^{S}C_{38})\ [\ {}^{S}C_{14}]\ ({}^{S}C_{10})\}$
${}^{S}\beta_{12}$	$\{({}^{S}C_{29})\ [\ {}^{S}C_{12}]\ ({}^{S}C_{10})\}$

Chapter 6

Conclusions

CONCLUSIONS

A new method, called singular point tracking (SPT), has been presented and tested. The result has proved that this method reveals information useful for the improvement of the understanding of the behaviour and the analysis of sliding bifurcations in nonsmooth systems.

After the research, we conclude that the vectors associated to each point of the trajectories in the DB have sufficient information to help in the identification of the dynamics or the changes in it in a way that allows us to study local bifurcations.

Research was expanded to include the classification of 36 singular and special points on the DB. This expansion becomes the stone to support the method. Over the DB, the sequences of singular and special points with segments composed of crossing and sliding points become closely related with the dynamics of the neighbourhood of the DB. Through Poincaré maps we observed that when the parameter values were changed, the singular points transformed into curves between areas in which all points had the same dynamics. These dynamics differed from those of the next area. On the basis of this information, we could find the parameter values that generate bifurcations. Each element, point, line, or surface has been illustrated with didactic symbols and colours with good results. Taking a look at the symbols in the figures and taking into account the labels nonlinear phenomena can be easily analyzed.

The results of the SPT method are promising and have been presented at international events and published in specialized journals. To demonstrate the way to use this method, these articles include illustrative examples. The classical oscillator used for demonstrating most of the new methods in nonsmooth systems has been used in this study. The Casini model (2), (1), has been demonstrated to be useful in validating methods related to systems with multiple DBs, has also been used. New models have also been developed to explore other fields of analysis. The complete catalogue of the local bifurcation for Filippov systems (a smaller class of nonsmooth systems) presented by Kuznetsov (3) was used for validating the SPT method. The result was hopeful, and since then, new features have been

added to this method in order to increase its applications. The extension discussed in Chapter 5 to track the sliding cycles for detecting bifurcations has been proved as a new way to study global dynamics in nonsmooth systems. An advantage of the SPT method is that it is based on previously collected information, and the performance gets better with an increase in the amount of information in the database. In this document, we have presented a large group of sequences of the dynamics in the DB and the sequences of periodic solutions that enrich the database and were recollected in the format of a catalogue. It is useful to know how this method was developed in a sequential form in order to carry out in-depth consultations.

The SPT method has some interesting features: when the tracking was implemented in the numerical solutions, the computing time decreased in comparison with brute- force methods. Further, the new way to present the results, using maps and the symbolic representation to indicate sequences of element and dynamics, opens a door to a wider graphic tool for analysing complex systems. Finally, as a support to validate the SPT method, a toolbox of numerical functions that correspond to the equations presented in this document was developed. Therefore, it was possible to perform comparisons, develop new examples, and confirm the utility and the power of the SPT method.

Finally, we can conclude that the SPT method is very precise regarding direct detection of most of local bifurcations catalogued and helps in the location of initial points used for tracking global bifurcations. Some cases require a study of the analytical differences in which the dynamics shows the same code, especially when the special points involved are saddle-nodes. Now, in the detection of global bifurcations, the singular and special points are a part of the sequences that determine each class of orbits or cycles

At this moment, the SPT method needs to be improved further in order to obtain the benefit of a tool which runs fast in your research. Addition of more information to the database is needed which will increase the reliability and performance of the method. This information would consist of new sequence codes for local dynamics and for sequences of periodic solutions. As the method runs with more information, the user needs to perform less previous analysis to study any system. One branch of applications where we found many possibilities is the one associated with mixed phenomena such as dry friction and impact. From the geometrical and numerical points of view, there are also improvements to be made in the software application that is used for driving the method. It is mandatory to expand the analysis to systems with a DB composed of non flat surfaces. Finally, a comparison of some known models simulated with the SPT method has revealed that some numerical problems could be improved.

The SPT method differs from other approaches in terms of the identification and manipulation of the system information. While the methods in (9) consider a system as one entity to be solved by a group of equations, the SPT method is based on previously collected information of the orbit in the evolution of the system. For that reason the first step it to break the orbit into entities. Then, the method identifies the entities of the orbits evaluating several characteristics such as its position, order, number, and ownership. This

information is compared with the previously collected information and determines which situation is presented and how to change parameters and initial conditions in order to obtain more results. In the case of the first group of methods that uses boundary value problems as the main tool, the main difficulty is that practical systems can have many DBs and a high dimension. This situation leads to a large group of equations that need to be manipulated and solved. The SPT method was conceptualized for systems with many discontinuity boundaries and a high dimension; the procedure for the comparison is the same for both simple and complex systems.

Another difference of the SPT method is the tool used for validating the approach developed through this work. Other approaches (14), (9) have used Auto (10), which has provided a considerable number of successful results in the treatment of problems in the field of smooth bifurcations. However, some characteristics are restricted: the graphical user interface (GUI) is not very friendly. Another approach (15) uses the environment of Matlab®, which is a tool with all the characteristics required for developing any application. This includes a group of toolboxes that allow one to solve differential equations using various methods, a group of functions to detect events, and a wide GUI with a good set of graphics that aid the comprehension of the results. This would have been also a good tool selected for solving boundary value problems. The SPT method was tested using a slow, high-level language, Labview®, graphically programmed, with similar characteristics to Matlab and with the possibility to share codes. Labview was selected because the highly developed editing and debugging functions, allowing the shortest development times.

APPENDIX A

The results of the method have been presented in international events, published in specialized magazines. In the publications listed and presented below only little format are changed in comparison with the original published versions.

I. Integration-free Analysis of Nonsmooth Local Dynamics in Planar Filippov Systems. International Journal of Bifurcations and Chaos. Year: 2009 Vol: 19 Issue: 3, March 2009 Page: 947 – 975.

II. Continuation of Nonsmooth Bifurcations in Filippov Systems Using Singular Point Tracking. International Journal of Applied Mathematics and Informatics Issue 1, Volume 1, 2007, Pages. 36-49.

III. Characteristic Point Sequences in Local and Global Bifurcation Analysis of Filippov Systems. Wseas Transactions on Systems, Issue 10, Volume 7, October 2008, Pages 840-854.

IV. Numerical Analysis of Sliding Dynamics in Three-Dimensional Filippov Systems using SPT Method. International Journal of Mathematical Models and Methods in Applied Sciences. Issue 1, Volume 2, 2008, Pages 342 354.

V. Characterizing Points on Discontinuity Boundary of Filippov Systems. Modelling, Identification, and Control (MIC 2008). Febrero 2008, Innsbruck, Austria.

VI. Analyzing Sliding Bifurcations on Discontinuity Boundary of Filippov Systems. Recent Advances on Applied Mathematics. Proceedings of the American Conference on Applied Mathematics (MATH '08). Cambridge, Massachusetts, USA, March 24-26, 2008.

VII. SPTCont 1.0: A LabView Toolbox for Bifurcation Analysis of Filippov Systems. New Aspects of Systems. Proceedings of the 12th WSEAS International Conference on Systems. Heraklion, Greece, July 22-24, 2008

VIII. Detecting Sliding Areas in Three-Dimensional Filippov Systems using an Integration-Free Method. New aspects on Computing Research. Proceedings of the 2nd European Computing Conference. (ECC‘08) Malta, September 11-13, 2008, Pages 160-166.

IX. Localization of sliding bifurcations in systems with multiple discontinuity boundaries. Submitted to the journal Dyna of the National university of Colombia in Medellin.

1. Integration-free Analysis of Nonsmooth Local Dynamics in Planar Filippov Systems.

International Journal of Bifurcations and Chaos. Year: 2009 Vol: 19 Issue: 3, March 2009 Page: 947 – 975.

Integration-Free Analysis of Nonsmooth Local Dynamics in Planar Filippov Systems

Iván Arango
Department of Mechanics Engineering, Universidad EAFIT, Medellín, Colombia

John Alexander Taborda
Department of Electrical and Electronics Engineering, and Computer Science, Universidad Nacional de Colombia, Sede Manizales Campus La Nubia, Manizales, Colombia

Abstract

In this paper, we present a novel method to analyze the behavior of discontinuous piecewise-smooth autonomous systems (denominated Filippov systems) in the planar neighborhood of the discontinuity boundary (DB). The method uses the evaluation of the vector fields on DB to analyze the nonsmooth local dynamics of the Filippov system without the integration of the ODE sets. The method is useful in detection of nonsmooth bifurcations in Filippov systems. We propose a classification of the points, events and events combinations on DB. This classification is more complete in comparison with the others previously reported. Additional characteristics as flow direction and sliding stability are included explicitly. The lines and the points are characterized with didactic symbols and the exclusive conditions for their existence based in geometric criterions. Boolean-valued functions are used to formulate the conditions of existence. Different problems are analyzed with the proposed methodology.

Key words: Bifurcation theory, sliding bifurcations, piecewise-smooth systems, Filippov systems.

1. Introduction

Most physical systems can be considered nonsmooth dynamical system. For this reason, realler models should include the inherent nonsmooth phenomena, such as impact, friction, hysteresis, saturation, switching or sliding. In the last years, the dynamical and bifurcation behavior of nonsmooth systems has been studied widely [Brogliato, 1999], [Brogliato, 2000], [Kunze, 2000], [Leine, 2006], [Utkin, 1992]. From 2000, the nonsmooth dynamical systems have been divided in three classes depending on the discontinuity type of their orbits and vector fields [Leine, 2000]. In each class, there have been theoretical and empirical works to improve the knowledge about these systems.

- **Class A.** Systems without sliding where the orbits are continuously-differentiable but with discontinuities in the first or higher derivatives of vector fields [Merillas, 2006], [di Bernardo et al , 2001],[Savi et al , 2007].
- **Class B.** Systems with sliding (type-Filippov) where the orbits are continuous but not continuously differentiable [Galvanetto, 2004], [Kuznetsov et al, 2003], [Leine, 2000], [Luo , 2005], [Kowalczyk & di Bernardo, 2005], [Kowalczyk et al , 2006].
- **Class C.** Systems with impact or hybrid systems. These are systems with discontinuous

Email addresses: iarango@eafit.edu.co (Iván Arango), yatabordag@unal.edu.co (John Alexander Taborda).

2. Continuation of Nonsmooth Bifurcations in Filippov Systems Using Singular Point Tracking.

International Journal of Applied Mathematics and Informatics Issue 1, Volume 1, 2007, Pages. 36-49.

Continuation of Nonsmooth Bifurcations in Filippov Systems Using Singular Point Tracking

Iván Arango and John Alexander Taborda

***Abstract*—**In this paper, we propose a novel numerical method for continuation of nonsmooth bifurcations in discontinuous piecewise smooth systems or Filippov systems which we are denominated Singular Point Tracking (SPT). We use the evaluation of the vector fields on the discontinuity boundary (DB) to analyze the dynamics of the systems without integration of the ODE sets or integrating only in the points computed by the SPT algorithm. We apply a classification of points and events on DB recently proposed. Local and global nonsmooth bifurcations are detected using the SPT method. Two principal advantages have the SPT method with other numerical methods to continuation of bifurcations in nonsmooth systems. First, the integration-free algorithms in the SPT avoid the well know numerical problems of these algorithms. Second, when the integration is unavoidable the SPT method computes the initial condition of the simulation to reduce the compute time. A piecewise smooth prey-predator model is used as illustrative example of the SPT method.

***Keywords*—**Bifurcation theory, continuation techniques, filippov systems, nonsmooth bifurcations.

I. Introduction

NONSMOOTH characteristics as sliding, switching or impact cause many mathematical and numerical difficulties in modeling, simulation and analysis stages [1-4]. The bifurcation theory and the piecewise smooth approach have been used widely to analyze the dynamics of nonsmooth systems as power converters [5,6], friction oscillator [7-9], impact oscillators [10-12] or ecological models [13,14].

The focus of this paper is in the discontinuous piecewise smooth systems (PWS) or Filippov systems. A lot of papers have been restricted to continuous PWS systems or Filippov systems not involving sliding motion because of the analysis is more simplified [15,16]. When the sliding motion on the discontinuity boundary is possible, the analysis is more complicated.

The number of specialized software in nonsmooth dynamics is reduced [17,18]. In [19] and [20,21], they are presented two toolboxes for analysis and continuation of nonsmooth bifurcations in Filippov systems. In this paper, we propose a novel numerical method for continuation of nonsmooth bifurcations which we are denominated: *Singular Point Tracking (SPT)*.

We use the evaluation of the vector fields on the discontinuity boundary (DB) to analyze the dynamics of the Filippov systems without integration of the ODE sets or integrating only in the points computed for the SPT algorithm. We apply a classification of forty-one points and seven events on DB recently proposed in [22-24]. The existence conditions of the two crossing points (C class), the four non-singular sliding points (S class) and the thirty-five singular sliding points (Ω class) are formulate using *Boolean-valued functions* B(.) based on integration-free geometric criterions. These conditions are easily programmable and they can be used directly in the detection of nonsmooth bifurcations

The existence of singular sliding points on DB determines the existence of the events on DB and the change of the event type when a parameter is varied determines the existence of nonsmooth bifurcations. Therefore, a continuation technique of nonsmooth bifurcations can be defined tracking the singular points on DB under variation in the parameters. Local and global nonsmooth bifurcations are detected using SPT method.

Two principal advantages have the SPT method with other numerical methods to continuation of bifurcations in nonsmooth systems. First, the integration-free algorithms in the SPT avoid the well know numerical problems of these algorithms. Second, when the integration is unavoidable the SPT method computes the initial condition of the simulation to reduce the compute time.

The paper is organized as follows. In section II we present the background concepts of Filippov systems. The type of points and events on DB are summarized in the sections III and IV, respectively. In the sections V and VI we present the SPT method to continuation of local and global nonsmooth bifurcation in Filippov systems, respectively. An illustrative example based on ecological models is presented in the section VII. Finally, the conclusions and future work are discussed in the section VIII.

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Iván Arango is with the EAFIT University, Medellín, Colombia (corresponding author to provide phone: 009-56-8861466; e-mail: iarango@eafit.edu.co).

John Alexander Taborda is with the Electrical and Electronics Engineering Department, National University of Colombia, Manizales, Colombia, (e-mail: yatabordag@unal.edu.co).

3. Characteristic Point Sequences in Local and Global Bifurcation Analysis of Filippov Systems.

Wseas Transactions on Systems, Issue 10, Volume 7, October 2008, Pages 840-854.

Characteristic Point Sequences in Local and Global Bifurcation Analysis of Filippov Systems

Ivan Arango
Universidad EAFIT
Department of Mechanics Engineering
Kra 42d 50 sur 40 Medellín
Colombia
iarango@eafit.edu.co

John Alexander Taborda
Universidad Nacional de Colombia
Department Electronics Engineering
Campus La Nubia Manizales
Colombia
yatabordag@unal.edu.co

Abstract: We explain the set of rules behind of the *LabView* toolbox for bifurcation analysis of Filippov systems denominated SPTCont 1.0. This software can detect nonsmooth bifurcations in n-dimensional systems using integration-free algorithms based on the evaluation of the vector fields on the discontinuity boundary (DB). In this paper, we present the characteristic point sequences that the software detects to guarantee the existence of local and global nonsmooth bifurcations in planar Filippov systems ($n = 2$). These sequences can be extended to three-dimensional or higher dimension Filippov systems. Boolean-valued functions are used to formulate the conditions of existence for each point defined in the sequences. Dynamics on DB and cycles are defined in function of the set of points. The full catalog of codim 1 local and global bifurcations is used to define the characteristic point sequence when the bifurcation parameter is varied. Finally, an illustrative example is analyzed using step-by-step routines of SPTCont 1.0.

Key–Words: Bifurcation theory, sliding bifurcations, piecewise-smooth systems, Filippov systems.

1 Introduction

Nonsmooth dynamics can be found in many physical systems such as electric motors, power converters, brakes, gears and joints. Switchings in electrical systems [1], [2], impacting motion in mechanical systems [3], [4], stick-slip motion in oscillators with friction [5], [6] and hybrid dynamics in control systems [7] are being studied with nonsmooth dynamical theory.

Nonsmooth characteristics were often neglected in mathematical models due to their difficulties of analysis and simulation. Actually, there is great interest to understand nonsmooth effects to try to minimise undesirable consequences such as wear of components, surface damage, fatigue failure and noise [8]. Many considerations based on bifurcations theory for smooth systems are violated and many new phenomena are observed such as sliding, crossing or grazing bifurcations.

Figure 1(a) shows a basic example of nonsmooth system. The undamped dry friction oscillator comprises a block resting on a belt, moving with velocity v. The motion of block with mass m along the belt is opposed by a spring (with stiffness k) connected to a fixed support. The spring exerts a restoring force on the block that is opposed by the friction force created by the belt. This system can be modelled with the

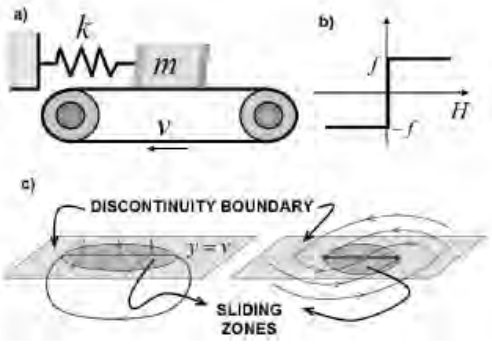

Figure 1: Basic example of nonsmooth systems: (a). Undamped dry friction oscillator on a rotating belt. (b). Coulomb friction model (H is the relative velocity). (c). Nonsmooth dynamics with sliding motion.

equation given in (1).

$$m\ddot{x} + kx = f\,\mathrm{sgn}\,(v - \dot{x}) \tag{1}$$

In an equivalent way, this system can be expressed with the non-dimensionless system of first order ODEs given by equation (2).

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + f\,\mathrm{sgn}\,H \quad where \quad H = (v - y) \end{cases} \tag{2}$$

4. Numerical Analysis of Sliding Dynamics in Three-Dimensional Filippov Systems using SPT Method.

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Numerical Analysis of Sliding Dynamics in Three-Dimensional Filippov Systems using SPT Method

IVAN ARANGO
Universidad EAFIT
Department of Mechanics Engineering
Kra 42d 50 sur 40 Medellín
Colombia
iarango@eafit.edu.co

JOHN ALEXANDER TABORDA
Universidad Nacional de Colombia
Department Electronics Engineering
Campus La Nubia Manizales
Colombia
yatabordag@unal.edu.co

Abstract: We present the numerical analysis of sliding dynamics on the discontinuity boundary (DB) of three-dimensional (3D) Filippov systems using an integration-free method denominated *Singular Point Tracking* (SPT). Many physical applications in engineering can be modelled as Filippov systems. Sliding dynamics due to non-smooth phenomena such as friction, hysteresis or switching are inherent to Filippov systems. The analysis of sliding dynamics have many mathematical and numerical difficulties. Several well-known numerical problems can be avoid using integration-free methods. Three-dimensional Filippov systems are being studied extensively because these systems still have many open problems. In this paper, we present a first attempt to extend the SPT method to 3D Filippov systems. The discontinuity boundary (DB) is characterized using geometric criterions based on angular evaluations. Eighteen basic points on DB are distinguished and eight basic scenarios on DB are defined. Finally, local and global bifurcation scenarios are conceptualized with the SPT method and some illustrative examples are given.

Key–Words: Bifurcation theory, numerical analysis, non-smooth bifurcations, Filippov systems.

1 Introduction

Many engineering applications such as electric motors, power converters, brakes, gears and joints can be modelled as Filippov systems. Switchings in electrical systems [1], [2], impacting motion in mechanical systems [3], [4], stick-slip motion in oscillators with friction [5], [6], [7] and hybrid dynamics in control systems [8] are being studied with the nonsmooth dynamical systems theory.

The classical dynamical systems theory restricts the analysis to smooth phenomena such as Hopf, flip and pitchfork bifurcations and excludes the analysis of nonsmooth phenomena such as grazing, sliding and border-collision bifurcations. Understanding nonsmooth characteristics of the physical systems, we could minimize undesirable effects such as wear of components, surface damage, fatigue failure, vibration and noise [9], [10].

Figure $1(a)$ shows a block representation of many nonsmooth systems. In these systems, we can determine an energy source, an oscillator and a switching mechanism triggered by the oscillator (see figure $1(b)$) [11]. Figure $1(c)$ presents an example of sliding cycle in planar state-space where the discontinuity boundary is the line $y = 0$. The DB in 3D Filippov systems is a 2D or 3D surface; for example the plane Σ presented in figure $1(d)$.

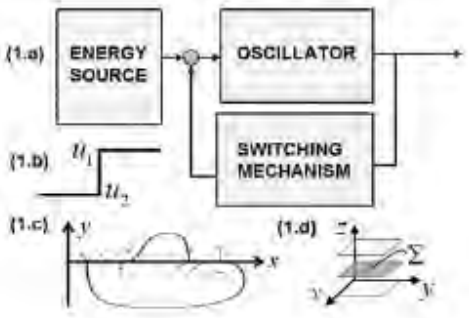

Figure 1: $(1.a)$, Generic block diagram of nonsmooth system. Three fundamental parts: Energy source, oscillator and switching mechanism. $(1.b)$, Switched signal with two positions. $(1.c)$, Example of sliding dynamic. $(1.d)$, Switching manifold or discontinuity boundary (Σ) in 3D space

Schemes of electrical and mechanical Filippov systems are presented in figure 2. The boost converter has the three elements described in figure 1. Two basic states can be distinguished: *On-state* where the inductor current increases and *Off-state* where the energy accumulated during the *On-state* is transferred to the load. Sliding dynamics can be detected when the cur-

5. Analyzing Sliding Bifurcations on Discontinuity Boundary of Filippov Systems.

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ANALYZING SLIDING BIFURCATIONS ON DISCONTINUITY BOUNDARY OF FILIPPOV SYSTEMS

IVÀN ARANGO
Universidad EAFIT
Department of Mechanics Engineering
Kra 42d 50 sur 40 Medellín
Colombia
iarango@eafit.edu.co

JOHN ALEXANDER TABORDA
Universidad Nacional de Colombia
Department Electronics Engineering
Campus La Nubia Manizales
Colombia
yatabordag@unal.edu.co

Abstract: In this paper, we propose a novel method to analyze sliding bifurcations in discontinuous piecewise smooth autonomous systems (denominated Filippov systems) on the planar neighborhood of the discontinuity boundary (DB). We use a classification recently proposed of points and events on DB to characterize the one-parameter sliding bifurcations. For each parameter value, crossing and sliding segments on DB are determined by means of existence conditions of two crossing points (C), four non-singular sliding points (S) and thirty-five singular sliding points (T, V, Π, Ψ, Q or Φ). Boolean-valued functions are used to formulate these conditions based on geometric criterions. This method was proven with the full catalog of local bifurcations that it was proposed recently. A topological normal form is used as illustrative example of the method.

Key–Words: Bifurcation theory, sliding bifurcations, piecewise-smooth systems, Filippov systems.

1 Introduction

Most of physical systems can be considered nonsmooth dynamical system. For this reason, realler models should include the inherent nonsmooth phenomena, such as impact, friction, hysteresis, saturation, switching or sliding. In the last years, the dynamical and bifurcational behavior of nonsmooth systems has been studied widely [1], [2], [3], [4], [5].

From 2000, the nonsmooth dynamical systems are divided in three class depending on the discontinuity type of their orbits and vector fields [6]. In each class, they have been generated theoretical and empirical works to improve the knowledge about these systems.

- **Class A.** Systems with impact or hybrid systems. They are systems with discontinuous jumps because of hard boundaries [2], [7], [8], [9], [10], [11].
- **Class B.** Systems with sliding (type-Filippov) where the orbits are continuous but not continuously differentiable [12], [13], [6], [14], [15], [16].
- **Class C.** Systems without sliding where the orbits are continuously-differentiable but discontinuities in the first or higher derivatives of vector fields [17], [18], [19], [20].

The focus of this paper is in the *class B* systems or Filippov systems. A lot of papers have been restricted to *class C* systems or Filippov systems not involving sliding motion because of the analysis is more simplified [13]. When the sliding motion on the discontinuity boundary is possible, the analysis is more complicated.

In this paper, we propose a novel method to analyze sliding bifurcations in Filippov systems. We use the classification of points and events on DB recently proposed in [21] to characterize the one-parameter sliding bifurcations. For each parameter value, crossing and sliding segments on DB are determined by means of existence conditions of two crossing points, four non-singular sliding points and thirty-five singular sliding points. Boolean-valued functions are used to formulate these conditions based on geometric criterions. This method was proven with the full catalog of local bifurcations that it was proposed recently in [13].

2 Background

The *Filippov systems* are modelled as piecewise-smooth systems (PWS) where the state space contains two kinds of entities: *Smooth Zones* (Z_i) and *Discontinuity Boundaries* (Σ). In the equation (1) is presented the state-space representation where the vector fields $\mathbf{F}_i$ determine the type of behaviors on the dis-

6. Characterizing Points on Discontinuity Boundary of Filippov Systems.

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CHARACTERIZING POINTS ON DISCONTINUITY BOUNDARY OF FILIPPOV SYSTEMS

Iván Arango
Universidad EAFIT
Kra 42d 50 sur 40
Medellín, Colombia
email: iarango@eafit.edu.co

John Alexander Taborda
Universidad Nacional de Colombia
Campus La Nubia
Manizales, Colombia
email: yatabordag@unal.edu.co

ABSTRACT
In this paper, we presented a basic methodology to understand the behavior of discontinuous piecewise smooth autonomous systems (denominated Filippov systems) in the planar neighborhood of the discontinuity boundary (DB). This methodology is useful in detection of nonsmooth bifurcations in Filippov systems. We propose a classification of the points and events on DB. This classification is more complete in comparison with the reported papers previously. The lines and the points are characterized with didactic symbols and the exclusive conditions for their existence based in geometric criterions. Boolean-valued functions are used to formulate the conditions. An illustrative example with a friction oscillator is presented.

KEY WORDS
Modelling, simulation, numerical methods, nonsmooth systems and bifurcation theory.

1. Introduction

Most of physical systems can be considered nonsmooth dynamical system. For this reason, realler models should include the inherent nonsmooth phenomena, such as impact, friction, hysteresis, saturation, switching or sliding. In the last years, the dynamical and bifurcational behavior of nonsmooth systems has been studied widely [1], [2], [3], [4], [5]. From 2000, the nonsmooth dynamical systems are divided in three class depending on the discontinuity type of their orbits and vector fields [6]. In each class, they have been generated theoretical and empirical works to improve the knowledge about these systems.

- **Class A.** Systems with impact or hybrid systems. They are systems with discontinuous jumps because of hard boundaries [2], [7], [8], [9], [10], [11].
- **Class B.** Systems with sliding (type-Filippov) where the orbits are continuous but not continuously differentiable [12], [13], [6], [14], [15], [16].
- **Class C.** Systems without sliding where the orbits are continuously-differentiable but discontinuities in the first or higher derivatives of vector fields [17], [18], [19].

The focus of this paper is in the *class B* systems or Filippov systems. A lot of papers have been restricted to *class C* systems or Filippov systems not involving sliding motion because of the analysis is more simplified [13]. When the sliding motion on the discontinuity boundary is possible, the analysis is more complicated. In this paper, a basic methodology to understand the behavior of Filippov systems on DB is presented. Great richness of DB points types and event types are detected. In consequence, we propose a wider classification of the points and events types on DB. This work is motivated in the necessity to define methodologies more general for the study of systems with sliding. The proposed methodology can be used in the detection of nonsmooth bifurcations, in numerical simulation based in event-driven methods or can be extended to more dimensions or to higher codimensions. The initial classification was realized in 1988 by *Filippov* [20]. He classified singular points in planar discontinuous systems and identified all one-codimension local singularities. In 2003, *Kuznetsov et al* [13] gave an overview of all codim 1 bifurcations in generic planar Filippov systems. In 2005, Luo [14] characterized the DB in connectable and accessible sub-domains. In this paper, additional characteristics as flow direction and sliding stability are included explicitly. These inclusions are oriented to the development of more versatile and more efficient numeric methods for the detection of nonsmooth bifurcations. The number of specialized software in nonsmooth dynamics is reduced [21], [22]. In [23] and [24], they are presented two toolbox for bifurcation analysis in Filippov systems. Soon, in [25], will be explained an analysis tool based in the methodology proposed here. The points and lines on DB are characterized using symbols and the necessary and sufficient conditions based in geometric criterions. Boolean-valued functions are used to formulate the conditions. This way, the conditions are easily programmable and they could be used directly in the detection of nonsmooth bifurcations. The methodology was proven in the full catalog of local bifurcations that it was proposed in [13].

2. Methodology of DB Analysis

The *Filippov systems* are modelled as piecewise-smooth systems (PWS) where the state space contains two kinds

7. Detecting Sliding Areas in Three-Dimensional Filippov Systems using an Integration-Free Method

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Detecting Sliding Areas in Three-Dimensional Filippov Systems using an Integration-Free Method

IVÁN ARANGO
Universidad EAFIT
Department of Mechanics Engineering
Kra 42d 50 sur 40 Medellin
Colombia
iarango@eafit.edu.co

JOHN ALEXANDER TABORDA
Universidad Nacional de Colombia
Department Electronics Engineering
Campus La Nubia Manizales
Colombia
yatabordag@unal.edu.co

Abstract: In this paper, we detect sliding areas in three-dimensional (3D) Filippov systems using an integration-free method denominated *Singular Point Tracking* (SPT). Many physical applications in engineering can be modelled as Filippov systems. Sliding dynamics due to nonsmooth phenomena as friction, hysteresis or switching are inherent to Filippov systems. The analysis of sliding dynamics has many mathematical and numerical difficulties. Several well-known numerical problems can be avoid using integration-free methods. In this paper, we extend the SPT method to 3D Filippov systems. In comparison with the 2D case, the evaluation of the vector fields on the discontinuity boundary (DB) should be reformulated and new dynamics on DB should be characterized.

Key–Words: Bifurcation theory, numerical analysis, non-smooth bifurcations, Filippov systems.

1 Introduction

Nonsmooth characteristics as sliding, switching or impact cause many mathematical and numerical difficulties in modeling, simulation and analysis stages [1], [2], [3]. The bifurcation theory and the piecewise-smooth approach have been used widely to analyze the dynamics of nonsmooth systems as power converters [4], [5], [6], [7], friction oscillator [8], [9], [3], [10], [11] or impact oscillators [1], [12], [13], [14].

In this paper we concentrate in Filippov systems of three dimensions (3D). A lot of papers have been restricted to 2D Filippov systems or Filippov systems not involving sliding motion because of the analysis is more simplified [9]. When the sliding motion exist and the Filippov system is 3D then the analysis is more complicated.

The analysis of sliding dynamics has many mathematical and numerical difficulties. The number of specialized software in nonsmooth dynamics is reduced [15], [16]. In [17] and [18], two toolboxes are presented for analysis and continuation of nonsmooth bifurcations in Filippov systems. Several well-known numerical problems can be avoid using integration-free methods. In this paper, we analyze sliding dynamics in three-dimensional (3D) Filippov systems using an integration-free method denominated *Singular Point Tracking* (SPT).

We use the evaluation of the vector fields on the discontinuity boundary (DB) to analyze the dynamics of the Filippov systems without integration of the ODE sets or integrating only in the points computed for the SPT algorithm. We apply a classification of points and events on DB recently proposed in [19], [20], [21].

In comparison with the 2D case, the evaluation of the vector fields on the discontinuity boundary (DB) should be reformulated and new dynamics on DB should be characterized. The existence conditions of the *crossing areas*, *sliding areas* and *singular sliding lines* are formulate using *Boolean-valued functions* $B(.)$ based on integration-free geometric criterions. These conditions are easily programmable and they can be used directly in the detection of nonsmooth bifurcations.

The *Boolean-valued functions* $B(.)$ return `TRUE` or `FALSE` when their arguments are evaluated. The logical functions are composed of logical connectives: *AND*, *OR* and *NOT* denoted by $\wedge$, $\vee$ and $\neg$, respectively.

The paper is organized as follows. In section II we present the background concepts of Filippov systems and the SPT numerical method. The type of areas on DB and singular lines on DB are summarized in the sections III and IV, respectively. The basic dynamics on DB are presented in the section V while an illustrative example is presented in section VI. Finally, the conclusions are presented in the section VII.

8. SPTCont 1.0: A LabView Toolbox for Bifurcation Analysis of Filippov Systems.

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SPTCont 1.0: A LabView Toolbox for Bifurcation Analysis of Filippov Systems

IVÁN ARANGO
Universidad EAFIT
Department of Mechanics Engineering
Kra 42d 50 sur 40 Medellín
Colombia

JOHN ALEXANDER TABORDA
Universidad Nacional de Colombia
Department Electronics Engineering
Campus La Nubia Manizales
Colombia

Abstract: In this paper, we report the development of *LabView* toolbox for bifurcation analysis of Filippov systems, which we are denominated SPTCont 1.0. This toolbox is included in the GAONDYSY software developed in language G for analysis of non-smooth dynamical systems. The SPTCont 1.0 uses integration-free algorithms based on the evaluation of the vector fields on the discontinuity boundary (DB). The routines apply the classification of points and events on DB recently proposed. Local and global bifurcations can be detected using the numerical method *Singular Point Tracking* or *SPT*. The lines and the points on DB are characterized with didactic symbols and the exclusive conditions for their existence based on geometric criterions. Boolean-valued functions are used to formulate the existence conditions. Two-dimensional and three-dimensional problems can be studied with the software SPTCont 1.0. Two principal advantages have the software in the analysis of nonsmooth systems. First, the software has educational and didactic subroutines for amateur users. Second, the software has functions for specialyzed users where the integration-free algorithms in the SPT avoid the well know numerical problems of these algorithms. If the integration is unavoidable, for example in detection of global bifurcations, the SPT method computes the initial condition of the simulation to reduce the compute time.

Key–Words: Bifurcation theory, educational software, non-smooth bifurcations, Filippov systems.

1 Introduction

Nonsmooth dynamical systems are divided in three class depending on the discontinuity type of their orbits and vector fields [1]. In each class, nonsmooth characteristics as sliding, switching or impact cause many mathematical and numerical difficulties in modeling, simulation and analysis stages.

- Systems with impact or hybrid systems (Class A). They are systems with discontinuous jumps because of hard boundaries [2], [3], [4], [5], [6].
- Systems with sliding or Filippov systems (Class B). In these systems the orbits are continuous but not continuously differentiable [7], [8], [1], [9], [10].
- Systems without sliding (Class C) where the orbits are continuously-differentiable but discontinuities in the first or higher derivatives of vector fields [11], [12], [13], [14].

In this paper we concentrate in Filippov systems of two and three dimensions (2D and 3D) [15]. A lot of papers have been restricted to *class C* systems or Filippov systems not involving sliding motion because of the analysis is more simplified [8]. When the sliding motion on the discontinuity boundary is possible, the analysis is more complicated.

The number of specialized software in nonsmooth dynamics is reduced [16], [17]. In [18] and [19], they are presented two toolboxes for analysis and continuation of nonsmooth bifurcations in Filippov systems. The platforms used in these toolboxes are Matlab and AUTO97. We have selected as development enviroment: LabVIEW. This platform allows the creation of user interface with graphical controls and indicators easily. Also, the fully object-oriented character of LabVIEW code allows functions reuse without modifications.

In this paper, we report the development of *LabView* toolbox for bifurcation analysis of Filippov systems, which we are denominated SPTCont 1.0. This toolbox is included in the GAONDYSY software developed in language G for analysis of non-smooth dynamical systems [20]. Also, we have used many libraries of LabVIEW with functions of graphics generation, mathematics and data analysis [21].

9. Localization of sliding bifurcations in a rotational oscillator with double cam

Journal Dyna. Nro 167, pag 160-168, 2011.

(83)

LOCALIZACIÓN DE BIFURCACIONES DESLIZANTES EN SISTEMAS CON MÚLTIPLES LÍMITES DE DISCONTINUIDAD

LOCALIZATION OF SLIDING BIFURCATIONS IN SYSTEMS WITH MULTIPLE DISCONTINUITY BOUNDARIES

IVAN ARANGO

Ingeniero Mecánico, Universidad Eafit. iarango@eafit.edu.co.

JHON ALEXANDER TABORDA

Ingeniero Electrónico, Universidad Nacional de Colombia, vataborda@unal.edu.co.

GERARD OLIVAR

Matemático, Universidad Nacional de Colombia, golivart@unal.edu.co

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RESUMEN: En este trabajo se analizan las bifurcaciones no suaves no convencionales también llamadas deslizantes en sistemas con múltiples límites de discontinuidad. El método de seguimiento de puntos singulares (SPT) es probado para localizar las bifurcaciones. Un oscilador rotativo de doble leva que exhibe múltiples límites de discontinuidad es utilizado como modelo. Los resultados indican que esta es una metodologia útil para analizar sistemas no suaves complejos que presentan dinámicas deslizantes.

PALABRAS CLAVE: Sistemas no suaves, sistemas de Filippov, bifurcaciones deslizantes, limite de discontinuidad, osciladores, fricción.

ABSTRACT: In this work, nonsmooth, non conventional bifurcations, also called sliding bifurcations are analyzed in systems with multiple discontinuity boundaries. The singular point tracking (SPT) method is used to locate the bifurcations. As model, is used a rotational oscillator with double cam which shows multiple discontinuity boundaries. The results indicate that this is a useful methodology to analyze complex nonsmooth systems with sliding dynamics.

KEYWORDS: Nonsmooth systems, sliding bifurcation, Filippov systems, discontinuity boundary, oscillators, friction.

10. Detection and Continuation of Filippov Sliding Limit Cycles Bifurcation in Multiple DB Systems

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DETECTION AND CONTINUATION OF FILIPPOV SLIDING LIMIT CYCLES BIFURCATION IN MULTIPLE DB SYSTEMS

Arango, Ivan
Mechanical Engineering Department, EAFIT University
Medelln, Colombia
iarango@eafit.edu.co

Pineda, Fabio
Mechanical Engineering Department, EAFIT University
Medelln, Colombia
fpineda@eafit.edu.co

Gerard, Olivar
Department of Electrical and Electronics Engineering & Computer Sciences, Universidad Nacional de Colombia
Manizales, Colombia
golivart@unal.edu.co

In this paper, we present a method for detecting bifurcations of limit cycles with sliding pieces in Filippov systems. These bifurcations, also called sliding bifurcations, appear when, before or after the bifurcation value, the limit cycle has different pieces corresponding to different vector fields and some of the pieces lie in the discontinuity boundaries (DB). The number of pieces of the cycle, the order in which they are generated, the position of the sliding piece in the sequence, the way the cycle reaches and leaves the sliding region, the evolution to a special or isolated point (saddle, pseudo-saddle or anti-collinear) over a sliding piece are cycle topological characteristics. They are used to find differences between two cycles when a parameter changes, in order to find topologically-nonequivalent systems. The different classes of points and curves that are found in the cycles are coded. The cycles are expressed as a code sequences. While the integration process is running, a comparison is done with the cycle codes before and after a change in a parameter, in order to detect a bifurcation. Following this procedure, the method is extended to do continuation of sliding bifurcations. Thus, a sort of sequences theory is used in order to numerically support our method. We illustrate our procedure with an example of a double band friction oscillator that shows multiple discontinuity boundaries and simultaneous bifurcations.

Keywords: Filippov systems, nonsmooth systems, limit cycles, bifurcation, continuation

1. Introduction

Events as impact, friction, hysteresis, saturation and commutation are common in nonsmooth systems. Bifurcations in nonsmooth systems have been intensively studied [5], [9], [15], [17], [19]. Nonsmooth systems can be classified according to the degree of nonsmoothness or discontinuity which the orbits and vector fields show [14]. Theoretical and numerical works have been reported, improving the understanding of the associated phenomena. An interesting and studied level of nonsmoothnes includes systems with continuous vector fields in the discontinuity boundary, having discontinuities in the derivatives of first or higher order. As an example we can consider a mechanical system with a single mass, a spring, a damping element and an elastic limit support [15]. In the next level of nonsmoothness we have systems described by differential equations with discontinuous right-hand which are also called Filippov systems [11]. In these systems the vector field is discontinuous in the switching boundary. This is usual for mechanical systems with dry friction [17]. The last level includes systems with jumps in the state variables. They are typically found in systems with impacts where the impact phenomenon is modeled assuming no deformation and a very short impact-time duration [5].

Regarding tools for analyzing bifurcations, a set of mathematical results are available if the systems are smooth enough. Smooth bifurcations generated in nonsmooth systems have also received considerable attention and as a consequence, the number of tools used for the analysis of bifurcations is increasing. Nut non-standard or sliding bifurcations generated in nonsmooth systems have been more difficult to analyze due to the conceptual differences

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