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# Symmetries and reductions of Dirac-Jacobi Structures

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A mis dos grandes amores:
mi esposa Ángela y mi madre.

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# Abstract

**Title: Symmetries and reductions of Dirac-Jacobi structures**

In this manuscript we study reductions of Dirac-Jacobi structures on a smooth manifold under the presence of symmetries given by the action of a connected Lie group. The main tools we used are the called "homogenization trick" and the well known reduction of Dirac structures. We show two particular cases, namely, reduction by moment map and the case when the base manifold is endowed with a contact 1-form.

**Keywords: $\mathcal{E}^1(M)$-Dirac structures; Dirac-Jacobi structures; Dirac structures; homogenization trick; Dirac-ization trick; reduction.**

# Resumen

**Título: Simetrías y reducciones de estructuras Dirac-Jacobi**

En el presente texto se estudia la reducción de estructuras Dirac-Jacobi sobre una variedad diferenciable bajo la presencia de simetrías dadas por la acción de un grupo de Lie conexo. La herramienta principal que se usa es el llamado "truco de homogenización" y las reducciones de Dirac ya conocidas. Se muestran dos casos particulares de reducción Dirac-Jacobi; cuando hay presencia de una aplicación momento y el caso cuando la variedad base está dotada de una 1-forma de contacto.

**Palabras clave: estructuras $\mathcal{E}^1(M)$; estructuras Dirac-Jacobi; estructuras Dirac; truco de homogenización; truco de Dirac-ización; reducción.**

**Contents**

## Chapter 1

## INTRODUCTION

Since the middle of the last century and in recent years, there have been an increasing interest and technical developments on several geometrical structures related to mathematical physics. This approach began with symplectic and Poisson structures on differential manifolds, and later it appeared some others, such as Dirac structures, Courant algebroids, Lie (bi)algebroids and Lie groupoids, (see [5], [3], [11] and references therein).

Relevant attention was addressed to Dirac geometry, that unifies symplectic, Poisson geometry, regular foliation theory and others ([2]). This geometry can be understood as even-dimensional geometry because any symplectic manifold must have even dimension and Poisson manifolds have a symplectic (singular) foliation. Dirac structures in general are studied as special subundles of Courant algebroids. In the realm of Courant algebroids there are also several areas of research, such as integration, bialgebroid splittings, local forms, generalized complex geometry, and in particular the reduction theory developed in [3]. Reduction theory also works for reduction of Dirac structures.

The odd-dimensional counterpart of these symplectic, pre-symplectic and Poisson geometries are the contact, locally conformally symplectic, and Jacobi structures in its classical definition via trivial line bundles. Also, there is another interpolating structure for the previous odd-geometries, Dirac-Jacobi structures. These last structures appear naturally in the study of geometric prequantization of Dirac structures (see [15], [16]), and also lie naturally in an extension of Courant algebroids, the Courant-Jacobi bundle, or

$\mathcal{E}^1(M)$-bundle ([13], [4] ). In spite of the well known relation between odd-geometries and even-geometries, by the use of the symplectization or Poissonization trick, research on the odd-viewpoint is far from the research on the even counterpart. As examples of this, we see that there is no reduction theorem for Dirac-Jacobi structures. Indeed this is the main goal of this manuscript.

This type of result can be achieved in different ways, for example trying to axiomatize Courant-Jacobi algebroids and to construct its reduction, and as a second step to consider the Dirac-Jacobi reduction as part of this procedure. This way requires extensive work and time, for this reason is far from the aim of this thesis. However, there is another way to construct a Dirac-Jacobi reduction, using the homogenization trick, an extension of the Poissonization procedure, so we can translate Dirac-Jacobi to honest Dirac structures inside a Courant algebroid and from there we proceed with the well known reduction theorem as in [3].

In this thesis we begin the study of the reduction of Dirac-Jacobi structures by using the theory of reduction of Dirac structures inside the reduced Courant algebroid. For this purpose, the document is organized as follows: Chapter 1 contains the definitions and main examples of Dirac and Dirac-Jacobi structures and it describes the first technical tool, the homogenization trick. The second chapter is devoted to describe the reduction procedure as in [3] and the reduction of the canonical Courant algebroid on the manifold $M \times \mathbb{R}$ is presented. Finally, the third chapter deals with the main results of reduction in two interesting cases, the contact/symplectic case and the case for which we have the existence of an equivariant moment map, both defining the extended action for an $\mathcal{E}^1(M)$-

structure. An important situation that can be considered for the Dirac-Jacobi reduction is the isotropicity condition of the $G$-action, for this reason we will study the contact case with a non-isotropic action while the general situation of moment map will be studied as an isotropic action. These two conditions will give us a different type of Dirac-Jacobi reduction: Theorem 4.2.5 and Theorem 4.2.7.

# Chapter 2

# PRELIMINARIES

In this chapter we will define the main differential structures associated to the object of the work. We begin with the general description of the Courant algebroids and Dirac structures. For the case of extended Courant algebroid we just give its definition in the trivial bundle which is the structure we need in order to define the Dirac-Jacobi structures. In the literature there is a lack of an axiomatization for extended Courant algebroids in the same sense as Courant algebroids, but this topic is beyond of the scope of this manuscript. We close this chapter with a construction, named *homogenization trick*, that will allow us to translate some information from extended Courant algebroids (in particular Dirac-Jacobi data) to Courant algebroids (to the particular data of Dirac structures).

## 2.1 COURANT ALGEBROIDS AND DIRAC STRUCTURES

### 2.1.1 Courant Algebroids

The results and definitions in this subsection where almost transcribed from [3], where the reader can found more details, as well as in [6] and [17]. We just want to mention some facts concerning Courant algebroids and Dirac structures that will be used in this text.

**Definition 2.1.1.** A **Courant algebroid** over a manifold $M$ is a structure $(E, \pi, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$, where $E$ is a vector bundle over $M$ equipped with a fiber-wise non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, $[\cdot,\cdot]$ is a bilinear bracket on smooth sections of $E$, and $\pi : E \to TM$ is a bundle map called the **anchor** of the Courant algebroid, all these data satisfying the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

C1) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,

C2) $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$,

C3) $[e_1, fe_2] = f[e_1, e_2] + (\pi(e_1)f)e_2$,

C4) $\pi(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,

C5) $[e_1, e_1] = \mathcal{D}\langle e_1, e_1\rangle$,

where $\mathcal{D} = \frac{1}{2}\pi^* \circ d : C^\infty(M) \to \Gamma(E)$(Using $\langle \cdot, \cdot \rangle$ to identify $E$ with $E^*$).

We see from axiom C5) that the bracket is not skew-symmetric, but rather satisfies

$$[e_1, e_2] = -[e_2, e_1] + 2\mathcal{D}\langle e_1, e_2\rangle. \tag{2.1}$$

**Example 2.1.1.** Let $M$ be a connected 0-dimensional smooth manifold, i.e. $M$ consists of only a single point. In this case we have that a $k$-vector bundle $E$ over $M$ is just a $k$-vector space for some field $k$. Suppose we have a Courant algebroid structure on $M$ with bracket $[,]$ and pairing $\langle , \rangle$. Axioms C2) and C3) are trivial, axiom C1) means that $[,]$ satisfies the Jacobi identity and C5) tell us that it is also skew-symmetric so $(E, [,])$ is a Lie algebra. Axiom C4) becomes $\langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle = 0$ for all $e_1, e_2, e_3 \in E$, so, $E$ is in fact a Lie algebra where is defined an invariant nondegenerate symmetric bilinear form. This kind of structures are interesting in the study of differential equations, see for example [1], but it is not part of this document.

**Example 2.1.2** (Standard Courant algebroid)**.** In this case we will consider the vector bundle $E = TM \oplus T^*M$ over $M$, with pairing

$$\langle X \oplus \alpha, Y \oplus \beta\rangle = \frac{1}{2}(\beta(X) + \alpha(Y)), \tag{2.2}$$

and bracket

$$[\![X \oplus \alpha, Y \oplus \beta]\!] = [X, Y] \oplus (\mathcal{L}_X \beta - i_Y d\alpha). \tag{2.3}$$

This bracket is known as the **Courant bracket** on $M$. We will refer to $(E, \pi, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!])$ as the **standard Courant algebroid** on $M$. Here $\pi$ is the projection on $TM$.

The proof that $E$, with the previous pairing and bracket on sections, is a Courant algebroid, is a straightforward computation on Lie brackets and Cartan calculus on the manifold $E$. From now on, we will use the symbol $\mathbb{T}M$ for the standard Courant algebroid over the base manifold $M$.

**Example 2.1.3.** We can twist the previous structure with a 3-form $H \in \Omega^3(M)$ in the following sense:

$$[X \oplus \alpha, Y \oplus \beta]_H = [X, Y] \oplus \mathcal{L}_X \beta - i_Y d\alpha + i_Y i_X H, \tag{2.4}$$

and the first axiom is equivalent to $dH = 0$. We refer this bracket as the $H$**-twisted bracket** on $M$.

**Definition 2.1.2.** A Courant algebroid is **exact** if the following sequence is exact:

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \tag{2.5}$$

In this case it is possible to verify that any exact Courant algebroid is isomorphic to a twisted structure as in Example 2.1.3 with the 3-form given explicitly via a splitting of the $\pi$ bundle map (for more details we refeer to [3]).

The symmetry group of an exact Courant algebroid, that is, the group of bundle automorphisms preserving the Courant algebroid structure, is given by the following definition.

**Definition 2.1.3.** The **automorphism group** $\mathrm{Aut}(E)$ of a Courant algebroid $E$ is the group of bundle automorphisms $F : E \to E$ covering diffeomorphisms $\varphi : M \to M$ such that

**(i)** $\varphi^*\langle F\cdot\,,\, F\cdot\rangle = \langle\cdot\,,\,\cdot\rangle$, i.e. $F$ is orthogonal,

**(ii)** $[\,F\cdot\,,\, F\cdot\,] = F[\,\cdot\,,\,\cdot\,]$, i.e. $F$ is bracket-preserving,

**(iii)** $\pi \circ F = \varphi_* \circ \pi$, i.e. $F$ is compatible with the anchor.

Let $M$ be a smooth manifold and consider the the $H$-twisted Courant algebroid $\mathbb{T}M$, $H \in \Omega^3_{\mathrm{cl}}(M)$. Let $\varphi$ be a diffeomorphism of $M$ preserving $H$, that is, $\varphi^*(H) = H$. Then we can see that $\varphi$ preserves the structure of $\mathbb{T}M$ in the sense that the vector bundle morphism $d\varphi \oplus (\varphi^{-1})^*$ from $\mathbb{T}M$ into itself satisfies the conditions of the last definition i.e. $d\varphi \oplus (\varphi^{-1})^* : \mathbb{T}M \to \mathbb{T}M$ is an automorphism of the Courant algebroid $\mathbb{T}M$.

On the other hand, the Courant bracket has an additional symmetry given by a **$B$-field transformation**, that is, given by the action of a closed 2-form $B$ via $e^B : X \oplus \alpha \mapsto X \oplus \alpha + i_X B$, (see [6], Proposition 3.23. for more details).

Indeed, an automorphism of an $H$-twisted Courant bracket must be the composition of a diffeomorphism $\varphi$ of $M$ and a transformation given by a 2-form $B$ satisfying $H - \varphi^* H = dB$. The following propositions concern this fact, and it is a transcript of proposition 2.3 in [3].

**Proposition 2.1.4.** *The automorphism group of an exact Courant algebroid $E$ is an extension of the diffeomorphisms preserving the cohomology class $[H]$ by the abelian group of* *closed 2-forms:*

$$0 \to \Omega^2_{\mathrm{cl}}(M) \to \mathrm{Aut}(E) \to \mathrm{Diff}_H(M) \to 0.$$

The infinitesimal version of an automorphism for an $H$-twisted exact Courant algebroid $E$ is given by differentiating a 1-parameter family of automorphisms $F_t = \varphi_t e^{tB}$, $F_0 = \mathrm{Id}$. Then we get that the Lie algebra $\mathrm{Der}(E)$ consists of pairs $(X, B) \in \Gamma(TM) \oplus \Omega^2(M)$ such that $\mathcal{L}_X H = dB$, which acts via

$$(X, B) \cdot (Y + \alpha) = \mathcal{L}_X(Y + \alpha) + i_Y B.$$

This remarks and the following proposition was taken from [3].

**Proposition 2.1.5.** *The Lie algebra of infinitesimal symmetries of an exact Courant algebroid $E$ is an abelian extension of the Lie algebra of smooth vector fields by the closed 2-forms:*

$$0 \to \Omega^2_{\mathrm{cl}}(M) \to \mathrm{Der}(E) \to \Gamma(TM) \to 0.$$

### 2.1.2 Dirac structures

Throughout this subsection we will restrict our attention to exact Courant algebroids which, as we have mentioned in the previous subsection, are isomorphic to a twisted Courant algebroid $\mathbb{T}M$ with pairing and bracket on sections given by (2.2) and (2.4) for some $H \in \Omega^3_{\mathrm{cl}}(M)$.

**Definition 2.1.6.** A **Dirac structure** on $M$ is a vector subbundle $L \subset \mathbb{T}M$ such that:

$D1)$ $L$ is lagrangian with respect to the pairing $\langle \cdot\, , \cdot \rangle$, i.e. $L^\perp = L$, and

$D2)$ $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$.

**Remark 2.1.7.** An isotropic subbundle $L$ is a subbundle such that $\langle \cdot, \cdot \rangle|_L = 0$. Condition D1) is equivalent to say that $L$ is maximally isotropic, which means that $L$ is isotropic and there is no $L' \subset \mathbb{T}M$ isotropic subbundle such that $L$ is strictly contained in $L'$. Because the pairing has splitting signature, the maximality of $L$ is equivalent to $\operatorname{rank} L = \dim M$.

### 2.1.3 Examples

In this subsection we want to see how some important structures can be characterized by means of Dirac structures.

Given any covariant 2-tensor $\omega$ on $M$, we can always define the map $\omega^\sharp : TM \to T^*M$, $X \mapsto i_X\omega$. We will denote by $L_\omega$ the graph of $\omega^\sharp$, which can be seen as a subbundle of $\mathbb{T}M$:

$$L_\omega = \{X \oplus \alpha \mid \omega^\sharp(X) = \alpha\} \subset \mathbb{T}M. \tag{2.6}$$

Suppose we have a covariant 2-tensor $\omega$ on $M$ such that $L_\omega$ is isotropic with respect to $\langle \cdot\,,\,\cdot \rangle_{\mathbb{T}M}$. Then, for all vector fields $X, Y$ on $M$ we have:

$$\begin{aligned} 0 = \langle X \oplus \omega^\sharp(X)\,,\, Y \oplus \omega^\sharp(Y) \rangle_{\mathbb{T}M} &= \frac{1}{2}(\omega^\sharp(Y)(X) + \omega^\sharp(X)(Y)) \\ &= \frac{1}{2}(\omega(Y, X) + \omega(X, Y)). \end{aligned}$$

This implies that $\omega$ is in fact a 2-form on $M$. We can see from the above calculation that the converse is also true, that is, if $\omega$ is a 2-form, the graph $L_\omega$ is isotropic with respect to $\langle \cdot\,,\,\cdot \rangle_{\mathbb{T}M}$.

Now suppose we have a 2-form $\omega$ defined on $M$. Let $X, Y$ be two vector fields on $M$, so we have $X \oplus \omega^\sharp(X)$ and $Y \oplus \omega^\sharp(Y)$ are two sections of $L_\omega$. The standard Courant bracket on $\mathbb{T}M$ computed for $X \oplus \omega^\sharp(X)$ and $Y \oplus \omega^\sharp(Y)$ gives us:

$$[\![\, X \oplus \omega^\sharp(X) \,,\, Y \oplus \omega^\sharp(Y) \,]\!]_{\mathbb{T}M} = [\, X \,,\, Y \,] \oplus \mathcal{L}_X i_Y \omega - i_Y d i_X \omega.$$

This last expression is again a section of $L_\omega$ if and only if

$$\mathcal{L}_X i_Y \omega - i_Y d i_X \omega = i_{[\, X \,,\, Y \,]}\omega. \tag{2.7}$$

Therefore, by Cartan calculus we can see that (2.7) is equivalent to $i_Y i_X d\omega = 0$.

Because this happens for all vector fields $X, Y$ on $M$, we can conclude that sections of $L_\omega$ are closed under the standard Courant bracket on $\mathbb{T}M$ if and only if $\omega$ is closed.

**Example 2.1.4.** Let $\pi$ be a contravariant 2-tensor on $M$. Then we have the bundle map $\pi^\sharp : T^*M \to TM$ given by $\pi^\sharp(\alpha) : C^\infty(M) \to C^\infty(M),\ f \mapsto \pi(\alpha, df)$, where $\pi^\sharp(\alpha)$ is seen as a derivation on the algebra of smooth functions on $M$. Following an analogous way described above, we can prove that $\pi$ is a bivector field on $M$ if and only if $L_\pi := \mathrm{graph}(\pi^\sharp)$ is an isotropic subbundle of $\mathbb{T}M$.

**PRESYMPLECTIC STRUCTURES** A **presymplectic structure** on a smooth manifold $M$ is a closed 2-form $\omega \in \Omega^2_{\mathrm{cl}}(M)$ with constant rank. By the above remarks we have made in this subsection, we can see that if we are given a covariant 2-tensor field $\omega$ such that $L_\omega$ is Dirac, then $\omega$ is a presymplectic structure on $M$.

We had seen that if a 2-form $\omega$ on $M$ is closed, then we have that $L_\omega$ is an isotropic subbundle of $\mathbb{T}M$ whose sections are closed under the standard Courant bracket. The fact that $L_\omega$ is actually a lagrangian subbundle of $\mathbb{T}M$ follows from the fact that $\mathrm{rank}(L_\omega) = \mathrm{rank}(TM)$, which in turns is true because $L_\omega$ is the graph of the map $\omega^\sharp : TM \to T^*M$. Hence, we have $L_\omega$ is a Dirac structure on $M$.

The following proposition shows how presymplectic structures are characterized by Dirac structures.

**Proposition 2.1.8.** *Presymplectic structures on a manifold $M$ are in a one-to-one correspondence with Dirac structures $L$ on $M$ satisfying*

$$L \cap (\{0\} \oplus T^*M) = \{0\}. \tag{2.8}$$

**POISSON STRUCTURES** Let $\pi \in \Gamma(\bigwedge^2 TM)$ be a bivector field on $M$. Define a bracket $\{\cdot, \cdot\}$ on the algebra of smooth functions over $M$ by $\{f, g\} := \pi(df, dg)$ for all $f, g \in C^\infty(M)$. We say that $\{\ ,\ \}$ is a **Poisson structure on** $M$ if it satisfies the Jacobi identity for all $f, g, h \in C^\infty(M)$:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

As we mentioned in Example 2.1.4, a 2-tensor field $\pi$ on $M$ is a bivector field on $M$ if and only if graph$(\pi^\sharp)$ defines an isotropic subbundle of $\mathbb{T}M$.

Suppose $\pi$ is a bivector field on $M$ and consider graph$(\pi^\sharp) \subset \mathbb{T}M$. To simplify notation, let us put $X_f := \pi^\sharp(df)$ for $f \in C^\infty(M)$. Let $f, g \in C^\infty(M)$. Because $d^2 = 0$ we have

$$[\![ X_f \oplus df \,,\, X_g \oplus dg ]\!]_{\mathbb{T}M} = [\, X_f \,,\, X_g \,] \oplus \mathcal{L}_{X_f} dg.$$

We see that the last expression gives a section of $L_\pi$ if and only if

$$[\, X_f \,,\, X_g \,] = \pi^\sharp(\mathcal{L}_{X_f} dg) = \pi^\sharp(d\{f, g\}) = \{\{f, g\}, \cdot\} \tag{2.9}$$

where we have used the fact that $X_f(g) = \{f, g\}$. To say that (2.9) holds is equivalent to say that if we compute each side at an arbitrary function, the result must be equal. Let

$h \in C^{\infty}(M)$. The left side of (2.9) at $h$ is

$$[X_f\,,\, X_g]h = X_f X_g h - X_g X_f h = \{f,\{g,h\}\} - \{g,\{f,h\}\} = \{f,\{g,h\}\} + \{g,\{h,f\}\}.$$

where we have used the skew symmetry of $\{,\}$.

Using again the skew symmetry of $\{,\}$, the right side of (2.9) evaluated at $h$ is equal to $-\{h,\{f,g\}\}$. Therefore, expression (2.9) is equivalent to get $\{,\}$ satisfies the Jacobi identity.

Summarizing, we have seen that $L_\pi$ is an involutive subbundle of $\mathbb{T}M$ under the standard Courant bracket if and only if $\{,\}$ satisfies the Jacobi identity, that is, $\pi$ is a Poisson structure on $M$.

In the following proposition we can see how Poisson structures can be characterized via Dirac structures.

**Proposition 2.1.9.** *Poisson structures $\pi$ on a manifold $M$ are in a one-to-one correspondence with Dirac structures $L$ satisfying*

$$L \cap (TM \oplus \{0\}) = \{0\}. \tag{2.10}$$

**SYMPLECTIC STRUCTURES** A **symplectic structure** on a smooth manifold $M$ is a differentiable closed 2-form $\omega \in \Omega^2(M)$ which is non-degenerate, which means that $\omega$ satisfy one of the following two equivalent conditions:

- The map $\omega^\sharp : TM \to T^*M$, $X \mapsto i_X\omega$, is a bundle isomorphism.
- If $\omega(X,Y) = 0$ for each vector field $Y$ on $M$, then we must have $X = 0$.

By definition, symplectic structures are presymplectic structures. Consequently, to every symplectic structure $\omega$ on $M$ there is associated a Dirac structure on $M$, namely $L_\omega \subset \mathbb{T}M$, where $L_\omega$ is given by (2.6). Furthermore, we have the following proposition.

**Proposition 2.1.10.** *There is a one-to-one correspondence between symplectic structures on $M$ and Dirac structures $L$ on $M$ satisfying the two following conditions*

$$L \cap (\{0\} \oplus T^*M) = \{0\} \quad \textit{and} \tag{2.11}$$

$$L \cap (TM \oplus \{0\}) = \{0\} \tag{2.12}$$

**REGULAR FOLIATIONS** Let $V \subset TM$ be a regular distribution. Consider the subbundle $L = V \oplus V^\circ$ of $\mathbb{T}M$, where $V^\circ$ is the anihilator of $V$. It follows immediately that $L$ is a lagrangian subbundle. We have that the Courant bracket restricted to sections of $L$ reduces to the Lie bracket restricted to $V$. Then, by Frobenis' Theorem, we have that $L$ is involutive with respect to the Courant bracket (hence a Dirac structure) if and only if $V$ is an integrable distribution. Therefore we can see regular foliations as Dirac structures.

**TWISTED SYMPLECTIC STRUCTURES** Consider the $H$-twisted Courant bracket $[\![\ ,\ ]\!]_H$ on $M$, where $H \in \Omega^3_{\mathrm{cl}}(M)$. Let $L \subset \mathbb{T}M$ be a twisted Dirac structure such that $L$ satisfies (2.20) and (2.21). From the proof of Proposition 2.1.10, we can construct a nondegenerate 2-form $\omega$ on $M$ such that $L_\omega = L$, i.e. $L$ is the graph of the map $\omega^\sharp : TM \to T^*M$, $X \mapsto i_X\omega$.

Let $X, Y$ be two vector fields on $M$. We have that

$$[\![\, X \oplus \omega^\sharp(X)\, ,\, Y \oplus \omega^\sharp(Y)\, ]\!]_H = [\, X\, ,\, Y\, ] \oplus \mathcal{L}_X i_Y\omega - i_Y d i_X\omega + i_X i_Y H.$$

Using Cartan calculus we can deduce that the last expression is equivalent to $i_Y i_X d\omega = i_Y i_X H$. Because this happens for all $X, Y \in \mathfrak{X}(M)$, it follows that $L$ is involutive (hence a twisted Dirac structure) if and only if $d\omega = H$.

A nondegenerate 2-form $\omega$ satisfying $d\omega = H$ is called a **twisted symplectic structure** on $M$. Therefore, we have just proved that twisted symplectic structures can be seen as twisted Dirac structures.

## 2.2 EXTENDED COURANT BRACKET AND DIRAC-JACOBI STRUCTURES

In the previous sections we define Dirac structures as Lagrangian subbundles which are involutive with respect to the Courant bracket. In this section we present Dirac-Jacobi structures as ivolutive lagrangian subbundles of a suitable vector bundle endowed with a natural pairing and a bracket on sections which is called the extended Courant bracket. Dirac-Jacobi structures are in certain sense the odd-analogous structures to Dirac structures.

### 2.2.1 The extended Courant bracket

Aïssa Wade introduced the extened Courant bracket in [14]. This extension lets us see some important structures such as contact, Jacobi and localy conformal presymplectic structures as Dirac structures in a suitable vector bundle on $M$. In this subsection we present the extended Courant bracket as in [8].

Consider the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ over $M$. In the same way as in $\mathbb{T}M$, there is a natural fiberwise bilinear form on $\mathcal{E}^1(M)$ defined by:

$$\langle (X_1, f_1) \oplus (\alpha_1, g_1)\,,\,(X_2, f_2) \oplus (\alpha_2, g_2)\rangle = \frac{1}{2}(i_{X_2}\alpha_1 + i_{X_1}\alpha_2 + f_1 g_2 + f_2 g_1)$$

for any $(X_j, f_j) \oplus (\alpha_j, g_j) \in \mathcal{E}^1(M)$, with $j = 1, 2$.

The **extended Courant bracket** on sections of $\mathcal{E}^1(M)$ is given by the non-skewsymmetric operation

$$\begin{aligned}\llbracket\,(X_1, f_1) \oplus (\alpha_1, g_1)\,,\,(X_2, f_2) \oplus (\alpha_2, g_2)\,\rrbracket_{\mathcal{E}^1(M)} &= ([\,X_1\,,\,X_2\,], X_1 f_2 - X_2 f_1)\\ &\oplus (\mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1 + g_2 df_1 + f_2 dg_1 + f_1\alpha_2 - f_2\alpha_1, f_1 g_2 + X_1 g_2 - X_2 g_1 + i_{X_2}\alpha_1),\end{aligned} \tag{2.13}$$

for all sections $(X_j, f_j) \oplus (\alpha_j, g_j)$ of $\mathcal{E}^1(M)$, with $j = 1, 2$. In the next section we will see how this bracket is related to the Courant bracket on $M$ by means of the *homogenization trick*.

We will use the notation $\llbracket\cdot,\cdot\rrbracket_{\mathcal{E}^1(M)}$ when some confusion is possible, but when no confusion can arise we will just write $\llbracket\cdot,\cdot\rrbracket$ or $[\cdot,\cdot]$ for the extended Courant bracket.

We want to give a more familiar expression for this bracket. In order to do this, first we mention the fact that in the more general vector bundle $\bigwedge^k T^*M \times \bigwedge^{k-1} T^*M$ there is a Cartan calculus given by the following exterior, interior and Lie derivatives:

The exterior derivative is given by

$$\widetilde{d} : \Omega^k(M) \times \Omega^{k-1}(M) \to \Omega^{k+1}(M) \times \Omega^k(M)$$

$$\widetilde{d}(\alpha, \beta) = (d\alpha, (-1)^k\alpha + d\beta), \tag{2.14}$$

for any $\alpha \in \Omega^k(M)$, $\beta \in \Omega^{k-1}(M)$, where $d$ is the exterior differentiation operator. When $k=0$, the derivative is defined by $\widetilde{d}f = (df, f)$. As a direct consequence of (2.14) we have $(\widetilde{d})^2 = 0$.

There is also a contraction map given by

$$\widetilde{i} : (TM \times \mathbb{R}) \times \Big(\bigwedge^k(T^*M) \times \bigwedge^{k-1}(T^*M)\Big) \to \bigwedge^{k-1}(T^*M) \times \bigwedge^{k-2}(T^*M)$$

$$\widetilde{i}_{(X,f)}(\alpha, \beta) = (i_X\alpha + (-1)^{k+1} f\beta, i_X\beta), \tag{2.15}$$

that also extends by linearity to the respective space of sections.

From these two operators, we can define the Lie derivative as

$$\widetilde{\mathcal{L}}_{(X,f)} := \widetilde{i}_{(X,f)} \circ \widetilde{d} + \widetilde{d} \circ \widetilde{i}_{(X,f)}. \tag{2.16}$$

In particular for $k=1$ these operators on $\mathcal{E}^1(M)$ allow us to write (2.13) as follows. Let $e_j = (X_j, f_j) \oplus (\alpha_j, g_j) \in \Gamma(\mathcal{E}^1(M))$ $(j = 1, 2)$. Then after a straightforward calculation we obtain

$$[\![\, e_1\,,\, e_2 \,]\!] = ([\, X_1\,,\, X_2\,], X_1f_2 - X_2f_1) \oplus (\widetilde{\mathcal{L}}_{(X_1,f_1)}(\alpha_2, f_2) - \widetilde{i}_{(X_2,f_2)}\widetilde{d}(\alpha_1, g_1)), \tag{2.17}$$

which, as we can see, is similar to the expression for the Courant bracket in sections of $\mathbb{T}M$ (2.3).

**Remark 2.2.1.** The bracket we have just defined is the non-skew-symmetric version of the extended Courant bracket. The skew-symmetric version can be found in [14]. The skew and non-skew symmetric versions coincide on lagrangian subbundles.

We now state a proposition with some properties of the extended Courant bracket, but we will give the proof in Section 2.3.

**Proposition 2.2.2.** *Consider the projection map* $\pi : \mathcal{E}^1(M) \to TM$, $(X, f) \oplus (\alpha, g) \mapsto X$ *and the inclusion map* $\pi^* : T^*M \times \mathbb{R} \to \mathcal{E}^1(M)$, $(\alpha, g) \mapsto (\alpha, g) \oplus 0$. *Then the extended Courant bracket on sections of* $\mathcal{E}^1(M)$ *satisfies the following properties for all sections* $e_1, e_2, e_2 \in \Gamma(\mathcal{E}^1(M))$ *and* $f \in C^\infty(M)$*:*

*E1)* $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$

*E2)* $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$,

*E3)* $[e_1, fe_2] = f[e_1, e_2] + (\pi(e_1)f)e_2$.

*E4) Define* $\widetilde{\mathcal{D}} := \frac{1}{2}\pi^* \circ \widetilde{d} : C^\infty(M) \to \Gamma(\mathcal{E}^1(M))$ *(where we identify* $\mathcal{E}^1(M)^*$ *with* $\mathcal{E}^1(M)$ *via* $\langle \cdot, \cdot \rangle_{\mathcal{E}^1(M)}$*). Then we have*

$$[\![ e_1, e_1 ]\!]_{\mathcal{E}^1(M)} = \widetilde{\mathcal{D}}\langle e_1, e_1 \rangle_{\mathcal{E}^1(M)}.$$

### 2.2.2 Dirac-Jacobi structures(DJ-structures)

**Definition 2.2.3.** A **Dirac-Jacobi structure** on a manifold $M$ is a subbundle $L \subset \mathcal{E}^1(M)$ over $M$, which satisfies the following two conditions.

DJ1) $L$ is lagrangian with respect to $\langle \cdot, \cdot \rangle_{\mathcal{E}^1(M)}$.

DJ2) $[\![ \Gamma(L), \Gamma(L) ]\!]_{\mathcal{E}^1(M)} \subset \Gamma(L)$.

**Remark 2.2.4.** Note that at each fiber, the pairing $\langle \cdot, \cdot \rangle_{\mathcal{E}^1(M)}$, is the natural pairing on the direct sum of a vector space and its dual. Then we have that the condition DJ1) is equivalent to $\langle \cdot, \cdot \rangle|_L = 0$ and $\operatorname{rank} L = \dim M + 1$ as in Remark 2.1.7.

Let us mention some examples of Dirac-Jacobi structures.

### 2.2.3 Examples

**LOCALLY CONFORMAL PRESYMPLECTIC STRUCTURES** See [7] and [14]. A **locally conformal presymplectic structure** on a manifold $M$ is a pair $(\omega, \eta)$, where $\omega$ and $\eta$ are respectively a 2-form and a closed 1-form on $M$, satisfying $d\omega = \eta \wedge \omega$.

The characterization by the extended Courant bracket is given in the following proposition which is part of Proposition 2.2 in [8].

**Proposition 2.2.5.** *Locally conformal presymplectic structures $(\omega, \eta)$ on $M$ corresponds to Dirac-Jacobi structures $L \subset \mathcal{E}^1(M)$ such that*

$$L_x \cap (\{0\} \oplus (T_x^* \times \mathbb{R})) = \{0\}$$

*for all $x \in M$.*

**JACOBI STRUCTURES** A **Jacobi structure** on a manifold $M$ is given by a pair $(\pi, X)$ formed by a bivector field $\pi$ and a vector field $X$ such that

$$[X, \pi]_s = 0, \ [\pi, \pi]_s = 2X \wedge \pi,$$

where $[\ ,\ ]_s$ is the Schouten-Nijenhuis bracket on the space of multi-vector fields. A manifold endowed with a Jacobi structure is called a **Jacobi manifold**.

When $X = 0$, we get a Poisson structure.

In general, if we have a bivector field $\pi$ and a vector field $X$ defined on $M$, we can define the following map

$$\begin{aligned} (\pi, X)^\sharp : T^*M \times \mathbb{R} &\rightarrow TM \times \mathbb{R} \\ (\alpha, f) &\mapsto (\pi^\sharp(\alpha) + fX, -i_X\alpha), \end{aligned} \tag{2.18}$$

where $\pi^\sharp : T^*M \rightarrow TM$ is given by $\pi^\sharp(\alpha) : C^\infty(M) \rightarrow C^\infty(M); g \mapsto \pi(\alpha, dg)$ (viewing $TM$ as a space of derivations of $C^\infty(M)$).

We have that the graph $L_{(\pi,X)}$ of $(\pi, X)^\sharp$ is a Dirac-Jacobi structure if and only if $(\pi, X)$ is a Jacobi structure. See Proposition 2.2 in [8] and references there in.

**Proposition 2.2.6.** *There is a one-to-one correspondence between Jacobi structures and Dirac-Jacobi structures $L$ on $M$ satisfying the following property.*

$$L \cap ((TM \times \mathbb{R}) \oplus \{0\}) = \{0\}. \tag{2.19}$$

**CONTACT STRUCTURES** Let $M$ be a $(2n+1)$-dimensional smooth manifold. A 1-form $\eta$ on $M$ is **contact** if $\eta \wedge (d\eta)^n \neq 0$ at every point. We call $(M, \eta)$ a **contact manifold**.

Given a contact 1-form $\eta$ on $M$ we construct a Jacobi structure on $M$ as follows. First, we have the isomorphism $\flat_\eta : TM \rightarrow T^*M$ given by $\flat_\eta(X) = i_X d\eta + (\eta(X))\eta$. Then the Jacobi structure is given by

$$\pi(\alpha, \beta) = d\eta(\flat_\eta^{-1}(\alpha), \flat_\eta^{-1}(\beta)), \quad \text{for all } \alpha, \beta \in T^*M,$$

$$E = \flat_\eta^{-1}(\eta),$$

which satisfies that $((\pi, E)^\sharp)^{-1}(X, f) = (-i_X d\eta - f\eta, \eta(X))$, where $((\pi, E)^\sharp)^{-1}$ is given by 2.18. Therefore, as for the case of Jacobi structures, we have that the graph

graph$((\pi, E)^\sharp)^{-1} < \mathcal{E}^1(M)$ defines a Dirac-Jacobi structure on $M$. So, When $\eta$ is a contact form on $M$, the corresponding Dirac-Jacobi structure on $M$ is given by a subbundle of $\mathcal{E}^1(M)$ whose sections are of the form $(X, f) \oplus (-i_X d\eta - f\eta, \eta(X))$.

Finally, we have an analogous to Proposition 2.1.10 for contact structures.

**Proposition 2.2.7.** *There is a one-to-one correspondence between contact 1-forms on a* $(2n+1)$*-dimensional manifold and Dirac-Jacobi structures* $L$ *on* $M$ *satisfying the properties*

$$L \cap ((TM \times \mathbb{R}) \oplus \{0\}) = \{0\} \tag{2.20}$$

$$L \cap (\{0\} \oplus (T^*M \times \mathbb{R})) = \{0\}. \tag{2.21}$$

For more details see [8].

## 2.3 THE HOMOGENIZATION TRICK

In this section we present a very useful tool for deriving properties of Dirac-Jacobi structures by using known properties of Dirac structures. It consists on a way to relate the extended Courant algebroid on $\mathcal{E}^1(M)$ with the standard Courant algebroid on a suitable manifold. This construction begins with introducing a new real coordinate in the base manifold, that is, we first transform $M$ into $\widetilde{M} := M \times I$ for an open interval $I \subset \mathbb{R}$.

For a positive integer $k$, consider the vector bundle

$$\mathcal{E}^k(M) := (TM \times \mathbb{R}) \oplus (\bigwedge^k T^*M \times \bigwedge^{k-1} T^*M)$$

over $M$, and $L$ any subbundle on it. Also, we will use the following vector bundle over $\widetilde{M}$.

$$\mathbb{T}^k(\widetilde{M}) := T\widetilde{M} \oplus \bigwedge^k T^*\widetilde{M}.$$

Then, we define the subbundle of $\mathbb{T}^k(\widetilde{M})$ whose fibers at each point $(p,s) \in \widetilde{M}$ are given by

$$\tau(L)_{(p,s)} := \{(X + f\partial_s t) \oplus e^s(\alpha + \beta \wedge d_s t) \mid (X,f) \oplus (\alpha,\beta) \in L_p\} \leq \mathbb{T}^k(\widetilde{M})_{(p,s)}. \quad (2.22)$$

It follows immediately that the fiber of $L$ at an arbitrary point $p \in M$ is isomorphic as a vector space to the fiber of $\tau(L)_{(p,s)}$ for each $s \in \mathbb{R}$. In particular, the dimensions of $\mathcal{E}^k(M)_p$ and $\mathbb{T}^k(\widetilde{M})_{(p,s)}$ are equal for all $(p,s) \in \widetilde{M}$.

When we fix a subbundle $L \subset T\widetilde{M} \oplus \bigwedge^k T^*\widetilde{M}$, it is useful to regard $\tau$ as a map of sections of $L$ to sections of $\tau(L)$. That is, to consider the $\mathbb{R}$-linear map (denoted by the same symbol $\tau$),

$$\begin{aligned} \tau : \Gamma(L) &\to \Gamma(\mathbb{T}^k(\widetilde{M})) \\ (X,f) \oplus (\alpha,\beta) &\mapsto X + f\partial t \oplus e^t(\alpha + \beta \wedge dt) \end{aligned} \quad (2.23)$$

for all $(X,f) \oplus (\alpha,\beta) \in \Gamma(L)$.

This last map is actually a morphism of $C^\infty(M)$-modules because when we consider a function $h \in \mathbb{C}^\infty(M)$ we get

$$\tau(he) = h\tau(e), \quad e \in \Gamma(L).$$

The meaning of the symbol $\tau$ will be clear from the context.

**Definition 2.3.1.** A subbundle $A \leq \mathbb{T}^k\widetilde{M}$ is **homogeneous** if there exists a subbundle $A_0 \subseteq \mathcal{E}^k(M)$ such that $\tau(A_0) = A$. Sections of $\mathbb{T}^k\widetilde{M}$ having the form $s = \tau(s_0)$ for some section $s_0 \in \Gamma(\mathcal{E}^k(M))$ will be called **homogeneous sections**.

Following this ideas, the $C^\infty(M)$-submodule of $\mathbb{T}^k\widetilde{M}$ that consists of all the homogeneous sections of a homogeneous subbundle $A \leq \mathbb{T}^k\widetilde{M}$ will be denoted as $\Gamma_{\text{hom}}(A)$.

Moreover, note that the homogeneous sections of a subbundle $A$ of $\mathcal{E}^1(M)$ form a $C^\infty(M)$ module.

Using the construction in (2.22), we get the following inclusion:

$$\tau(\Gamma(L)) \subset \Gamma(\tau(L)), \tag{2.24}$$

but in general we do not have the equality: Just take $A = \mathcal{E}^k(M)$, so we have $\tau(\mathcal{E}^k(M)) = \mathbb{T}^k\widetilde{M}$, but it is clear that $\tau(\Gamma(\mathcal{E}^k(M))) \subsetneq \Gamma(\mathbb{T}^k\widetilde{M})$.

Before we proceed with Proposition 2.3.4, let us state some properties of the correspondence $\tau$.

**Lemma 2.3.2.** *The following properties are satisfied by $\tau$.*

*i)* $d \circ \tau = \tau \circ \widetilde{d}$

*ii)* $i \circ \tau = \tau \circ \widetilde{i}$

*iii) If $S_1, S_2 \in \Gamma(TM \times \mathbb{R})$, then $\tau S_1, \tau S_2 \in \Gamma(T\widetilde{M})$ and $\tau[\![S_1, S_2]\!]_{\mathcal{E}^1(M)} = [\,\tau S_1, \tau S_2\,]$, where the bracket $[\,,\,]$ is the Lie bracket on sections of $T\widetilde{M}$.*

*Proof.* The proof is straightforward, but to exemplify the use of this operations we present the proof of i).

Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. Then on the one hand we have.

$$\begin{aligned} d \circ \tau(\alpha, \beta) &= d(e^t(\alpha + \beta \wedge dt)) = e^t\, dt \wedge (\alpha + \beta \wedge dt) + e^t\, d(\alpha + \beta \wedge dt) \\ &= e^t(dt \wedge \alpha + d\alpha + d\beta \wedge dt) = e^t(d\alpha + (-1)^k \alpha \wedge dt + d\beta \wedge dt) \\ &= e^t(d\alpha + ((-1)^k \alpha + d\beta) \wedge dt). \end{aligned}$$

Here we have used the Leibniz property of $d$ and the properties of the wedge product. On the other hand we have.

$$\begin{aligned}\tau \circ \widetilde{d}(\alpha, \beta) &= \tau(d\alpha, (-1)^k \alpha + d\beta) \\ &= e^t(d\alpha + ((-1)^k \alpha + d\beta) \wedge dt),\end{aligned}$$

which is the same as $d \circ \tau(\alpha, \beta)$. Therefore we can conclude i). $\square$

**Corollary 2.3.3.** *Consider the extended Courant bracket on $\mathcal{E}^1(M)$, the Courant bracket on $\mathbb{T}M$ and the $\tau$ map defined on sections of $\mathcal{E}^1(M)$. Then we have*

$$\tau(\langle \cdot, \cdot \rangle_{\mathcal{E}^1(M)}) = \langle \tau \cdot, \tau \cdot \rangle_{\mathbb{T}\widetilde{M}} \quad \textit{and} \quad \tau\left(\llbracket \cdot, \cdot \rrbracket_{\mathcal{E}^1(M)}\right) = \llbracket \tau \cdot\, , \tau \cdot \rrbracket_{\mathbb{T}\widetilde{M}}. \tag{2.25}$$

*Proof.* The first identity is easy to compute taking in account that $\tau(f) = e^t f \in C^\infty(\widetilde{M})$ for all $f \in C^\infty(M)$. For the second identity, let $S, S'$ be two sections of $\mathcal{E}^1(M)$. Suppose $S = S_T \oplus S_C$ and $S' = S'_T \oplus S'_C$, with $S_T, S'_T \in TM \times \mathbb{R}$ and $S_C, S'_C \in T^*M \times \mathbb{R}$. In this way we have

$$\begin{aligned}\tau\llbracket S\, , S' \rrbracket_{\mathcal{E}^1(M)} &= \tau(\llbracket S_T\, , S'_T \rrbracket_{\mathcal{E}^1(M)}) \oplus \tau(\widetilde{\mathcal{L}}_{S_T} S'_C - \widetilde{i}_{S'_T} \widetilde{d} S_C) \\ &= [\tau S_T\, , \tau S'_T] \oplus \mathcal{L}_{\tau S_T} \tau S'_C - i_{\tau S'_T} d\tau S_C \\ &= \llbracket \tau S\, , \tau S' \rrbracket_{\mathbb{T}\widetilde{M}}\end{aligned}$$

where we have used equation (2.17), the preceding lemma and the fact that $\tau S = \tau S_T \oplus \tau S_C$. $\square$

As a corollary of Lemma 2.3.2, we prove Proposition 2.2.2. But before starting the proof, note that if $\widetilde{\pi} : \mathbb{T}\widetilde{M} \to T\widetilde{M}$ is the projection on $T\widetilde{M}$, then, for all $f \in C^\infty(M)$ and for all sections $e \in \Gamma(\mathcal{E}^1(M))$, it holds $\mathcal{L}_{\tau(\pi e)} f = \mathcal{L}_{\pi e} f = \mathcal{L}_{\widetilde{\pi}(\tau e)} f$, where we are regarding all the functions as depending only on $M$.

*proof of Proposition 2.2.2.* In this proof we use the properties of the definition of a Courant bracket that satisfies $[\![\, , \,]\!]_{\mathbb{T}\widetilde{M}}$.

E1) For every $e_1, e_2, e_3 \in \Gamma(\mathcal{E}^1(M))$ we have

$$\begin{aligned}
\tau[\![e_1 \, , \, [\![e_2 \, , \, e_3]\!]]\!]_{\mathcal{E}^1(M)} &= [\![\tau e_1 \, , \, [\![\tau e_2 \, , \, \tau e_3]\!]_{\mathbb{T}\widetilde{M}}]\!]_{\mathbb{T}\widetilde{M}} \\
&= [\![[\![\tau e_1 \, , \, \tau e_2]\!]_{\mathbb{T}\widetilde{M}} \, , \, \tau e_3]\!]_{\mathbb{T}\widetilde{M}} + [\![\tau e_2 \, , \, [\![\tau e_1 \, , \, \tau e_3]\!]_{\mathbb{T}\widetilde{M}}]\!]_{\mathbb{T}\widetilde{M}} \\
&= \tau([\![[\![e_1 \, , \, e_2]\!]_{\mathcal{E}^1(M)} \, , \, e_3]\!]_{\mathcal{E}^1(M)} + [\![e_2 \, , \, [\![e_1 \, , \, e_3]\!]_{\mathcal{E}^1(M)}]\!]_{\mathcal{E}^1(M)}).
\end{aligned}$$

Since $\tau : \Gamma(\mathcal{E}^1(M)) \to \Gamma(\mathbb{T}\widetilde{M})$ is injective, we can conclude E1).

E2) Let $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(\mathcal{E}^1(M))$, then we have

$$\begin{aligned}
\pi([\![e_1 \, , \, e_2]\!]_{\mathcal{E}^1(M)}) \cdot f &= \widetilde{\pi}[\![\tau e_1 \, , \, \tau e_2]\!]_{\mathbb{T}\widetilde{M}} \cdot f \\
&= [\![\widetilde{\pi}\tau e_1 \, , \, \widetilde{\pi}\tau e_2]\!]_{\mathbb{T}\widetilde{M}} \cdot f \\
&= [\![\pi e_1 \, , \pi e_2 \,]\!]_{\mathcal{E}^1(M)} \cdot f.
\end{aligned}$$

Because this is true for every $f \in C^\infty(M)$ we conclude that $\pi([\![e_1 \, , \, e_2]\!])_{\mathcal{E}^1(M)} = [\![\pi e_1 \, , \, \pi e_2]\!]_{\mathcal{E}^1(M)}$

E3) For $e_1, e_2 \in \Gamma(\mathcal{E}^1(M))$ and $f \in C^\infty(M)$, then

$$\begin{aligned}
\tau[\![\, e_1 \, , \, f e_2 \,]\!]_{\mathcal{E}^1(M)} &= [\![\, \tau e_1 \, , \, \tau(f e_2) \,]\!]_{\mathbb{T}\widetilde{M}} \\
&= f[\![\, \tau e_1 \, , \, \tau e_2 \,]\!]_{\mathbb{T}\widetilde{M}} + (\widetilde{\pi}(\tau e_1) \cdot f)(\tau e_2) \\
&= \tau(f[\![\, e_1 \, , \, e_2 \,]\!]_{\mathcal{E}^1(M)} + (\pi e_1 \cdot f) e_2).
\end{aligned}$$

By the injectivity of $\tau$ we obtain E3).

E4) Notice that $\widetilde{\pi}^* \circ \tau = \tau \circ \pi^*$. Then we have the following computation.

$$\begin{aligned}
\tau([\![e_1, e_1]\!]_{\mathcal{E}^1(M)}) = [\![\tau e_1, \tau e_1]\!]_{\mathbb{T}\widetilde{M}} &= \mathcal{D}\langle \tau e_1, \tau e_1\rangle_{\mathbb{T}\widetilde{M}} \\
&= \mathcal{D}\tau\langle e_1, e_1\rangle_{\mathcal{E}^1(M)} \\
&= \frac{1}{2}\widetilde{\pi}^* \circ d(\tau\langle e_1, e_1\rangle_{\mathcal{E}^1(M)}) \\
&= \tau\left(\frac{1}{2}\pi^* \circ \widetilde{d}\langle e_1, e_1\rangle_{\mathcal{E}^1(M)}\right) \\
&= \tau(\widetilde{\mathcal{D}}\langle e_1, e_1\rangle_{\mathcal{E}^1(M)}).
\end{aligned}$$

Again, by the injectivity of $\tau$ in sections we deduce E4).

$\square$

Now, we provide a proof of the second claim in Remark 2.5 of [9].

**Proposition 2.3.4.** *Let $L$ be a subbundle of the bundle $\mathcal{E}^1(M)$ and consider $\tau(L) \subset \mathbb{T}\widetilde{M}$. Then we have that $L$ is a Dirac-Jacobi structure on $M$ if and only if $\tau(L)$ is a Dirac structure on $\widetilde{M}$.*

*Proof.* First note that from Corollary 2.3.3 we have that $L$ is Lagrangian with respect to $\langle\cdot,\cdot\rangle_{\mathcal{E}^1(M)}$ if and only if $\tau(L)$ is Lagrangian with respect to $\langle\cdot,\cdot\rangle_{\mathbb{T}\widetilde{M}}$.

Again by Corollary 2.3.3 we have that if $\Gamma(\tau(L))$ is closed under $[\![\cdot,\cdot]\!]_{\mathbb{T}\widetilde{M}}$, then $\Gamma(L)$ is closed under $[\![\cdot,\cdot]\!]_{\mathcal{E}^1(M)}$. Note that by the paragraph made after Definition 2.3.1, Corollary 2.3.3 does not imply the converse, but rather it implies that if $\Gamma(L)$ is closed under $[\![\cdot,\cdot]\!]_{\mathbb{T}\widetilde{M}}$ then the homogeneous sections of $\tau(L)$ are closed under $[\![\cdot,\cdot]\!]_{\mathcal{E}^1(M)}$. So, assuming that $L$ is Dirac-Jacobi we have $\tau(L)$ is Lagrangian and homogeneous sections of $\tau(L)$ are closed under $[\![\cdot,\cdot]\!]_{\mathcal{E}^1(M)}$. Let $e_1, e \in \tau(\Gamma(L)) \subseteq \Gamma(\tau(L))$ be two homogeneous sections of $\tau(L)$ and $e_2 \in \Gamma(\tau(L))$. Then by the property C4) in Definition 2.1.1 we have

$$\langle [\![e_1, e_2]\!]_{\mathbb{T}\widetilde{M}}, e\rangle = \widetilde{\pi}(e_1)\langle e_2, e\rangle - \langle e_2, [\![e_1, e]\!]_{\mathbb{T}\widetilde{M}}\rangle = 0,$$

where the two terms of the right hand side are zero because we have that $[\![e_1, e]\!]_{\mathbb{T}\widetilde{M}} \in \Gamma(\tau(L))$ and $\tau(L)$ is Lagrangian. Because $e$ was arbitrary, then at each point $(p, s) \in \widetilde{M}$ we have $\langle ([\![e_1, e_2]\!]_{\mathbb{T}\widetilde{M}})_{(p,s)}\, ,\, e_{(p,s)}\rangle = 0$ for all $e_{(p,s)} \in \tau(L)_{(p,s)}$. Because $\tau(L)$ is Lagrangian we have $[\![e_1, e_2]\!]_{\mathbb{T}\widetilde{M}} \in \Gamma(\tau(L))$ for every homogeneous section $e_1$ of $\tau(L)$ and for every section $e_1$ of $\tau(L)$.

Now, take $e_1, e_2 \in \Gamma(\tau(L))$ and $e$ an homogeneous section of $\tau(L)$, then, as before we have

$$\langle [\![e_1, e_2]\!]_{\mathbb{T}\widetilde{M}}, e\rangle = \widetilde{\pi}(e_1)\langle e_2, e\rangle - \langle e_2, [\![e_1, e]\!]_{\mathbb{T}\widetilde{M}}\rangle = 0,$$

where the last terms are zero because by the previous analysis $[\![e_1, e]\!]_{\mathbb{T}\widetilde{M}} \in \Gamma(\tau(L))$ and $\tau(L)$ is Lagrangian. As before, because $e$ was arbitrary and $\tau(L)$ is Lagrangian, we can conclude that $[\![e_1, e_2]\!]_{\mathbb{T}\widetilde{M}} \in \Gamma(\tau(L))$ for arbitrary sections of $\tau(L)$, and this completes the proof.

$\square$

The trick presented so far is the key idea to develop the reduction procedure for extended Courant algebroids and Dirac-Jacobi structures in terms of the well known reduction as in [3].

# Chapter 3

# REDUCTION OF COURANT ALGEBROIDS

## 3.1 EXTENDED ACTIONS

In order to obtain a reduction of a structure on a manifold $M$ we should first consider a $G$-action on the manifold. But in the case of Courant Algebroids it is not enough because there are more on its symmetries. For this reason it was introduced the so called extended actions (see [3]).

**Definition 3.1.1.** A **Courant algebra** over a Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ equipped with a bilinear bracket $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ and a map $\pi : \mathfrak{a} \to \mathfrak{g}$, which satisfy the following conditions for all $a_1, a_2, a_3 \in \mathfrak{a}$:

c1) $[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [[a_2, a_1], a_3]$,

c2) $\pi([a_1, a_2]) = [\pi(a_1), \pi(a_2)]$.

An **exact Courant algebra** is one for which $\pi$ is surjective and $\mathfrak{h} = \ker \pi$ is abelian, i.e. $[h_1, h_2] = 0$ for all $h_1, h_2 \in \mathfrak{h}$.

**Example 3.1.1.** Consider the direct sum $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$, where $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is a $\mathfrak{g}$-module. Define over $\mathfrak{a}$ the following bracket

$$[\,(v_1, h_1)\,,\,(v_2, h_2)\,] = ([\,v_1\,,\,v_2\,], v_1 \cdot h_2), \quad \text{for all } (v_i, h_i) \in \mathfrak{a}, i = 1, 2, \tag{3.1}$$

where $v \cdot h$ denotes the $\mathfrak{g}$-action. This bracket is called the **hemisemidirect product** of $\mathfrak{g}$ with $\mathfrak{h}$ and $(\mathfrak{a}, [,], \mathrm{pr}_1 : \mathfrak{a} \to \mathfrak{g})$ becomes a Courant algebra over $\mathfrak{g}$, where $\mathrm{pr}_1$ is the projection onto the fist component. We will use the symbol $\mathfrak{g} \ltimes \mathfrak{h}$ for the Courant algebra $(\mathfrak{a}, [,], \mathrm{pr}_1)$.

**Definition 3.1.2.** Let $G$ be a connected Lie group acting on a manifold $M$ with infinitesimal action $\psi : \mathfrak{g} \to \Gamma(TM)$. An **extended action to an exact Courant algebroid** $E$ over $M$, of the infinitesimal action $\psi$, is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, "\kappa"] & \mathfrak{a} \arrow[r] \arrow[d, "\xi"] & \mathfrak{g} \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \Gamma(T^*M) \arrow[r] & \Gamma(E) \arrow[r] & \Gamma(TM) \arrow[r] & 0 \end{tikzcd}$$

where $\mathfrak{a} \to \mathfrak{g}$ is an exact Courant algebra over $\mathfrak{g}$ with kernel $\mathfrak{h}$ and with a Courant algebra morphism $\xi : \mathfrak{a} \to \Gamma(E)$ such that $\mathfrak{h}$ acts trivially on sections of $E$, that is, $\mathrm{ad} \circ \xi(\mathfrak{h}) = \mathrm{ad} \circ \kappa(\mathfrak{h}) = 0$ where ad stands the adjoint action on $\Gamma(E)$.

**Remark 3.1.3.** *i)* The condition that $\mathfrak{h}$ acts trivially on $\Gamma(E)$ is equivalent to $\kappa(\mathfrak{h}) \subset \Omega^1_{\mathrm{cl}}(M)$. We can see this as follows. Take $a \in \mathfrak{h}$, because $\mathfrak{h}$ is the kernel of $\mathfrak{a} \to \mathfrak{g}$ and by the commutativity of the last diagram, we have that the cotangent part of $\kappa(a)$ is zero, hence $\kappa(a) \in \Omega^1(M)$. Then, for $X \oplus \alpha \in \Gamma(E)$ we have that the adjoint action of $\kappa(a)$ on $X \oplus \alpha$ is,

$$\begin{aligned} \mathrm{ad} \circ \kappa(a)(X \oplus \alpha) &= [\![0 \oplus \kappa(a), X \oplus \alpha]\!] \\ &= [0, X] \oplus \mathcal{L}_0 \alpha - i_X d\kappa(a) + i_X i_0 H \\ &= -i_X d\kappa(a). \end{aligned}$$

Therefore, if $\mathrm{ad} \circ \kappa(a) = 0$ we have that $-i_X d\kappa(a) = 0$ holds for every $X \in \Gamma(TM)$, that is, $\kappa(a)$ is closed. The converse is clear.

*ii)* with the hypothesis in definition 3.1.2 we have an induced infinitesimal action on $\Gamma(E)$ by defining for all $v \in \mathfrak{g}$ and for all $e \in \Gamma(E)$

$$v \cdot e = [\![\xi(v'), e]\!], \tag{3.2}$$

where $v' \in \mathfrak{a}$ is such that $\pi(v') = v$. Furthermore, this infinitesimal action integrates to an action of $G$ on the total space $E$, acting by Courant automorphisms (see [17] reference for details).

We show some important cases of extended actions which will be used to construct the desired reduction.

**Example 3.1.2.** Let $G$ be a Lie group acting on a (pre)symplectic manifold $(M, \omega)$. Define $\kappa(v) = i_{\psi(v)}\omega$ for all $v \in \mathfrak{g}$. Then we have that $\xi = \psi \times \kappa : \mathfrak{g} \oplus \mathfrak{g} \to \Gamma(\mathbb{T}M)$ defines an extended action, where we are using the Courant algebra structure on $\mathfrak{g} \oplus \mathfrak{g}$ of the Example 3.1.1 ($\mathfrak{g}$ acting on itself by the adjoint representation). The verification of Courant morphism of $\xi$ and the condition $\kappa(\mathfrak{h}) \subset \Omega^1_{cl}(M)$ just follows from the facts that $\omega$ is closed and $\mathcal{L}_{\psi(u)}\omega = 0$ for all $u \in \mathfrak{g}$.

**Example 3.1.3.** Suppose $G$ is a Lie group acting on a manifold $M$ with infinitesimal action $\psi : \mathfrak{g} \to TM$. Let $\mathfrak{h}$ be a $\mathfrak{g}$-module. Consider a smooth map $\mu : M \to \mathfrak{h}^*$. For every $h \in \mathfrak{h}$ define the map $\mu_h$ by

$$\mu_h : M \to \mathbb{R}; \quad \mu_h(p) = \mu(p)(h). \tag{3.3}$$

It follows that $\mu_h$ is a smooth map. We say that a map $\mu : M \to \mathfrak{h}^*$ is $\mathfrak{g}$-equivariant if the following equation holds for all $v \in \mathfrak{g}$ and all $h \in \mathfrak{h}$:

$$\mu_{v \cdot h} = \psi(v)\mu_h,$$

where $v \cdot h$ means the $\mathfrak{g}$-action of $v$ over $h$.

Under the equivariance of $\mu$ we get that the map $\xi$ defined as follows,

$$\begin{aligned} \xi : \mathfrak{g} \ltimes \mathfrak{h} &\rightarrow \mathbb{T}M \\ (v,h) &\mapsto \psi(v) + d(\mu_h), \end{aligned} \tag{3.4}$$

is an extended action for the non-twisted Courant algebroid $\mathbb{T}M$. Note that the verification follows from the following identity.

$$[\psi(v_1), \psi(v_2)] \oplus \mathcal{L}_{\psi(v_1)} d\mu(h_2) = \psi([v_1, v_2]) \oplus d\mu_{v_1 \cdot h_2}.$$

For $v_1, v_2 \in \mathfrak{g}$ and $h_2 \in \mathfrak{h}$.

It is worth to mention that both examples are the same when the $G$-action on the (pre)symplectic manifold $(M, \omega)$ is equipped with a honest moment map for $\omega$, in particular $\kappa(u) = d\mu(u)$ and the $\mathfrak{g}$ action on $\mathfrak{g}$ is given by the Lie bracket.

## 3.2 REDUCTION PROCEDURE

Let $G$ be a connected Lie group and $E$ be an exact Courant algebroid over $M$ as it was defined in the previous sections. Let $\xi : \mathfrak{a} \rightarrow \Gamma(E)$ be an extended action. Then we have two natural distributions in $E$: the image of $\xi$, $K = \xi(\mathfrak{a})$, and its orthogonal, $K^{\perp}$, taken with respect to the pairing in $E$.

Using the action of $\mathfrak{g}$ on $\Gamma(E)$, given by (3.2), but for sections of $K$, we obtain for all $g \in \mathfrak{g}$ and $a \in \mathfrak{a}$,

$$g \cdot \xi(a) = [\,\xi(\overline{g})\,,\, \xi(a)\,] = \xi([\,\overline{g}\,,\, a]) \in \Gamma(K), \tag{3.5}$$

where $\overline{g} \in \mathfrak{a}$ is any lift of $g$, i.e. $\pi(\overline{g}) = g$. It follows that $K$ is a $G$-invariant distribution and, since the $G$-action on $E$ preserves the symmetric pairing $\langle \cdot\,,\,\cdot \rangle$, $K^{\perp}$ is also a $G$-invariant distribution.

**Definition 3.2.1.** Given an extended action with image distribution $\xi(\mathfrak{a}) = K \subset E$, define the **big distribution** $\Delta_b = \pi(K + K^{\perp}) \subset TM$ and the **small distribution** $\Delta_s = \pi(K^{\perp}) \subset TM$. These are $G$-invariant distributions.

To continue with the reduction procedure it is necessary to consider leaves of the big distribution $\Delta_b$. So we are going to suppose that the distribution $\Delta_b$ is integrable.

**Proposition 3.2.2** (See [3])**.** *Let $P \subset M$ be a leaf of the big distribution $\Delta_b$ on which $G$ acts freely and properly, and suppose $\xi(\mathfrak{h})$ has constant rank along $P$. Then $K$ and $K \cap K^{\perp}$ both have constant rank along $P$.*

Hence, under the assumptions of the above proposition we have that $K$ and $K \cap K^{\perp}$ are $G$-invariant vector bundles over $P$. So we can consider the quotient vector bundle

$$E_{\text{red}} := \left. \frac{K^{\perp}|_P}{K \cap K^{\perp}|_P} \right/ G \tag{3.6}$$

over the reduced manifold $M_{\text{red}} := P/G$.

**Theorem 3.2.3** (Theorem 3.3 in [3])**.** *Let $E$ be an exact Courant algebroid over $M$ and $\xi : \mathfrak{a} \to \Gamma(E)$ be an extended $G$-action. Let $P \subset M$ be a leaf of $\Delta_b$ on which $G$ acts freely an properly, and over which $\xi(\mathfrak{h})$ has constant rank. Then the Courant bracket on $E$ descends to $E_{red}$ and makes it into a Courant algebroid over $M_{red} = P/G$ with surjective anchor. It is exact if and only if the following holds along $P$:*

$$\pi(K) \cap \pi(K^{\perp}) = \pi(K \cap K^{\perp}). \tag{3.7}$$

A particular case of an exact reduced Courant algebroid is when $K$ is isotropic because (3.7) is trivially satisfied.

Note that $E_{\mathrm{red}}$ is define by taking two kinds of quotient structures, namely the one concerning the action by $G$, and the one given by taking classes modulo $K \cap K^{\perp}|_P$. So, sections of $E_{\mathrm{red}}$ will be seen as $G$-invariant sections of $K^{\perp}|_P$ modulo $K \cap K^{\perp}|_P$. From now on we will not mention that we are taking sections modulo $K \cap K^{\perp}|_P$ but it will be understood from the context.

The way we compute the bracket on sections of $E_{\mathrm{red}}$ is done by extensions of $G$-invariant sections of $K^{\perp}|_P$. This means that if $e_1, e_2$ are $G$-invariant sections of $K^{\perp}|_P$, we choose $\hat{e}_1, \hat{e}_2$ to be sections of $E$ (along $M$) such that restricted to $P$ are $e_1$ and $e_2$, and then we compute the bracket

$$[\![e_1, e_2]\!]_{E_{\mathrm{red}}} = ([\![\hat{e}_1, \hat{e}_2]\!]_E)|_P. \tag{3.8}$$

The proof of Theorem 3.3 in [3] shows that in fact this computation gives a $G$ invariant section of $K^{\perp}|_P$. Moreover, this construction is well defined modulo $\Gamma(K \cap K^{\perp})^G$, that is, it does not depend on the extensions $\hat{e}_1, \hat{e}_2$ chosen when it takes values on section of $E_{red}$. Last claim allows us to consider

$$[\![\cdot, \cdot]\!]_{E_{\mathrm{red}}} : \Gamma(K^{\perp})^G \times \Gamma(K^{\perp})^G \to \Gamma(E_{red}).$$

Also in the proof of the theorem it is showed that this operation descends to $\Gamma(E_{red})$ because is independent of any element of $\Gamma(K \cap K^{\perp})^G$. This last operation will also be written as $[\![\cdot, \cdot]\!]_{E_{\mathrm{red}}}$.

## 3.3 REDUCTION OF THE COURANT ALGEBROID $\mathbb{T}\widetilde{M}$

Let $M$ be a $G$-manifold with smooth action $\varphi : G \times M \to M$ and infinitesimal action $\psi : \mathfrak{g} \to TM$. Define

$$\widetilde{\varphi} : G \times \widetilde{M} \to \widetilde{M};\ \widetilde{\varphi}(a, (p, t)) := (\varphi_a p, t), \quad a \in G,\ (p, t) \in \widetilde{M}. \tag{3.9}$$

Clearly this map defines an action on $\widetilde{M}$ whose infinitesimal action, denoted by $\widetilde{\psi}$, is

$$\widetilde{\psi}(v) = \psi(v) + 0\partial t \in T\widetilde{M}.$$

Let us define some notation for the $\mathbb{T}\widetilde{M}$ case. An extended action for $\mathbb{T}\widetilde{M}$ is denoted by $\widetilde{\rho} : \mathfrak{a} \to \mathbb{T}\widetilde{M}$ [1] and the usual distributions are

$$\widetilde{K} = \widetilde{\rho}(\mathfrak{a}) \quad \text{and} \quad \widetilde{K}^{\perp} \tag{3.10}$$

where the second distribution is computed with respect to the natural pairing

$$\begin{aligned} \langle e_1, e_2 \rangle &= i_{X_1 + f_1\,\partial t}(\alpha_2 + g_2\,dt) + i_{X_2 + f_2\,\partial t}(\alpha_1 + g_1\,dt) \\ &= i_{X_1}\alpha_2 + i_{X_2}\alpha_1 + f_1 g_2 + f_2 g_1 \end{aligned}$$

for $e_i = X_i + f_i\,\partial t \oplus \alpha_i + g_i\,dt \in \mathbb{T}\widetilde{M}$, $i = 1, 2$.

Following the reduction theorem Theorem 3.2.3, we obtain that for a leaf $\widetilde{P}$ of the big distribution $\widetilde{\Delta}_b = \pi(\widetilde{K} + \widetilde{K}^{\perp})$, the reduction of the exact Courant algebroid $\mathbb{T}\widetilde{M}$ is

$$(\mathbb{T}\widetilde{M})_{\text{red}} := \left. \frac{\widetilde{K}^{\perp}|_{\widetilde{P}}}{\widetilde{K} \cap \widetilde{K}^{\perp}|_{\widetilde{P}}} \middle/ G \right. \tag{3.11}$$

when $G$ acts freely and properly on $\widetilde{P}$.

[1] The use of the symbol $\widetilde{\rho}$ for this extended action (and no $\widetilde{\xi}$) will be clear later.

Our next goal is to compare the reduction $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$ with the exact Courant algebroid over a manifold of the form $Q \times (a, b)$. For this reason we shall consider the following assumption:

**Assumption 3.3.1.** *There exists a submanifold $P \subset M$ satisfying the following conditions:*

- *$TP = \Delta_b|_P$, i.e. $P$ is a leaf of the big distribution $\Delta_b \subset TM$ for an extended action on $\mathbb{T}M$.*
- *$P \times (a, b) \subset \widetilde{M}$ is a leaf of the big distribution $\widetilde{\Delta}_b \subset T\widetilde{M}$.*

In order to get a well defined reduction (along the leaves of the big distribution), we should note that the assumption inmediately implies the following lemma.

**Lemma 3.3.2.** *If we suppose that $G$ acts freely and properly on $P$, then we have that $G$ acts freely and properly on the leaf $\widetilde{P}$ of the big distribution $\widetilde{\Delta}_b$.*

which is a direct consequence of the following result:

**Proposition 3.3.3.** *Let $N \subset M$ be an arbitrary sub-manifold of the manifold $M$. Then we have:*

*i) If the action given by $\varphi$ acts freely on $N$, the action given by $\widetilde{\varphi}$ acts freely on the sub-manifold $N \times (a, b) \subset \widetilde{M}$.*

*ii) If the action $\varphi$ acts properly on $N$, the action given by $\widetilde{\varphi}$ acts properly on the sub-manifold $N \times (a, b) \subset \widetilde{M}$.*

*Proof.* *i)* Suppose the action $\varphi$ acts freely on $N$. Then we have that $a \cdot q = q$ for some $q \in N$ implies $a = e$, where $e$ is the identity element of $G$. Suppose that $a \cdot (q, t) = (q, t)$

for some $(q,t) \in N\times\mathbb{R}$, then by the definition of the action on $\widetilde{M}$ we have that $a\cdot q = q$, then by hypothesis we have that $a = e$.

$ii)$ Suppose the action given by $\varphi$ is proper. Let $\{(p_i,t_i)\}_i$ be a convergent sequence in $N\times\mathbb{R}$ and $\{a_i\}_i$ be a sequence in $G$ such that $\{a_i\cdot(p_i,t_i)\}_i = \{(a_i\cdot p_i, t_i)\}_i$ converges in $N\times\mathbb{R}$. By the continuity of the projection on the first component $\pi_N : N\times\mathbb{R}\to N$ we have that $\{p_i\}_i$ converges in $N$, so, because $\varphi$ is a proper action there exists a sub-sequence $\{a_{i_k}\}_k$ of the sequence $\{a_i\}_i$ that is convergent in $G$. This implies that $G$ acts properly on $N\times\mathbb{R}$.

□

**Remark 3.3.4.** It is worth to mention that a reduction of an exact Courant bracket with no twist (i.e. $H = 0$) may yield to a non-trivial twisted bracket (see example 3.12 in [3]). In this case, we will suppose that in the reduction of $\mathbb{T}\widetilde{M}$, the exact reduced Courant bracket has a twist. This says that there exists a 3-form $\widetilde{H} \subset \Omega^3(\widetilde{M}_{\mathrm{red}})$ such that $[\![,]\!]$ has the form (2.4). Note that $\widetilde{H}$ can be expressed as $H_2 + H_1\wedge dt$ for $H_1 \in \Omega^2(\widetilde{M}_{\mathrm{red}})$ and $H_2 \in \Omega^3(\widetilde{M}_{\mathrm{red}})$. If we assume that $\widetilde{M}_{\mathrm{red}} = M_{\mathrm{red}}\times I$ and $\mathcal{L}_{\partial t}\widetilde{H} = \widetilde{H}$, that is, $\widetilde{H}$ is homogeneous, then we have that $\widetilde{H} = e^t(H_2 + H_1\wedge dt)$, where in this case $H_1, H_2$ are forms on $M_{red}$. So, for this situation we can define a twist for the extended Courant bracket on $\mathcal{E}^1(M_{\mathrm{red}})$, namely $(H_1,H_2) \in \Omega^2(M_{\mathrm{red}})\times\Omega^3(M_{\mathrm{red}})$, and the extended Courant bracket, for sections $e_i = (X_i,f_i)\oplus(\alpha_i,g_i)$, $i = 1,2$, becomes

$$\begin{aligned}[\![e_1,e_2]\!]_{\mathcal{E}^1(M_{\mathrm{red}})} &= ([\,X_1\,,\,X_2\,], X_1f_2 - X_2f_1)\\ &\oplus\ \widetilde{\mathcal{L}}_{(X_1,f_1)}(\alpha_2,f_2) - \widetilde{i}_{(X_2,f_2)}\widetilde{d}(\alpha_1,g_1) + \widetilde{i}_{(X_2,f_2)}\widetilde{i}_{(X_1,f_1)}(H_2,H_1),\end{aligned}$$

Nevertheless, the study of twisted $\mathcal{E}^1(M)$-structures is far from the scope of this manuscript, so this situation will not be considered.

## 3.4 EXAMPLES

We will exemplify how this machinery works in the important cases of this odd-geometry: the contact case and the existence of moment map. These two examples will be studied in several steps, first the reduction of $\mathbb{T}\widetilde{M}$, later in Section 4.1.2 the reduction of the respective $\mathcal{E}^1(M)$ and in Section 4.2 the condition to reduce Dirac-Jacobi structures.

### The contact manifold $M$

If $M$ is a (co-oriented) contact manifold with contact form $\eta$ and the Lie group $G$ acts on $M$ preserving the form $\eta$, then $\widetilde{\rho} : \mathfrak{g} \oplus \mathfrak{g} \to \Gamma(\mathbb{T}\widetilde{M})$ defined by

$$\widetilde{\rho}(u, v) = \psi(u) \oplus e^t(d(i_{\psi(v)}\eta) + i_{\psi(v)}\eta dt) \tag{3.12}$$

is an extended action for $\mathbb{T}\widetilde{M}$. Note that this is a particular example for the $G$-action on the symplectic manifold $(\widetilde{M}, -d(e^t\eta))$ as in Example 3.1.2.

Moreover, if the $G$-action is symplectic on $(\widetilde{M}, \omega = -d(e^t\eta))$ then we recover the reduction procedure as in [3, Example 3.14], in particular we have

$$\widetilde{K} = \psi(\mathfrak{g}) \oplus \omega(\psi(\mathfrak{g})) \ \text{ and } \ \widetilde{K}^\perp = \psi(\mathfrak{g})^\omega \oplus \mathrm{Ann}(\mathfrak{g})^\omega,$$

with its respective small and big distributions

$$\widetilde{\Delta}_s = \psi(\mathfrak{g})^\omega \ \text{ and } \ \widetilde{\Delta}_b = \psi(\mathfrak{g})^\omega + \psi(\mathfrak{g}).$$

**Remark 3.4.1.** Recall that under the existence of a honest moment map for a symplectic manifold $(Q, \omega)$ endowed with a hamiltonian $G$-action, it was proved that the reduction of

$\mathbb{T}Q$ is exact and coincides with the untwisted standard Courant algebroid of $(\mathbb{T}Q)_{\mathrm{red}}$. Such result is a consequence of the reduction of level sets of the symplectic moment map as it is commented on Example 3.14 in the reference [3].

Following the idea of the previous remark on the special case of $Q = \widetilde{M}$ with $\omega = -d(e^t\eta)$ such moment map always exists for any $G$-action preserving the 1-form $\eta$. Assumption 3.3.1 leads us to identify the reduced manifold $M_{\mathrm{red}}$ with $\widetilde{P} = P \times (a, b)$. These two ideas yield the following result:

**Corollary 3.4.2** (Corollary of Theorem 3.3 in [3]). *Let $(M, \eta)$ be a contact manifold and the $G$-action preserves the contact form $\alpha$. Under the Assumption 3.3.1, we get*

$$(\mathbb{T}\widetilde{M}, [\![,]\!], \langle,\rangle)_{\mathrm{red}} = (\mathbb{T}\widetilde{M_{\mathrm{red}}}, [\![,]\!], \langle,\rangle).$$

### The $\mathfrak{h}$-moment map

From now on we will consider a $\mathfrak{g}$-module $\mathfrak{h}$ as in Example 3.1.3, and a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{h}^*$, i.e $\mu_{g\cdot h} = \mathcal{L}_{\psi(g)}\mu_h$ for any $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$, where $\mu_h$ denotes the evaluation map $\langle \mu, h\rangle$.

**Proposition 3.4.3.** *The map $\widetilde{\mu} : \widetilde{M} \to \mathfrak{h}^*$ defined by $\widetilde{\mu}(p,t) = e^t\mu(p)$ for all $(p,t) \in \widetilde{M}$, is a $\mathfrak{g}$-equivariant map.*

*Proof.* We have to prove that

$$\widetilde{\mu}_{v\cdot h} = \widetilde{\psi}(v)\widetilde{\mu}_h,$$

for all $v \in \mathfrak{g}$ and all $h \in \mathfrak{h}$.

First, observe that for all $h \in \mathfrak{h}$ and $(p,t) \in \widetilde{M}$ it holds

$$\widetilde{\mu}_h(p,t) = \widetilde{\mu}(p,t)(h) = e^t\mu(p)(h) = e^t\mu_h(p),$$

that is, $\widetilde{\mu}_h = e^t\mu_h$ for all $h \in \mathfrak{h}$. So, for all $v \in \mathfrak{g}$ and $h \in \mathfrak{h}$ we have:

$$\widetilde{\mu}_{v\cdot h}e^t\mu_{v\cdot h} = e^t(\psi(v)\mu(h)) = \widetilde{\psi}(v)(e^t\mu(h)) = \widetilde{\psi}(v)\widetilde{\mu}(h).$$

which completes the proof. $\square$

By Proposition 3.4.3 and the fact mentioned in Example 3.1.3, it follows immediately that the map defined by

$$\begin{aligned} \widetilde{\rho} : \mathfrak{g} \ltimes \mathfrak{h} &\to \Gamma(\mathbb{T}\widetilde{M}) \\ (v,h) &\mapsto \psi(v) + 0\partial t \oplus d\widetilde{\mu}_h, \end{aligned} \tag{3.13}$$

is an extended action on $\mathbb{T}\widetilde{M}$.

Moreover, observe that for all $h \in \mathfrak{h}$ we have

$$d(\widetilde{\mu}_h) = d(e^t\mu_h) = e^t d\mu_h + e^t\mu_h dt = e^t(d(\mu_h) + \mu_h dt)$$

So, $\widetilde{\rho}$ can also be written as

$$\widetilde{\rho}(v,h) = \psi(v) + 0\partial t \oplus e^t(d(\mu_h) + \mu_h dt). \tag{3.14}$$

In this way, we have the following commutative diagram for the extended action $\widetilde{\rho}$

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, "\widetilde{\nu}"] & \mathfrak{g} \ltimes \mathfrak{h} \arrow[r] \arrow[d, "\widetilde{\rho}"] & \mathfrak{g} \arrow[r] \arrow[d, "\widetilde{\psi}"] & 0 \\ 0 \arrow[r] & \Gamma(T^*\widetilde{M}) \arrow[r] & \Gamma(\mathbb{T}\widetilde{M}) \arrow[r] & \Gamma(T\widetilde{M}) \arrow[r] & 0 \end{tikzcd} \tag{3.15}$$

where $\widetilde{\nu} : \mathfrak{h} \to \Gamma(T^*\widetilde{M})$ is the restriction of $\widetilde{\rho}$ to $\mathfrak{h}$, i.e. $\widetilde{\nu}(h) = \widetilde{\rho}(h)$ for all $h \in \mathfrak{h}$.

Following the reduction procedure for exact Courant algebroids we consider two distributions in $\mathbb{T}\widetilde{M}$:

$$\widetilde{K} = \widetilde{\rho}(\mathfrak{g} \ltimes \mathfrak{h}) \quad \text{and} \quad \widetilde{K}^{\perp} \tag{3.16}$$

From the definition of $\widetilde{\rho}$ (equation (3.13)) we have that

$$\widetilde{K} = \psi(\mathfrak{g}) \oplus \widetilde{\nu}(\mathfrak{h}). \tag{3.17}$$

Now take an element $e = X + f\,\partial t \oplus \alpha + g\,dt \in \mathbb{T}\widetilde{M}$. If $e' = \psi(v) \oplus e^t(d\mu_h + \mu_h\,dt)$ is an arbitrary element of $\widetilde{K}$, then it should hold

$$\begin{aligned} 0 &= \langle e, e' \rangle = i_{(X+f\,\partial t)}(e^t(d\mu_h + \mu_h\,dt)) + i_{\psi(v)}(\alpha + g\,dt) \\ &= e^t(i_X d\mu_h + f\mu_h) + i_{\psi(v)}(\alpha). \end{aligned}$$

In particular this holds for all elements of the form $(v, 0)$ and $(0, h)$ of $\mathfrak{g} \ltimes \mathfrak{h}$. Then we can see that

$$\widetilde{K}^{\perp} = \ker(\widetilde{\nu}(\mathfrak{h})) \oplus \mathrm{Ann}(\psi(\mathfrak{g})) \times \mathbb{R} \tag{3.18}$$

**Remark 3.4.4.** Note that if zero is a regular value of $\widetilde{\mu}$, then over the submanifold $\widetilde{P} = \widetilde{\mu}^{-1}(0)$ we have $\widetilde{\nu} = d\mu$ and $\mathrm{Ker}(\widetilde{\nu}(\mathfrak{h})) = \mathrm{Ker}(d\mu) \times \mathbb{R}$. So, in this case we obtain the relations

$$\begin{aligned} \widetilde{K} &= \psi(\mathfrak{g}) + 0\,\partial t \oplus d\mu + 0\,dt \\ \widetilde{K}^{\perp} &= \mathrm{Ker}(d\mu) \times \mathbb{R} \oplus \mathrm{Ann}(\psi(\mathfrak{g})) \times \mathbb{R}. \end{aligned}$$

Equation (3.5) works for the action of $\mathfrak{g}$ on sections of $\widetilde{K}$, then the distributions $\widetilde{K}$ and $\widetilde{K}^{\perp}$ are $G$-invariant distributions.

Now we define the big and small distributions as $\widetilde{\Delta}_b = \widetilde{\pi}(\widetilde{K} + \widetilde{K}^{\perp}) \subset T\widetilde{M}$ and $\widetilde{\Delta}_s = \widetilde{\pi}(\widetilde{K}^{\perp}) \subset T\widetilde{M}$ respectively, where $\widetilde{\pi} : \mathbb{T}\widetilde{M} \to T\widetilde{M}$ is the natural projection. Thus, we have that both $\widetilde{\Delta}_b$ and $\widetilde{\Delta}_s$ are $G$-invariant distributions.

For this reason, under these assumptions and assuming that $\widetilde{\rho}(\mathfrak{h})$ has constant rank along $\widetilde{P}$, we can use Proposition 3.2.2 and the discussion in this subsection to conclude that $\widetilde{K}$ and $\widetilde{K} \cap \widetilde{K}^{\perp}$ are $G$-invariant vector bundles, and hence the quotient

$$(\mathbb{T}\widetilde{M})_{\mathrm{red}} := \frac{\widetilde{K}^{\perp}|_{\widetilde{P}}}{\widetilde{K} \cap \widetilde{K}^{\perp}|_{\widetilde{P}}} \Big/ G \tag{3.19}$$

is a vector bundle over the reduced manifold $\widetilde{M}_{\mathrm{red}} = \widetilde{P}/G$, which, by Theorem 3.2.3 is a reduction of the standard Courant algebroid $\mathbb{T}\widetilde{M}$.

If we assume that the extended action on $\mathbb{T}\widetilde{M}$ is isotropic, then the reduced structure is exact and can be identified with

$$(\mathbb{T}\widetilde{M})_{\mathrm{red}} = \frac{\widetilde{K}^{\perp}|_{\widetilde{P}}}{\widetilde{K}|_{\widetilde{P}}} \Big/ G \simeq \mathbb{T}(\widetilde{M}_{\mathrm{red}}) \tag{3.20}$$

as a vector bundle over $\widetilde{M}_{\mathrm{red}}$. Note that in the presence of a moment map $\widetilde{\mu}$ such that the induced extended action $\widetilde{\rho}$ is isotropic, we have that the leaves of the big distribution $\widetilde{\Delta}_b$ are the level sets of regular values of $\widetilde{\mu}$, and in this case we can get the condition of the constant rank of $\widetilde{\rho}(\mathfrak{h})$ along $\widetilde{P}$ by assuming that $\xi(\mathfrak{h})$ has constant rank along $P$.

Finally, considering Assumption 3.3.1 it follows that integrability of $\widetilde{\Delta}_b$ is a consequence of the integrability of the big distribution $\Delta_b$ in $\mathbb{T}M$, moreover, we can write $\widetilde{M}_{\mathrm{red}} = P/G \times (a, b)$, which finally yields the following result:

**Corollary 3.4.5** (Corollary of Theorem 3.3 in [3]). *Let $\mu : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map on the $G$-manifold $M$, where $\mathfrak{h}$ is a $\mathfrak{g}$-module. Let $P \subset M$ and $\widetilde{P}$ as in Assumption 3.3.1.*

*In addition suppose that $G$ acts freely and properly on $P$ and that $\xi(\mathfrak{h})$ has constant rank there. Then the standard Courant algebroid $(\mathbb{T}\widetilde{M}, [\![,]\!], \langle,\rangle)$ has a Courant reduction which is exact when the extended action $\rho$ is isotropic, and in that case we get:*

$$(\mathbb{T}\widetilde{M}, [\![,]\!], \langle,\rangle)_{\mathrm{red}} \simeq (\mathbb{T}\widetilde{M_{\mathrm{red}}}, [\![,]\!]_{\mathrm{red}}, \langle,\rangle_{\mathrm{red}}).$$

Note that we can combine equations (3.17), (3.18) and (3.20) to obtain

$$(\mathbb{T}\widetilde{M})_{\mathrm{red}} = \frac{\ker(\widetilde{\nu}(\mathfrak{h})) \oplus \mathrm{Anh}(\psi(\mathfrak{g}))|_{\widetilde{P}}}{\psi(\mathfrak{g}) \oplus \widetilde{\nu}(\mathfrak{h})|_{\widetilde{P}}} \Big/ G = \frac{\ker(\widetilde{\nu}(\mathfrak{h}))|_{\widetilde{P}}}{\psi(\mathfrak{g})|_{\widetilde{P}}} \Big/ G \oplus \frac{\mathrm{Anh}(\psi(\mathfrak{g}))_{\widetilde{P}}}{\widetilde{\nu}(\mathfrak{h})|_{\widetilde{P}}} \Big/ G \quad (3.21)$$

Where the last two terms are the tangent and cotangent parts of the reduced manifold, namely they are $T(\widetilde{M}_{\mathrm{red}})$ and $T^*(\widetilde{M}_{\mathrm{red}})$. Also note that under assumption Assumption 3.3.1 and the hypothesis $P = \mu^{-1}(0)$, we obtain

$$(\mathbb{T}\widetilde{M})_{\mathrm{red}} = \frac{\mathrm{Ker}(\nu(\mathfrak{h}))|_P \times \mathbb{R}}{\psi(\mathfrak{g})|_P} \Big/ G \oplus \frac{\mathrm{Ann}(\psi(\mathfrak{g}))|_P \times \mathbb{R}}{\widetilde{\nu}(\mathfrak{h})|_P} \Big/ G. \quad (3.22)$$

Then, when this happens, sections of $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$ can be written in the form $X_{\widetilde{P}} \oplus \alpha_{\widetilde{P}}$, where $X_{\widetilde{P}}$ and $\alpha_{\widetilde{P}}$ are $G$-invariant sections of $\mathrm{Ker}(\widetilde{\nu}(\mathfrak{h}))|_{\widetilde{P}}$ (modulo $\psi(\mathfrak{g})|_P$) and $\mathrm{Anh}(\psi(\mathfrak{g}))|_{\widetilde{P}}$ (modulo $\widetilde{\nu}(\mathfrak{h})|_P$ ) respectively.

In the case we have a moment map and the induced extended action is isotropic, we can induce a splitting on the exact Courant reduction (this is a particular case of Example 3.13 in [3] with the additional hypothesis that $\widetilde{\rho}$ is isotropic). If we take the isotropic splitting on $\mathbb{T}\widetilde{M}$ given by $\widetilde{\nabla} : T\widetilde{M} \to \mathbb{T}\widetilde{M}$, $X \mapsto X \oplus 0$, we get that the induced splitting on $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$ is just $\widetilde{\nabla}_{\mathrm{red}} : T\widetilde{M}_{\mathrm{red}} \to \mathbb{T}\widetilde{M}_{\mathrm{red}}$, $X_{\widetilde{P}} \mapsto X_{\widetilde{P}} \oplus 0_{\widetilde{P}}$, and then, because the initial Courant algebroid $\mathbb{T}\widetilde{M}$ has no twisted 3-form, we have a non-twisted reduction. Moreover, if we denote the anchor map of $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$ by $\widetilde{\pi}_{\mathrm{red}}$, then we have that $\widetilde{\pi}^*_{\mathrm{red}} : T^*\widetilde{M} \to \mathbb{T}\widetilde{M}$ is given by $\alpha_{\widetilde{P}} \mapsto 0_{\widetilde{P}} \oplus \alpha_{\widetilde{P}}$ and thus we obtain the isomorphism of Courant algebroids:

$$\widetilde{\nabla}_{\mathrm{red}} \oplus \widetilde{\pi}^*_{\mathrm{red}} : T\widetilde{M}_{\mathrm{red}} \oplus T^*\widetilde{M}_{\mathrm{red}} \to (\mathbb{T}\widetilde{M})_{\mathrm{red}} \tag{3.23}$$

# Chapter 4

# REDUCTION OF HOMOGENEOUS STRUCTURES: THE $\mathcal{E}^1(M)$ AND DIRAC-JACOBI CASES

Our main goal is to reduce Dirac-Jacobi structures which are particular subbundles $L$ of $\mathcal{E}^1(M)$. We have already seen that $\tau(L)$ is a subbundle of the standard Courant algebroid $\mathbb{T}\widetilde{M}$ from which, as we saw in the last subsection, we know some reductions. To achieve this construction we will study some reduced structures related to the homogenization trick on $\mathcal{E}^1(M)$.

## 4.1 REDUCTION OF THE EXTENDED COURANT BRACKET $\mathcal{E}^1(M)$

In a first sight, in order to construct the reduction of a Dirac-Jacobi structure $L$, we should consider the reduction of $\tau(L)$ inside the reduction of $\mathbb{T}\widetilde{M}$, but at the same time we must construct the reduction of $\mathbb{T}\widetilde{M}$.

Following the construction of the reduction of Courant algebroids, to reduce the structure $(\mathcal{E}^1(M), \langle,\rangle, [\![,]\!])$ we need to reduce the Courant algebroid $(\mathbb{T}\widetilde{M}, \langle,\rangle, [\![,]\!])$ to a suitable exact Courant algebroid and also $\mathcal{E}^1(M)$ has some extended action similar to the situation in $\mathbb{T}M$ . This is our purpose in this section.

### 4.1.1 Extended Action for $\mathcal{E}^1(M)$

**Definition 4.1.1.** An exteded action for a structure $\mathcal{E}^1(M)$ is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, "\nu"] & \mathfrak{a} \arrow[r] \arrow[d, "\rho"] & \mathfrak{g} \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \Gamma(T^*M \times \mathbb{R}) \arrow[r] & \Gamma(\mathcal{E}^1(M)) \arrow[r] & \Gamma(TM \times \mathbb{R}) \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

where $\pi : \mathfrak{a} \to \mathfrak{g}$ is an exact Courant algebra over $\mathfrak{g}$ with kernel $\mathfrak{h}$, $\nu$ is the restriction of $\rho$ to $\mathfrak{h}$ and the map $\rho$ preserves the bracket. We require also that $\widetilde{d}(\nu(h)) = 0$ for all $h \in \mathfrak{h}$ (equivalently $\mathfrak{h}$ acts trivially on $\Gamma(\mathcal{E}^1(M))$; equivalently each element of $\widetilde{\nu}$ is of the form $(df, f)$ for some $f \in C^\infty(M)$).

The property E1) in Proposition 2.2.2 allows us to consider an action of $\mathfrak{g}$ on $\Gamma(\mathcal{E}^1(M))$ by means of the adjoint action, i.e. we consider the action

$$v \cdot e = [\![\rho(a), e]\!], \tag{4.2}$$

where $a \in \mathfrak{a}$ is such that $\pi(a) = v$. As in the case of Courant algebroids this action is well define as we show now. Take $a, a' \in \mathfrak{a}$ such that $\pi(a) = \pi(a') = v$, then we have that $\pi(a'_a) = 0$ and so $a - a' \in \mathfrak{h}$ so we have

$$[\![\rho(a), e]\!] - [\![\rho(a').e]\!] = [\![\rho(a - a'), e]\!] = [\![\nu(a - a'), e]\!] = 0,$$

where we have used the fact that $\nu(\mathfrak{h})$ acts trivially on $\mathcal{E}^1(M)$. This shows that (4.2) is well defined.

**Remark 4.1.2.** If we have an extended action like in the previous definition and we consider the action on $\widetilde{M}$ given by the map $\widetilde{\varphi}$ defined in (3.9), then we have by Lemma 2.3.2 and Corollary 2.25 that the map $\widetilde{\rho} := \tau \circ \rho : \mathfrak{a} \to \Gamma(\mathbb{T}\widetilde{M})$ is an extended action as in Definition 3.1.2. Recall the constructions $\widetilde{K} = \widetilde{\rho}(\mathfrak{a})$ and $\widetilde{K}^\perp$, and the related constructions to them as before.

Motivated by the Courant algebroid case we define $K = \rho(\mathfrak{a})$ and its perpendicular $K^\perp$ taken with respect to the natural pairing $\langle , \rangle_{\mathcal{E}^1(M)}$ given by equation (**??**). By a direct verification of the orthogonality condition on elements belonging to $K^{\perp}$ we can derive the following proposition.

**Proposition 4.1.3.** *The distributions $K$ and $K^{\perp}$ satisfy*

- $\tau(K) = \widetilde{K}$, *and*
- $\tau(K^{\perp}) = \widetilde{K}^{\perp}$

The proof is a direct consequence of the fact that $K$ is the image of $\rho$ , the map $\tau$ is a fiberwise linear isomorphism and $e \in K^{\perp}$ if and only if $\tau(e) \in \widetilde{K}^{\perp}$.

**Corollary 4.1.4.** *Assume an extended action $\rho$ for $\mathcal{E}^1(M)$ and the induced extended action $\widetilde{\rho} = \tau \circ \rho$ for $\mathbb{T}\widetilde{M}$. Then we have that $\pi(K \cap K^{\perp}) = \pi(K) \cap \pi(K^{\perp})$ implies*

$$\widetilde{\pi}(\widetilde{K} \cap \widetilde{K}^{\perp}) = \widetilde{\pi}(\widetilde{K}) \cap \widetilde{\pi}(\widetilde{K}^{\perp}).$$

*Proof.* The proof follows by an easy computation taking in account Proposition 4.1.3, also $\tau(\pi(\cdot)) = \widetilde{\pi}(\tau(\cdot))$, and that $\tau$ preserves intersections of subbundles of $\mathcal{E}^1(M)$. $\square$

### 4.1.2 Reduction of $\mathcal{E}^1(M)$

For a complete approach for the reduction procedure of the extended Courant bracket in its general setting it is needed a more axiomatic definition of the $\mathcal{E}^1(M)$ structures and this is out of the scope of this work. However we can consider this reduction procedure in two examples of extended action, the contact case and $\mathfrak{h}$-moment map as in the previous chapter. It is worth to mention that the contact case is a particular case of moment map by taking $\mathfrak{h} = \mathfrak{g}$ and $\mu(h) = i_{\psi(h)}\alpha$; but we will adopt two different approaches: (a) the contact case is not isotropic while in (b) we will impose isotropicity for general moment map $\mu$ on any $\mathfrak{g}$-module $\mathfrak{h}$. These two different considerations will give some different results on the reduction of Dirac-Jacobi structures as can be seen in theorems 4.2.5 and 4.2.7.

Before starting with the two cases mentioned, first note that assuming Assumption 3.3.1, we can see that an extended action for $\mathcal{E}^1(M)$ such that $\pi(K \cap K^\perp) = \pi(K) \cap \pi(K^\perp)$ implies, by Corollary 4.1.4, that the extended action $\widetilde{\rho}$ in Remark 4.1.2 gives us an exact reduction of the extended Courant algebroid $\mathbb{T}\widetilde{M}$ given by 3.19.

**Contact case**

Following the same notation as in Section 3.4, here we can define the extended action for $\mathcal{E}^1(M)$ as $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathcal{E}^1(M)$ given by

$$\rho(u, v) = (\psi(u), 0) \oplus (di_{\psi(v)}\eta, i_{\psi(v)}\eta). \tag{4.3}$$

By an easy computation one can see that the last map defines an extended action for $\mathcal{E}^1(M)$ as in Definition 4.1.1. So, the map $\widetilde{\rho} := \tau \circ \rho$ defines an extended action for $\mathbb{T}\widetilde{M}$ which is in fact the same as the one given by equation (3.12).

Using the reduction construction for $\mathbb{T}\widetilde{M}$, the Assumption 3.3.1 and Proposition 4.1.3 we obtain

$$(\mathbb{T}\widetilde{M})_{\text{red}} = \tau \left( \left. \frac{K^\perp}{K \cap K^\perp} \right|_P \Big/ G \right).$$

This fact is consequence of $\widetilde{K} \cap \widetilde{K}^\perp = \tau(K \cap K^\perp)$

**The $\mathfrak{h}$-moment map**

Let $\mu : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map, with $\mathfrak{h}$ a $\mathfrak{g}$-module. We will study the case of $\mathfrak{h}$-moment map given by

$$\begin{aligned} \rho : \mathfrak{g} \ltimes \mathfrak{h} &\to \Gamma(\mathcal{E}^1(M)) \\ (v,h) &\mapsto (\psi(v),0) \oplus (d\mu_h, \mu_h), \end{aligned} \tag{4.4}$$

where $\mu_h$ is given by (3.3). We denote by $\nu$ the restriction of $\rho$ to $\mathfrak{h}$.

We can easily check that $\rho$ is an extended action for $\mathcal{E}^1(M)$ as in Definition 4.1.1. So we have that $\widetilde{\rho} = \tau \circ \rho$ defines an extended action for $\widetilde{\mathbb{T}M}$ whose restriction to $\mathfrak{h}$ is $\widetilde{\nu} = \tau \circ \nu$. In fact, by equation (3.14), it is the same extended action we obtained in Section 3.4 by using the moment map $\widetilde{\mu}$.

As it was done for the reduction of exact Courant algebroids, we define two natural distributions of $\mathcal{E}^1(M)$: $K = \rho(\mathfrak{g} \ltimes \mathfrak{h})$ and $K^{\perp}$ which satisfy the properties of Proposition 4.1.3.

**Remark 4.1.5.** We want to see how the distribution $K^{\perp}$ is. Let $(X,f) \oplus (\alpha, g) \in \Gamma(K^{\perp})$. It must hold

$$\langle (X,f) \oplus (\alpha,g)\,,\, (\psi(v),0) \oplus (d\mu_h,\mu_h)\rangle = \alpha(\psi(v)) + d\mu_h(X) + f\mu_h = 0$$

for all $(v,h) \in \mathfrak{g} \ltimes \mathfrak{h}$. In particular if $v = 0$, we have

$$d\mu_h(X) + f\mu_h = 0.$$

This is equivalent to $\widetilde{i}_{(X,f)}(d\mu_h,\mu_h) = 0$, that is, $(X,f) \in \mathrm{Ker}(d\mu_h,\mu_h)$ for all $h \in \mathfrak{h}$, where the kernel is taken with respect to the operator $\widetilde{i}$ defined by (2.15). On the other hand, if we consider $h = 0$, then $\alpha(\psi(v)) = 0$ for all $v \in \mathfrak{g}$, that is, we must have $\alpha \in \mathrm{Ann}(\psi(\mathfrak{g}))$.

Conversely, if $e = (X, f) \oplus (\alpha, g) \in \Gamma(\mathrm{Ker}(\nu(\mathfrak{h})) \oplus \mathrm{Ann}(\psi(\mathfrak{g})) \times \mathbb{R})$ then we can easily see that $e \in K^{\perp}$. We conclude that

$$K^{\perp} = \mathrm{Ker}(\nu(\mathfrak{h})) \oplus \mathrm{Ann}(\psi(\mathfrak{g})) \times \mathbb{R}. \tag{4.5}$$

where the kernel is with respect to the pairing in $\mathcal{E}^1(M)$. Notice that this expression is very similar to (3.18).

**Remark 4.1.6.** As in Remark 3.4.4, if zero is a regular value of $\mu$, then on the submanifold $\mu^{-1}(0)$ the tangent part in the last equation is just $\mathrm{Ker}(d\mu) \times \mathbb{R}$, and in this case the kernel is taken with respect to the operator $i$.

Recall the big and small distributions on $M$, $\Delta_b = \pi(K + K^{\perp}) \subset TM$ and $\Delta_s = \pi(K^{\perp}) \subset TM$. For a leave $\widetilde{P}$ of the big distribution $\widetilde{\Delta}_b$ satisfying Assumption 3.3.1 we have that if $\rho(\mathfrak{h})$ has constant rank along $P$, then $\widetilde{\rho}(\mathfrak{h})$ has constant rank along $\widetilde{P}$. this is because the fibers $\mathcal{E}^1(M)_p$ and $\mathbb{T}\widetilde{M}_{(p,t)}$ are linearly isomorphic through $\tau$ for all $p \in M$ and $t \in \mathbb{R}$. Now suppose that $\rho$ is isotropic so it holds $K = \rho(\mathfrak{g} \ltimes \mathfrak{h}) \subset K^{\perp}$. Then we can use Proposition 4.1.3 to conclude that $\widetilde{K}$ is isotropic as well:

$$\widetilde{K} = \tau(K) \subset \tau(K^{\perp}) = \widetilde{K}^{\perp}.$$

Therefore, if $\rho$ is isotropic, we have that $\widetilde{\Delta_b} = \widetilde{\pi}(\widetilde{K} + \widetilde{K}^{\perp}) = \widetilde{\pi}(\widetilde{K}^{\perp}) = \widetilde{\Delta_s}$, and then we will use the symbol $\widetilde{\Delta}$ for the big and small distributions. On the other hand, by Corollary 4.1.4 the reduction in Corollary 3.4.5 is an exact Courant algebroid.

From now on we will assume that $\rho$ is isotropic and the Assumption 3.3.1. By equation

(3.11), for this special case we have

$$(\mathbb{T}\widetilde{M})_{\mathrm{red}} := \frac{\widetilde{K}^{\perp}|_{\widetilde{P}}}{\widetilde{K}|_{\widetilde{P}}} \Big/ G. \tag{4.6}$$

**Definition 4.1.7.** We define the reduction of $\mathcal{E}^1(M)$ as being

$$\mathcal{E}^1(M)_{\mathrm{red}} := \left.\frac{K^{\perp}}{K}\right|_{P} \Big/ G. \tag{4.7}$$

Which is a honest vector bundle because $K$ and $K^{\perp}$ are $G$-invariant.

We end this section studying the case $P = \mu^{-1}(0)$ where we can write

$$K^{\perp} = \mathrm{Ker}(\nu(\mathfrak{h})) \times \mathbb{R} \oplus \mathrm{Ann}(\psi(\mathfrak{g})) \times \mathbb{R}$$

which will imply a special situation for $(\mathcal{E}^1(M))_{\mathrm{red}}$.

First we state the following lemma which is a consequence of Lemma 10.29 in [10].

**Lemma 4.1.8.** *Let $V$ be an arbitrary vector bundle over an arbitrary smooth manifold $N$. Consider $\widetilde{N} = N \times \mathbb{R}$ and the vector bundles $\mathcal{E}^1(N) \to N$ and $\mathbb{T}\widetilde{N} \to \widetilde{N}$. Suppose we have an isomorphism of $C^\infty(N)$-modules between $\Gamma(V)$ and $\Gamma_{\mathrm{hom}}(\mathbb{T}\widetilde{N})$. Then we have an isomorphism of vector bundles between $V$ and $\mathcal{E}^1(N)$ over the manifold $N$.*

*Proof.* Under the hypothesis of the Lemma and the fact that $\Gamma(\mathcal{E}^1(N))$ and $\Gamma_{\mathrm{hom}}(\mathbb{T}\widetilde{N})$ are isomorphic as $C^\infty(N)$-modules by means of $\tau$, it follows that $\Gamma(\mathcal{E}^1(N))$ and $\Gamma(V)$ are isomorphic as $C^\infty(N)$-modules. Then the isomorphism between $\mathcal{E}^1(N)$ and $V$ follows by Lemma 10.29 in [10]. □

We apply Lemma 4.1.8 to the isomorphism of $C^\infty(M_{\mathrm{red}})$-modules between $\Gamma(\mathcal{E}^1(M)_{\mathrm{red}})$ and $\Gamma_{\mathrm{hom}}((\mathbb{T}\widetilde{M})_{\mathrm{red}})$. To do this recall that sections of $\mathcal{E}^1(M)_{\mathrm{red}}$ are represented by $G$-invariant sections of $K^\perp|_P$ and sections of $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$ are represented by $G$-invariant sections of $\widetilde{K}^\perp|_{\widetilde{P}}$. In this case we define a map $\overline{\tau} : \Gamma(\mathcal{E}^1(M)_{\mathrm{red}}) \to \Gamma(\mathbb{T}\widetilde{M}_{\mathrm{red}})$ as follows: let $e_P$ be a $G$-invariant section of $K_P^\perp$ and let $\hat{e}$ be an extension of $e_P$ with $\hat{e} \in \Gamma(\mathcal{E}^1(M))$. Then

$$\overline{\tau}(e_P) := \tau(\hat{e})|_{\widetilde{P}}. \tag{4.8}$$

**Proposition 4.1.9.** *Under previous considerations on $\widetilde{P}$, the map $\overline{\tau}$ in the last definition is well defined.*

*Proof.* By Assumption 3.3.1 we know that $\widetilde{P} = P \times \mathbb{R}$. The proof of the proposition is done by showing the following two claims.

- First, we show that $\tau(\hat{e})|_{\widetilde{P}}$ is in fact a $G$-invariant section of $\widetilde{K}|_{\widetilde{P}}$ no matter the extension $\hat{e} \in \Gamma(\mathcal{E}^1(M))$ we choose.

- Then we show that $\overline{\tau}(e_P)$ is well defined modulo $\Gamma(\widetilde{K}|_{\widetilde{P}})^G$ when $e_P$ is chosen modulo $\Gamma(K^\perp|_P)^G$.

To show the first claim note that for all $(p,t) \in \widetilde{M}$ we have that $\tau$ is a linear isomorphism between $\mathcal{E}^1(M)_p$ and $(\mathbb{T}\widetilde{M})_{(p,t)}$ such that $\langle \tau\cdot, \tau\cdot\rangle_{\mathbb{T}\widetilde{M}} = \tau(\langle\cdot,\cdot\rangle_{\mathcal{E}^1(M)}) = e^t\langle\cdot,\cdot\rangle_{\mathcal{E}^1(M)}$. In particular this happens for all $(p,t) \in \widetilde{P}$. So, if $(\hat{e})_p = (e_P)_p \in (K^\perp)_p$ then (using Proposition 4.1.3) it follows $(\tau(\hat{e}))_{(p,t)} \in (\widetilde{K}^\perp)_{(p,t)}$ for all $(p,t) \in \widetilde{P}$, hence we conclude that $\overline{\tau}(e_P) = \tau(\hat{e})|_{\widetilde{P}}$ is a section of $\widetilde{K}^\perp|_{\widetilde{P}}$ and it does not depend on the extension $\hat{e} \in \Gamma(\mathcal{E}^1(M))$. The fact that $\tau(\hat{e})|_{\widetilde{P}}$ is $G$-invariant follows from the properties of $\tau$ and that we can choose $\hat{e}$ in $\Gamma(\widetilde{K}^\perp)^G$.

To verify the second claim it is enough to see that every section of $K|_P$ is sent to a section of $\widetilde{K}|_{\widetilde{P}}$, but by the first claim, if $e_P \in \Gamma(K|_P)$ we can choose $\hat{e} \in \Gamma(K)$ and then the claim follows as a consequence of the identity $\widetilde{K} = \tau(K)$. □

From the definition of $\overline{\tau}$ we get that $\tau$ is injective, surjective onto the homogeneous sections of $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$ and $C^\infty(M_{\mathrm{red}})$-linear. So the map $\overline{\tau}$ is an isomorphism of $C^\infty(M_{\mathrm{red}})$-modules between between $\Gamma(\mathcal{E}^1(M)_{\mathrm{red}})$ and $\Gamma_{\mathrm{hom}}((\mathbb{T}\widetilde{M})_{\mathrm{red}})$. Then, we can apply Lemma 4.1.8 to conclude that $\mathcal{E}^1(M)_{\mathrm{red}}$ is isomorphic to $\mathcal{E}^1(M_{\mathrm{red}})$.

Finally, we summarize previous remarks in the following theorem.

**Theorem 4.1.10.** *Under the presence of a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{g}^*$ we have,*

1. $(\mathbb{T}\widetilde{M})_{\mathrm{red}} = \tau(\mathcal{E}^1(M)_{\mathrm{red}})$*, and*
2. *for $P = \mu^{-1}(0)$ we have that $\mathcal{E}^1(M)_{\mathrm{red}}$ is an $\mathcal{E}^1$-reduced structure, i.e. $\mathcal{E}^1(M)_{\mathrm{red}} = \mathcal{E}^1(M_{\mathrm{red}})$.*

## 4.2 REDUCTION OF DIRAC-JACOBI STRUCTURES

We begin this section presenting the reduction procedure for Dirac structures:

**Definition 4.2.1.** A Dirac structure $D \subset E$ is preserved by an extended action $\rho$ if and only if $[\![\, \rho(\mathfrak{a}) \,, \Gamma(D) \,]\!] \subset \Gamma(D)$.

It worth to mention that previous condition on symmetries of the bracket on a Dirac structure yields a well defined bracket on the quotient (3.6), thus it is possible to reduce the whole structure in $D$ as the following result states (see [3, Theorem 4.2].)

**Theorem 4.2.2.** *Let $E$, $\rho$ and $P$ be as in theorem 3.2.3, and such that $E_{\text{red}}$ is exact over $M_{\text{red}} = P/G$. Let the action $\rho$ preserve a Dirac structure $D \subset E$. Define*

$$D_{\text{red}} := \left. \frac{(D \cap K_b^{\perp} + K_b)|_P}{K_b|_P} \middle/ G \subset E_{\text{red}} \right. \tag{4.9}$$

*with $K_b = K \cap (K^{\perp} + T^*M)$. Then if $D_{\text{red}}$ is a smooth subbundle (e.g. if $D \cap K_b$ has constant rank), it defines a Dirac structure on the reduction $M_{\text{red}}$.*

In the case that the extended action is isotropic, the reduced bundle simplifies to

$$D_{\text{red}} = \left. \frac{(D \cap K^{\perp} + K)|_P}{K|_P} \middle/ G. \right.$$

The main goal of this section is to derive a similar reduction in the context of Dirac-Jacobi structures. To achieve this result we should define the notion of a Dirac-Jacobi structure preserved by an extended action.

**Definition 4.2.3.** We say that a Dirac-Jacobi structure $L \subset \mathcal{E}^1(M)$ is preserved by the action $\rho$ if:

$$[\![\, \rho(\mathfrak{a})\, ,\, \Gamma(L)\, ]\!]_{\mathcal{E}^1(M)} \subset \Gamma(L), \tag{4.10}$$

As an example take a contact form $\eta$ on $M$ and an action of a connected Lie group $G$ on $M$ preserving the contact form. Take $\mathfrak{a} = \mathfrak{g} \ltimes \mathfrak{g}$ and the extended action defined in equation (4.3). Because the action of $G$ on $M$ preserves the contact structure we have $di_{\psi(v)}\eta = \mathcal{L}_{\psi(v)}\eta - i_{\psi(v)}d\eta = -i_{\psi(v)}d\eta$, and then we can notice that the elements of $\rho(\mathfrak{a})$ are sections of the Dirac-Jacobi structure associated to the contact form $\eta$ (see in Subsection 2.2.3 the example about contact structures). Therefore $\rho$ preserve the Dirac-Jacobi structure in this case.

Assuming a extended action $\rho : \mathfrak{a} \to \Gamma(\mathcal{E}^1(M))$ for $\mathcal{E}^1(M)$ such that it preserves the Dirac structure $L$, we can see that the extended action $\widetilde{\rho} = \tau \circ \rho$ for $\mathbb{T}\widetilde{M}$ is closed for homogeneous sections of the Dirac structure $\tau(L)$:

$$\begin{aligned} [\![ \widetilde{\rho}(\mathfrak{a}) \, , \, \Gamma_{\text{hom}}(\tau(L)) ]\!]_{\mathbb{T}\widetilde{M}} &= [\![ \tau(\rho(\mathfrak{a})) \, , \, \tau(\Gamma(L)) ]\!]_{\mathbb{T}\widetilde{M}} \\ &= \tau\left( [\![ \rho(\mathfrak{a}) \, , \, \Gamma(L) ]\!]_{\mathcal{E}^1(M)} \right) \subset \tau\left(\Gamma(L)\right) = \Gamma_{\text{hom}}(\tau(L)). \end{aligned}$$

By applying a similar argument as in the proof of Proposition 2.3.4 we can actually see that $\widetilde{\rho}$ preserves the Dirac structure $\tau(L)$. This happens in particular if $\rho$ is given by an equivariant moment map $\mu : M \to \mathfrak{h}^*$.

Hence, once we begin with a preserved Dirac-Jacobi structure $L$, we have that $\tau(L)$ is preserved by the extended action as Dirac structure and we can proceed to the reduction of $\tau(L)$ as in (4.9), which gives,

$$\tau(L)_{red} = \left. \frac{\tau(L) \cap (\widetilde{K})_b^{\perp} + (\widetilde{K})_b}{(\widetilde{K})_b} \right|_{\widetilde{P}} \Big/ G,$$

as Dirac structure in $(\mathbb{T}\widetilde{M})_{\text{red}}$. Finally, as a corollary of the reduction theorem, Theorem 4.2.2, and the facts on the map $\tau$ we get

**Corollary 4.2.4.** *If there exists $D$ subbundle of $\mathcal{E}^1(M)_{\text{red}}$ with $\tau(D) = \tau(L)_{\text{red}}$, then $D$ is Dirac-Jacobi*

Now we give the reduction for Dirac-Jacobi structures for the two cases we have been working on, namely, the contact case and the moment map case.

### Contact case

We close this example by showing the reduction theorem of Dirac-Jacobi structures preserved by an extended action $\rho = \psi \oplus \nu : \mathfrak{g} \times \mathfrak{g} \to \mathcal{E}^1(M)$. Let $L \subset \mathcal{E}^1(M)$ be a Dirac-Jacobi structure on a $G$-manifold $M$ which is preserved by the extended action $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathcal{E}^1(M)$ and also consider $\widetilde{M}_{\mathrm{red}} = M_{\mathrm{red}} \times \mathbb{R}$ the base manifold for the reduction of the Courant algebroid $\mathbb{T}\widetilde{M}$ as in Corollary 3.4.2, then we have.

**Theorem 4.2.5.** *If the Dirac-Jacobi structure $L$ satisfies the following conditions:*

1. *The projection on the tangent part of $L \cap K_b^{\perp}$ along $M_{\mathrm{red}}$ fits inside $TM_{\mathrm{red}}$.*

2. $\mathrm{Ann}(\psi(\mathfrak{g})^{d\eta})|_{[p]} \subset \mathrm{Ann}(T_pG \cdot p)$ *at each $p \in P$.*

*then* $D_{\mathrm{red}} = \dfrac{L \cap K_b^{\perp} + K_b}{K_b}|_{M_{\mathrm{red}}}$ *is Dirac-Jacobi in $\mathcal{E}^1(M_{\mathrm{red}})$ reduction of $L$*

*Proof.* First we begin with the proof that $D_{\mathrm{red}}$ is a subbundle of $\mathcal{E}^1(M_{\mathrm{red}})$. For this, note that first condition implies that the tangent part is inside $TM_{\mathrm{red}}$, while the definition of $D_{\mathrm{red}}$ implies that the cotangent part belongs to $\mathrm{Ann}(\psi(\mathfrak{g})^{d\eta})|_{[p]}$, but by general theory of $G$-actions we know $T^*_{[p]}M_{\mathrm{red}} = \mathrm{Ann}(T_pG \cdot p)$ which is the second condition. Now, a direct consequence of the Proposition 4.1.3 is that $\tau(K_b) = \widetilde{K}_b$ and $\tau(L) \cap \widetilde{K}_b^{\perp} = \tau(L \cap K_b^{\perp})$. These two facts, together with the $G$-equivariance of the map $\tau$, lead us to verify that

$$\tau(L)_{\mathrm{red}} = \left( \frac{\tau(L) \cap \widetilde{K}_b^{\perp} + \widetilde{K}_b}{\widetilde{K}_b} \right)\Bigg|_{\widetilde{P}} \Big/ G = \tau \left( \frac{L \cap K_b^{\perp} + K_b}{K_b}\Bigg|_{P} \Big/ G \right).$$

Finally, the conclusion follows from the Corollary 4.2.4. $\square$

### The $\mathfrak{h}$-moment map

We close this example by showing the reduction theorem of Dirac-Jacobi structures preserved by an isotropic extended action. So let $\rho : \mathfrak{g} \ltimes \mathfrak{h} \to \mathcal{E}^1(M)$ be an isotropic extended action for $\mathcal{E}^1(M)$ given by (4.4). Hence we have that $\widetilde{\rho} = \tau \circ \rho$ is an extended action for

$\mathbb{T}\widetilde{M}$. Recall we assume $P$ and $\widetilde{P}$ satisfy Assumption 3.3.1. Following the reduction theorem for $\mathcal{E}^1(M)$, Theorem 4.1.10, we get our third main result.

**Theorem 4.2.6.** *Let $G$ be a connected Lie group acting on a smooth manifold $M$. Let $\mu : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map such that the induced extended action on $\mathcal{E}^1(M)$ is isotropic. Also suppose the Assumption 3.3.1 and that $\rho(\mathfrak{h})$ has constant rank on $P$ where $G$ acts freely and properly. Then, if we have that $L \subset \mathcal{E}^1(M)$ is a Dirac-Jacobi structure on $M$, is preserved by the extended action $\rho$ and $\pi_T(L) \subset TM$, where $\pi_T : \mathcal{E}^1(M) \to TM \times \mathbb{R}$ is the projection onto the tangent component, we have that $L$ has a Dirac-Jacobi reduction given by*

$$L_{\mathrm{red}} := \left.\frac{L \cap K^{\perp} + K}{K}\right|_P \Big/ G \tag{4.11}$$

*over the reduced manifold $M_{\mathrm{red}} = P/G$.*

*Proof.* By the hypothesis, $L$ is preserved by the extended action $\rho$, then we have that the Dirac structure $\tau(L)$ is preserved by $\widetilde{\rho}$. By Theorem 4.2.2 and by the fact that $\widetilde{\rho}$ is isotropic (because $\rho$ is), we have that $\tau(L)$ has a Dirac reduction which simplifies to

$$\tau(L)_{\mathrm{red}} = \left.\frac{\tau(L) \cap \widetilde{K}^{\perp} + \widetilde{K}}{\widetilde{K}}\right|_{\widetilde{P}} \Big/ G,$$

over $\widetilde{M}_{\mathrm{red}}$.

From the definition of $\tau$ we have $\tau(L) \subset \mathbb{T}\widetilde{M}$ thus $\tau(L) \cap \widetilde{K}^{\perp} = \tau(L \cap K^{\perp})$. This fact together with Assumption 3.3.1 yield to

$$\tau(L)_{\mathrm{red}} = \frac{\tau(L \cap K^{\perp} + K)|_P}{\tau(K)|_P} \Big/ G.$$

Also note that, from Remark 4.1.5 and the condition $\pi_T(L) \subset TM$, it follows that the tangent part of $L_{\text{red}}$ fits inside $\dfrac{\operatorname{Ker}(\nu(\mathfrak{h}))|_P \times \mathbb{R}}{\psi(\mathfrak{g})|_P}/G$ while the cotangent in $\dfrac{\operatorname{Ann}(\psi(\mathfrak{g})|_P \times \mathbb{R}}{\nu(\mathfrak{h})|_P}/G$, in other words $L_{\text{red}}$ is a subbundle of $\mathcal{E}^1(M_{\text{red}})$ and we obtain the Dirac structure

$$\tau(L)_{\text{red}} = \tau\left(\left.\frac{L \cap K^\perp + K}{K}\right|_P\right)\Big/ G$$

over $\widetilde{M}_{red} = (P \times (a,b))/G$. Finally, the conclusion of the theorem comes from Corollary 4.2.4 for $D = L_{\text{red}}$. $\square$

We end with the reduction theorem for the case $P = \mu^{-1}(0)$ where in particular we get $\mathcal{E}^1(M)_{\text{red}} = \mathcal{E}^1(M_{\text{red}})$.

**Theorem 4.2.7.** *Let $G$ be a connected Lie group acting on a smooth manifold $M$. Let $\mu : M \to \mathfrak{h}^*$ be a $\mathfrak{g}$-equivariant map such that the induced extended action on $\mathcal{E}^1(M)$ is isotropic. Also suppose the Assumption 3.3.1 for $P = \mu^{-1}(0)$ and that $\rho(\mathfrak{h})$ has constant rank on $P$ where $G$ acts freely and properly. Then, if we have that $L \subset \mathcal{E}^1(M)$ is a Dirac-Jacobi structure on $M$ which is preserved by the extended action $\rho$, we have that $L$ has a Dirac-Jacobi reduction given by*

$$L_{\text{red}} := \left.\frac{L \cap K^\perp + K}{K}\right|_P \Big/ G \tag{4.12}$$

*over the reduced manifold $M_{\text{red}} = P/G$.*

*Proof.* Firstly, recall that with these hypothesis we have already seen that $\mathbb{T}\widetilde{M}$ can be reduced to (4.6) over $\widetilde{M}_{\text{red}} = \widetilde{P}/G \simeq M_{\text{red}} \times \mathbb{R} = P/G \times \mathbb{R}$, and $\mathcal{E}^1(M)_{\text{red}}$ can be reduced to (4.7).

Now, because $L$ is preserved by the extended action $\rho$ we have that the Dirac structure $\tau(L)$ is preserved by $\widetilde{\rho}$. Then by Theorem 4.2.2 and by the fact that $\widetilde{\rho}$ is isotropic (because $\rho$ is), we have that $\tau(L)$ has a Dirac reduction which simplifies to

$$\tau(L)_{\mathrm{red}} := \left. \frac{\tau(L) \cap \widetilde{K}^{\perp} + \widetilde{K}}{\widetilde{K}} \right|_{\widetilde{P}} \Big/ G, \tag{4.13}$$

over $\widetilde{M}_{\mathrm{red}}$. Clearly $\tau(L)_{\mathrm{red}}$ is a subbundle of $(\mathbb{T}\widetilde{M})_{\mathrm{red}}$.

Notice that $L_{\mathrm{red}}$ is a subbundle of $\mathcal{E}^1(M)_{\mathrm{red}} \simeq \mathcal{E}^1(M_{\mathrm{red}})$, and that $\overline{\tau}(L) = \tau(L)_{\mathrm{red}}$. So, we have the following commutative diagram on which the brackets are preserved.

$$\begin{tikzcd}
\Gamma(\mathcal{E}^1(M)_{\mathrm{red}}) \arrow[r, "\overline{\tau}"] \arrow[d, leftrightarrow] & \Gamma((\mathbb{T}\widetilde{M})_{\mathrm{red}}) \arrow[d, leftrightarrow] \\
\Gamma(\mathcal{E}^1(M_{\mathrm{red}})) \arrow[r, "\tau"] & \Gamma(\tau(\mathcal{E}^1(M_{\mathrm{red}})))
\end{tikzcd}$$

Then we can conclude by the properties of $\tau$ that $L_{\mathrm{red}}$ is a Dirac-Jacobi structure on $M_{\mathrm{red}}$ because $\tau(L)_{\mathrm{red}}$ is a Dirac structure on $\tau(\mathcal{E}^1(M_{\mathrm{red}}))$.

$\square$

# Chapter 5

# OUTLOOK ON THE PROJECT

- We just work on two main examples, but such construction can be extended to other situations such as Jacobi and locally conformal presymplectic structures, however the general reduction procedure is a bigger project because, as we comment along the text, there is a lack on the axiomatization of $\mathcal{E}^1(M)$-structures. This is part of the problems showed in Section 4.1.

- We only consider trivial actions on the real part, i.e $G$-action on $\widetilde{M}$ is of the form $\widetilde{\varphi}_g(m, r) = (\varphi_g(m), r)$. Such trivial actions are related to strick contact morphism, but known research as [12] shows that these are not enough. To consider non-strick action we must consider a more axiomatized structure of the $\mathcal{E}^1$-structures.

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