Gesture Theory: Topos-Theoretic Perspectives and Philosophical Framework

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Tesis presentada como requisito parcial para optar al título de
Doctor en Matemáticas

Director:
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Línea de Investigación:
Teoría de Categorías

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Bogotá, Colombia
Abril de 2018

Agradecimientos

A los miembros de mi familia quienes procuraron rodearme de las mejores condiciones para que yo pudiera dedicarme a esta tesis, especialmente mi mamá, María Isabel, y mi papá, Germán. *A ellos dos está dedicada esta tesis.* Les agradezco infinitamente el amor que me han expresado de tantas maneras.

Al profesor Fernando Zalamea, por su inmensa generosidad tanto intelectual como humana y su diáfano ejemplo. Si a alguien puedo considerar un maestro es a él, quien con serenidad me mostró, como a muchos de mis compañeros, un camino intelectual fecundo y a la vez aliviador. Si no hubiera sido por esa manera tan distinta de ver las matemáticas y el conocimiento, esta tesis no existiría. Siempre estaré agradecido con Fernando por haber confiado en mí, y más en medio de esta empresa a la que muy pocos le hubieran apostado.

A los profesores Octavio Agustín-Aquino, Emilio Lluis-Puebla y Guerino Mazzola, por haberme adoptado en el mundo de la teoría matemática de la música. Importantísimos fueron esos dos viajes a México en los que tuve la oportunidad de compartir con ellos, siempre con la memorable guía de Fernando Zalamea.

A mis compañeros y profes loquitos de música, especialmente a los de la pedagógica (El Nogal), a quienes de manera consciente o inconsciente debo muchas reflexiones en esta tesis. Muchas gracias también por haber sido, aún si de una manera accidentada y abruptamente interrumpida, mi refugio emocional.

Finalmente, agradezco la ayuda recibida de COLCIENCIAS, enmarcada en la convocatoria 727 de 2015 para doctorados nacionales; esta ayuda se efectuó entre enero de 2016 y junio de 2017.

Abstract

Using tools from category theory and 2-category theory, in particular from topos theory, we present the rudiments of a theory of mathematical gestures and hypergestures, including diverse incarnations thereof: topological gestures, gestures on locales, gestures on topological categories, gestures in elementary topoi, diagrams in categories, Mazzola's formulas, and gestures on Grothendieck topoi. In particular, we address the problem of the exponential presentation of certain objects of gestures and hypergestures in some of these instances, for example, in the representative cases of gestures on topological spaces and diagrams in categories. After the mathematical work, we present a philosophical and methodological reflection on gesture theory, mainly based on Galois, Riemann, Kan, Grothendieck, Peirce, Merleau-Ponty, Zalamea, and Mazzola.

Keywords: gesture theory, mathematical music theory, locales, topos theory, simplicial and semi-simplicial sets, Kan extensions, 2-category theory, weighted limits and bilimits, mathematical philosophy, mathematical musicology.

Contents

Introduction

The idea of *mathematical gestures* has an astonishing versatility. It is motivated from a musical point of view, since music is mainly a gestural action of a performer, and hence from the point of view of painting and of any form of expression produced by the *movement of the body*. Also, it has a philosophical motivation, because gestures are at the core of the problem of knowledge: according to Cavaillès [26, p. 25], understanding is a process of assimilation and continuation of gestures: 'Comprendre est attraper le geste et pouvoir continuer.' Moreover, from a mathematical point of view, diagrammatic thinking, incarnated in *category theory* effectively, suggests an underlying movement and intuition: according to Châtelet [29, p. 10], 'Diagrams are in a degree the accomplices of poetic metaphor', and recalling Mazzola [26, p. 25], '*The gesture is a morphism, where the linkage is a real movement and not only a symbolic arrow without bridging substance.*'

Based on these motivations and on Saint-Victor's philosophical definition of gesture as *the movement and configuration of the body's limbs, towards an action and having a modality* (see Section 6.1), Mazzola defines, mathematically, a gesture as a diagram of curves in a topological space (see Section 1.1). This simple definition is the *germ* of this thesis. And, though it was, in principle, conceived within mathematical music theory (MaMuTh), this thesis intends to demonstrate that it, and its associated archetype, the concept of *abstract gestures*, have a considerable interest as *mathematical concepts*; and on the other hand, to contribute to the ineluctable (even more after Grothendieck's extensive reflections on his mathematical work) task of exploring the philosophical consequences of its mathematical richness. For, though the initial idea of the author was to escape from the task of doing an excessively specialized thesis, in a vindication of his inclination for music, into the task of constructing a thesis where mathematics and music were combined, he realized that it is impossible to carry out such an enterprise without a philosophical mediation. In this way, beyond the conceptual adjunction between mathematics and music established by Mazzola (Section 6.1), the natural mediation is, of course, philosophy, so a Peircean triad mathematics/music/philosophy is involved: mathematics is firstness: the realm of possibility; music is secondness: according to Merleau-Ponty, music is activity-reactivity in the world since [32, p. 14] 'La musique, à l'inverse, est trop en deçà du monde et du désignable pour figurer autre chose que des épures de l'Être, son flux et son reflux, sa croissance, ses éclatements, ses tourbillons'; and philosophy is thirdness: knowledge.

However, the mathematical research on Mazzola's gestures does not confine to this thesis, though, to a great extent, after Mazzola's foundational articles [26, 27], the developments of *gesture theory* (excepting those in this thesis) are basically due to Mazzola and some of his collaborators, and are expected to appear in a new version of *The Topos of Music* [24], namely *The Topos of Music III: Gestures*, which to date (2017) has not been published. Certainly, gesture theory has meant a revolution for mathematical music theory and the mathematical musicology established by Mazzola in his first version of *The Topos of Music*, in a remarkable inversion of point of view, from the markedly intellectual and algebraic[1] to the eminently bodily and topological. On the other hand, the orientation and the motivations of this thesis and Mazzola's work have a common root (the foundational articles) but differ in some respects. Whereas Mazzola's developments seek a constant musical motivation and interpretation, and their developments have been gravitating around the concept of topological category (Chapter 5), in this thesis we have sought the location of the concept of gestures, in a very precise way, within category theory: namely, as a particular branch of the theory of *Kan extensions*, rendering this concept very free so as to incarnate in a variety of categories, and even releasing it from its spatiality, which at bottom is not an inherent feature of gestures—but always taking into account the original instance (topological spaces), the previous theory (for example, topological categories), and the motivations.

In this thesis we will be concerned with two important branches of mathematical music theory. On the one hand, we have a discrete branch, an algebraic one, that is, formulas. On the other hand, we have a continuous branch, a topological one, that is, gestures. The algebraic branch contains diagrams in the category of modules over a commutative ring with affine transformations, and in particular, networks, which are just the limits of these diagrams. A particular case of the theory of networks [25] is Lewin's transformational theory, which is an important tool in musical analysis. The topological branch contains the different definitions of gestures in different notions of space: topological gestures and gestures on topological categories (also with an algebraic flavor). It is remarkable that gestures and formulas are at the core of the relation between mathematics and music, which has been proposed by Mazzola in the form of a conceptual mathematical adjunction between formulas and gestures, where music transforms formulas into gestures to produce sounds and mathematics transforms gestures into formulas (Section 6.1). Thus, as suggested by the use of an adjunction as model, mathematics and music should have an effect on each other. The mathematical part of this thesis was motivated by two specific questions that are related to this dialectical relation. Is it possible to unify the formulaic aspects and the gestural aspects of mathematical music theory? Is there a gestural analogue of the Yoneda embedding? These problems were stated in the foundational articles [26, 27], where they were christened *the diamond conjecture* and *the gestural Yoneda embedding* respectively.

The diamond conjecture deals with the formulation in precise mathematical terms of the dialectic between mathematics and music proposed by Mazzola in the form of an adjunction. A first formulation [26] of the conjecture is the following. We unfold the common substance of the category of digraphs in two branches. The first branch is the category of gestures and the second one the category of formulas. Mazzola has constructed a category of gestoids

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[1] Here, it is important to note that, though the concept of topos was essential in Mazzola's monograph, his use was inclined to its conception as a generalized universe of sets (Lawvere), which has a lot of advantages, rather than to its original conception as a generalized space (Grothendieck). This thesis is more concerned with the latter conception.

(algebraic topology flavor) as a way to obtain algebraic data from gestures, and a category of formoids (algebraic geometry flavor) to obtain topological data from formulas. Mazzola conjectures that there exists a universe X that harmonizes the two branches in the sense that the categories involved are connected by suitable pairs of adjoint functors as follows:

$$\begin{tikzcd}
 & X \arrow[dl, shift right] \arrow[dr, shift left] & & . \\
Formoid \arrow[ur, shift right] \arrow[d, shift right] & & Gestoid \arrow[ul, shift left] \arrow[d, shift right] & \\
Formula \arrow[u, shift right] \arrow[dr, shift left] & & Gesture \arrow[u, shift right] \arrow[dl, shift right] & \\
 & Digraph \arrow[ul, shift left] \arrow[ur, shift right] & &
\end{tikzcd}$$

This category would express a unified comprehension of music and would clarify its relationship with mathematics. On the other hand, the gestural Yoneda embedding would consist of a representation of an abstract category $\mathcal{C}$ into an associated category of gestures, instead of the associated category of presheaves $\mathbf{Set}^{\mathcal{C}^{op}}$. As Mazzola has noted in [27], it would help to reconstruct 'the motion, which is intuitively happening, when moving from the argument x of a function f to the value $f(x)$.'

A first approach to answer both problems was given by Mazzola in [27], where the bicategory of gestures was postulated as a good candidate for the conjectural universe X. However the questions remain unsolved since full solutions have not been given yet. The construction of the bicategory of gestures passed through the construction of gestures on topological categories. Topological categories are internal categories in the category of topological spaces and constitute a possible way of generalizing topological ideas to abstract categories; indeed they are regarded by Mazzola as generalized spaces to define gestures on them.

The existence of more natural topologies on categories (Grothendieck topologies), and the fact that we are trying to unify discrete and continuous perspectives, specifically ideas from algebraic topology and algebraic geometry by means of the construction of gestures on generalized spaces, makes us think that the generalization of topological gestures to gestures on Grothendieck topoi (categories of sheaves on a site) are a good way to address the two fundamental problems mentioned above. The pertinence of Grothendieck topoi as a profound metamorphosis of the notion of space and a synthesis of algebraic geometry, topology, and arithmetic, was stressed by Grothendieck himself; see [31]. Thus, the main concern of this thesis arises: the definition of gestures on (in) topoi. Nevertheless, as suggested and summarized by the alternatives on (in), there are two main perspectives on this concern. The first one is that of Grothendieck, in which *a Grothendieck topos is a generalized space*. The second one is that of Lawvere, in which *an elementary topos is a generalized universe of sets*, whose objects (generalized sets) vary continuously over a base[2]. Both perspectives

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[2] Though, elementary topoi that are cocomplete (equivalently, complete) can also be regarded as gener-

are explored in this thesis, though there is an inclination for the first one. In this way, so as to reach this general setting, we have identified a path of intermediate structures between topological spaces and Grothendieck topoi, namely: sober spaces, locales, and localic topoi. This thesis is mainly structured following this path.

In the first chapter we recapitulate Mazzola's foundational article on topological gestures [26], though by no means this chapter is confined to a mere account. We quickly move towards the relation between gestures and geometric realization (called spatialization in [26]) and to the formulation of a problem that seems fundamental in gesture theory: the presentation of the space of gestures $\Gamma@X$ as the exponential $X^{|\Gamma|}$, where $|\Gamma|$ denotes the geometric realization of the skeleton Γ. We show that this presentation holds whenever $|\Gamma|$ is exponentiable, that is, whenever Γ satisfies a certain property of local finiteness. The proof of this fact is obtained from a more general characterization of the space of gestures as a function space, namely $\mathbf{Top}(|\Gamma|, X)$ equipped with the compact-open topology. In turn, it is based on an explicit presentation of the limit from which $\Gamma@X$ is obtained, this presentation being valid in categories other than that of topological spaces. A final discussion points at the generalization of the concept of space of gestures to locales.

The second chapter starts with a review of our article on gestures on locales [21], in which it is shown that the constructions of the space of gestures and the space of hypergestures (iterations of the form $\Gamma_1@\Gamma_2@X$, an essential feature of Mazzola's gestures) can be generalized to locales. Then an important feature arises. Locales of gestures, unlike spaces of gestures, are not made up of points, that is, of individual gestures (=diagrams of curves), but of abstract packs or opens of gestures[3]. This problem is at the core of our understanding of the notion of space. Is it made up from atoms? Or, is it more related to notions of enveloping and covering as suggested by our experience of the world? Thus, the space of points of the locale of gestures is characterized, and then the problem of the spatiality and the non-spatiality of the locale of gestures is addressed. Also, the exponential presentation of the locale of gestures is shown, though the strategy is different from that used in the first chapter; now the strategy is based on the theory of injective locales, which is essential to the theory of exponentiable locales and can be regarded, in a wide sense, as an interesting dual of the theory of projective modules[4]. Finally, as a very useful example (for the theory), we study the locale of gestures $\Gamma@\mathbf{3}$, where $\mathbf{3}$ is the simplest injective locale (the cogenerator of the category of locales), and observe its volatility when Γ varies.

The definition of the locale of gestures leads to the disembodiment of the concepts of gestures and hypergestures since this particular instance shows that the definition does not depend on the properties of a particular category, but it can be formulated in any category $\mathcal{C}$ with certain minimal properties; this was first noted in [21]. In chapter 3, we develop this idea. After giving the basic definitions and motivations, we define *the object of gestures* $\Gamma@S$

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alized spaces, since they are a generalization of complete Heyting algebras (locales), an elementary topos being a generalized Heyting algebra.

[3] Nevertheless, as we will show later in Chapter 3, there is a simple presentation of the generalized elements of the locale of gestures.

[4] For example, the fact that every injective locale can be expressed as a retract of some free frame corresponds to the fact that every projective module is a direct summand of a free module.

with skeleton Γ with respect to a contravariant functor[5] $S : G_1^{op} \longrightarrow \mathcal{C}$ (which just plays the role of the topological digraph of a space) as the limit of the functor

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G_1^{op} \xrightarrow{S} \mathcal{C},$$

if it exists. This seemingly dry formalization is the fact that opens up the location of the concept of gestures within category theory. At the same time, it shows that the truncated category G_1 can be replaced by the whole *semi-simplicial category* G, thus defining the object of gestures with skeleton a semi-simplicial set Γ with respect to a semi-simplicial object $S : G^{op} \longrightarrow \mathcal{C}$. This further generalization is not, after all, a whim of the author: this is very useful for the theory of hypergestures, specifically for the problem of their exponential presentation. And, of course, this generalization includes the previous theory for digraphs, which are particular instances of semi-simplicial sets. Therefore, from now on Γ denotes a semi-simplicial set and S denotes a semi-simplicial object in $\mathcal{C}$.

Under these new perspectives, the limit defining $\Gamma@S$ is just the *tensor product* [15, §VII.2] of functors $\Gamma \otimes_G S^{op}$, that is, in a more natural terminology, a *cotensor product* of functors, namely $\Gamma \pitchfork S$. This seemingly trivial inversion, hardly detectable if we are confined to a particular category, is the cornerstone of the theory of gestures proposed in this thesis. Tensor products of functors, which are so important in topos theory (they yield the inverse images of many geometric morphisms), have an associated fundamental adjunction, namely the Hom-tensor adjunction, so, by duality, we obtain *the fundamental adjunction for gestures*. More precisely, the concept of object of gestures is the dual of that of realization of semi-simplicial sets with respect to functors $T : G \longrightarrow \mathcal{C}$, the latter concept being based on tensor products.

Besides this basic observation, many other issues are explored in Chapter 3: Mazzola's gestures (the case when S is of the form C^T, where C is an object of $\mathcal{C}$), which lead to the construction of hypergestures, and their exponential presentations; generalized elements of the object of gestures, which have a simple presentation as natural transformations thanks to the fundamental adjunction for gestures; the characterization of the gesture functors of the form $\_@S$ as Kan extensions, thus locating the concept within category theory ('all concepts are Kan extensions' according to MacLane [14, X.7]); the generalized Escher theorem; the relation of realization and gestures to their analogues for *simplicial sets*, which leads to the emergence of the *geometric product* of semi-simplicial sets, the exponential formulas for spaces of hypergestures and their characterization as spaces of gestures on skeleta of (possibly) higher dimensions (which fully justifies the introduction of semi-simplicial and simplicial sets in gesture theory); and a simple but central fact: contravariant gesture functors are sheaves with values in $\mathcal{C}$ (in Grothendieck's sense) for natural topologies on the category of semi-simplicial sets. Also, the major contribution of the thesis to the understanding of the diamond conjecture is placed here. It is shown that our general definition of gestures includes the topological perspectives discussed so far, namely topological gestures and gestures on locales; and, at the same time, that it includes some important algebraic perspectives of

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[5] The category G_1 consists of just two parallel arrows between two different vertices plus identities.

mathematical music theory, namely the concept of diagrams with shapes given by digraphs, and gestures on linear categories, which are essentially Mazzola's formulas[6]. It is important to clarify that we are not claiming a solution for the diamond conjecture, since our unification is a concept, not a category. Instead, we consider that the conjecture has not been formulated in a correct way yet.

Nevertheless, the theory of *abstract gestures* could be an appropriate theoretical background for a correct formulation of this conjecture. The author hopes that the results of the third chapter, together with others in the thesis, turn out to be the rudiments of gesture theory, which can be described as the dual of the theory of realizations, or even of the theory of tensor products of functors. Since the objects that in this thesis are called tensor and cotensor products of functors are called indexed limits and indexed colimits in *enriched category theory* (see, for example, Kelly's classic text), it is necessary to determine, in further research, to what extent the theory presented in this thesis is a particular case of enriched category theory, as well as the contributions of the latter to gesture theory, though probably the ideas around Mazzola's (hyper)gestures and their exponential presentations are not discussed in the classical categorical literature. However we are aware of the relation of our concepts to enriched category, as can be seen in the fourth chapter, where we define gestures in 2-categories, which are enriched categories over the category of categories.

The fourth chapter is devoted to the original problem of this thesis: the appropriate definition of gestures on (in) topoi. We start with the definition *in* a particular (co)complete elementary topos, which is simply a particular case of that given in the third chapter. Consequently, given the excellent behavior of these objects regarding gestures (limits and cartesian closedness), the previous theory can be applied in this particular case in an optimal way. In particular, the exponential formulas for Mazzola's gestures and hypergestures are valid. Then we throw ourselves into the more difficult task of defining gestures *on* Grothendieck topoi, that is to say, of defining the Grothendieck topos $\Gamma@\mathcal{E}$ of gestures with skeleton Γ and body in a Grothendieck topos $\mathcal{E}$. However, it is not possible to use the definition of the third chapter, since the natural structure of the universe of all Grothendieck topoi is that of a 2-category, rather than that of a plain category. Thus, the correct definition is achieved by means of *bilimits* of suitable *pseudofunctors*, which of course are analogous to these of the categorical case. This definition yields, without effort, a definition of the object of gestures in an arbitrary 2-category, which can be applied in the case of the 2-categories of locales and localic topoi—some work is made in that direction at the end of the chapter. Also, some analogues of the essential facts of the previous chapters are obtained: fundamental adjunction (now in terms of weighted bilimits), points, and gesture pseudofunctors and their possible (bi)adjoints. It is necessary to make clear that the problem of the gestural Yoneda embedding is not addressed in this thesis, because the author was not able to detect its correct formulation with the tools here explored—which need not mean that such a formulation is not possible.

We end the mathematical part with the fifth chapter. This chapter is inspired by some questions by Mazzola that were communicated to the author in his presentation of the article

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[6] Warning: these formulas are not to be confused with the formulas of mathematical logic.

[21] at the *International Congress on Music and Mathematics* (Puerto Vallarta, 2014). In this chapter we explore the relation of topological categories to internal categories in the category of locales (localic categories). Also, we recapitulate, under the perspective given by the third chapter, Mazzola's definition of topological categories of gestures on topological categories (in the case when the skeleta are digraphs). To this end, we give explicit characterizations of limits and exponentials in categories of internal categories—which are likely well known, though the author was not able to find them in the categorical literature. A very remarkable consequence of this recapitulation is that we finally complete the proof that all the notions of gestures in the diverse notions of space considered in previous stages by Mazzola and the author (topological spaces, locales, and topological categories), obey the same archetype, thus establishing the pertinence of our generalizations. Moreover, thanks to our framework of abstract gestures, the morphisms of the topological category of gestures (in Mazzola's sense) can be characterized as individual abstract gestures, thus justifying the introduction of arbitrary semi-simplicial objects in gesture theory replacing interval objects. We end the chapter by showing the existence of analogous localic categories of gestures on localic categories, though we do not emphasize this issue.

Then we enter the philosophical part of this thesis, which corresponds to the sixth chapter. This chapter preserves the original idea of being an extension of Zalamea's article [37], though it also includes several reflections on the results obtained in the mathematical part—a procedure that, as we have already said, is almost an ethic responsibility after Grothendieck's reflections on his mathematical work [31]. And despite the author has not the authority on philosophy that is required, he is convinced that mathematical ideas have the power to speak to us and have a life of its own—precisely, this is one of the points of gesture theory. Our discussion gravitates around the following main ideas:

i) Mazzola's philosophical framework for gesture theory, which includes some references to French gestural philosophers, including Cavaillès, Châtelet, and Alunni.

ii) Peirce's Cenopythagorean categories, whose use is suggested by the pragmaticist character of Saint-Victor's definition of gesture, thus allowing a modalization of this definition.

iii) Peirce's triadic sign (object-representamen-interpretant), especially, as recast by Zalamea in his book on Peirce's continuum [38].

iv) Merleau-Ponty's *chiasm* [33], which postulates a gluing of subject/object, self/world, and mind/body, where the chiasm (crossing of optic nerves on the brain) helps to explain passages between visibility and non-visibility.

v) Galois connections/adjunctions, which can be related metaphorically to dialectics, as in the case of Mazzola's conceptual adjunction mathematics/music between the poles formulas/gestures. Thus, a main feature (fundamental theorem of Galois theory) of adjunctions, helps to establish further mediations between these polarities, in a true mathematical expression of Merleau-Ponty's chiasm and Zalamea's horosis.

vi) Kan extensions, which at bottom are a different manifestation of Galois adjunctions. This issue is very important, as we observed before, for gesture functors are Kan extensions. Probably, the entire chapter can be regarded, metaphorically, as a Kan extension of Zalamea's article, along a functor that represents the possibilities that the author, with all his deficiencies, can bring under his actual circumstances.

vii) Riemann surfaces, whose dual processes uniformization/ramification can be related to Mazzola's functors in his triad of Western musical interpretation (see the diagram below). Also, branch points can be related to the idea of chiasm. Furthermore, Riemann's analytic continuation, strongly related to Riemann surfaces, gives rise (Leray) to *sheaves*. Certainly, there is a continuation of gestures, which can be expressed in a precise way as the fact that the contravariant gesture functor $\_@S$ is a sheaf with values in $\mathcal{C}$.

viii) Grothendieck's realm of sheaves and topoi, which, as stressed by Zalamea in his seminars at UN, comprises, in an astonishing way, Galois and Riemann ideas. Besides, since the main objective of the thesis was to define gestures on (in) topoi, a discussion on the spatial character of Grothendieck topoi, spatiality that reverberates through the mathematical part, is necessary.

To fix ideas and to make clear a point of arrival, the sixth chapter intends to extend Mazzola's diagram of Western musical interpretation

$$\begin{tikzcd}
\text{gestures} \arrow[rr, "\mathit{instrumentalize}"] \arrow[dr, shift left, "\mathit{freeze}"] & & \text{sounds}\ , \\
& \text{scores} \arrow[ul, shift left, "\mathit{thaw}"] \arrow[ur, "\mathit{perform}"'] &
\end{tikzcd}$$

using the previous account of ideas. However, the discussion does not confine to this, and some other concerns, dear to the author, are addressed.

Finally, in the seventh chapter we record some final notes on the thesis, including auxiliary results, remarks, and perspectives on further research. Also, the author encourages the reader to look at the table of basic notation used in this thesis, which is placed at the end of the document, before the bibliography.

Chapter 1

Gestures on Topological Spaces

In this chapter we expose the basic theory of gestures on topological spaces and present some new results.

1.1 Definition

A *directed graph* (or *digraph*, for short) Γ is a tuple (A, V, t, h), where A and V are sets and $t, h : A \longrightarrow V$ are functions. For a more detailed discussion on digraphs, see Subsection 3.1.1.

The construction of gestures, along the lines of the original definition given by Mazzola in [26], runs as follows. Let Γ be a *digraph*, X a topological *space*, and I the unit *interval* $[0,1]$ in $\mathbb{R}$. In the sequel, we will denote the set of opens of the topological space X by $\mathcal{O}(X)$.

First, we construct the space X^I of *paths* in X. In fact, the space I is an exponentiable object in **Top** by Theorem [3, 5.3]: it is a *locally compact space*[1], so $\mathcal{O}(I)$ is a *continuous lattice*[2] by Lemma [9, VII.4.2]. Furthermore, the exponential X^I is the set $\mathbf{Top}(I, X)$ of continuous maps from I to X, endowed with the compact-open topology. The subbasic opens of this topology are those of the form

$$W(K, U) := \{c \in \mathbf{Top}(I, X) \mid c(K) \subseteq U\},$$

where K is compact in I and U is open in X. For a detailed exposition about topologies for function spaces in **Top**, see [3].

Second, we consider the *spatial digraph* $\overrightarrow{X}$ of the space X. It is the tuple (X^I, X, e_0, e_1), where e_0 and e_1 are the *evaluation maps* at 0 and 1 respectively, that is, $e_0(c) = c(0)$

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[1] The definition of local compactness used through this monograph is the following: a topological space X is said to be *locally compact* if for each point $x \in X$ and each open neighborhood $U \ni x$, there is a compact neighborhood of x contained in U. In the case when X is a Hausdorff space, this definition is equivalent to saying that each point in X has a compact neighborhood. In this way, every compact Hausdorff space is locally compact.

[2] Or core-compact, according to the terminology in [3, §5]. The lattice of opens $\mathcal{O}(X)$ of a topological space X is continuous if any given open neighborhood V of a point x contains an open neighborhood U of x with the property that every open cover of V has a finite subcover of U.

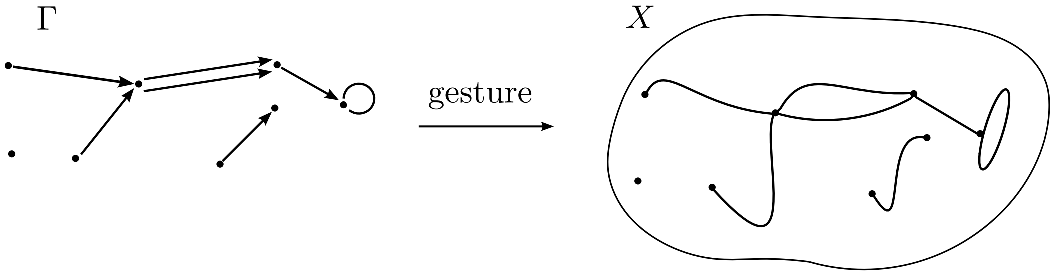

Figure 1.1: A gesture is the embodiment of a skeleton in a space.

and $e_1(c) = c(1)$ for each $c \in X^I$. Actually, both e_0 and e_1 are continuous maps; in fact, $e_0^{-1}(U) = \{c \in X^I \mid c(0) \in U\} = W(\{0\}, U)$, and similarly $e_1^{-1}(U) = W(\{1\}, U)$, which are subbasics of the topology of X^I. Note also that e_0 and e_1 are essentially obtained by applying the functor $X^{(-)}$ to the inclusions $i_0, i_1 : \{*\} \longrightarrow I$ of the endpoints.

Third, we define a *gesture with skeleton* Γ *and body in* X (Figure 1.1) to be a morphism of digraphs from Γ to $\overrightarrow{X}$.

Example 1.1.1. Consider the case when $X = \mathbb{R}^2$. In this case, the spatial digraph $\overrightarrow{\mathbb{R}^2}$ of $\mathbb{R}^2$ is the tuple

$$(\mathbf{Top}(I, \mathbb{R}^2), \mathbb{R}^2, e_0, e_1).$$

In this way, the digraph $\overrightarrow{\mathbb{R}^2}$ has as arrows all continuous curves in $\mathbb{R}^2$ and as vertices all points in $\mathbb{R}^2$.

Now suppose that Γ is the digraph

$$\begin{tikzcd} \bullet x \arrow[loop left, "a"] \arrow[r, "b"] & \bullet y \end{tikzcd} ,$$

that is, $\Gamma = (\{a, b\}, \{x, y\}, t, h)$, where $t(a) = h(a) = t(b) = x$ and $h(b) = y$. Then a gesture $\delta : \Gamma \longrightarrow \overrightarrow{\mathbb{R}^2}$ is a pair (u, v), where $u : \{a, b\} \longrightarrow \mathbf{Top}(I, \mathbb{R}^2)$ and $v : \{x, y\} \longrightarrow \mathbb{R}^2$ are functions satisfying the conditions $u(a)(0) = u(a)(1) = u(b)(0) = v(x)$ and $u(b)(1) = v(y)$. In words, it is simply a diagram of curves that match according to the configuration of Γ. Concretely, we can consider the case when $u(a)$ is the parametrization $(cos(2\pi t), sin(2\pi t))$ (for $0 \leq t \leq 1$) of the unit circle, $u(b)$ is the segment parametrization $(t+1, 0)$ (for $0 \leq t \leq 1$), $v(x) = (1, 0)$, and $v(y) = (2, 0)$. ◇

1.2 Hypergestures

One of the main features of Mazzola's construction is the possibility of building gestures of gestures, namely *hypergestures*. In fact, the set of gestures $Digraph(\Gamma, \overrightarrow{X})$ has a natural topology so that we can iterate the preceding construction. To see this, first check that Γ is the colimit of the diagram $\mathcal{D}$ of digraphs obtained by taking an arrow digraph $\bullet \xrightarrow{a} \bullet$ for each

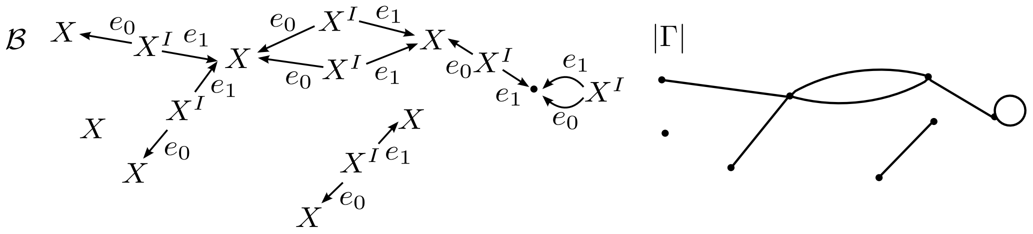

Figure 1.2: Both $\mathcal{B}$ and $|\Gamma|$ (see Section 1.3) are related to the digraph Γ in Figure 1.1.

arrow a in A, a vertex digraph $\bullet x$ for each vertex x in V, and an inclusion morphism $\bullet x \hookrightarrow \bullet \xrightarrow{a} \bullet$ whenever $x = t(a)$ or $x = h(a)$. The contravariant *Hom* functor $Digraph(\_, \overrightarrow{X})$, from the category of digraphs to **Set**, carries colimits of digraphs to limits in **Set** (see [14], isomorphism (3) at p. 117), so $Digraph(\Gamma, \overrightarrow{X})$ is the limit in **Set** of the image diagram $Digraph(\mathcal{D}, \overrightarrow{X})$ of $\mathcal{D}$ under $Digraph(\_, \overrightarrow{X})$. But we can identify $Digraph(\mathcal{D}, \overrightarrow{X})$ with the following diagram $\mathcal{B}$ in **Top** (Figure 1.2, left-hand side): assign to each arrow a in A the space X^I, to each vertex x in V the space X, take a copy of the morphism $e_0 : X^I \to X$ whenever $t(a) = x$, and a copy of the morphism $e_1 : X^I \to X$ whenever $h(a) = x$. In this way, since limits in **Top** have as underlying sets the respective limits in **Set**, we have the bijections

$$Digraph(\Gamma, \overrightarrow{X}) \cong Lim\ Digraph(\mathcal{D}, \overrightarrow{X}) \cong Lim_{\mathbf{Top}}\mathcal{B},$$

and therefore $Digraph(\Gamma, \overrightarrow{X})$ is equipped with the topology transferred from the right-hand space. We thus define the *space of gestures with skeleton* Γ *and body in* X, denoted by $\Gamma@X$, as the space $Digraph(\Gamma, \overrightarrow{X})$. Consequently, if Γ_1 is another digraph, we can construct the space of hypergestures $\Gamma_1@\Gamma@X$ and so on, since **Top** is complete. A gesture with skeleton Γ_1 and body in $\Gamma@X$ is called a *hypergesture*.

Example 1.2.1. Let Γ be as in Example 1.1.1. The space $\Gamma@\mathbb{R}^2$ is isomorphic to the limit in **Top** of the diagram

$$\begin{tikzcd}
\mathbf{Top}(I, \mathbb{R}^2) \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & \mathbb{R}^2 & \mathbf{Top}(I, \mathbb{R}^2) \arrow[l, "e_0"'] \arrow[r, "e_1"] & \mathbb{R}^2 .
\end{tikzcd}$$

According to the construction of limits (by means of products and equalizers [14, V.2]) in **Top**, this limit is the subspace of the Cartesian product (equipped with the Tychonoff topology)

$$\mathbf{Top}(I, \mathbb{R}^2) \times \mathbf{Top}(I, \mathbb{R}^2) \times \mathbb{R}^2 \times \mathbb{R}^2$$

consisting of all tuples (c_a, c_b, p_x, p_y) satisfying the conditions $c_a(0) = c_a(1) = c_b(0) = p_x$ and $c_b(1) = p_y$ (also, see Theorem 1.4.1 below). Note that such a tuple is essentially the same as a gesture δ, as described in Example 1.1.1. $\diamond$

Example 1.2.2. Let Γ be the arrow digraph $\bullet \longrightarrow \bullet$. Note that in this case a gesture δ with skeleton Γ and body in a topological space X is just a continuous path in X. Moreover, the space $\Gamma@X$ can be identified with the space of paths X^I. Thus, a hypergesture with skeleton

Γ and body in $\Gamma @ X$ is just a path F in X^I, that is, a homotopy between the two curves $F(0)$ and $F(1)$, which need not have the same endpoints. Also, the space of hypergestures $\Gamma @ \Gamma @ X$ can be identified with the exponential $(X^I)^I$, which is homeomorphic to $X^{I \times I}$. $\diamond$

1.3 Geometric realization and gestures

Given a digraph Γ corresponding to the tuple (A, V, t, h), its *geometric realization* $|\Gamma|$ is the colimit of the following diagram $\mathcal{D}_1$ of topological spaces (see the right-hand side of Figure 1.2): assign to each arrow a in A the space I, to each vertex x in V a singleton $\{*\}$, take a copy of the inclusion $i_0 : \{*\} \longrightarrow I$ whenever $t(a) = x$, and a copy of the inclusion $i_1 : \{*\} \longrightarrow I$ whenever $h(a) = x$.

According to the construction of colimits (via coproducts and coequalizers) in **Top**, the colimit $|\Gamma|$ can be presented as the quotient of the disjoint union[3]

$$\bigcup_{a \in A} I \times \{a\} \cup \bigcup_{z \in V} \{z\}$$

by the relation $\sim$ defined by

$$(\alpha(f), a) \sim f(a) \text{ whenever } a \in A \text{ and } f \in \{t, h\},$$

where $\alpha : \{t, h\} \longrightarrow \{0, 1\} \subseteq I$ is defined by $\alpha(t) = 0$ and $\alpha(h) = 1$.

By the definition of quotient topology, a subset $U \subseteq |\Gamma|$ is open if and only if the inverse image of U under the natural projection q onto the quotient is open in the disjoint union. In this way, $q^{-1}(\_)$ is a bijective correspondence between opens U in $|\Gamma|$ and sequences of the form $\{U_a\}_{a \in A} \cup \{V_z\}_{z \in V}$, where each U_a is open in I and each V_z is contained in $\{z\}$, satisfying the condition that $\alpha(f) \in U_a$ if and only if $f(a) \in V_{f(a)}$, for each $a \in A$ and $f \in \{t, h\}$.

Example 1.3.1. Consider the digraph Γ of Example 1.1.1. The geometric realization $|\Gamma|$ is the colimit in **Top** of the diagram

$$\begin{tikzcd} I & \{*\} \arrow[l, hook', shift right, "i_0"'] \arrow[l, hook, shift left, "i_1"] \arrow[r, hook, "i_0"] & I & \{*\} \arrow[l, hook', "i_1"'] \end{tikzcd} .$$

Thus, the geometric realization $|\Gamma|$ is the quotient of the disjoint union

$$(I \times \{a\}) \cup (I \times \{b\}) \cup \{x\} \cup \{y\}$$

by the relation $\sim$ defined by $(0, a) \sim (1, a) \sim (0, b) \sim x$ and $(1, b) \sim y$. The resulting object corresponds to Figure 1.3. In this way, an open of the quotient topology on $|\Gamma|$ corresponds to a tuple

$$(U_a, U_b, V_x, V_y),$$

where $U_a, U_b \in \mathcal{O}(I)$, $V_x \subseteq \{x\}$, and $V_y \subseteq \{y\}$, satisfying the conditions i) $0 \in U_a$ if and only if $1 \in U_a$ if and only if $0 \in U_b$ if and only if $x \in V_x$ and ii) $1 \in U_b$ if and only if $y \in V_y$. $\diamond$

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[3] This disjoint union coincides, of course, with $(I \times A) \cup V$.

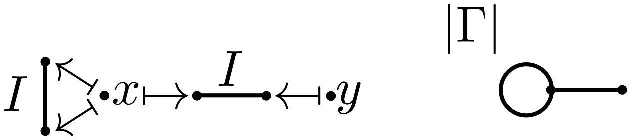

Figure 1.3: The way of identifying the points of the disjoint union (left-hand side) and the realization of the digraph of Example 1.1.1 (right-hand side).

Moreover, the geometric realization is a *CW-complex*, with adjunction of 1-cells given by the pushout

$$\begin{tikzcd}
\bigcup_{a\in A}\{0,1\}\times\{a\} \arrow[r, hook] \arrow[d] & \bigcup_{a\in A} I\times\{a\} \arrow[d, "q"] \\
V \arrow[r, hook, "q"] & {|\Gamma|}
\end{tikzcd} ,$$

where the left-hand map sends $(\alpha(f), a)$ to $f(a)$ whenever $f \in \{t, h\}$. An open (and also closed) 0-cell of $|\Gamma|$ is the image under q of a singleton $\{z\}$ with $z \in V$, a closed 1-cell is the image under q of a set of the form $I \times \{a\}$ with $a \in A$, and an open 1-cell is the image under q of a set of the form $(0,1) \times \{a\}$ with $a \in A$.

Now, from Figure 1.1 and Figure 1.2, we immediately observe that, after all, a gesture $\Gamma \longrightarrow \overrightarrow{X}$ is simply a continuous map $|\Gamma| \longrightarrow X$. In fact, once again, since the *Hom* functor $\mathbf{Top}(\_, X)$ carries colimits in **Top** to limits in **Set**, and the diagram $\mathbf{Top}(\mathcal{D}_1, X)$ is essentially $\mathcal{B}$, we have the bijections

$$\mathbf{Top}(|\Gamma|, X) \cong \mathit{Lim}\ \mathbf{Top}(\mathcal{D}_1, X) \cong \mathit{Lim}\ \mathcal{B} \cong \Gamma@X. \tag{1.1}$$

There is another interesting way of obtaining this correspondence between gestures and continuous maps. The construction of the spatial digraph of a space (Section 1.1) yields a functor $\overrightarrow{(\_)} : \mathbf{Top} \longrightarrow \mathit{Digraph}$. Also, the geometric realization of a digraph yields a functor $|\_| : \mathit{Digraph} \longrightarrow \mathbf{Top}$, and this functor is left adjoint to $\overrightarrow{(\_)}$. This fact is a simple consequence of the more general Hom-tensor adjunction, which will be examined in detail in Subsection 3.4.1 (also, see Example 3.4.1). The adjunction provides a bijection

$$\mathbf{Top}(|\Gamma|, X) \cong \mathit{Digraph}(\Gamma, \overrightarrow{X}),$$

natural in both arguments Γ and X. Moreover, the unit

$$\eta : id_{Digraph} \longrightarrow \overrightarrow{|\_|}$$

of this adjunction consists of, for each digraph Γ, the morphism of digraphs $\eta_\Gamma : \Gamma \longrightarrow \overrightarrow{|\Gamma|}$, which is the pair (u, v), where the morphisms $u(a) : I \longrightarrow |\Gamma|$ (for $a \in A$) and $u(x) : \{*\} \longrightarrow |\Gamma|$ (for $x \in V$) are the legs of the colimiting cone with vertex $|\Gamma|$. Note that η_Γ is a gesture with body in the space $|\Gamma|$. In [26, §5], this canonical gesture is called *the natural gesture associated with* Γ.

Thus, a natural question arises: are $\mathbf{Top}(|\Gamma|, X)$ and $\Gamma@X$ homeomorphic? It is important to clarify that we are placing the compact-open topology on the set $\mathbf{Top}(|\Gamma|, X)$, which is the most natural choice. The answer to that question is affirmative and will be justified in Section 1.5; but before entering the proof, we need an explicit computation of the space of gestures.

1.4 Computation of the space of gestures

Let Γ be a digraph corresponding to the tuple (A, V, t, h), X a topological space, and $\mathcal{B}$ the diagram in **Top** whose limit is (isomorphic to) the space of gestures $\Gamma@X$ (Section 1.2).

From [14, V.2], the limit $\Gamma@X$ can be obtained in terms of products and equalizers in **Top**, specifically as the equalizer of the two pointed arrows in the diagram

$$\begin{tikzcd}
 & X \\
(X^I)^A \times X^V \arrow[ur, shift left, bend left, "p_{h(a)}"] \arrow[ur, shift right, bend left, "p_{t(a)}"'] \arrow[r, shift left, dotted, "\exists! f"] \arrow[r, shift right, dotted, "\exists! g"'] \arrow[d, "p_a"'] & X^A \times X^A \arrow[u, shift left, "p_{a,0}"] \arrow[u, shift right, "p_{a,1}"'] \arrow[d, shift right, "p_{a,0}"'] \arrow[d, shift left, "p_{a,1}"] \\
X^I \arrow[r, shift left, "e_0\ (x{=}t(a))"] \arrow[r, shift right, "e_1\ (y{=}h(a))"'] & X
\end{tikzcd} ,$$

where $p_{a,0}$ and $p_{a,1}$ denote the projections from the product $X^A \times X^A$ of codomains labelled by morphisms in $\mathcal{B}$, $p_{t(a)}$ and $p_{h(a)}$ denote the projections from the product $(X^I)^A \times X^V$ of nodes in $\mathcal{B}$ on the components of $X^A \times X^A$ labelled by $(a, 0)$ and $(a, 1)$ respectively, and p_a is the projection from $(X^I)^A \times X^V$ on the component X^I labelled by a. The morphism f is obtained from the universal property of $X^A \times X^A$ and the morphisms of the top triangles, and g is obtained from the universal property of $X^A \times X^A$ and the morphisms of the bottom squares.

By the computation of products and equalizers in **Top** we have the following theorem.

Theorem 1.4.1. *Let Γ be a digraph corresponding to the tuple (A, V, t, h) and X a topological space. The space of gestures $\Gamma@X$ is the subspace (regular subobject) of $(X^I)^A \times X^V$ consisting of all sequences*

$$\{c_a\}_{a\in A} \cup \{x_z\}_{z\in V} \text{ such that } c_a(\alpha(f)) = x_{f(a)} \text{ whenever } a \in A \text{ and } f \in \{t, h\},$$

where α is the function in the construction of $|\Gamma|$ (Section 1.3).

The remaining part of this section is devoted to an additional computation that will be useful in the next chapter. It is important to stress that the following observations are valid in any small-complete category.

First, check that the equalizer above is a pullback, namely the right-hand square of the following diagram, the morphisms F and G from the left-hand triangle and the middle square being obtained by the universal property of $X^A \times X^A$:

$$\begin{tikzcd}
 & X \\
X^V \arrow[ur, bend left=40, "p_{h(a)}"] \arrow[ur, bend left=20, "p_{t(a)}"'] \arrow[r, dotted, "\exists ! F"] & X^A \times X^A \arrow[u, shift right, "p_{a,0}"] \arrow[u, shift left, "p_{a,1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(X^I)^A \arrow[r, dotted, "\exists ! G"] \arrow[d, "p_a"] & X^A \times X^A \arrow[d, shift right, "p_{a,0}"'] \arrow[d, shift left, "p_{a,1}"] \\
X^I \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
(X^I)^A \arrow[r, "G"] & X^A \times X^A \ . \\
\Gamma @ X \arrow[u] \arrow[r] & X^V \arrow[u, "F"']
\end{tikzcd}$$

At the same time, this pullback coincides with the pullback

$$\begin{tikzcd}
(X^I)^A \times X^{V'} \arrow[r, "G'"] & X^A \times X^A \ , \\
\Gamma @ X \arrow[u] \arrow[r] & X^{V \setminus V'} \arrow[u, "F'"']
\end{tikzcd}$$

where V' is the set of isolated vertices of Γ, and F' and G' are analogous to F and G respectively. We claim that F' is a split monomorphism, so that F' and its pullback along G' are regular monomorphisms. To prove this, we will show that F' is a product of diagonal morphisms, which are split monomorphisms; hence F' must be a split monomorphism. Let $z \in V \setminus V'$. Define

$$A_z = \{(a,0) | t(a) = z\} \cup \{(a,1) | h(a) = z\}.$$

Note that

$$\bigcup_{z \in V \setminus V'} A_z = A \sqcup A$$

and hence

$$\prod_{z \in V \setminus V'} X^{A_z} = X^A \times X^A.$$

Let $diag_z : X \longrightarrow X^{A_z}$ be the diagonal morphism characterized by the equations of the form $\pi_{a,i} diag_z = id_X$, where $\pi_{a,i}$ denotes a typical product projection and $i = 0, 1$. In this way, $F' = H$ ($H := \prod_{z \in V \setminus V'} diag_z$) since $p_{a,0} F' = p_{t(a)}$ and $p_{a,0} H = \pi_{a,0} p_{A_{t(a)}} H = \pi_{a,0} diag_{t(a)} p_{t(a)} = id_X p_{t(a)} = p_{t(a)}$ (similarly $p_{a,1} F' = p_{a,1} H = p_{h(a)}$), where $p_{A_z} : X^A \times X^A \longrightarrow X^{A_z}$ is the projection.

A further computation depends on the following lemma.

Lemma 1.4.2. *Let $\mathcal{C}$ be a small-complete category and $\{\alpha_i : C_i \longrightarrow B_i | i \in \mathcal{I}\}$ a family of regular monomorphisms. Let $f : A \longrightarrow \prod_{i \in \mathcal{I}} B_i$ be a morphism of $\mathcal{C}$, which can be written as $\langle f_i \rangle_i$, where $f_i = \pi_i f$. The pullback of $\prod_{i \in \mathcal{I}} \alpha_i$ along f coincides with the intersection of regular monomorphisms $\bigwedge_{i \in \mathcal{I}} \beta_i$, where β_i is the pullback of α_i along f_i.*

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "diag"] & \prod_{i \in \mathcal{I}} A \arrow[r, "\prod_{i \in \mathcal{I}} f_i"] & \prod_{i \in \mathcal{I}} B_i \\
D \arrow[u, "g"] \arrow[r] & \prod_{i \in \mathcal{I}} D_i \arrow[u, "\prod_{i \in \mathcal{I}} \beta_i"] \arrow[r] & \prod_{i \in \mathcal{I}} C_i, \arrow[u, "\prod_{i \in \mathcal{I}} \alpha_i"]
\end{tikzcd}$$

where $g := \bigwedge_{i\in\mathcal{I}} \beta_i$. We can assume that the right-hand square is a pullback since double limits commute. Moreover, we can assume that the left-hand square is a pullback by the definition of g (check). In this way, the outer rectangle is a pullback. But the upper row is f, so the result follows. □

Thus, if for each $z \in V \setminus V'$ we define β_z as the pullback of $diag_z$ along $p_{A_z}G'$, then

$$\Gamma @ X = \bigwedge_{z\in V\setminus V'} \beta_z.$$

Furthermore, note that if A_z is a singleton, that is, if there is a unique non-loop with z as one of its vertices, then $\beta_z = id$; so we can omit in the meet above each β_z with $z \in V''$, where $V'' := \{z \in V \mid |A_z| = 1\}$. This discussion can be summarized as follows.

Proposition 1.4.3. *Let Γ be a digraph and X an object of* **Top**. *For each $z \in V \setminus (V' \cup V'')$ define β_z to be the pullback of $diag_z$ along $p_{A_z}G'$. The object of gestures $\Gamma@X$ is equal to the meet of regular subobjects*

$$\bigwedge_{z\in V\setminus(V'\cup V'')} \beta_z.$$

The preceding characterizations are not only valid in the category **Top**, but also in any small-complete category. In fact, it is possible to define the object of gestures in more general cases. In the following chapter we carry out this definition in the category of locales.

1.5 Spaces of gestures as function spaces

In this section, we study the compact-open topology on the set $\mathbf{Top}(|\Gamma|, X)$, aiming to prove Theorem 1.5.4. We start with a characterization of the compact subsets of the geometric realization of a digraph.

Since $|\Gamma|$ is a CW-complex, we have the following result (see Proposition [7, 1.2.1]).

Proposition 1.5.1. *If Γ is a digraph, then $|\Gamma|$ is a Hausdorff space.*

Before proving the following result, note that a subset $K \subseteq |\Gamma|$ can be identified with the sequence $q^{-1}(K)$, which can be written in the form $\{K_a\}_{a\in A} \cup \{K_z\}_{z\in V}$ and satisfies the same conditions as the opens of $|\Gamma|$. (See Section 1.3, where the quotient map q is defined and the opens of $|\Gamma|$ are identified with sequences.)

Proposition 1.5.2. *A subset K of $|\Gamma|$ is compact if and only if it is the image under q of a finite sequence of closed sets.*

Proof. If K is a compact subset of $|\Gamma|$, then K is closed since $|\Gamma|$ is Hausdorff by 1.5.1, and hence each K_i is closed. Also, since $|\Gamma|$ is a CW-complex; by Proposition [7, 1.5.2], K is contained in a finite union of open cells. Let $A_0 \subseteq A$ and $V_0 \subseteq V$ be the sets of indices of these open cells (see Section 1.3 for the presentation of the cells of $|\Gamma|$). Denote by S the finite sequence $\{K_a\}_{a\in A_0} \cup \{K_z\}_{z\in V_0}$. Note that its image under q is equal to K: since K is contained in the union of the open cells indexed by the elements of $A_0 \cup V_0$, in particular, it is contained in the union of the closed cells indexed by the elements of the same set, which means that $K \subseteq q((I \times A_0) \cup V_0)$; that is[4], $K = q(S)$.

On the other hand, suppose that K is the image under q of such a finite closed sequence. Each K_i is closed in I or a singleton, and hence compact. Therefore, the sequence is compact since it is a finite disjoint union of compacts, and hence its image under the continuous projection q is also compact. □

Lemma 1.5.3. *Suppose that K is a compact subset of $|\Gamma|$ and that S is the finite sequence $\{K_a\}_{a\in A_0} \cup \{K_z\}_{z\in V_0}$ from 1.5.2. If U is an open in $|\Gamma|$, then $K \subseteq U$ if and only if $K_i \subseteq q^{-1}(U)$ for all $i \in A_0 \cup V_0$.*

Proof. First, if $K \subseteq U$, then $S \subseteq q^{-1}(K) \subseteq q^{-1}(U)$, so $K_i \subseteq q^{-1}(U)$ for all $i \in A_0 \cup V_0$. On the other hand, if $K_i \subseteq q^{-1}(U)$ for all $i \in A_0 \cup V_0$, then $S \subseteq q^{-1}(U)$, so $K = q(S) \subseteq q(q^{-1}(U)) = U$. □

The following theorem uses the computation of the space of gestures established in Theorem 1.4.1.

Theorem 1.5.4. *If X is a topological space and Γ is a digraph, then the space of gestures $\Gamma@X$ is homeomorphic to the function space $\mathbf{Top}(|\Gamma|, X)$, endowed with the compact-open topology.*

Proof. First, check that the image of a sequence $\delta = \{c_a\}_{a\in A} \cup \{x_z\}_{z\in V}$ in $\Gamma@X$ under the natural bijection $\phi : \Gamma@X \longrightarrow \mathbf{Top}(|\Gamma|, X)$ from expression 1.1, is given by

$$\begin{array}{rccccl} \phi(\delta): & |\Gamma| & \longrightarrow & X & & \\ & q(r_a) & \longmapsto & c_a(r_a) & (r_a \in I_a) & . \\ & q(z) & \longmapsto & x_z & (z \in V) & \end{array}$$

Also, the image under ϕ^{-1} of a continuous map $f : |\Gamma| \longrightarrow X$ is the sequence $\{c_a\}_{a\in A} \cup \{x_z\}_{z\in V}$, where $c_a : I \longrightarrow X$ is given by $c_a(r_a) = f(q(r_a))$ and $x_z = f(q(z))$.

We must see that ϕ is continuous and open. It is enough to show that both ϕ^{-1} and ϕ transform subbasic opens into opens, since bijections preserve unions and intersections.

A subbasic open B in $\mathbf{Top}(|\Gamma|, X)$ is of the form $\{f \in \mathbf{Top}(|\Gamma|, X)|\ K \subseteq f^{-1}(U)\}$, where K is a compact subset of $|\Gamma|$ and U an open in X, so by 1.5.3,

$$\phi^{-1}(B) = \{\phi^{-1}(f) | K_i \subseteq q^{-1}(f^{-1}(U)) \text{ for all } i \in A_0 \cup V_0 \subseteq_{fin} A \cup V\} =$$

$$\{\{c_a\}_{a\in A} \cup \{x_z\}_{z\in V} \in \Gamma@X | K_a \subseteq c_a^{-1}(U) \text{ and } K_z \subseteq x_z^{-1}(U), \text{ for all } a \in A_0 \text{ and } z \in V_0\} =$$

$$\Gamma@X \cap \bigcap_{i\in A_0\cup V_0} \pi_i^{-1}(W(K_i, U)),$$

---

[4] *Proof of this detail.* The affirmation $K \subseteq q((I \times A_0) \cup V_0)$ implies $K = K \cap q((I \times A_0) \cup V_0)$. But $K \cap q((I \times A_0) \cup V_0) = K \cap q(q^{-1}(K) \cap ((I \times A_0) \cup V_0)) = K \cap q(S)$. This means that $K = K \cap q(S)$, that is, $K \subseteq q(S)$. On the other hand $q(S) \subseteq q(q^{-1}(K)) = K$.

where the $W(K_a, U)$ (respectively $W(K_z, U)$) are subbasic opens in the compact-open topology on X^I (respectively[5] $X^{\{z\}}$) since each K_a (respectively K_z) is closed in I (respectively $\{z\}$). Thus, $\phi^{-1}(B)$ is the intersection of $\Gamma@X$ and a finite intersection of opens in $(X^I)^A \times X^V$, and hence it is open.

On the other hand, a subbasic open in $\Gamma@X$ is of the form $\pi_i^{-1}(W(C,U)) \cap \Gamma@X$ where $i = a \in A$ and $C \subseteq I$ is closed, or $i = z \in V$ and $C = \{z\}$, so

$$\begin{aligned}\phi(\pi_i^{-1}(W(C,U)) \cap \Gamma@X) &= \phi(\{\delta \in \Gamma@X | \delta_i(C) \subseteq U\}) \\ &= \{\phi(\delta) | \phi(\delta)(q(C)) \subseteq U\} \\ &= W(q(C), U).\end{aligned}$$

In fact, the latter is a subbasic open in $\mathbf{Top}(|\Gamma|, X)$ since the image $q(C)$ under the projection onto the quotient $|\Gamma|$ is compact by 1.5.2. $\square$

1.6 Spaces of gestures as exponentials

Further, we may consider the case when $|\Gamma|$ is exponentiable in **Top**. By Theorem [3, 5.3], $|\Gamma|$ is exponentiable if and only if the lattice of opens of $|\Gamma|$ is continuous (or core-compact). The following result determines when $\mathcal{O}(|\Gamma|)$ is continuous.

A digraph Γ is said to be *locally finite* if each vertex is the tail or head of finitely many arrows.

Proposition 1.6.1. *Suppose that Γ is a digraph. Then $\mathcal{O}(|\Gamma|)$ is a continuous lattice if and only if Γ is a locally finite digraph.*

Proof. Since $|\Gamma|$ is a Hausdorff space by 1.5.1, in particular, it is sober. Therefore, by Corollary [9, VII.4.5], $\mathcal{O}(|\Gamma|)$ is a continuous lattice if and only if $|\Gamma|$ is locally compact. Moreover, by Corollary 3.5.10 below, $|\Gamma|$ is locally compact if and only if Γ is locally finite. $\square$

Corollary 1.6.2. *The space $|\Gamma|$ is exponentiable in* **Top** *if and only if Γ is locally finite.*

Thus, in the case when Γ is locally finite, it is natural to ask whether $X^{|\Gamma|}$ is homeomorphic to $\Gamma@X$. In that case, since $|\Gamma|$ is Hausdorff, by the final paragraph of [3, §5], the exponential $X^{|\Gamma|}$ coincides with the set of functions $\mathbf{Top}(|\Gamma|, X)$ equipped with the compact-open topology, so by Theorem 1.5.4, we obtain the following affirmative answer.

Theorem 1.6.3. *If X is a topological space and Γ is a locally finite digraph, then there exists a homeomorphism*

$$\Gamma@X \cong X^{|\Gamma|}.$$

---

[5] Note that the compact-open topology on $X^{\{z\}}$ is just the topology on X.

1.7 Another point of view

Our main goal in this monograph is to generalize the concept of gestures to topoi, in particular, to Grothendieck topoi, which was claimed as a generalization of the concept of space by Grothendieck himself. But Grothendieck topoi need not have points to operate them as classical topological spaces, the notion of cover being the essential feature of these structures.

Now, there are two intermediate steps that we introduce hoping for a better understanding of the topological flavor of our generalization: sober spaces and locales. To give an idea of the former, we reinterpret the topological digraph of a topological space in terms of correspondences between sets of opens.

Each path c in X^I induces a correspondence $c^{-1}(\_)$ from $\mathcal{O}(X)$ to $\mathcal{O}(I)$.

From the evaluation maps e_0 and e_1 we obtain correspondences $e_0', e_1' : \mathcal{O}(X) \to \mathcal{O}(X^I)$ defined by $e_0'(U) = W(\{0\}, U) = \{c \in X^I \mid 0 \in c^{-1}(U)\}$, and $e_1'(U) = W(\{1\}, U) = \{c \in X^I \mid 1 \in c^{-1}(U)\}$. Note also that e_0 and e_1 are the composites of continuous maps

$$X^I \xrightarrow{\cong} X^I \times \{0\} \xrightarrow{id \times i_0} X^I \times I \xrightarrow{e} X$$

and

$$X^I \xrightarrow{\cong} X^I \times \{1\} \xrightarrow{id \times i_1} X^I \times I \xrightarrow{e} X$$

respectively, where i_0 and i_1 are the inclusions of the endpoints of I, and e is the evaluation map. Thus, e_0' and e_1' are also the composites, in the reverse order, of the correspondences induced by the maps in the diagrams above.

In general, the map from $\mathbf{Top}(X, Y)$ to the set $\mathbf{Frm}(\mathcal{O}(Y), \mathcal{O}(X))$ of all the functions from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ that preserve finite intersections and arbitrary unions, assigning to each continuous function from X to Y the associated inverse image function from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$, is not bijective. But if the space Y is sober, this correspondence is a bijection; see [18, II.1.3]. Recall that Y is *sober* if it is a T_0-space and every irreducible[6] closed set C is the closure of a point. Moreover, if X is a sober space, then we can recover X from $\mathcal{O}(X)$. We will discuss in Section 2.4 the concept of sobriety in the language of adjoints.

In this way, sober spaces are an important intermediate step between topological spaces and Grothendieck topoi since they and their morphisms can be regarded both as spaces of points with continuous functions between them and as suitable lattices of opens with their natural correspondences (which preserve covers, that is, unions). But, rather than giving the definition of gestures in the category of sober spaces, we formulate the definition in the category of locales, which comprises the former. The advantages of this procedure will become apparent through the following chapter.

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[6] A closed set C is said to be *irreducible* if it is nonempty and $C = C_1 \cup C_2$, for C_1 and C_2 closed, implies $C = C_1$ or $C = C_2$.

Chapter 2

Gestures on Locales

2.1 Locales and frames

The category of *frames*, denoted by **Frm**, is obtained by taking the essential algebraic properties of both the sets of opens $\mathcal{O}(X)$ in topological spaces X and the correspondences $f^{-1}(\_) : \mathcal{O}(Y) \to \mathcal{O}(X)$ induced by continuous maps $f : X \to Y$. Formally, the objects of **Frm** are *complete Heyting algebras*, that is, complete lattices L satisfying the infinite distributive law

$$a \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} a \wedge s,$$

for all $a \in L$ and $S \subseteq L$. Morphisms of frames are the functions that preserve finite meets including $\mathbf{1}$ and arbitrary joins including $\mathbf{0}$. In particular these functions preserve the order.

The category **Loc** of *locales* is the opposite of **Frm**. If $f : L \to M$ is a morphism of locales, we denote the corresponding morphism of frames by $f^* : M \to L$. The category **Loc** is complete and cocomplete (see [18, II.3]), the final object $\mathbf{2}$ being the locale $\{\emptyset, \{*\}\}$ of opens of the singleton.

For our computations, we will need an explicit presentation of *the product of two locales*. The formal construction based on C-ideals, as exposed in [9, II.2.12], runs as follows. Given two locales L and M, their Cartesian product $L \times M$ is a meet-semilattice with the pointwise meet. Define a coverage[1] C on $L \times M$ by taking $C(a, b)$ to be the collection of all covers of the form

$$\{(s, b) | s \in S\} \text{ for } S \subseteq L \text{ and } a = \vee S, \text{ and } \{(a, t) | t \in T\} \text{ for } T \subseteq M \text{ and } b = \vee T.$$

The product $L \times_l M$ is defined to be the locale of C-ideals of $L \times M$. The locale $L \times_l M$ is a sublocale of the frame $D(L \times M)$ consisting of all lower subsets of $L \times M$, and has an associated nucleus $j : D(L \times M) \longrightarrow L \times_l M$ which sends each lower set to the least C-ideal

---

[1] It is interesting to observe that a coverage on a meet-semilattice is the analogue of a Grothendieck topology in this context. The difference is that Johnstone's definition only preserves the base change axiom, since, as observed by him [11, C.2.1], this axiom allows the definition of sheaves, without any further requirement. For the definition of C-ideal and coverage, see [9, II.2.11]

containing it. The projections $\pi_1 : L \times_l M \longrightarrow L$ and $\pi_2 : L \times_l M \longrightarrow M$ correspond to the frame morphisms

$$\pi_1^* : a \mapsto j(\downarrow (a,1)) \text{ and } \pi_2^* : b \mapsto j(\downarrow (1,b)).$$

Therefore, by writing $a * b$ for the C-ideal $j(\downarrow (a,b))$, we can check the identities

$$(\bigvee_{a\in S} a) * b = \bigvee_{a\in S} (a * b) \text{ for all } S \subseteq L \text{ and } b \in M,$$

and

$$a * (\bigvee_{b\in T} b) = \bigvee_{b\in T} (a * b) \text{ for all } T \subseteq M \text{ and } a \in L.$$

In this way, we may regard the elements of the form $a*1$ and $1*b$ as analogues of the subbasic opens generating the Tychonoff topology in the case of topological spaces. In particular, note that $0 * b = (\bigvee \emptyset) * b = \bigvee \emptyset = 0 * 0$ and similarly $a * 0 = 0 * 0$ for all $a \in L$ and $b \in M$.

Also, a presentation of the colimit of locales is required. The following one is based on the construction of products and equalizers of frames, as discussed in [18, II.3.1]. *The colimit of a diagram* $E : J \longrightarrow$ **Loc** *is the subframe of* $\prod_{i\in Ob(J)} E_i$ (product of frames) consisting of all sequences $\{u_i\}_{i\in Ob(J)}$ such that $E_x^*(u_j) = u_i$ for each morphism $x : i \longrightarrow j$ of J. Of course, the frame projections on each component correspond to the legs of the colimiting cone. In the case when E is the diagram defining *the geometric realization* in **Loc** of a digraph Γ, that is, when E is the composite of $\mathcal{O}$ with the diagram $\mathcal{D}_1$ from Section 1.3, note that our description of the colimit of E coincides with the early description of the geometric realization from Section 1.3 in terms of sequences. This is because the functor $\mathcal{O}$, as a left adjoint (Section 2.4), preserves colimits.

2.2 Motivation

As we have already seen, the construction of the space of gestures with skeleton Γ and body in X is done in the following three steps.

1. Construction of the *space* X^I *of paths* in X.
2. Construction of the *spatial digraph* $\overrightarrow{X}$ of X.
3. *Gluing* of spatial digraphs and copies of X according to the skeleton Γ.

Thus, if we want to extend the construction to the category of locales, we should try to follow these steps. Note that step 3 only depends on the existence of both objects in the two preceding steps and limits in the category, so we will focus on the first two steps.

From now on, we identify the locale $\mathcal{O}(I)$ with I.

Regarding paths, in the first instance, one is tempted to define the space of paths in a locale L as the set $\mathbf{Loc}(I, L) = \mathbf{Frm}(L, I)$ equipped with an analogue of the compact-open topology. But the right notion is given by the exponential L^I in the category of locales. Indeed, I is exponentiable in **Loc** since I is locally compact (Theorem [9, VII 4.11]), the morphisms of locales from I to L being the points of the exponential L^I. In this way, we call the exponential L^I the *locale of paths in* L, whose generators are the analogues of the subbasic opens generating the compact-open topology[2]. The difference between this construction and the construction for topological spaces is that the exponential L^I need not be spatial[3], that is, it need not coincide with the natural topology on $\mathbf{Loc}(I, L)$, even in the case when L is spatial: for example, take the order topology on the long segment[4]. Moreover, it could have no points at all, for example, if L has no points; see the discussion after Proposition 2.5.4.

Analogously to the case of topological spaces, we define the localic digraph $\overrightarrow{L}$ of L by means of the evaluation maps $e_0, e_1 : L^I \to L$ corresponding to morphisms of frames e_0^* and e_1^* sending an element $a \in L$ to 'all the paths whose inverse images of a contain the respective endpoint':

$$e_0^* : \begin{array}{ccc} L & \longrightarrow & L^I \\ a & \longmapsto & \bigvee_{0 \in U \in \mathcal{O}(I)} [W(U,a)] \end{array}, \quad e_1^* : \begin{array}{ccc} L & \longrightarrow & L^I \\ a & \longmapsto & \bigvee_{1 \in U \in \mathcal{O}(I)} [W(U,a)] \end{array}.$$

The objects $[W(U,a)]$ ($[U \ll f^*(a)]$ in Hyland's terminology) are the equivalence classes of the symbols $W(U,a)$ (respectively $U \ll f^*(a)$) in the presentation of the exponential L^I; see [8, p. 270] and [9, VII 4.11]. In fact, the morphisms e_0 and e_1 so defined are morphisms of frames, because they are the composites (analogous to those in Section 1.7)

$$L^I \xrightarrow{\cong} L^I \times \mathbf{2} \xrightarrow{id \times i_0} L^I \times I \xrightarrow{e} L$$

and

$$L^I \xrightarrow{\cong} L^I \times \mathbf{2} \xrightarrow{id \times i_1} L^I \times I \xrightarrow{e} L$$

respectively, where i_0 and i_1 are the morphisms induced by the inclusions of the endpoints of I and e is the evaluation map, defined by $e^*(a) = \bigvee_{U \in \mathcal{O}(I)} [U \ll f^*(a)] * U$ according to [8, p. 275]. Also, as in Section 1.1, the morphisms e_0 and e_1 are essentially obtained by applying the functor $L^{(-)}$ to i_0 and i_1, since $L^{\mathbf{2}} \cong L$.

2.3 Construction

Following the steps in Section 2.2, we next carry out the construction of gestures on locales. The key point that enables us to formulate the concept is the possibility of constructing both the locale of paths in a locale and arbitrary limits in $\mathbf{Loc}$.

Let Γ be a digraph corresponding to the tuple (A, V, t, h) and L a locale.

As we have already noted, the locale I is a continuous lattice since I is locally compact, and therefore I is exponentiable in $\mathbf{Loc}$. Thus, we have the *locale* L^I *of paths* in L.

The *localic digraph* $\overrightarrow{L}$ of L is the tuple (L^I, L, e_0, e_1) where e_0 and e_1 are 'the evaluation at the endpoints morphisms'.

---

[2] See [9, VII.4.11] for an explicit presentation.

[3] Spatial locales are those isomorphic to the locale of opens of some topological space. If a locale is spatial, we say that it *has enough points*. Spatial locales will be discussed in Section 2.4 in more detail.

[4] See note 7.2.1 for further explanation.

Since L need not have enough points, it is not possible to give a satisfactory definition of gestures on locales by defining them as morphisms of digraphs, as was made in Section 1.1. Rather than defining a gesture, we define the *locale of gestures with skeleton* Γ *and body in* L, denoted by $\Gamma@L$, to be the limit of the diagram $\mathcal{B}$ (analogous to that in Section 1.2) defined as follows: assign to each arrow a in A the locale L^I, to each vertex x in V the locale L, take a copy of the morphism $e_0 : L^I \to L$ whenever $x = t(a)$, and a copy of $e_1 : L^I \to L$ whenever $x = h(a)$.

2.4 Points and gestures

The construction of an object of gestures that we have done is a little more abstract than those presented in [26, 27], in the sense that we are not defining a particular gesture by patching curves with matching endpoints in L according to the digraph Γ; in fact, as noted above, L need not have enough points and the exponential L^I need not coincide with $\mathbf{Loc}(I, L)$.

However, we next show, by characterizing the points of the locale $\Gamma@L$, how it is possible to regard topological gestures on reasonable spaces (sober spaces) in terms of localic gestures. In this way, we are showing a possible way to represent gestural movements inside an algebraic context; see the question posed in [27, p. 33 (4)].

From [9, pp. 41-44] (or [15, IX.3]), we have the functors

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "\mathcal{O}"] & \mathbf{Loc} & \mathbf{Frm}, & \mathbf{Loc} \arrow[r, "pt"] & \mathbf{Top} & ; \\
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}(X) \arrow[d, "(f^*)^{op}"] & \mathcal{O}(X) & L \arrow[r, mapsto] \arrow[d, "f"'] & pt(L) = \mathbf{Loc}(\mathbf{2}, L) \arrow[d, "f\circ\_"] & \\
Y \arrow[r, mapsto] & \mathcal{O}(Y) & \mathcal{O}(Y) \arrow[u, "f^*=f^{-1}(\_)"'] & M \arrow[r, mapsto] & pt(M) = \mathbf{Loc}(\mathbf{2}, M) &
\end{tikzcd}$$

where pt is right adjoint to $\mathcal{O}$. Therefore, we have a natural correspondence

$$\mathbf{Loc}(\mathcal{O}(X), L) \cong \mathbf{Top}(X, pt(L))$$

for each topological space X and each locale L. Moreover, the adjunction restricts to an equivalence between the full subcategories **Sob** of sober spaces and **Sloc** of spatial locales (cf. Proposition 7.0.1); specifically,

> **Sob**=‘spaces isomorphic to $pt(L)$ for some locale L’=‘spaces X such that $pt(\mathcal{O}(X)) \cong X$’=‘fixed points of $pt\mathcal{O}$’, and
>
> **SLoc**=‘locales isomorphic to $\mathcal{O}(X)$ for some space X’=‘locales L such that $\mathcal{O}(pt(L)) \cong L$’=‘fixed points of $\mathcal{O}pt$’.

Note that pt preserves limits since it is a right adjoint; also, we have the following proposition regarding the preservation of exponentials.

Proposition 2.4.1. *Let L be a locale and E a locally compact space. The space $pt(L^{\mathcal{O}(E)})$ is homeomorphic to the exponential $pt(L)^E$ in* **Top**.

Proof. Since E is locally compact, $\mathcal{O}(E)$ is a continuous lattice (Lemma [9, VII.4.2]), E is exponentiable in **Top** (Theorem [3, 5.3]), and $\mathcal{O}(E)$ is exponentiable in **Loc** (Theorem [9, VII.4.11]); so we have the following diagrams of adjoint functors:

$$\mathbf{Top} \underset{\mathcal{O}}{\overset{pt}{\leftrightarrows}} \mathbf{Loc} \underset{\_\times\mathcal{O}(E)}{\overset{(\_)^{\mathcal{O}(E)}}{\leftrightarrows}} \mathbf{Loc}, \quad \mathbf{Top} \underset{\_\times E}{\overset{(\_)^{E}}{\leftrightarrows}} \mathbf{Top} \underset{\mathcal{O}}{\overset{pt}{\leftrightarrows}} \mathbf{Loc}.$$

But $\mathcal{O}(X) \times_l \mathcal{O}(E) \cong \mathcal{O}(X \times E)$ for all spaces X since E is locally compact; see Proposition [9, II.2.13]. Also, it can be checked that this isomorphism is natural in X. Thus, by the uniqueness of adjoints up to isomorphism, $pt(L^{\mathcal{O}(E)}) \cong pt(L)^E$ for any locale L. ☐

The isomorphism $pt(L^I) \cong pt(L)^I$ is obtained as follows. Let $\mathbf{2} \to L^I$ be a point of L^I. We have a corresponding morphism $\mathcal{O}(I) \cong \mathbf{2} \times \mathcal{O}(I) \to L$ given by the universal property of exponentials. Then, the latter arrow induces a path $I \to pt(L)$ by the adjunction between pt and $\mathcal{O}$.

Also, it can be checked that pt carries the evaluation morphisms $e_0, e_1 : L^I \to L$ to the evaluation morphisms $e_0, e_1 : pt(L)^I \to pt(L)$ by means of the isomorphism $pt(L^I) \cong pt(L)^I$. Therefore, pt carries the diagram $\mathcal{B}(L)$ of Section 2.3 to a diagram naturally isomorphic to the corresponding diagram $\mathcal{B}(pt(L))$ for $pt(L)$ in **Top**. Thus, since pt preserves limits, we have the following proposition.

Proposition 2.4.2. *$pt(\Gamma@L) \cong \Gamma@pt(L)$ in* **Top**.

In other words, we can say that pt preserves the gestural structure. An explicit computation of the isomorphism $pt(\Gamma@L) \cong \Gamma@pt(L)$ is the following. Every point of $\Gamma@L$ is a morphism $\mathbf{2} \to \Gamma@L$, which induces a cone

$$\{p_a : \mathbf{2} \to L^I\}_{a\in A} \cup \{p_z : \mathbf{2} \to L\}_{z\in V}$$

over the diagram $\mathcal{B}(L)$ with vertex $\mathbf{2}$, where A and V are the sets of arrows and vertices of Γ respectively. In turn, by transposing this cone across the adjunction of the exponential and the adjunction $\mathcal{O} \dashv pt$, it induces a cone

$$\{p'_a : \{*\} \to pt(L)^I\}_{a\in A} \cup \{p'_z : \{*\} \to pt(L)\}_{z\in V}$$

over $\mathcal{B}(pt(L))$, which is just a sequence

$$\{c_a : I \to pt(L)\}_{a\in A} \cup \{x_z\}_{z\in V} \in \Gamma@pt(L)$$

as in Theorem 1.4.1.

By taking $L = \mathcal{O}(X)$, for X a sober space, we obtain the following corollary.

Corollary 2.4.3. *If X is a sober space, then*

$$\Gamma@X \cong pt(\Gamma@\mathcal{O}(X)).$$

Thus, every gesture with skeleton Γ and body in a sober space X is a point of a locale, namely the locale of gestures with skeleton Γ and body in $\mathcal{O}(X)$. Moreover, since the components of the counit of the adjunction $\mathcal{O} \dashv pt$ are always regular monomorphisms (that is, their associated frame homomorphisms are surjective; see the proof of Proposition [15, IX.3.3]), in particular, the natural morphism $\mathcal{O}pt(\Gamma@\mathcal{O}(X)) \longrightarrow \Gamma@\mathcal{O}(X)$ is a regular monomorphism. Hence, by Corollary 2.4.3, there is a regular monomorphism

$$\mathcal{O}(\Gamma@X) \longrightarrow \Gamma@\mathcal{O}(X).$$

This means that the topology of every space of gestures on a sober space is a sublocale of a locale of gestures, in other words, *the topology of every space of gestures on a sober space is embedded in a locale of gestures.* These facts help to regard gestures on sufficiently well-behaved spaces (for example Hausdorff spaces) in 'purely' algebraic terms, and hence could be a way to codify gestures in a computational setting. These characterizations of topological gestures in the category of locales are a contribution of the present monograph to the theory of gestures. The inverse problem, that is, that of rebuilding algebraic structures in gestural terms, is discussed in Section [26, 6.1] for finitely generated abelian groups.

On the other hand, by Proposition [15, IX.3.3], the locale of gestures $\Gamma@L$ is spatial if and only if the natural map from the locale $\mathcal{O}pt(\Gamma@L)$ (which is isomorphic to $\mathcal{O}(\Gamma@pt(L))$ to $\Gamma@L$ is an isomorphism if and only if $\mathcal{O}(\Gamma@pt(L)) \cong \Gamma@L$. The problem of whether the latter is an isomorphism seems to be difficult in the case when L is spatial; for example, if Γ is the digraph $\bullet \to \bullet$, the locale of gestures coincides with the locale of paths, which, as we have discussed, need not be isomorphic to the topology of the space of paths. In the case when L is non-spatial[5], we will see in the next section that $\Gamma@L$ is non-spatial whenever the digraph is not the initial digraph.

2.5 Gestures on non-spatial locales

The essential feature of locales in our approach to gestures is the fact that they enlarge the notion of space to define gestures. The category of locales, which has the category of sober spaces as a full subcategory, does not capture the non-sober spaces (on which we have already defined gestures in Section 1.1); however it contains the non-spatial locales, on which we can now define the notion of gestures. In this section we deal with the problem of whether the locale of gestures $\Gamma@L$ on a non-spatial locale L is again non-spatial.

The following two results aim to determine when L is a retract of $\Gamma@L$.

Lemma 2.5.1. *The exponential transpose $cts : L \longrightarrow L^I$ of the projection $\pi_1 : L \times I \longrightarrow L$ ('the process of taking constant paths') is a common right inverse for the evaluation at x morphisms $e_x : L^I \longrightarrow L$, where $x \in I$. In particular, L is a retract of L^I.*

Proof. The map e_x is the composite

$$L^I \xrightarrow{p_1^{-1}} L^I \times \mathbf{2} \xrightarrow{id \times i_x} L^I \times I \xrightarrow{e} L,$$

---

[5] A locale is non-spatial if it is not spatial.

where p_1^{-1} is the two-sided inverse for the projection p_1 on the first component of $L^I \times \mathbf{2}$, and i_x corresponds to the inclusion $\{x\} \hookrightarrow I$. Thus, from the commutative diagram

$$\begin{tikzcd}
L^I \arrow[r, "p_1^{-1}"] & L^I \times \mathbf{2} \arrow[r, "id\times i_x"] & L^I \times I \arrow[r, "e"] & L \\
L \arrow[u, "cts"] \arrow[r, "p_1^{-1}"'] & L \times \mathbf{2} \arrow[u, "cts\times id"'] \arrow[r, "id\times i_x"'] & L \times I \arrow[u, "cts\times id"'] \arrow[ur, "\pi_1"] & L \times \mathbf{2} \arrow[l, "id\times i_x"] \arrow[u, "p_1"']
\end{tikzcd}$$

we obtain that

$$e_x cts = \pi_1(id \times i_x)p_1^{-1} = p_1 p_1^{-1} = id.$$

□

Proposition 2.5.2. *If Γ is a non-initial digraph and L is a locale, then L is a retract of $\Gamma@L$.*

Proof. Consider the cone over the diagram defining $\Gamma@L$

$$\begin{tikzcd}
 & & L & t(a) \arrow[dd, "a"] & , \\
L \arrow[r, "cts"] \arrow[urr, "id"] \arrow[drr, "id"'] \arrow[ddrr, bend right, "id"'] & L^I \arrow[ur, "e_0"'] \arrow[dr, "e_1"] & & & a \in A \\
 & & L & h(a) & \\
 & & L & z \in V &
\end{tikzcd}$$

where cts is the right inverse for e_x from Lemma 2.5.1. In fact, it is a cone since $e_0 cts = e_1 cts = id$. By the universal property of $\Gamma@L$, there exists a suitable $m : L \longrightarrow \Gamma@L$. Since Γ is non-initial, it has at least a vertex z, so the projection $p_z : \Gamma@L \longrightarrow L_z$ is a left inverse for m; hence L is a retract of $\Gamma@L$. □

Now note that the epimorphic image of a spatial locale is spatial: in fact, a frame (locale) L is isomorphic to $\mathcal{O}(X)$ for some topological space X if and only if there is an injective morphism of frames from L into $\mathcal{P}(X)$ (the condition of being a frame morphism is exactly the fact that the elements of the image of L are the opens of a topology) if and only if there is an epimorphism of locales from the discrete space $\mathcal{P}(X)$ onto L, so an epimorphic image of a spatial locale is spatial again since epimorphisms are closed under composition. In particular, spatiality is inherited by retracts of spatial locales because they are epimorphic images. Therefore, we have the following result.

Corollary 2.5.3. *If Γ is a non-initial digraph and L is a non-spatial locale, then $\Gamma@L$ is non-spatial.*

Thus, the locale of gestures on a non-spatial locale is again non-spatial if the skeleton is not trivial, so the notion of gestures on locales is irreducible to its spatial analogue. Moreover, we next show an example of a nontrivial locale of gestures without points at all! On the other hand, if Γ is the initial digraph without vertices, note that from the definition of limit, $\Gamma@L$ is the final object $\mathbf{2}$ in **Loc**, which is spatial.

Among the most accessible examples of non-spatial locales are the complete non-atomic boolean algebras (that is, complete boolean algebras that are not isomorphic to any discrete space). For instance, for each locale L we have the double negation nucleus $\neg\neg : L \longrightarrow L$, which sends an element of $a \in L$ to $\neg\neg a$, where $\neg a$ is the pseudocomplement $a \to 0$ of a. This nucleus induces a sublocale $L_{\neg\neg}$ of L, namely the sublocale of fixed points of $\neg\neg$. The elements $\mathbf{0}$ and $\mathbf{1}$ as well as the meet in $L_{\neg\neg}$ are the same of L; but the join in $L_{\neg\neg}$ is $\neg\neg$ applied to the join in L. It can be shown[6] that $L_{\neg\neg}$ is the least dense sublocale[7] of L, and that it is a boolean algebra; in particular, it is a complete boolean algebra. The locales of the form $L_{\neg\neg}$ are rarely spatial (that is, atomic), so they provide us with many examples of non-atomic boolean algebras.

For example, consider the locale $\mathcal{O}(\mathbb{R})$ of opens of the real numbers. The elements of $\mathcal{O}(\mathbb{R})_{\neg\neg}$ are the opens $U \in \mathcal{O}(\mathbb{R})$ for which $Int(\overline{U}) = U$. This locale has no points, as the following result establishes.

Proposition 2.5.4. *The locale $\mathcal{O}(\mathbb{R})_{\neg\neg}$ has no points.*

Proof. First, note that every prime element[8] in a sublocale $S \subseteq L$ is a prime element in the whole locale L. Indeed, let $j : L \longrightarrow L$ be the nucleus associated with S; if $c \in S$ is a prime element in S and $a \wedge b \leq c$, then $j(a) \wedge j(b) \leq j(c) = c$ since nuclei preserve binary meets. Thus, since c is prime, $a \leq j(a) \leq c$ or $b \leq j(b) \leq c$.

Now, the prime elements of $\mathcal{O}(\mathbb{R})$ are, of course, those of the form $\mathbb{R} \setminus \{x\}$ where $x \in \mathbb{R}$; but they are not in $\mathcal{O}(\mathbb{R})_{\neg\neg}$ since $Int(\overline{\mathbb{R} \setminus \{x\}}) = \mathbb{R}$, so $\mathcal{O}(\mathbb{R})_{\neg\neg}$ has no prime elements, and hence has no points. □

Consequently, we have an example of *a locale for which the locale of gestures is nontrivial and has no points* provided the skeleton is not the initial graph. In fact, by Proposition 2.4.2, we have a homeomorphism $pt(\Gamma@\mathcal{O}(\mathbb{R})_{\neg\neg}) \cong \Gamma@pt(\mathcal{O}(\mathbb{R})_{\neg\neg}) = \Gamma@\emptyset$, the latter being empty (if Γ is non-initial) or a singleton. Moreover, by Proposition 2.5.2, if Γ is non-initial, $\mathcal{O}(\mathbb{R})_{\neg\neg}$ is a retract of $\Gamma@\mathcal{O}(\mathbb{R})_{\neg\neg}$ and therefore the latter is nontrivial.

In particular, if Γ is the digraph $\bullet \to \bullet$, the locale $\Gamma@L$, which coincides with the locale of paths $(\mathcal{O}(\mathbb{R})_{\neg\neg})^I$, has no points.

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[6] See [9, II.2.4] and [18, II.2.13], for the proofs of our assertions.

[7] A sublocale L_j relative to the nucleus j is said to be *dense* if $j(0) = 0$. Indeed, the existence of a least dense sublocale is a distinctive feature between locales and spaces: a space need not have a least dense subspace.

[8] See [9, II.1.3] (or Lemma [15, IX.2.1]) for the definition of prime elements and their identification with points.

2.6 Locales of gestures as exponentials

Inspired by Theorem 1.6.3, in this section, given a digraph Γ, we address the problem of determining when the locale of gestures $\Gamma@L$ is isomorphic to the exponential $L^{\mathcal{O}(|\Gamma|)}$ for each locale L. We first show that this reduces to the problem of determining when the functor F preserves regular monomorphisms, where F is the left adjoint to $\Gamma@\_$ from Theorem 3.2.4 (cf. Theorem 3.9.6). We might also try to show directly that the condition of Proposition 3.2.2 holds for a suitable E, but this is a harder enterprise and offers no considerable conceptual gain. In contrast, our approach deals with the concept of *injectivity*, which has a central role in the theory of exponentiable locales and, in general, in category theory.

Suppose that F preserves regular monomorphisms. We claim that $\Gamma@\_$ preserves injectives. In fact, given a diagram of the form

$$\begin{tikzcd}
 & \Gamma@C \text{ ,} \\
M & L \arrow[l, tail, "\gamma"'] \arrow[u, "\phi"]
\end{tikzcd}$$

with C injective and γ regular monomorphism; by transposing across the adjunction and by the injectivity of C, it induces the commutative diagram

$$\begin{tikzcd}
 & C & \text{,} \\
F(M) \arrow[ur, "\exists f"] & F(L) \arrow[l, tail, "F(\gamma)"'] \arrow[u, "\phi_0"] &
\end{tikzcd}$$

with $F(\gamma)$ regular monomorphism. Moreover, the transpose $f' : M \longrightarrow \Gamma@C$ of f satisfies $f'\gamma = \phi$, that is, it completes the first diagram to yield a commutative triangle. Certainly, this follows from the identity

$$\epsilon_C F(f'\gamma) = \epsilon_C F(f')F(\gamma) = fF(\gamma) = \phi_0 = \epsilon_C F(\phi),$$

where ϵ_C corresponds to the counit of the adjunction, and the universal property related to the latter. Hence, $\Gamma@\_$ preserves injectives. Therefore,

> *if F, the left adjoint to $\Gamma@\_$, preserves regular monomorphisms, then $\Gamma@\_$ preserves injectives.*

Let $\mathbf{3}$ denote the topology of the Sierpiński space[9] $\mathbb{S}$, and identify $|\Gamma|$ with the locale $\mathcal{O}(|\Gamma|)$. Now we proceed to show that the preservation of injectives implies the existence of the exponential $\mathbf{3}^{|\Gamma|}$ and the isomorphism $\Gamma@\mathbf{3} \cong \mathbf{3}^{|\Gamma|}$. Suppose that $\Gamma@\_$ preserves injectives. Since $\mathbf{3}$ is injective, $\Gamma@\mathbf{3}$ is, and therefore it is spatial and $pt(\Gamma@\mathbf{3})$ is an injective T_0-space (Corollary [9, VII.4.9]). By Corollary 2.4.3, $pt(\Gamma@\mathbf{3}) \cong \Gamma@\mathbb{S}$. Also, by Theorem

---

[9] The Sierpiński space and its locale of opens $\mathbf{3}$, which is the cogenerator in the category of locales, are frequently used in theory of locales; see [9].

1.5.4, $\Gamma@\mathbb{S} \cong \mathbb{S}^{|\Gamma|} \cong \mathcal{O}(|\Gamma|)$, and a subbasic for the compact-open topology on $\mathcal{O}(|\Gamma|)$ is of the form

$$\{U \in \mathcal{O}(|\Gamma|) \mid K \subseteq U\}$$

with K compact in $|\Gamma|$. Thus, $\mathcal{O}(|\Gamma|)$ is an injective T_0-space, that is, a continuous lattice with the Scott topology (Proposition [9, VII.4.8]), the lattice structure being that given by the specialization order, according to [9, II.1.9] and [9, II.1.8]. Further, we can determine the specialization order on $\mathcal{O}(|\Gamma|)$.

Lemma 2.6.1. *The specialization order for the compact-open topology on $\mathcal{O}(|\Gamma|)$ coincides with the inclusion order.*

Proof. Let $U, V \in \mathcal{O}(|\Gamma|)$. If $U \in \overline{\{V\}}$, then $x \in U$ implies $x \in V$ since $\{x\}$ is compact. Thus, $U \subseteq V$. On the other hand, if $U \subseteq V$, then $K \subseteq U$ implies $K \subseteq V$, so $U \in \overline{\{V\}}$. □

Therefore, $\mathcal{O}(|\Gamma|)$ is a continuous lattice, so the exponential $\mathbf{3}^{|\Gamma|}$ exists and is equal to $\Sigma(\mathcal{O}(|\Gamma|))$ by Theorem [11, C.4.1.9]; in particular, $\mathbf{3}^{|\Gamma|}$ is spatial. Therefore,

if $\Gamma@\_$ preserves injectives, then $\Gamma@\mathbf{3}$ is spatial and the exponential $\mathbf{3}^{|\Gamma|}$ exists.

Suppose that $\Gamma@\mathbf{3}$ is spatial and that the exponential $\mathbf{3}^{|\Gamma|}$ exists. Let us show that the natural morphism $\tau_{\mathbf{3}} : \mathbf{3}^{|\Gamma|} \longrightarrow \Gamma@\mathbf{3}$ from Theorem 3.2.4 is an isomorphism. By the commutative square

$$\begin{tikzcd}
\mathbf{3}^{|\Gamma|} \arrow[d, "\tau_{\mathbf{3}}"] & \mathcal{O}pt(\mathbf{3}^{|\Gamma|}) \arrow[l] \arrow[d, "\mathcal{O}pt(\tau_{\mathbf{3}})"] \\
\Gamma@\mathbf{3} & \mathcal{O}pt(\Gamma@\mathbf{3}) \arrow[l]
\end{tikzcd}$$

given by the counit of the adjunction between $\mathcal{O}$ and pt (Section 2.4), whose rows are isomorphisms since the left-hand locales are spatial, it suffices to show that $pt(\tau_{\mathbf{3}})$ is an isomorphism. Consider the commutative diagram

$$\begin{tikzcd}
pt(\mathbf{3}^{|\Gamma|}) \arrow[r] \arrow[d, "pt(\tau_{\mathbf{3}})"] & pt(\mathbf{3})^{|\Gamma|} \arrow[d, "\theta"] \\
pt(\Gamma@\mathbf{3}) \arrow[r] & \Gamma@pt(\mathbf{3})
\end{tikzcd},$$

whose rows are the isomorphisms described in the lines after propositions 2.4.1 and 2.4.2. Note that θ is uniquely determined. Moreover, we claim that θ is equal to the isomorphism ϕ^{-1} from Theorem 1.5.4. In fact, let $\{r_i | i \in J\}$ be the colimiting cone with vertex $|\Gamma|$ in **Top**. The inverse of the top isomorphism sends a continuous map $f : |\Gamma| \to pt(\mathbf{3})$ to a point $p : \mathbf{2} \to \mathbf{3}^{|\Gamma|}$ by transposition across the adjunctions $\mathcal{O} \dashv pt$ and that of the exponential. Then $pt(\tau_{\mathbf{3}})$ sends p to the point $\mathbf{2} \longrightarrow \Gamma@\mathbf{3}$ determined by the sequence of points $\{\mathbf{3}^{\mathcal{O}(r_i)}p | i \in J\}$. Finally, the bottom isomorphism sends this sequence to the sequence of maps $\{fr_i\}_{i \in J}$, so $\theta = \phi^{-1}$. Thus, $pt(\tau_{\mathbf{3}})$ is an isomorphism. Therefore,

> *if* $\Gamma@\mathbf{3}$ *is spatial and the exponential* $\mathbf{3}^{|\Gamma|}$ *exists, then* $\Gamma@\mathbf{3} \cong \mathbf{3}^{|\Gamma|}$.

Suppose that $\mathbf{3}^{|\Gamma|}$ exists and that $\tau_{\mathbf{3}} : \mathbf{3}^{|\Gamma|} \longrightarrow \Gamma@\mathbf{3}$ is an isomorphism. First, note that $|\Gamma|$ is an exponentiable locale by the proof of Theorem [11, C.4.1.9]. We next show that the natural morphism $\tau_L : L^{|\Gamma|} \longrightarrow \Gamma@L$ is an isomorphism for every locale L. Given a locale L, it can be embedded in a power of the cogenerator $\mathbf{3}$ in **Loc**. The embedding f is the unique making the diagrams of the form

$$\begin{tikzcd}
& \mathbf{3} \\
L \arrow[ur, "f_a"] \arrow[r, "f"'] & \prod_{a\in L}\mathbf{3} \arrow[u, "\pi_a"']
\end{tikzcd}$$

commute, where f_a is the morphism determined by a. In particular, note that $f^*(\pi_a^*(m)) = f_a^*(m) = a$ where $\mathbf{0} < m < \mathbf{1}$ in $\mathbf{3}$, so f^* is surjective and hence f is a regular monomorphism. This means that f is the equalizer of a pair $g', h' : \prod_{a\in L}\mathbf{3} \longrightarrow M$ of morphisms of locales. But, in turn, there is an embedding $f' : M \rightarrowtail \prod_{b\in M}\mathbf{3}$, which is a monomorphism. This implies (check) that f is the equalizer of the pair $g, h : \prod_{a\in L}\mathbf{3} \longrightarrow \prod_{b\in M}\mathbf{3}$, where $g = f'g'$ and $h = f'h'$.

In this way, by the naturality of $\tau : (\_)^C \longrightarrow Lim\ (\_)^E$ (Theorem 3.2.4), and since both $\Gamma@\_$ and $(\_)^{|\Gamma|}$ preserve limits, we have the following diagram with inner and outer squares commuting, and whose rows are equalizers:

$$\begin{tikzcd}
L^{|\Gamma|} \arrow[r, "f^{|\Gamma|}"] \arrow[d] & (\prod_{a\in L}\mathbf{3})^{|\Gamma|} \arrow[r, shift left, "g^{|\Gamma|}"] \arrow[r, shift right, "h^{|\Gamma|}"'] \arrow[d] & (\prod_{b\in M}\mathbf{3})^{|\Gamma|} \arrow[d] \\
\Gamma@L \arrow[r, "\Gamma@f"] & \Gamma@\prod_{a\in L}\mathbf{3} \arrow[r, shift left, "\Gamma@h"] \arrow[r, shift right, "\Gamma@g"'] & \Gamma@\prod_{b\in M}\mathbf{3}
\end{tikzcd}.$$

Furthermore, the middle and right-hand columns are isomorphisms since they are suitable products of the isomorphism $\Gamma@\mathbf{3} \cong \mathbf{3}^{|\Gamma|}$ (this follows from the naturality of τ and the fact that $\Gamma@\_$ and $(\_)^{|\Gamma|}$ preserve the respective product projections), so the left-hand column is an isomorphism. Therefore,

> *if the exponential* $\mathbf{3}^{|\Gamma|}$ *exists and* $\tau_{\mathbf{3}} : \Gamma@\mathbf{3} \longrightarrow \mathbf{3}^{|\Gamma|}$ *is an isomorphism, then* $|\Gamma|$ *is an exponentiable locale and* $\tau_L : \Gamma@L \longrightarrow L^{|\Gamma|}$ *is an isomorphism for every locale* L.

Hence, we have the following proposition.

Proposition 2.6.2. *Let* Γ *be a digraph. The following statements are equivalent:*

i) The locale $|\Gamma|$ *is exponentiable and for each locale* L *there is a natural isomomorphism*

$$\Gamma@L \cong L^{|\Gamma|}.$$

ii) *The locale* $|\Gamma|$ *is exponentiable and the functors* $\Gamma@\_$ *and* $(\_)^{|\Gamma|}$ *are naturally isomorphic.*

iii) *The locale* $\Gamma@\mathbf{3}$ *is spatial and the exponential* $\mathbf{3}^{|\Gamma|}$ *exists.*

iv) *The locale* $\Gamma@\mathbf{3}$ *is injective.*

v) *The functor* $\Gamma@\_$ *preserves injectives.*

vi) *The left adjoint* F *to* $\Gamma@\_$ *preserves regular monomorphisms.*

Proof. The equivalence *i)* $\Leftrightarrow$ *ii)* is simply the definition of natural isomorphism. *iv)* is a particular case of *v)*. The implications *vi)* $\Rightarrow$ *v)*, *iv)* $\Rightarrow$ *iii)*, and *iii)* $\Rightarrow$ *ii)* were already described. To show *ii)* $\Rightarrow$ *vi)*, note that if $\Gamma@\_$ and $(\_)^{|\Gamma|}$ are naturally isomorphic, then F is naturally isomorphic to $|\Gamma| \times_l \_$, which preserves regular monomorphisms (check). ☐

Thus, we must determine the condition on Γ under which the left adjoint F preserves regular monomorphisms. By the latter proposition, if F preserves regular morphisms, then $|\Gamma|$ is exponentiable in **Loc**, so $\mathcal{O}(|\Gamma|)$ is a continuous lattice (Theorem [9, VII.4.11]) and hence Γ is locally finite (Corollary 1.6.1). This means that locally finiteness of Γ is a necessary condition. We claim that it is also a sufficient condition. The remaining part of this section is devoted to show this claim. Let Γ be a locally finite digraph, E the diagram defining the geometric realization of Γ in **Loc** (Section 2.1) and $f : L \longrightarrow M$ a regular monomorphism of locales. Since the commutative diagram

$$\begin{tikzcd}
F(L) = Colim\ L \times_l E \arrow[r, "\psi_{L,E}"] \arrow[d, "F(f)=Colim\ f\times id"'] & L \times_l |\Gamma| \arrow[d, "f\times id"] \\
F(M) = Colim\ M \times_l E \arrow[r, "\psi_{M,E}"] & M \times_l |\Gamma|
\end{tikzcd}$$

where ψ comes from Proposition 3.2.3, has a regular monomorphism as right-hand arrow, it can be shown that if the horizontal arrows are regular monomorphisms, then the left-hand morphism is. In this way, *we have reduced the problem to show that the natural map*

$$\psi_{L,E} : Colim\ L \times_l E \longrightarrow L \times_l |\Gamma|$$

is a regular monomorphism for each locale L. We next focus our efforts on the latter property. We first note that the property of being a regular monomorphism can be tested locally (Lemma 2.6.3).

Recall that a *cover of a locale* L *by open sublocales* is a collection $\{\downarrow (a_i)\}_{i\in\mathcal{I}}$ where $a_i \in L$ and $\bigvee_{i\in\mathcal{I}} a_i = 1_L$; in that case we say that *the family* $\{a_i\}_{i\in\mathcal{I}}$ *covers* L. Indeed, every open sublocale of a locale L is isomorphic to a locale of the form $\downarrow (a)$ where $a \in L$, the corresponding frame homomorphism $u^* : L \longrightarrow \downarrow (a)$ being defined by $u^*(d) = (a \wedge d)$.

Lemma 2.6.3. *Let $f : L \longrightarrow M$ be a morphism of locales and $\{\downarrow(b_i)\}_{i\in\mathcal{I}}$ a cover of L by open sublocales. If for each $i \in \mathcal{I}$ the morphism f_i in the pullback diagram (see Lemma [9, II.2.8])*

$$\begin{tikzcd}
L \arrow[r, "f"] & M \\
\downarrow(f^*(b_i)) \arrow[u, tail] \arrow[r, "f_i"] & \downarrow(b_i) \arrow[u, tail]
\end{tikzcd} ,$$

where the vertical arrows are the respective open sublocale inclusions, is a regular monomorphism of locales, then f is a regular monomorphism.

Proof. First, note that f_i^* is uniquely determined and corresponds to the restriction of f^* for each i: from the square we obtain that $f_i^*(b \wedge b_i) = f^*(b) \wedge f^*(b_i) = f^*(b \wedge b_i)$ for all $b \in M$, so in particular, if $c \leq b_i$, then $f_i^*(c) = f^*(c)$.

Put $a_i = f^*(b_i)$ for each i. Suppose that each f_i^* is surjective. Then f^* is surjective: if $a \in L$, then for each i there exists $c_i \leq b_i$ such that $a \wedge a_i = f_i^*(c_i) = f^*(c_i)$, and hence $a = a \wedge \bigvee_{i\in\mathcal{I}} a_i = \bigvee_{i\in\mathcal{I}} a \wedge a_i = \bigvee_{i\in\mathcal{I}} f^*(c_i) = f^*(\bigvee_{i\in\mathcal{I}} c_i)$. ☐

In this way, we need a suitable cover of the codomain of $\psi_{L,E}$ by open sublocales. Let $E : J \longrightarrow$ **Loc** be an arbitrary diagram and L a locale. Now we proceed to show that each cover of C, where C is the colimit of E, induces a cover of the codomain $L \times_l C$ of the morphism $\psi_{L,E}$ from Proposition 3.2.3 such that the pullbacks above are commutative squares associated with the natural transformation $\psi_{L,\_}$. We need the following lemma.

Lemma 2.6.4. *Let L and M be locales, $a \in L$, and $b \in M$. The frame homomorphism*

$$\begin{array}{rccc} e^* : & \downarrow(a)\times_l \downarrow(b) & \longrightarrow & \downarrow(a * b) \\ & a' * b' & \longmapsto & a' * b' \end{array}$$

defines an isomorphism e of locales and the triangle

$$\begin{tikzcd}
 & L \times_l M \\
\downarrow(a)\times_l \downarrow(b) \arrow[ur, tail, "f\times g"] & \downarrow(a * b) \arrow[u, tail, "h"'] \arrow[l, "e"']
\end{tikzcd}$$

commutes, where f, g, and h are the respective open sublocale inclusions. In other words, $f \times g$ is the inclusion of the open sublocale associated with $a * b$.

Proof. Consider the following pullback diagrams:

$$\begin{tikzcd}
L \arrow[r, "\chi_a"] & \mathbf{3} & M \arrow[r, "\chi_b"] & \mathbf{3} & L \times_l M \arrow[r, "\chi_a\times\chi_b"] & \mathbf{3}\times_l \mathbf{3} \\
\downarrow(a) \arrow[u, tail, "f"] \arrow[r] & \mathbf{2} \arrow[u, tail, "T"'] & \downarrow(b) \arrow[u, tail, "g"] \arrow[r] & \mathbf{2} \arrow[u, tail, "T"'] & \downarrow(a)\times_l \downarrow(b) \arrow[u, tail, "f\times g"] \arrow[r] & \mathbf{2}\times_l \mathbf{2} \arrow[u, tail, "T\times T"']
\end{tikzcd} ,$$

where the left-hand and middle squares correspond to the fact that **3** classifies the open sublocales of a given locale (certainly this follows from the identities[10] $\chi_a^*(m) = a$ and $\chi_b^*(m) = b$ and Lemma [9, II.2.8]), and the right-hand square, which is the product of the preceding squares, corresponds to the fact that double limits commute. Now according to Lemma 1.4.2, the latter pullback is the intersection of the pullbacks β_1 and β_2 of the open sublocale T along $\chi_a\pi_1$ and $\chi_b\pi_2$ respectively, where π_1 and π_2 are the projections from $L \times_l M$. Since pullbacks preserve open sublocales (Lemma [9, II.2.8]) and

$$\pi_1^*\chi_a^*(m) = a * 1 \text{ and } \pi_2^*\chi_b^*(m) = 1 * b,$$

the sublocales β_1 and β_2 are the open ones associated with $a * 1$ and $1 * b$. Moreover, since the intersection of two open sublocales is open, the intersection $\beta_1 \wedge \beta_2$ corresponds to the open sublocale associated with $a * b$ (which is the meet $(a*1) \wedge (1*b)$); see [18, II.2.6]. Thus, the pullback $f \times g$ is the open sublocale associated with $a * b$. □

Also, we need the following fact about $\psi_{L,E}$. From the computation of colimits (last paragraph in Section 2.1), given any diagram $E : J \longrightarrow$ **Loc** with colimit C, an explicit calculation of the natural map ψ_E is

$$\begin{array}{rccc} \psi_{L,E}^* : & L \times_l C & \longrightarrow & Colim\ L \times_l E \\ & a * \{u_i\}_{i \in Ob(J)} & \longmapsto & \{a * u_i\}_{i \in Ob(J)} \end{array}$$

at the generators.

An element U in $Colim\ E$ is a sequence $\{u_i\}_{i \in Ob(J)}$ such that $E_x^*(u_j) = u_i$ for each morphism $x : i \longrightarrow j$ of J. Note that U induces a natural transformation $\mu : E' \longrightarrow E$, where $E'_i :=\downarrow (u_i)$, $(E'_x)^*$ is the restriction of $(E_x)^*$, and the μ_i are the open sublocale inclusions. In fact, $(E'_x)^*$ is well-defined since $a \leq u_j$ implies $(E'_x)^*(a) \leq (E'_x)^*(u_j) = u_i$. Also, μ is natural since $E_x^*(c) \wedge u_i = E_x^*(c) \wedge E_x^*(u_j) = E_x^*(c \wedge u_j)$ for each $c \in E_j$ and each morphism $x : i \longrightarrow j$ of J. Moreover, the commutative squares of the form

$$\begin{tikzcd} E'_i \arrow[r, tail, "\mu_i"] \arrow[d, "E'_x"'] & E_i \arrow[d, "E_x"] \\ E'_j \arrow[r, tail, "\mu_j"] & E_j \end{tikzcd}$$

are pullbacks since $E_x^*(u_j) = u_i$. Conversely, given a natural transformation $\mu : E' \longrightarrow E$ such that each $\mu_i : E'_i \longrightarrow E_i$ corresponds to the inclusion of the open sublocale associated with an element u_i in E_i, and such that each square as above is a pullback (that is, $E_x^*(u_j) = u_i$), corresponds to an element U in $Colim\ E$, namely the sequence $\{u_i\}_{i \in Ob(J)}$. In the following lemma we investigate the interplay between this situation and $\psi_{L,E}$.

---

[10] As before, m is the middle element of **3**.

Lemma 2.6.5. *Let L be a locale, $\mu : E' \longrightarrow E$ the natural transformation associated with an element U in the colimit C of E, and C' the colimit of E'. The square*

$$\begin{tikzcd}
Colim\ L \times_l E \arrow[r, "\psi_{L,E}"] & L \times_l C \\
Colim\ L \times_l E' \arrow[r, "\psi_{L,E'}"] \arrow[u, "Colim\ \ L\times\mu"] & L \times_l C' \arrow[u, "L\times Colim\ \ \mu"']
\end{tikzcd}$$

is the pullback of the inclusion of the open sublocale $\downarrow(1 * U)$. In particular, a cover $\{\downarrow(U_i)\}_{i\in\mathcal{I}}$ of C by open sublocales induces a cover $\{\downarrow(1 * U_i)\}_{i\in\mathcal{I}}$ of $L \times_l C$ by open sublocales where each U_i has an associated pullback square as above.

Proof. We begin by proving that the columns are open sublocale inclusions.

The right-hand side is an open sublocale inclusion. By our presentation of colimits of locales (last paragraph in Section 2.1), C' is precisely the subframe $\downarrow(\{u_i\}_{i\in Ob(J)})$ of C and $Colim\ \mu : C' \longrightarrow C$ is the open sublocale inclusion. Thus, by Lemma 2.6.4, the right-hand side is the inclusion of the open sublocale associated with $1 * U$.

The left-hand side is an open sublocale inclusion. The commutative squares

$$\begin{tikzcd}
L \times_l E'_i \arrow[r, tail, "id\times\mu_i"] \arrow[d, "id\times E'_x"'] & L \times_l E_i \arrow[d, "id\times E_x"] \\
L \times_l E'_j \arrow[r, tail, "id\times\mu_j"] & L \times_l E_j
\end{tikzcd} \quad ,$$

where, according to Lemma 2.6.4, $id \times \mu_i$ is the inclusion of the open sublocale associated with $1*u_i$, are pullbacks since $(id\times E_x)^*(1*u_j) = 1*E_x^*(u_j) = 1*u_i$. Therefore, $Colim\ L\times_l E'$ can be identified with the subframe $\downarrow(\{1*u_i\}_{i\in Ob(J)})$ of $Colim\ L\times_l E$ and hence $Colim\ L\times\mu$ is the inclusion of the open sublocale associated with $\{1 * u_i\}_{i\in Ob(J)}$.

On the other hand, $\psi^*_{L,E}(1 * U) = \{1 * u_i\}_{i\in Ob(J)}$, so the former square is a pullback. □

After the following two lemmas, we can proceed to check that $\psi_{L,E} : Colim\ L \times_l E \longrightarrow L \times_l |\Gamma|$ is locally a regular morphism whenever Γ is a locally finite digraph.

Lemma 2.6.6. *A locally closed sublocale (that is, an intersection of an open sublocale and a closed sublocale) of a spatial sublocale is spatial.*

Proof. Let L be a spatial locale and consider a sublocale that is the intersection of an open sublocale $S \rightarrowtail L$ and a closed sublocale $T \rightarrowtail L$. This intersection $S \wedge T$ is, by definition, the pullback of $S \rightarrowtail L$ along $T \rightarrowtail L$. Thus, T is spatial (see examples [11, C.1.2.6]) and $S \wedge T$ is an open sublocale of T (Lemma [9, II.2.8]); hence $S \wedge T$ is spatial [11, C.1.2.6]. □

Lemma 2.6.7. *If Γ is a finite digraph, then $\Gamma@\mathbf{3}$ is spatial.*

Proof. According to Proposition 1.4.3, $\Gamma@\mathbf{3}$ is a sublocale of an injective (and hence spatial) locale L, namely[11]

$$(\mathbf{3}^I)^A \times_l \mathbf{3}^{V'},$$

---

[11] This locale is injective since the functor $(\_)^I$ preserves injectives (Lemma [9, VII.4.10]) and products of injectives are injective.

and is a finite intersection of pullbacks of diagonal sublocales of the form $\mathbf{3} \rightarrowtail \mathbf{3}^n$, where n is a positive integer. Moreover, by Proposition 2.7.2 and the discussion before it, each diagonal is the join of an open sublocale (finite intersection of open sublocales) and a closed sublocale. In this way, since pullbacks preserve open and closed sublocales (Lemma [9, II.2.8]) as well as finite joins ([9, p. 55] and [18, II.3.8]), $\Gamma@\mathbf{3}$ is a finite intersection of sublocales of L of the form $S \vee T$ with S open sublocale of L and T closed sublocale of L. In this way, by distributive laws in the coframe of sublocales plus the fact that finite intersections of closed (respectively open) sublocales are closed (respectively open)[12], $\Gamma@\mathbf{3}$ is a finite join of locally closed sublocales of L, which are spatial by Lemma 2.6.6. This implies that $\Gamma@\mathbf{3}$ is spatial since joins of spatial sublocales are spatial ([18, II.2.14], final paragraph). □

Proposition 2.6.8 (Local argument)**.** *Let L be a locale and Γ a finite digraph. The natural map*

$$\psi_{L,E} : Colim\ L \times E \longrightarrow L \times |\Gamma|,$$

where E is the diagram defining $|\Gamma|$, is an isomorphism.

Proof. According to the preceding lemma, $\Gamma@\mathbf{3}$ is spatial. Also, $|\Gamma|$ is exponentiable in **Loc** since it is locally compact. In this way, by Proposition 2.6.2, the natural transformation $\tau : (\_)^{|\Gamma|} \longrightarrow \Gamma@\_$ from Theorem 3.2.4 is an isomorphism. Moreover, by Corollary 3.2.6, ψ_E is an isomorphism. □

Finally, in the following proposition we patch the local information to obtain a whole regular morphism.

Proposition 2.6.9. *Let L be a locale and Γ a locally finite digraph. The natural map*

$$\psi_{L,E} : Colim\ L \times E \longrightarrow L \times |\Gamma|,$$

where E is the diagram defining $|\Gamma|$, is a regular monomorphism.

Proof. For each $z \in V$, define

$$U_z = \{[0,1)_a\}_{a\in T} \cup \{(0,1]_a\}_{b\in H} \cup \{[0,1]_c\}_{c\in O} \cup \{z\},$$

where T is the set of non-loops with tail z, H is the set of non-loops with head z, and O is the set of loops. Note that U_z is an open in $|\Gamma|$. Therefore, $|\Gamma|$ is covered by the collection of all the U_z with $z \in V$ and all the $(0,1)_a$ with $a \in A$. Consequently, if for each open U in the cover we prove that $\psi_{L,E'}$ is a regular monomorphism, where E' is the diagram associated with U, then by lemmas 2.6.5 and 2.6.3, $\psi_{L,E}$ is a regular monomorphism.

In fact, for each open U in the cover, there is a *finite* subdigraph Σ of Γ such that U is an open of $|\Sigma|$ (take the digraph with the arrows involved in U plus their vertices). By Proposition 2.6.8, $\psi_{L,S} : Colim\ L \times S \longrightarrow L \times |\Sigma|$ is an isomorphism, where S is the diagram

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[12] See Proposition [18, II.2.6]

defining $|\Sigma|$. In particular, by Lemma 2.6.5, $\psi_{L,E'}$ is an isomorphism (the pullback of an isomorphism)[13]. ☐

Thus, if Γ is locally finite, then the left adjoint F to $\Gamma@\_$ preserves regular monomorphisms, so Proposition 2.6.2 implies the central theorem of this section.

Theorem 2.6.10. *Let Γ be a digraph. The following statements are equivalent:*

i) The locale $|\Gamma|$ is exponentiable in **Loc** *and for each locale L there is a natural isomomorphism*

$$\Gamma@L \cong L^{|\Gamma|}.$$

ii) The locale $|\Gamma|$ is exponentiable and the functors $\Gamma@\_$ and $(\_)^{|\Gamma|}$ are naturally isomorphic.

iii) The locale $\Gamma@\mathbf{3}$ is injective.

iv) The functor $\Gamma@\_$ preserves injectives.

v) The left adjoint F to $\Gamma@\_$ preserves regular monomorphisms.

vi) The digraph Γ is locally finite.

Proof. As we have just seen, $vi)$ implies $v)$. By the paragraph after Proposition 2.6.2, $|\Gamma|$ exponentiable implies Γ locally finite, so $i)$ implies $vi)$. ☐

In particular, when $L = \mathbf{3}$, we obtain the following corollary.

Corollary 2.6.11. *Let Γ be a locally finite digraph. The locale $\Gamma@\mathbf{3}$ is spatial, where $\mathbf{3}$ is the locale of opens of the Sierpiński space $\mathbb{S}$. In particular, $\Gamma@\mathbf{3}$ is the locale of opens of the space of gestures $\Gamma@\mathbb{S}$, which is homeomorphic to $\mathcal{O}(|\Gamma|)$ equipped with the topology whose subbasics are of the form*

$$\{U \in \mathcal{O}(|\Gamma|) |\ K \subseteq U\}$$

with K compact in $|\Gamma|$ and U open in $|\Gamma|$ (Theorem 1.5.4).

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[13] There is a subtlety here: we have established that $\psi_{L,E'}$ is an isomorphism when we regard E' as a diagram whose domain coincides with that of S, which need not coincide with the domain of E. However, the interpretations of E' as a diagram with the same domain as S and as a diagram with the same domain as E lead to definitions of $\psi_{L,E}$ that are essentially equal; see the explicit calculation of $\psi_{L,E}$ after Lemma 2.6.4.

2.7 Gestures on the triadic locale

Since the contravariant gesture functor $\Gamma@\_$ coincides with the exponential functor $(\_)^{|\Gamma|}$ if and only if Γ is locally finite (Theorem 2.6.10), it is worth studying which properties of the exponential functor remain valid when Γ is not locally finite, that is, when Γ has at least a vertex which is the tail or head of infinitely many arrows. The fundamental property of the exponential functors in **Loc** is that they preserve injective locales, which are, in particular, spatial.

To test injectivity and spatiality, we will take Γ to be the digraph consisting of a set of arrows having the same tail and different heads. Formally, it is the tuple

$$\Gamma = (\{0\} \times P, \{0\} \cup P, \pi_1, \pi_2),$$

where P is a set that does not contain 0. On the other hand, we will take L to be the simplest nontrivial injective locale, that is, the locale $\mathbf{3}$ of opens of the Sierpiński space $\mathbb{S}$. To begin with, by Theorem 2.6.10, $\Gamma@\mathbf{3}$ is not injective if P is infinite.

Our aim is the following theorem.

Theorem 2.7.1. *If P is countable, $\Gamma@\mathbf{3}$ is spatial. If P is uncountable, $\Gamma@\mathbf{3}$ is non-spatial.*

Nevertheless, we will need a considerable amount of work to obtain it.

By Proposition 1.4.3 (which is also valid in **Loc**), $\Gamma@\mathbf{3}$ is the pullback

$$\begin{tikzcd}
(\mathbf{3}^I)^P \arrow[r, "e_0^P"] & \mathbf{3}^P \\
\Gamma@\mathbf{3} \arrow[u] \arrow[r] & \mathbf{3} \arrow[u, "diag"']
\end{tikzcd} .$$

That is, $\Gamma@\mathbf{3}$ is the inverse image under e_0^P of the diagonal $\mathbf{3} \rightarrowtail \mathbf{3}^P$ (which is a split monomorphism, and hence regular), and therefore $\Gamma@\mathbf{3}$ is a sublocale of $(\mathbf{3}^I)^P$ since regular monomorphisms are preserved under pullback.

The locale $\mathbf{3} = \mathcal{O}(\mathbb{S})$ is spatial and injective, so $\mathbf{3}^P$ is injective (and hence spatial) since products of injective locales are injective. On the other hand, the functor *pt*, which preserves products, sends the diagonal $\mathbf{3} \rightarrowtail \mathbf{3}^P$ to the diagonal $\mathbb{S} \rightarrowtail \mathbb{S}^P$ in **Top**, and we essentially get $diag : \mathbf{3} \rightarrowtail \mathbf{3}^P$ back by applying $\mathcal{O}$ to $diag : \mathbb{S} \rightarrowtail \mathbb{S}^P$ because $\mathbf{3}$ and $\mathbf{3}^P$ are spatial. Consequently, the diagonal corresponds to the subspace of $\mathbb{S}^P$ consisting of the two constant sequences related to 0 and 1 respectively. Let $S_i \rightarrowtail \mathcal{O}(\mathbb{S}^P)$ be the sublocale corresponding to the subspace whose unique point is the constant sequence with value i, for $i \in \{0, 1\}$. Since the sublocale induced by a union of subspaces is the join, in the coframe of sublocales, of the induced sublocales ([18, II.2.14], final paragraph), $diag = S_0 \vee S_1$. Thus, according to [9, p. 55] and [18, II.3.8], the inverse image of the diagonal is the join, in the coframe of sublocales of $(\mathbf{3}^I)^P$, of the pullbacks of S_0 and S_1.

The reason to express the diagonal as the join of S_0 and S_1 is the following. As we will show shortly, S_0 is a closed sublocale and S_1 is an intersection of open sublocales. In this way, the pullback of S_0 is a closed sublocale of $(\mathbf{3}^I)^P$ (and hence spatial, given the spatiality of the injective locale $(\mathbf{3}^I)^P$) and the pullback of S_1 is the intersection of the pullbacks of those open sublocales since the inverse image is a coframe homomorphism. But, as we will show, the pullback of S_1 is non-spatial if P is uncountable and this implies that $\Gamma@\mathbf{3}$, the join of the pullbacks of S_0 and S_1, is non-spatial. In contrast, if P is countable, both S_1 and $\Gamma@\mathbf{3}$ are spatial.

To show that S_0 is closed, it is enough to check that the associated subspace is closed (see [9, II.2.4 (a)]). As in [9, VII.4.8], the space $\mathbb{S}^P$, which is equipped with the Tychonoff topology, is isomorphic to the lattice $\wp(P)$ endowed with the Scott topology, whose basics are the upper sets of the form $\uparrow(F)$, where F is a finite subset of P. Also, under the isomorphism $\mathbb{S}^P \cong \wp(P)$, the constant sequence of zeros corresponds to the empty set. In this way, if a subset X of P is nonempty, it belongs to some $\uparrow(\{*\})$, which does not contain $\emptyset$, so the complement of $\{\emptyset\}$ in $\wp(P)$ is open and hence $\{\emptyset\}$ is closed.

On the other hand, in the next proposition we express S_1 as a meet of open sublocales.

Proposition 2.7.2. *The sublocale $S_1 \rightarrowtail \mathcal{O}(\mathbb{S}^P)$ is the meet $\bigwedge_{F\subseteq_{fin}P} S_F$, where S_F is the open sublocale associated with the Scott-open $\uparrow(F)$.*

Proof. We make the proof in terms of sublocale sets (see the exercise 3 in [18, p. 68] for the definition). First, note that the sublocale set induced by an intersection of subspaces is always contained in the intersection of the induced sublocale sets, and that $\bigcap_{F\subseteq_{fin}P}\uparrow(F) = \{P\}$, so it remains to show that $\bigcap_{F\subseteq_{fin}P} S_F \subseteq S_1$. We start by computing the sublocale sets S_1 and S_F.

The sublocale set S_1 is $\{\emptyset, \wp(P)\}$. In fact, the nucleus j associated with S_1 sends a Scott-open a in $\wp(P)$ to $\bigcup\{b \in \Sigma\wp(P) | b \cap \{P\} \subseteq a\}$. If $a = \emptyset$, then $j(a) = \emptyset$. If a is nonempty, then $P \in a$, so $j(a) = \wp(P)$. Thus the image of j is $\{\emptyset, \wp(P)\}$.

Now, we compute the sublocale set S_F. The nucleus j associated with S_F sends an element a of the form $\bigcup_{i\in J}\uparrow(X_i)$, with $X_i \subseteq_{fin} P$, to the element $j(a)$, which satisfies

$$
\begin{aligned}
j(a) &= \bigcup\{\uparrow(Y) \in \Sigma\wp(P) |\ \uparrow(Y) \cap \uparrow(F) \subseteq \bigcup_{i\in J}\uparrow(X_i)\} \\
&= \bigcup\{\uparrow(Y) |\ \uparrow(Y \cup F) \subseteq \bigcup_{i\in J}\uparrow(X_i)\} \\
&= \bigcup\{\uparrow(Y) | \ Y \cup F \in \bigcup_{i\in J}\uparrow(X_i)\} = \bigcup\{\uparrow(Y) | \ Y \cup F \supseteq X_i \text{ for some } i\} \\
&= \bigcup\{\uparrow(Y) | \ Y \supseteq X_i \setminus F \text{ for some } i\} = \bigcup_{i\in J}\uparrow(X_i \setminus F),
\end{aligned}
$$

so an element of the sublocale set S_F is of the form $\bigcup_{k\in K}\uparrow(Y_k)$, with $Y_k \subseteq_{fin} P \setminus F$ for all k.

Finally, we show the inclusion. Let $a \in \bigcap_{F\subseteq_{fin}P} S_F$. If $a = \emptyset$, then $a \in S_1 = \{\emptyset, \wp(P)\}$. Now suppose that a is nonempty. If $a = \uparrow(\emptyset) = \wp(P)$, then $a \in S_1$. The remaining case leads us to a contradiction. Indeed, if a is neither the whole space nor empty, then a is of the form $\bigcup_{i\in J}\uparrow(X_i)$ $(X_i \subseteq_{fin} P)$, where $X_i \neq \emptyset$ for some $i \in J$. But, by hypothesis, $a \in S_{X_i}$, so it is also of the form $\bigcup_{k\in K}\uparrow(Y_k)$, where $Y_k \subseteq P\setminus X_i$ for all k. Thus, $X_i \supseteq Y_k$ for some $Y_k \subseteq P\setminus X_i$, which implies $Y_k=\emptyset$ and $a=\wp(P)$, a contradiction. $\square$

Also, S_1 is the meet $\bigwedge_{p\in P} S_{\{p\}}$, as the following proposition establishes.

Proposition 2.7.3. *The equality* $\bigwedge_{p\in P} S_{\{p\}} = \bigwedge_{F\subseteq_{fin} P} S_F$ *holds.*

Proof. First, note that $\bigwedge_{p\in F} S_{\{p\}} = S_F$ for all $F\subseteq_{fin} P$ since the sublocale induced by a finite intersection of open subspaces is the intersection of the induced sublocales; here, $\uparrow(F)=\bigcap_{p\in F}\uparrow(\{p\})$. Thus $\bigwedge_{p\in P} S_{\{p\}} \leq \bigwedge_{F\subseteq_{fin} P} S_F$.

Second, $\bigwedge_{F\subseteq_{fin} P} S_F \leq S_{\{p\}}$ for all $\{p\}\subseteq P$, so $\bigwedge_{F\subseteq_{fin} P} S_F \leq \bigwedge_{p\in P} S_{\{p\}}$. $\square$

Now, by Theorem [11, C.4.1.9], $\mathbf{3}^I$ is the locale $\Sigma\mathcal{O}(I)$ of Scott-opens in the continuous lattice $\mathcal{O}(I)$, and hence $\mathbf{3}^I$ is injective and spatial, so $(\mathbf{3}^I)^P$ is also injective and is equal to the locale $(\Sigma\mathcal{O}(I))^P$, which satisfies

$$(\Sigma\mathcal{O}(I))^P \cong \mathcal{O}pt((\Sigma\mathcal{O}(I))^P) \cong \mathcal{O}((pt\Sigma\mathcal{O}(I))^P) \cong \mathcal{O}(\mathcal{O}(I)^P)$$

by its spatiality and the fact that the Scott topology on continuous lattices is sober (Theorem [9, VII.2.6]); hence $(\Sigma\mathcal{O}(I))^P$ is essentially the Tychonoff topology on the product $\mathcal{O}(I)^P$. Moreover, since $e:\mathbf{3}^I\times_l I\longrightarrow\mathbf{3}$ corresponds to the open $\bigvee\{\uparrow(u)*u|\ u\in\mathcal{O}(I)\}$ (proof of Theorem [11, C.4.1.9]) so that

$$e_0:\mathbf{3}^I \xrightarrow{\pi_1^{-1}} \mathbf{3}^I\times_l \mathbf{2} \xrightarrow{id\times i_0} \mathbf{3}^I\times_l I \xrightarrow{e} \mathbf{3}$$

corresponds to the open $\bigcup\{\uparrow(u)|0\in u\}$, which is just[14] $\{u|0\in u\}\in\Sigma\mathcal{O}(I)$, the morphism of locales $e_0^P:(\mathbf{3}^I)^P\longrightarrow\mathbf{3}^P$ corresponds to the morphism of frames defined by

$$\begin{array}{rccc} (e_0^P)^*: & \Sigma\wp(P) & \longrightarrow & (\Sigma\mathcal{O}(I))^P \\ & \uparrow(F) & \longmapsto & \{(u_p)_{p\in P}\in\mathcal{O}(I)^P|\ 0\in u_p \text{ for all } p\in F\} \end{array}$$

at the basics of the form $\uparrow(F)$ with $F\subseteq_{fin} P$. Consider the following definitions. Let L be the locale $\mathcal{O}(\mathcal{O}(I)^P)$ (which is isomorphic to $(\Sigma\mathcal{O}(I))^P$), L_F (for $F\subseteq_{fin} P$) the open sublocale of L induced by the open T_F, where $T_F:=(e_0^P)^*(\uparrow(F))$, and M the sublocale of L induced by the subspace T, where $T:=\bigcap_{F\subseteq_{fin} P} T_F$. In this way, by [9, p. 55] or [18, II.3.8], and the fact that $S_1=\bigwedge_{F\subseteq_{fin} P} S_F=\bigwedge_{p\in P} S_{\{p\}}$ (propositions 2.7.2 and 2.7.3), the pullback of S_1 along e_0^P is equal to the meets $\bigwedge_{F\subseteq_{fin} P} L_F$ and $\bigwedge_{p\in P} L_{\{p\}}$ of pullbacks.

The following two propositions are the core of our discussion and the most difficult ones.

For the first proposition, we need the following notations. The basic opens of the product topology on $\mathcal{O}(I)^P$ are finite intersections of opens of the form $\pi_p^{-1}(\uparrow(U_p))$, where $U_p\in\mathcal{O}(I)$ and $p\in P$. In this way, a basic open depends on a sequence U, with $U=(U_p)_{p\in P}\in\mathcal{O}(I)^P$, such that there exists $F\subseteq_{fin} P$ such that $U_p=\emptyset$ whenever $p\in P\setminus F$; in that case we say that U *has finite support.* Given such a sequence U, we define $\uparrow(U)=\bigcap_{p\in F}\pi_p^{-1}(\uparrow(U_p))$ so that the basics of the product topology are of this form. Also, given $U,V\in\mathcal{O}(I)^P$ with U having finite support, we write $V\gg U$ if $V_p\gg U_p$ for all $p\in P$. Thus, with this notation, $\uparrow(U)=\{V\in\mathcal{O}(I)^P|\ V\gg U\}$.

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[14] Since $\mathcal{O}(I)$ is a continuous poset, $u=\bigcup\{v|v\ll u\}$, so if $0\in u$, then there is $v\ll u$ such that $0\in v$.

Lemma 2.7.4. *Let $U \in \mathcal{O}(I)^P$ be a sequence with finite support and $V' \gg U$ such that $0 \in V'_p$ for some $p \in P$. There exists a sequence V, with finite support, satisfying $0 \in V_p$ and $U \ll V \ll V'$.*

Proof. By the density of the pointwise way below relation (Lemma [9, VII.2.4]), there exists V, with finite support, such that $V' \gg V \gg U$. Also, since $0 \in V'_p$ and V'_p is the join of all the elements that are way below it, $0 \in w$ for some $w \ll V'_p$. Therefore, by replacing V_p by $V_p \cup w$, which is way below V'_p by Lemma [9, VII.2.2] and satisfies $U_p \ll V_p \cup w$, we obtain the result. □

Proposition 2.7.5. *If P is countable, then $\bigwedge_{F \subseteq_{fin} P} L_F = M$, that is, the inverse image of S_1 is spatial.*

Proof. We make the proof in terms of sublocale sets. This is a sort of diagonal argument. First, observe that $M \subseteq \bigcap_{F \subseteq_{fin} P} L_F$. It remains to show the other inclusion.

Let $a \in \bigcap_{F \subseteq_{fin} P} L_F$. To show that $a \in M$, it is enough to prove that for each basic $b \in L$ of the form $\Uparrow (U)$, $b \cap T \subseteq a$ implies $b \subseteq a$. Let b be such a basic and suppose that $b \nsubseteq a$, we will show that this implies that $b \cap T \nsubseteq a$.

We can assume that $P = \mathbb{N}$ without loss of generality. Define a sequence $(f_n)_{n \in \mathbb{N}}$ by recursion (plus choice), where $f_n \in \mathcal{O}(I)^{\mathbb{N}}$ for each $n \in \mathbb{N}$, in the following way.

i) Define f_0 to be some sequence $V \in b$ such that $0 \in V_0$ and $\Uparrow (V) \nsubseteq a$. Let us see that such a sequence V exists. If $b \cap T_{\{0\}} \subseteq a$, then $b \subseteq a$ since $a \in L_{\{0\}}$, which is a contradiction; so there is $V' \gg U$ with $0 \in V'_0$ satisfying $V' \notin a$, and hence, by Lemma 2.7.4, there exists V such that $0 \in V_0$ and $U \ll V \ll V'$.

ii) Suppose we have defined f_{n-1} such that $0 \in f_{n-1,n-1}$ and $\Uparrow (f_{n-1}) \nsubseteq a$. If $\Uparrow (f_{n-1}) \cap T_{\{n\}} \subseteq a$, then $\Uparrow (f_{n-1}) \subseteq a$ since $a \in L_{\{n\}}$, which is a contradiction; so there exists $W \gg f_{n-1}$ with $0 \in W_n$ such that $W \notin a$. Define f_n to be some sequence satisfying $0 \in f_{n,n}$ and $f_{n-1} \ll f_n \ll W$ (Lemma 2.7.4). In this way, $\Uparrow (f_n) \nsubseteq a$.

Consequently, $(f_{n,n})_{n \in \mathbb{N}} \in b$ since $f_n \gg f_{n-1} \gg \cdots \gg f_0 \gg U$ so that $f_{n,n} \gg U_n$ for all n and $(f_{n,n})_{n \in \mathbb{N}} \gg U$.

Moreover, we claim that the sequence $(f_{n,n})_{n \in \mathbb{N}}$ is not in a. For if $\Uparrow (W)$ is a basic open contained in a, then there exists t such that $W_n = \emptyset$ for all $n > t$. But $f_t \notin a$, so there exists $k \in \{0, \ldots, t\}$ such that $W_k \not\ll f_{t,k}$ and hence $W_k \not\ll f_{k,k}$, that is, $(f_{n,n})_{n \in \mathbb{N}} \notin \Uparrow (W)$.

Finally, $(f_{n,n})_{n \in \mathbb{N}}$ is in T by construction, but is not in a, so $\Uparrow (U) \cap T \nsubseteq a$. □

Therefore, in this case, $\Gamma@\mathbf{3}$ is the join of two locales induced by subspaces, so by the final paragraph in [18, II.2.14], $\Gamma@\mathbf{3}$ is induced by a subspace and hence is spatial.

Proposition 2.7.6. *If P is uncountable, then $\bigwedge_{p \in P} L_{\{p\}}$ is not a sublocale of M and therefore the inverse image of S_1 is non-spatial.*

Proof. Again, we make the proof in terms of sublocale sets. We will show that there is an element a in the intersection that is not contained in M.

Define $a = \bigcup_{n\in\mathbb{Z}^+} A_n$, where

$$A_n := \bigcup\{\bigcap_{p\in F} \pi_p^{-1}(\uparrow([0,1/n]))\mid F \subseteq_{fin} P \text{ and } |F| = n\}.$$

First, let us show that $a \in \bigcap_{p\in P} L_{\{p\}}$. Let $k \in P$ and suppose that $b \cap T_{\{k\}} \subseteq a$, where b is a basic open of the form $\uparrow(U)$. We must show that $b \subseteq a$. By the notation established above, $U_p = \emptyset$ if $p \in P \setminus F$, for some $F \subseteq_{fin} P$.

Let $(V_p)_{p\in P} \in b$, that is, $V_p \supseteq \overline{U_p}$ for all $p \in F$. Note that, since a is an upper set, we can assume that $V_p = \emptyset$ for all $p \in P \setminus F$. Let $F' \subseteq F$ be the subset of indices p satisfying $0 \in V_p$. If $k \in F'$, then $(V_p)_{p\in P} \in b \cap T_{\{k\}} \subseteq a$. If $k \notin F'$, then $0 \notin V_k$. Therefore, by separation properties[15], and the fact that a is an upper set, we can assume that $0 \notin \overline{V_k}$. Let $m > |F'|$ $(m \geq 1)$ such that[16] $[0, 1/m) \cap \overline{V_k} = \emptyset$, and W the sequence with all its entries equal to those of V except for the kth, which is equal to $V_k \cup [0, 1/m)$. Thus $W \in b \cap T_{\{k\}} \subseteq a$, so it belongs to some $\bigcap_{p\in G} \pi_p^{-1}(\uparrow([0, 1/n]))$ with $|G| = n$, that is, $[0, 1/n] \subseteq W_p$ for all $p \in G$. If $k \notin G$, then $V \in a$. The remaining possibility leads to a contradiction. In fact, if $k \in G$, that is, if $[0, 1/n] \subseteq W_k = V_k \cup [0, 1/m)$, by connectedness, $[0, 1/n] \subseteq [0, 1/m)$ and hence $n \geq m + 1 > |F'| + 1$. This implies that there exists some V_p with $0 \in V_p$ and $p \notin F'$, a contradiction.

Now, we claim that $j_M(a) = \mathcal{O}(I)^P$, where $j_M : L \to L$ is the nucleus associated with M, so that $a \notin M$ since $a \subsetneq \mathcal{O}(I)^P$ (for example, $(\emptyset)_{p\in P} \notin a$). To prove this, let $(U_p)_{p\in P} \in \mathcal{O}(I)^P$ such that $0 \in U_p$ for all p. We know that for each p, there exists $n(p) \in \mathbb{Z}^+$ such that $[0, 1/n(p)] \subseteq U_p$. Since P is uncountable, there exists an *infinite* subset Y of P such that $n(p) = q$ for all $p \in Y$, where q is fixed. Let $F \subseteq_{fin} Y$ satisfying $|F| = q$. Thus, $(U_p)_{p\in P} \in \bigcap_{p\in F} \pi_p^{-1}(\uparrow([0, 1/q])) \subseteq a$. This proves that $\mathcal{O}(I)^P \cap T \subseteq a$, so $\mathcal{O}(I)^P \subseteq j_M(a) \subseteq \mathcal{O}(I)^P$.

Finally, $\bigwedge_{p\in P} L_{\{p\}}$ is non-spatial since it does not coincide with M, the locale induced by the subspace T, which satisfies $pt(\bigwedge_{p\in P} L_{\{p\}}) = \bigwedge_{p\in P} pt(L_{\{p\}}) = \bigcap_{p\in P} T_{\{p\}} = T$. Certainly, the functor pt preserves meets of sublocales since they are limits, and $pt(L_{\{p\}}) = T_{\{p\}}$ since $T_{\{p\}}$ is an open subspace of a sober space, and hence sober[17]. □

Before showing that $\Gamma@\mathbf{3}$ is non-spatial in this case, we need to prove the following lemma.

Lemma 2.7.7. *Let $f : L \longrightarrow S$ be a morphism of locales and S_0 and S_1 sublocales of S such that $S_0 \wedge S_1 = \mathbf{0}$ in the co-frame of sublocales of S. If the inverse image $f^{-1}[S_1]$ is non-spatial, then $f^{-1}[S_0 \vee S_1]$ is non-spatial.*

Proof. If S_0 and S_1 are sublocales of S such that $S_0 \wedge S_1 = \mathbf{0}$, then $f^{-1}[S_0 \vee S_1] = f^{-1}[S_0] \vee f^{-1}[S_1]$ and $f^{-1}[S_0] \wedge f^{-1}[S_1] = f^{-1}[S_0 \wedge S_1] = f^{-1}[\mathbf{0}] = \mathbf{0}$ since the inverse image is a morphism of co-frames according to [9, p. 55] and [18, II.3.8]. Therefore, $f^{-1}[S_0]$ and

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[15] By the density of the way below relation, there exists u such that $V_p \supseteq \overline{u} \supseteq u \supseteq \overline{U_p}$.

[16] Such an interval $[0, 1/m)$ exists since I, as a metric space, is *normal*.

[17] This is a consequence of Lemma [11, C.1.2.5], since every open subset of a space coincides with its subclosure.

$f^{-1}[S_1]$ are complemented sublocales of $f^{-1}[S \vee T]$. But complemented sublocales of spatial locales are spatial (see [11, C1.2.13]), so $f^{-1}[S \vee T]$ cannot be spatial. □

Note that $S_0 = \{\wp(P) \setminus \{\emptyset\}, \wp(P)\}$ and $S_1 = \{\emptyset, \wp(P)\}$ as sublocale sets, so $S_0 \wedge S_1 = \mathbf{0}$. Thus, by 2.7.6 and 2.7.7, $\Gamma @ \mathbf{3}$ is non-spatial if P is uncountable.

2.8 Gestures on locales of subsheaves

Locales are complete Heyting algebras viewed from a topological point of view. On the other hand, given a sheaf F on a site, the lattice $Sub(F)$ of all subsheaves of F is a complete Heyting algebra [15, p. 146]. Moreover, every complete Heyting algebra L is isomorphic to $Sub(\mathbf{1})$, where $\mathbf{1}$ is the final sheaf on the canonical site associated with the locale L [15, p. 149]. Therefore, the locales of gestures on locales are precisely the locales of gestures on the complete Heyting algebras of subobjects of sheaves. This remarkable fact opens the way to a possible definition of gestures in a Grothendieck topos.

We have the following result concerning the spatiality of locales of subpresheaves.

Proposition 2.8.1. *Let $\mathcal{C}$ be a small category. The locale $Sub(F)$ of subfunctors of a presheaf F of $\widehat{\mathcal{C}}$ is spatial.*

Proof. Let S_1 and S_2 be subfunctors of F such that $S_1 \nleq S_2$. According to [9, II.1.5], we must find a prime element P of $Sub(F)$ satisfying $S_1 \nleq P$ and $S_2 \leq P$.

Let $x \in S_1(C) \setminus S_2(C)$ for some $C \in Ob(\mathcal{C})$. Consider the set $\Sigma = \{R \in Sub(F) | \ x \notin R(C) \text{ and } S_2 \leq R\}$. Note that Σ is nonempty since $S_2 \in \Sigma$. On the other hand, if $\{R_i\}_{i\in\mathcal{I}}$ is a chain in Σ, then $\bigcup_{i\in\mathcal{I}} R_i$ is an upper bound for $\{R_i\}_{i\in\mathcal{I}}$ in Σ. In fact, $x \notin \bigcup_{i\in\mathcal{I}} R_i(C)$ and $S_2 \leq \bigcup_{i\in\mathcal{I}} R_i$. Hence, by Zorn's Lemma Σ has a maximal element P.

Note that $x \notin P(C)$ and consequently $P \lneq F$. Let us show that P is a prime element of $Sub(F)$. If $Q \cap R \subseteq P$ for $Q, R \in Sub(F)$ and we suppose that $Q \nleq P$ and $R \nleq P$, then since $x \notin Q \cap R(C) \subseteq P(C)$, we have two possibilities: i) $x \notin Q(C)$, in which case $x \notin Q(C) \cup P(C)$ and $S_2 \leq Q \vee P$, or ii) $x \notin R(C)$, in which case $x \notin R(C) \cup P(C)$ and $S_2 \leq Q \vee P$. In both cases we contradict the maximality of P in Σ. Thus $Q \leq P$ or $R \leq P$.

Finally, note that $S_1 \nleq P$ and $S_2 \leq P$, since $x \in S_1(C)$ and $P \in \Sigma$. □

Nevertheless, note that the same result need not be valid for the locale of subsheaves of a given sheaf on a site; for instance, take the site associated with a non-spatial locale L (for example, the locale $\mathcal{O}(\mathbb{R})_{\neg\neg}$ from Section 2.5), so that L is isomorphic to the locale of subsheaves of the final sheaf [15, p. 149].

Proposition 2.8.1 asserts that we do not obtain new non-spatial locales as a result of locales of subpresheaves (assuming Zorn's lemma), which is useful in the following case. Given a digraph Σ, it can be regarded as a presheaf on G_1 (see Subsection 3.1.1), so we deduce from Proposition 2.8.1 that the locale of subpresheaves of Σ, that is, the locale of subdigraphs of Σ, is spatial. This observation is of some interest if we want to regard a digraph as a space—an idea that makes sense, given the close relation between spaces, simplicial sets, and semi-simplicial sets (of which digraphs are particular examples), relation that we will study in the next chapter. In this way, we can regard the locale of subdigraphs of Σ both as a topological space and as a locale, so that we can define spaces of gestures with body in the space of points of the locale of subdigraphs of Σ, and we can also study the locales of gestures with body in $Sub(\Sigma)$.

Example 2.8.2. Let Γ be a digraph or skeleton for gestures and Σ the loop digraph $(\{a\}, \{x\})$ with just an arrow and a vertex (we omit the tail and head functions, for which there is a unique possibility). In this case, the locale $Sub(\Sigma)$ consists of the three digraphs $(\emptyset, \emptyset)$, $(\emptyset, \{x\})$, and Σ. Thus, $Sub(\Sigma)$ is isomorphic to $\mathbf{3}$. The prime elements of $Sub(\Sigma)$ are $(\emptyset, \emptyset)$ and $(\emptyset, \{x\})$ and hence the two points of $Sub(\Sigma)$ correspond to these prime elements. Intuitively, $(\emptyset, \emptyset)$ can be related to the vertex x and $(\emptyset, \{x\})$ can be related to the arrow a. Moreover, the locale of gestures $\Gamma @ Sub(\Sigma)$, which is isomorphic to $\Gamma @ \mathbf{3}$, was studied in Section 2.7; see Theorem 2.7.1. □

2.9 Summary of results and examples

Besides result 2.4.2, relating gestures on locales to topological gestures, that says that the space of points of the locale of gestures with skeleton Γ and body in L is the space of gestures with skeleton Γ and body in the space of points of L, there were three main properties of the locale of gestures $\Gamma @ L$ studied in this chapter: spatiality, injectivity, and exponential presentation. As to spatiality, we showed (Corollary 2.5.3) that the locale of gestures is non-spatial whenever L is non-spatial and Γ is not the digraph without vertices (initial digraph); if Γ is the initial digraph, then the locale of gestures is the final object $\mathbf{2}$ in **Loc**, which is spatial. A concrete example of non-spatial locale is the sublocale $\mathcal{O}(\mathbb{R})_{\neg\neg}$ of the usual topology on the real numbers, which consists of all opens that coincide with the interior of their closure (see the lines before Proposition 2.5.4). A concrete example of non-spatial locale of gestures is the locale $\Gamma @ \mathcal{O}(\mathbb{R})_{\neg\neg}$, where Γ is the digraph with just an arrow; this locale coincides with the locale of paths on $\mathcal{O}(\mathbb{R})_{\neg\neg}$. On the other hand, the exponential presentation $\Gamma @ L \cong L^{\mathcal{O}(|\Gamma|)}$ holds if and only if $\mathcal{O}(|\Gamma|)$ is exponentiable if and only if Γ is locally finite if and only if $\Gamma @ \mathbf{3}$ is injective if and only if $\Gamma @\_$ preserves injectives (Theorem 2.6.10). Also, the functor $\Gamma @\_$ can preserve spatial locales or send spatial locales to non-spatial ones, whether Γ is locally finite or not. Indeed, if Γ is a locally finite digraph, $\Gamma @ \mathbf{3}$ is injective and hence spatial (Theorem 2.6.10); if Γ is the arrow digraph and L is the order topology on the long segment (in this case L is spatial), then $\Gamma @ L \cong L^{\mathcal{O}(I)}$ and hence the space of gestures is non-spatial (see Subsection 7.2.1); and if Γ is the digraph consisting of a set of arrows having the same tail and different heads, then $\Gamma @ \mathbf{3}$ is spatial whenever there is a countable infinity of arrows, and is non-spatial whenever there is an uncountable infinity of arrows (Theorem 2.7.1). Then, we observed that the gestures on locales are just the gestures on the complete Heyting algebras of subsheaves of the form $Sub(F)$ with F sheaf on a site. In particular, when we consider locales of subpresheaves (presheaves are sheaves on the trivial topology), they turn out to be spatial under Zorn's lemma (Theorem 2.7.1). Thus, we can think of the locale of gestures $\Gamma @ Sub(\Sigma)$, where Σ is a digraph (presheaf), and in particular we can consider the case when Σ is a loop (Example 2.8.2), in which case the locale of gestures $\Gamma @ Sub(\Sigma)$ is isomorphic to $\Gamma @ \mathbf{3}$.

Chapter 3

Abstract Gestures

It is important to stress that the construction of gestures is not exclusively topological. Although we have defined gestures on spaces and locales, they are incarnations of a general archetype that is flexible enough to include both branches (the algebraic and the topological) of mathematical music theory.

This chapter intends to give an exposition of the general theory of gestures in categories satisfying few requirements, and to support the statement made in the introduction that gestures are a mathematical concept that is worth studying, besides its implications for mathematical music theory.

Through this chapter we use the following conventions. Consider the following properties on a category $\mathcal{C}$:

(C) It is cartesian, that is, has all finite limits; in particular, it has a final object $\mathbf{1}$.

(L) It is small-complete [14, p. 109]; in particular, it is cartesian.

(CC) It is cartesian closed, that is, it is cartesian and all its objects are exponentiable.

(COL) It is small-cocomplete.

(H) It has small hom-sets.

We start with the case of gestures whose skeleta are directed graphs, the first ones studied in mathematical music theory[1] in the particular case of topological spaces.

3.1 The case of directed graphs

3.1.1 Directed graphs and internal digraphs

Let G_1 be the category with two parallel arrows between two vertices $[0]$ and $[1]$ plus the identities; it can be depicted as follows:

$$\begin{tikzcd}
{[0]} \arrow[loop left, "id"] \arrow[r, shift left, "\epsilon_1"] \arrow[r, shift right, "\epsilon_0"'] & {[1]} \arrow[loop right, "id"]
\end{tikzcd} .$$

---

[1] See the foundational article [26].

A *directed graph* (or *digraph*, for short) Γ is a tuple (A, V, t, h), where A and V are sets and $t, h : A \longrightarrow V$ are functions. Digraphs correspond bijectively to presheaves on the category G_1 so from now on we identify a digraph Γ as above with its associated presheaf $\Gamma : G_1^{op} \longrightarrow \mathbf{Set}$ defined by $\Gamma([1]) = A$, $\Gamma([0]) = V$, $\Gamma(\epsilon_0) = t$, and $\Gamma(\epsilon_1) = h$. In this way, there is a topos *Digraph* of digraphs, namely the Grothendieck topos[2]

$$\mathbf{Set}^{G_1^{op}}.$$

Thus, a morphism of digraphs from (A_1, V_1, t_1, h_1) to (A_2, V_2, t_2, h_2) (that is, a natural transformation) corresponds to a pair of functions (u, v), with $u : A_1 \longrightarrow A_2$ and $v : V_1 \longrightarrow V_2$, satisfying the identities

$$vt_1 = t_2 u \text{ and } vh_1 = h_2 u.$$

Similarly, if $\mathcal{C}$ is an arbitrary category, a functor $S : G_1^{op} \longrightarrow \mathcal{C}$ can be identified with a tuple (S_1, S_0, e_0, e_1), that is, with the diagram

$$\begin{tikzcd} S_1 \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_0"'] & S_0 \end{tikzcd}$$

of morphism of $\mathcal{C}$, by defining $S_1 = S([1])$, $S_0 = S([0])$, $e_0 = S(\epsilon_0)$, and $e_1 = S(\epsilon_1)$. A tuple (S_1, S_0, e_0, e_1), where $e_0, e_1 : S_1 \longrightarrow S_0$ are morphisms of $\mathcal{C}$, is called an *internal digraph* in $\mathcal{C}$. In this way, functors $S : G_1^{op} \longrightarrow \mathcal{C}$ can be identified with internal digraphs in $\mathcal{C}$.

On the other hand, limits and colimits of digraphs are computed pointwise in the sense of [15, p. 22].

3.1.2 The category of elements

Given a presheaf $P : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ on a category $\mathcal{C}$, *the category of elements of* P, denoted by $\int P$, is defined as follows. Its objects are pairs (C, p) where C is an object of $\mathcal{C}$ and $p \in P(C)$, and a morphism from (C', p') to (C, p) is a morphism $u : C' \longrightarrow C$ of $\mathcal{C}$ such that $P(u)(p) = p'$. Also, there is a projection functor $\pi_P : \int P \longrightarrow \mathcal{C}$ sending $u : (C', p') \longrightarrow (C, p)$ to its underlying morphism $u : C' \longrightarrow C$.

In the case when $\mathcal{C} = G_1$, note that the category $\int \Gamma$ of elements of a digraph $\Gamma = (A, V, t, h)$ can be identified with the category whose set of objects is $V \sqcup A$ and whose morphisms are the identities and the pairs of the form $(t(a), a)$ or $(h(a), a)$ where $a \in A$, domains and codomains being the first and second projections respectively. With this identification the projection $\int \Gamma \xrightarrow{\pi_\Gamma} G_1$ sends the vertices in V to $[0]$, the arrows in A to $[1]$, $(t(a), a)$ to ϵ_0, and $(h(a), a)$ to ϵ_1.

Similarly, $(\int \Gamma)^{op}$ can be identified with the category whose set of objects is $V \sqcup A$ and whose morphisms are the identities and pairs of the form $(a, t(a))$ or $(a, h(a))$, domains and codomains being the first and second projections respectively. In other words, it is the free category on the digraph[3] $(t \sqcup h, V \sqcup A, \pi_1, \pi_2)$.

---

[2] Any category of presheaves on a small category is a Grothendieck topos. In fact, given a category of presheaves on a small category $\mathcal{C}$, it is a category of sheaves if we consider on $\mathcal{C}$ the *trivial topology*, whose unique covering sieve for each object of $\mathcal{C}$ is the maximal sieve.

[3] Regarding the functions t and h as sets of ordered pairs.

3.1.3 First abstraction

Now we present a first abstraction of the definitions studied in the preceding chapters. If $\mathcal{C}$ satisfies (C), consider two additional properties:

(B) There is an object B of $\mathcal{C}$ with two morphisms $i_0, i_1 : \mathbf{1} \longrightarrow B$. Such a tuple $(B, \mathbf{1}, i_0, i_1)$ will be referred to as a *base for gestures* in $\mathcal{C}$ and can be identified with a functor

$$T : G_1 \longrightarrow \mathcal{C}$$

defined by $T([0]) = \mathbf{1}$, $T([1]) = B$, $T(\epsilon_0) = i_0$, and $T(\epsilon_1) = i_1$.

(BE) The base object B from (B) is exponentiable in $\mathcal{C}$.

Suppose that $\mathcal{C}$ satisfies (C), (B), and (BE). If C is an object of $\mathcal{C}$, then we have the 'evaluation at endpoints morphisms'

$$e_k : C^B \overset{\cong}{\to} C^B \times \mathbf{1} \xrightarrow{id \times i_k} C^B \times B \overset{e}{\to} C \ (k = 0, 1),$$

where e is the evaluation map associated with the exponential. Note that e_0 and e_1 are essentially the morphisms C^{i_0} and C^{i_1} respectively, which are obtained by applying the functor $C^{(\_)}$ to i_0 and i_1. In this way, we define the *internal digraph of* C in $\mathcal{C}$ with respect to the base $(B, \mathbf{1}, i_0, i_1)$ as the tuple (C^B, C, e_0, e_1), which can be identified with a functor

$$S_C : G_1^{op} \longrightarrow \mathcal{C}.$$

Let Γ be a digraph and $F : (\int \Gamma)^{op} \longrightarrow \mathcal{C}$ the functor that sends each vertex in V to C, each arrow in A to C^B, each morphism $(a, t(a))$ to e_0, and each morphism $(a, h(a))$ to e_1. Note that the functor F is precisely the composite

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G_1^{op} \xrightarrow{S_C} \mathcal{C}.$$

We define *the object of gestures with base* $(B, \mathbf{1}, i_0, i_1)$*, skeleton* Γ*, and body in* C, denoted[4] by $\Gamma @_B C$, as the limit of the diagram F whenever it exists.

This generalization was made taking into account the possibility of building *hypergestures* from $\Gamma @_B C$, which is an object of $\mathcal{C}$. Certainly, if Γ' is another digraph we can construct the object $\Gamma' @_B \Gamma @_B C$, and so on, depending on the existence of suitable limits in $\mathcal{C}$. For example, though it is not a necessary condition, the constructions of hypergestures are always possible if $\mathcal{C}$ satisfies (B), (BE), and (L).

Figure 3.1 shows how the definitions of gestures in several categories are examples of our general definition. The usual properties to define gestures are tested in each category, namely (B), (BE), and (L) . There, in all the cases, the morphisms i_0 and i_1 of the base are the naturally induced by the inclusions $i_0, i_1 : \{*\} \longrightarrow I$ of the endpoints of the unit interval in $\mathbb{R}$.

---

[4] If the base is clear from the context, we simply write $\Gamma @ C$

cat. property	**Top**	**Loc**	$\mathfrak{LTop}$	$\mathfrak{BTop}$	$\mathfrak{Top}$	**Cat**(**Top**)
(B)	$I = [0,1]$	$\mathcal{O}(I)$	$Sh(\mathcal{O}(I))$	$Sh(\mathcal{O}(I))$	?	$\mathbb{I} = (C_1, C_0)$ cat. of the poset $(I, \leq)$
(BE)	yes loc. compact	yes cont. lattice	yes	yes continuous cat.	?	yes C_0, C_1, C_2 loc. compact
(L)	yes	yes	yes	some	?	yes

Figure 3.1: Different incarnations of the general definition.

The particular constructions corresponding to the second and third columns are addressed in Chapter 1 and Chapter 2 respectively; those of the fourth and fifth columns, which must be recast in 2-categorical terms, in Chapter 4; and that of the seventh column, in Chapter 5. The construction of gestures in the 2-category of elementary topoi with geometric morphisms (sixth column), where there is little knowledge of the properties, exceeds the purposes of this monograph and is not discussed here.

Nevertheless, the definition above may be restrictive for many interesting examples in mathematical music theory, for example, in the case of formulas in spectroids (which will be discussed in Subsection 3.11.2). For this reason, we give a more encompassing definition. In fact, the internal digraph of an object can be replaced by any internal digraph (S_1, S_0, e_0, e_1) in the category $\mathcal{C}$, that is, by any functor $S : G_1^{op} \longrightarrow \mathcal{C}$. In this way, given a digraph Γ, we define *the object* $\Gamma @ S$ *of gestures with skeleton* Γ *with respect to* S as the limit of the functor

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G_1^{op} \xrightarrow{S} \mathcal{C},$$

whenever it exists.

Following this definition, since Γ corresponds to a tuple (A, V, t, h) and S corresponds to an internal digraph (S_1, S_0, e_0, e_1) in $\mathcal{C}$, the object of gestures with skeleton Γ with respect to S is the limit of the following diagram in $\mathcal{C}$: take a copy of S_1 for each $a \in A$, a copy of S_0 for each $x \in V$, a copy of $e_0 : S_1 \longrightarrow S_0$ whenever $t(a) = x$, and a copy of $e_1 : S_1 \longrightarrow S_0$ whenever $h(a) = x$. Note that if S is the internal digraph (C^B, C, e_0, e_1) of C, then we recover the definition of gestures with body in C.

A deeper discussion for semi-simplicial sets, instead of digraphs, which generalizes this instance will be addressed in Section 3.3. The reasons for such general treatment is that it encompasses both the topological branch (gestures on spaces, locales, and topological categories) and the algebraic branch (diagrams and formulas) of mathematical music theory, with the advantage of combinatorial manipulation provided by semi-simplicial sets; see for instance the formula of representation of spaces of hypergestures as exponentials and as spaces of gestures whose skeleta are semi-simplicial sets of possibly higher dimension (Theorem 3.5.16) and the analogue of this formula for diagrams (Theorem 3.11.2).

3.1.4 Realizations

As we will see through this chapter, the concept of *realization* of a digraph is closely related to that of gestures. In fact, *realization and gestures are dual concepts of each other* (see Subsection 3.4.3).

Let $\mathcal{C}$ be a category, $\Gamma : G_1^{op} \longrightarrow \mathbf{Set}$ a digraph, and $T : G_1 \longrightarrow \mathcal{C}$ a functor. We define *the realization of* Γ *with respect to* T, denoted by $|\Gamma|_T$, as the colimit in $\mathcal{C}$ of the functor

$$\int \Gamma \xrightarrow{\pi_\Gamma} G_1 \xrightarrow{T} \mathcal{C},$$

whenever it exists.

Since Γ corresponds to a tuple (A, V, t, h) and T can be identified with a pair of morphisms $i_0, i_1 : T_0 \longrightarrow T_1$ of $\mathcal{C}$, the realization $|\Gamma|_T$ is the colimit of the following diagram in $\mathcal{C}$: take a copy of T_1 for each $a \in A$, a copy of T_0 for each $x \in V$, a copy of i_0 whenever $t(a) = x$, and a copy of i_1 whenever $h(a) = x$. In particular, we have given the definition of the geometric realization of a digraph with respect to a base $(B, \mathbf{1}, i_0, i_1)$ for gestures (Subsection 3.1.3). In this case we use the notation $|\Gamma|_B$ for the realization with respect to the base.

Example 3.1.1. Let Γ be a digraph.

The base that was used in Chapter 1 was the tuple $(I, \{*\}, i_0, i_1)$, where i_0 and i_1 are the inclusions of the endpoints of the unit interval I. The resulting realization $|\Gamma|_I$ is the geometric realization, as discussed in Section 1.3.

The base used in Chapter 2 was the tuple $(\mathcal{O}(I), \mathbf{2}, \mathcal{O}(i_0), \mathcal{O}(i_1))$, which is obtained by applying the functor $\mathcal{O}$ to $(I, \{*\}, i_0, i_1)$. The resulting realization $|\Gamma|_{\mathcal{O}(I)}$ coincides with $\mathcal{O}(|\Gamma|_I)$ since the functor $\mathcal{O}$, as a left adjoint (Section 2.4), preserves colimits. Thus, $|\Gamma|_{\mathcal{O}(I)}$ is the geometric realization defined in Section 2.1. ◇

3.2 Exponential presentation

Suppose that $\mathcal{C}$ satisfies (C), (B), and (BE). As suggested by theorems 1.6.3 and 2.6.10, a fundamental problem in gesture theory is to determine when the object of gestures $\Gamma @_B C$ is isomorphic to the exponential $C^{|\Gamma|_B}$, if these objects exist. In this section we propose some tools for addressing this problem. These tools were already used in Chapter 2.

First, we examine the ideal case of cartesian closed categories. Note that if $\mathcal{C}$ satisfies (CC), then it satisfies (BE) for any base. Regarding this situation, we have the following result, which allows to define the object of gestures $\Gamma @_B C$ via the exponential $C^{|\Gamma|_B}$, that is, without explicit use of limits.

Proposition 3.2.1. *Let $\mathcal{C}$ be a category satisfying (CC) and (COL), $(B, \mathbf{1}, i_0, i_1)$ a base for gestures (Subsection 3.1.3), C an object of $\mathcal{C}$, and Γ a digraph. The existence of the exponential $C^{|\Gamma|_B}$ implies the existence of $\Gamma @_B C$ and*

$$\Gamma @_B C \cong C^{|\Gamma|_B}.$$

Proof. First, note that the image under $C^{(\_)}$ of the functor $T\pi_\Gamma$ defining the realization (Subsection 3.1.4) is naturally isomorphic to the functor $S_C \pi_\Gamma^{op}$ defining the object of gestures (Subsection 3.1.3). Therefore, by Proposition 3.2.2 below, we only have to show that the natural morphism

$$Colim\ D \times (T\pi_\Gamma) \longrightarrow D \times |\Gamma|_B$$

is an isomorphism for each object D of $\mathcal{C}$, but this follows from the fact that the functor $D \times \_$ preserves colimits since it is left adjoint to the exponential functor $(\_)^D$. □

Proposition 3.2.2. *Let $\mathcal{C}$ be a category satisfying (C) and (COL), and $E : J \longrightarrow \mathcal{C}$ a diagram*[5] *of exponentiable objects of $\mathcal{C}$ whose colimit C is exponentiable in $\mathcal{C}$. Let A be an object of $\mathcal{C}$ and consider the diagram $A^E : J^{op} \longrightarrow \mathcal{C}$ obtained by applying $A^{(\_)}$ to E. If the natural map*

$$Colim\ D \times E \longrightarrow D \times C$$

is an isomorphism for each object D of $\mathcal{C}$, then the limit of A^E exists in $\mathcal{C}$ and

$$Lim\ A^E = A^C.$$

Proof. The proof is long but natural.

Let $\{f_i : E_i \longrightarrow C\}_{i \in J}$ be the colimiting cone associated with C. Since $A^{(\_)}$ is a contravariant functor, $\{A^{f_i} : A^C \longrightarrow A^{E_i}\}_{i \in J}$ is a cone over the diagram A^E. We will show that it is a limiting cone.

Let $\{p_i : D \longrightarrow A^{E_i}\}_{i \in J}$ be a cone over A^E, that is, $A^{E_\alpha} p_j = p_i$ for each morphism $\alpha : i \longrightarrow j$ of J. We need a suitable morphism $\psi : D \longrightarrow A^C$, so we will first construct $\overline{\psi} : D \times C \longrightarrow A$. Since the natural map δ in the commutative diagram

$$\begin{tikzcd}
D \times E_i \arrow[dd, "D\times E_\alpha"'] \arrow[dr] \arrow[drr, bend left, "D\times f_i"] & & \\
& Colim\ D \times E \arrow[r, "\exists!\delta"] & D \times C \\
D \times E_j \arrow[ur] \arrow[urr, bend right, "D\times f_j"'] & &
\end{tikzcd}$$

is an isomorphism, it is enough to construct a suitable cocone over $D \times E$ with vertex A. In fact, the $\overline{p_i}$ defined as the composites $D \times E_i \xrightarrow{p_i \times id} A^{E_i} \times E_i \xrightarrow{e} A$, where e is the evaluation

---

[5] From now on, given an object i of the indexing category J we write $i \in J$ instead of $i \in Ob(J)$. Of course, J is assumed to be a small category.

map, form such a cocone since

$$\begin{tikzcd}
A^{E_i} \times E_i \arrow[rr, "e"] & & A \\
A^{E_j} \times E_i \arrow[u, "A^{E_\alpha} \times id"] \arrow[r, "id \times E_\alpha"] & A^{E_j} \times E_j \arrow[ur, "e"] & \\
D \times E_i \arrow[u, "p_j \times id"] \arrow[r, "id \times E_\alpha"'] & D \times E_j \arrow[u, "p_j \times id"] \arrow[uur, "\overline{p_j}"'] &
\end{tikzcd}$$

commutes, that is, $\overline{p_j}(id \times E_\alpha) = e(A^{E_\alpha} p_j \times id) = e(p_i \times id) = \overline{p_i}$.

Consequently, we have the diagrams

$$\begin{tikzcd}
D \times E_i \arrow[dd, "D \times E_\alpha"'] \arrow[dr, "D \times f_i"] \arrow[drr, bend left, "\overline{p_i}"] & & \\
& D \times C \arrow[r, "\exists ! \overline{\psi}"] & A, \\
D \times E_j \arrow[ur, "D \times f_j"'] \arrow[urr, bend right, "\overline{p_j}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^C \times C \arrow[r, "e"] & A \\
D \times C \arrow[u, "\psi \times id"] \arrow[ur, "\overline{\psi}"'] &
\end{tikzcd}$$

given by the universal properties of the colimit and the exponential respectively. Thus, we must show that ψ is the unique morphism making the diagrams of the form

$$\begin{tikzcd}
& A^{E_i} \\
D \arrow[ur, "p_i"] \arrow[r, "\psi"'] & A^C \arrow[u, "A^{f_i}"']
\end{tikzcd}$$

commute.

The commutativity is proved as follows. The diagram

$$\begin{tikzcd}
A^{E_i} \times E_i \arrow[rr, "e"] & & A \\
A^C \times E_i \arrow[u, "A^{f_i} \times id"] \arrow[r, "id \times f_i"] & A^C \times C \arrow[ur, "e"] & \\
D \times E_i \arrow[u, "\psi \times id"] \arrow[r, "id \times f_i"'] & D \times C \arrow[u, "\psi \times id"] \arrow[uur, "\overline{\psi}"'] &
\end{tikzcd}$$

commutes, that is, $\overline{p_i} = \overline{\psi}(id \times f_i) = e(A^{f_i}\psi \times id)$, so $A^{f_i}\psi = p_i$.

To prove the uniqueness, suppose that ψ' satisfies $A^{f_i}\psi' = p_i$ for all $i \in J$; then the diagrams of the form

$$\begin{tikzcd}
& A^{E_i} \times E_i \arrow[r, "e"] & A \\
D \times E_i \arrow[ur, "p_i \times id"] \arrow[r, "\psi' \times id"'] & A^C \times E_i \arrow[u, "A^{f_i} \times id"'] \arrow[r, "id \times f_i"'] & A^C \times C \arrow[u, "e"']
\end{tikzcd}$$

commute, that is, $\overline{p_i} = e(id \times f_i)(\psi' \times id) = e(\psi' \times id)(id \times f_i) = \overline{\psi'}(id \times f_i)$. Thus, by the universal property of $D \times C$, which is isomorphic to $Colim\ D \times E$, $\overline{\psi'} = \overline{\psi}$ and hence $\psi' = \psi$. □

The following result justifies the adjective *natural* in the proposition above. This result was also used in Section 2.6.

Proposition 3.2.3. *Let $\mathcal{C}$ be a category satisfying (C) and (COL), D an object of $\mathcal{C}$, and E a diagram of $\mathcal{C}^J$. The canonical morphism $\psi_{D,E} : Colim\ D \times E \longrightarrow D \times Colim\ E$ defines a natural transformation in both arguments D and E.*

Proof. Let $\tau : E' \longrightarrow E$ be a natural transformation between diagrams of $\mathcal{C}^J$. We must show that the diagram

$$\begin{tikzcd}
Colim\ D \times E \arrow[r, "\psi_E"] & D \times Colim\ E \\
Colim\ D \times E' \arrow[u, "Colim\ D \times \tau"] \arrow[r, "\psi_{E'}"] & D \times Colim\ E' \arrow[u, "D \times Colim\ \tau"']
\end{tikzcd}$$

commutes, where the natural transformation $D \times \tau : D \times E' \longrightarrow D \times E$ is defined by $(D \times \tau)_i = D \times \tau_i$. Therefore, using the universal property of $Colim\ D \times E'$, it is enough to show that

$$\psi_E(Colim\ D \times \tau)q'_i = (D \times Colim\ \tau)\psi_{E'}q'_i$$

for each $i \in J$, where q'_i is the ith leg of the colimiting cone with vertex $Colim\ D \times E'$. In fact,

$$\psi_E(Colim\ D \times \tau)q'_i = \psi_E q_i(D \times \tau_i) = (D \times r_i)(D \times \tau_i) = D \times r_i\tau_i$$

and

$$(D \times Colim\ \tau)\psi_{E'}q'_i = (D \times Colim\ \tau)(D \times r'_i) = D \times (Colim\ \tau)r'_i = D \times r_i\tau_i,$$

where q_i is the ith leg of the colimiting cone with vertex $Colim\ D \times E$, and r_i (respectively r'_i) is the ith leg of the colimiting cone with vertex $Colim\ E$ (respectively $Colim\ E'$).

On the other hand, given a morphism $f : D \longrightarrow D'$ of $\mathcal{C}$, we must check the commutativity of

$$\begin{tikzcd}
Colim\ D \times E \arrow[r, "\psi_D"] \arrow[d, "Colim\ (f \times id)"'] & D \times Colim\ E\ . \arrow[d, "f \times id"] \\
Colim\ D' \times E \arrow[r, "\psi_{D'}"] & D' \times Colim\ E
\end{tikzcd}$$

In fact, if $\{s_i | i \in J\}$ is the colimiting cone with vertex $Colim\ D' \times E$, then the equality $(f \times id)\psi_D = \psi_{D'}Colim\ (f \times id)$ follows from the identities

$$(f \times id)\psi_D q_i = (f \times id)(id \times r_i) = f \times r_i$$

and

$$\psi_{D'}Colim\ (f \times id)q_i = \psi_{D'}s_i(f \times id) = (id \times r_i)(f \times id) = f \times r_i.$$

□

The object of gestures and the geometric realization are constructions of a very special kind, whose particulars will be examined in the following sections, but, as Proposition 3.2.2 shows, it is useful to deal with general colimits instead of those defining realizations. The following theorem illustrates that the general situation is ruled by an adjunction, which, in the particular case when E is the diagram defining the geometric realization of a digraph Γ, shows that the gesture functor sending an object C to the object of gestures with skeleton Γ and body in C, has a left adjoint. On the other hand, the adjunction supports, in a fairly transparent way, the idea that the object gestures (with respect to a base) is a *quantization* of the concept of exponential: we have deformed exponentials obtaining an interesting new object.

Theorem 3.2.4. *Let $\mathcal{C}$ be a category satisfying (C) and (COL), and $E : J \longrightarrow \mathcal{C}$ a diagram of exponentiable objects in $\mathcal{C}$ whose colimit C is also exponentiable. Suppose that for each object A of $\mathcal{C}$, the limit $Lim\ A^E$ exists so that the functor $Lim\ (\_)^E$ exists. Then*

i) $Lim\ (\_)^E$ is right adjoint to $Colim\ \_ \times E$, and

ii) the canonical natural transformation

$$\tau : (\_)^C \longrightarrow Lim\ (\_)^E$$

is the conjugate of

$$\psi_E : Colim\ \_ \times E \longrightarrow \_ \times C.$$

Proof. i) Note that there is a chain of isomorphisms

$$\mathcal{C}(D, Lim\ A^E) \cong \mathcal{C}^{J^{op}}(K_D, A^E) \cong \mathcal{C}^J(D \times E, K_A) \cong \mathcal{C}(Colim\ D \times E, A)$$

that are natural in D and A, where $K_{(\_)}$ denotes the respective diagonal functor [15, p. 20]. In fact, the right-hand isomorphism holds since the functor $Colim$ is left adjoint to the diagonal functor $K_{(\_)}$ (see [15, p. 22]) and the left-hand isomorphism holds in a similar way, since the limit $Lim\ A^E$ exists for each object A of $\mathcal{C}$. Moreover, the middle isomorphism sends a cone $\{p_i : D \longrightarrow A^{E_i}\}_{i \in J}$ to the cocone $\{\overline{p_i} : D \times E_i \longrightarrow A\}_{i \in J}$ defined as in the proof of Proposition 3.2.2 and this assignment is bijective and natural in each variable since each of its components corresponds to an isomorphism of the form $\mathcal{C}(D, A^{E_i}) \cong \mathcal{C}(D \times E_i, A)$ given by the adjunction $\_ \times E_i \dashv (\_)^{E_i}$.

By the way, let us compute the transpose of a morphism $f : Colim\ X \times E \longrightarrow A$ across the adjunction above. It is the unique morphism $f' : X \longrightarrow Lim\ A^E$ making the diagrams of the form

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[dr, "\widetilde{fq_i}"'] & Lim\ A^E \arrow[d, "p_i"] \\
 & A^{E_i}
\end{tikzcd}$$

commute, where $\{q_i\}_{i \in J}$ is the colimiting cone with vertex $Colim\ X \times E$, $\widetilde{fq_i}$ is the exponential transpose of fq_i, and $\{p_i\}_{i \in J}$ is the limiting cone associated with $Lim\ A^E$.

ii) The component τ_A of the natural transformation τ is obtained in the following manner. Let $\{r_i : E_i \longrightarrow C\}_{i\in J}$ be the colimiting cone associated with C. Since $A^{(\_)}$ is a contravariant functor, $\{A^{r_i} : A^C \longrightarrow A^{E_i}\}_{i\in J}$ is a cone over the diagram A^E. Therefore, by the universal property of $Lim\ A^E$, there exists a unique $\tau_A : A^C \longrightarrow Lim\ A^E$ such that $p_i\tau_A = A^{r_i}$ for each $i \in J$.

It remains to show that τ is the conjugate of ψ_E. To this end, we will use the definition of conjugation given in [14, pp. 99-100]. We need to verify that for each morphism $h : X \times C \longrightarrow A$ with exponential transpose $\widetilde{h}$, the morphism $\tau_A\widetilde{h}$ is equal to the transpose of $h\psi_{X,E}$ across the adjunction from i). According to the triangle above, we must see that for each $i \in J$

$$p_i\tau_A\widetilde{h} = \widetilde{h\psi_{X,E}}q_i.$$

Certainly, this follows from the identities of the form

$$p_i\tau_A\widetilde{h} = A^{r_i}\widetilde{h} \text{ and } e(A^{r_i}\widetilde{h} \times id) = e(\widetilde{h} \times r_i) = h(id \times r_i) = h\psi_{X,E}q_i.$$

□

To obtain a further consequence we need the following lemma concerning the case when one of the two natural transformations of a conjugate pair is an isomorphism.

Lemma 3.2.5. *Suppose $F, F' : \mathcal{C} \longrightarrow \mathcal{D}$ and $G, G' : \mathcal{D} \longrightarrow \mathcal{C}$ are pairs of functors such that F is left adjoint to G, F' is left adjoint to G', and $(\sigma : F \longrightarrow F', \tau : G' \longrightarrow G)$ is a conjugate pair. Then τ is an isomorphism if and only if σ is an isomorphism.*

Proof. First, suppose that σ is an isomorphism, that is, that there exists a two-sided inverse σ' for σ. By Theorem [14, IV.7.2], there exists a (unique) τ' such that (σ', τ') is a conjugate pair. Thus, since conjugate pairs are closed under vertical composition [14, p. 101], $\tau'\tau : G' \longrightarrow G'$ is the conjugate of $Id_{F'}$ and $\tau\tau' : G \longrightarrow G$ is the conjugate of Id_F. This implies that both composites are identities and hence τ is an isomorphism. The other direction is similar. □

Corollary 3.2.6. *Suppose that τ and ψ_E are as in Theorem 3.2.4. Then τ is an isomorphism if and only if ψ_E is an isomorphism.*

3.3 The case of semi-simplicial sets

In this section, we give a general definition of the objects of gestures whose skeleta are semi-simplicial sets instead of digraphs, with respect to general semi-simplicial objects S in a category $\mathcal{C}$ rather than internal digraphs therein. In most cases, this general definition contains the definition given at the end of Subsection 3.1.3, since digraphs are just semi-simplicial sets of dimension less than or equal to 1 and internal digraphs in a category $\mathcal{C}$ can be regarded as semi-simplicial objects whenever $\mathcal{C}$ has an initial object (Subsection 3.3.2). This perspective is particularly interesting for the theory of hypergestures since, in some cases, there is a reduction of hypergestures made from digraphs to plain gestures made from semi-simplicial sets of higher dimensions. Further, we hope that the following formal approach provides a general mathematical framework for gesture theory where each construction is fully identified as a categorical concept.

3.3.1 Simplicial and semi-simplicial objects

Let Δ be the *simplicial category* from [15, VIII.7]: its objects are ordered sets (ordinals) $[n]$ ($n \in \mathbb{N}$), where $[n] = \{0, 1, 2, \dots, n\}$, and its morphisms are functions $\alpha : [n] \longrightarrow [m]$ that preserve the usual order $\leq$. Let $\mathcal{C}$ be a category; recall that a *simplicial object* in $\mathcal{C}$ is a functor $S : \Delta^{op} \longrightarrow \mathcal{C}$.

Similarly, the *semi-simplicial category* G is the subcategory of Δ having the same objects and with morphisms all the injective ones inherited from Δ. For each $n \in \mathbb{N}$, define G_n to be the full subcategory of G whose objects are all the $[k]$ with $k \leq n$. Note that G_n can be presented as the family of morphisms in the diagram

$$\begin{tikzcd}
{[0]} \arrow[r, shift left, "\epsilon_1"] \arrow[r, shift right, "\epsilon_0"'] & {[1]} \arrow[r, shift left=2, "\epsilon_2"] \arrow[r, "\epsilon_1" description] \arrow[r, shift right=2, "\epsilon_0"'] & {[2]} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & {[3]} & \dots & {[n]} ,
\end{tikzcd}$$

which has $n+1$ vertices and $k+1$ face maps ϵ_i from $k-1$ to k , for $k = 1, \dots, n$; the latter satisfying the relations

$$\epsilon_j \epsilon_i = \epsilon_i \epsilon_{j-1} \quad (i < j).$$

Certainly, every injective map of G_n (and G) is the composite of face maps (see the lemma in [14, VII.5]). Also, G is the union of all the G_n such that $n \in \mathbb{N}$. As above, given a category $\mathcal{C}$, a *semi-simplicial object* in $\mathcal{C}$ is a functor $S : G^{op} \longrightarrow \mathcal{C}$, and an *$n$-truncated semi-simplicial object* in $\mathcal{C}$ is a functor $S : G_n^{op} \longrightarrow \mathcal{C}$.

On the other hand, a *cosimplicial object* in a category $\mathcal{C}$ is a functor $T : \Delta \longrightarrow \mathcal{C}$ and a *semi-cosimplicial object* in $\mathcal{C}$ is a functor $T : G \longrightarrow \mathcal{C}$.

3.3.2 Truncation and extension

The inclusion $K_n : G_n^{op} \hookrightarrow G^{op}$ between small categories induces a *truncation* functor tr_n, namely $\mathbf{Set}^{K_n} : \widehat{G} \longrightarrow \widehat{G_n}$, which has a left adjoint (see [14, X.3.1]), namely the *extension* functor ex_n, defined by

$$ex_n(\Gamma)([m]) = Colim(K_n \downarrow [m] \xrightarrow{P} G_n^{op} \xrightarrow{\Gamma} \mathbf{Set}) = \begin{cases} \Gamma([m]) & \text{if } m \leq n, \\ \emptyset & \text{if } m > n, \end{cases}$$

for an n-truncated semi-simplicial set Γ. In fact, if $m \leq n$, since the identity on $[m]$ is the final object of $K_n \downarrow [m]$, the colimit above exists and is equal to $\Gamma([m])$; and if $m > n$, since there is no morphism $f : [m] \longrightarrow [k]$ in G with $k \leq n$, $K_n \downarrow [m]$ is empty and hence $\Gamma([m])$ is the initial object $\emptyset$ in $\mathbf{Set}$.

The image $Sk_n(\Gamma)$ of a semi-simplicial set Γ under the functor Sk_n, where $Sk_n := ex_n tr_n$, is called the *n-skeleton* of Γ. Note that

$$Sk_n(\Gamma)([m]) = \begin{cases} \Gamma([m]) & \text{if } m \leq n, \\ \emptyset & \text{if } m > n. \end{cases}$$

The adjunction $tr_n \vdash ex_n$ above induces an equivalence. On the one hand, the fixed points of $tr_n ex_n$ are exactly the objects of $\widehat{G_n}$. Further, the fixed points of Sk_n, that is, the semi-simplicial sets of *dimension less than or equal to* n, are those Γ such that $\Gamma([m]) = \emptyset$ for $m > n$, and we write $dim(\Gamma) \leq n$. Therefore, by Proposition 7.0.1 below, *there is an equivalence between n-truncated semi-simplicial sets and semi-simplicial sets of dimension less than or equal to n.*

Consequently, define an *n-skeleton* to be a semi-simplicial set Γ with $dim(\Gamma) \leq n$. Note that it corresponds to a diagram of sets

$$\begin{tikzcd}
\Gamma_0 & \Gamma_1 \arrow[l, shift right, "d_1"'] \arrow[l, shift left, "d_0"] & \Gamma_2 \arrow[l, shift right=2, "d_2"'] \arrow[l, "d_1" description] \arrow[l, shift left=2, "d_0"] & \Gamma_3 \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] & \cdots & \Gamma_n ,
\end{tikzcd}$$

with morphisms $\{d_i^k : \Gamma_k \longrightarrow \Gamma_{k-1} |\ i = 0, \ldots, k\}$ for $k = 1, \ldots, n$, where $d_i^k := \Gamma(\epsilon_i^k)$, satisfying the relations

$$d_i d_j = d_{j-1} d_i \quad (i < j).$$

In the case when $n = 1$, note that a 1-skeleton is precisely a digraph $\Gamma = (A, V, t, h)$. On the one hand, we have the necessary identifications $A = \Gamma_1$ and $V = \Gamma_0$. But, there are two possible identifications for the tail and head functions; either $t = d_0$ and $h = d_1$ or $t = d_1$ and $h = d_0$. *The choice will depend on the particular example and will be shown explicitly in each case.*

Note the importance of this discussion: digraphs are a particular instance of semi-simplicial sets. In this way, the general theory to be presented through this chapter applies to digraphs. Moreover, if $\mathcal{C}$ is a category with an initial object $\mathbf{0}$, the same reasoning as above shows that the truncation functor $\mathcal{C}^{K_1} : \mathcal{C}^{G^{op}} \longrightarrow \mathcal{C}^{G_1^{op}}$ has a left adjoint L_1 defined by

$$L_1(S)([m]) = \begin{cases} S([m]) & \text{if } m \leq 1, \\ \mathbf{0} & \text{if } m > 1, \end{cases}$$

for each internal digraph in $\mathcal{C}$, or 1-truncated semi-simplicial object, $S : G_1^{op} \longrightarrow \mathcal{C}$. In this way, every internal digraph in a category $\mathcal{C}$ with an initial object can be regarded as a semi-simplicial object in $\mathcal{C}$, and hence, as we will see shortly, the following definition of gestures contains the definition at the end of Subsection 3.1.3, in particular the definitions of gestures in **Top** (Chapter 1), **Loc** (Chapter 2), and **Cat**(**Top**) (Chapter 5), since these categories have an initial object.

3.3.3 Definition: gestures whose skeleta are semi-simplicial sets

Let $\mathcal{C}$ be a category. Given a semi-simplicial set $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ and a semi-simplicial object $S : G^{op} \longrightarrow \mathcal{C}$ in $\mathcal{C}$, we define *the object of $\mathcal{C}$ of gestures with skeleton Γ with respect to S*, denoted by $\Gamma@S$, as the limit of the functor

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G^{op} \xrightarrow{S} \mathcal{C},$$

whenever it exists.

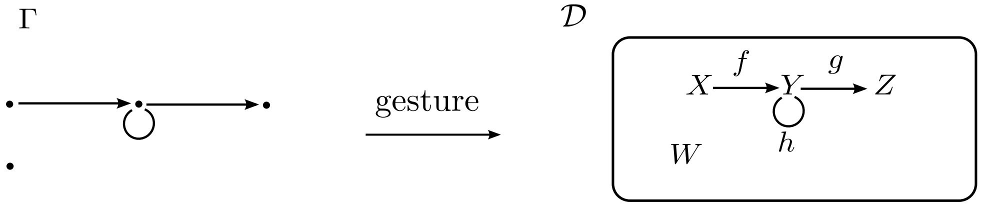

Figure 3.2: A gesture with respect to the nerve (see Example 3.3.1 below).

Following this definition, the object of gestures with n-skeleton Γ with respect to a semi-simplicial object $S : G^{op} \longrightarrow \mathcal{C}$ is the limit of the following diagram in $\mathcal{C}$: take a copy of S_k for each element in Γ_k and $k = 0, \ldots, n$, and a copy of $S(\epsilon_i^k) : S_k \longrightarrow S_{k-1}$ whenever $d_i^k(a_k) = a_{k-1}$, $1 \le k \le n$, $0 \le i \le k$, $a_k \in \Gamma_k$, and $a_{k-1} \in \Gamma_{k-1}$.

In particular, if Γ is a 1-skeleton corresponding to the digraph (A, V, t, h) (with the identifications $t = d_0$ and $h = d_1$) and the semi-simplicial object is the extension $L_1(S)$ of an internal digraph $S : G_1^{op} \longrightarrow \mathcal{C}$ (Subsection 3.3.2), then we recover the definition at the end of Subsection 3.1.3.

Note that given a skeleton Γ and a simplicial object $S : \Delta^{op} \longrightarrow \mathcal{C}$ in $\mathcal{C}$, there is a canonical semi-simplicial object $SK : G^{op} \longrightarrow \mathcal{C}$, namely the restriction induced by the inclusion $K : G^{op} \hookrightarrow \Delta^{op}$. In this way we can construct *the object of gestures with skeleton* Γ *with respect to* S, which we denote by $\Gamma@S$ rather than by $\Gamma@SK$. It is a very important remark since many examples of semi-simplicial objects used in this monograph come from simplicial ones.

Example 3.3.1 (*Diagrams are gestures*)**.** Suppose that $\mathcal{C} = \mathbf{Set}$ and that (D_1, D_0, d_0, d_1) is a small category $\mathcal{D}$ regarded as an internal category in **Set**. Let $N : \Delta^{op} \longrightarrow \mathbf{Set}$ be the *nerve* of $\mathcal{D}$ (see [14, XII.2]) and Γ a 1-skeleton, that is, a digraph (A, V, t, h) under the identifications $t = \Gamma(\epsilon_0)$ and $h = \Gamma(\epsilon_1)$.

As in Section 3.1, the category $(\int \Gamma)^{op}$ corresponds to the free category on the digraph $(t \sqcup h, V \sqcup A, \pi_1, \pi_2)$; and the functor $N\pi_\Gamma : (\int \Gamma)^{op} \longrightarrow \mathbf{Set}$ sends each object a in A to D_1, each object $v \in V$ to D_0, each arrow $(a, t(a))$ to the domain map d_0, and each arrow $(a, h(a))$ to the codomain map d_1.

By the computation of limits in **Set**, the set of gestures $\Gamma@N$ consists of all the sequences

$$\{F_i | i \in V \sqcup A\} \in \prod_{x \in V} D_0 \times \prod_{a \in A} D_1$$

such that $d_0(F_a) = F_{t(a)}$ and $d_1(F_a) = F_{h(a)}$ for each $a \in A$. Thus, an element of the set of gestures with skeleton Γ with respect to the nerve of $\mathcal{D}$ is just a diagram of the shape Γ in $\mathcal{D}$ (definition in [14, p. 51]), as illustrated in Figure 3.2. ◇

The following two examples show how the constructions of gestures on topological spaces and locales given for digraphs in the first two chapters can be extended to semi-simplicial sets, in a more natural way than by using the extension L_1 applied to the internal digraph of a space or a locale. First, note that the standard n-simplex Δ^n, where

$$\Delta^n = \{(t_1, \ldots, t_n) | \; 0 \le t_1 \le \cdots \le t_n \le 1\} \subseteq I^n,$$

is a locally compact space since it is a compact Hausdorff space[6]. For this reason, $\mathcal{O}(\Delta^n)$ is a continuous lattice, so Δ^n is exponentiable in **Top** and $\mathcal{O}(\Delta^n)$ is exponentiable in **Loc**.

Example 3.3.2 (Gestures on topological spaces and locales)**.** The standard simplex functor [15, VIII.7]

$$\Delta^{(\_)} : \Delta \longrightarrow \mathbf{Top}$$

induces a simplicial object in **Top** for each space X, namely the *singular complex functor* (which is not merely a simplicial set)

$$\mathbf{Top}(\Delta^{(\_)}, X) : \Delta \xrightarrow{\Delta^{(\_)}} \mathbf{Top} \xrightarrow{X^{(\_)}} \mathbf{Top}.$$

Given a 1-skeleton Γ, that is, a digraph (A, V, t, h) under the identifications $t = \Gamma(\epsilon_1)$ and $h = \Gamma(\epsilon_0)$, the functor $\mathbf{Top}(\Delta^{(\_)}, X)\pi_\Gamma : (\int \Gamma)^{op} \longrightarrow \mathbf{Top}$ sends each object a in A to X^I, each object $v \in V$ to X, each arrow $(a, t(a))$ to X^{ϵ_1}, and each arrow $(a, h(a))$ to X^{ϵ_0}. Note that the face maps $\epsilon_1, \epsilon_0 : \Delta^0 = \{*\} \longrightarrow \Delta^1 = I$, with $\epsilon_1 = i_0$ and $\epsilon_0 = i_1$, induce the evaluation at endpoints maps $e_0 = X^{\epsilon_1}, e_1 = X^{\epsilon_0} : X^I \longrightarrow X$, so the space of gestures with 1-skeleton Γ with respect to the singular complex coincides with $\Gamma@X$, as defined in Chapter 1.

Similarly, for each locale L, we have a simplicial object

$$\Delta \xrightarrow{\Delta^{(\_)}} \mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Loc} \xrightarrow{L^{(\_)}} \mathbf{Loc}$$

in **Loc**. In this case, the face maps $i_0, i_1 : \mathbf{2} \longrightarrow \mathcal{O}(I)$ (where $\mathbf{2}$ is the final object in **Loc**) induce the evaluation at endpoints maps $e_0, e_1 : L^{\mathcal{O}(I)} \longrightarrow L$ (note that $L^{\mathbf{2}} \cong L$), and the locale of gestures with 1-skeleton Γ with respect to this simplicial object coincides with $\Gamma@L$, as defined in Chapter 2. We call the composite $\mathcal{O}(\Delta^{(\_)}) : \Delta \longrightarrow \mathbf{Loc}$ *the standard simplex functor in the category of locales.*

But the importance of the singular complex functor and of its localic analogue is that they allow us to define new objects of gestures, when the skeleton Γ is not a digraph. For instance, in Example 3.5.17 below, we will give a characterization of the space of gestures with respect to a combinatorial model of the torus (Example 3.5.8), which is not a digraph. ◇

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The latter example suggests the following definition. Let C be an object of a category $\mathcal{C}$ and $T : G \longrightarrow \mathcal{C}$ a functor whose images are exponentiable in $\mathcal{C}$. We define the *semi-simplicial object S_C of C with respect to T* as the composite

$$G \xrightarrow{T} \mathcal{C} \xrightarrow{C^{(\_)}} \mathcal{C},$$

which is, of course, a contravariant functor. In this case, given a semi-simplicial set Γ, we write $\Gamma@C$ instead of $\Gamma@S_C$, and call it *the object of gestures with skeleton Γ and body in*

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[6] See the first footnote in Section 1.1

C, whenever the limit exists. Once again, this construction implies that of *hypergestures*: if Γ' is another skeleton, we can construct the object $\Gamma' @ \Gamma @ C$, and so on, depending on the existence of suitable limits in $\mathcal{C}$.

The construction of hypergestures is the main reason for defining the object $\Gamma @ C$ of gestures with skeleton Γ and body in C since the key point of the construction of hypergestures is that $\Gamma @ C$ is an object of $\mathcal{C}$ again so that it can be regarded as a new body for gestures and we can iterate the construction. Moreover, as we have seen, this definition led to the general definition of the object of gestures $\Gamma @ S$, by successive abstraction.

On the other hand, note that we have defined an object of gestures rather than an individual gesture. The reason is that, as was exemplified in the case of gestures on locales (Section 2.4), the points of the locale of gestures $\Gamma @ L$, which correspond to morphism from Γ to the underlying digraph of the internal digraph (L^I, L, e_0, e_1) (see Theorem 3.4.7), do not characterize the object $\Gamma @ L$, which is a *necessary construction for the theory of hypergestures on locales.*

3.4 Realization and gestures

The concept of *realization* of a semi-simplicial set is the dual of that of the object of gestures as discussed in the preceding section. Moreover, regarding Mazzola's gestures, realizations are at the core of the relation between objects of gestures and their exponential presentations. In this section we develop these ideas. We start by giving a short account of tensor products, of which realizations are particular cases.

3.4.1 The Hom-tensor adjunction

Let $\mathcal{D}$ and $\mathcal{E}$ be categories, where $\mathcal{E}$ satisfies (H), and $A : \mathcal{D} \longrightarrow \mathcal{E}$ a functor. Consider the functor $R : \mathcal{E} \longrightarrow \widehat{\mathcal{D}}$ with

$$R(E) := \mathcal{E}(A(\_), E)$$

for each object E of $\mathcal{E}$ and $R(f)$ the natural transformation $f \circ (\_)$. The functor R has a left adjoint[7]

$$\_ \otimes_{\mathcal{D}} A : \widehat{\mathcal{D}} \longrightarrow \mathcal{E}$$

sending each presheaf $P : \mathcal{D}^{op} \longrightarrow \mathbf{Set}$ to $P \otimes_{\mathcal{D}} A$, where

$$P \otimes_{\mathcal{D}} A := Colim \left(\int P \xrightarrow{\pi_P} \mathcal{D} \xrightarrow{A} \mathcal{E} \right),$$

whenever the colimit exists for each presheaf P on $\mathcal{D}$ (this will hold, for example, if we assume (COL) in $\mathcal{E}$).

In this way, we have a bijection

$$\mathcal{E}(P \otimes_{\mathcal{D}} A, E) \cong Nat(P, \mathcal{E}(A(\_), E))$$

---

[7] See the theorem at [15, p. 47], which holds for cocomplete categories. This theorem remains valid if we only assume the existence of the colimits involved in the definition of L.

for each presheaf P and each object E of $\mathcal{E}$, which is natural in both arguments P and E. According to the proof of Theorem [15, I.5.2], this bijection is obtained as follows. First, given a natural transformation $\tau : P \longrightarrow \mathcal{E}(A(\_), E)$, the family

$$\{\tau_D(p) : A(D) \longrightarrow E|\ (D, p) \text{ is an object of } \int P\}$$

is a cocone over $A\pi_P$ with vertex E, so there is a unique morphism $f : P \otimes_{\mathcal{D}} A \longrightarrow E$ such that $f\lambda_{(D,p)} = \tau_D(p)$ for each (D, p), where

$$\{\lambda_{(D,p)} : A(D) \longrightarrow P \otimes_{\mathcal{D}} A|(D, p) \text{ is an object of } \int P\}$$

is the colimiting cocone over $A\pi_P$ with vertex $P\otimes_{\mathcal{D}}A$. In the other direction, the isomorphism sends a morphism $f : P \otimes_{\mathcal{D}} A \longrightarrow E$ to the natural transformation τ defined by $\tau_D(p) = f\lambda_{(D,p)}$ for each object (D, p) of $\int P$.

Consequently, the unit of this adjunction is the natural transformation

$$\eta : id_{\widehat{\mathcal{D}}} \longrightarrow R \circ (\_ \otimes_{\mathcal{D}} A),$$

where $\eta_P : P \longrightarrow \mathcal{E}(A(\_), P \otimes_{\mathcal{D}} A)$ (the transpose of the identity on $P \otimes_{\mathcal{D}} A$) is defined by $\eta_{P,D}(p) = \lambda_{(D,p)}$ for each object D of $\mathcal{D}$ and $p \in P(D)$.

3.4.2 Realization

Let $\mathcal{C}$ be a category, $\Gamma : G^{op} \longrightarrow$ **Set** a semi-simplicial set, and $T : G \longrightarrow \mathcal{C}$ a functor, that is, a semi-cosimplicial object in $\mathcal{C}$. We define *the realization of* Γ *with respect to* T, denoted by[8] $|\Gamma|_T$, as the tensor product $\Gamma \otimes_G T$, where

$$\Gamma \otimes_G T = Colim\ \left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{T} \mathcal{C}\right),$$

whenever it exists.

According to the Hom-tensor adjunction (Subsection 3.4.1), in the case when $\mathcal{C}$ is a category satisfying (H), if the realization $|\Gamma|_T$ exists for each skeleton Γ, then there is a functor

$$|\_|_T : \widehat{G} \longrightarrow \mathcal{C},$$

which is left adjoint to the functor $\mathcal{C}(T, \_)$ that sends each object C of $\mathcal{C}$ to the semi-simplicial set $\mathcal{C}(T(\_), C)$. This means that for each semi-simplicial set Γ and each object of C there is a bijection

$$\mathcal{C}(|\Gamma|_T, C) \cong Nat(\Gamma, \mathcal{C}(T(\_), C)),$$

natural in both arguments Γ and C. As we will show, this adjunction is very useful in the theory of gestures.

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[8] Once again, if T is clear from the context, we write $|\Gamma|$.

In the case when $T : \Delta \longrightarrow \mathcal{C}$ is a cosimplicial object in $\mathcal{C}$, we define *the realization of* Γ *with respect to* T, denoted by $|\Gamma|_T$, as the tensor product $\Gamma \otimes_G T$, where T is restricted to G, if it exists. Once again this is an important remark since most of the realizations to be considered in this monograph come from cosimplicial objects.

Example 3.4.1. Let $\Delta^{(\_)} : \Delta \longrightarrow \mathbf{Top}$ be the standard simplex functor and Γ a 1-skeleton, that is, a digraph (A, V, t, h) under the identifications $t = \Gamma(\epsilon_1)$ and $h = \Gamma(\epsilon_0)$. Recall the description of the category of elements of Γ made in Section 3.1. The functor $\Delta^{(\_)}\pi_\Gamma$ sends each object a in A to $\Delta^1 = I$, each object $v \in V$ to $\Delta^0 = \{*\}$, each arrow $(t(a), a)$ to $i_0 : \{*\} \longrightarrow I$, and each arrow $(h(a), a)$ to $i_1 : \{*\} \longrightarrow I$. In this way, $|\Gamma|$ coincides with the definition of the geometric realization of Γ given in Section 1.3 and the corresponding bijection is

$$\mathbf{Top}(|\Gamma|, X) \cong Nat(\Gamma, \mathbf{Top}(\Delta^{(\_)}, X)) = Digraph(\Gamma, \overrightarrow{X}),$$

which establishes the correspondence between gestures[9] and continuous functions from the geometric realization already discussed in Section 1.3. ◇

Realizations from adjoints

Let $\mathcal{D}$ and $\mathcal{E}$ be categories with $\mathcal{D}$ small and $F : \widehat{\mathcal{D}} \longrightarrow \mathcal{E}$ a functor that preserves small limits; for example, let F be a left adjoint. Given a presheaf $P : \mathcal{D}^{op} \longrightarrow \mathbf{Set}$, according to [14, §III.7],

$$P = Colim\left(\int P \xrightarrow{\pi_P} \mathcal{D} \xrightarrow{\mathbf{y}} \widehat{\mathcal{D}}\right),$$

where $\mathbf{y}$ is the Yoneda embedding, and therefore $F(P) = Colim(F \circ \mathbf{y} \circ \pi_P)$. Hence $F(P) = P \otimes_{\mathcal{D}} F\mathbf{y}$ for every presheaf P and $F = \_ \otimes_{\mathcal{D}} F\mathbf{y}$.

In particular, if $\mathcal{D} = G$, then F is the realization functor $|\_|_{F\mathbf{y}}$.

Example 3.4.2. Recall that (Subsection 3.1.1) the category of digraphs *Digraph* is just the category of presheaves $\widehat{G_1}$ on G_1. According to [14, §II.7], the functor $Path : \widehat{G_1} \longrightarrow \mathbf{Cat}$ that sends a digraph to its associated free category is a left adjoint. In this way, if $\mathbf{y} : G_1 \longrightarrow \widehat{G_1}$ is the Yoneda embedding, then *Path* is the realization functor $|\_|_{Path \circ \mathbf{y}}$, related to the realization defined in Subsection 3.1.4. Moreover, $Path \circ \mathbf{y}$ sends $[0]$ to the final category (just an object and an arrow); $[1]$ to the category with just two objects, their identities, and a unique arrow between them; and the morphisms ϵ_0 and ϵ_1 to the two inclusions of the final category. ◇

3.4.3 The duality realization/gestures

The definition of the object of gestures is the dual of that of realization. To see this, change $\mathcal{C}$ for $\mathcal{C}^{op}$ in the definition of the realization of Γ with respect to T (Subsection 3.4.2). In

---

[9] Note that the latter equality is erroneous in a strict sense. Certainly, a morphism of digraphs $\delta : \Gamma \longrightarrow \overrightarrow{X}$ is different from a natural transformation from Γ to $\mathbf{Top}(\Delta^{(\_)}, X)$, but this difference is only between the codomains and is not substantial.

this way, we obtain that the realization of a semi-simplicial set Γ with respect to a functor $T : G \longrightarrow \mathcal{C}^{op}$ (which corresponds uniquely to a functor $S : G^{op} \longrightarrow \mathcal{C}$, by applying $(\_)^{op}$) is

$$Colim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{T} \mathcal{C}^{op}\right) = Lim\left(\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G^{op} \xrightarrow{S} \mathcal{C}\right) = \Gamma @ S.$$

Therefore we have the following delicate and fundamental fact:

The concept of gestures is the dual of that of realization.

Thus, we have the dual statement of the Hom-tensor adjunction; we study it in the following subsection.

3.4.4 The fundamental adjunction for gestures

Let $\mathcal{D}$ and $\mathcal{E}$ be categories, where $\mathcal{E}$ satisfies (H), and $A : \mathcal{D}^{op} \longrightarrow \mathcal{E}$ a functor. Consider the functor $L : \mathcal{E} \longrightarrow \left(\widehat{\mathcal{D}}\right)^{op}$ with

$$L(E) := \mathcal{E}(E, A(\_))$$

for each object E of $\mathcal{E}$ and $L(f)$ the natural transformation $(\_) \circ f$. By dualizing the Hom-tensor adjunction (Subsection 3.4.1), L has a right adjoint

$$\_ \pitchfork A : \left(\widehat{\mathcal{D}}\right)^{op} \longrightarrow \mathcal{E}$$

that sends every presheaf $P : \mathcal{D}^{op} \longrightarrow \mathbf{Set}$ to $P \pitchfork A$, where

$$P \pitchfork A := Lim\left(\left(\int P\right)^{op} \xrightarrow{\pi_P^{op}} \mathcal{D}^{op} \xrightarrow{A} \mathcal{E}\right),$$

whenever the limit exists for each presheaf P on $\mathcal{D}$.

This means that there is a bijection

$$\mathcal{E}(E, P \pitchfork A) \cong Nat(P, \mathcal{E}(E, A(\_)))$$

natural in both arguments P and E. The object $P \pitchfork A$ is called the *cotensor product*[10] of the presheaf P and the functor A. By dualizing the results in Subsection 3.4.1, we obtain the following description of this bijection. Given a natural transformation $\tau : P \longrightarrow \mathcal{E}(E, A(\_))$, the family

$$\{\tau_D(p) : E \longrightarrow A(D) |\ (D, p) \text{ is an object of } \int P\}$$

is a cone over $A(\pi_P^{op})$ with vertex E, so there is a unique morphism $f : E \longrightarrow P \pitchfork A$ such that $\mu_{(D,p)} f = \tau_D(p)$ for each (D, p), where

$$\{\mu_{(D,p)} : P \pitchfork A \longrightarrow A(D) | (D, p) \text{ is an object of } \int P\}$$

---

[10] The pitchfork notation $\pitchfork$ is adopted by analogy with the standard notation for cotensor products in enriched category theory (cf. [11, B.1.1.4 (b)]).

is the limiting cone over $A(\pi_P^{op})$ with vertex $P \pitchfork A$. Conversely, the isomorphism sends a morphism $f : E \longrightarrow P \pitchfork A$ to the natural transformation τ defined by $\tau_D(p) = \mu_{(D,p)} f$ for each object (D, p) of $\int P$.

The counit of this adjunction is the natural transformation

$$\epsilon^{op} : L \circ (\_ \pitchfork A) \Longrightarrow id : \left(\widehat{\mathcal{D}}\right)^{op} \longrightarrow \left(\widehat{\mathcal{D}}\right)^{op},$$

where $\epsilon_P : P \longrightarrow \mathcal{E}(P \pitchfork A, A(\_))$ is defined by $\epsilon_{P,D}(p) = \mu_{(D,p)}$ for each object D of $\mathcal{D}$ and $p \in P(D)$.

In particular, consider a semi-simplicial object $S : G^{op} \longrightarrow \mathcal{C}$ in a category $\mathcal{C}$ satisfying (H). By applying the adjunction above in the case when $\mathcal{E} = \mathcal{C}$ and $\mathcal{D} = G$, we obtain that if the object of gestures $\Gamma@S$ (which is, of course, the cotensor product $\Gamma \pitchfork S$) exists for each semi-simplicial set Γ, then there is a functor

$$\_@S : \left(\widehat{G}\right)^{op} \longrightarrow \mathcal{C},$$

which is right adjoint to the functor $\mathcal{C}(\_, S)$ that sends each object C of $\mathcal{C}$ to the semi-simplicial object $\mathcal{C}(C, S(\_))$. This means that for each skeleton Γ and each object C of $\mathcal{C}$ there is a bijection

$$\mathcal{C}(C, \Gamma@S) \cong Nat(\Gamma, \mathcal{C}(C, S(\_))), \tag{3.1}$$

natural in both arguments Γ and C.

If the category $\mathcal{C}$ has a terminal object $\mathbf{1}$, then we obtain a bijection between the set $\mathcal{C}(\mathbf{1}, \Gamma@S)$ of points of $\Gamma@S$ and

$$Nat(\Gamma, \mathcal{C}(\mathbf{1}, S(\_))).$$

The semi-simplicial set $\mathcal{C}(\mathbf{1}, S(\_))$ is called the *underlying semi-simplicial set* of the semi-simplicial object S.

One of the main interests of the fundamental bijection 3.1, is that it characterizes, in a transparent way, the generalized elements[11] of the object of gestures. In this way, *a gesture with skeleton Γ with respect to S over A, or an A-addressed gesture with skeleton Γ with respect to S*, can be identified with a natural transformation

$$\tau : \Gamma \longrightarrow \mathcal{C}(A, S(\_)).$$

In the case when S is the semi-simplicial object S_C of C with respect to a functor $T : G \longrightarrow \mathcal{C}$ whose images are exponentiable in $\mathcal{C}$ (see the construction of Mazzola's gestures in Subsection 3.3.3) we have that

$$\mathcal{C}(A, S_C(\_)) = \mathcal{C}(A, C^{T(\_)}) \cong \mathcal{C}(A \times T(\_), C),$$

---

[11] Given a category $\mathcal{C}$ and an object X of $\mathcal{C}$, an element of X over A (cf. the definition of generalized element in a topos in [15, pp. 236-237]), or, in Mazzola's terminology [24], an A-addressed point of X, is a morphism $A \longrightarrow X$.

so *an A-addressed gesture with skeleton* Γ *and body in* C can be identified with a natural transformation

$$\tau : \Gamma \longrightarrow \mathcal{C}(A \times T(\_), C).$$

Note that this also allows to define addressed gestures even if the images of T are not exponentiable.

Example 3.4.3. Suppose that Γ is a digraph (A, V, t, h) ($t = \Gamma(\epsilon_1)$, $h = \Gamma(\epsilon_0)$) and that $T : G \longrightarrow \mathbf{Top}$ is the restriction of the standard simplex functor in **Top**. In this case, an I-addressed gesture with skeleton Γ and body in a topological space X is just a morphism of digraphs from Γ to the digraph whose arrows are continuous maps $f : I \times I \longrightarrow X$ (that is, homotopies), whose vertices are paths in X, and with tail and head functions given by the restriction of homotopies to $I \times \{0\}$ and $I \times \{1\}$ respectively. ◇

Example 3.4.4. Let Γ be a semi-simplicial set and $T : G \longrightarrow \mathbf{Loc}$ the restriction of the standard simplex functor in **Loc** (Example 3.3.2). If L and M are locales, then the set of M-addressed gestures with skeleton Γ and body in L is $Nat(\Gamma, \mathbf{Loc}(M \times_l \mathcal{O}(\Delta^{(\_)}), L))$. In particular, if $M = \mathcal{O}(X)$ for some topological space X, then

$$\begin{aligned}\mathbf{Loc}(M \times_l \mathcal{O}(\Delta^{(\_)}), L) &= \mathbf{Loc}(\mathcal{O}(X) \times_l \mathcal{O}(\Delta^{(\_)}), L)\\ &\cong \mathbf{Loc}(\mathcal{O}(X \times \Delta^{(\_)}), L)\\ &\cong \mathbf{Top}(X \times \Delta^{(\_)}, pt(L)).\end{aligned}$$

In fact, the isomorphism of the second line follows from Proposition [9, II.2.13] and the isomorphism of the third line follows from the adjunction $\mathcal{O} \dashv pt$ (Section 2.4). Thus, $\mathcal{O}(X)$-addressed gestures with skeleton Γ and body in L correspond bijectively to X-addressed gestures with skeleton Γ and body in $pt(L)$. ◇

3.4.5 Reduction to exponentials

Let $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ be a skeleton and $T : G \longrightarrow \mathcal{C}$ a functor whose images are exponentiable in $\mathcal{C}$. Recall from the definition of Mazzola's gestures (Subsection 3.3.3) that $\Gamma@C$ is the limit

$$Lim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{T} \mathcal{C} \xrightarrow{C^{(\_)}} \mathcal{C}\right).$$

In particular, if the colimit

$$Colim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{T} \mathcal{C}\right)$$

defining $|\Gamma|$ exists and is exponentiable in $\mathcal{C}$ and the functor $C^{(\_)}$ transforms the colimit $|\Gamma|$ into a limit (for example, if the conditions of 3.2.2 are satisfied), then

$$C^{|\Gamma|} = Lim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{T} \mathcal{C} \xrightarrow{C^{(\_)}} \mathcal{C}\right) = \Gamma@C.$$

Consequently, we have the following result, which is analogous to Proposition 3.2.1.

Proposition 3.4.5 (Reduction to exponentials)**.** *Let $\mathcal{C}$ be a category satisfying (CC) and (COL), and $T : G \longrightarrow \mathcal{C}$ a semi-cosimplicial object. The object of gestures $\Gamma@C$ exists and*

$$\Gamma@C = C^{|\Gamma|}$$

for each skeleton $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ and each object C of $\mathcal{C}$.

3.4.6 Gestures from semi-simplicial sets

The construction of hypergestures (Subsection 3.3.3) relies on the existence of suitable exponentials. However, the construction of exponentials is not always available so we introduce the following notion of *semi-simplicial set of an object.* This construction allows us to give the notion of *a gesture* with body in an object of the category involved, in contrast to our preceding definition of the object of gestures.

Let $\mathcal{C}$ be a category satisfying (H) and $T : G \longrightarrow \mathcal{C}$ a functor such that the realization functor $|\_|_T$ exists. Given an object C of $\mathcal{C}$, we define the *semi-simplicial set s_C of C* as the composite

$$G \xrightarrow{T} \mathcal{C} \xrightarrow{\mathcal{C}(\_,C)} \mathbf{Set},$$

which coincides with its underlying semi-simplicial set (Subsection 3.4.4) since it is a functor to **Set**. Therefore, according to Subsection 3.4.4, we have a bijection between the points of $\Gamma@s_C$ (that is, its elements) and the set

$$Nat(\Gamma, \mathcal{C}(T(\_), C)).$$

Consequently, in this case, we can define *a gesture with skeleton Γ and body in C with respect to T* as a natural transformation

$$\delta : \Gamma \longrightarrow s_C.$$

In this way, the set of gestures $\Gamma@s_C$ is completely determined by all the individual gestures δ, in contrast to the case of locales of gestures, which need not be characterized by their points (see Section 2.5).

Note that in turn, if the realization functor $|\_|_T$ exists, then s_C coincides with the value at C of the right adjoint to the realization functor (Subsection 3.4.2) and hence there is a bijection

$$\mathcal{C}(|\Gamma|_T, C) \cong Nat(\Gamma, s_C).$$

Thus, individual gestures with skeleton Γ and body in C correspond bijectively to morphisms from the realization $|\Gamma|_T$ to C.

This discussion is recorded in the following theorem.

Theorem 3.4.6. *Let $\mathcal{C}$ be a category satisfying (H), $T : G \longrightarrow \mathcal{C}$ a functor such that the realization functor $|\_|_T$ exists, and C an object of $\mathcal{C}$. There are natural bijections*

$$\Gamma@s_C \cong Nat(\Gamma, s_C) \cong \mathcal{C}(|\Gamma|_T, C).$$

Thus, a gesture with skeleton Γ and body in C with respect to T can be presented in any of the following forms:

i) As an element of the limit $\Gamma@s_C$*.*

ii) As a natural transformation $\delta : \Gamma \longrightarrow s_C$*.*

iii) As a morphism of $\mathcal{C}$ *from* $|\Gamma|_T$ *to* C*.*

At this point we have given two definitions of gestures with body in an object C of $\mathcal{C}$: the object of $\mathcal{C}$ of gestures $\Gamma@C$ (Subsection 3.3.3) and, on the other hand, an individual gesture (Theorem 3.4.6). The following result shows that the two definitions are compatible in a natural way: the individual gestures are just the points of the objects of gestures.

Theorem 3.4.7. *Let* $\mathcal{C}$ *be a category satisfying (H) and (C),* $T : G \longrightarrow \mathcal{C}$ *a functor with all its images exponentiable in* $\mathcal{C}$ *such that the realization functor* $|\_|_T$ *exists,* C *an object of* $\mathcal{C}$*, and* Γ *a semi-simplicial set. The underlying semi-simplicial set of* S_C *(see Subsection 3.4.4) is naturally isomorphic to* s_C*. Moreover, there is a bijection between*

i) points of $\Gamma@C$*, that is, morphisms of* $\mathcal{C}$ *from* $\mathbf{1}$ *to* $\Gamma@C$*; and*

ii) gestures with skeleton Γ *and body in* $\mathcal{C}$*, in the sense of Theorem 3.4.6.*

Proof. First, note that

$$\mathcal{C}(\mathbf{1}, S_C(\_)) = \mathcal{C}(\mathbf{1}, C^{T(\_)}) \cong \mathcal{C}(T(\_), C) = s_C$$

by the bijections associated with the exponentials of the form $C^{T([n])}$ and the fact that $\mathbf{1} \times X \cong X$ for each object X of $\mathcal{C}$. Thus, the underlying semi-simplicial set of S_C is naturally isomorphic to s_C.

Moreover, there are natural bijections

$$\mathcal{C}(\mathbf{1}, \Gamma@S_C) \cong Nat(\Gamma, \mathcal{C}(\mathbf{1}, S_C(\_))) \cong Nat(\Gamma, s_C).$$

The first bijection is given by the fundamental adjunction for gestures (Subsection 3.4.4), and the second one is induced by the bijection of the first paragraph. □

3.5 A geometric approach to realization

So far, we have been concerned with the definition of the realization of a semi-simplicial set with respect to a semi-cosimplicial object and, on the other hand, with the definition of the object of gestures with skeleton a semi-simplicial set with respect to a semi-simplicial object. However, we may think of the possibility of defining these concepts when we drop the prefix semi-. There is a strong reason for doing this: the resulting notions have a more geometric behavior in the sense that simplicial sets are more similar to spaces than semi-simplicial sets. In this section, we discuss some perspectives of this approach.

The main ideas of this subsection are exposed in [19] in the case of the category **Top**. We will review these ideas in the general case and apply them to gestures.

3.5.1 Semi-simplicial sets vs. simplicial sets

The inclusion $K : G \hookrightarrow \Delta$, which is not full, induces a restriction functor $\_ \circ K^{op} : \widehat{\Delta} \longrightarrow \widehat{G}$, which we denote by F. As in the case of the truncation functor (Subsection 3.3.2), the functor F has a left adjoint L. Although the existence of L is easily proved using the theory of Kan extensions, we will use an explicit presentation of it (check): if $[n]$ is an object of Δ,

$$L(\Gamma)([n]) = \{(f, a) | f : [n] \twoheadrightarrow [p] \text{ and } a \in \Gamma([p])\},$$

and if $\alpha : [n] \longrightarrow [m]$ is a morphism of Δ,

$$L(\alpha)(g, b) = (e, \Gamma(h)(b)),$$

where he is the unique epi-mono factorization of $g\alpha$.

For each semi-simplicial set Γ, the component $\eta_\Gamma : \Gamma \longrightarrow L(\Gamma) \circ K^{op}$ of the unit of the adjunction $L \dashv F$ is defined by $\eta_\Gamma([n])(a) = (id_n, a)$ for each object $[n]$ of G and each $a \in \Gamma([n])$.

3.5.2 Realization as a Milnor realization

Let $\mathcal{C}$ be a category satisfying (COL) and $T : \Delta \longrightarrow \mathcal{C}$ a cosimplicial object in $\mathcal{C}$. We define the *Milnor realization* ${}_M|P|_T$ *of a simplicial set* $P : \Delta^{op} \longrightarrow$ **Set** *with respect to* T as the tensor product $P \otimes_\Delta T$ (Subsection 3.4.1).

As we will see shortly, the Milnor geometric realization (in **Top**) has a fundamental advantage over the geometric realization for semi-simplicial sets: under suitable hypotheses, it transforms the product of two simplicial sets into a product of spaces. For this reason, we need a translation of realization (closely related to gestures) in terms of Milnor realization so that the geometric properties of gestures are better grasped. Certainly, this approach produces, under suitable hypotheses, an interesting formula expressing hypergestures as gestures; this is the main goal of Subsection 3.5.6.

In the following proposition we establish the isomorphism between the realization of a semi-simplicial set Γ with respect to a cosimplicial set and the Milnor realization ${}_M|L\Gamma|$ of its associated simplicial set. For simplicity, we will assume (H).

Proposition 3.5.1. *Let $\mathcal{C}$ be a category satisfying (H) and (COL), and $T : \Delta \longrightarrow \mathcal{C}$ a cosimplicial object. If Γ is a semi-simplicial set, then there is an isomorphism*

$$|\Gamma| \cong {}_M|L\Gamma|.$$

Here, $|\Gamma|$ denotes the realization of Γ with respect to T (Subsection 3.4.2).

Proof. The two pairs of adjoint functors

$$L : \widehat{G} \rightleftarrows \widehat{\Delta} : \_ \circ K^{op} \qquad \text{and} \qquad {}_M|\_| : \widehat{\Delta} \rightleftarrows \mathcal{C} : \mathcal{C}(T, \_)$$

from subsections 3.5.1 and 3.4.1, with left adjoints on the left, compose to yield the pair of adjoint functors

$$_M|L| : \widehat{G} \rightleftarrows \mathcal{C} : \mathcal{C}(TK, \_) \ .$$

But the right-hand functor is precisely the right adjoint to the realization functor $|\_| : \widehat{G} \longrightarrow \mathcal{C}$ (see Subsection 3.4.2), so $_M|L|$ is naturally isomophic to $|\_|$ by the uniqueness of adjoints up to isomorphism. □

According to this proposition, given a pair Γ_1, Γ_2 of semi-simplicial sets, we have a pair of associated simplicial sets $L\Gamma_1, L\Gamma_2$ whose Milnor realizations coincide with the realizations of the initial pair. In this way, the product $L\Gamma_1 \times L\Gamma_2$ can be regarded as an improved product of Γ_1 and Γ_2, but it is a simplicial set rather than a semi-simplicial set. For this reason, we must associate a semi-simplicial set to this product. The resulting object should be an improved product of Γ_1 and Γ_2 in $\widehat{G}$. Thus, in Subsection 3.5.3 we proceed to construct a suitable semi-simplicial set from each simplicial set.

Before that, we write the dual of Proposition 3.5.1, which is one about objects of gestures.

Proposition 3.5.2 (Dual of Proposition 3.5.1)**.** *Let $\mathcal{C}$ be a category satisfying (H) and (L), and $S : \Delta^{op} \longrightarrow \mathcal{C}$ a simplicial object. If Γ is a semi-simplicial set, then there is an isomorphism*

$$\Gamma @ S \cong (L\Gamma) @ S.$$

Here, $\Gamma @ S$ is the object of gestures with respect to the simplicial object S defined in Subsection 3.3.3, the object $(L\Gamma) @ S$ is the cotensor product $(L\Gamma) \pitchfork S$ defined in Subsection 3.4.4, and L is as in the proof of Proposition 3.5.1.

The importance of this proposition is that it opens up a more spatial definition of gestures in what concerns the skeleta; in fact, as in the case of realization, given a semi-simplicial set Γ we have an associated simplicial set $L\Gamma$, which is more similar to a space (in fact, a simplicial set is a good combinatorial model of an appropriate CW-complex), and a notion of object of gestures (the cotensor product) $(L\Gamma) @ S$ that is isomorphic to $\Gamma @ S$. In this way, given a simplicial set Σ, it is worth defining the object of *Milnor's gestures* with skeleton $\Sigma : \Delta^{op} \longrightarrow$ **Set** with respect to a simplicial object $S : \Delta \longrightarrow \mathcal{C}$ as the cotensor product $\Sigma \pitchfork S$ defined in Subsection 3.4.4. However, this definition will not be used in what follows. Here, our unique purpose is to point out a more spatial treatment of skeleta for gestures, and we prefer the use of semi-simplicial sets since they carry less information than simplicial sets so that they simplify many computations; for example, for digraphs, we are not forced to consider the identity arrows of a digraph.

3.5.3 The core of a simplicial set

Recall that an n-simplex of a simplicial set P (respectively semi-simplicial set Γ) is an element $a \in P([n])$ (respectively $a \in \Gamma([n])$). In the case of a simplicial set P, we say that a is *degenerate* if there exists $f : [n] \twoheadrightarrow [m]$ with $m < n$ such that $P(f)(b) = a$ for some

$b \in P([m])$. A *non-degenerate* simplex is a simplex that is not degenerate, and we denote by NP_n the set of all n-simplices of P that are non-degenerate.

Given a simplicial set P, we define an associated semi-simplicial set as follows. The *core of* P, denoted by $Core(P)$, is the least subpresheaf of the restriction $P \circ K^{op}$ containing all non-degenerate simplices of P; that is,

$$Core(P)([n]) = NP_n \cup \{a \in P([n]) | \text{ there is } f : [n] \rightarrowtail [p] \text{ and } b \in NP_p \text{ s.t. } a = Pf(b)\}.$$

In words, the core consists of all non-degenerate simplices of P and their faces, the face maps being the restrictions of those of P. We say that a simplicial set P is non-degenerate if $Core(P)$ contains no degenerate simplices (the faces of non-degenerate simplices may be degenerate); in that case, $Core(P)([n]) = NP_n$ for each object $[n]$ of G, and hence we write NP for $Core(P)$.

With these definitions, it can be shown that $L\Gamma$ is non-degenerate, and that the image of the component $\eta_\Gamma : \Gamma \longrightarrow L(\Gamma) \circ K^{op}$ of the unit of the adjunction $L \dashv F$ (Subsection 3.5.1), which is monomorphic, is exactly $Core(L\Gamma)$, so

$$Core(L\Gamma)([n]) = \{(id_{[n]}, a) | a \in \Gamma([n])\} = NL\Gamma_n$$

and hence $\Gamma \cong Core(L\Gamma)$.

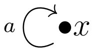
3.5.4 Geometric product of semi-simplicial sets

When we deal with geometric realizations and their products, the Milnor realization has a better behavior than the geometric realization for semi-simplicial sets. Certainly, if P and Q are simplicial sets, then

$$_M|P \times Q| = {}_M|P| \times_{Ke} {}_M|Q|,$$

where $\times_{Ke}$ is the product in the category **CGHaus** of compactly generated Hausdorff spaces (or Kelley product). But the same is not true for semi-simplicial sets, as shown in the following example.

Example 3.5.3. Let Γ be a loop digraph as in the picture



Formally, Γ is the tuple $(\{a\}, \{x\}, !, !)$. The product of semi-simplicial sets $\Gamma \times \Gamma$ is the tuple $(\{(a, a)\}, \{(x, x)\}, !, !)$, that is, a loop again. The realization of Γ in **CGHaus** (with respect to the standard simplex) is just the circle $\mathbf{S}^1$; see the discussion before Corollary 3.5.7 below. The realization of $\Gamma \times \Gamma$ is also $\mathbf{S}^1$. In this way,

$$|\Gamma \times \Gamma| = \mathbf{S}^1 \neq \mathbf{S}^1 \times \mathbf{S}^1 = |\Gamma| \times |\Gamma|.$$

Moreover, since $\mathbf{S}^1$ is locally compact, the product of topological spaces above is also the Kelley product [5, III.2.1.1]. ◇

For this reason, given two semi-simplicial sets Γ_1 and Γ_2, we define their *geometric product*, denoted by $\Gamma_1 \times_g \Gamma_2$, as

$$Core(L\Gamma_1 \times L\Gamma_2),$$

which is a semi-simplicial set. This product is due to the authors of [19]. Certainly, this product has a good behavior regarding the geometric realization, but before proving this, we need some work.

Let P be a simplicial set. Consider the natural transformation

$$\theta : LCore(P) \longrightarrow P,$$

defined by $\theta_{[n]}(f, a) = Pf(a)$. By the Eilenberg-Zilber lemma [13], it is an epimorphism. Moreover, by the same result, it is an isomorphism if P is non-degenerate. Thus, given two semi-simplicial sets Γ_1 and Γ_2, we have an epimorphism

$$\theta : L(\Gamma_1 \times_g \Gamma_2) \longrightarrow L\Gamma_1 \times L\Gamma_2,$$

which would be an isomorphism if we were able to show that $L\Gamma_1 \times L\Gamma_2$ is non-degenerate. The following lemma clarifies the situation.

Lemma 3.5.4. *If Γ_1 and Γ_2 are semi-simplicial sets, then $L\Gamma_1 \times L\Gamma_2$ is non-degenerate.*

Proof. First, note that $((f, a), (g, b)) \in L\Gamma_1 \times L\Gamma_2([n])$ is degenerate if and only if there is i $(0 \leq i \leq n-1)$ such that both f and g factor through the degeneracy $\sigma_i : [n] \longrightarrow [n-1]$ if and only if there is i such that $f(i) = f(i+1)$ and $g(i) = g(i+1)$.

On the other hand, $L\Gamma_1\epsilon_i \times L\Gamma_2\epsilon_i((f, a), (g, b)) = ((f', \Gamma_1\alpha(a)), (g', \Gamma_2\beta(b)))$, where

$$\begin{tikzcd}
{[n-1]} \arrow[r, two heads, "f'"] \arrow[d, tail, "\epsilon_i"] & \bullet , \arrow[d, tail, "\alpha"] & & {[n-1]} \arrow[r, two heads, "g'"] \arrow[d, tail, "\epsilon_i"] & \bullet \arrow[d, tail, "\beta"] \\
{[n]} \arrow[r, two heads, "f"] & {[p]} & & {[n]} \arrow[r, two heads, "g"] & {[q]}
\end{tikzcd}$$

are epi-mono factorizations. Thus, if $((f', \Gamma_1\alpha(a)), (g', \Gamma_2\beta(b)))$ is degenerate, then there exists j $(0 \leq j \leq n-2)$ such that $f'(j) = f'(j+1)$ and $g'(j) = g'(j+1)$, and hence $f\epsilon_i(j) = f\epsilon_i(j+1)$ and $g\epsilon_i(j) = g\epsilon_i(j+1)$. This implies that $f\epsilon_i(j) = f(\epsilon_i(j)+1)$ and $g\epsilon_i(j) = g(\epsilon_i(j)+1)$, since $\epsilon_i(j) \leq \epsilon_i(j)+1 \leq \epsilon_i(j+1)$; that is, $((f, a), (g, b))$ is degenerate.

Thus, the faces of non-degenerate simplices in $L\Gamma_1 \times L\Gamma_2$ are non-degenerate.

□

Consequently, there is an isomorphism

$$L(\Gamma_1 \times_g \Gamma_2) \cong L\Gamma_1 \times L\Gamma_2. \tag{3.2}$$

Finally, we come to the desired results.

Theorem 3.5.5. *Let $T : \Delta \longrightarrow \mathcal{C}$ be a cosimplicial object in a category satisfying (COL) and (H). If the Milnor realization preserves binary products, then the realization satisfies*

$$|\Gamma_1 \times_g \Gamma_2| = |\Gamma_1| \times |\Gamma_2|,$$

for each pair Γ_1, Γ_2 of semi-simplicial sets.

Proof. If Γ_1 and Γ_2 are semi-simplicial sets, then

$$\begin{aligned} |\Gamma_1 \times_g \Gamma_2| &\cong {}_M|L(\Gamma_1 \times_g \Gamma_2)| \\ &\cong {}_M|L\Gamma_1 \times L\Gamma_2| \\ &\cong {}_M|L\Gamma_1| \times {}_M|L\Gamma_2| \\ &\cong |\Gamma_1| \times |\Gamma_2|. \end{aligned}$$

The isomorphisms of the first and fourth lines are given by Proposition 3.5.1. The isomorphism of the second line corresponds to isomorphism 3.2, and that of the third line is given by hypothesis. □

In particular, if $\mathcal{C}$ is the category **CGHaus** of compactly generated Hausdorff spaces [14, VII.8], also called Kelley spaces in [5, III.4], we obtain the following result. Recall that the standard simplex functor in **CGHaus** is the standard simplex functor in **Top**, except for its codomain.

Corollary 3.5.6. *The realization in* **CGHaus** *satisfies*

$$|\Gamma_1 \times_g \Gamma_2| = |\Gamma_1| \times_{Ke} |\Gamma_2|,$$

for each pair Γ_1, Γ_2 of semi-simplicial sets.

Proof. The category **CGHaus** has small-sets and is small-cocomplete by [14, VII.8.2] . Also, the Milnor realization in **CGHaus** preserves binary products according to [5, III.3.5]. □

The realization of a semi-simplicial set Γ in **Top** is a compactly generated Hausdorff space (Corollary 3.5.10) and hence it coincides with the realization of Γ in **CGHaus** (see the construction of colimits in **CGHaus** in [14, VII.8.2]). In this way, the realizations in Corollary 3.5.6 can be regarded as realizations in **Top**.

The following corollary is essential for the theory of gestures. Recall that a semi-simplicial set Γ is *locally finite* if each simplex in Γ is the face of only finitely many simplices in Γ.

Corollary 3.5.7. *If Γ_1 or Γ_2 is a locally finite semi-simplicial set, then the realization in* **Top** *satisfies*

$$|\Gamma_1 \times_g \Gamma_2| = |\Gamma_1| \times |\Gamma_2|.$$

Proof. This result follows from Corollary 3.5.6. In fact, as observed, the realizations can be assumed to be realizations in **Top**. Moreover, if Γ_i is locally finite, then $|\Gamma_i|$ is a locally compact space (Corollary 3.5.10 below) and hence the product $|\Gamma_1| \times_{Ke} |\Gamma_2|$ coincides with the product $|\Gamma_1| \times |\Gamma_2|$ in **Top** [5, III.2.1.1]. □

We end this discussion with an example. For our computations, note that according to the proof of Lemma 3.5.4, the geometric product $\Gamma_1 \times_g \Gamma_2$ of two semi-simplicial sets can be computed using the formula

$$\Gamma_1 \times_g \Gamma_2([n]) = \{[(f,a),(g,b)] \in L\Gamma_1 \times L\Gamma_2 | \forall i \in [n-1] \ f(i) \neq f(i+1) \text{ or } g(i) \neq g(i+1)\}$$

for each object $[n]$ of G.

Example 3.5.8 (The torus as a geometric product)**.** Let Γ be as in Example 3.5.3. Let us compute the geometric product $\Gamma \times_g \Gamma$:

$\Gamma \times_g \Gamma([0]) = \{[(id,x),(id,x)]\}$.

$\Gamma \times_g \Gamma([1]) = \{[(id,a),(id,a)],[(id,a),(!,x)],[(!,x),(id,a)]\}$.

$\Gamma \times_g \Gamma([2]) = \{[(\sigma_0,a),(\sigma_1,a)],[(\sigma_1,a),(\sigma_0,a)]\}$.

$\Gamma \times_g \Gamma([n]) = \emptyset$ for $n \geq 3$.

Here, σ_i denotes the ith degeneracy (see [15, p. 452]). Moreover, the face operators (Subsection 3.3.2) of $\Gamma \times_g \Gamma$ are defined as follows:

There is a unique possibility for d^1_0 and d^1_1 since there is a unique vertex.

$d_0([(\sigma_0,a),(\sigma_1,a)]) = [(id,a),(!,x)]$ and $d_0([(\sigma_1,a),(\sigma_0,a)]) = [(!,x),(id,a)]$.

$d_1([(\sigma_0,a),(\sigma_1,a)]) = [(id,a),(id,a)]$ and $d_1([(\sigma_1,a),(\sigma_0,a)]) = [(id,a),(id,a)]$.

$d_2([(\sigma_0,a),(\sigma_1,a)]) = [(!,x),(id,a)]$ and $d_2([(\sigma_1,a),(\sigma_0,a)]) = [(id,a),(!,x)]$.

In this way, the semi-simplicial set $\Gamma \times_g \Gamma$ has dimension 2 and is a *combinatorial model of the torus*. Its geometric realization is, in fact, isomorphic to the torus $\mathbf{S}^1 \times \mathbf{S}^1$, which coincides with $|\Gamma| \times |\Gamma|$. ◇

3.5.5 Geometric realizations as CW-complexes

Now we review the characterization of the geometric realization of a semi-simplicial set as a CW-complex, a feature that allows to relate properties of the geometric realization to properties of the semi-simplicial set.

Proposition 3.5.9. *If T is the standard simplex functor $\Delta^{(-)} : \Delta \longrightarrow$ **Top** and Γ is a semi-simplicial set, then $|\Gamma|$ is a CW-complex.*

Proof. The n-skeleton of the geometric realization is defined to be the geometric realization of the n-skeleton of the semi-simplicial set. We confine ourselves to exhibit the adjunction of n-cells (for $n \geq 1$): it is given by the pushout

$$\begin{tikzcd}
\bigsqcup_{a\in\Gamma([n])} \partial\Delta^n \arrow[r, hook] \arrow[d] & \bigsqcup_{a\in\Gamma([n])} \Delta^n \arrow[d] \\
|Sk_{n-1}(\Gamma)| \arrow[r, hook] & |Sk_n(\Gamma)|
\end{tikzcd}.$$

Here, the boundary $\partial\Delta^n$ is the union of all the faces of the standard n-simplex and the left-hand arrow is induced by the attaching maps $|a| : \partial\Delta^n \longrightarrow |Sk_{n-1}(\Gamma)|$ for $a \in \Gamma([n])$, which are obtained in the following way. For each $a \in \Gamma([n])$, consider the natural transformation $a : \mathbf{y}([n]) \longrightarrow Sk_n(\Gamma)$ corresponding to the element $a \in Sk_n(\Gamma)([n]) = \Gamma([n])$ (Yoneda lemma, $\mathbf{y} : G \longrightarrow \widehat{G}$ the Yoneda embedding), which restricts to a natural transformation $a : \partial\mathbf{y}([n]) \longrightarrow Sk_{n-1}(\Gamma)$, where $\partial\mathbf{y}([n])$ is the boundary of $\mathbf{y}([n])$ and $\partial\mathbf{y}([n]) = Sk_{n-1}(\mathbf{y}([n]))$. The geometric realization of $\partial\mathbf{y}([n])$ is $\partial\Delta^n$ (check), so a induces the morphism $|a| : \partial\Delta^n \longrightarrow |Sk_{n-1}(\Gamma)|$, which is the unique making the diagrams of the form

$$\begin{tikzcd}
|\partial\mathbf{y}([n])| \arrow[r, "|a|"] & |Sk_{n-1}(\Gamma)| \\
\Delta^m \arrow[u, "q_{([m],h)}"] \arrow[ur, "r_{([m],a\cdot h)}"] &
\end{tikzcd}$$

commute, where q_i and r_j denote the legs of the respective colimiting cones and $([m],h) \in \int \partial\mathbf{y}([n])$. In particular, if h is the ith face map $\epsilon_i : [n-1] \longrightarrow [n]$, $|a|q_{([n-1],\epsilon_i)} = r_{([n-1],a\cdot\epsilon_i)}$; so $|a|$ identifies the ith face of the standard n-simplex with the projection of the copy of Δ^{n-1} labelled by $a \cdot \epsilon_i = d_i(a)$ on the quotient $|Sk_{n-1}(\Gamma)|$.

A detailed argument in the case of simplicial sets can be found in [13]. The case of semi-simplicial sets can be proved along the same lines. □

By the well-known properties of CW-complexes we have the following corollary.

Corollary 3.5.10. *If Γ is a semi-simplicial set, then $|\Gamma|$ has the following properties.*

i) It is a Hausdorff k-space[12] *(in the sense of [7]). Equivalently, it is a compactly generated Hausdorff space.*

ii) It is locally compact if and only if Γ is locally finite

Proof. i) This is a consequence of Proposition [7, 1.2.1]. ii) According to Proposition [7, 1.5.10], $|\Gamma|$ is locally compact if and only if every open cell of $|\Gamma|$ meets only finitely many closed cells of $|\Gamma|$. But open (and closed) cells are in correspondence with simplices of Γ and an open cell relative to a meets a closed cell relative to b if and only if a is a face of b. Thus, $|\Gamma|$ is locally compact if and only if Γ is locally finite. □

3.5.6 Some presentations of hypergestures as exponentials

In this subsection, we present some formulas that express spaces and locales of hypergestures as exponentials and as spaces of gestures on skeleta of possibly higher dimensions. To a great extent, *they summarize all the preceding work.*

If Γ_1 and Γ_2 are locally finite digraphs and X is a topological space, then

$$\Gamma_1 @ \Gamma_2 @ X \cong X^{|\Gamma_1| \times |\Gamma_2|} \cong X^{|\Gamma_1 \times_g \Gamma_2|}$$

---

[12] See the definition in [7, p. 242].

in **Top**. This follows from Theorem 1.6.3, the laws of exponents, and Corollary 3.5.7.

Moreover, if L is a locale, then

$$\Gamma_1 @ \Gamma_2 @ L \cong L^{\mathcal{O}(|\Gamma_1|)\times_l \mathcal{O}(|\Gamma_2|)} \cong L^{\mathcal{O}(|\Gamma_1|\times|\Gamma_2|)} \cong L^{\mathcal{O}(|\Gamma_1 \times_g \Gamma_2|)}$$

in **Loc**. This follows from Theorem 2.6.10, the laws of exponents, Proposition [9, II.2.13], and Corollary 3.5.7.

It is possible to obtain a further refinement of the first formula by using semi-simplicial sets, but before proving it, we need to generalize the presentation of spaces of gestures as function spaces exposed in Section 1.5. First, we state the analogue of Theorem 1.4.1, which gives an explicit presentation of the space of gestures.

Theorem 3.5.11. *Let Γ be a semi-simplicial set and X a topological space. The space of Mazzola's gestures $\Gamma@X$ with respect to the standard simplex functor in* **Top** *(Subsection 3.3.3) is the subspace (regular subobject) of*

$$\prod_{n\in\mathbb{N}} (X^{\Delta^n})^{\Gamma([n])}$$

consisting of all sequences

$$\{C_n\}_{n\in\mathbb{N}} \text{ such that } c_{k,a}\epsilon_i = c_{k-1,d_i(a)} \text{ whenever } \epsilon_i : [k-1] \to [k] \text{ is a face and } a \in \Gamma([k]),$$

where $C_n = \{\Delta^n \xrightarrow{c_{n,a}} X\}_{a\in\Gamma([n])}$ for each $n \in \mathbb{N}$, $d_i : \Gamma([k]) \longrightarrow \Gamma([k-1])$ is the morphism $\Gamma(\epsilon_i)$, and the same notation is used for the face $\epsilon_i : [k-1] \to [k]$ and the induced face map $\epsilon_i : \Delta^{k-1} \longrightarrow \Delta^k$.

Proof. The result follows from three essential points: the presentation of limits in terms of products and equalizers, the computation of products and equalizers in **Top**, and the fact that every map $\alpha : [n] \longrightarrow [m]$ in the semi-simplicial category is a composite of face maps. □

Second, we give an explicit presentation of the usual realization of a skeleton in **Top**. Given a semi-simplicial set Γ, by the computation of colimits in **Top**, the realization $|\Gamma|$ with respect to the standard simplex is the quotient of the disjoint union of standard simplices

$$\bigcup_{n\in\mathbb{N}} \Delta^n \times \Gamma([n])$$

by the relation $\sim$ defined by

$$(\epsilon_i(\mathbf{t}), a) \sim (\mathbf{t}, d_i(a)) \text{ whenever } \epsilon_i : [k-1] \to [k] \text{ is a face, } a \in \Gamma([k]), \text{ and } \mathbf{t} \in \Delta^{k-1},$$

where we use the same conventions of the preceding theorem. Let q be the quotient map associated with this construction. An open U of $|\Delta|$ can be identified[13] with a sequence

$$\{U_{n,a} |\ n \in \mathbb{N}, a \in \Gamma([n])\}$$

---

[13] This is a consequence of the fact that the functor $\mathcal{O}$ from spaces to locales, as a left adjoint, preserves colimits, and of the computation of colimits in **Loc** commented in Section 2.1.

satisfying $\epsilon_i^*(U_{k,a}) = U_{k-1,d_i(a)}$ for each face map ϵ_i and each $a \in \Gamma([k])$. Note also that we can obtain the sequence by applying q^{-1}, that is, $q^{-1}(U) = \{U_{n,a} | \ n \in \mathbb{N}, a \in \Gamma([n])\}$.

Third, as in Section 1.5, we characterize the compact subsets of the geometric realization of a semi-simplicial set. The proof of the following result relies on the observation that a subset $K \subseteq |\Gamma|$ can be identified with the sequence $q^{-1}(K)$, which can be written in the form $\{K_{n,a} | \ n \in \mathbb{N}, a \in \Gamma([n])\}$ and satisfies the same conditions as the opens of $|\Gamma|$. The proof of the following proposition is almost a copy of that of 1.5.2.

Proposition 3.5.12. *Let Γ be a semi-simplicial set and $|\Gamma|$ its realization with respect to the standard simplex in* **Top**. *A subset K of $|\Gamma|$ is compact if and only if it is the image under q of a finite sequence of closed sets.*

Proof. If K is a compact subset of $|\Gamma|$, K is closed since $|\Gamma|$ is Hausdorff by Corollary 3.5.10, and hence each K_i is closed. Also, since $|\Gamma|$ is a CW-complex, by Proposition [7, 1.5.2], K is contained in a finite union of open cells. Since open cells are in correspondence with simplices of Γ, there is a natural number k and sets $A_0, \dots, A_k$, with $A_n \subseteq_{fin} \Gamma([n])$ for each $n \leq k$, such that the simplices in their union correspond to the finite open cells in which K is contained. Denote by S the finite sequence

$$\{K_{0,a}\}_{a \in A_0} \cup \{K_{1,a}\}_{a \in A_1} \cup \cdots \cup \{K_{k,a}\}_{a \in A_k}.$$

Note that its image under q is equal to K: since K is contained in the union of the open cells indexed by the elements of $A_0 \cup \cdots \cup A_k$, in particular, it is contained in the union of the closed cells indexed by the elements of the same set, which means that

$$K \subseteq q(\bigcup_{0 \leq n \leq k} \Delta^n \times A_n),$$

that is[14], $K = q(S)$.

On the other hand, suppose that K is the image under q of such a finite closed sequence. Each K_i is closed in a certain simplex Δ^n (which is compact), and hence compact. Therefore, the sequence is compact since it is a finite disjoint union of compacts, and hence its image under the continuous projection q is also compact. □

For simplicity, in the next two results we will write F for the union $A_0 \cup \cdots \cup A_k$ obtained in Proposition 3.5.12, which is a finite set, and $\{K_i\}_{i \in F}$ for the related sequence S. This identification is valid since each element in A_n is implicitly associated with n.

Lemma 3.5.13. *(Cf. Lemma 1.5.3) Suppose that K is a compact subset of $|\Gamma|$ and $S = \{K_i\}_{i \in F}$ is the finite sequence $\{K_i\}_{i \in F}$ from 3.5.12. If U is an open in $|\Gamma|$, then $K \subseteq U$ if and only if $K_i \subseteq q^{-1}(U)$ for all $i \in F$.*

Proof. First, if $K \subseteq U$, then $S \subseteq q^{-1}(K) \subseteq q^{-1}(U)$, so $K_i \subseteq q^{-1}(U)$ for all $i \in F$. On the other hand, if $K_i \subseteq q^{-1}(U)$ for all $i \in F$, then $S \subseteq q^{-1}(U)$, so $K = q(S) \subseteq q(q^{-1}(U)) = U$. □

---

[14] See the footnote in the proof of Proposition 1.5.2.

Now we have all the ingredients for proving the following central result.

Theorem 3.5.14 (Cf. 1.5.4)**.** *Let Γ be a semi-simplicial set and $|\Gamma|$ its realization with respect to the standard simplex in* **Top***. If X is a topological space, then the space of gestures $\Gamma@X$ is homeomorphic to* $\mathbf{Top}(|\Gamma|, X)$ *endowed with the compact-open topology.*

Proof. First, since the representable functor $\mathbf{Top}(\_, X)$ transforms colimits in **Top** into limits in **Set**, we have the isomorphism

$$\mathbf{Top}(|\Gamma|, X) \cong Lim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{\mathbf{Top}(\Delta^{(\_)}, X)} \mathbf{Set}\right).$$

But the right-hand side is just the limit (in **Set**) of the diagram defining the space of gestures $\Gamma@X$. In this way, since limits in **Top** have as underlying sets the synonymous limits in **Set**, we have a natural bijection $\phi : \Gamma@X \longrightarrow \mathbf{Top}(|\Gamma|, X)$. Further, it can be shown that the image of a sequence $\delta = \{c_{n,a} | n \in \mathbb{N}, a \in \Gamma([n])\}$ in $\Gamma@X$ under ϕ is given by

$$\begin{array}{cccc} \phi(\delta): & |\Gamma| & \longrightarrow & X \\ & q(\mathbf{t}, a) & \longmapsto & c_{n,a}(\mathbf{t}) \quad (\mathbf{t} \in \Delta^n, a \in \Gamma([n])) \end{array}.$$

Also, the image under ϕ^{-1} of a continuous map $f : |\Gamma| \longrightarrow X$ is the sequence $\{c_{n,a} | n \in \mathbb{N}, a \in \Gamma([n])\}$, where $c_{n,a} : \Delta^n \longrightarrow X$ is given by $c_{n,a}(\mathbf{t}) = f(q(\mathbf{t}, a))$.

We must see that ϕ is continuous and open. As in Theorem 1.5.4, it is enough to show that both ϕ^{-1} and ϕ transform subbasic opens into opens.

A subbasic open B in $\mathbf{Top}(|\Gamma|, X)$ is of the form $\{f \in \mathbf{Top}(|\Gamma|, X) |\ K \subseteq f^{-1}(U)\}$, where K is a compact subset[15] of $|\Gamma|$ and U an open in X, so by 3.5.13,

$$\phi^{-1}(B) = \{\phi^{-1}(f) | K_i \subseteq q^{-1} f^{-1}(U) \text{ for all } i \in F\} =$$
$$\{\{c_{n,a}\}_{n,a} \in \Gamma@X | K_a \subseteq c_{n,a}^{-1}(U) \text{ for all } a \in F\} =$$
$$\Gamma@X \cap \bigcap_{i \in F} \pi_i^{-1}(W(K_i, U)),$$

where each $W(K_i, U)$ is a subbasic open in the compact-open topology of some space of the form X^{Δ^n} since K_i is closed in Δ^n (and hence compact). Thus, $\phi^{-1}(B)$ is the intersection of $\Gamma@X$ and a finite intersection of opens in $\prod_{n \in \mathbb{N}} (X^{\Delta^n})^{\Gamma([n])}$, and hence it is open.

On the other hand, a subbasic open in $\Gamma@X$ is of the form $\pi_i^{-1}(W(C, U)) \cap \Gamma@X$ where $i = a \in \Gamma([n])$ for some n and $C \subseteq \Delta^n$ is closed, so

$$\begin{aligned} \phi(\pi_i^{-1}(W(C, U)) \cap \Gamma@X) &= \phi(\{\delta \in \Gamma@X | c_{n,a}(C) \subseteq U\}) \\ &= \{\phi(\delta) | \phi(\delta)(q(C)) \subseteq U\} \\ &= W(q(C), U). \end{aligned}$$

In fact, the latter is a subbasic open in $\mathbf{Top}(|\Gamma|, X)$ since the image $q(C)$ under the projection onto the quotient $|\Gamma|$ is compact by 3.5.12. □

---

[15] Recall that this compact has an associated sequence $\{K_i\}_{i \in F}$, where F is a finite set, as in Lemma 3.5.13.

Consequently, using the same arguments of Section 1.6, we obtain the following exponential presentation for gestures with locally finite skeleta.

Theorem 3.5.15. *If X is a topological space and Γ is a locally finite semi-simplicial set, then there exists a homeomorphism*

$$\Gamma@X \cong X^{|\Gamma|}.$$

Thus, we have the following refinement of the initial formula in this section.

Theorem 3.5.16. *Consider the realization $|\_|$ with respect to the standard simplex in* **Top**. *If Γ_1 and Γ_2 are locally finite semi-simplicial sets and X is a topological space, then*

$$\Gamma_1@\Gamma_2@X \cong X^{|\Gamma_1|\times|\Gamma_2|} \cong X^{|\Gamma_1\times_g\Gamma_2|} \cong (\Gamma_1 \times_g \Gamma_2)@X$$

in **Top**.

Proof. The first homeomorphism (from left to right) is given by Theorem 3.5.15 and the laws of exponents, the second one, by Corollary 3.5.7. As to the last one, note that $|\Gamma_1 \times_g \Gamma_2|$ is exponentiable and hence $\Gamma_1 \times_g \Gamma_2$ is locally finite (cf. Corollary 1.6.2). Thus, the homeomorphism follows from Theorem 3.5.15. □

This theorem, which is one of the main contributions of this thesis to the theory of topological gestures, deserves some reflection. First, it fully justifies the introduction of semi-simplicial sets and the geometric product in gesture theory because if Γ_1 and Γ_2 are locally finite *digraphs* (which are particular cases of semi-simplicial sets), then we have a homeomorphism between the space of hypergestures $\Gamma_1@\Gamma_2@X$ and the space of *gestures* $(\Gamma_1 \times_g \Gamma_2)@X$, where the skeleton is usually a semi-simplicial set of dimension greater than 1, not a digraph. Second, this theorem shows that @ is a sort of action of the 'monoid' of locally finite semi-simplicial sets (with the geometric product) on the category of topological spaces (cf. Corollary [27, 2.5]). Third, this theorem shows that the nature of the spaces of hypergestures whose skeleta are locally finite is combinatorial: they essentially depend on the digraphs, not on the particular space.

Example 3.5.17. Let Γ be a loop. The geometric product $\Gamma \times_g \Gamma$ is a combinatorial model of the torus and its realization is $\mathbf{S}^1 \times \mathbf{S}^1$, as shown in Example 3.5.8. Theorem 3.5.16 asserts that both the space of hypergestures $\Gamma@\Gamma@X$ and the space of gestures $(\Gamma \times_g \Gamma)@X$ are isomorphic to the exponential $X^{\mathbf{S}^1\times\mathbf{S}^1}$, that is, the set of continuous maps from the torus to X equipped with the compact-open topology. ◇

3.6 Gesture functors

Let $\mathcal{C}$ be a category satisfying (L), **Cat** the category of all small categories, and J an object of the latter. Recall from [14, pp. 114-115] the *covariant and contravariant Lim functors*

$$\mathcal{C}^J \xrightarrow{Lim} \mathcal{C} \text{ and } \mathbf{Cat}/\mathcal{C} \xrightarrow{Lim} \mathcal{C}.$$

Given a natural transformation $\tau : F \longrightarrow G$ and a morphism $H : K \longrightarrow J$ (from $F' : K \longrightarrow \mathcal{C}$ to $F : J \longrightarrow \mathcal{C}$), their images $Lim\ \tau$ and $Lim\ H$ are defined, respectively, as sketched in the diagrams

$$\begin{tikzcd}
Lim\ F \arrow[r, "\exists! Lim\ \tau"] \arrow[d, "p_j"'] & Lim\ G \arrow[d, "q_j"] \\
F(j) \arrow[r, "\tau_j"'] & G(j)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & F(H(k)) = F'(k)\ , \\
Lim\ F \arrow[ur, "p_{H(k)}"] \arrow[r, "\exists! Lim\ H"'] & Lim\ F' \arrow[u, "r_k"']
\end{tikzcd}$$

where p_j, q_j, and r_k denote typical legs of the limiting cones from $Lim\ F$, $Lim\ G$, and $Lim\ F'$ respectively.

Also, given a small category $\mathcal{D}$, the categories of elements of presheaves on $\mathcal{D}$ yield the functors

$$\begin{tikzcd}
\widehat{\mathcal{D}} \arrow[r, "\int"] & \mathbf{Cat} & & \int P_1 \arrow[r, "\int \tau"] & \int P_2 \\
P_1 \arrow[r, mapsto] \arrow[d, "\tau"', "\bullet"] & \int P_1 \arrow[d, "\int \tau"] & & (D,p) \arrow[r, mapsto] \arrow[d, "u:D\longrightarrow D'"', "p'\cdot u=p"] & (D,\tau_D(p)) \arrow[d, "u:D\longrightarrow D'"', "\tau_{D'}(p')\cdot u=\tau_D(p)"] \\
P_2 \arrow[r, mapsto] & \int P_2 & & (D',p') \arrow[r, mapsto] & (D',\tau_{D'}(p'))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\widehat{\mathcal{D}} \arrow[r, "\pi"] & \mathbf{Cat}/\mathcal{D} \quad . \\
P_1 \arrow[r, mapsto] \arrow[d, "\tau"', "\bullet"] & \pi_{P_1} : \int P_1 \longrightarrow \mathcal{D} \arrow[d, "\int \tau"] \\
P_2 \arrow[r, mapsto] & \pi_{P_2} : \int P_2 \longrightarrow \mathcal{D}
\end{tikzcd}$$

Let Γ_1 and Γ_2 be semi-simplicial sets, S_1 and S_2 semi-simplicial objects in $\mathcal{C}$, and $\mu : S_2 \longrightarrow S_1$ and $\tau : \Gamma_1 \longrightarrow \Gamma_2$ natural transformations. The morphism $\Gamma_1@\mu : \Gamma_1@S_2 \longrightarrow \Gamma_1@S_1$ obtained by applying the covariant Lim to the natural transformation

$$\begin{tikzcd}
(\int \Gamma_1)^{op} \arrow[r, "\pi_{\Gamma_1}"] & G^{op} \arrow[r, "S_2"] & \mathcal{C}\ , \\
(\int \Gamma_1)^{op} \arrow[r, "\pi_{\Gamma_1}"'] & G^{op} \arrow[r, "S_1"'] & \mathcal{C}
\arrow[from=1-1, to=2-1, phantom, ""]
\arrow[from=1-2, to=2-2, phantom, ""]
\end{tikzcd}$$

with the 2-cells $\bullet\big\downarrow id$ between the arrows π_{Γ_1} and $\bullet\big\downarrow \mu$ between the arrows S_2 and S_1,

and the morphism $\tau@S_2 : \Gamma_2@S_2 \longrightarrow \Gamma_1@S_2$ obtained by applying the contravariant Lim to the functor $\int \tau$ in the commutative diagram

$$\begin{tikzcd}
(\int \Gamma_1)^{op} \arrow[r, "\pi_{\Gamma_1}^{op}"] \arrow[d, "(\int \tau)^{op}"'] & G^{op} \arrow[r, "S_2"] & \mathcal{C}\ , \\
(\int \Gamma_2)^{op} \arrow[ur, "\pi_{\Gamma_2}^{op}"'] & &
\end{tikzcd}$$

define the *covariant and contravariant gesture functors*

$$\Gamma_1@\_ : \mathcal{C}^{G^{op}} \longrightarrow \mathcal{C} \text{ and } \_@S_2 : \widehat{G} \longrightarrow \mathcal{C}$$

respectively. Certainly, they are the composites

$$\mathcal{C}^{G^{op}} \xrightarrow{\mathcal{C}^{\pi\Gamma_1}} \mathcal{C}^{(\int\Gamma_1)^{op}} \xrightarrow{Lim} \mathcal{C}$$

and

$$\widehat{G} \xrightarrow{\pi} \mathbf{Cat}/G \xrightarrow{op} \mathbf{Cat}/G^{op} \xrightarrow{S_2\circ} \mathbf{Cat}/\mathcal{C} \xrightarrow{Lim} \mathcal{C}$$

respectively.

Moreover, they fit together to form a bifunctor, as the following proposition establishes.

Proposition 3.6.1. *Let $\mathcal{C}$ be a category satisfying (L). There is a bifunctor*

$$(\widehat{G})^{op} \times \mathcal{C}^{G^{op}} \xrightarrow{\_@\_} \mathcal{C} \qquad .$$

$$\begin{tikzcd}
(\Gamma_2, S_2) \arrow[r, mapsto] \arrow[d, "{(\tau^{op},\mu)}"'] & \Gamma_2@S_2 \arrow[d, "\tau^{op}@\mu"] & & \Gamma_2@S_2 \arrow[r, "\tau@S_2"] \arrow[d, "\Gamma_2@\mu"'] \arrow[dr, "\tau^{op}@\mu"] & \Gamma_1@S_2 \arrow[d, "\Gamma_1@\mu"] \\
(\Gamma_1, S_1) \arrow[r, mapsto] & \Gamma_1@S_1 & & \Gamma_2@S_1 \arrow[r, "\tau@S_1"'] & \Gamma_1@S_1
\end{tikzcd}$$

Proof. According to Proposition [14, II.3.1], it is necessary to check that the right-hand square above commutes. If $([n], a_n) \in Ob(\int\Gamma_1)$ and

$$p_{(n,a_n)} : \Gamma_1@S_1 \longrightarrow S_1([n]),\ q_{(n,a_n)} : \Gamma_1@S_2 \longrightarrow S_2([n]),$$

$$r_{(n,b_n)} : \Gamma_2@S_2 \longrightarrow S_2([n]),\ \text{and } s_{(n,b_n)} : \Gamma_2@S_1 \longrightarrow S_1([n])$$

are the projections from the limits, then

$$p_{(n,a_n)}(\Gamma_1@\mu)(\tau@S_2) = \mu_{[n]}q_{(n,a_n)}(\tau@S_2) = \mu_{[n]}r_{(n,\tau_{[n]}(a_n))}$$

and

$$p_{(n,a_n)}(\tau@S_1)(\Gamma_2@\mu) = s_{(n,\tau_{[n]}(a_n))}(\Gamma_2@\mu) = \mu_{[n]}r_{(n,\tau_{[n]}(a_n))}.$$

Since this is valid for all objects of $\int\Gamma_1$, $(\Gamma_1@\mu)(\tau@S_2) = (\tau@S_1)(\Gamma_2@\mu)$ and hence $\_@\_$ is a bifunctor. □

Mazzola's gesture bifunctor

Suppose given a functor $T : G \longrightarrow \mathcal{C}$ with all its images exponentiable in $\mathcal{C}$. We thus have a functor

$$S : \mathcal{C} \longrightarrow \mathcal{C}^{G^{op}}$$

that sends each object A of $\mathcal{C}$ to the semi-simplicial object

$$S_A : G^{op} \xrightarrow{T^{op}} \mathcal{C}^{op} \xrightarrow{A^{(\_)}} \mathcal{C}$$

of A (Subsection 3.3.3), and each morphism $f : A \longrightarrow B$ to the natural transformation $\mu(f)$ as in the diagram

$$\begin{tikzcd}
{[n]} \arrow[d, "\alpha"'] & A^{T_n} \arrow[r, "\mu(f)_{[n]}=f^{T_n}"] & B^{T_n} \\
{[m]} & A^{T_m} \arrow[u, "A^{T(\alpha)}"] \arrow[r, "\mu(f)_{[m]}=f^{T_m}"'] & B^{T_m} \arrow[u, "B^{T(\alpha)}"']
\end{tikzcd}.$$

In fact, the latter square commutes since

$$e(B^{T(\alpha)} f^{T_m} \times id) = e(id \times T(\alpha))(f^{T_m} \times id) = e(f^{T_m} \times id)(id \times T(\alpha)) = fe(id \times T(\alpha))$$

and

$$e(f^{T_n} A^{T(\alpha)} \times id) = fe(A^{T(\alpha)} \times id) = fe(id \times T(\alpha)),$$

where e denotes the evaluation map for exponentials. The functoriality is given by that of the exponential functors of the form $(\_)^{T_n}$. In this way, we obtain a bifunctor

$$\_@\_ : (\widehat{G})^{op} \times \mathcal{C} \xrightarrow{id \times S} (\widehat{G})^{op} \times \mathcal{C}^{G^{op}} \xrightarrow{\_@\_} \mathcal{C}$$

according to the notation at the end of Subsection 3.3.3: we write $\Gamma@A$ and $\tau^{op}@f$ instead of $\Gamma@S_A$ and $\tau^{op}@\mu(f)$ respectively.

In particular, by fixing the skeleton Γ, we obtain *Mazzola's gesture functor*:

$$\Gamma@\_ : \mathcal{C} \xrightarrow{S} \mathcal{C}^{G^{op}} \xrightarrow{\Gamma@} \mathcal{C}.$$

3.7 Gestures, realization, and Kan extensions

Let $\mathcal{C}$ be a category, $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ a semi-simplicial set, $S : G^{op} \longrightarrow \mathcal{C}$ a semi-simplicial object in $\mathcal{C}$, and $T : G \longrightarrow \mathcal{C}$ a functor.

By Corollary [14, X.3.2], since G is small and $\mathbf{Set}$ has small hom-sets, if $\mathcal{C}$ satisfies (L), then there is a right adjoint Ran_Γ to

$$\mathcal{C}^\Gamma : \mathcal{C}^{\mathbf{Set}} \longrightarrow \mathcal{C}^{G^{op}};$$

and if $\mathcal{C}$ satisfies (COL), then there is a left adjoint Lan_Γ to

$$\mathcal{C}^{\Gamma^{op}} : \mathcal{C}^{\mathbf{Set}^{op}} \longrightarrow \mathcal{C}^G,$$

where $\Gamma^{op} : G \longrightarrow \mathbf{Set}^{op}$.

In particular, according to Theorem [14, X.3.1],

$Lim\ (\{*\} \downarrow \Gamma \xrightarrow{Q} G^{op} \xrightarrow{S} \mathcal{C})$ is the value at $\{*\}$ of the right Kan extension of S along Γ, and

$Colim\ (\Gamma^{op} \downarrow \{*\} \xrightarrow{P} G \xrightarrow{T} \mathcal{C})$ is the value at $\{*\}$ of the left Kan extension of T along Γ^{op},

where P and Q are the projections. Moreover, since $\{*\} \downarrow \Gamma = (\int \Gamma)^{op}$ and $\Gamma^{op} \downarrow \{*\} = \int \Gamma$, by the definitions of the object of gestures with skeleton Γ (Subsection 3.3.3) and the realization of Γ (Subsection 3.4.2) we have the identities

$$Ran_{\Gamma}(S)(\{*\}) = \Gamma @ S \text{ and } Lan_{\Gamma^{op}}(T)(\{*\}) = \Gamma \otimes_G T = |\Gamma|_T$$

for a category $\mathcal{C}$ satisfying (L) and (COL) respectively.

Now suppose that $\mathcal{C}$ satisfies (H). Note that the category of elements $\int \Gamma$ is isomorphic to the comma category $\mathbf{y} \downarrow \Gamma$ and that $(\int \Gamma)^{op}$ is isomorphic to $\Gamma \downarrow \mathbf{y}^{op}$, where $\mathbf{y} : G \longrightarrow \widehat{G}$ is the Yoneda embedding. Therefore,

$\Gamma @ S = Lim\ (\Gamma \downarrow \mathbf{y}^{op} \xrightarrow{Q} G^{op} \xrightarrow{S} \mathcal{C}) = Ran_{\mathbf{y}^{op}}(S)(\Gamma)$, that is, $\Gamma @ S$ is the value at Γ of the right Kan extension of S along $\mathbf{y}^{op}$, and

$|\Gamma|_T = Colim\ (\mathbf{y} \downarrow \Gamma \xrightarrow{P} G \xrightarrow{T} \mathcal{C}) = Lan_{\mathbf{y}}(T)(\Gamma)$, that is, $|\Gamma|_T$ is the value at Γ of the left Kan extension of T along $\mathbf{y}$.

In this way, *the gesture functor* $\_@S$ *is the right Kan extension of* S *along* $\mathbf{y}^{op}$ and *the realization functor* $|\_|_T$ *is the left Kan extension of* T *along* $\mathbf{y}$, whenever the limit (respectively colimit) above exists.

In the case when all small limits and colimits exist in $\mathcal{C}$, since G is small and $\widehat{G}$ has small hom-sets, the formulas above define the right adjoint $Ran_{\mathbf{y}^{op}}$ and the left adjoint $Lan_{\mathbf{y}}$ to the functors

$$\mathcal{C}^{\mathbf{y}^{op}} : \mathcal{C}^{(\widehat{G})^{op}} \longrightarrow \mathcal{C}^{G^{op}} \text{ and } \mathcal{C}^{\mathbf{y}} : \mathcal{C}^{\widehat{G}} \longrightarrow \mathcal{C}^{G}$$

respectively.

In this way, we can give an additional characterization of the gesture functor $\_@S$ as a free object associated with the construction of a right adjoint to the restriction functor $\mathcal{C}^{\mathbf{y}^{op}}$. This can be expressed by saying that there is a natural transformation $\epsilon : \mathbf{y}^{op}@S \longrightarrow S$ satisfying the usual universal property:

$$\begin{tikzcd}
\mathcal{C} & S & \_@S & \mathbf{y}^{op}@S \arrow[r, "\epsilon"] & S . \\
\left(\widehat{G}\right)^{op} \arrow[u, "\forall T"] & T \circ (\mathbf{y}^{op}) \arrow[u, "\forall \alpha", "\bullet"'] & T \arrow[u, "\exists ! \sigma", "\bullet"'] & T \circ (\mathbf{y}^{op}) \arrow[u, "\sigma_{\mathbf{y}^{op}}", "\bullet"'] \arrow[ru, "\alpha"'] &
\end{tikzcd}$$

This definition has the advantage that it does not rely on limits (though we need the property (H) for the Yoneda embedding); nevertheless, the important cases are when the right Kan extension $\_@S$ is computed via limits, that is, when it is pointwise [14, X.5]—of course, there are examples of non-pointwise Kan extensions, but they tend to be pathological.

3.8 Changing the base category

Let $\mathcal{C}$ be a category and $S : G^{op} \longrightarrow \mathcal{C}$ a semi-simplicial object in $\mathcal{C}$. Suppose that all limits of diagrams defining objects of gestures with skeleta in $\widehat{G}$ with respect to S exist in $\mathcal{C}$, that is, that the right Kan extension $\_@S$ of S along $\mathbf{y}^{op}$ exists. Then we say that the pair $(\mathcal{C}, \_@S)$ is a *gestural structure*. A morphism of gestural structures from $(\mathcal{C}, \_@S)$ to $(\mathcal{D}, \_@R)$ is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ satisfying $FS \cong R$ and preserving all limits of functors of the form

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma} G^{op} \xrightarrow{S} \mathcal{C}$$

for Γ in $\widehat{G}$, that is, all those defining objects of gestures. This means that the image of $\Gamma@S$ under F is (isomorphic to) $\Gamma@R$. In view of these definitions, we have a *category of gestural structures.*

Example 3.8.1. Take $\mathcal{C} = \mathbf{Loc}$, $\mathcal{D} = \mathbf{Top}$, $S : \Delta^{op} \longrightarrow \mathbf{Loc}$ and $R : \Delta^{op} \longrightarrow \mathbf{Top}$ the simplicial objects from Example 3.3.2 with $X = pt(L)$, and F the functor pt (Section 2.4). By Proposition 2.4.1, $FS \cong R$. Moreover, pt preserves all limits since it is a right adjoint (Section 2.4). Thus, for each skeleton Γ we obtain the identity

$$pt(\Gamma@S) = pt(\Gamma@L) \cong \Gamma@pt(L) = \Gamma@R,$$

of which Proposition 2.4.2 is an instance. ◇

We christened the pairs of the form $(\mathcal{C}, \_@S)$ gestural structures in view of the preservation of the operator @ in Proposition 2.4.2, however note that the notion of a morphism of gestural structures is basically that of preservation of Kan extensions [14, X.5].

3.9 Preservation of limits

Let Γ be a skeleton and $\mathcal{C}$ a category satisfying (L). By unraveling the computation of Ran_Γ by limits (Theorem [14, X.3.1], Section 3.7), we note that the covariant gesture functor $\Gamma@\_ : \mathcal{C}^{G^{op}} \longrightarrow \mathcal{C}$ coincides with the composite

$$\mathcal{C}^{G^{op}} \xrightarrow{Ran_\Gamma} \mathcal{C}^{\mathbf{Set}} \xrightarrow{e_{\{*\}}} \mathcal{C},$$

where $e_{\{*\}}$ is the evaluation at $\{*\}$ functor. Therefore, $\Gamma@\_$ preserves limits since Ran_Γ is a right adjoint and limits in $\mathcal{C}^{\mathbf{Set}}$ are computed pointwise. We have just proved the following proposition.

Proposition 3.9.1. *Let Γ be a skeleton and $\mathcal{C}$ a category satisfying (L). The covariant gesture functor $\Gamma@\_$ preserves limits.*

Furthermore, under (COL), we have a strong result.

Proposition 3.9.2. *Let Γ be a skeleton and $\mathcal{C}$ a category satisfying (L) and (COL). The covariant gesture functor $\Gamma@\_$ has a left adjoint.*

Proof. First, recall from Section 3.6 that $\Gamma@\_$ is the composite

$$\mathcal{C}^{G^{op}} \xrightarrow{\mathcal{C}^{\pi_{\Gamma_1}}} \mathcal{C}^{(\int \Gamma_1)^{op}} \xrightarrow{Lim} \mathcal{C}.$$

We will show that both functors have a left adjoint so that the composite of these adjoints is the desired one. Certainly, the diagonal functor from [15, p. 21] is the left adjoint to Lim and, on the other hand, by Corollary [14, X.3.2], $\mathcal{C}^{\pi_\Gamma}$ has a left adjoint since $\mathcal{C}$ is cocomplete and both G and $\int \Gamma$ are small categories. □

For the subsequent discussion we need the following result on preservation of limits for functors between functor categories.

Lemma 3.9.3. *Let $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{C}$ be categories, where $\mathcal{A}$ satisfies (L), and $K : \mathcal{M} \longrightarrow \mathcal{C}$ a functor. The induced functor $\mathcal{A}^K : \mathcal{A}^{\mathcal{C}} \longrightarrow \mathcal{A}^{\mathcal{M}}$ preserves small limits.*

Proof. Let $F : J \longrightarrow \mathcal{A}^{\mathcal{C}}$ be a diagram in $\mathcal{A}^{\mathcal{C}}$ with limit $Lim\ F$. Limits in functor categories are computed pointwise, so for each object C of $\mathcal{C}$, the diagram

$$J \xrightarrow{F} \mathcal{A}^{\mathcal{C}} \xrightarrow{e_C} \mathcal{A},$$

where e_C is the evaluation at C functor, has a limit $Lim\ F(C)$; and for each morphism $f : C \longrightarrow C'$ of $\mathcal{C}$, $Lim\ F(f)$ is the unique making the diagrams of the form

$$\begin{tikzcd}
Lim\ F(C) \arrow[r, "Lim\ F(f)"] \arrow[d, "p_j"] & Lim\ F(C') \arrow[d, "q_j"] \\
F_j(C) \arrow[r, "F_j(f)"] & F_j(C')
\end{tikzcd}$$

commute, where p_j and q_j are the projections from the limits. In particular, this is true if $C = K(M)$ with M object of $\mathcal{M}$ and $f = K(g)$ with $g : M \longrightarrow M'$ morphism of $\mathcal{M}$. This means that $\mathcal{A}^K F$ has a limit in $\mathcal{A}^{\mathcal{M}}$ and this limit coincides with $(Lim\ F)K$. □

Suppose that $\mathcal{C}$ satisfies (L). Now let us examine the case of the functor

$$\Gamma@\_ : \mathcal{C} \xrightarrow{S} \mathcal{C}^{G^{op}} \xrightarrow{\Gamma@} \mathcal{C}$$

from the end of Section 3.6, which is induced by a functor $T : G \longrightarrow \mathcal{C}$ with all its images exponentiable. By Proposition 3.9.1, the problem of whether $\Gamma@\_$ preserves limits reduces to the same problem for S. But S is the composite

$$\mathcal{C} \xrightarrow{exp} \mathcal{C}^{\mathcal{C}_0^{op}} \xrightarrow{\mathcal{C}^{T^{op}}} \mathcal{C}^{G^{op}},$$

where $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ consisting of all exponentiable objects of $\mathcal{C}$, and exp is the functor assigning to each object A of $\mathcal{C}$ the exponential functor $A^{(\_)}$ and to each morphism $f : A \longrightarrow B$ of $\mathcal{C}$ the natural transformation $\mu_f : A^{(\_)} \longrightarrow B^{(\_)}$ with $\mu_f(E) = f^E$ for each object E of $\mathcal{C}_0$. (The argument that shows that *exp* is a functor is similar to that used for S in Section 3.6.) By Lemma 3.9.3, $\mathcal{C}^{T^{op}}$ preserves small limits, and hence we need to determine whether *exp* preserves limits.

Proposition 3.9.4. *Let $\mathcal{C}$ be an arbitrary category. The functor $exp : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{C}_0}$ preserves limits.*

Proof. Let $F : J \longrightarrow \mathcal{C}$ be a diagram and suppose that its limit $Lim\ F$ exists in $\mathcal{C}$. We need to show that the limit of $exp \circ F$ exists in $\mathcal{C}^{\mathcal{C}_0}$ and is equal to $exp(Lim\ F)$.

Since limits in the functor category $\mathcal{C}^{\mathcal{C}_0}$ are computed pointwise, we start by showing that the limit of the functor

$$F^E : J \xrightarrow{F} \mathcal{C} \xrightarrow{(\_)^E} \mathcal{C}$$

exists for each object E of $\mathcal{C}_0$. In fact, this follows from the fact that $(\_)^E$ is a right adjoint. Moreover, $Lim\ F^E$ can be computed as $(Lim\ F)^E = exp \circ Lim\ F(E)$ with limiting cone $\{p_i^E : (Lim\ F)^E \longrightarrow F_i^E\}_{i \in J}$, where $\{p_i : Lim\ F \longrightarrow F_i\}_{i \in J}$ is the limiting cone over F.

It remains to show that the morphism $exp \circ Lim\ F(f) = (Lim\ F)^f : (Lim\ F)^{E'} \longrightarrow (Lim\ F)^E$ induced by a morphism $f : E \longrightarrow E'$ of $\mathcal{C}_0$ corresponds to the unique making the diagrams of the form

$$\begin{tikzcd}
(Lim\ F)^{E'} \arrow[r, "(Lim\ F)^f"] \arrow[d, "p_i^{E'}"] & (Lim\ F)^E \arrow[d, "p_i^E"] \\
F_i^{E'} \arrow[r, "F_i^f"] & F_i^E
\end{tikzcd}$$

commute. This follows from the identities (for e the evaluation maps)

$$e(p_i^E (Lim\ F)^f \times id) = e(p_i^E \times id)((Lim\ F)^f \times id) = p_i e((Lim\ F)^f \times id) = p_i e(id \times f) \text{ and}$$

$$e(F_i^f p_i^{E'} \times id) = e(F_i^f \times id)(p_i^{E'} \times id) = e(id \times f)(p_i^{E'} \times id) = e(p_i^{E'} \times id)(id \times f) = p_i e(id \times f)$$

for each $i \in J$. □

This discussion yields the following theorem.

Theorem 3.9.5. *Let $\mathcal{C}$ be a category satisfying (L), Γ a skeleton, and $T : G \longrightarrow \mathcal{C}$ a functor with all its images exponentiable in $\mathcal{C}$. Mazzola's gesture functor $\Gamma@\_$ preserves limits.*

Moreover, under (COL), we have the following central result, which is a direct consequence of part *i*) of Theorem 3.2.4.

Theorem 3.9.6. *Let $\mathcal{C}$ be a category satisfying (L) and (COL), Γ a skeleton, and $T : G \longrightarrow \mathcal{C}$ a functor with all its images exponentiable in $\mathcal{C}$. Mazzola's gesture functor $\Gamma@\_$ has a left adjoint.*

Proof. Take $E = T\pi_\Gamma$ in Theorem 3.2.4. In this way $Lim\ (\_)^E = \Gamma@\_$ and $\Gamma@\_$ has a left adjoint $Colim\ \_ \times T\pi_\Gamma$. □

Regarding the contravariant gesture functors we have the following result.

Proposition 3.9.7. *Let $\mathcal{C}$ be a category satisfying (H) and $S : G^{op} \longrightarrow \mathcal{C}$ a functor. If the contravariant gesture functor $\_@S$ exists, then it transforms colimits of skeleta into limits in $\mathcal{C}$.*

Proof. This is a direct consequence of the fundamental adjunction for gestures (Section 3.4.4). ☐

3.10 Gestures and ends, the Escher theorem

In this section we introduce the language of ends to give a proof of a general result that has been christened the Escher theorem by Mazzola in the case of spaces and topological categories [27]. The following discussion does not appear in the literature on gesture theory.

Let $\mathcal{C}$ be a category satisfying (H), $T : G \longrightarrow \mathcal{C}$ a functor, and $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ a skeleton. By the expression of left Kan extensions as coends (Theorem [14, X.4.1]) and the Yoneda lemma, we have the following formula for the realization of Γ:

$$|\Gamma|_T = \Gamma \otimes_G T = Lan_{\mathbf{y}} T(\Gamma) = \int^{[n]} Nat(\mathbf{y}([n]), \Gamma) \cdot T_n \cong \int^{[n]} \Gamma_n \cdot T_n$$

whenever the right-hand coend exists. Here $\Gamma_n \cdot T_n$ denotes the copower $\coprod_{a \in \Gamma_n} T_n$.

In the same way, given a functor $S : G^{op} \longrightarrow \mathcal{C}$ and a skeleton Γ, by the expression of right Kan extensions as ends [14, p. 242] and the Yoneda lemma, we have the following formula for the object of gestures $\Gamma@S$:

$$\Gamma@T = Ran_{\mathbf{y}^{op}} S(\Gamma) = \int_{[n]} S_n^{Nat(\mathbf{y}([n]), \Gamma)} \cong \int_{[n]} S_n^{\Gamma_n}$$

whenever the right-hand end exists. Here $S_n^{\Gamma_n}$ denotes the power $\prod_{a \in \Gamma_n} S_n$.

In particular, if C is an object of $\mathcal{C}$ and $T : G \longrightarrow \mathcal{C}$ is a functor with all its images exponentiable in $\mathcal{C}$, then we have the following formula for the object of gestures with body in C:

$$\Gamma@C = \int_{[n]} (C^{T_n})^{\Gamma_n}.$$

Once again, the end is required to exist.

Before the proof of the theorem, we need the following lemma, which states that Mazzola's gesture functor $\Gamma@\_$ commutes with exponentials.

Lemma 3.10.1. *Let $\mathcal{C}$ be a category satisfying (L). If $T : G \longrightarrow \mathcal{C}$ is a functor with all its images exponentiable in $\mathcal{C}$ and E is exponentiable in $\mathcal{E}$, then the functors*

$$\Gamma@(\_)^E \text{ and } (\Gamma@\_)^E$$

are naturally isomorphic.

Proof. For each object C of $\mathcal{C}$, we have the isomorphisms

$$\Gamma@(C^E) = Lim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{(C^E)^{T(\_)}} \mathcal{C}\right) \cong Lim\left(\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{(C^{T(\_)})^E} \mathcal{C}\right) \cong (\Gamma@C)^E$$

by the properties of exponentials and because the functor $(\_)^E$ preserves limits (right adjoint). Moreover, this isomorphism is natural in C since the isomorphisms involved are. ☐

Finally, the Escher theorem is a consequence of Fubini's theorem for ends (corollary in [14, IX.8]).

Theorem 3.10.2 (The Escher Theorem)**.** *Let $\mathcal{C}$ be a category satisfying*[16] *(L) and (H). If $T : G \longrightarrow \mathcal{C}$ is a functor with all its images exponentiable in $\mathcal{C}$, C is an object of $\mathcal{C}$, and Γ, Γ' is a pair of skeleta, then there is a canonical isomorphism*

$$\Gamma@\Gamma'@C \cong \Gamma'@\Gamma@C.$$

Proof. First, consider the functor from $G^{op} \times G \times G^{op} \times G$ to $\mathcal{C}$ that sends the quadruple $([n], [m], [p], [q])$ to the object

$$(C^{T_n \times T_p})^{\Gamma_m \times \Gamma'_q}.$$

Second, note that since powers and exponentials commute and products are commutative (up to natural isomorphism), there are natural isomorphisms

$$(((C^{T_n})^{\Gamma_m})^{T_p})^{\Gamma'_q} \cong (C^{T_n \times T_p})^{\Gamma_m \times \Gamma'_q} \cong (((C^{T_p})^{\Gamma'_q})^{T_n})^{\Gamma_m}.$$

In this way, using the coend formula for the object of gestures and applying Fubini's theorem for ends, we obtain a canonical isomorphism

$$\int_{[n]} \Gamma'@(C^{T_n})^{\Gamma_n} \cong \int_{[p]} \Gamma@(C^{T_p})^{\Gamma'_p}.$$

Finally, using Lemma 3.10.1 and the fact that Mazzola's gesture functor preserves products (Theorem 3.9.5), the preceding isomorphism implies that

$$\Gamma@\Gamma'@C = \int_{[n]} ((\Gamma'@C)^{T_n})^{\Gamma_n} \cong \int_{[p]} ((\Gamma@C)^{T_p})^{\Gamma'_p} = \Gamma'@\Gamma@C.$$

☐

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[16] Note that (L) ensures the existence of all small ends in $\mathcal{C}$ (Corollary [14, IX.5.2]).

3.11 Formulas and diagrams as gestures

3.11.1 Diagrams: gestures in the category of (small) categories

Let **Cat** be the category of all small categories. Consider the semi-cosimplicial category $T : G \longrightarrow \mathbf{Cat}$ that sends $[n]$ to the category associated with the poset $[n] = \{0 < 1 < \cdots < n\}$ and an order-preserving function to the respective functor.

Since **Cat** satisfies (COL) [2, §4], we know that the realization functor $|\_|_T$ exists (Subsection 3.4.2), but we require a more explicit presentation. Recall that $|\_|_T$ is left adjoint to the functor $\mathbf{Cat}(T, \_) : \mathbf{Cat} \longrightarrow \widehat{G}$, that is, to the (semi-)nerve functor N. Indeed, N is defined as suggested in the diagram

$$\begin{tikzcd} G \arrow[r, "N(\mathcal{C})"] & \mathbf{Set} \\ {[n-1]} \arrow[r, mapsto] \arrow[d, "\epsilon_i"] & \mathbf{Cat}([n-1],\mathcal{C}) \\ {[n]} \arrow[r, mapsto] & \mathbf{Cat}([n],\mathcal{C}) \arrow[u, "\_\circ\epsilon_i"'] \end{tikzcd}$$

for ϵ_i the face maps $(0 \leq i \leq n)$. Now by a similar argument to that used to make the explicit computation of the fundamental category of a simplicial set[17] (see Section [13, 1.3] and especially the description before Corollary 1.3.2 there), we have that for each semi-simplicial set Γ, $|\Gamma|_T$ can be assumed to be the quotient of $Path \circ tr_1(\Gamma)$ by the relations[18]

$$(d_0p)(d_2p) \sim d_1p \text{ for } p \in \Gamma_2,$$

where $Path$ is the free category functor[19], left adjoint to the forgetful functor from categories to digraphs, and tr_1 is the truncation functor (Subsection 3.3.2). Following the terminology in [13], we call the category $|\Gamma|_T$ the *fundamental category of* Γ and denote it by $\mu(\Gamma)$. Similarly, we denote the realization functor $|\_|_T$ by μ. In the case when Γ is a digraph, note that $\mu(\Gamma) = Path(\Gamma)$.

On the other hand, given a category $\mathcal{C}$ and a semi-simplicial set Γ, consider the category $\Gamma@\mathcal{C}$ of gestures with skeleton Γ and body in $\mathcal{C}$ (Subsection 3.3.3). Since **Cat** also satisfies (CC), the categories of functors being the exponentials, the reduction to exponentials (Proposition 3.4.5) holds for objects of gestures, that is,

$$\Gamma@\mathcal{C} = \mathcal{C}^{|\Gamma|_T} = \mathcal{C}^{\mu(\Gamma)}.$$

The following result summarizes our discussion.

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[17] The argument is simpler since there are no degeneracies. By the same reason the explicit description of the fundamental category for semi-simplicial sets is simpler.

[18] See the notation in Subsection 3.3.2.

[19] Since the free category is defined in [14] for digraphs of the form (A, V, t, h), so as to identify them with semi-simplicial sets, through this section we follow the conventions $t = \Gamma(\epsilon_1)$ and $h = \Gamma(\epsilon_0)$.

> *Given a category $\mathcal{C}$, the category $\Gamma@\mathcal{C}$ of gestures with skeleton Γ and body in $\mathcal{C}$ (T as above) can be identified with the category of all functors from the fundamental category $\mu(\Gamma)$ of Γ to $\mathcal{C}$.*

Example 3.11.1. Let Γ be the digraph $\bullet x \overset{a}{\to} \bullet y$. Its realization with respect to T is its free category, which is the category with just an arrow plus identities and can be depicted as

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[loop left, "id_x"] & y \arrow[loop right, "id_y"] \end{tikzcd} .$$

Note that this category is isomorphic to the category of the poset $\{0 < 1\}$. Moreover, given a small category $\mathcal{C}$, the category $\Gamma@\mathcal{C}$ is precisely the category of functors from the category of the poset $\{0 < 1\}$ to $\mathcal{C}$. Thus, the objects of $\Gamma@\mathcal{C}$ are essentially morphisms of $\mathcal{C}$ and a morphisms of $\Gamma@\mathcal{C}$ from $f : A \longrightarrow B$ to $g : C \longrightarrow D$ is just a pair of morphisms $(h : A \longrightarrow C, k : B \longrightarrow D)$ such that $kf = gh$. Thus, our category of gestures is the *category of morphisms of* $\mathcal{C}$. ◇

Note also that the functor $s_{\mathcal{C}}$ (Subsection 3.4.6) is precisely $N(\mathcal{C})$. By the theorems in 3.4.6 we have the following result; cf. Example 3.3.1.

> *A gesture with skeleton Γ and body in $\mathcal{C}$ with respect to T (T as above) can be presented in any of the following forms:*
>
> *i) As an object of the category $\Gamma@\mathcal{C}$.*
> *ii) As a natural transformation $\delta : \Gamma \longrightarrow N(\mathcal{C})$.*
> *iii) As a functor from the fundamental category $\mu(\Gamma)$ of Γ to $\mathcal{C}$.*

At this point, it is important to stress the importance of this discussion in mathematical music theory. The concept of abstract gestures embraces the classical notion of gestures on topological spaces (Chapter 1), that is, a serious attempt to model the body's movements, and the notion of diagram of the shape Γ (item *ii*) above and [14, p. 51]), which is one of the basic constructions of category theory. In particular, we can think of diagrams in the small category associated with the affine endomorphisms of the abelian group $\mathbb{Z}_{12}$ (a model of the twelve tones of the chromatic scale), or more generally, diagrams in the category[20] $\mathbf{ModAf}_{\mathbb{Z}}$, which are the basic input to transformational theory and network theory (see [25, §3]). Note that $\mathbf{ModAf}_{\mathbb{Z}}$, the category of abelian groups regarded as $\mathbb{Z}$-modules, is not small, so we cannot apply the preceding results; nevertheless, this problem can easily be fixed by enlarging the universe **Set** so that the families of objects and morphisms of $\mathbf{ModAf}_{\mathbb{Z}}$ be objects of the new universe. But the important point is that in this way the concept of abstract gestures includes both a model of the performer's movements and of the transformational diagrams of musical analysis. Therefore, this concept unifies continuous and discrete aspects of mathematical music theory; see a further discussion in Chapter 6.

---

[20] Let R be a commutative ring with unity. The category $\mathbf{ModAf}_R$ has as objects all R-modules over R and as morphisms affine transformation between them. Given two R-modules M and N, an *affine transformation* from M to N is of the form $f + y$, where $f : M \longrightarrow N$ is an R-homomorphism, y is an element of N, and $(f + y)(x) := f(x) + y$ for each $x \in M$.

Exponential presentation for diagrammatic hypergestures

Now we give a exponential presentation of the category of hypergestures and a reduction of the latter to a category of gestures, in a similar vein that in 3.5.6.

Theorem 3.11.2. *If* Γ_1 *and* Γ_2 *are arbitrary semi-simplicial sets and* $\mathcal{C}$ *is a category, then*

$$\Gamma_1 @ \Gamma_2 @ \mathcal{C} \cong \mathcal{C}^{\mu(\Gamma_1)\times\mu(\Gamma_2)} \cong \mathcal{C}^{\mu(\Gamma_1 \times_g \Gamma_2)} \cong (\Gamma_1 \times_g \Gamma_2) @ \mathcal{C}$$

in **Cat**. *Here,* $\Gamma_1 \times_g \Gamma_2$ *is the geometric product defined in Subsection 3.5.4.*

Proof. The first and third isomorphisms (from left to right) are given by the fact that categories of gestures, as discussed above, are suitable exponentials, and the laws of exponents. The second isomorphism is given by Theorem 3.5.5. In fact, the fundamental category of a product of *simplicial* sets (Section [13, 1.3]) is the product of the respective fundamental categories (Corollary [13, 1.3.3]), and it coincides with the Milnor realization with respect to the functor $\Delta \longrightarrow$ **Cat** described in the last paragraph of [13, p. 10] (the associated realization being our realization $|\_|_T$, which is just μ), so the hypotheses of Theorem 3.5.5 are satisfied. □

Example 3.11.3. Let Γ be a loop digraph. The geometric product $\Gamma \times_g \Gamma$ was computed in Example 3.5.8. On the one hand, since Γ is a digraph, $\mu(\Gamma)$ is the free category on Γ, which can be identified with the free monoid $\langle a \rangle$ on a generator a (which is just $\{1, a, a^2, \dots\}$). Thus, $\mu(\Gamma) \times \mu(\Gamma)$ can be identified with the product of monoids $\langle a \rangle \times \langle a \rangle$, which has two commuting generators, namely $(a, 1)$ and $(1, a)$. On the other hand, the fundamental category of $\Gamma \times_g \Gamma$ is the quotient of the free category on the digraph $(\{b, c, d\}, \{x\}, !, !)$ (three loops around the same vertex) by the relations

$$cd \sim b \text{ and } dc \sim b.$$

Hence, as asserted in Theorem 3.5.5, $\mu(\Gamma) \times \mu(\Gamma)$ is isomorphic to $\mu(\Gamma \times_g \Gamma)$. The isomorphism sends the generators c and d of the latter to $(a, 1)$ and $(1, a)$ respectively.

Moreover, by Theorem 3.11.2, for each small category $\mathcal{C}$, the categories $\Gamma @ \Gamma @ \mathcal{C}$ and $(\Gamma \times_g \Gamma) @ \mathcal{C}$ are isomorphic to the category of functors from the monoid $\langle a \rangle \times \langle a \rangle$ (regarded as a category) to $\mathcal{C}$. ◇

3.11.2 A relation to Mazzola's formulas: gestures on linear categories

Let R be a commutative ring with unity. We define an *R-linear category* to be a category $\mathcal{M}$ enriched over the category of R-modules $\mathbf{Mod}_R$. This means that for each pair A, B of objects of $\mathcal{M}$ the set of morphisms $\mathcal{M}(A, B)$ is an R-module and that for each triple A, B, C of objects of $\mathcal{M}$ the composition $\circ : \mathcal{M}(B, C) \times \mathcal{M}(A, B) \longrightarrow \mathcal{M}(A, C)$ is R-bilinear. Given two R-linear categories $\mathcal{M}$ and $\mathcal{N}$, an R-linear functor from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ such that for each pair A, B of objects of $\mathcal{M}$, the function $F : \mathcal{M}(A, B) \longrightarrow$

$\mathcal{N}(F(A), F(B))$ is an R-homomorphism of modules. In this way, we have *the category* $\mathbf{Cat}_R$ *of all small R-linear categories and R-linear functors between them.* Moreover, if we add the natural transformations between R-linear functors, we observe that $\mathbf{Cat}_R$ is actually a 2-category[21]. On the other hand, an ideal $\mathcal{I}$ of an R-linear category consists of a family of subgroups $\mathcal{I}(A,B) \leqslant \mathcal{M}(A,B)$ indexed by all pairs of objects of $\mathcal{M}$ such that $f \in \mathcal{M}(A,B)$ implies $gfe \in \mathcal{M}(D,C)$ for all $e \in \mathcal{M}(D,A)$ and $g \in \mathcal{M}(B,C)$.

Now let k be a commutative field. We say that a small k-linear category $\mathcal{M}$ is a *spectroid* (as defined in [26]) if the non-invertible morphisms of $\mathcal{M}$ form an ideal $Rad(\mathcal{M})$ of $\mathcal{M}$ and if distinct objects of $\mathcal{M}$ are not isomorphic. It can be shown that the first requirement is equivalent to saying that the k-algebras $\mathcal{M}(A,A)$ are local[22] for all $A \in Ob(\mathcal{M})$.

Gestures on R-linear categories

A construction of free R-linear categories is possible in much the same way that in the case of free modules in $\mathbf{Mod}_R$. In fact, there is a functor $R(\_) : \mathbf{Cat} \longrightarrow \mathbf{Cat}_R$ that is left adjoint to the forgetful functor U from $\mathbf{Cat}_R$ to $\mathbf{Cat}$. Given a small category $\mathcal{C}$, the R-linear category $R\mathcal{C}$ has as objects the objects of $\mathcal{C}$, for each pair of objects A, B the set $R\mathcal{C}(A,B)$ is the free module $R^{\mathcal{C}(A,B)}$ on $\mathcal{C}(A,B)$, and the composition is the linear extension of the composition in $\mathcal{C}$, that is, for each triple A, B, C the composition is defined by means of the R-homomorphism

$$R^{\mathcal{C}(B,C)} \otimes R^{\mathcal{C}(A,B)} \cong R^{\mathcal{C}(B,C)\times\mathcal{C}(A,B)} \longrightarrow R^{\mathcal{C}(A,C)},$$

where the right-hand homomorphism is the free module functor applied to the composition. We thus have the semi-cosimplicial R-category

$$RT : G \xrightarrow{T} \mathbf{Cat} \xrightarrow{R(\_)} \mathbf{Cat}_R$$

where T is the semi-cosimplicial category in Subsection 3.11.1. So as to compute the realization of a semi-simplicial set Γ with respect to RT, recall (Corollary [14, X.3.3]) that $\mu \circ \mathbf{y}$ and T are naturally isomorphic since μ is the left Kan extension of T along $\mathbf{y}$ (Section 3.7) and the Yoneda embedding $\mathbf{y}$ is full and faithful. Therefore, RT is naturally isomorphic to $R(\_) \circ \mu \circ \mathbf{y}$, so by our discussion on realizations from adjoints (Subsection 3.4.2), we can assume that the realization $|\_|_{RT}$ coincides with $R(\_) \circ \mu$.

In contrast to the case of $\mathbf{Cat}$, $\mathbf{Cat}_R$ is not a cartesian closed category[23]. For this reason, the most natural semi-simplicial object that we can associate with an R-linear category $\mathcal{M}$ is *its semi-simplicial set* (Subsection 3.4.6)

$$s_{\mathcal{M}} : G \xrightarrow{RT} \mathbf{Cat}_R \xrightarrow{\mathbf{Cat}_R(\_,\mathcal{M})} \mathbf{Set}.$$

Therefore, according to Theorem 3.4.6 and the adjunction $R(\_) \circ \mu \dashv N \circ U$,

$$\Gamma @ s_{\mathcal{M}} \cong \mathbf{Cat}_R(|\Gamma|, \mathcal{M}) = \mathbf{Cat}_R(R\mu(\Gamma), \mathcal{M}) \cong Nat(\Gamma, NU(\mathcal{M})). \tag{3.3}$$

---

[21] For the definition of 2-category, see [14, XII.3].

[22] That is, local rings: all non-invertible elements form a two-sided ideal.

[23] Exercise!

Thus, a gesture with skeleton Γ and body in $\mathcal{M}$ with respect to RT is just a natural transformation

$$\delta : \Gamma \longrightarrow NU(\mathcal{M}).$$

If Γ is a digraph, then a gesture $\delta : \Gamma \longrightarrow NU(\mathcal{M})$ is simply a morphism of digraphs from Γ to the underlying digraph of $\mathcal{M}$ regarded as a category. This case of digraphs is very important since the resulting gestures are strongly related to formulas in the sense of [26, §7]. The difference is that formulas are defined for spectroids $\mathcal{M}$ and that the arrows of the codomain of a formula are only allowed to be non-invertible morphisms of $\mathcal{M}$. A similar result to bijection 3.3 should express formulas as gestures. The better situation would be when the functor Rad (see [26, p. 40]) from spectroids to digraphs has a left adjoint[24]; in such a case, using a reasoning similar to that above, the associated set of gestures with skeleton Γ and body in a spectroid $\mathcal{M}$ would be isomorphic (as in isomorphism 3.3) to

$$Nat(\Gamma, coex_1(Rad(\mathcal{M}))),$$

where the coextension $coex_1$ is the right adjoint of tr_1 (obtained by right Kan extensions, cf. 3.3.2). Moreover, in the case of the extension $ex_1(\Gamma)$ of a digraph Γ, by the adjunctions $ex_1 \dashv tr_1 \dashv coex_1$, we would have the natural isomorphism

$$Nat(ex_1(\Gamma), coex_1(Rad(\mathcal{M}))) \cong Nat(tr_1(ex_1(\Gamma)), Rad(\mathcal{M})) = Digraph(\Gamma, Rad(\mathcal{M})).$$

This means that a gesture with skeleton Γ would be precisely a morphisms of digraphs

$$\delta : \Gamma \longrightarrow Rad(\mathcal{M}),$$

that is, a Γ-formula in $\mathcal{M}$ according to Mazzola [26, §7]. Now the functor $R(\_) \circ Path$ is a naive candidate for such a left adjoint of Rad, but the images of the functor $R(\_) \circ Path$ are not spectroids in general as discussed in the following example and hence we discard it.

Example 3.11.4. If Γ is a loop (see Example 5.3.3), then the realization $RPath(\Gamma)$ is isomorphic to the polynomial ring $R[x]$, which is never local since $1 - x$ and x are non-invertible with $1 = 1 - x + x$ invertible. This shows that $RPath(\Gamma)$ is not a spectroid.

However, if R is a field k, the quotient algebra $k[x]/\langle x^2\rangle$, which can be identified with the algebra of dual numbers, is local with ideal of non-invertible elements generated by the equivalence class of x. Thus, $k[x]/\langle x^2\rangle$, regarded as the set of morphisms of a category with just an object, is a spectroid.

In this way, a gesture with skeleton a loop and body in the linear category $k[x]/\langle x^2\rangle$ is just the choice of an equivalence class $[a + bx]$ in $k[x]/\langle x^2\rangle$. In contrast, a formula in the spectroid $k[x]/\langle x^2\rangle$ is the choice of a class of the form $[bx]$. For instance, the element $[x]$ is a formula, which can be interpreted as the element x subject to the condition $x^2 = 0$; and hence the relation with the intuitive idea of a formula. Finally, note that the class of the unity of k is a gesture that is not a formula. ◇

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[24] The author does not know whether or not such a left adjoint exists.

3.12 Sheaves of gestures

Now we discuss the relation between gestures and sheaves in Grothendieck's sense. This section intends to show that given a semi-simplicial object $S : G^{op} \longrightarrow \mathcal{C}$ in a suitable category $\mathcal{C}$, the gesture functor $\_@S : \left(\widehat{G}\right)^{op} \longrightarrow \mathcal{C}$ is a sheaf with values in $\mathcal{C}$ (instead of **Set**; see [1, II.6]) with respect to the canonical Grothendieck topology on $\widehat{G}$.

First, we rewrite the characterization of sheaves in terms of equalizers within our language of cotensor products and gestures. Let $(\mathcal{D}, J)$ be a site and $\mathcal{C}$ a category satisfying (L). Note that a presheaf $F : \mathcal{D}^{op} \longrightarrow \mathcal{C}$ with values in $\mathcal{C}$ is a sheaf if and only if for each object D of $\mathcal{D}$ and each covering sieve R in $J(D)$ we have the identity

$$R \pitchfork F = F(D),$$

where the cotensor product $R \pitchfork F$ is defined as (Subsection 3.4.4)

$$Lim\left(\left(\int R\right)^{op} \xrightarrow{\pi_R^{op}} \mathcal{D}^{op} \xrightarrow{F} \mathcal{C}\right).$$

In fact, using the presentation of this limit as an equalizer, the identity $R \pitchfork F = F(D)$ means that the diagram

$$F(D) \longrightarrow \prod_{(f:E\to D)\in R} F(E) \rightrightarrows \prod_{\substack{(f:E\to D)\in R \\ m:E'\to E}} F(E'),$$

with appropriate arrows, is an equalizer [15, p. 122]. Moreover, note that the definition in terms of limits (cotensors) is more general since, if we drop the hypothesis (L), then the limit above can exist even if the equalizer above does not exist; in this way, for the following general discussion we adopt the definition by limits. Note also that the condition $R \pitchfork F = F(D)$ above just says that for each object A of $\mathcal{C}$, every matching family

$$\{x_f : A \longrightarrow F(E) |\ (f : E \to D) \in R\}$$

for R of generalized elements of F has a unique amalgamation $x : A \longrightarrow F(D)$.

Second, we are interested in the case when $\mathcal{D}$ is the category of presheaves $\widehat{G}$, so we need an appropriate Grothendieck topology on it[25]. The topology that we will use is the canonical one. Recall that [15, p. 126] a Grothendieck topology on $\mathcal{D}$ is *subcanonical* if all representable presheaves on D are sheaves, and that [1, II.2.5] the *canonical topology* on a category $\mathcal{D}$ is the greatest subcanonical topology. We can rewrite this definition using the category of elements as follows. A topology J on $\mathcal{D}$ is subcanonical if and only if for each object D of $\mathcal{D}$ and each covering sieve R in $J(D)$ the identity

$$Colim\left(\int R \xrightarrow{\pi_R} \mathcal{D}\right) = D$$

---

[25] There is a subtlety here: the category $\widehat{G}$ is not small, so the definition of topology and site given in [15, p. 110] cannot be used. But this is easily corrected by *changing the universe*, as suggested in [1].

holds. In fact, this equality just says that for each object E of $\mathcal{D}$, every matching family for R of elements of $\mathcal{D}(\_, E)$ has a unique amalgamation. In the case when $\mathcal{D}$ is a category of sheaves on a site, the canonical topology on $\mathcal{D}$ is that having as covering sieves all generated by epimorphic families; to see this, note that epimorphic families define a topology in every elementary topos (check), that every sieve of the canonical topology is an epimorphic family, and recall that, conversely, in a category of sheaves every epimorphic sieve is a sieve of the canonical topology by part 2) of Proposition [1, II.4.3]. In this way, in the case when $\mathcal{D} = \widehat{G}$, the canonical topology is given by all sieves that are epimorphic families; concretely, a sieve R on a skeleton Γ is an epimorphic family if and only if for each $[n]$ and each $a \in \Gamma([n])$ there is a natural transformation $\tau : \Gamma' \longrightarrow \Gamma$ in R and an element $x \in \Gamma'([n])$ such that $\tau_{[n]}(x) = a$.

Now we can prove the main result of this section.

Theorem 3.12.1. *Let $\mathcal{C}$ be a category satisfying (H) and (L), and $S : G^{op} \longrightarrow \mathcal{C}$ a semi-simplicial object in $\mathcal{C}$. The gesture functor $\_@S : \left(\widehat{G}\right)^{op} \longrightarrow \mathcal{C}$ is a sheaf with values in $\mathcal{C}$ for any subcanonical topology on $\widehat{G}$, in particular for the canonical one consisting of all sieves that are epimorphic families.*

Proof. Consider a subcanonical topology J on $\widehat{G}$. According to the characterization of subcanonical topologies above, for each object Γ of $\widehat{G}$ and each sieve R in $J(\Gamma)$ we have the identity

$$Colim \left(\int R \xrightarrow{\pi_R} \widehat{G} \right) = \Gamma.$$

Moreover, since the gesture functor, as a right adjoint, transforms colimits in $\widehat{G}$ into limits in $\mathcal{C}$ (Subsection 3.4.4), by applying $\_@S$ to the identity above, we obtain that

$$Lim \left(\left(\int R \right)^{op} \xrightarrow{\pi_R^{op}} \widehat{G}^{op} \xrightarrow{\_@S} \mathcal{C} \right) = \Gamma @S.$$

According to the characterization of sheaves above, this means that $\_@S$ is a sheaf. □

Chapter 4

Gestures, Topoi, and 2-Categories

The fundamental duality of topoi is that they can be regarded both as generalized spaces and as generalized universes of sets (prologue of [15]). The first approach was developed under the strong influence of Grothendieck (the creator of the concept of topos) and his school and the second one corresponds to Lawvere (the creator of the concept of elementary topos). However, as noted by Johnstone in the preface of [11], the elephant can be appreciated from many other points of view given the richness of the topic.

This chapter is the core of this monograph. Following the initial duality geometry/logic of topoi, it intends to introduce the notion of gestures *in a topos* and, on the other hand, *on a topos*. The first notion approaches topoi as generalized universes of sets, and the second one approaches topoi as generalized spaces. The difference between the two constructions is illustrated in Figure 4.1.

The notion of gestures *in* a particular elementary topos $\mathcal{E}$ is easier to state (it is simply a particular case of the general notion given in Subsection 3.3.3) and leads to a simplification and softening of the notion of gesture: the objects of Mazzola's gestures (end of Subsection 3.3.3) are exponentials. The reason for that simplification is that elementary topoi are cartesian closed categories. However, we need to add the axiom of completeness (equivalent to cocompleteness) so as to guarantee the existence of all objects of gestures for skeleta of infinite type. As an instance of Mazzola's gestures in a topos, we study the case when the gestures are constructed from linear orders, obtaining a further refinement of the above simplification: in this case, hypergestures reduce to gestures (for skeleta of higher dimensions). The notion of gestures in an elementary topos is a very interesting perspective regarding a possible gestural embedding since every locally small category $\mathcal{C}$ can be embedded (Yoneda embedding) in the topos $\mathbf{Set}^{\mathcal{C}^{op}}$ of presheaves on $\mathcal{C}$, which is both complete and cocomplete.

On the other hand, the notion of gestures *on* a Grothendieck topos (stressed as a generalized space by Grothendieck himself) is the natural consequence of the constructions made in the first two chapters: we just have followed the track of continuity from spaces to topoi, by successive generalizations, as shown in Figure 4.2. There, sober spaces simplify topological spaces, in the other cases, each concept is a generalization of that on its left. However, in this stage we face a problem that was invisible in the previous chapters: rather than a plain category, Grothendieck topoi and geometric morphisms (and natural transformations

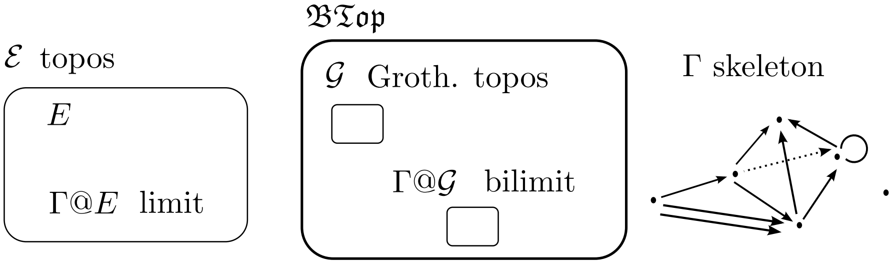

Figure 4.1: The object of gestures $\Gamma@E$ in a topos (left-hand box) and the Grothendieck topos $\Gamma@\mathcal{G}$ of gestures on a topos (middle box).

cat.	ob.	topological spaces	sober spaces	locales	sites	Grothendieck topoi	Lawvere topoi
	mor.	cont. functions	=inv. image maps	morphisms of locales	morphisms of sites	geometric morphisms	

Figure 4.2: The extended realm of topology.

between them) form a 2-category and the constructions of limits and exponentials obey this 2-categorical nature, a plain categorical formulation being impossible. It is important to stress that this notion of gestures is not so easily extended to a general elementary topos since the existence of suitable finite limits and exponentials in the 2-category of all topoi cannot be guaranteed. In contrast, we can ensure the existence of the Grothendieck topos of gestures $\Gamma@\mathcal{G}$, which is a 2-categorical *bilimit*, whenever the skeleton Γ is finite, since the 2-category $\mathfrak{BTop}$ of all Grothendieck topoi has all finite bilimits.

4.1 Elementary gestures: gestures in Lawvere topoi

We start this chapter with the discussion on gestures in an elementary topos. In particular we make a study of the cases of topoi of presheaves, topoi of sheaves on a site, and the case when the gestures come from a standard simplex functor based on a linear order.

4.1.1 An infinitary condition

Recall that an elementary topos (or Lawvere topos) $\mathcal{E}$ is a cartesian closed category having finite limits and a subobject classifier [15, p. 167]. In this way, given a semi-simplicial object $S : G^{op} \longrightarrow \mathcal{E}$ in a topos $\mathcal{E}$, the single definition of an elementary topos does not guarantee the existence of the limit defining the object $\Gamma@S$ when Γ is an infinite semi-simplicial set, though $\Gamma@S$ exists when Γ is finite. In particular, since $\mathcal{E}$ has all exponentials, given an object E of $\mathcal{E}$ we ensure the existence of the object $\Gamma@E$ of gestures with skeleton Γ and body in E (see the end of Subsection 3.3.3) whenever Γ is finite.

Thus, we need to add infinitary conditions to $\mathcal{E}$. A first possibility is to decree that $\mathcal{E}$ has all small limits so that all objects of gestures of the form $\Gamma@S$ exist. Moreover, this implies, in the same way that the existence of finite limits in $\mathcal{E}$ implies the existence of finite colimits in $\mathcal{E}$ (Corollary [15, IV.5.4], result due to Paré), that $\mathcal{E}$ is small-cocomplete. Another possibility is to start with the hypothesis that $\mathcal{E}$ is cocomplete. In this case, the product $\prod_{i\in\mathcal{I}} E_i$ of an arbitrary family of objects of $\mathcal{E}$ can be computed as the image P of $h : Y \longrightarrow B$ ($Y := \coprod_{i\in\mathcal{I}} E_i$, $B := \coprod_{i\in\mathcal{I}} \mathbf{1}$, $h := \coprod_{i\in\mathcal{I}} !$) under the right adjoint to $\_ \times B : \mathcal{E} \longrightarrow \mathcal{E}/B$ (constructed in the proof of Theorem [15, I.9.4]); in fact, we have natural isomorphisms

$$\mathcal{E}(E,P) \cong \mathcal{E}/B(E \times B \xrightarrow{\pi_2} B, h) \cong \prod_{i\in\mathcal{I}} \mathcal{E}(E, E_i)$$

where the left-hand one is given by the mentioned adjunction and the right-hand one sends a morphism f (from π_2 to h) to the family $(f_i)_{i\in\mathcal{I}}$ obtained by means of the pullbacks of the form

$$\begin{tikzcd}
\coprod_{i\in\mathcal{I}} E \cong E \times B \arrow[r, "f"] & \coprod_{i\in\mathcal{I}} E_i \ , \\
E \arrow[u, "r_i"] \arrow[r, "f_i"'] & E_i \arrow[u, "q_i"']
\end{tikzcd}$$

where r_i and q_i denote typical coproduct injections. To see the existence of this pullback we need the following lemma.

Lemma 4.1.1. *In an elementary topos $\mathcal{E}$ every diagram of the form*

$$\begin{tikzcd}
\coprod_{i\in\mathcal{I}} X_i \arrow[r, "\coprod_{i\in\mathcal{I}} g_i"] & \coprod_{i\in\mathcal{I}} Y_i \ , \\
X_j \arrow[u, "r_j"] \arrow[r, "g_j"'] & Y_j \arrow[u, "q_j"']
\end{tikzcd}$$

where q_j and r_j are the jth coproduct injections, is a pullback.

Proof. Define $Y = \coprod_{i\in\mathcal{I}} Y_i$. Since the pullback functor $q_j^* : \mathcal{E}/Y \longrightarrow \mathcal{E}/Y_j$ from [15, pp. 192-193], as a left adjoint, preserves colimits, and since coproducts in a topos are disjoint[1],

$$q_j^*(\coprod_{i\in\mathcal{I}} g_i) \cong \coprod_{i\in\mathcal{I}} q_j^*((\coprod_{i\in\mathcal{I}} g_i)r_i) = \coprod_{i\in\mathcal{I}} q_j^*(q_i g_i) = g_j.$$

□

Thus, the existence of the desired pullback follows from this lemma and the fact that $hf = \pi_2$. The existence of products in $\mathcal{E}$ implies that $\mathcal{E}$ is small-complete since it is already finitely complete. We have just reminded the fact that *an elementary topos is small-cocomplete if and only if it is small-complete.* In fact, there is not a long distance between small-cocomplete (small-complete) topoi and Grothendieck topoi, the only difference is that the former need not have a small family of generators (proposition in [15, p. 593]). Finally, note that our infinitary condition (cocompleteness or completeness) ensures that the Heyting algebras of subobjects of a given object are complete (joins of subobjects are computed via coproducts and images), so they are locales, and we are in the realm of topological intuition.

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[1] See the definition in [15, p. 574] and the proof of the proposition in [15, p. 593].

4.1.2 Gestures in topoi of presheaves

It is worth studying the case of topos of presheaves since in this instance it is possible to give explicit computations of limits and exponentials in a reasonable way.

Let $\mathcal{C}$ be a category with small hom-sets, $\mathbf{Set}^{\mathcal{C}^{op}}$ its associated category of presheaves, and $S : G^{op} \longrightarrow \mathbf{Set}^{\mathcal{C}^{op}}$ a functor. Given any skeleton Γ, by the computation of limits via products and equalizers [14, V.2], the object of gestures $\Gamma@S$ exists and can be computed as a suitable subpresheaf of the product of presheaves

$$\prod_{n\in\mathbb{N}} \prod_{a\in\Gamma([n])} S_n.$$

In particular, if $\Gamma = (A, V, t, h)$ ($t = \Gamma(\epsilon_0)$, $h = \Gamma(\epsilon_1)$) is a digraph, then $\Gamma@S$ is the subpresheaf of the product of presheaves

$$S_0^V \times S_1^A$$

defined by

$$\Gamma@S([i]) = \{\{s_x\}_{x\in V} \cup \{s_a\}_{x\in A} \in S_0^V[i] \times S_1^A[i] |\ S(\epsilon_0)_i(s_a) = s_{t(a)} \text{ and } S(\epsilon_1)_i(s_a) = s_{h(a)}\}$$

for $i = 0, 1$.

Now consider the case when $T : G \longrightarrow \mathbf{Set}^{\mathcal{C}^{op}}$ is a functor. Given a presheaf P on $\mathcal{C}$ and a skeleton Γ, the object $\Gamma@P$, which is by definition $\Gamma@S_P$ (see Subsection 3.3.3), can also be computed since each exponential of the form P^{T_n} is defined (up to isomorphism) by

$$E^{T_n}([k]) = Nat(\mathbf{y}([k]) \times T_n, P).$$

Then one applies the preceding computation of the limit defining the object of gestures when $S = S_P$. On the other hand, according to the reduction to exponentials for cartesian closed categories (Proposition 3.4.5), $\Gamma@P$ can also be computed as the exponential $P^{|\Gamma|_T}$, where $|\Gamma|_T$ is the realization (Subsection 3.4.2) of Γ with respect to T, which is a suitable colimit computed pointwise in $\mathbf{Set}^{\mathcal{C}^{op}}$.

Example 4.1.2. The category $\widehat{G}$ of semi-simplicial sets is the topos of presheaves on the semi-simplicial category G defined in Subsection 3.3.1. This topos is very important in our discussion on gestures in topoi because, beyond its interpretation as a universe of sets, it can be regarded as a very abstract category of spaces; certainly, semi-simplicial sets (and simplicial ones) are combinatorial models for spaces: to obtain a space from a semi-simplicial set, just apply the realization functor with respect to the standard simplex functor in **Top**. In this way, there is an analogy between gestures on topological spaces and gestures on semi-simplicial objects, and this analogy justifies to a great extent our definition of gestures inside a topos. For this reason it is worth commenting the following definition of gestures in $\widehat{G}$.

We will be concerned with Mazzola's gestures (Subsection 3.3.3) with respect to the most natural functor from G to $\widehat{G}$: the Yoneda embedding $\mathbf{y}$. Let Γ be a skeleton (that is, a semi-simplicial set) and Σ a semi-simplicial set. In this case, the semi-simplicial set of gestures $\Gamma @ \Sigma$ coincides with the exponential $\Sigma^{|\Gamma|_{\mathbf{y}}}$. But the realization $|\Gamma|_{\mathbf{y}}$ is precisely the tensor product $\Gamma \otimes_G \mathbf{y}$, which is equal to Γ since this tensor product is the presentation of Γ as a colimit of representable functors. Thus, $\Gamma @ \Sigma = \Sigma^{\Gamma}$, and hence exponentials of semi-simplicial sets are objects of gestures.

Nevertheless, as we observed in Subsection 3.5, simplicial sets have a better behavior as combinatorial models of spaces than semi-simplicial sets. For this reason, it is to be studied whether or not the theory of gestures in the category of simplicial sets has some application in the theory of topological gestures (likely using the geometric realization functor). In this case, we can consider Mazzola's gestures respect to the restriction to G of the Yoneda embedding $\mathbf{y} : \Delta \longrightarrow \widehat{\Delta}$. As before, if P is a simplicial set, the simplicial set $\Gamma @ P$ coincides with $P^{|\Gamma|_{\mathbf{y}}}$. Moreover, by Proposition 3.5.1,

$$|\Gamma|_{\mathbf{y}} \cong {}_M|L\Gamma| = L\Gamma \otimes_{\Delta} \mathbf{y} = L\Gamma,$$

and therefore $\Gamma @ P \cong P^{L\Gamma}$, where L is as in Subsection 3.5.1. ◇

4.1.3 Gestures in topoi of sheaves

In the case of topoi of sheaves it is also possible to give explicit computations.

Let $(\mathcal{C}, J)$ be a (small) site [15, p. 110], $Sh(\mathcal{C}, J)$ its associated Grothendieck topos of sheaves, and $S : G^{op} \longrightarrow Sh(\mathcal{C}, J)$ a functor. Given any skeleton Γ, by the computation of limits in categories of sheaves [15, p. 134], the object of gestures $\Gamma @ S$ can be computed as a limit of presheaves, regarding all sheaves involved as presheaves. In particular, the presentation given in the preceding subsection in the case when Γ is a digraph remains valid.

When $T : G \longrightarrow Sh(\mathcal{C}, J)$ is a functor, given a sheaf F on $(\mathcal{C}, J)$ and a skeleton Γ, the object $\Gamma @ F$ can be computed since each exponential of the form F^{T_n} is defined as in the preceding subsection, regarding F and T_n as presheaves. Also, in this case, according to the computation of colimits in $Sh(\mathcal{C}, J)$ [15, p. 135] the exponential reduction of $\Gamma @ F$ takes the form

$$F^{\mathbf{a}(|\Gamma|_T)},$$

where $|\Gamma|_T$ is the realization of Γ regarding T as a functor to presheaves, and $\mathbf{a}$ is the associated sheaf functor. This exponential is computed as an exponential of presheaves.

4.1.4 Gestures from linear orders

Let $\mathcal{E}$ be an elementary topos and (I, R, b, t) an order in $\mathcal{E}$ as exposed in [15, VIII.8]. We can construct a standard simplex functor

$$\Delta_I^{(\_)} : \Delta \longrightarrow \mathcal{E}$$

by defining Δ_I^n to be the intersection of subobjects of I^n

$$I^k \times R \times I^{n-k-2} \rightarrowtail I^k \times I^2 \times I^{n-k-2}$$

for $k = 0, \dots, n-2$. This means that for every morphism f, with $f = (f_1, \dots, f_n) : E \longrightarrow I^n$, the condition $f_1 \leq \cdots \leq f_n$ is satisfied if and only if f factors through Δ_I^n. Now given an order-preserving map $\alpha : [n] \longrightarrow [m]$, there is a morphism

$$I^\alpha : I^n \longrightarrow I^m$$

between cubes, which is the unique with the property that $\pi_j I^\alpha = I^n \xrightarrow{!} T \xrightarrow{b} I$ if $\overline{\alpha}(j) = 0$, $\pi_j I^\alpha : I^n \xrightarrow{!} T \xrightarrow{t} I$ if $\overline{\alpha}(j) = n+1$, and $\pi_j I^\alpha = \pi_{\overline{\alpha}(j)}$ if $0 < \overline{\alpha}(j) < n+1$, where

$$\overline{\alpha}(j) := n + 1 - \widetilde{\alpha}(m + 1 - j)$$

for $j \in [m+1]$ (note that $\overline{\alpha}$ is a sort of reflection of $\widetilde{\alpha}$). Here, $\widetilde{\alpha}$ is defined as in [15, VII.7, p. 456]. In this way, since I^α preserves the order on Δ_I^n (because $\overline{\alpha}$ does), we have the factorization

$$\begin{tikzcd}
I^n \arrow[r, "I^\alpha"] & I^m . \\
\Delta_I^n \arrow[u, tail] \arrow[r, "\exists ! \Delta_I^\alpha"] & \Delta_I^m \arrow[u, tail]
\end{tikzcd}$$

Note that the presentation of the standard simplex as an analogue of the presentation of the standard simplex functor in **Top** in terms of barycentric coordinates is the key fact that allows to do this generalization: in that setting, the components of I^α correspond to projections, the top element, or the bottom element; which can be codified categorically.

Therefore, for each object E of $\mathcal{E}$, there is a simplicial object in $\mathcal{E}$ given by the composite

$$\Delta \xrightarrow{\Delta_I^{(\_)}} \mathcal{E} \xrightarrow{E^{(\_)}} \mathcal{E}.$$

This object is well defined since every object is exponentiable in $\mathcal{E}$ (in particular, the standard n-simplex Δ_I^n is). By Proposition 3.4.5, to have a plentiful supply of Mazzola's gestures with excellent properties in $\mathcal{E}$, we may use the small cocompleteness of $\mathcal{E}$. With the definitions above, given a semi-simplicial set $\Gamma : G^{op} \longrightarrow \mathbf{Set}$, we thus have the object $\Gamma@E$ of gestures with skeleton Γ and body in E (see the end of Subsection 3.3.3). Its properties are summarized as follows:

i) $\Gamma@E = E^{|\Gamma|}$ (Proposition 3.4.5).

ii) *The Escher theorem* (see Proposition [27, 2.4] and Section 3.10). There is a canonical isomorphism
$$\Gamma_1@\Gamma_2@E \cong \Gamma_2@\Gamma_1@E,$$
for each pair of skeleta Γ_1, Γ_2. This result is a direct consequence of i), the properties of exponentials, and the commutativity of products. More generally, an analogous result holds for any permutation of the indices of a finite list $\Gamma_1, \dots, \Gamma_n$ of skeleta.

iii) *Hypergestures reduce to plain gestures.* If $\mathcal{E}$ has small hom-sets, then there is an isomorphism

$$\Gamma_1 @ \Gamma_2 @ E \cong E^{|\Gamma_1| \times |\Gamma_2|} \cong E^{|\Gamma_1 \times_g \Gamma_2|} \cong (\Gamma_1 \times_g \Gamma_2) @ E,$$

for each pair of skeleta Γ_1, Γ_2, where $\times_g$ is the geometric product from Section 3.5.1. The reason is that the geometric realization functor preserves finite limits (that is, it is left exact; see [15, VIII.8, p. 463]), so the isomorphism follows from i) and Theorem 3.5.5.

Remark 4.1.3. Note that the formula in [15, VIII.7, p. 456] for $\Delta^\alpha(t)_j$ does not send, in general, the ith vertex of Δ^n to the $\alpha(i)$th vertex of Δ^m. For example, if $\alpha : [2] \longrightarrow [2]$ is defined by $\alpha(0) = 1$, $\alpha(1) = 1$, and $\alpha(2) = 2$, then

$$\begin{tikzcd}
\alpha : & & & \widetilde{\alpha} : & & & \overline{\alpha} : & & . \\
0 \arrow[dr, mapsto] & 0 & & 0 \arrow[r, mapsto] & 0 & & 0 \arrow[r, mapsto] & 0 & \\
1 \arrow[r, mapsto] & 1 & & 1 \arrow[ur, mapsto] & 1 & & 1 \arrow[r, mapsto] & 1 & \\
2 \arrow[r, mapsto] & 2 & & 2 \arrow[r, mapsto] & 2 & & 2 \arrow[dr, mapsto] & 2 & \\
 & & & 3 \arrow[r, mapsto] & 3 & & 3 \arrow[r, mapsto] & 3 &
\end{tikzcd}$$

In this way, the pair $t = (t_1, t_2)$ is sent to $\Delta^\alpha(t) = (0, t_2)$ according to the formula in [15], so the 0th vertex $(0, 0)$ is fixed by Δ^α, though we should have $\Delta^\alpha(v_0) = v_{\alpha(0)} = v_1 = (0, 1)$. For this reason, we need to replace $\widetilde{\alpha}$ by the reflection $\overline{\alpha} : [m+1] \longrightarrow [n+1]$. Certainly, with the formula

$$\Delta^\alpha(t)_j = \begin{cases} 0 & \text{if } \overline{\alpha}(j) = 0, \\ 1 & \text{if } \overline{\alpha}(j) = n+1, \\ t_{\overline{\alpha}(j)} & \text{if } 0 < \overline{\alpha}(j) < n+1; \end{cases}$$

we have the identities $\Delta^\alpha(t_1, t_2) = (t_1, 1)$, $\Delta^\alpha(v_0) = \Delta^\alpha(0, 0) = (0, 1) = v_1 = v_{\alpha(0)}$, $\Delta^\alpha(v_1) = \Delta^\alpha(0, 1) = (0, 1) = v_1 = v_{\alpha(1)}$, and $\Delta^\alpha(v_2) = \Delta^\alpha(1, 1) = (1, 1) = v_2 = v_{\alpha(2)}$.

Example 4.1.4. Consider the topos $\widehat{\Delta}$ of simplicial sets. The representable simplicial set $\mathbf{y}([1])$ is a linear order in $\widehat{\Delta}$ since for each $n \in \mathbb{N}$, $\Delta([n], [1])$ can be identified with the set of vertices of the standard $(n+1)$-simplex, which is linearly ordered, the constant sequence with value 1 (respectively 0) being the top (respectively bottom) element (see Example [15, VIII.8.1]). Moreover, according to Lemma [15, VIII.8.6], the standard simplex functor induced by this linear order is naturally isomorphic to the Yoneda embedding $\mathbf{y} : \Delta \longrightarrow \widehat{\Delta}$. Thus, the object of Mazzola's gestures $\Gamma @ P$ with respect to the standard simplex functor induced by this linear order is essentially that computed in Example 4.1.2, that is, the exponential $P^{L\Gamma}$. ◇

4.2 Gestures on Grothendieck topoi

In this section we define and give some lines of study of Grothendieck topoi of gestures. In particular, we study the (pseudo)functors arising from this notion and some of their basic properties. The new feature of this section is that now we need to consider the 2-categorical aspects of the objects involved. This 2-categorical language has been introduced by the very need of defining gestures on Grothendieck topoi and is absolutely necessary to give a correct definition. However, several results are given for general 2-categories, though always giving the particular statement in the case when the 2-category is that of Grothendieck topoi, since the treatment of the general case is simpler. Despite this, we have decided not to write an independent section (or even a chapter) with the general theory of gestures in 2-categories since in this monograph we are interested in this language in so far as it is useful for the study of gestures on Grothendieck topoi. The main reference used in this monograph for 2-category theory is [6] and we follow its notation. The complementary references are [4] and [11].

4.2.1 Sites and Grothendieck topoi

Given a small category $\mathcal{C}$, recall that a site ([15, p. 112], [1, II.1.1.5]) on $\mathcal{C}$ is a pair $(\mathcal{C}, J)$, where J is a Grothendieck topology on $\mathcal{C}$. A *Grothendieck topos* is a category equivalent to a category $Sh(\mathcal{C}, J)$ of sheaves on a site $(\mathcal{C}, J)$. Given two topoi $\mathcal{E}$ and $\mathcal{F}$, *a geometric morphism* from $\mathcal{E}$ to $\mathcal{F}$ is a pair (f^*, f_*), where $f^* : \mathcal{F} \longrightarrow \mathcal{E}$ is left adjoint to f_* and f_* is a left exact functor. In other words, these geometric morphisms can be identified with the functors $f^* : \mathcal{F} \longrightarrow \mathcal{E}$ that preserve finite limits and small colimits (Corollaire [1, IV.1.6]). Note the *fundamental* analogy with the inverse image maps of locales and topological spaces. Certainly, the *notion of topos*, as conceived by Grothendieck, was inspired by the notion of topological space, as commented in [1, IV.0.3]. Moreover, according to Grothendieck's school, *the object of topology is the study of topoi*[2] [1, IV.0.4]. Rather than a category, Grothendieck topoi and geometric morphisms between them (plus natural transformations between geometric morphisms) form a 2-category ([1, IV.3.3.2], [15, p. 352]), which we denote by $\mathfrak{BTop}$ following the notation used in [11].

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be sites and $\phi : \mathcal{D} \longrightarrow \mathcal{C}$ a functor. According to [1, IV.4.9.1], we say that ϕ is *continuous* if the functor

$$\_ \circ \phi^{op} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}}$$

restricts to a functor f_* between the respective categories of sheaves $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$. In this case, according to the proof of Proposition [1, III.1.2], the induced functor f_* has a left adjoint, namely the composite

$$f^* : \widetilde{\mathcal{D}} \hookrightarrow \widehat{\mathcal{D}} \xrightarrow{\_\otimes_{\mathcal{D}} \mathbf{y}\phi} \widehat{\mathcal{C}} \xrightarrow{\mathbf{a}} \widetilde{\mathcal{C}},$$

where $\mathbf{y} : \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ is the Yoneda embedding and $\mathbf{a}$ is the associated sheaf functor [15, III.5]. In this way, we say that $\phi : \mathcal{D} \longrightarrow \mathcal{C}$ is a *morphism of sites* from $(\mathcal{C}, J)$ to $(\mathcal{D}, K)$ if it is continuous and f^* is left exact, in which case there is an induced geometric morphism $(f^*, f_*) : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$. Also, given two morphisms $\phi, \psi : \mathcal{D} \longrightarrow \mathcal{C}$ from $(\mathcal{C}, J)$ to $(\mathcal{D}, K)$, we

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[2] Though this conception has changed since the SGA4 years. In fact, there are other interesting perspectives on topology, also due to Grothendieck, like moderate spaces and geometry of shapes.

define[3] a morphisms from ϕ to ψ to be a natural transformation from ϕ to ψ. These data yield the 2-category **Site** of small sites with 1-cells being morphisms of sites and 2-cells being the morphisms of morphisms of sites discussed above.

Given a morphism of locales $f : L \longrightarrow M$, note that by Theorem [15, VII.10.2], the inverse image f^* is a morphism of sites from L to M, where L and M are equipped with their canonical topologies. Moreover, given two morphisms of locales $f, g : L \longrightarrow M$, there is a morphism of morphisms of sites from f^* to g^* if and only if $f^*(b) \leq g^*(b)$ for each $b \in M$.

These 2-categorical remarks have profound implications for topos theory, so they must be taken into account if we want to construct gestures along the lines of the preceding chapter.

4.2.2 Gestures with body in a Grothendieck topos: motivation

For instance, so as to define the Grothendieck topos of gestures with skeleton a semi-simplicial set $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ and body in a Grothendieck topos $\mathcal{E}$ (see the end of Subsection 3.3.3), we should start with a *standard simplex functor* $\mathcal{T} : \Delta \longrightarrow \mathfrak{BTop}$. Then we need to mimic the construction of the *power functor* associated with the body of the gestures and ensure the existence of the *limit* of a suitable diagram defining the topos of gestures. However, the emphasized terms deserve a further clarification since they must be defined in a different way in the 2-categorical case for structural reasons. Next we give a motivation of these new definitions.

The standard simplex pseudofunctor for topoi and cosimplicial topoi

As to the standard simplex functor, a first choice, following our classical approach based on simplices, would be the composite of the standard simplex functor from Δ to the category of locales with the functor Sh from the latter to $\mathfrak{BTop}$. But we must be careful since the correct definition of Sh is that of a *pseudofunctor* from the 2-category **Site** of sites to $\mathfrak{BTop}$. The pseudofunctor Sh sends a morphism $\phi : (\mathcal{C}, J) \longrightarrow (\mathcal{D}, K)$ of sites to its induced geometric morphism. We can illustrate the definition of the pseudofunctor Sh on 2-cells with the diagram

$$\begin{array}{ccc}
\mathbf{Site}((\mathcal{C}, J), (\mathcal{D}, K)) & \longrightarrow & \mathfrak{BTop}(\widetilde{C}, \widetilde{D}) \\
\tau : \phi \Rightarrow \psi & \longmapsto & Sh(\tau) : \mathbf{a}(\_ \otimes_{\mathcal{D}} \mathbf{y}\phi) \overset{\bullet}{\to} \mathbf{a}(\_ \otimes_{\mathcal{D}} \mathbf{y}\psi) \\
\tau : \phi \overset{\bullet}{\to} \psi : \mathcal{D} \longrightarrow \mathcal{C} & & Sh(\tau)_G = \mathbf{a}Colim\ (\mathbf{y} * \tau * \pi_G) : \mathbf{a}(G \otimes_{\mathcal{D}} \mathbf{y}\phi) \overset{\bullet}{\to} \mathbf{a}(G \otimes_{\mathcal{D}} \mathbf{y}\psi)
\end{array} .$$

Note that by the choice of the 1-cells and 2-cells of **Site**, Sh is covariant in 1-cells and 2-cells. In the case when the 2-cells in $\mathfrak{BTop}$ are taken to be the natural transformations between

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[3] This presentation is different from that given in SGA4. There, a morphism from ϕ to ψ is defined to be a natural transformation from ψ to ϕ. The reason is that in SGA4 the definition of the 2-cells of $\mathfrak{BTop}$ is given in terms of the direct images of geometric morphisms so that the functor Sh is covariant. Instead, following the work made on locales, we define the 2-cells of $\mathfrak{BTop}$ in terms of the inverse images. This, besides our definition of **Site**, renders Sh covariant (see the definition below).

the *direct images*, we can illustrate the definition on 2-cells with the diagram

$$\begin{array}{ccc} \mathbf{Site}((\mathcal{C},J),(\mathcal{D},K)) & \longrightarrow & \mathfrak{BTop}(\widetilde{C},\widetilde{D}) \\ \tau:\phi\Rightarrow\psi & \longmapsto & Sh(\tau):\_\circ\psi^{op}\overset{\bullet}{\to}\_\circ\phi^{op} \\ \tau:\phi\overset{\bullet}{\to}\psi:\mathcal{D}\longrightarrow\mathcal{C} & & Sh(\tau)_F=F\circ\tau^{op}:F\circ\psi^{op}\overset{\bullet}{\to}F\circ\phi^{op} \end{array}.$$

In this case the pseudofunctor Sh is covariant in 1-cells and contravariant in 2-cells. Note that Sh is not a strict 2-functor: if $\psi:(\mathcal{B},H)\longrightarrow(\mathcal{C},J)$ and $\phi:(\mathcal{C},J)\longrightarrow(\mathcal{D},K)$ are morphisms of sites, then

$$Sh(\psi\phi)^*=\mathbf{a}(\_\otimes_{\mathcal{D}}\mathbf{y}\psi\phi)\text{ and }Sh(\psi)^*Sh(\phi)^*=\mathbf{a}(\_\otimes_{\mathcal{C}}\mathbf{y}\psi)\mathbf{a}(\_\otimes_{\mathcal{D}}\mathbf{y}\phi),$$

so $Sh(\psi\phi)^*$ is naturally isomorphic[4] to $Sh(\psi)^*Sh(\phi)^*$ rather than equal to it. However, Sh can be regarded as a *unital*[5] pseudofunctor. In fact, first note that for each site $(\mathcal{C},J)$, $Sh(id_{\mathcal{C}})_*=\_\circ id^{op}=id_{\widetilde{\mathcal{C}}}$. Second, $Sh(id_{\mathcal{C}})^*=\mathbf{a}(\_\otimes_{\mathcal{C}}\mathbf{y})$, so $Sh(id_{\mathcal{C}})^*$ depends on the particular choice of a tensor product (colimit) $F\otimes_{\mathcal{C}}\mathbf{y}$ for each sheaf F on $(\mathcal{C},J)$. In this way, we can take the colimits $F\otimes_{\mathcal{C}}\mathbf{y}$ and $\mathbf{a}(F\otimes_{\mathcal{C}}\mathbf{y})$ to be F, and hence $Sh(id_{\mathcal{C}})^*=id_{\widetilde{\mathcal{C}}}$.

Now regard Δ as a 2-category with just the identities as 2-cells. In this way, the correct choice of the desired standard simplex functor is the *pseudofunctor*

$$\mathcal{T}:\Delta\xrightarrow{\mathcal{O}(\Delta^{(\_)})}\mathfrak{Loc}\xrightarrow{Sh}\mathfrak{BTop},$$

where $\mathcal{O}(\Delta^{(\_)})$ is the pseudofunctor (sending identity 2-cells to identity 2-cells) associated with the standard simplex functor $\mathcal{O}(\Delta^{(\_)}):\Delta\longrightarrow\mathbf{Loc}$. This pseudofunctor $\mathcal{T}$ may be called *the standard simplex pseudofunctor in the 2-category of Grothendieck topoi*. This explains why a cosimplicial topos must be defined to be a pseudofunctor

$$\mathcal{T}:\Delta\longrightarrow\mathfrak{BTop}.$$

Finite products of Grothendieck topoi

In the case of categories, the construction of the power functor $C^{(\_)}$ (of an object C) relies on the exponential functors, which in turn rely on the construction of the product functors (and hence on products). But the right definition of products in $\mathfrak{BTop}$ is not the same given for plain categories. To explain this, let us examine the elementary case of *the final Grothendieck topos*, which in turn is useful for the discussion of points of a given Grothendieck topos. The final topos is the category **Set** of sets, but we cannot expect the universal property to be the same as in the categorical case; the correct definition is expressed by saying that for each Grothendieck topos $\mathcal{E}$, the category $\mathfrak{BTop}(\mathcal{E},\mathbf{Set})$ is *equivalent* to the final category (having just an object and an arrow). See [1, IV.4.3] for details.

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[4] The natural isomorphism follows from the fact that $Sh(\phi)_*Sh(\psi)_*=(\_\circ\phi^{op})(\_\circ\psi^{op})=\_\circ(\psi\phi)^{op}=Sh(\psi\phi)_*$, which implies that $Sh(\psi)^*Sh(\phi)^*$ and $Sh(\psi\phi)^*$ are left adjoint functors to the same functor and hence are naturally isomorphic.

[5] A pseudofunctor $F:\mathscr{K}\longrightarrow\mathscr{K}'$ between 2-categories is said to be unital if for each object A of $\mathscr{K}$ $F(id_A)=id_{F(A)}$ and the respective coherence 2-cell $\delta:id_{F(A)}\longrightarrow F(id_A)$ is the identity.

In a similar way, given two Grothendieck topoi $\mathcal{E}$ and $\mathcal{F}$, their product $\mathcal{E}\times\mathcal{F}$ always exists and their universal property is expressed by saying that there are two geometric morphisms $\pi_1 : \mathcal{E}\times\mathcal{F} \longrightarrow \mathcal{E}$ and $\pi_2 : \mathcal{E}\times\mathcal{F} \longrightarrow \mathcal{F}$ such that for each Grothendieck topos $\mathcal{G}$, the canonical functor

$$\begin{array}{ccc} \mathfrak{BTop}(\mathcal{G},\mathcal{E}\times\mathcal{F}) & \longrightarrow & \mathfrak{BTop}(\mathcal{G},\mathcal{E})\times\mathfrak{BTop}(\mathcal{G},\mathcal{F}) \\ \alpha : g \Rightarrow h & \longmapsto & (Id_{\pi_1} * \alpha, Id_{\pi_2} * \alpha) : (\pi_1\circ g, \pi_2\circ g) \Rightarrow (\pi_1\circ h, \pi_2\circ h) \end{array}$$

is an equivalence of categories.

In the case when $\mathcal{E} = Sh(\mathcal{C}, J)$ and $\mathcal{F} = Sh(\mathcal{D}, K)$, the product $\mathcal{E}\times\mathcal{F}$ is the topos $Sh(\mathcal{C}\times\mathcal{D}, J\times K)$, where $J\times K$ is the least Grothendieck topology such that for each pair $(C, D) \in Ob(\mathcal{C}\times\mathcal{D})$ the sieves of the form $R\times S$, where $R \in J(C)$ and $S \in K(D)$, are covering sieves of (C, D). The projections

$$\pi_1 : \widetilde{\mathcal{C}\times\mathcal{D}} \longrightarrow \widetilde{\mathcal{C}},\ \pi_2 : \widetilde{\mathcal{C}\times\mathcal{D}} \longrightarrow \widetilde{\mathcal{D}},$$

whose inverse images π_1^* and π_2^* are the functors

$$\mathbf{a}(\_\circ p_1^{op}) : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}\times\mathcal{D}},\ \mathbf{a}(\_\circ p_2^{op}) : \widetilde{\mathcal{D}} \longrightarrow \widetilde{\mathcal{C}\times\mathcal{D}},$$

are induced by the product projections $p_1 : \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{C}$ and $p_2 : \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{D}$, which satisfy the covering lifting property (cocontinuity); see Theorem [15, VII.10.5]. For each $\mathcal{G}$, the equivalence that ensures that $\widetilde{\mathcal{C}\times\mathcal{D}}$ is the product $\widetilde{\mathcal{C}}\times\widetilde{\mathcal{D}}$ is obtained from the equivalence

$$\mathbf{ConFlat}(\mathcal{C}\times\mathcal{D},\mathcal{G}) \approx \mathbf{ConFlat}(\mathcal{C},\mathcal{G})\times\mathbf{ConFlat}(\mathcal{D},\mathcal{G})$$

that sends each continuous flat functor $F : \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{G}$ to the pair (A_F, B_F) where A_F and B_F are the continuous flat functors defined by $A_F(C) = Colim\ F(C,\_)$ and $B_F(D) = Colim\ F(\_, D)$, and that, on the other direction, sends a pair (A, B) of continuous flat functors (from $\mathcal{C}$ and $\mathcal{D}$ respectively) to the continuous flat functor $A\times B$ defined by $(A\times B)(C, D) = A(C)\times B(D)$ for each pair $(C, D) \in Ob(\mathcal{C}\times\mathcal{D})$. The former direction corresponds to the canonical functor (induced by the projections π_1 and π_2) under appropriate equivalences between geometric morphisms and continuous flat functors (Corollary [15, VII.7.4]).

Exponentials of Grothendieck topoi and the power pseudofunctor

From the universal property of the product we deduce, for each Grothendieck topos $\mathcal{E}$, the product *pseudofunctor* $\_\times\mathcal{E} : \mathfrak{BTop} \longrightarrow \mathfrak{BTop}$. This gives rise to the notion of *exponentiable Grothendieck topos*: we say that $\mathcal{E}$ is an exponentiable topos if the product pseudofunctor $\_\times\mathcal{E}$ *has* a right *biadjoint*, following the terminology of [6, Ch. 9] (or pseudo-adjoint, according to the terminology of [12, §4]). This means that for each Grothendieck topos $\mathcal{F}$, the (contravariant) pseudofunctor $\mathfrak{BTop}(\_\times\mathcal{E},\mathcal{F})$ is *representable*, that is, there exist a Grothendieck topos $\mathcal{F}^{\mathcal{E}}$ and a pseudonatural transformation

$$\Omega : \mathfrak{BTop}(\_,\mathcal{F}^{\mathcal{E}}) \Rightarrow \mathfrak{BTop}(\_\times\mathcal{E},\mathcal{F}) : \mathfrak{BTop}^{op} \longrightarrow \mathfrak{Cat}$$

such that for each Grothendieck topos $\mathcal{G}$, the component

$$\Omega_{\mathcal{G}} : \mathfrak{BTop}(\mathcal{G}, \mathcal{F}^{\mathcal{E}}) \longrightarrow \mathfrak{BTop}(\mathcal{G} \times \mathcal{E}, \mathcal{F})$$

is an equivalence of categories (note that $\mathfrak{BTop}(\_, \mathcal{F}^{\mathcal{E}})$ is a strict 2-functor). Further, the pseudonatural transformation Ω can be chosen in a canonical way; in fact, for each $\mathcal{G}$, there is a functor

$$\begin{array}{ccc} \mathfrak{BTop}(\mathcal{G}, \mathcal{F}^{\mathcal{E}}) & \longrightarrow & \mathfrak{BTop}(\mathcal{G} \times \mathcal{E}, \mathcal{F}) \\ \alpha : f \Rightarrow g & \longmapsto & \epsilon_{\mathcal{F}} * (\alpha \times \mathcal{E}) : \epsilon_{\mathcal{F}} \circ (f \times \mathcal{E}) \Rightarrow \epsilon_{\mathcal{F}} \circ (g \times \mathcal{E}) \end{array},$$

where $\epsilon_{\mathcal{F}} : \mathcal{F}^{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathcal{F}$ is known as a *biuniversal arrow*. For this reason, as in the categorical case, exponentials $\mathcal{F}^{\mathcal{E}}$ of Grothendieck topoi can be characterized in terms of biuniversal arrows (see Theorem [6, 9.18]), so by the (bi)universal property of exponentials, we deduce for each exponentiable Grothendieck topos $\mathcal{E}$, an *exponential pseudofunctor* $(\_)^{\mathcal{E}} : \mathfrak{BTop} \longrightarrow \mathfrak{BTop}$. A very remarkable result due to Johnstone and Joyal (Theorem [12, 4.10]) establishes that a Grothendieck topos is exponentiable if and only if it is a continuous category, analogously to Hyland's result that a locale is exponentiable if and only if it is a continuous lattice. A category $\mathcal{E}$ is said to be *continuous* [12, p. 264] if it has small filtered colimits and the filtered colimit functor

$$Colim : Ind - \mathcal{E} \longrightarrow \mathcal{E}$$

has a left adjoint, where $Ind - \mathcal{E}$ is the category of all small filtered diagrams in $\mathcal{E}$ and suitable morphisms as defined in [12, p. 260]. Note that, in particular, all the localic topoi of the form $Sh(\mathcal{O}(\Delta^n))$ for $n \in \mathbb{N}$ are exponentiable since the Δ^n are compact Hausdorff spaces (see the lines before Example 3.3.2 and Corollary [12, 5.10]), so the standard simplex pseudofunctor

$$Sh(\mathcal{O}(\Delta^{(\_)})) : \Delta \longrightarrow \mathfrak{BTop}$$

has all its images exponentiable in $\mathfrak{BTop}$.

Now we discuss the power pseudofunctors. Denote by $\mathfrak{BTop}_0$ the full sub-2-category of $\mathfrak{BTop}$ consisting of all Grothendieck topoi that are exponentiable, that is, all Grothendieck topoi that are continuous categories as defined in the paragraph above. For each object $\mathcal{F}$ of $\mathfrak{BTop}$, from all the possible exponentials of the form $\mathcal{F}^{\mathcal{E}}$, we deduce the *power pseudofunctor*

$$\mathcal{F}^{(\_)} : \mathfrak{BTop}_0^{op} \longrightarrow \mathfrak{BTop}$$

which is contravariant in 1-cells and covariant in 2-cells[6]. Note that it is possible to compose the power pseudofunctor $\mathcal{F}^{(\_)}$ with $\mathcal{T}$ so that we have an analogue of the functor defining Mazzola's gestures (end of Subsection 3.3.3).

For the sake of completeness, we describe the data of a typical power pseudofunctor. Let $\mathscr{K}$ be a 2-category with binary products, $\mathscr{K}_0$ the full sub-2-category of all exponentiable

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[6] In fact, given a 2-category $\mathscr{K}$, we define $\mathscr{K}^{op}$ to be the 2-category obtained by reversing the direction of the 1-cells of $\mathscr{K}$; the direction of the 2-cells is preserved.

objects in $\mathscr{K}$ in the sense discussed above, and A an object of $\mathscr{K}$. First, as mentioned before, it sends an exponentiable object E of $\mathscr{K}$ to A^E. Second, given two exponentiable objects E and E', we have the functor

$$\begin{array}{rcl} \mathscr{K}(E,E') & \longrightarrow & \mathscr{K}(A^{E'},A^E) \\ \alpha : f \Rightarrow g & \longmapsto & A^\alpha : A^f \Rightarrow A^g \end{array},$$

where A^f and A^α are defined according to the diagrams

$$\begin{tikzcd}
A^{E'} \arrow[d, "\exists A^f"'] & A^{E'} \times E \arrow[r, "A^{E'}\times f"] \arrow[d, "A^f\times E"'] & A^{E'} \times E', \arrow[d, "\epsilon"] & A^f \arrow[d, Rightarrow, "\exists! A^\alpha"'] & \epsilon \circ (A^f \times E) \arrow[r, Rightarrow, "\mu"] \arrow[d, Rightarrow, "\epsilon*(A^\alpha\times E)"'] & \epsilon \circ (A^{E'} \times f) \arrow[d, Rightarrow, "\epsilon*(A^{E'}\times\alpha)"] & , \\
A^E & A^E \times E \arrow[r, "\epsilon"'] \arrow[ur, Rightarrow, "\mu"'] & A & A^g & \epsilon \circ (A^g \times E) \arrow[r, Rightarrow, "\mu"'] & \epsilon \circ (A^{E'} \times g) &
\end{tikzcd}$$

which are obtained from the fact that ϵ is a biuniversal arrow. Finally, we describe the coherence 2-cells. Given two 1-cells $f : E \longrightarrow E'$ and $g : E' \longrightarrow E''$, there is a 2-cell $\gamma_{g,f} : A^f \circ A^g \Rightarrow A^{g\circ f}$ obtained according to the diagram (relative to a biuniversal arrow again)

$$\begin{tikzcd}
A^f \circ A^g \arrow[d, Rightarrow, "\exists!\gamma_{g,f}"'] & \epsilon \circ (A^f \circ A^g \times E) \arrow[r, Rightarrow, "\epsilon*\gamma^{-1}_{A^f,A^g}"] \arrow[d, Rightarrow, "\epsilon*(\gamma_{g,f}\times E)"'] & \epsilon \circ (A^f \times E) \circ (A^g \times E), \arrow[d, Rightarrow] \\
A^{g\circ f} & \epsilon \circ (A^{g\circ f} \times E) \arrow[r, Rightarrow, "\mu"'] & \epsilon \circ (A^{E''} \times g \circ f)
\end{tikzcd}$$

where the right-hand 2-cell is the composite

$$\epsilon \circ (A^f \times E) \circ (A^g \times E) \xRightarrow{\mu * Id} \epsilon \circ (A^{E'} \times f) \circ (A^g \times E) \xRightarrow{Id_\epsilon * \lambda} \epsilon \circ (A^g \times f)$$

$$\xRightarrow{Id_\epsilon * \lambda'^{-1}} \epsilon \circ (A^g \times E') \circ (A^{E''} \times f) \xRightarrow{\mu * Id} \epsilon \circ (A^{E''} \times g) \circ (A^{E''} \times f) \xRightarrow{Id_\epsilon * \lambda''} \epsilon \circ (A^{E''} \times g \circ f),$$

the 2-cells λ, λ', and λ'' being appropriate coherence 2-cells of the product pseudofunctor $(\_) \times (\_)$ in two arguments. Also, for each E there is a 2-cell $\delta : id_{A^E} \Rightarrow A^{id_E}$, which is the unique making the triangle

$$\begin{tikzcd}
\epsilon \circ (id_{A^E} \times E) \arrow[d, Rightarrow, "\epsilon*(\delta\times E)"'] \arrow[dr, equal] & \\
\epsilon \circ (A^{id_E} \times E) \arrow[r, Rightarrow, "\mu"'] & \epsilon \circ (A^E \times id_E)
\end{tikzcd}$$

commute.

Bilimits

Finally, the right definition of limit, that is, that of (conical) *bilimit* of a pseudofunctor runs as follows (Definition [6, 3.13]). Let J be a small category regarded as a 2-category with just identity 2-cells and $\mathscr{K}$ a 2-category. Also, consider for each object C of $\mathscr{K}$ the constant strict 2-functor $K_C : J \longrightarrow \mathscr{K}$. Given a pseudofunctor $F : J \longrightarrow \mathscr{K}$, a *bilimit* of F consists of an object L of $\mathscr{K}$ and a pseudonatural transformation $\pi : K_L \Rightarrow F$ such that for each object C of $\mathscr{K}$ the functor

$$\begin{array}{ccc} \mathscr{K}(C, L) & \longrightarrow & PsNat(K_C, F) \\ \alpha : G \Rightarrow H & \longmapsto & \pi * K_\alpha : \pi \circ K_G \rightsquigarrow \pi \circ K_H \end{array}$$

is an equivalence of categories. Here $PsNat(K_C, F)$ is the category whose objects are pseudo natural transformations from K_C to F and whose morphism are all modifications between them, K_G is the 2-natural transformation from K_C to K_L naturally induced by G, and the modification $\pi * K_\alpha$ consists for each object i of J of the 2-cell of $\mathscr{K}$

$$Id_{\pi_i} * \alpha : \pi_i \circ G \Rightarrow \pi_i \circ H.$$

In this situation, the analogue of the 1-categorical fact that limits can be computed via products and equalizers is slightly different. The bilimit of the pseudofunctor F can be computed via products and *descent objects* (see [20] for the definition). Descent objects play the role that equalizers played in the 1-categorical case and they are *weighted bilimits* (see definitions [6, 3.24] and [11, B.1.1.3]) rather than plain bilimits. More specifically, the bilimit of F can be computed as the descent object of an appropriate diagram X of the form

$$\begin{tikzcd} \prod\limits_{i \in J} F_i \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \prod\limits_{i \xrightarrow{\alpha} j} F_j \arrow[l, "s" description] \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & \prod\limits_{i \xrightarrow{\alpha} j \xrightarrow{\beta} k} F_k \, . \end{tikzcd}$$

This computation is a simple consequence of Street's computation of weighted bilimits in terms of products, cotensors, and descent objects, which is commented in [20, pp. 55-56]. The reason why equalizers are not enough is that the axioms for the coherence 2-cells of pseudocones over F cannot be codified by means of the (bi)equalizer of the pair (d_0, d_1) from the left-hand of the diagram above, rather they are codified in a suitable way thanks to the extra relations of the descent object; in fact, for each object C of $\mathscr{K}$ there is an equivalence of categories between the category of descent $Desc(\mathscr{K}(C, X))$ and the category $PsNat(K_C, F)$ of pseudocones over F with vertex C.

Further, this computation of the bilimit of F can be refined since descent objects can be obtained via (bi)equalizers and (bi)equifiers of invertible 2-cells (see [20, p. 55]) and the latter can be computed as suitable biidentifiers (as defined in [20, p. 53]) of automorphic 2-cells. In turn, biidentifiers can be computed via cotensors and (bi)equalizers (see [20, p. 54] and Example [11, B.1.1.4 (f)]). Thus, note that *weighted bilimits are necessary in the computation of the (conical) bilimit of F, whether in the form of descent objects, equifiers, or cotensors. Hence, weighted bilimits are necessary in 2-category theory even from the point of view of (conical) bilimits.*

As usual, if the category J has finitely many vertices and finitely many arrows, then we say that the bilimit of F is finite, whenever it exists. According to the preceding discussion, in this case, the bilimit of F can be obtained via finite products, (bi)equalizers, and (bi)equifiers. Equivalently, finite products and (bi)equalizers can be replaced by finite products and (bi)pullbacks, or by (bi)pullbacks and a terminal object. In this way the 2-category $\mathfrak{BTop}$ (which is just $\mathfrak{BTop}/\mathbf{Set}$) has finite bilimits. In fact, as we observed before it has finite products, it has bi(pullbacks) by the second paragraph in [11, p. 415], and it has equifiers by Corollary [11, B.4.1.7].

The definition of bicolimit is given dually. According to [16, §2], the 2-category $\mathfrak{BTop}$ has all weighted bicolimits[7], and in particular, it has all bicolimits.

4.2.3 Definition of gestures on Grothendieck topoi

We thus have all the ingredients for the definition of gestures with body in a Grothendieck topos, which is one of the main purposes of this monograph.

Definition 4.2.1. *Let* $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ *be a semi-simplicial set and* $\mathcal{E}$ *a Grothendieck topos. The topos of gestures with skeleton* Γ *and body in* $\mathcal{E}$*, denoted by* $\Gamma@\mathcal{E}$*, is the bilimit of the pseudofunctor*

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G^{op} \xrightarrow{\mathcal{T}^{op}} \mathfrak{BTop}_0^{op} \xrightarrow{\mathcal{E}^{(\_)}} \mathfrak{BTop},$$

whenever it exists*. Here* π_Γ *is the induced strict* 2*-functor between the respective strict* 2*-categories with trivial* 2*-cells,* $\mathcal{T}$ *is the standard simplex pseudofunctor in the category of Grothendieck topoi (Subsection 4.2.2), and the operator* $(\_)^{op}$ *refers to the process of inversion of* 1*-cells.*

In this way, if $\mathscr{K}$ is a 2-category, it makes sense to define a *semi-simplicial object in* $\mathscr{K}$ as a pseudofunctor $S : G^{op} \longrightarrow \mathscr{K}$ and to give the following more flexible definition.

Definition 4.2.2. *Let* $\mathscr{K}$ *be a 2-category,* $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ *a semi-simplicial set, and* $S : G^{op} \longrightarrow \mathscr{K}$ *a semi-simplicial object in* $\mathscr{K}$*. The object of gestures with skeleton* Γ *with respect to* S*, denoted by* $\Gamma@S$*, is the bilimit of the pseudofunctor*

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G^{op} \xrightarrow{S} \mathscr{K},$$

whenever it exists*.*

Therefore, we have a notion of gestures in any 2-category.

Remark 4.2.3 (Existence of Grothendieck topoi of gestures for finite skeleta)**.** It is important to stress that in the 2-category $\mathfrak{BTop}$ of Grothendieck topoi, we can only ensure the existence of the object of gestures $\Gamma@S$ if the skeleton is finite. We say that a semi-simplicial set Γ is *finite* if $\int \Gamma$ is a finite category, that is, if Γ is truncated ($\Gamma([n])$ vanishes for each $n \geq p$ for some fixed p) and each value of the form $\Gamma([n])$ is a finite set. In this way, since $\mathfrak{BTop}$ has finite bilimits (see the discussion in Subsection 4.2.2), if Γ is finite, then $\Gamma@S$

---

[7] Dualize the definition of weighted bilimit so as to obtain that of weighted bicolimit. Of course, bi(co)limits are particular instances of weighted bi(co)limits.

exists. In particular, if Γ is finite and $\mathcal{E}$ is a Grothendieck topos, then the object of gestures $\Gamma @ \mathcal{E}$ from Definition 4.2.1 exists. If we apply this to the case when Γ is a digraph (A, V, t, h), then we can ensure the existence of $\Gamma @ S$ and $\Gamma @ \mathcal{E}$ whenever A and V are finite.

Finally, we note that as in the case of categories, we can construct objects of gestures from cosimplicial objects by abstracting Definition 4.2.1 giving rise to the following definition of *Mazzola's gestures.*

Definition 4.2.4. *Given a cosimplicial object $\mathcal{T} : \Delta \longrightarrow \mathscr{K}$ in a 2-category (that is, a pseudofunctor) with all its images exponentiable in $\mathscr{K}$ in the sense discussed in the motivation above, a skeleton Γ, and an object C of $\mathscr{K}$, we define the object of gestures with skeleton Γ and body in C, denoted by $\Gamma @ C$, to be $\Gamma @ S_C$ (Definition 4.2.2), where S_C is the composite*

$$\Delta^{op} \xrightarrow{\mathcal{T}^{op}} \mathscr{K}_0^{op} \xrightarrow{C^{(\_)}} \mathscr{K}.$$

Once again, this construction implies that of *hypergestures*: if Γ' is another skeleton, we can construct the object $\Gamma' @ \Gamma @ C$, and so on, depending on the existence of suitable bilimits in $\mathscr{K}$.

4.2.4 Gestures and weighted bilimits

In this subsection we express the bilimit in Definition 4.2.2, which defines objects of gestures in a 2-category, as a weighted bilimit (Definition [6, 3.24]). This fact is the 2-categorical analogue of the fundamental adjunction in Subsection 3.4.4.

Let $F : J \longrightarrow \mathscr{K}$ be a pseudofunctor from a small category J to a 2-category $\mathscr{K}$. Let $W : J^{op} \longrightarrow$ **Set** be a functor regarded as a strict 2-functor $W : J^{op} \longrightarrow$ **Cat** by considering each set as a discrete category[8]. The main result of this section is that if the bicolimit $W \otimes_J F$ of the pseudofunctor

$$\int W \xrightarrow{\pi_W} J \xrightarrow{F} \mathscr{K}$$

exists, then there is an equivalence of categories

$$PsNat(W, \mathscr{K}(F(\_), C)) \approx \mathscr{K}(W \otimes_J F, C)$$

that is (pseudo)natural in C. Note that this result is a sort of generalization of the hom-tensor adjunction from Subsection 3.4.1, so for that reason we use the tensor $\otimes$ notation. The proof of this result can be made in much the same way that in the categorical proof (Theorem [15, I.5.2]); we next give a sketch of the proof. It basically relies on the following proposition, which says that the pseudonatural transformations from W to $\mathscr{K}(F(\_), C)$ can be identified with pseudo-cocones over $F \circ \pi_W$ with vertex C.

---

[8] The discrete category of a set X has as objects the elements of X and their morphisms are just the identities.

Proposition 4.2.5. *With the preceding notation, there is an isomorphism of categories*

$$\lambda_C : PsNat(W, \mathscr{K}(F(\_), C)) \longrightarrow PsNat(F \circ \pi_W, K_C)$$

that is 2*-natural in* C*.*

Proof. The plan of this proof is simply to unravel the definitions of the objects and the morphisms of the left-hand category above so as to figure out that they correspond to the objects and the morphisms of the right-hand category.

First, consider a pseudonatural transformation $\psi : W \Longrightarrow \mathscr{K}(F(\_), C)$. It consists for each $i \in J$ of a functor $\psi_i : W_i \longrightarrow \mathscr{K}(F_i, C)$, which is essentially a function since W_i is a discrete category. In this way, ψ can be identified with a correspondence that assigns to each object (i, a) of $\int W$ the morphism $\psi_i(a) : F_i \longrightarrow C$. Also, ψ is equipped with a natural transformation τ_α as in the diagram

$$\begin{tikzcd}
W_i \arrow[r, "\psi_i"] & \mathscr{K}(F_i, C) \\
W_j \arrow[u, "W_\alpha"] \arrow[r, "\psi_j"] & \mathscr{K}(F_j, C) \arrow[u, "\_\circ F_\alpha"'] \arrow[ul, Rightarrow, "\tau_\alpha"']
\end{tikzcd}$$

for each morphism $\alpha : i \longrightarrow j$ of J, which is essentially a correspondence that assigns to each morphism $\alpha : (i, a) \longrightarrow (j, b)$ of $\int W$ (which satisfies $a = W_\alpha(b)$) the 2-cell $\tau_\alpha(b) : \psi_j(b) \circ F_\alpha \Longrightarrow \psi_i(a)$ of $\mathscr{K}$. Further, the pseudonatural transformation axioms exactly say that (check) we have constructed a pseudonatural transformation from $F \circ \pi_W$ to K_C consisting for each (i, a) of the morphism $\psi_i(a) : F_i \longrightarrow C$ and with coherence 2-cells of the form $\tau_\alpha(b)^{-1}$, as shown in the diagram

$$\begin{tikzcd}
(i,a) \arrow[dd, "\alpha"] & F_i \arrow[dr, "\psi_i(a)"] \arrow[dd, "F_\alpha"'] & \\
& & C \arrow[dl, Rightarrow, shorten=12pt, "\tau_\alpha(b)^{-1}"'] \\
(j,b) & F_j \arrow[ur, "\psi_j(b)"'] &
\end{tikzcd} .$$

Second, a morphism from ψ to ϕ is just a modification $\Theta : \psi \rightsquigarrow \phi : W \Longrightarrow (F(\_), C)$, which consists for each $i \in J$ of a natural transformation $\Theta_i : \psi_i \longrightarrow \phi_i$, which in turn is the same as giving for each $a \in W_i$ a 2-cell $\Theta_i(a) : \psi_i(a) \Longrightarrow \phi_i(a)$ of $\mathscr{K}$ (no commutativity is required since W_i is discrete). Moreover, the modification axiom for Θ exactly states that the correspondence that assigns to each object (i, a) the 2-cell $\Theta_i(a) : \psi_i(a) \Longrightarrow \phi_i(a)$ is a modification between the pseudonatural transformations (from $F \circ \pi_W$ to K_C) induced by ψ and ϕ.

The two preceding paragraphs justify the existence of the claimed isomorphism of categories. It remains to show that this isomorphism is 2-natural in C, that is, that for each 2-cell $\sigma : f \Longrightarrow g : C \longrightarrow D$ of $\mathscr{K}$, we have the equality

$$PsNat(F \circ \pi_W, K_\sigma) * Id_{\lambda_C} = Id_{\lambda_D} * PsNat(W, \mathscr{K}(F(\_), \sigma))$$

of natural transformations between functors from $PsNat(W, \mathscr{K}(F(\_), C))$ to $PsNat(F \circ \pi_W, K_D)$. In fact, both these natural transformations consist for each pseudonatural transformation $\psi : W \Longrightarrow \mathscr{K}(F(\_), C)$ of a modification from $F \circ \pi_W$ to K_D, which in turn consists for each object (i, a) of $\int W$ of the 2-cell

$$\sigma * Id_{\psi_i(a)} : f \circ \psi_i(a) \Longrightarrow g \circ \psi_i(a) : F_i \longrightarrow D$$

of $\mathscr{K}$. We omit these verifications. □

Now, following our discussion, if the bicolimit $W \otimes_J F$ exists, then, by the definition of bicolimit (dual of that of bilimit, see Subsection 4.2.2), there exists an equivalence of categories

$$PsNat(F \circ \pi_W, K(C)) \approx \mathscr{K}(W \otimes_J F, C)$$

pseudonatural in C. Thus, according to the preceding proposition, there is an equivalence of categories

$$\mathscr{K}(W \otimes_J F, C) \approx PsNat(W, \mathscr{K}(F(\_), C))$$

pseudonatural in C. This implies (dual of Definition [6, 3.24]) that *the bicolimit* $W \otimes_J F$ *is the* W*-weighted bicolimit of* F.

Dually (cf. Subsection 3.4.4), consider a pseudofunctor $F : J^{op} \longrightarrow \mathscr{K}$, where J is a small category and $\mathscr{K}$ is a 2-category. Let $W : J^{op} \longrightarrow \mathbf{Set}$ be a functor regarded as a strict 2-functor $W : J^{op} \longrightarrow \mathbf{Cat}$ again. Moreover, suppose that the bilimit of the pseudofunctor

$$\left(\int W\right)^{op} \xrightarrow{\pi_W^{op}} J^{op} \xrightarrow{F} \mathscr{K}$$

exists in $\mathscr{K}$. Then there is an equivalence of categories

$$\mathscr{K}(C, Bilim\ F \circ \pi_W^{op}) \approx PsNat(W, \mathscr{K}(C, F(\_)))$$

pseudonatural in C and hence the bilimit $Bilim\ F \circ \pi_W^{op}$ *is the* W*-weighted bilimit of* F. In particular, we have the following result regarding realization and gestures.

Theorem 4.2.6. *Let* $\mathscr{K}$ *be a 2-category,* $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ *a semi-simplicial set, and* $S : G^{op} \longrightarrow \mathscr{K}$ *a semi-simplicial object in* $\mathscr{K}$*. Regard* Γ *as a strict 2-functor from* G^{op} *to* $\mathbf{Cat}$*. If the object of gestures* $\Gamma@S$ *exists, then it is the* Γ*-weighted bilimit of* S*. This means that there is an equivalence of categories*

$$\mathscr{K}(C, \Gamma@S) \approx PsNat(\Gamma, \mathscr{K}(C, S(\_)))$$

pseudonatural in C*.*

In a similar way, if $T : G \longrightarrow \mathscr{K}$ *is a pseudofunctor and the bicolimit* $\Gamma \otimes_G T$ *exists, then it is the* Γ*-weighted bicolimit of* T*. This means that there is an equivalence of categories*

$$\mathscr{K}(W \otimes_J T, C) \approx PsNat(\Gamma, \mathscr{K}(T(\_), C))$$

pseudonatural in C*.*

This theorem is very important since it provides a characterization of the categories of generalized elements of the object of gestures: given an object C, the category of generalized elements $\mathscr{K}(C, \Gamma@S)$ of $\Gamma@S$ is equivalent to the category of pseudonatural transformations from Γ to $\mathscr{K}(C, S(\_))$. Thus, we can define *a C-addressed gesture with skeleton Γ with respect to S* as a pseudonatural transformation $\delta : \Gamma \Longrightarrow \mathscr{K}(C, S(\_))$. In particular, this theorem allows us to give a characterization of the points of a Grothendieck topos of gestures. In fact, let Γ be a semi-simplicial set and $\mathcal{E}$ a Grothendieck topos. Consider the Grothendieck topos of gestures $\Gamma@\mathcal{E}$ (Definition 4.2.1) obtained from the standard simplex in $\mathfrak{BTop}$. Using Theorem 4.2.6, the biadjunction related to exponentials, and the fact that **Set** is the final object of $\mathfrak{BTop}$, we obtain the equivalence

$$\begin{aligned}\mathfrak{BTop}(\mathbf{Set}, \Gamma@\mathcal{E}) \approx\ & PsNat(\Gamma, \mathfrak{BTop}(\mathbf{Set}, \mathcal{E}^{Sh(\Delta^{(\_)})})) \\ \approx\ & PsNat(\Gamma, \mathfrak{BTop}(Sh(\Delta^{(\_)}), \mathcal{E})),\end{aligned} \tag{4.1}$$

which says that the category of points of $\Gamma@\mathcal{E}$ (left-hand category) is equivalent to a category whose objects are pseudonatural transformations of the form

$$g : \Gamma \Longrightarrow \mathfrak{BTop}(Sh(\Delta^{(\_)}), \mathcal{E}) : G^{op} \longrightarrow \mathbf{Cat}. \tag{4.2}$$

In the following proposition, we deal with the particular case when $\mathcal{E}$ is the topos of sheaves on a locale.

Proposition 4.2.7. *Let Γ be a finite semi-simplicial set. If L is a locale equipped with its canonical topology, then the category $\mathfrak{BTop}(\mathbf{Set}, \Gamma@Sh(L))$ of points of the Grothendieck topos $\Gamma@Sh(L)$ of gestures (Definition 4.2.1) is a category equivalent to the category whose objects are the individual gestures with skeleton Γ and body in $pt(L)$ (Theorem 3.4.6) and in which there is just a morphism $\delta \rightsquigarrow \delta' : \Gamma \to s_{pt(L)}$ if and only if for each $(n, a) \in \int\Gamma$, $\delta_{[n]}(a) \leq \delta'_{[n]}(a)$ with respect to the specialization order on continuous maps [9, II.1.11].*

Proof. First, note that the Grothendieck topos $\Gamma@\mathcal{E}$ exists by Remark 4.2.3. Also, we have the following equations:

$$\begin{aligned}\mathfrak{BTop}(\mathbf{Set}, \Gamma@Sh(L)) \approx\ & PsNat(\Gamma, \mathfrak{Loc}(\mathcal{O}\Delta^{(\_)}, L)) \\ \cong\ & Nat(\Gamma, \mathbf{Loc}(\mathcal{O}\Delta^{(\_)}, L)) \\ \cong\ & Nat(\Gamma, \mathbf{Top}(\Delta^{(\_)}, pt(L))) \\ =\ & Nat(\Gamma, s_{pt(L)}).\end{aligned}$$

The equivalence of the first row holds by equivalence 4.1 and because

$$\mathfrak{BTop}(Sh(\mathcal{O}(\Delta^n)), Sh(L)) \approx \mathfrak{Loc}(\mathcal{O}(\Delta^n), L)$$

for each positive integer n (details in Subsection 4.3.1). Also, by the proof of Proposition 4.2.5, the category

$$PsNat(\Gamma, \mathfrak{Loc}(\mathcal{O}\Delta^{(\_)}, L))$$

is isomorphic to the category with set of objects $Nat(\Gamma, \mathbf{Loc}(\mathcal{O}\Delta^{(\_)}, L))$ and with just a morphism $\delta \rightsquigarrow \delta' : \Gamma \to \mathbf{Loc}(\mathcal{O}\Delta^{(\_)}, L)$ if and only if for each $(n, a) \in \int\Gamma$, $\delta_{[n]}(a) \leq \delta'_{[n]}(a)$, where

$\leq$ denotes the order on morphisms of locales (Subsection 4.3.1). The isomorphism of the third row is obtained as follows. The correspondence on objects holds by the adjunction $\mathcal{O} \dashv pt$ (Section 2.4). Moreover, under transposition across the adjunction $\mathcal{O} \dashv pt$, a morphism $\delta \rightsquigarrow \delta' : \Gamma \to \mathbf{Loc}(\mathcal{O}\Delta^{(-)}, L)$ corresponds to a morphism $\delta \rightsquigarrow \delta' : \Gamma \to \mathbf{Top}(\Delta^{(-)}, pt(L))$ as in the statement of this proposition (check). Finally, the equality of the fourth row holds by the definition of individual gestures on the space $pt(L)$ (Subsection 3.4.6). $\square$

Example 4.2.8. In particular, if L is the locale $\mathcal{O}(\mathbb{R})_{\neg\neg}$ induced by the double negation nucleus on $\mathcal{O}(\mathbb{R})$, we have that the category of points of $\Gamma@Sh(\mathcal{O}(\mathbb{R})_{\neg\neg})$ is equivalent to the category associated with the set of gestures with skeleton Γ and body in the empty space (Proposition 2.5.4). This set of gestures is the empty set if Γ is non-initial, and is a singleton if Γ is the initial semi-simplicial set (constant presheaf with value $\emptyset$). This implies that the category of points of $\Gamma@Sh(\mathcal{O}(\mathbb{R})_{\neg\neg})$ is empty whenever Γ has at least a vertex. $\diamond$

Individual gestures

According to expression 4.1, if we are using the standard simplex pseudofunctor in the category of Grothendieck topoi, *a gesture* with skeleton Γ (semi-simplicial set) and body in a Grothendieck topos $\mathcal{E}$ can be defined as in expression 4.2. Let us write in detail what that definition means. For simplicity, we use the equivalent presentation of a pseudonatural transformation of the form 4.2 as one from $Sh(\Delta^{(-)}) \circ \pi_\Gamma$ to $K_\mathcal{E}$, given by Proposition 4.2.5.

Thus, according to the proof of Proposition 4.2.5, a gesture g consists of:

i) For each object $([n], a)$ of $\int \Gamma$, a geometric morphism $g_a : Sh(\Delta^n) \longrightarrow \mathcal{E}$.

ii) For each morphism $\alpha : ([n], a) \longrightarrow ([m], b)$ of $\int \Gamma$, an invertible natural transformation (between inverse images) as in the diagram

$$\begin{tikzcd}
Sh(\Delta^n) \arrow[dr, "g_a", ""{name=U, below}] \arrow[dd, "Sh(\Delta^\alpha)"'] & \\
& \mathcal{E} \\
Sh(\Delta^m) \arrow[ur, "g_b"', ""{name=D, above}] &
\arrow[Rightarrow, from=U, to=D, "\tau_\alpha"']
\end{tikzcd} .$$

These data are required to satisfy the pseudonatural transformation axioms. However, $\int \Gamma$ is a plain category, Sh can be regarded as a unital functor (Subsection 4.2.2), and $K_\mathcal{E}$ is a strict 2-functor. Therefore, the axioms reduce to the following requirements:

A1) If $\alpha : ([n], a) \longrightarrow ([m], b)$ and $\beta : ([m], b) \longrightarrow ([l], c)$ are morphisms of $\int \Gamma$, then the diagram

$$\begin{tikzcd}
g_a \arrow[r, Rightarrow, "\tau_{\beta\alpha}"] \arrow[d, Rightarrow, "\tau_\alpha"] & g_c Sh(\Delta^{\beta\alpha}) \arrow[r, equal] & g_c Sh(\Delta^\beta \Delta^\alpha) \\
g_b Sh(\Delta^\alpha) \arrow[r, Rightarrow, "\tau_\beta * Id"] & g_c Sh(\Delta^\beta) Sh(\Delta^\alpha) \arrow[u, Rightarrow, "Id*\gamma"] &
\end{tikzcd}$$

commutes, where γ denotes a suitable coherence 2-cell of Sh.

A2) If $id_{[n]} : ([n], a) \longrightarrow ([n], a)$ is an identity in $\int \Gamma$, then $\tau_{id_{[n]}} = Id_{g_a}$.

In particular, in the case when Γ is a digraph, say $\Gamma = (A, V, t, h)$ under the identifications $t = \Gamma(\epsilon_1)$ and $h = \Gamma(\epsilon_0)$, a gesture g consists of:

i) For each arrow $a \in A$ a geometric morphism $g_a : Sh(I) \longrightarrow \mathcal{E}$.

ii) For each vertex $x \in V$ a geometric morphism $g_x : Sh(\{*\}) \longrightarrow \mathcal{E}$, that is, a point **Set** $\longrightarrow \mathcal{E}$ of $\mathcal{E}$.

iii) An invertible natural transformation $\tau_{\epsilon_1} : g_x \Rightarrow g_a Sh(i_0)$ whenever $x = t(a)$, that is, whenever $\epsilon_1 : ([0], x) \longrightarrow ([1], a)$ is a morphism of $\int \Gamma$. Here $i_0 = \Delta^{\epsilon_1}$ is the inclusion of the endpoint 0 in I.

iv) An invertible natural transformation $\tau_{\epsilon_0} : g_x \Rightarrow g_a Sh(i_1)$ whenever $x = h(a)$, that is, whenever $\epsilon_0 : ([0], x) \longrightarrow ([1], a)$ is a morphism of $\int \Gamma$. Here $i_1 = \Delta^{\epsilon_0}$ is the inclusion of the endpoint 1 in I.

In this case, it is understood that the axiom A2 is satisfied without any allusion to the objects involved. Moreover, since there are no compositions in $\int \Gamma$ besides those with identities, the axiom A1 is satisfied.

Example 4.2.9. Let Γ be a digraph identified with a quadruple (A, V, t, h) as above, X a topological space, and $\delta : \Gamma \longrightarrow \overrightarrow{X}$ a classical gesture as defined in Section 1.1. In this way, we have for each arrow $a \in A$ a continuous path $\delta_a : I \longrightarrow X$, and for each arrow $x \in V$ a point $\delta_x : \{*\} \longrightarrow X$, in such a way that $\delta_a i_0 = \delta_x$ whenever $x = t(a)$ and $\delta_a i_1 = \delta_x$ whenever $x = h(a)$. We obtain a gesture with skeleton Γ and body in $Sh(X)$ as follows. Define $g_a = Sh(\delta_a)$ and $g_x = Sh(\delta_x)$ for each $a \in A$ and $x \in V$. The natural transformations from iii) and iv) above are obtained by the natural isomorphisms

$$Sh(\delta_a)Sh(i_0) \cong Sh(\delta_a i_0) = Sh(\delta_x) \text{ and } Sh(\delta_a)Sh(i_1) \cong Sh(\delta_a i_1) = Sh(\delta_y)$$

whenever $x = t(a)$ and $y = h(a)$. These isomorphisms are suitable coherence 2-cells of Sh. ◇

4.2.5 Gesture pseudofunctors

Now we discuss the gesture pseudofunctors. We start with the contravariant ones.

The contravariant gesture pseudofunctors

Let $F : \mathcal{M} \longrightarrow \mathcal{C}$ be a functor from a small category $\mathcal{M}$ to a category $\mathcal{C}$ with small hom-sets and $S : \mathcal{M} \longrightarrow \mathscr{K}$ a pseudofunctor, where $\mathscr{K}$ is a 2-category. Suppose that for each object C of $\mathcal{C}$ the pseudofunctor

$$C \downarrow F \xrightarrow{Q} \mathcal{M} \xrightarrow{S} \mathscr{K},$$

where Q is the projection from the comma category, has a bilimit, which we denote by $R(C)$. We claim that this assignment is the action on objects of a pseudofunctor $R : \mathcal{C} \longrightarrow \mathscr{K}$.

To see this, first note that given a morphism $g : C \longrightarrow C'$ of $\mathcal{C}$, we have a 1-cell $R(g) : R(C) \longrightarrow R(C')$ obtained as follows. Each bilimit $R(C)$ has an associated pseudonatural transformation $\pi : K_{R(C)} \longrightarrow SQ$ such that the induced canonical functor

$$\mathscr{K}(A, R(C)) \longrightarrow PsNat(K_A, SQ)$$

is an equivalence of categories for each object A of $\mathscr{K}$. In turn, π can be pictured as a pseudocone

$$\begin{tikzcd}
R(C) \arrow[r, "\pi_f"] \arrow[dr, "\pi_{f'}"'] \arrow[dr, phantom, "\Leftarrow \tau_h"] & S(m) \arrow[d, "S(h)"] & C \arrow[r, "f"] \arrow[d, equal] & F(m) \arrow[d, "F(h)"] & m \arrow[d, "h"] \\
 & S(m') & C \arrow[r, "f'"] & F(m') & m'
\end{tikzcd}$$

over SQ, where the left-hand triangle commutes up to a coherence 2-cell τ_h. Thus, it can be shown that the family of all the π_{fg} with $f : C' \longrightarrow F(m)$ and the 2-cells of the form τ_h form a pseudonatural transformation from $K_{R(C)}$ to SQ (relative to C') and therefore, by the canonical equivalence

$$\mathscr{K}(R(C), R(C')) \longrightarrow PsNat(K_{RC}, SQ),$$

there is a 1-cell $R(g) : R(C) \longrightarrow R(C')$ of $\mathscr{K}$ and an invertible modification Ξ^g with components as in the diagram

$$\begin{tikzcd}
 & S(m) \, , \\
R(C) \arrow[ur, "\pi_{fg}"] \arrow[r, "R(g)"'] & R(C') \arrow[u, "\pi'_f"'] \arrow[ul, phantom, "\overset{\Xi^g_f}{\Leftarrow}"]
\end{tikzcd}$$

where the π'_f are the components of the pseudonatural transformation π' associated with the bilimit $R(C')$.

Second, we describe the coherence 2-cells of the pseudofunctor R. Let $g : C \longrightarrow C'$ and $g' : C' \longrightarrow C''$ be morphisms of $\mathcal{C}$. There is a 2-cell

$$\gamma_{g',g} : R(g') \circ R(g) \Longrightarrow R(g'g)$$

induced by the canonical equivalence

$$\mathscr{K}(R(C), R(C'')) \longrightarrow PsNat(K_{RC}, SQ)$$

and the modification with components of the form[9]

$$\pi''_f \circ R(g') \circ R(g) \xRightarrow{\Xi^{g'}_f * Id_{R(g)}} \pi'_{fg'} \circ R(g) \xRightarrow{\Xi^g_{fg'}} \pi_{fg'g} \xRightarrow{(\Xi^{g'g}_f)^{-1}} \pi''_f \circ R(g'g),$$

---

[9] We omit the computations that show that these components yield a modification.

where $\pi''_f : R(C'') \longrightarrow S(m)$ is a typical projection from the bilimit $R(C'')$. Also, for each object of C, there is a 2-cell

$$\delta : id_{R(C)} \Longrightarrow R(id_C)$$

induced by the modification

$$(\Xi^{id_C})^{-1} : \pi \circ K_{id_{R(C)}} = \pi \rightsquigarrow \pi \circ K_{R(id_C)}.$$

Finally, it can be checked that these data satisfy the pseudofunctor axioms; we omit this computation.

The important point of this construction is that it can be regarded as a 2-categorical Kan extension. We call the pseudofunctor R the *right Kan extension of* S *along* F. In the case when F is the opposite

$$\mathbf{y}^{op} : G^{op} \longrightarrow \left(\widehat{G}\right)^{op}$$

of the Yoneda embedding relative to the semi-simplicial category G and $S : G^{op} \longrightarrow \mathscr{K}$ is a semi-simplicial object in $\mathscr{K}$, note that the bilimit $R(\Gamma)$, where $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ is a skeleton, is precisely the bilimit of the pseudofunctor of Definition 4.2.2 of the object of gestures (cf. Section 3.7). Thus, in the 2-categorical case we have a connection between gestures and Kan extensions analogously to the categorical case (Section 3.7), and hence if for each skeleton Γ the bilimit $\Gamma@S$ exists, then we have the *contravariant gesture pseudofunctor*

$$\_@S : \left(\widehat{G}\right)^{op} \longrightarrow \mathscr{K}.$$

Let us consider the case of the 2-category of Grothendieck topoi. When $S : G^{op} \longrightarrow \mathfrak{BTop}$ is a semi-simplicial topos, we cannot guarantee the existence of all bilimits defining R, that is, of all objects of gestures for arbitrary skeleta (cf. Definition 4.2.2) since only the existence of bilimits of finite type can be ensured in $\mathfrak{BTop}$ (see Remark 4.2.3), though some objects of gestures for skeleta of infinite type may exist; for an example, see the final example of gestures on Grothendieck topoi in Subsection 4.3.3 below. In this way, the same argument above shows that there is a contravariant pseudofunctor

$$\_@S : \left(\overline{G}\right)^{op} \longrightarrow \mathfrak{BTop},$$

where $\overline{G}$ is the full subcategory of the category of semi-simplicial sets consisting of all skeleta such that the bilimit $\Gamma@S$ exists. We call this pseudofunctor the *contravariant gesture pseudofunctor* associated with S. It need not be a 2-categorical Kan extension, though could be so for a suitable S; see the final discussion in Subsection 4.3.3.

The covariant gesture pseudofunctors

Let $\mathscr{K}$ be a 2-category such that the bilimit of each pseudofunctor $F : J \longrightarrow \mathscr{K}$ with J a small fixed category exists in $\mathscr{K}$. By the definition of bilimit, there is a canonical equivalence of categories

$$\mathscr{K}(C, Bilim\ F) \approx PsNat(K_C, F)$$

for each pseudofunctor F as above and each object C of $\mathscr{K}$. This precisely means (see Theorem [6, 9.18]) that we can construct a pseudofunctor

$$Bilim : PsFun(J, \mathscr{K}) \longrightarrow \mathscr{K}$$

that is right biadjoint to the diagonal strict 2-functor

$$K_{(\_)} : \mathscr{K} \longrightarrow PsFun(J, \mathscr{K}).$$

Suppose that $\Gamma : G^{op} \longrightarrow \mathbf{Set}$ is a skeleton such that all the bilimits of pseudofunctors with domain $(\int \Gamma)^{op}$ and codomain $\mathscr{K}$ exist. Then the *covariant gesture pseudofunctor* $\Gamma@\_$ is the composite

$$PsFun(G^{op}, \mathscr{K}) \xrightarrow{PsFun(\pi_\Gamma^{op}, \mathscr{K})} PsFun\left(\left(\int \Gamma\right)^{op}, \mathscr{K}\right) \xrightarrow{Bilim} \mathscr{K},$$

where $PsFun(\pi_\Gamma^{op}, \mathscr{K})$ is the strict 2-functor obtained by composing with π_Γ^{op} (see Remark [4, 2.2.17 (i)]). Note that this pseudofunctor sends a semi-simplicial object $S : G^{op} \longrightarrow \mathscr{K}$ to the object of gestures $\Gamma@S$.

In the case when $\mathscr{K}$ is the 2-category $\mathfrak{BTop}$, if Γ is a finite skeleton, then the pseudofunctor *Bilim* above exists (see Remark 4.2.3) and hence we ensure the existence of the covariant gesture pseudofunctor

$$\Gamma@\_ : PsFun(G^{op}, \mathscr{K}) \longrightarrow \mathfrak{BTop}$$

associated with Γ.

Next we study Mazzola's gesture pseudofunctor. We first construct the analogue of the functor *exp* of Section 3.9. Let $\mathscr{K}$ be a 2-category with binary products and $\mathscr{K}_0$ the full sub-2-category of $\mathscr{K}$ consisting of all its exponentiable objects in the sense of Subsection 4.2.2. There is a pseudofunctor

$$exp : \mathscr{K} \longrightarrow PsFun(\mathscr{K}_0^{op}, \mathscr{K})$$

built from the following data:

i) The correspondence on objects sends an object C of $\mathscr{K}$ to the power pseudofunctor

$$exp(A) = A^{(\_)} : \mathscr{K}_0^{op} \longrightarrow \mathscr{K}.$$

ii) For each pair A, B of objects of $\mathscr{K}$ there is a functor

$$\begin{array}{ccc} \mathscr{K}(A,B) & \longrightarrow & PsNat(A^{(\_)}, B^{(\_)}) \\ \alpha : F \Rightarrow G & \longmapsto & exp(\alpha) : F^{(\_)} \rightsquigarrow G^{(\_)} \end{array}.$$

The pseudonatural transformation $F^{(\_)} : A^{(\_)} \Rightarrow B^{(\_)}$ consists for each object C of $\mathscr{K}_0$ of the 1-cell

$$F^C : A^C \longrightarrow B^C,$$

which is the image of F under the respective exponential pseudofunctor, and for each 1-cell $H : C \longrightarrow C'$ of $\mathscr{K}_0$ of a 2-cell $\tau_H : B^H \circ F^{C'} \Rightarrow F^C \circ A^H$ obtained by means of a suitable iso 2-cell $\epsilon \circ (B^H \circ F^{C'} \times C) \Rightarrow \epsilon \circ (F^C \circ A^H \times C)$, where $\epsilon : B^C \times C \longrightarrow C$ is the evaluation map, and the canonical equivalence $\mathscr{K}(A^{C'}, B^C) \approx \mathscr{K}(A^{C'} \times C, B)$ associated with the exponential B^C. We omit the long verifications of the pseudonatural transformation axioms.

The modification $exp(\alpha)$ consists for each object C of $\mathscr{K}_0$ of the 2-cell α^C, which is the image of α under the exponential pseudofunctor $(\_)^C$. We omit again the verification that this defines a modification. The functoriality of this definition is a consequence of the functoriality in each component given by that of the exponential pseudofunctors.

iii) Now we describe the coherence 2-cells, which are modifications. We confine ourselves to exhibiting their components. Given two 1-cells $F : A \longrightarrow A'$ and $F' : A' \longrightarrow A''$ of $\mathscr{K}$, there is a modification $F'^{(\_)} \circ F^{(\_)} \rightsquigarrow (F' \circ F)^{(\_)}$ consisting for each object C of $\mathscr{K}_0$ of the coherence 2-cell $\gamma_{F',F} : F'^C \circ F^C \Rightarrow (F' \circ F)^C$ of the exponential pseudofunctor $(\_)^C$. On the other hand, for each object A of $\mathscr{K}$ we have a modification $id_{A^{(\_)}} \rightsquigarrow (id_A)^{(\_)}$ consisting for each C of the coherence 2-cell $\delta : id_{A^C} \Rightarrow (id_A)^C$ of the exponential pseudofunctor $(\_)^C$.

Finally, the fact that these coherence modifications satisfy the pseudofunctor axioms is a consequence of the fact that exponentiation is pseudo-functorial for each object C of $\mathscr{K}$.

In this way, we can define *Mazzola's gesture pseudofunctor*. Given a pseudofunctor $T : G \longrightarrow \mathscr{K}$ with all its images exponentiable in $\mathscr{K}$ and a skeleton Γ as above, it is the composite of pseudofunctors

$$\mathscr{K} \xrightarrow{S_{(\_)}} PsFun(G^{op}, \mathscr{K}) \xrightarrow{\Gamma @} \mathscr{K},$$

where $S_{(\_)}$ is the composite

$$\mathscr{K} \xrightarrow{exp} PsFun(\mathscr{K}_0^{op}, \mathscr{K}) \xrightarrow{PsFun(T^{op},\mathscr{K})} PsFun(G^{op}, \mathscr{K}).$$

The left biadjoint to Mazzola's gesture pseudofunctor

Now we study the left biadjoint to Mazzola's gesture pseudofunctor. The existence of such a biadjoint is a consequence of the general fact that given a small category J, a 2-category $\mathscr{K}$ with binary products, and a pseudofunctor $F : J \longrightarrow \mathscr{K}$ with all its images exponentiable in $\mathscr{K}$ (in the sense of Subsection 4.2.2), the composite of pseudofunctors

$$\mathscr{K} \xrightarrow{exp} PsFun(\mathscr{K}_0^{op}, \mathscr{K}) \xrightarrow{PsFun(F^{op},\mathscr{K})} PsFun(J^{op}, \mathscr{K}) \xrightarrow{Bilim} \mathscr{K}$$

has as left biadjoint the composite of pseudofunctors

$$\mathscr{K} \xrightarrow{prod} PsFun(\mathscr{K}, \mathscr{K}) \xrightarrow{PsFun(F,\mathscr{K})} PsFun(J, \mathscr{K}) \xrightarrow{Bicolim} \mathscr{K},$$

where *prod* and *Bicolim* will be studied through this section. In fact, given a skeleton Γ, in the case when F is the composite pseudofunctor

$$\int \Gamma \xrightarrow{\pi_\Gamma} G \xrightarrow{T} \mathscr{K}_0,$$

we obtain that Mazzola's gesture pseudofunctor $\Gamma@\_ : \mathscr{K} \longrightarrow \mathscr{K}$ (whenever it exists) has a left biadjoint provided that all the bicolimits of pseudofunctors with domain $\int \Gamma$ and codomain $\mathscr{K}$ exist. This observation is useful in the case when $\mathscr{K}$ is the 2-category $\mathfrak{BTop}$ of Grothendieck topoi because it has all finite bilimits and all bicolimits (see the discussion in Subsection 4.2.2). In this way, Mazzola's gesture functor exists and has a left biadjoint if T is the standard simplex in $\mathfrak{BTop}$ (Subsection 4.2.2) and Γ is a finite digraph. We proceed to exhibit the pseudofunctors involved in our general result and to give a sketch of its proof.

First note that the product pseudofunctor $\_ \times \_ : \mathscr{K} \times \mathscr{K} \longrightarrow \mathscr{K}$ is, by definition, left biadjoint to the *strict* diagonal 2-functor, given the existence of binary products in $\mathscr{K}$. In fact, $\_ \times \_$ is precisely the pseudofunctor *Bilim* in the case when J is the category with just two objects plus identities. Moreover, this functor is unital. *The pseudofunctor prod* is defined as folllows. It assigns to each 2-cell $\alpha : G \Longrightarrow H : A \longrightarrow B$ of $\mathscr{K}$ a modification $\alpha \times \_ : G \times \_ \rightsquigarrow H \times \_ : A \times \_ \longrightarrow B \times \_$, where:

i) The pseudofunctor $A \times \_ : \mathscr{K} \longrightarrow \mathscr{K}$ is obtained by fixing the first variable of $\_ \times \_$.

ii) The pseudonatural transformation $G \times \_ : A \times \_ \longrightarrow B \times \_$ consists for each 0-cell X of $\mathscr{K}$ of the 1-cell $G \times X : A \times X \longrightarrow B \times X$ (obtained by applying $\_ \times \_$ to (G, id_X)) and for each 1-cell $F : X \longrightarrow Y$ of $\mathscr{K}$ of the composite 2-cell

$$(B \times F) \circ (G \times X) \overset{\gamma}{\Rightarrow} G \times F \overset{\beta^{-1}}{\Longrightarrow} (G \times Y) \circ (A \times F),$$

where β and γ denote appropriate coherence 2-cells of the pseuofunctor $\_ \times \_$.

iii) The modification $\alpha \times \_$ consists for each 0-cell X of $\mathscr{K}$ of the 2-cell

$$\alpha \times X : G \times X \Longrightarrow H \times X : A \times X \longrightarrow B \times X.$$

iv) Given two 1-cells $F : A \longrightarrow B$ and $F' : B \longrightarrow C$ of $\mathscr{K}$, the respective coherence 2-cell $\gamma_{F',F}$ for composition is the modification $(F' \times \_) \circ (F \times \_) \rightsquigarrow (F' \circ F) \times \_$ consisting for each 0-cell X of $\mathscr{K}$ of the coherence 2-cell

$$(F' \times X) \circ (F \times X) \Longrightarrow (F' \circ F \times X)$$

of the pseudofunctor $\_ \times X$.

It can be shown that these definitions (we omit the verification of the axioms involved) satisfy the axioms of a unital (the pseudonatural transformation $id_A \times \_$ is the identity on $A \times \_$ for each object A of $\mathscr{K}$) pseudofunctor *prod*, which hold since $\_ \times X$ is a unital pseudofunctor.

Second, we discuss the pseudofunctor *Bicolim*, the dual of *Bilim*. Let $\mathscr{K}$ be a 2-category such that the bicolimit of each pseudofunctor $F : J \longrightarrow \mathscr{K}$, with J a fixed small category, exists in $\mathscr{K}$. By the definition of bicolimit, there is a canonical equivalence of categories

$$\mathscr{K}(Bicolim\ F, C) \approx PsNat(F, K_C)$$

for each pseudofunctor F as above and each object C of $\mathscr{K}$. This means that we have a biadjunction

$$Bicolim : PsFun(J, \mathscr{K}) \rightleftarrows \mathscr{K} : K_{(\_)},$$

where *Bicolim* is left biadjoint to the diagonal strict 2-functor $K_{(\_)}$.

Third, we give a sketch of the proof of the desired biadjunction. Suppose that we have the following equivalences of categories

$$\mathscr{K}(C, Bilim\ D^F) \approx PsNat(K_C, D^F) \approx PsNat(C \times F, K_D) \approx \mathscr{K}(Bicolim\ C \times F, D)$$

and that they are pseudonatural in both arguments C and D; then according to Definition [6, 9.8] we have the desired biadjunction. We first observe that since *Bilim* is a right biadjoint and *Bicolim* is a left biadjoint, the left-hand and the right-hand equivalences exist and for this reason it is enough to show the existence of the middle equivalence. Now we exhibit the construction of a functor from $PsNat(K_C, D^F)$ to $PsNat(C \times F, K_D)$ that is an equivalence pseudonatural in C and D:

i) Consider a pseudonatual transformation (that is, a pseudo cone) $\Omega : K_C \Longrightarrow D^F$. It can be pictured as

$$\begin{tikzcd}
 & D^{F_i} & i \arrow[dd, "\alpha"] \\
C \arrow[ur, "\Omega_i"] \arrow[dr, "\Omega_j"'] & & \\
 & D^{F_j} \arrow[uu, "D^{F_\alpha}"'] \arrow[ul, Rightarrow, "\tau_\alpha"] & j
\end{tikzcd} ,$$

where τ_α corresponds to a typical coherence 2-cell of Ω. The functor sends Ω to a pseudonatural transformation $\overline{\Omega} : C \times F \Longrightarrow K_D$ consisting for each $i \in J$ of the composite 1-cell

$$C \times F_i \xrightarrow{\Omega_i \times F_i} D^{F_i} \times F_i \xrightarrow{\epsilon} D,$$

where ϵ is the biuniversal arrow related to D^{F_i}. Also, given a morphism $\alpha : i \longrightarrow j$ of J, its associated coherence 2-cell $\overline{\tau}_\alpha$ is the composite

$$\epsilon \circ (\Omega_i \times F_i) \xRightarrow{Id_\epsilon * (\tau_\alpha^{-1} \times F_i)} \epsilon \circ (D^{F_\alpha} \Omega_j \times F_i) \xRightarrow{Id_\epsilon * \lambda^{-1}} \epsilon \circ (D^{F_\alpha} \times F_i) \circ (\Omega_j \times F_i) \xRightarrow{\mu * Id_{\Omega_j \times F_i}}$$

$$\epsilon \circ (D^{F_j} \times F_\alpha) \circ (\Omega_j \times F_i) \xRightarrow{Id_\epsilon * \lambda_0} \epsilon \circ (\Omega_j \times F_\alpha) \xRightarrow{Id_\epsilon * \lambda_1^{-1}} \epsilon \circ (\Omega_j \times F_j) \circ (C \times F_\alpha),$$

where λ is an appropriate coherence 2-cell of $\_ \times F_i$, μ is the 2-cell used for defining D^{F_α}, and λ_0 and λ_1 are suitable coherence 2-cells of $\_ \times \_$. We omit the large computations that show that $\overline{\Omega}$ is a pseudonatural transformation. Finally, we illustrate $\overline{\Omega}$ with the diagram

$$\begin{tikzcd}
i \arrow[dd, "\alpha"'] & C \times F_i \arrow[dd, "C\times F_\alpha"'] \arrow[dr, "\overline{\Omega}_i", ""{name=U, below}] & & . \\
& & D & \\
j & C \times F_j \arrow[ur, "\overline{\Omega}_j"'] & &
\arrow[Rightarrow, from=U, to=3-2, "\overline{\tau}_\alpha"']
\end{tikzcd}$$

ii) Now we give the definition on morphisms. Let $\Xi : \Omega \rightsquigarrow \Psi : K_C \Longrightarrow D^F$ be a modification consisting for each $i \in J$ of the 2-cell $\Xi_i : \Omega_i \Longrightarrow \Psi_i : C \longrightarrow D^{F_i}$. The functor sends Ξ to the modification $\overline{\Xi} : \overline{\Omega} \rightsquigarrow \overline{\Psi}$ consisting for each $i \in J$ of the 2-cell

$$\overline{\Omega}_i = \epsilon \circ (\Omega_i \times F_i) \xRightarrow{Id_\epsilon * (\Xi_i \times F_i)} \epsilon \circ (\Psi_i \times F_i) = \overline{\Psi}_i.$$

iii) The functoriality follows from the fact that each component is functorial, the functor of the ith component being the equivalence $\mathscr{K}(C, D^{F_i}) \approx \mathscr{K}(C \times F_i, D)$ associated with the exponential D^{F_i}, and the fact that the composition is pointwise as well as the identity modification.

iv) Now let $\Psi : C \times F \Longrightarrow K_D$ be a pseudonatural transformation with coherence 2-cell $\tau_\alpha : \Psi_i \Longrightarrow \Psi_j \circ (C \times F_\alpha)$ for each $\alpha : i \longrightarrow j$. We must see that Ψ is isomorphic to $\overline{\Omega}$ for some $\Omega : K_C \Longrightarrow D^F$. In fact, given the canonical equivalence $\mathscr{K}(C, D^{F_i}) \approx \mathscr{K}(C \times F_i, D)$ for each $i \in J$, we know that there is $\Omega_i : C \longrightarrow D^{F_i}$ and an invertible 2-cell $\theta_i : \epsilon \circ (\Omega_i \times F_i) \Longrightarrow \Psi_i$ for each i. The coherence 2-cell $\tilde{\tau}_\alpha : D^{F_\alpha} \circ \Omega_j \Longrightarrow \Omega_i$ is the unique such that $Id_\epsilon * (\tilde{\tau}_\alpha \times F_i)$ is equal to the 2-cell

$$\epsilon \circ (D^{F_\alpha} \circ \Omega_j \times F_i) \xRightarrow{Id_\epsilon * \lambda^{-1}} \epsilon \circ (D^{F_\alpha} \times F_i) \circ (\Omega_j \times F_i) \xRightarrow{\mu * Id_{\Omega_j \times F_i}} \epsilon \circ (D^{F_j} \times F_\alpha) \circ (\Omega_j \times F_i)$$

$$\xRightarrow{Id_\epsilon * (\lambda_1^{-1} \lambda_0)} \epsilon \circ (\Omega_j \times F_j) \circ (C \times F_\alpha) \xRightarrow{\theta_i^{-1} \odot \tau_\alpha^{-1} \odot (\theta_j * Id_{C \times F_\alpha})} \epsilon \circ (\Omega_i \times F_j).$$

With these definitions, it can be shown that Ω is a pseudonatural transformation and that the collection of all θ_i with $i \in J$ is a modification $\theta : \overline{\Omega} \rightsquigarrow \Psi$, which is an isomorphism since all its components are.

v) Moreover, our functor is full and faithful. This is basically a consequence of the fact that each equivalence of the form $\mathscr{K}(C, D^{F_i}) \approx \mathscr{K}(C \times F_i, D)$ is full and faithful. Thus, by iv) above, the functor is an equivalence of categories.

vi) It remains to show that the equivalence is pseudonatural in C and D. We omit the long calculations justifying this claim, which to a great extent follows from the fact that each equivalence of the form $\mathscr{K}(C, D^{F_i}) \approx \mathscr{K}(C \times F_i, D)$ is pseudonatural in C and D.

In this way, we have the following result concerning the existence of a left biadjoint to Mazzola's gesture pseudofunctor.

Theorem 4.2.10. *Suppose that* $\mathscr{K}$ *is a* 2*-category with binary products, that* $T : G \longrightarrow \mathscr{K}$ *is a pseudofunctor whose images are exponentiable in* $\mathscr{K}$*, and that* Γ *is a semi-simplicial set such that the bilimits of all pseudofunctors with domain* $(\int \Gamma)^{op}$ *exist in* $\mathscr{K}$ *and such that the bicolimits of all pseudofunctors with domain* $\int \Gamma$ *exist in* $\mathscr{K}$*. Then Mazzola's gesture pseudofunctor* $\Gamma@\_$ *has a left biadjoint. This means that there is an equivalence of categories*

$$\mathscr{K}(C, \Gamma@D) \approx \mathscr{K}(Bicolim\ C \times (T\pi_\Gamma), D)$$

pseudonatural in C *and* D*.*

In particular, since $\mathfrak{BTop}$ has all finite bilimits and all finite bicolimits (see our discussion in Subsection 4.2.2), we have the following result.

Theorem 4.2.11. *Suppose that* $T : \Delta \longrightarrow \mathfrak{BTop}$ *is the standard simplex pseudofunctor in* $\mathfrak{BTop}$ *(Subsection 4.2.2) and that* Γ *is a finite semi-simplicial set (Remark 4.2.3). Then Mazzola's gesture pseudofunctor* $\Gamma@\_$ *has a left biadjoint. This means that there is an equivalence of categories*

$$\mathfrak{BTop}(\mathcal{F}, \Gamma@\mathcal{E}) \approx \mathfrak{BTop}(Bicolim\ \mathcal{F} \times (T\pi_\Gamma), \mathcal{E})$$

pseudonatural in $\mathcal{E}$ *and* $\mathcal{F}$*.*

This allows us to make another characterization of the points of the Grothendieck topos of gestures $\Gamma@\mathcal{E}$ with a skeleton Γ of finite type and body in $\mathcal{E}$. Next, we record this characterization, which is a direct consequence of the preceding theorem.

Corollary 4.2.12. *Suppose that* $\mathcal{E}$ *is a Grothendieck topos, that* $T : \Delta \longrightarrow \mathfrak{BTop}$ *is the standard simplex pseudofunctor in* $\mathfrak{BTop}$*, and that* Γ *is a finite semi-simplicial set (Remark 4.2.3). Then there is an equivalence of categories*

$$\mathfrak{BTop}(\mathbf{Set}, \Gamma@\mathcal{E}) \approx \mathfrak{BTop}(\Gamma \otimes_G T, \mathcal{E}).$$

Here, $\Gamma \otimes_G T$ *is the tensor product defined in Subsection 4.2.4.*

Finally, note that we have two different characterizations of the category of points of the Grothendieck topos $\Gamma@\mathcal{E}$, namely that given by the previous corollary and that given by expression 4.1. Thus, we have the equivalences

$$PsNat(\Gamma, \mathfrak{BTop}(Sh(\Delta^{(\_)}), \mathcal{E})) \approx \mathfrak{BTop}(\mathbf{Set}, \Gamma@\mathcal{E}) \approx \mathfrak{BTop}(\Gamma \otimes_G Sh(\Delta^{(\_)}), \mathcal{E}),$$

where $\Gamma \otimes_G Sh(\Delta^{(\_)})$ deserves, of course, the name of *geometric realization of* Γ. These equivalences make up an analogue of Theorem 3.4.7.

4.3 Gestures on localic topoi

One of the main purposes of this monograph has been to give an appropriate definition of gestures on Grothendieck topoi (definitions 4.2.1 and 4.2.2) by successive generalizations from the first definition for topological spaces (sections 1.1 and 1.2). In this way, after considering the definition of gestures in the case of locales (Chapter 2, definitions at Section 2.3; Chapter 3, Subsection 3.3.3) a natural intermediate stage between it and the definition of gestures on Grothendieck topoi is the exploration of the definition in the 2-category of localic topoi. This exploration is the purpose of this section.

4.3.1 The localic reflection

At this point of the monograph our paradigm of the definition of gestures has changed dramatically in many respects: now there is a definition in arbitrary categories and 2-categories, whose data are semi-simplicial objects that generalize interval objects and semi-simplicial sets that generalize digraphs. As a consequence of these changes, our conception of the category of locales may be enriched by the fact that it is a 2-category in a natural way. Certainly, the 2-category of locales, denoted by $\mathfrak{Loc}$, is obtained from **Loc** by observing that for each pair of locales L, M, the set of morphisms of locales from L to M is a poset by defining

$$f \leq g \text{ if and only if } f^*(b) \leq g^*(b) \text{ for each } b \in M,$$

or equivalently, by properties of adjoints (conjugation), by defining

$$f \leq g \text{ if and only if } g_*(a) \leq f_*(a) \text{ for each } a \in L.$$

This new feature of the category of locales is not introduced by a whim of the author, but by a structural necessity. In fact, this feature will be helpful for giving some examples of bilimits (related to gestures) both in the 2-category $\mathfrak{LTop}$ of localic topoi (full sub-2-category of $\mathfrak{BTop}$ of all topoi equivalent to $Sh(L)$ for some locale L) and the 2-category of Grothendieck topoi, bilimits being the correct notion of conical limits in 2-categories according to the preceding section. Also, the existence of Mazzola's gestures in $\mathfrak{LTop}$ is a consequence of the fact that this 2-category is equivalent, in a suitable sense to be discussed later, to the 2-category of locales $\mathfrak{Loc}$. This equivalence is precisely that induced by a fundamental biadjunction in topos theory: the biadjunction associated with the *localic reflection.* A first motivation for this biadjunction can be found in [15, IX.5].

This biadjunction states that the composite pseudofunctor

$$\mathfrak{Loc} \hookrightarrow \textbf{Site} \xrightarrow{Sh} \mathfrak{BTop} \hookrightarrow \mathfrak{CTop},$$

where $\mathfrak{CTop}$ is the 2-category of all cocomplete (elementary) topoi with geometric morphisms between them and 2-cells as in $\mathfrak{BTop}$, is right biadjoint to the strict 2-functor $Sub_{(\_)}(\mathbf{1})$ defined by

$$\begin{array}{ccc} \mathfrak{CTop}(\mathcal{E}, \mathcal{F}) & \longrightarrow & \mathfrak{Loc}(Sub_{\mathcal{E}}(\mathbf{1}), Sub_{\mathcal{F}}(\mathbf{1})) \\ \tau : f^* \overset{\bullet}{\to} g^* & \longmapsto & f^* \leq g^* \end{array},$$

for each pair $\mathcal{E}, \mathcal{F}$ of cocomplete topoi[10]. In fact, the counit ϵ (which is a pseudonatural transformation) of this biadjunction consists for each locale L of the isomorphism $\epsilon_L : Sub_{\widetilde{L}}(\mathbf{1}) \longrightarrow L$ that sends each subsheaf F of the final sheaf on L, that is, each principal ideal $\downarrow(a)$ in L, to its generator a. The locale $Sub_{\mathcal{E}}(\mathbf{1})$ is known as the *localic reflection* of the cocomplete topos $\mathcal{E}$.

This biadjunction restricts to an equivalence of 2-categories (see Definition [4, 2.4.9] and Corollary [4, 2.4.30])

$$Sh : \mathfrak{Loc} \rightleftarrows \mathfrak{LTop} : Sub_{(\_)}(\mathbf{1})$$

between locales and localic topoi. In fact, the counit

$$\epsilon : Sub_{Sh(\_)}(\mathbf{1}) \Longrightarrow id_{\mathfrak{Loc}}$$

has isomorphisms as components and the restriction

$$\eta : id_{\mathfrak{LTop}} \Longrightarrow Sh(Sub_{(\_)}(\mathbf{1}))$$

of the unit of the adjunction above has equivalences as components, so they are *pseudo-natural equivalences* (see Theorem [4, 2.4.12]), that is, equivalences in the 2-categories $PsFun(\mathfrak{Loc}, \mathfrak{Loc})$ and $PsFun(\mathfrak{LTop}, \mathfrak{LTop})$ respectively. In particular, this equivalence remains a biadjunction (Theorem [4, 2.4.24]). In much the same way as equivalences of plain categories, this equivalence of 2-categories can be described by saying that *i*) for each localic topoi $\mathcal{E}$ there is a locale whose category of sheaves is equivalent to $\mathcal{E}$ and that *ii*) for each pair L, M of locales the functor

$$Sh : \mathfrak{Loc}(L, M) \longrightarrow \mathfrak{BTop}(\widetilde{L}, \widetilde{M})$$

is an equivalence of categories. For an explicit description of this equivalence, see the discussion before Proposition [15, IX.5.2].

4.3.2 Some bilimits and exponentials of locales and localic topoi

It is well known that the category of locales has all small limits [18, §II.3]. For this reason it is natural to ask whether these limits become strict 2-limits in the 2-category $\mathfrak{Loc}$ of locales. The answer is affirmative and they are also bilimits as the following lemma asserts. This lemma is very useful because the pseudofunctor Sh, as a right biadjoint, preserves bilimits. This is the starting point to compute the bilimits needed for gestures in $\mathfrak{LTop}$.

Lemma 4.3.1. *Limits in* **Loc** *induce strict* 2*-limits in* $\mathfrak{Loc}$ *and they are also bilimits. Moreover, since pseudofunctors* $F : J \longrightarrow \mathfrak{Loc}$*, where* J *is a small category, are just functors* $F : J \longrightarrow \mathbf{Loc}$*, all bilimits in* $\mathfrak{Loc}$ *of these pseudofunctors can be obtained in this way.*

---

[10] Note that since the 2-functor $Sub_{(\_)}(\mathbf{1})$ is defined, in a natural way, in terms of inverse images, it is convenient to use the description of the pseudofunctor Sh in terms of inverse images. For this reason, in this case, it is more reasonable to define the 2-cells of $\mathfrak{BTop}$ to be the natural transformations between the inverse images of pairs of geometric morphisms and to define a 2-cell $\phi \Rightarrow \psi$ of morphism of sites to be a natural transformation from ϕ to ψ, since this renders Sh covariant. Hence the definition of Sh given in Subsection 4.2.2.

Proof. We must show that for each locale C the canonical functor

$$\mathfrak{Loc}(C, Lim\ F) \longrightarrow 2\text{-}Nat(K_C, F),$$

which we already know is bijective on objects, is an *isomorphism* of categories, that is, that it is bijective on sets of morphisms. We prove the result for equalizers and products in **Loc** and then the general case.

First, consider an equalizer diagram

$$\begin{tikzcd} E \arrow[r, "e"] & L \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M \end{tikzcd}$$

in **Loc** and two morphisms of locales $\alpha, \beta : C \longrightarrow L$ such that $f\alpha = g\alpha$, $f\beta = g\beta$, and $\alpha \leq \beta$. By the universal property of the equalizer there exist α' and β' such that $e\alpha' = \alpha$ and $e\beta' = \beta$. We must see that $\alpha' \leq \beta'$. In fact, the equalizer e is a regular monomorphism in **Loc** and hence e^* is surjective, so if $b \in E$, there exists $a \in L$ such that $e^*(a) = b$ and therefore

$$\alpha'^*(b) = \alpha'^* e^*(a) = \alpha^*(a) \leq \beta^*(a) = \beta'^* e^*(a) = \beta'^*(b).$$

Second, consider a product P, with $P = \prod_{i \in \mathcal{I}} L_i$, of locales and suppose that $\{f_i : C \to L_i |\ i \in \mathcal{I}\}$ and $\{g_i : C \to L_i |\ i \in \mathcal{I}\}$ are cones with $f_i \leq g_i$ for each $i \in \mathcal{I}$. We must see that the uniquely determined arrows $f, g : C \to \prod_{i \in \mathcal{I}} L_i$ satisfy $f \leq g$. In fact, by the construction of products in **Loc** [9, II.2.12], each element a (a C-ideal) of a product $\prod_{i \in \mathcal{I}} L_i$ of locales with projections p_i ($i \in \mathcal{I}$) is a join of elements that are finite meets of elements of the form $p_i^*(a_i)$ with $a_i \in L_i$, so it is enough to show that $f^*(a) \leq g^*(a)$ for elements a of the form $p_{i_1}^*(a_{i_1}) \wedge \cdots \wedge p_{i_k}^*(a_{i_k})$:

$$\begin{aligned} f^*(a) &= f^*(p_{i_1}^*(a_{i_1})) \wedge \cdots \wedge f^*(p_{i_k}^*(a_{i_k})) \\ &= f_{i_1}^*(a_{i_1}) \wedge \cdots \wedge f_{i_k}^*(a_{i_k}) \\ &\leq g_{i_1}^*(a_{i_1}) \wedge \cdots \wedge g_{i_k}^*(a_{i_k}) = g^*(a). \end{aligned}$$

Third, the general case follows from the construction of small limits via products and equalizers [14, V.2]. In fact, if $\{f_i : C \to F_i |\ i \in J\}$ and $\{g_i : C \to F_i |\ i \in J\}$ are cones over a diagram $F : J \longrightarrow$ **Loc** such that $f_i \leq g_i$ for each $i \in J$, then by the discussion on products, the induced morphisms to the product satisfy $\langle f_i \rangle_{i \in J} \leq \langle g_i \rangle_{i \in J}$. Moreover, they equalize the two arrows of which the limit L of F is the equalizer, and hence the induced maps $f, g : C \to L$ satisfy $f \leq g$ by our discussion on equalizers.

Finally, these 2-limits are also bilimits since the 2-cells that are isomorphisms in $\mathfrak{Loc}$ are just the identities so that the 2-cones over F coincide with the pseudocones over F. □

We have an analogous result for exponentials.

Lemma 4.3.2. *If E is an exponentiable locale in* **Loc**, *then it is exponentiable in the 2-category $\mathfrak{Loc}$.*

Proof. Let E be an exponentiable locale. So as to see that it is exponentiable in $\mathfrak{Loc}$, it is enough to show that for each pair of locales M, L the functor

$$\begin{array}{ccc} \mathfrak{Loc}(M, L^E) & \longrightarrow & \mathfrak{Loc}(M \times E, L) \\ f \leq g & \longmapsto & e(f \times E) \leq e(g \times E) \end{array},$$

where $e : L^E \times E \to L$ is the evaluation map, is an isomorphism of categories; but by the universal property in **Loc**, this functor is bijective on objects, so it remains to show that it is bijective on 2-cells. In fact, if $\alpha, \beta : M \times E \longrightarrow L$ are two morphisms of locales such that $\alpha \leq \beta$, then their respective exponential transposes $f, g : M \longrightarrow L^E$ satisfy

$$\begin{aligned} f^*(W(a,b)) &= \bigvee \{x \in M |\ (\exists a' \gg a)((x, a') \in \alpha^*(b))\} \\ &\leq \bigvee \{x \in M |\ (\exists a' \gg a)((x, a') \in \beta^*(b))\} = g^*(W(a,b)) \end{aligned}$$

on the generators of the form $W(a,b)$ of L^E, according to [9, p. 320], and hence $f \leq g$. Moreover, this 2-cell is unique by the definition of the 2-cells of $\mathfrak{Loc}$. □

Example 4.3.3 (Gestures in the 2-category of locales)**.** Note that the constructions of limits and exponentials in $\mathfrak{Loc}$ needed to define Mazzola's gestures (with respect to the standard simplex from Example 3.3.2) are always available. Moreover, given a semi-simplicial set Γ and a pseudofunctor $S : G^{op} \longrightarrow \mathfrak{Loc}$ (which is essentially a functor) the object $\Gamma@S$ (of $\mathfrak{Loc}$) of gestures with skeleton Γ with respect to S (Definition 4.2.2) coincides with its synonymous object $\Gamma@S$ in **Loc**. In particular the same is true for Mazzola's gestures. ◇

4.3.3 Gestures in the 2-category of localic topoi

Now we discuss the diverse constructions of gestures in the case of the 2-category $\mathfrak{LTop}$. The fact that allows these constructions is that $\mathfrak{LTop}$ is equivalent to $\mathfrak{Loc}$. Certainly, by the constructions made in $\mathfrak{Loc}$ in the preceding subsection, this equivalence implies that $\mathfrak{LTop}$ has all small bilimits (indexed by small categories) and that all the localic topoi of the form $Sh(E)$, where E is an exponentiable locale, are exponentiable in $\mathfrak{LTop}$. Next, we give more explicit presentations.

First consider an arbitrary diagram $F : J \longrightarrow \mathfrak{LTop}$, where J is a small category. By a general argument on biadjoints that are equivalences, the bilimit of F can be computed as $Sh(L)$, where L is the bilimit of $Sh_F(\mathbf{1}) : J \longrightarrow \mathfrak{Loc}$, which we can compute according to Lemma 4.3.1. In particular, given a skeleton Γ and a pseudofunctor $S : G^{op} \longrightarrow \mathfrak{LTop}$, the localic topos of gestures $\Gamma@S$ (Definition 4.2.2) can be computed as $Sh(L)$, where L is the bilimit of the pseudofunctor (essentially a functor)

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G^{op} \xrightarrow{S} \mathfrak{LTop} \xrightarrow{Sub_{(\_)}(\mathbf{1})} \mathfrak{Loc}.$$

Now we proceed to study the particular case of Mazzola's gestures when the cosimplicial localic topos is (the same used for Grothendieck topoi; see Subsection 4.2.2)

$$\mathcal{T} : \Delta \xrightarrow{Sh(\mathcal{O}(\Delta^{(\_)}))} \mathfrak{LTop}_0,$$

where $\mathfrak{LTop}_0$ is the full sub-2-category of $\mathfrak{LTop}$ of all exponentiable localic topoi. If $\mathcal{E}$ is a localic topos, then the pseudofunctor $S_{\mathcal{E}}$ (Definition 4.2.4) used for the construction of gestures with body in $\mathcal{E}$ is the composite (contravariant pseudofunctor)

$$\Delta \xrightarrow{Sh(\mathcal{O}(\Delta^{(\_)}))} \mathfrak{LTop}_0 \xrightarrow{\mathcal{E}^{(\_)}} \mathfrak{LTop}.$$

But since $\mathcal{E}$ is equivalent to $Sh(L)$ for some locale L, the pseudofunctor $\mathcal{E}^{(\_)}$ is equivalent, in the 2-category $PsFun(\mathfrak{LTop}_0^{op}, \mathfrak{LTop})$, to $Sh(L)^{(\_)}$ and hence $S_{\mathcal{E}}$ is equivalent to

$$\Delta \xrightarrow{\mathcal{O}(Sh(\Delta^{(\_)}))} \mathfrak{LTop}_0 \xrightarrow{Sh(L)^{(\_)}} \mathfrak{LTop},$$

which in turn, since Sh (regarded as an equivalence) preserves exponentials so that

$$Sh(L^{\mathcal{O}(\Delta^n)}) = Sh(L)^{Sh(\mathcal{O}(\Delta^n))},$$

can be assumed to be

$$\Delta \xrightarrow{\mathcal{O}(\Delta^{(\_)})} \mathfrak{Loc}_0 \xrightarrow{L^{(\_)}} \mathfrak{Loc} \xrightarrow{Sh} \mathfrak{LTop}.$$

Finally, since Sh, as a right biadjoint, preserves bilimits, the bilimit of

$$\left(\int \Gamma\right) \xrightarrow{\pi_\Gamma} G \xrightarrow{\mathcal{O}(\Delta^{(\_)})} \mathfrak{Loc}_0 \xrightarrow{L^{(\_)}} \mathfrak{Loc} \xrightarrow{Sh} \mathfrak{LTop}$$

is precisely $Sh(\Gamma@L)$, where $\Gamma@L$ is the locale of gestures with skeleton Γ and body in L. Thus,

$$\Gamma@\mathcal{E} \approx Sh(\Gamma@L).$$

Consequently, according to a previous discussion on gestures on locales (Theorem 2.6.10), if Γ is a *locally finite digraph*, we have the identity

$$\Gamma@\mathcal{E} \approx Sh(L^{|\Gamma|}) = Sh(L)^{Sh(|\Gamma|)}$$

in the 2-category of localic topoi.

Another example of gestures on Grothendieck topoi

With the tools discussed in this section we can give another example of gestures for a semi-simplicial Grothendieck topos other than the associated with the standard simplex pseudofunctor in $\mathfrak{BTop}$.

In fact, first note that the biadjunction from the localic reflection, remains a biadjunction if we change $\mathfrak{CTop}$ for $\mathfrak{BTop}$, that is, there is a biadjunction

$$Sh : \mathfrak{Loc} \rightleftarrows \mathfrak{BTop} : Sub_{(\_)}(\mathbf{1})$$

with Sh right biadjoint to $Sub_{(\_)}(\mathbf{1})$. Second, note that for each locale L there is a simplicial Grothendieck topos

$$\Delta \xrightarrow{\mathcal{O}(\Delta^{(\_)})} \mathfrak{Loc}_0 \xrightarrow{L^{(\_)}} \mathfrak{Loc} \xrightarrow{Sh} \mathfrak{BTop},$$

which is obtained by composing the simplicial locale used to construct the locale of gestures with body in L with the pseudofunctor Sh. Therefore, as above, given a skeleton Γ the object of gestures with skeleton Γ with respect to this simplicial Grothendieck topos coincides with $Sh(\Gamma@L)$, where $\Gamma@L$ is the locale of gestures with skeleton Γ and body in L, since Sh preserves bilimits.

Finally, note that this simplicial Grothendieck topos does not coincide with that induced by the standard simplex pseudofunctor in $\mathfrak{BTop}$ since $Sh(L^E)$ need not coincide with the exponential $Sh(L)^{Sh(E)}$ in $\mathfrak{BTop}$ according to [12], though the coincidence holds in the 2-category of localic topoi.

Chapter 5

Gestures on Internal Categories

From the generalization of topological gestures to the category of locales made in Chapter 2, and more generally, to any suitable category (Chapter 3), several questions arise regarding the developments in the theory of categorical gestures due to G. Mazzola [27]. The gestures on topological categories proposed by him are another generalization of topological gestures, and are the basic input to give a partial answer to the diamond conjecture and the Yoneda gestural embedding, as was exposed in [27].

Thus, is there an analogue of topological categories for locales? If so, what is the relationship between topological categories and their localic analogues? Is there a way to recover morphisms in a topological category from the associated localic category, by means of an adjunction or embedding? Moreover, is it possible to get back topological functors from the associated localic ones? Is Mazzola's generalization essentially distinct from that made in Chapter 3? The following discussion deals with these questions.

The plan of this chapter is the following. We start with the fact that topological categories are internal categories in the category of topological spaces so that we can define localic analogues as internal categories in the category of locales. Next we explore the possibility of extending the fundamental adjunction between the category of topological spaces and the category of locales to an adjunction between the category of topological categories and the category of internal categories in the category of locales. Finally, we show that both gestures on topological categories and gestures on 'localic categories' are particular cases of the constructions made in Chapter 3. In this way, we end the mathematical part of this thesis by completing the picture of the different definitions of gestures on several notions of space (topological spaces, locales, Grothendieck topoi, and topological categories) and showing that they obey the same pattern, which, as an additional advantage, includes formulas and diagrams (Section 3.11).

5.1 Overview of internal categories

Through this chapter we only use the basic definitions concerning internal categories. These definitions are sketched in this section; see [15, V.7] for reference.

Given a category $\mathcal{C}$ with pullbacks, an *internal category* in $\mathcal{C}$ is a tuple

$$(C_1, C_0, e, d_0, d_1, m),$$

where C_1 and C_0 are objects of $\mathcal{C}$ representing morphisms and objects respectively; $e : C_0 \to C_1$, $d_0 : C_1 \to C_0$, $d_1 : C_1 \to C_0$, and $m : C_1 \times_{C_0} C_1 \to C_1$ are morphisms in $\mathcal{C}$ representing the identity, domain, codomain, and composition respectively; and C_1, C_0, e, d_0, d_1, and m satisfy the appropriate commutative diagrams reflecting the axioms for categories.

By an *internal functor* from $(C_1, C_0, e, d_0, d_1, m)$ to $(D_1, D_0, e', d'_0, d'_1, m')$ we mean a pair of morphisms (F_1, F_0), where $F_1 : C_1 \to D_1$ and $F_0 : C_0 \to D_0$, that satisfies the commutative squares expressing the preservation of domain, codomain, composition, and identity.

Thus, we have the *category of internal categories in* $\mathcal{C}$ and internal functors, which we denote by $\mathbf{Cat}(\mathcal{C})$. Actually it is a 2-category if we consider internal natural transformations between internal functors, but we will not be concerned with this additional structure here, though a definition of gestures in this 2-category can be studied following the work made in Chapter 4; see the definition 4.2.2 of the object of gestures in an arbitrary 2-category.

For a pullback-preserving functor $G : \mathcal{C} \to \mathcal{D}$ between categories with pullbacks, we have an induced functor $\overline{G} : \mathbf{Cat}(\mathcal{C}) \to \mathbf{Cat}(\mathcal{D})$ between the respective categories of internal categories sending an internal category $(C_1, C_0, e, d_0, d_1, m)$ in $\mathcal{C}$ to $(GC_1, GC_0, Ge, Gd_0, Gd_1, Gm)$ and an internal functor (F_1, F_0) to (GF_1, GF_0).

5.2 Topological categories versus localic categories

Recall from Section 2.4 that we have adjoint functors

$$\mathbf{Top} \underset{\mathcal{O}}{\overset{pt}{\leftrightarrows}} \mathbf{Loc},$$

where pt is the right adjoint to $\mathcal{O}$. Moreover, this adjunction restricts to an equivalence

$$\mathbf{Sob} \underset{\mathcal{O}}{\overset{pt}{\leftrightarrows}} \mathbf{SLoc}$$

between the full subcategories **Sob** of sober spaces and **SLoc** of spatial locales. The functor pt preserves limits, in particular the finite ones, so this functor preserves pullbacks. Thus, from Section 5.1 we have an induced functor

$$\mathbf{Cat}(\mathbf{Top}) \xleftarrow{\overline{pt}} \mathbf{Cat}(\mathbf{Loc})$$

from the 2-category $\mathbf{Cat}(\mathbf{Loc})$ of internal categories in **Loc** to the 2-category $\mathbf{Cat}(\mathbf{Top})$ of internal categories in **Top**, the latter also known as the category of *topological categories* [27]. On the other hand, it is well known that the functor $\mathcal{O}$ does not preserve all the binary products (see [9, II.2.14] for an example), and therefore is not pullback-preserving (otherwise it would preserve products since it preserves the final object), so if we want to define a reverse arrow to $\overline{pt}$, we must restrict the domain to a suitable subcategory of **Top**.

More precisely, we will show that there are two categories $\mathcal{C}$ and $\mathcal{D}$ with pullbacks, where $\mathcal{C}$ is a full subcategory of **Top** and $\mathcal{D}$ is a full subcategory of **Loc**, such that all the functors in the left-hand side of the following diagram are pullback-preserving so that they induce the right-hand side:

$$\begin{tikzcd}
\mathbf{Top} & \mathbf{Loc} & & \mathbf{Cat}(\mathbf{Top}) & \mathbf{Cat}(\mathbf{Loc}) . \\
\mathcal{C} \arrow[u, hook] \arrow[r, shift left, "\mathcal{O}"] & \mathcal{D} \arrow[u, hook] \arrow[l, shift left, "pt"] & & \mathbf{Cat}(\mathcal{C}) \arrow[u, hook] \arrow[r, shift left, "\overline{\mathcal{O}}"] & \mathbf{Cat}(\mathcal{D}) \arrow[u, hook] \arrow[l, shift left, "\overline{pt}"]
\end{tikzcd}$$

Further, we will choose $\mathcal{C}$ and $\mathcal{D}$ in such a way that the pair of functors $\overline{\mathcal{O}}$ and $\overline{pt}$ form an adjoint equivalence. In consequence, $\mathbf{Cat}(\mathcal{C})$ is embedded in $\mathbf{Cat}(\mathbf{Loc})$ since $\overline{\mathcal{O}}$, as part of an equivalence, is full and faithful and $\mathbf{Cat}(\mathcal{D})$ is a full subcategory of $\mathbf{Cat}(\mathbf{Loc})$.

First step: definition of $\mathcal{C}$ *and proof that the inclusion* $\mathcal{C} \hookrightarrow$ **Top** *preserves pullbacks.* Define $\mathcal{C}$ to be the full subcategory of **Top** consisting of all the locally compact Hausdorff spaces[1]. We must show that $\mathcal{C}$ has pullbacks and that they coincide with the respective pullbacks in **Top**. To do this, we need the following lemma, which we will not prove.

Lemma 5.2.1. *In a category with binary products, the pullback of a pair of morphisms*

$$f : L \longrightarrow N, \;\; g : M \longrightarrow N$$

exists if the pullback of the diagonal morphism $(id, id) : N \longrightarrow N \times N$ *along* $f \times g : L \times M \longrightarrow N \times N$ *exists, in which case they coincide.*

Proposition 5.2.2. *The category of locally compact Hausdorff spaces has pullbacks and they coincide with the respective pullbacks in* **Top**.

Proof. First, note that this category has binary products and that they coincide with the respective products in **Top**: the product of two locally compact spaces is locally compact and the product of two Hausdorff spaces is Hausdorff. In this way, by the preceding lemma, given two morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ between locally compact Hausdorff spaces, we can compute their pullback as the pullback P of the diagonal $(id, id) : Z \longrightarrow Z \times Z$ along $f \times g$, if the latter pullback exists. To show the existence of this pullback, note that it is enough to show that the respective pullback in **Top** is a locally compact space. In fact, by the computation of inverse images in **Top**, this pullback P is the inverse image under $f \times g$ of the diagonal set $\{(z, z) |\ z \in Z\}$ (the subspace associated with the diagonal (id, id), which is a regular subobject) and P is a subspace of $X \times Y$. But since Z is Hausdorff, the diagonal set is closed and hence the inverse image P is a closed subspace of the locally compact Hausdorff space $X \times Y$. Since a subspace of a Hausdorff space is Hausdorff and a *closed* subspace of a locally compact space is locally compact, P is locally compact Hausdorff. Moreover, by construction P was computed in **Top**. $\square$

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[1] See the first footnote in Chapter 1 for the definition of local compactness.

Second step: the functor $\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Loc}$ preserves all pullbacks in $\mathcal{C}$. The reason for the choice of $\mathcal{C}$ as the category of locally compact Hausdorff spaces is that $\mathcal{O}$ preserves all pullbacks in this category. To show this, we need the following two results.

Proposition 5.2.3 (Cf. Corollary [9, III.1.3])**.** *Let X, Y, and Z be topological spaces such that $\mathcal{O}(Z)$ is strongly Hausdorff*[2] *and X is locally compact. Given a pullback diagram*

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathcal{O}(X) \arrow[d, "\mathcal{O}(f)"] \\
\mathcal{O}(Y) \arrow[r, "\mathcal{O}(g)"'] & \mathcal{O}(Z)
\end{tikzcd}$$

in **Loc***, the pullback P is spatial.*

Proof. According to Lemma 5.2.1, the pullback can be obtained as the pullback of the diagonal $(id, id) : \mathcal{O}(Z) \longrightarrow \mathcal{O}(Z) \times_l \mathcal{O}(Z)$, which is a closed sublocale since $\mathcal{O}(Z)$ is strongly Hausdorff, along $\mathcal{O}(f) \times \mathcal{O}(g)$ and hence P is a closed sublocale of $\mathcal{O}(X) \times_l \mathcal{O}(Y)$ since closed sublocales are preserved by pullbacks [9, II.2.8]. On the other hand, $\mathcal{O}(X) \times_l \mathcal{O}(Y) \cong \mathcal{O}(X \times Y)$ since X is locally compact [9, II.2.13], so $\mathcal{O}(X) \times_l \mathcal{O}(Y)$ is spatial. Thus, P, a closed sublocale of a spatial locale, is spatial [11, C.1.2.6 (b)]. ☐

Lemma 5.2.4. *Let $F : J \longrightarrow \mathbf{Top}$ be a small diagram such that F_i is a sober space for each $i \in J$. If $Lim\ \mathcal{O}F$ is spatial, then $\mathcal{O}(Lim\ F) \cong Lim\ \mathcal{O}F$, that is, $\mathcal{O}$ preserves the limit of F.*

Proof. If $Lim\ \mathcal{O}F$ is spatial, then

$$Lim\ \mathcal{O}F \cong \mathcal{O}(pt(Lim\ \mathcal{O}F)) \cong \mathcal{O}(Lim\ pt\mathcal{O}F) \cong \mathcal{O}(Lim F).$$

The first isomorphism holds since spatial locales are just the fixed points of $\mathcal{O}pt$, the second one since pt preserves limits, and the third since sober spaces are just the fixed points of $pt\mathcal{O}$. ☐

With these ingredients we can show our claim.

Proposition 5.2.5. *The functor $\mathcal{O}$ preserves all pullbacks in the category of locally compact Hausdorff spaces.*

Proof. By Proposition 5.2.2, the pullbacks can be computed in **Top**. Suppose given two continuous maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ between locally compact Hausdorff spaces. First, note that $\mathcal{O}(Z)$ is *strongly Hausdorff*; in fact, since Z is locally compact, $\mathcal{O}(Z) \times_l \mathcal{O}(Z) \cong \mathcal{O}(Z \times Z)$ and hence (check) the diagonal $(id, id) : \mathcal{O}(Z) \longrightarrow \mathcal{O}(Z) \times_l \mathcal{O}(Z)$ corresponds to the closed sublocale of $\mathcal{O}(Z \times Z)$ induced by the diagonal $Z \longrightarrow Z \times Z$, which is identified with a closed subspace (Z is Hausdorff). In this way, by Proposition 5.2.3, the pullback P of the pair $\mathcal{O}(f), \mathcal{O}(g)$ is spatial. Moreover, as Hausdorff implies sober, by Lemma 5.2.4, $\mathcal{O}$ preserves the pullback of the pair f, g. ☐

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[2] A locale L is said to be *strongly Hausdorff* if the diagonal $(id, id) : L \longrightarrow L \times_l L$ is a closed sublocale; see the details of this definition in [9, III.1.3]. This notion, due to John Isbell, is probably the most natural extension of the Hausdorff property to locales.

*Third step: definition of $\mathcal{D}$ and proof that the inclusion $\mathcal{D} \hookrightarrow$ **Loc** preserves pullbacks.* A simple choice for $\mathcal{D}$ is the full subcategory of **Loc** consisting of all images under $\mathcal{O}$ of locally compact Hausdorff spaces. Once again, we must show that $\mathcal{D}$ has pullbacks and that they coincide with the respective pullbacks in **Loc**. In fact, this follows from Proposition 5.2.5, which says that $\mathcal{D}$ is closed under pullbacks in **Loc**.

Fourth step: the functors $\mathcal{O}$ and pt restrict to an adjoint equivalence between the categories $\mathcal{C}$ and $\mathcal{D}$. On the one hand, by the definition of $\mathcal{D}$, the functor $\mathcal{O}$ restricts to a functor from $\mathcal{C}$ to $\mathcal{D}$. On the other hand, an object of $\mathcal{D}$ is of the form $\mathcal{O}(X)$ where X is a locally compact Hausdorff space. Thus, $pt\mathcal{O}(X) \cong X$ because X is sober and $pt\mathcal{O}(X)$ is locally compact. This means that pt restricts to a functor from $\mathcal{D}$ to $\mathcal{C}$. Now we must show that these restrictions form an adjoint equivalence. Let $\eta : id_{\mathbf{Top}} \longrightarrow pt\mathcal{O}$ and $\epsilon : \mathcal{O}pt \longrightarrow id_{\mathbf{Loc}}$ be the unit and the counit of the adjunction $\mathcal{O} \dashv pt$. These natural transformations restrict to natural *isomorphisms* $\eta : id_{\mathcal{C}} \longrightarrow pt\mathcal{O}$ and $\epsilon : \mathcal{O}pt \longrightarrow id_{\mathcal{D}}$ since $\mathcal{C}$ is a full subcategory of **Sob** and $\mathcal{D}$ is a full subcategory of **SLoc**, and they satisfy the triangle identities because the original ones did, so we have the desired adjoint equivalence.

Fifth step: the adjoint equivalence of the preceding step induces an adjoint equivalence between $\mathbf{Cat}(\mathcal{C})$ and $\mathbf{Cat}(\mathcal{D})$, namely the functors $\overline{\mathcal{O}}$ and $\overline{pt}$. This assertion follows from the following general fact.

Proposition 5.2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with pullbacks, and functors*

$$\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D},$$

where F is left adjoint to G, and F is pullback-preserving. This adjunction extends to an adjunction between the respective categories of internal categories

$$\mathbf{Cat}(\mathcal{C}) \underset{\overline{F}}{\overset{\overline{G}}{\leftrightarrows}} \mathbf{Cat}(\mathcal{D}),$$

where $\overline{F}$ is left adjoint to $\overline{G}$. Moreover, if the original adjunction is an equivalence, then the induced adjunction is an equivalence.

Proof. We only give a sketch of the proof. So as to show that $\overline{F}$ is left adjoint to $\overline{G}$, we will exhibit two natural transformations $\overline{\eta} : id_{\mathbf{Cat}(\mathcal{C})} \longrightarrow \overline{GF}$ and $\overline{\epsilon} : \overline{FG} \longrightarrow id_{\mathbf{Cat}(\mathcal{D})}$ that satisfy the triangular identities.

Let $\eta : id_{\mathcal{C}} \longrightarrow GF$ and $\epsilon : FG \longrightarrow id_{\mathcal{D}}$ be the unit and the counit of the adjunction $F \dashv G$ respectively. Given an internal category $\mathbb{K}$ in $\mathcal{C}$ with object of morphisms C_1 and object of objects C_0, we define $\overline{\eta}_{\mathbb{K}} : \mathbb{K} \longrightarrow \overline{GF}(\mathbb{K})$ to be (η_{C_1}, η_{C_0}). Then, it can be shown that (η_{C_1}, η_{C_0}) is an internal functor and that $\overline{\eta}$ is natural. Symmetrically, given an internal category $\mathbb{D}$ in $\mathcal{D}$ with objects D_1 and D_0, we define $\overline{\epsilon}_{\mathbb{D}} : \overline{FG}(\mathbb{D}) \longrightarrow \mathbb{D}$ as the internal functor $(\epsilon_{D_1}, \epsilon_{D_0})$, and this defines a natural transformation $\overline{\epsilon}$. Moreover, the triangular identities are just the identities

$$\epsilon_{FC_1} \circ F\eta_{C_1} = id_{FC_1} \text{ and } \epsilon_{FC_0} \circ F\eta_{C_0} = id_{FC_0}$$

for all internal categories $\mathbb{K}$ in $\mathcal{C}$ with objects C_1 and C_0, and the identities

$$G\epsilon_{D_1} \circ \eta_{GD_1} = id_{GD_1} \text{ and } G\epsilon_{D_0} \circ \eta_{GD_0} = id_{GD_0}$$

for all internal categories $\mathbb{D}$ in $\mathcal{D}$ with objects D_1 and D_0. Thus, these identities follow from the triangular identities for η and ϵ.

Moreover, if the adjunction $F \dashv G$ is an equivalence, then η and ϵ are isomorphisms and hence each component of $\overline{\eta}$ and $\overline{\epsilon}$ is an invertible internal functor (check); thus, $\overline{\eta}$ and $\overline{\epsilon}$ are isomorphisms and the induced adjunction is an equivalence. □

Final step: the main result. Since the inclusion from the first step is full, the induced inclusion $\mathbf{Cat}(\mathcal{C}) \hookrightarrow \mathbf{Cat}(\mathbf{Top})$ is also a full inclusion. Similarly, the inclusion $\mathbf{Cat}(\mathcal{D}) \hookrightarrow \mathbf{Cat}(\mathbf{Loc})$ induced by the inclusion from the third step is full. Adding the adjoint equivalence from the fifth step we obtain the following diagram of full and faithful functors (that is, embeddings):

$$\begin{tikzcd}
\mathbf{Cat}(\mathbf{Top}) & \mathbf{Cat}(\mathbf{Loc}) . \\
\mathbf{Cat}(\mathcal{C}) \arrow[u, hook] \arrow[r, shift left, "\overline{\mathcal{O}}"] & \mathbf{Cat}(\mathcal{D}) \arrow[u, hook] \arrow[l, shift left, "\overline{pt}"]
\end{tikzcd}$$

Consequently, we have a full embedding of the category $\mathbf{Cat}(\mathcal{C})$, the full subcategory of $\mathbf{Cat}(\mathbf{Top})$ consisting of all internal categories in the category of locally compact Hausdorff spaces, in the category $\mathbf{Cat}(\mathbf{Loc})$:

$$\mathbf{Cat}(\mathcal{C}) \xrightarrow{\overline{\mathcal{O}}} \mathbf{Cat}(\mathcal{D}) \hookrightarrow \mathbf{Cat}(\mathbf{Loc}).$$

As a consequence of this categorical display, we obtain that the category of topological categories is not embedded in its localic analogue by means of the natural construction expected to become an adjunction, but a sufficiently wide interesting subcategory is. This means that we have a good representation of topological categories with spaces of objects and morphisms being locally compact Hausdorff inside the category of internal categories in $\mathbf{Loc}$. Moreover, given such a representation, we can get back the original topological category, up to isomorphism, by applying the functor $\overline{pt}$.

On the other hand, note that, unlike the case of topological categories, localic internal categories are not categories, but we can obtain genuine categories by applying the functor $\overline{pt}$, which in many cases preserve the original information; for example, if they are in $\mathbf{Cat}(\mathcal{D})$. However, the application of the functor $\overline{pt}$ usually implies a loss of information.

Though locales were not implemented in principle to construct localic internal categories generalizing topological categories, but as a bridge between classical topology and Grothendieck topoi, the category $\mathbf{Cat}(\mathbf{Loc})$ is strong enough to represent a wide spectrum of 'down-to-earth' topological categories in purely algebraic terms, which in the first place connects continuous perspectives with discrete ones, and second, could represent certain topological categories in a computational setting.

5.3 Gestures on topological categories

Now we proceed to show that the concept of gestures on topological categories is a particular case of the concept of abstract gestures defined in Chapter 3. Moreover, we show that there is a localic analogue of the concept of gestures on topological categories. Before explaining the constructions of gestures, we prove two basic results on limits and exponentials of internal categories that we will need. These results are probably well known, but the author was not able to find them in the bibliography.

Theorem 5.3.1. *If $\mathcal{C}$ is a small (respectively finitely) complete category, then $\mathbf{Cat}(\mathcal{C})$ is small (respectively finitely) complete.*

Proof. We will show that $\mathbf{Cat}(\mathcal{C})$ has small (respectively finite) products and equalizers. We only give the main lines.

Products. Given a family $\{\mathbb{A}_i\}_{i\in\mathcal{I}}$ of internal categories, their product is the internal category

$$(\prod_{i\in\mathcal{I}} A_{i,1}, \prod_{i\in\mathcal{I}} A_{i,0}, \prod_{i\in\mathcal{I}} e_i, \prod_{i\in\mathcal{I}} d_i, \prod_{i\in\mathcal{I}} c_i, \prod_{i\in\mathcal{I}} m_i),$$

where $\mathbb{A}_i = (A_{i,1}, A_{i,0}, e_i, d_i, c_i, m_i)$ for each $i \in \mathcal{I}$.

Equalizers. An equalizer diagram

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "(q_1{,}q_0)"] & \mathbb{A} \arrow[r, shift left, "(F_1{,}F_0)"] \arrow[r, shift right, "(G_1{,}G_0)"'] & \mathbb{B}
\end{tikzcd},$$

where $F_1, G_1 : A_1 \longrightarrow B_1$ and $F_0, G_0 : A_0 \longrightarrow B_0$, is obtained as follows. Let $q_0 : E_0 \longrightarrow A_0$ be the equalizer of the pair F_0, G_0. Now consider the intersection $\alpha : P \longrightarrow A_1$ of the two pullbacks

$$\begin{tikzcd}
A_1 \arrow[r, "d"] & A_0 & \text{and} & A_1 \arrow[r, "c"] & A_0 \\
P_1 \arrow[u, tail] \arrow[r] & E_0 \arrow[u, tail, "q_0"'] & & P_2 \arrow[u, tail] \arrow[r] & E_0 \arrow[u, tail, "q_0"']
\end{tikzcd},$$

where d and c are the domain and codomain morphisms of $\mathbb{A}$. We define E_1 as the domain of the equalizer β of the pair $F_1\alpha, G_1\alpha$, and define $q_1 = \alpha\beta$. Now we define the morphisms e', d', c', and m' of $\mathbb{E}$.

Definition of e'. Note that, by the universal properties of the pullbacks used to define P, the morphism $eq_0 : E_0 \longrightarrow A_1$ factors as αh for some morphism $h : E_0 \longrightarrow P$. Also, h equalizes the pair $F_1\alpha, G_1\alpha$; in fact, $F_1\alpha h = F_1 e q_0 = e F_0 q_0 = e G_0 q_0 = G_1 e q_0 = G_1 \alpha h$. In this way, by the universal property of the equalizer β, h factors as $\beta e'$ for some $e' : E_0 \longrightarrow E_1$.

Definition of c' and d'. The morphism $cq_1 : E_1 \longrightarrow A_0$, which coincides with $c\alpha\beta$, equalizes the pair F_0, G_0 because $F_0 c\alpha\beta = cF_1\alpha\beta = cG_1\alpha\beta = G_0 c\alpha\beta$. Hence, by the universal property of the equalizer q_0, there is $c' : E_1 \longrightarrow E_0$ such that $q_0 c' = cq_1$. The definition of d' is similar.

Definition of m'. As in the case of the definition of e', the composite

$$E_1 \times_{E_0} E_1 \xrightarrow{q_1 \times q_1} A_1 \times_{A_0} A_1 \xrightarrow{m} A_1$$

factors as αh for some $h : E_1 \times_{E_0} E_1 \longrightarrow P$ such that h equalizes the pair $F_1\alpha, G_1\alpha$. Thus, there is $m' : E_1 \times_{E_0} E_1 \longrightarrow E_1$ such that $\beta m' = h$. □

Theorem 5.3.2. *Let $\mathcal{C}$ be a finitely complete category. If $\mathbb{E}$ is an internal category in $\mathcal{C}$ (with components E_1, E_0, e', d', c', and m') such that E_0, E_1, and the object of composable arrows $E_1 \times_{E_0} E_1$ are exponentiable in $\mathcal{C}$, then $\mathbb{E}$ is exponentiable in $\mathbf{Cat}(\mathcal{C})$.*

Proof. First, note that the category $\mathbf{Cat}(\mathcal{C})$ has binary products by Theorem 5.3.1, so it makes sense to give the definition of exponential in this category. We will exhibit the exponential $\mathbb{A}^{\mathbb{E}}$ for a fixed internal category $\mathbb{A}$ (with components A_1, A_0, e, d, c, and m) and the evaluation internal functor $\overline{\epsilon} : \mathbb{A}^{\mathbb{E}} \times \mathbb{E} \longrightarrow \mathbb{A}$. We omit the remaining details.

The components P_1, P_0, $\overline{e}$, $\overline{d}$, $\overline{c}$, and $\overline{m}$ of $\mathbb{A}^{\mathbb{E}}$ are obtained as follows. This construction is an internal analogue of the construction of the category of functors between two categories, which is an exponential in $\mathbf{Cat}$.

Construction of P_0. First, consider the equalizer diagram

$$\begin{tikzcd}
S \arrow[r, "q"] & A_1^{E_1} \times A_0^{E_0} \arrow[r, shift left, "(d^{E_1}, c^{E_1}, A_1^{e'})\pi_1"] \arrow[r, shift right, "(A_0^{d'}, A_0^{c'}, e^{E_0})\pi_2"'] & A_0^{E_1} \times A_0^{E_1} \times A_1^{E_0} .
\end{tikzcd}$$

Define $E_2 = E_1 \times_{E_0} E_1$ and $A_2 = A_1 \times_{A_0} A_1$. Now, since E_2 is exponentiable, we can define a morphism $\tilde{\delta} : S \longrightarrow A_2^{E_2}$ as the exponential transpose of the morphism δ obtained by the universal property of A_2 as follows:

$$\begin{tikzcd}
 & A_1^{E_1} \times E_2 \arrow[r, "id\times q_1"] & A_1^{E_1} \times E_1 \arrow[r, "\epsilon"] & A_1 \arrow[dr, "d"] & . \\
S \times E_2 \arrow[ur, "\pi_1 q\times id"] \arrow[rr, "\exists!\delta"] \arrow[dr, "\pi_1 q\times id"'] & & A_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & A_0 \\
 & A_1^{E_1} \times E_2 \arrow[r, "id\times q_2"'] & A_1^{E_1} \times E_1 \arrow[r, "\epsilon"'] & A_1 \arrow[ur, "c"'] &
\end{tikzcd}$$

Here, q_1, q_2, p_1, and p_2 are the pullback projections and ϵ is the evaluation map. We define P_0 to be the domain of the equalizer α of the pair of morphisms $m^{E_2}\tilde{\delta}, A_1^{m'}\pi_1 q : S \longrightarrow A_1^{E_2}$. Intuitively, the subobject $q\alpha : P_0 \longrightarrow A_1^{E_1} \times A_0^{E_0}$ represents all functors from $\mathbb{E}$ to $\mathbb{A}$; the equalizer q being related to the conditions about preservation of domains, codomains, and identities; and α being related to the preservation of composition.

Construction of P_1. Consider the equalizer diagram

$$\begin{tikzcd}
T \arrow[r, "\beta"] & P_0 \times P_0 \times A_1^{E_0} \arrow[r, shift left, "(d^{E_0}, c^{E_0})\pi_3"] \arrow[r, shift right, "(\pi_2 q\alpha\pi_1, \pi_2 q\alpha\pi_2)"'] & A_0^{E_0} \times A_0^{E_0} .
\end{tikzcd}$$

We define a pair of morphisms $\tilde{f}, \tilde{g} : T \longrightarrow A_2^{E_1}$ as the exponential transposes of the morphisms f and g obtained by the universal property of A_2 as follows:

$$\begin{tikzcd}
 & P_0 \times E_1 \arrow[r, "\pi_1 q\alpha\times id"] & A_1^{E_1} \times E_1 \arrow[r, "\epsilon"] & A_1 \arrow[dr, "d"] & \\
T \times E_1 \arrow[ur, "\pi_2\beta\times id"] \arrow[rr, "\exists! f"] \arrow[dr, "id\times d'"'] & & A_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & A_0 \\
 & T \times E_0 \arrow[r, "\pi_3\beta\times id"'] & A_1^{E_0} \times E_0 \arrow[r, "\epsilon"'] & A_1 \arrow[ur, "c"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & T \times E_0 \arrow[r, "\pi_3\beta\times id"] & A_1^{E_0} \times E_0 \arrow[r, "\epsilon"] & A_1 \arrow[dr, "d"] & \\
T \times E_1 \arrow[ur, "id\times c'"] \arrow[rr, "\exists ! g"] \arrow[dr, "\pi_1\beta\times id"'] & & A_2 \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & & A_0 \\
 & P_0 \times E_1 \arrow[r, "\pi_1 q\alpha\times id"'] & A_1^{E_1} \times E_1 \arrow[r, "\epsilon"'] & A_1 \arrow[ur, "c"'] &
\end{tikzcd} .$$

We define P_1 as the domain of the equalizer γ of the pair $m^{E_1}\tilde{f}, m^{E_1}\tilde{g} : T \longrightarrow A_1^{E_1}$. In this case, the subobject $\beta\gamma : P_1 \longrightarrow P_0 \times P_0 \times A_1^{E_0}$ represents all natural transformations between functors from $\mathbb{E}$ to $\mathbb{A}$; the equalizer β being related to the appropriate definition of domains and codomains of natural transformations, and the equalizer γ being related to the commutative squares that express the naturality.

Construction of $\overline{e}$. Let $h : P_0 \longrightarrow A_1^{E_0}$ be the composite $e^{E_0}\pi_2 q\alpha$, which coincides with $A_1^{e'}\pi_1 q\alpha$. By the universal property of the equalizer β, the morphism $(id, id, h) : P_0 \longrightarrow P_0 \times P_0 \times A_1^{E_0}$ factors as $\beta h'$ for a unique $h' : P_0 \longrightarrow T$. Further, it can be shown that h' equalizes the pair $m^{E_1}\tilde{f}, m^{E_1}\tilde{g}$ so that there is a unique morphism $\overline{e}$ such that $\gamma\overline{e} = h'$.

Construction of $\overline{d}$ *and* $\overline{c}$. They are precisely the projections from P_1 to the first and second components, that is, the composites

$$P_1 \xrightarrow{\beta\gamma} P_0 \times P_0 \times A_1^{E_0} \xrightarrow{\pi_1} P_0 \text{ and } P_1 \xrightarrow{\beta\gamma} P_0 \times P_0 \times A_1^{E_0} \xrightarrow{\pi_2} P_0$$

respectively.

Construction of $\overline{m}$. Define $P_2 = P_1 \times_{P_0} P_1$ and let $r_1, r_2 : P_2 \longrightarrow P_1$ be the projections from the pullback. Consider the morphism σ obtained from the universal property of A_2 as follows:

$$\begin{tikzcd}
 & A_1^{E_0} \times E_0 \arrow[rr, "\epsilon"] & & A_1 \arrow[dr, "d"] & \\
P_2 \times E_0 \arrow[ur, "\pi_3\beta\gamma r_1\times id"] \arrow[r, "\exists ! \sigma"] \arrow[dr, "\pi_3\beta\gamma r_2\times id"'] & A_2 \arrow[urr, "p_1"'] \arrow[drr, "p_2"] & & & A_0 \\
 & A_1^{E_0} \times E_0 \arrow[rr, "\epsilon"'] & & A_1 \arrow[ur, "c"'] &
\end{tikzcd} .$$

In this way, we have the morphism $m\sigma : P_2 \times E_0 \longrightarrow A_1$ and its exponential transpose $\widetilde{m\sigma} : P_2 \longrightarrow A_1^{E_0}$. Also, we have a morphism $(\overline{d}r_2, \overline{c}r_1, \widetilde{m\sigma}) : P_2 \longrightarrow P_0 \times P_0 \times A_1^{E_0}$ which, by the universal property of β, yields a unique morphism $k : P_2 \longrightarrow T$ such that $\beta k = (\overline{d}r_2, \overline{c}r_1, \tilde{m\sigma})$. In turn, k equalizes the pair $m^{E_1}\tilde{f}, m^{E_1}\tilde{g}$ and hence there is a unique morphism $\overline{m} : P_2 \longrightarrow P_1$ satisfying $\gamma\overline{m} = k$.

Construction of $\overline{\epsilon}$. The components $\overline{\epsilon}_1$ and $\overline{\epsilon}_0$ of this internal functor are the composites

$$P_1 \times E_1 \xrightarrow{\gamma\times id} T \times E_1 \xrightarrow{f} A_2 \xrightarrow{m} A_1 \text{ and } P_0 \times E_0 \xrightarrow{\pi_2 q\alpha\times id} A_0^{E_0} \times E_0 \xrightarrow{\epsilon} A_0$$

respectively. □

With these two results at hand, we can show the existence of the constructions of gestures on *topological categories* (internal categories in **Top**) and on *localic categories* (internal categories in **Loc**), in the sense of Subsection 3.1.3. Moreover, since both **Cat**(**Top**) and **Cat**(**Loc**) have an initial object, the constructions of gestures also obey the general archetype of Subsection 3.3.3; see the final paragraph in Subsection 3.3.2. It is important to stress that our construction of gestures on topological categories coincides with Mazzola's construction in [27].

5.3.1 The case of topological categories

Let I be the unit interval in $\mathbb{R}$ and $\mathbb{I}$, with $\mathbb{I} = (E_1, E_0, e', d', c', m')$, the topological category of the poset $(I, \leq)$, that is,

i) $(E_1, E_0) = (\{(x, y)|\ x \leq y \text{ in } I\}, I)$;

ii) $e' : E_0 \longrightarrow E_1$ is the diagonal, that is, $e'(x) = (x, x)$;

iii) $d', c' : E_1 \longrightarrow E_0$ are the first and second projection respectively;

iv) $E_2 = E_1 \times_{E_0} E_1 = \{((w, z), (x, y)) \in I^2 \times I^2 |\ x \leq y = w \leq z\}$, and $m' : E_2 \longrightarrow E_1$ is defined by $m'((y, z), (x, y)) = (x, z)$; and

v) the set E_0 has the usual topology on I, E_1 is a subspace of $I \times I$ (product topology), and E_2 is a subspace of I^4, so e' (diagonal), d', c', and m' (projections) are continuous.

To show that $\mathbb{I}$ is exponentiable in **Cat**(**Top**) we check the conditions of Theorem 5.3.2: in fact, E_0, E_1, and E_2 are exponentiable in **Top**, that is, locally compact, since they are closed subsets of some finite power of I, the latter being locally compact since finite products of locally compact spaces are locally compact.

Also, there is a base $(\mathbb{I}, \mathbf{1}, \mathbf{i}, \mathbf{j})$ in **Cat**(**Top**) analogous to the base $(I, \{*\}, i_0, i_1)$ (see sections 3.1 and 1.1, and Figure 3.1) in **Top**. Note that the terminal category

$$(\{*\}, \{*\}, id, id, id, !)$$

is the final object $\mathbf{1}$ in **Cat**(**Top**). The internal functors $\mathbf{i}, \mathbf{j} : \mathbf{1} \longrightarrow \mathbb{I}$ are defined by $\mathbf{i}_0(*) = 0$, $\mathbf{j}_0(*) = 1$, $\mathbf{i}_1(*) = (0, 0)$, and $\mathbf{j}_1(*) = (1, 1)$.

Given a topological category $\mathbb{K}$, with $\mathbb{K} = (C_1, C_0, e, d, c, m)$, the corresponding internal digraph of $\mathbb{K}$ in **Cat**(**Top**) is the tuple $(\mathbb{K}^{\mathbb{I}}, \mathbb{K}, \mathbf{e}_0, \mathbf{e}_1)$, where $\mathbb{K}^{\mathbb{I}}$ is the category of all topological functors from $\mathbb{I}$ to $\mathbb{K}$ with its set of objects P_0 (that is, of topological functors) topologized as a subspace[3] of $C_1^{E_1} \times C_0^I$ and its set of morphisms P_1 (that is, of natural transformations) topologized as a subspace of $P_0 \times P_0 \times C_1^{E_0}$, and

$$\begin{tikzcd}
\mathbb{K}^{\mathbb{I}} \arrow[r, "\mathbf{e}_i"] & \mathbb{K} \\
F \arrow[r, mapsto] \arrow[d, "\tau"'] & F(i) \arrow[d, "\tau_i"] \\
G \arrow[r, mapsto] & G(i),
\end{tikzcd}$$

for $i = 0, 1$. Since **Cat**(**Top**) is small-complete by Theorem 5.3.1, for each digraph Γ, the topological category of gestures $\Gamma@\mathbb{K}$ with skeleton Γ and body in $\mathbb{K}$ (in the sense of Subsection 3.1.3) exists. This definition is essentially the same given in [27], where applications of gestures on topological categories in mathematical music theory are discussed.

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[3] Note that the subobject $P_0 \rightarrowtail C_1^{E_1} \times C_0^{E_0}$ (Theorem 5.3.2) was a composition of regular subobjects (equalizers) and hence, in this case, it is a composition of subspace inclusions, that is, a subspace inclusion.

Example 5.3.3. Let Γ be a loop digraph as in the picture

$$\begin{tikzcd} \bullet x \arrow[loop left, "a"] \end{tikzcd} .$$

Let us make an explicit computation of the topological category $\Gamma@\mathbb{K}$ for any topological category $\mathbb{K}$ with $\mathbb{K} = (C_1, C_0, e, d, c, m)$. First, note that according to the definition of $\Gamma@\mathbb{K}$, it is the equalizer of the diagram

$$\begin{tikzcd} \mathbb{K}^{\mathbb{I}} \arrow[r, shift left, "\mathbf{e}_1"] \arrow[r, shift right, "\mathbf{e}_0"'] & \mathbb{K} \end{tikzcd} .$$

Thus, according to our construction of equalizers in Theorem 5.3.1, $\Gamma@\mathbb{K}$ can be described as follows:

i) Its objects are topological functors $F : \mathbb{I} \longrightarrow \mathbb{K}$ that are loops, that is, pairs $(F_1, F_0) \in C_1^{E_1} \times C_0^I$ (correspondence on morphisms and on objects) satisfying the functor conditions and $F_0(0) = F_0(1)$. In this way, the set of objects of $\Gamma@\mathbb{K}$ is equipped with the subspace topology of the Tychonoff topology on the product $C_1^{E_1} \times C_0^I$. Here, $C_1^{E_1}$ and C_0^I are function spaces, which are endowed with the compact-open topology.

ii) A morphism from F to G, where F and G are topological functors as in i), is a triple (F, G, τ), where $\tau : F \longrightarrow G$ is a natural transformation such that $\tau_0 : F_0(0) \longrightarrow G_0(0)$ and $\tau_1 : F_0(1) \longrightarrow G_0(1)$ are the same morphism. Here we regard τ as a continuous map from I to C_1 satisfying the usual natural transformation conditions. In this way, the set of morphisms of $\Gamma@\mathbb{K}$ is endowed with the subspace topology of the Tychonoff topology on the product of function spaces

$$C_1^{E_1} \times C_0^I \times C_1^{E_1} \times C_0^I \times C_1^I.$$

◇

Morphisms between gestures on a topological category

On the other hand, note that the topological category $\Gamma@\mathbb{K}$ is in particular a small category. This underlying category can be presented in a way that is simpler than the above presentation as an involved limit. To do this, first recall that the underlying digraph of a small category $\mathcal{C}$ can be obtained as the presheaf (bijection in Subsection 3.1.1) $\mathbf{Cat}(T, \mathcal{C}) : G_1^{op} \longrightarrow \mathbf{Set}$, where the functor $T : G_1 \longrightarrow \mathbf{Cat}$ corresponds (correspondence in Subsection 3.1.4) to the pair of inclusion functors of the final category into the category $\mathbf{2}$ pictured as

$$\begin{tikzcd} x \arrow[loop left, "id_x"] \arrow[r, "a"] & y \arrow[loop right, "id_y"] \end{tikzcd} .$$

This suggests that we can recover the underlying category of $\Gamma@\mathbb{K}$ by a similar process, where the values of the functor T are associated topological categories. In fact, the terminal topological category $\mathbf{1}$ (discussed above) is just the final category, and the category[4] $\mathbf{2}$ can be

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[4] The notation $\mathbf{2}$ makes sense, since the category denoted is just the category of the poset (ordinal number) 2.

regarded as the topological category by equipping both its set of morphisms $\{a, id_x, id_y\}$ and its set of objects $\{x, y\}$ with the discrete topology so that the domain, codomain, and identity operations are continuous, and the composition is continuous since the set of composable arrows (a pullback) turns out to be equipped with the discrete topology. In this way, note that the objects of $\Gamma@\mathbb{K}$ are just functors from $\mathbf{1}$ to $\Gamma@\mathbb{K}$, and that the morphisms of $\Gamma@\mathbb{K}$ are just continuous functors from $\mathbf{2}$ to $\Gamma@\mathbb{K}$, or equivalently, plain functors from $\mathbf{2}$ to $\Gamma@\mathbb{K}$ since the topologies on objects and morphisms of $\mathbf{2}$ are discrete. Furthermore, this implies that if once again we denote by T the functor associated with the two (continuous) inclusions of $\mathbf{1}$ into $\mathbf{2}$, then the functor $\mathbf{Cat}(\mathbf{Top})(T, \Gamma@\mathbb{K}) : G_1 \longrightarrow \mathbf{Set}$ corresponds to the underlying digraph of $\Gamma@\mathbb{K}$. But, the fundamental adjunction for gestures (Subsection 3.4.4), which is also valid if we replace G by G_1, states that there is a bijection

$$\mathbf{Cat}(\mathbf{Top})(\mathbb{A}, \Gamma@\mathbb{K}) \cong \mathit{Digraph}(\Gamma, \mathbf{Cat}(\mathbf{Top})(\mathbb{A}, S_{\mathbb{K}}))$$

natural in $\mathbb{A}$, where $S_{\mathbb{K}}$ corresponds to the internal digraph $(\mathbb{K}^{\mathbb{I}}, \mathbb{K}, \mathbf{e}_0, \mathbf{e}_1)$ of $\mathbb{K}$ (notation in Subsection 3.1.3). Thus, by means of a suitable composition (put T instead of the argument $\mathbb{A}$), we obtain a natural isomorphism

$$\mathbf{Cat}(\mathbf{Top})(T, \Gamma@\mathbb{K}) \overset{\bullet}{\to} \mathit{Digraph}(\Gamma, \mathbf{Cat}(\mathbf{Top})(T, S_{\mathbb{K}}))$$

of functors from G_1^{op} to $\mathbf{Set}$. The left-hand side of this isomorphism is the underlying digraph of $\Gamma@\mathbb{K}$, that is, the digraph whose arrows and vertices are the morphisms and the objects of $\Gamma@\mathbb{K}$ respectively and whose tail and head functions are the domain and codomain functions respectively. The isomorphism ensures that this underlying digraph can be identified with the right-hand digraph, that is, one that has

i) as set of arrows all the morphisms of digraphs from Γ to the digraph $(P_1, C_1, \mathbf{e}_0, \mathbf{e}_1)$, where P_1 is the set of continuous natural transformations between continuous functors from $\mathbb{I}$ to $\mathbb{K}$, C_1 is the set of arrows of $\mathbb{K}$, and $\mathbf{e}_0$ and $\mathbf{e}_1$ are the (continuous) functions that assign to each natural transformation τ the morphisms τ_0 and τ_1 respectively;

ii) as set of vertices all the morphisms of digraphs from Γ to the digraph $(P_0, C_0, \mathbf{e}_0, \mathbf{e}_1)$, where P_0 is the set of continuous functors from $\mathbb{I}$ to $\mathbb{K}$, C_0 is the set of objects of $\mathbb{K}$, and $\mathbf{e}_0$ and $\mathbf{e}_1$ are the functions that assign to each continuous functor F the objects $F(0)$ and $F(1)$ respectively; and

iii) as tail and head functions the pairs of functions $(\overline{d}, d)$ and $(\overline{c}, c)$ respectively, where $\overline{d}, \overline{c} : P_1 \longrightarrow P_0$ are the domain and codomain functions from the exponential $\mathbb{K}^{\mathbb{I}}$ (Theorem 5.3.2) and $d, c : C_1 \longrightarrow C_0$ are the domain and codomain functions of $\mathbb{K}$.

Observe that in this case arrows and vertices can be identified, using the fundamental adjunction for gestures (Subsection 3.4.4), with points (elements) of the spaces of gestures with skeleton Γ with respect to functors from G_1^{op} to $\mathbf{Top}$ associated with the digraphs $(P_1, C_1, \mathbf{e}_0, \mathbf{e}_1)$ and $(P_0, C_0, \mathbf{e}_0, \mathbf{e}_1)$; see the definition at the end of Subsection 3.1.3. These are excellent examples of gestures in $\mathbf{Top}$ that are not induced by the internal digraph of some space as usual (see [26] and the first definition in Subsection 3.1.3)—certainly the elements of P_1 are natural transformations, not mere paths. This is a good reason to take into account abstract gestures. Now let us work a simple example of our characterization.

Example 5.3.4. Let Γ be the digraph as in the picture

$$x \xrightarrow{a} y \xrightarrow{b} z.$$

Let us compute explicitly the underlying category of the topological category $\Gamma @ \mathbb{K}$. An object is just a tuple $(F_a, F_b, C_x, C_y, C_z)$, where i) F_a and F_b are continuous functors from $\mathbb{I}$ to $\mathbb{K}$, ii) C_x, C_y, and C_z are objects of $\mathbb{K}$, and iii) $F_a(0) = C_x$, $F_a(1) = C_y = F_b(0)$, and $F_b(1) = C_z$. On the other hand, a morphism in $\Gamma @ \mathbb{K}$ from $(F_a, F_b, C_x, C_y, C_z)$ to $(G_a, G_b, D_x, D_y, D_z)$ is a tuple $(\tau_a, \tau_b, f_x, f_y, f_z)$, where i) $\tau_a : F_a \longrightarrow G_a$ and $\tau_b : F_b \longrightarrow G_b$ are continuous natural transformations (when they are regarded as suitable functions from I to C_1), ii) $f_x : C_x \longrightarrow D_x$, $f_y : C_y \longrightarrow D_y$, and $f_z : C_z \longrightarrow D_z$ are morphisms of $\mathbb{K}$, iii) $\tau_a(0) = f_x$, $\tau_a(1) = f_y = \tau_b(0)$, and $\tau_b(1) = f_z$. ◇

5.3.2 The case of localic categories

Again, consider the topological category $\mathbb{I}$ from Subsection 5.3.1. As we observed there, both E_0 and E_1 are locally compact spaces. Also, they are Hausdorff spaces because they are subspaces of the Hausdorff spaces I and I^2 respectively. Thus, $\mathbb{I}$ is an internal category in the category of locally compact Hausdorff spaces, and therefore, by our main result in Section 5.2 (see *Final step* there), $\overline{\mathcal{O}}(\mathbb{I})$ is an internal category in **Loc**.

Note that $\overline{\mathcal{O}}(\mathbb{I})$ is exponentiable in **Cat**(**Loc**). Certainly, since $\mathcal{O}$ preserves pullbacks in the category of locally compact Hausdorff spaces (Proposition 5.2.5), the object of composable arrows $\mathcal{O}(E_1) \times_{\mathcal{O}(E_0)} \mathcal{O}(E_1)$ coincides with $\mathcal{O}(E_1 \times_{E_0} E_1)$, which is just $\mathcal{O}(E_2)$. In this way, since E_2 is also locally compact (Proposition 5.2.2), $\mathcal{O}(E_0)$, $\mathcal{O}(E_1)$, and $\mathcal{O}(E_2)$ are continuous lattices by Lemma [9, VII.4.2], and hence they are exponentiable in **Loc** by Theorem [9, VII 4.11]. This means that $\overline{\mathcal{O}}(\mathbb{I})$ is exponentiable by Theorem 5.3.2.

The base $(\mathbb{I}, \mathbf{1}, \mathbf{i}, \mathbf{j})$ for gestures in **Cat**(**Top**) induces a base $(\overline{\mathcal{O}}(\mathbb{I}), \overline{\mathcal{O}}(\mathbf{1}), \overline{\mathcal{O}}(\mathbf{i}), \overline{\mathcal{O}}(\mathbf{j}))$ for gestures in **Cat**(**Loc**), where $\overline{\mathcal{O}}(\mathbf{1})$ is just the final localic category $(\mathbf{2}, \mathbf{2}, id, id, id, !)$. In this way, given a localic category $\mathbb{A}$, we have an internal digraph $(\mathbb{A}^{\overline{\mathcal{O}}(\mathbb{I})}, \mathbb{A}, \mathbb{A}^{\overline{\mathcal{O}}(\mathbf{i})}, \mathbb{A}^{\overline{\mathcal{O}}(\mathbf{j})})$ of $\mathbb{A}$ in **Cat**(**Loc**) obtained by exponentiation of the base. Finally, as the category **Cat**(**Loc**) is small-complete by Theorem 5.3.1, given a digraph Γ, we can ensure the existence of the localic category $\Gamma @ \mathbb{A}$ of gestures with skeleton Γ and body in $\mathbb{A}$, according to the definition given in Subsection 3.1.3.

Chapter 6

A Philosophical Framework for Gesture Theory

6.1 Mazzola: first visions and problems

Soon after the accomplishment of the first version of *The Topos of Music* [24], an enterprise that achieved a framework for musicology based on topos theory (a theory of performance included), and that gave a very complete account of the mathematical structures present in music, Mazzola became aware that his own activity as a free jazz pianist had little to do with the structures and procedures described in his monograph; see [23, Ch. 24]. *Gestures*, rather than formulas, were the essence of his performance. Certainly, free jazz improvisation is mainly determined by the movements of the body's limbs, that is, by a *dancing of the body*, the classical structures of Western music being secondary and auxiliary. Therefore, a rigorous reflection on gestures was necessary, and not only in the case of musical improvisation, but in music in general, since all its power and intensity relies in its realization in bodily terms, even in the Western classical tradition.

According to Mazzola [23, p. 146], a first diagram of Western musical performance is

$$\begin{tikzcd}
\text{gestures} \arrow[rr, "instrumentalize"] \arrow[dr, shift left, "freeze"] & & \text{sounds}, \\
& \text{scores} \arrow[ul, shift left, "thaw"] \arrow[ur, "perform"'] &
\end{tikzcd}$$

which expresses the performance of a score decomposed into two parts: the symbols of the score are thawed by the performer producing gestures, and then these gestures produce sounds when they interact with the instrument. A reverse action to *thaw* corresponds to freezing gestures producing scores by successive abstraction; this is a sort of MIDI device codifying the projected movements of the body in terms of symbols and formulas. However, it is necessary to take into account that the score, closely related to attributions of meaning and significance to music (if they are necessary at all) made by means of a clear analogy with prose and language, is not present in many musical expressions, so it is possible to make music by direct interaction with the instrument. In this respect, Mazzola stresses the presemiotic nature of gestures.

It is remarkable that gestures are at the core of the relation between mathematics and music. Mazzola has proposed a fundamental conceptual adjunction

$$\begin{tikzcd}\text{formulas} \arrow[r, shift left, "music"] & \text{gestures} \arrow[l, shift left, "mathematics"]\end{tikzcd} ,$$

where the arrows correspond to the activities of the disciplines: mathematicians take gestures (intuitions, mental movements, analogies with reality,...) to produce formulas, and musicians take formulas (scores, diagrams, musical notations,...) to produce gestures. The term adjunction refers to a relation that is more profound than a mere inversion or isomorphism, it corresponds to a true dialectic that is grasped formidably by the categorical concept of *adjunction* between functors. This hypothetical adjunction implies, as stressed by Mazzola, that every movement in mathematics should produce an effect on music (in fact, this partly corresponds to the effort to develop a mathematical music theory based on contemporary mathematics), and that every movement in music should produce an effect on mathematics. Mazzola then urges an emergence of gestures in mathematics in which diagrams and formulas would recover their gestural essence. The problem can be exemplified by a function

$$f : X \longrightarrow Y,$$

whose notation as an arrow between two terms suggests a sort of intermediate movement between a given argument $x \in X$ of the function and its image $f(x)$, a movement that is missed, as shown by the set-theoretic definition of function in terms of ordered pairs.

The point of departure towards a formal definition of gesture is the one given by Hugues de Saint-Victor in the chapter XII of *De Institutione Novitiorum* [34]:

> Gestus est motus et figuratio membrorum corporis, ad omnen agendi et habendi modum.
>
> Gesture is the movement and configuration of the body's limbs, towards an action and having a modality.[1]

Based on this definition[2], Mazzola gives the first mathematical definition of gesture as a diagram of curves in a topological space (see Section 1.1 for the precise definition, and Figure 1.1); here the diagram corresponds to the configuration of the body's limbs and the topological space corresponds to the space-time where the movement of the body occurs. The absence of a load of significance is deliberate in this definition and corresponds to Mazzola's presemiotic approach. Certainly, a gesture need not be related to a Saussure's sign of the type signifier-signified. It has an existence in its own right, regardless of whether or not it conveys any meaning, as in the cases of dancing or free improvisation. Further,

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[1] Our translation.

[2] Actually, Mazzola's definition is based on an English translation from a free translation into French made by J-C. Schmitt, which can be found in Schmitt's book *La Raison des gestes dans l'Occident médiéval* (Paris: Gallimard).

this definition is generalized to topological categories in [27] to include both algebraic and topological information in gestures, and then to locales (Chapter 2) as a first step to define gestures on generalized notions of space. These different instances of defining gestures belong, though not strictly, to the topological branch of the theory of gestures.

On the other hand, there is an algebraic counterpart of this. In [26, p. 39], Mazzola defines a formula in a spectroid[3] as a suitable *diagram* in this particular kind of linear category, which is the starting point to develop a mathematical framework for both the theory of nets and Lewin's transformational theory.

It is important to stress that all these different definitions rely on the notion of digraph: both gestures and formulas are morphisms of digraphs whose domains are given skeleta. Moreover, following Mazzola's ideas, these instances can be regarded as attempts to reanimate the implicit movement that the drawing of a digraph by means of arrows and nodes suggests. In Mazzola's own words [26, p. 25]:

> *The gesture is a morphism, where the linkage is a real movement and not only a symbolic arrow without bridging substance.*

Given the mathematical definition, we can state two main problems. The first one deals with the search for a common universe for the two branches, the formulaic and the gestural, of mathematical music theory: that is, the diamond conjecture. The second one corresponds to a gestural representation of categories in which composition of arrows can be manipulated at the level of gestural intuitions, in much the same way as the Yoneda embedding allows the representation of categories in topoi of presheaves. See also the introduction to this thesis for a more detailed discussion on theses two problems.

Following Mazzola's legacy, as a point of departure, we adhere to his formal definition of gesture, which was developed in some detail in the preceding chapters, and to the philosophical definition of gesture due to Saint-Victor. To fix ideas, the main objective of the following discussion is to extend Mazzola's diagram of Western musical performance using the ideas exposed in the introduction of this thesis. However, many other issues are discussed.

6.2 Peirce: phaneroscopy and orientation

According to Peirce (see [38, Ch. 3] and [35, §6, p. 74]), *phaneroscopy*, or *phenomenology*, is the study or description of the *phaneron* defined as the complete collective that is present to the mind. Phaneroscopy includes the study of the cenopythagorean categories, the *three modes of being*, or the tints occurring in phenomena. We use the synthesis of the three categories made by Zalamea in [38, Ch. 3], which relates these to keywords as follows:

> *Firstness*: immediacy, first impression, freshness, sensation, unary predicate, monad, chance, possibility.
>
> *Secondness*: action-reaction, effect, resistance, alterity, binary relation.

---

[3] See Subsection 3.11.2 for the definition of spectroid. Spectroids were introduced by Pierre Gabriel in representation theory of quivers or digraphs.

> *Thirdness*: mediation, order, law, continuity, knowledge, ternary relation, triad, generality, necessity.

In what follows, we regard these categories as the three fundamental *modes* from which we progressively stratify the concepts to study, and use them by successive iteration and recursion[4]. This procedure is the general orientation for the following discussion.

The fundamental observation of Zalamea (see the introduction to [37]) that Saint-Victor's definition of gesture as *the movement and configuration of the body's limbs, towards an action and having a modality* is fully pragmaticist in Peirce's sense, can initially be interpreted as the fact that it is marked by thirdness essentially. If a gesture is the *movement* and configuration of the *body* with an *aim*, then it is by definition a mediation (movement) between two states of the body, a former state or beginning, and a second state or aim. In turn, the movement implies *continuity* and it is an essential manifestation of thirdness, so *a gesture is marked by thirdness.*

On the other hand, we conceive the movement of the body and the body itself in relation to music performance, that is, as *actual* instances of being, so they correspond to secondness regarding being. It is necessary to clarify at this point that the aim is also actual[5], in fact, the aim is *an action* (omnen agendi). Therefore, this avoids the problem of whether the gesture achieves its aim or not; in fact it does, and moreover with a character of unity or totality (omnen), but in a particular way, that is, once the movement begins to occur, the aim also begins to take its form, by means of a gradual and continuous process. The aim is not suddenly achieved in a last moment, its unity is in gestation from the beginning and is open to generate new gestures. That unity is not only abstract but expressed in the configuration of the body's limbs, or rather is an entire entity composed of visibility and non-visibility. *The gesture determines itself in so far as it evolves*: the first state need not include a full awareness or intellectual operation of how the gesture will be. Recalling Châtelet [29, p. 9]:

> — the gesture is not substantial: it gains amplitude by determining itself. Its sovereignty is equal to its penetration and that is why we refer to the gesture's 'accuracy': the precision of the strike is proof of the reverberation of its skill.

The gesture is unimportant if it does not succeed in its searching for unity, its power lies in its capacity to take a form and a proper unity and, of course, its ability to multiply. Therefore, the body, the movement, and the aim are actual, and certainly, in the following discussions we will point at that identification between being and its acts, oriented by the phenomenological thought of Peirce and Merleau-Ponty. According to Peirce [35, p. 76]:

> For as long as things do not act upon one another there is no sense or meaning in saying that they have any being, unless it be that they are such in themselves that they may perhaps come into relation with others.

---

[4] Recursion has already been included as one of the main features of Mazzola's mathematical musicology [24], so probably our discussion could easily be included in such a framework.

[5] However, regarding a gesture as the movement of the body with a purpose that may be realized or not, the body is secondness (and refers to the aim, a purpose), the aim is firstness (a mere possibility), and the movement is thirdness (the mediation between the body and the objective). But since there may be no realization of the aim, this possibility does not seem to be so fecund.

As a beginning of a gesture, a first state of the body is firstness. We stress that the beginning (and the end) is not regarded in our discussion as a mere point in the space-time, which would correspond to a conception of topological spaces as composed of well-determined and separate atoms or points, but as a neighborhood or a curvature of space-time, in a *locale* or a *Grothendieck topos*, in which the beginning (or end) occurs. As the end of a gesture, the aim is secondness. It corresponds to the dyadic character of the achievement of that aim: the realization of the aim necessarily refers to a first state of the body. According to Peirce [35, p. 76]:

> *The beginning is first, the end second, the middle third. The end is second, the means third. The thread of life is a third; the fate that snips it, its second.*

To conclude, the gesture is thirdness since it is the mediation between the first state of the body (firstness) and the aim (secondness), and hence the presence of thirdness in the definition of gesture is unavoidable and constitutes one of the main pragmaticist features of this definition. This discussion is compatible with the conception of Mazzola (Section 6.1) that *the gesture is a morphism, where the linkage is a real movement and not only a symbolic arrow.* Certainly, in a morphism $f : A \longrightarrow B$, firstness (A), secondness (B), and the suggested thirdness ($\longrightarrow$) are quite explicit.

Now we add to our discussion a new term from Saint-Victor's definition: *modality* (modum). If the movement takes place having a modality, then we may relate this to the three categories of Peirce, thus finding a new trace of pragmaticism in the definition of gesture. We use this feature to give modal sub-determinations of gestures. This recursive possibility offered by the approach from Peirce's thought testifies a methodological richness to be exploited.

Let us consider the instance of piano performance. But, before embarking on the modalizations, there is an introductory *dialectic between the body of the performer and the instrument* that may be considered. In the case of the pianist, this dialectic can be framed in a wider dialectic *fixed/variable*. Certainly, in an initial stage, the piano, regarded as an external structure, is completely static if we ignore for a moment the vibrations and sounds that it produces thanks to the touch of the human body. The piano has forms that are perfectly defined, and concerning the keys, they are also of a great perfection in shape and flatness, and a rigorous pattern repeats along its little more than seven octaves. Though the keys have a fundamental possibility of movement, it occurs within a limited space and essentially in two directions. In contrast, the body could be described in almost an opposite way. The fingers are rough and sticky, so they can produce an effective touch of the flat surface of the keys, and they are utterly changing, elastic. When the performer faces the piano and touches it, the resistance of the piano to being dominated and partially modified cedes to the body. On the other hand, the fingers adapt to the forms of the keys, as well as the hands which, as many pianists can testify, change their shape and become structured in a more consistent way (a kind of claw of tiger or wolf) in so far as practice, or technical training, is carried out. These modifications of the body can reach the point of changing the configuration of the spine, as in the case of Glenn Gould. After this introduction, three initial modes of the gesture regarding the piano can be considered.

1. A raw movement, potential but not reactive. Not every movement of the performer produces an effective touch of the piano (and hence a sound), even though the gesture could be related to humming, which would open up a new recursive possibility regarding the voice as an instrument. These gestures in a state of firstness are those auxiliary gesticulations of the limbs that suggest moods, directions, waves, or continuities; and that are spontaneous and immediate. In some respect, they constitute an envelope made of flesh and thickness for further stages of sound. Here we recall the famous hummings and envelopes of Glenn Gould and Keith Jarrett.

2. The movement that effectively acts on the instrument and produces sounds. This is the active-reactive movement of the body, and is made of two fundamentals in Merleau-Ponty's philosophy, which are intertwined by a chiasm: *the touching* and *the touched* [33]. The body, in particular the hands, envelope the instrument and, at the same time, the body is touched by the instrument, that is, it feels the instrument. In a metaphorical mathematical sense, we could see *touch* as a functor from the body to the instrument, which, if interpreted as a presheaf on a monoid and if we make an analogy between monoid and body, could express an action of the body on the instrument. Besides touch, or the active character of the movement, and a first reaction or the feel of the instrument, there is another reactive character of that touch of the instrument: sound. When the body touches the instrument, the instrument vibrates and this vibration, when propagates through the thickness of air and space, is the sound itself. The situation of a pure active-reactive gesture on the piano could be roughly ciphered in a categorical way as the diagram of functors

$$\begin{tikzcd}
\text{body} \arrow[r, shift left=3, "touch"] & \text{instrument} \arrow[l, squiggly, "sound"'] \arrow[l, shift left=3, "feel"] \arrow[r, squiggly, "sound"'] & \bullet
\end{tikzcd}$$

where *touch* is as above, the functors *sound* can be regarded as different projections or prolongations of the instrument on its environment, and $feel$ corresponds to the counterpart of the functor *touch*, that is, the direct action of the instrument on the body. The pairs of functors *touch*/*feel* and *touch*/*sound* could be regarded as adjunctions (Section 6.4), which would express a dialectical relation between the pairs of terms, and moreover the possibility of a mediation between them, which could be related to a chiasm. Note that the mathematical fact that adjoints are unique up to natural isomorphism could express, in this situation, a certain identification between the *feel* of the instrument and the *sound* that affects the body. Also, it is important to stress that the functor *touch* corresponds to a genuine gesture of the body.

3. Coordination-performance: the movement mediates between the touch of the instrument and the sounds that are produced. A first mediating instance is *hearing*, or the reactive action of the sound on the body with a certain degree of consciousness[6]. The body is embedded in sound and is enveloped by it, but it reverberates inside the body, makes it vibrate,

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[6] As we will observe at the end of this chapter (Subsection 6.9.4), this definition is insufficient since perception cannot be reduced to the action of the perceived on the perceiver. Moreover the introduction of consciousness is also problematic since its definition is far from being clear. However, this first approximation is useful in these early stages of modalization.

in particular, it affects the brain. The sound is also an extension of the instrument, not a mere physical phenomenon, so to that extent it is accurate to say that we hear the instrument (recall our previous identification between sound and the feel of the instrument). The sound is a gesture, a movement, and the process of hearing is its active-reactive character in the body. Hearing may be conceived here in a wider sense than usual, and in fact, it can be included in the dialectic *hear/instrumentalize* that will appear in Section 6.4. By means of hearing and its diverse reverberations, the movement of the body is modified and hence the sound, so hearing mediates between touch and sound. It is difficult to conceive a performance without a coordination. Hearing and feeling the sound and the instrument are first instances of interaction or communication in the performance, and allow the performers to take their own decisions about how their movement should become so as to have the appropriate modality when fusing touch and sound. This is the essence of performance. These modalities are the particularities of each performer, their modes of being regarding the instrument. This opens up a new opportunity to apply the three Peircean modes.

In a vague way, three further subdeterminations of coordination-performance can be distinguished, according to the Peircean categories. In this case, the modality is introduced in the way that the sound is produced: a spontaneous one, one given by the opposition between forces, or one given by a law or design.

3.1. Free improvisation. Probably related to a strong back-and-forth between conscious and subconscious states of the body-mind, free improvisation is spontaneous and is hardly based on previous references or on limitations of musical resources (space, techniques, instrumentation). This is fully exemplified by Mazzola's vision of his activity as a free jazz pianist as related to immediate gestures rather than mediating formulas. As in the mathematical notion of free object, in free improvisation the structure is not totally absent, but is skeletal. Note the relation to (semi-)simplicial sets (which we have called skeleta in our mathematical discussion in Section 3.3) and, in particular, digraphs: these structures can be interpreted as sketches of spaces made by means of triangular approximations, their incarnation as spaces being obtained by the process of Milnor's *geometric realization* of (semi-)simplicial sets (see Subsection 3.5.2), which can be expressed as a left adjoint and hence consists of free objects. The skeletal configuration allows a high capacity of transformation and of internal movement (free objects have special properties of projection on the objects of the same category), which tend to *multiplicity* and *freshness*.

3.2. Thematic improvisation. Related to the dialectics *facts/concepts* and *visible/non-visible*, it is the improvisation that consciously uses a determined system of musical techniques or concepts. If we are located within jazz standards and academic traditions of music, we can define improvisation as the live creation with a wide knowledge of musical techniques (harmony, counterpoint, composition), though, especially in jazz, there may be important exceptions in which an extraordinary intuition can replace technique.

3.3. Mediated interpretation. Perhaps the most important mediation in music performance is vision, whether actual or as an image in the mind. Interpretation, in traditional sense, takes place according to a score, a prescription, a gestural discipline. At this point of considerable triadic elaboration, the fundamental sign of the following discussion emerges in a natural way. The sign is Mazzola's triad *sounds/scores/gestures* of Western performance,

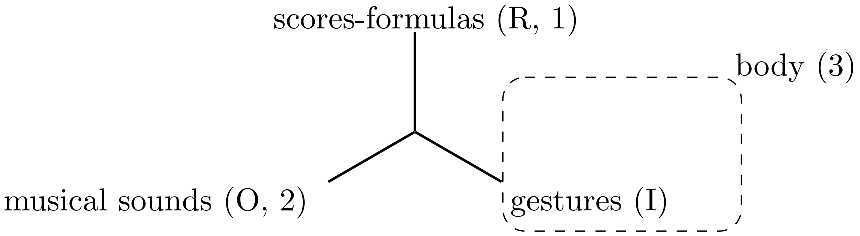

Figure 6.1: The triad of Western performance as a Peirce's sign.

which we will study in the next section.

6.3 Peirce: semeiotics and pragmatics

According to Peirce, knowledge occurs through signs. A *sign*, in Peirce's sense, is a ternary relation $S(1,2,3)$ (hence thirdness) consisting of an *object* O (or second 2), a *representamen* R (or first 1), and a *quasi-mind* (or third 3), such that the representamen replaces the object for the quasi-mind, in which the representamen is transformed into the *interpretant* I of the sign. A main feature of this definition, stressed by Peirce himself [35, p. 100], is that the triadic relation is not reduced to any complexus of dyadic relations between pairs of terms. In the case when such a reduction is given, we say that the triad is *degenerate*. On the other hand, the quasi-mind need not be related to a human mind; rather it corresponds to the interpretation context where the *semeiosis*, that is, the transformation of the representamen into the interpretant, occurs. Similarly, the terms O, R, and I are quite flexible: they can be things, concepts or even signs, giving place to an endless semeiosis.

According to our general orientation (Section 6.2), we can locate *Western musical performance* as part of what we have called *mediated interpretation*, which has been obtained as an iterated thirdness. For this reason, there is no surprise in the fact that there is a sign associated with Western musical performance. This sign can initially be schematized by means of Figure 6.1. There, the object (2) of the sign is *musical sounds*, the representamen (1) is *scores-formulas*, and we regard the *body* as the quasi-mind (3) where the scores-formulas are transformed into *gestures*. These terms deserve a further clarification. First, the term (musical) *sounds* refers to sounds as real objects, acoustical phenomena, or as musical ideas. Second, the term *scores-formulas* refers to the real scores or musical notations used by musicians, but can also be ciphered mathematically as diagrams in suitable categories; after all, a score is a diagram in a space, containing traces of both discontinuity (for example, notes) and continuity (for example, dynamics). In fact, this identification between formulas and diagrams makes sense since diagrams can be regarded as generalized systems of equations (formulas), their limits being generalized sets of solutions. Third, the definition of gestures is that given by Saint-Victor (Section 6.1), on which we have based the different notions of gestures for several notions of space (topological spaces, locales, and Grothendieck topoi) along this monograph.

The exhibition of the sign is only the point of departure for the process of endless semeiosis. Certainly, formulas and gestures are different representations of sounds, and more dramatically, the terms O, R, and I are transmutations of each other: scores-formulas produce gestures in the body, which produce new sounds by means of the interaction with an instrument, which in turn produce scores in an appropriate (quasi-)mind (think of the process of transcription of a musical piece, or some process of codification of sound), and so on, in an spiral that is repeated endlessly. Within this logic of endless semeiosis, there are more possibilities. First, the musical sounds considered can also be representations of previous concepts; in fact, musical ideas can be related to previous ones or even to non-musical ideas like those of poetry or some kind of text. Also, scores can be the interpretant of previous ideas that represent musical sounds; for example, motives, or the subjects on which fugues are developed.

Endless semeiosis can be related to a mathematical fact studied in Chapter 3: formulas-diagrams and topological gestures are particular instances of the notion of *Mazzola's gestures* and hence of the more general notion of *abstract gestures* (see Subsection 3.3.3). For this reason, many possible representation-interpretation contexts of musical sounds could be ciphered in the *category of gestural structures* (Section 3.8), where the different transmutations of musical sounds should happen. But there is another concept hidden behind that of gestures, which is useful for our discussion. The concept of *abstract gestures* is precisely the dual (in a precise categorical sense) of that of *realization* (see subsections 3.4.2 and 3.4.3), which is a well-established concept in mathematics, and was one of the first examples of left adjoints given by Kan; see his foundational article on adjoints and the *realization/nerve* adjunction in Subsection 3.4.2. It is surprising how the mathematical category-theoretic term *realization* is also a very musical term (we speak of the realization of a motive or a musical idea) and fits in our setting for gestures. Besides the categorical duality *gestures/realization*, the concept of Mazzola's gestures is more strongly related to realization since, for example, topological gestures and formulas can be characterized as morphisms whose domains are suitable realizations: a topological gesture is essentially a continuous map from the geometric realization of a digraph Γ to a space X (Theorem 1.5.4), and a diagram-formula is a functor from the free category (a suitable realization) on a digraph to a small category $\mathcal{C}$ (see Subsection 3.11.1). This means that topological gestures and formulas correspond to two particular realizations. In this way, we can regard endless semeiosis as the process of interpretation-representation of musical sounds in diverse contexts (gestural structures) associated with different mathematical realizations together with (functorial) comparison of these interpretations-representations. Thus, musical analysis made by means of nets, which is related to Lewin's transformational theory, can also be regarded as a semeiotic process since nets are the points of the limits of diagrams of abelian groups (or more generally, modules on a commutative ring) and affine transformations between them. These diagrams are gestures in a wide sense, are related to a gestural structure[7], and are new interpretants for the representamen scores-formulas, thus forming a new sign.

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[7] Though there is a subtle point here. The category of abelian groups is not small, so in principle we cannot view diagrams in this category as gestures since our result (Subsection 3.11.1) only applies to small categories. This is not a serious problem, which could be fixed by considering universes larger than that of sets satisfying the ZFC axioms.

Thanks to this discussion, we realize that it is not possible to grasp the richness of our sign *sounds/scores/gestures* by the exclusive examination of the dyadic relations of the triad, though an approximation by pairs of adjoints between the terms of the sign will be studied in Section 6.4. At this point, it is important to ask what is the concept or feature that keeps the triad indivisible. On the one hand, we can relate this to the pragmaticist maxim and to the Yoneda lemma, which assert that to understand an object (in this case musical sounds and musical ideas) is to view it from all possible perspectives, discarding the preference for biased ones. On the other hand, we have Merleau Ponty's *chiasm*, which we will study in Section 6.6.

6.4 Galois: adjunctions and dialectics

The idea of *adjunction*, one of the most important in category theory, has its origin in the works of Galois on solubility of polynomial equations by radicals. Instead of using long calculations to approach the problem, Galois discovers the structural reasons behind the impossibility of obtaining a formula that provides the zeros of a polynomial of degree greater than four, giving rise to group theory and Galois theory. The structural essence relies on the *Galois connection* between the poset of intermediate fields of a field extension $F \subset E$ and the opposite of the poset of subgroups of the Galois group of the extension; this being useful for the problem of solubility of a polynomial when F is the ground field and E is the splitting field of the polynomial. This concept is so powerful that it has a life of its own, and appears in other important places in mathematics whether in its form of Galois connection (case posets) or of adjunction (in general categories, concept written down for the first time by Kan in 1958): Galois connection between the poset of subsets of the affine n-space k^n (for k field) and the opposite of the poset of ideals of the polynomial ring $k[x_1, \ldots, x_n]$ as a foundational fact of classical algebraic geometry, free objects in diverse categories (free groups, free modules, etc.) are fragments of adjunctions, tensor products (of rings, modules and functors), adjunction $pt \vdash \mathcal{O}$ between locales and topological spaces (see Section 2.4), localic reflection (Subsection 4.3.1) as link between locales and Grothendieck topoi, etc. Adjunctions express relations more profound and dynamic than those of mere inversions or bijections, establishing a precise and fine tool for expressing dialectical relations between poles. Though this concept will be used as a mathematical metaphor in the following discussion, it is important to get acquainted with the formal definition as exposed, for example, in [14, Ch. IV].

Once the adjunction is established, the crucial task is to make explicit the *underlying equivalence*. According to Proposition 7.0.1, this equivalence is the restriction of the original adjunction to the subcategories of fixed points or invariants under the back-and-forth of the adjunction; see Figure 6.2. This feature is the mediation, the ambiguity of concepts behind the opposition expressed by the adjunction, the way of dialogue between opposites; thirdness in Peirce's sense. This task is thus associated with the characterization of these subcategories of fixed points, which are usually related to important concepts: in the case of Galois theory, the mediation is given by conditions of *finiteness*, *separability*, and *normality*, and gives rise to the fundamental theorem of Galois theory; in the case of algebraic geometry,

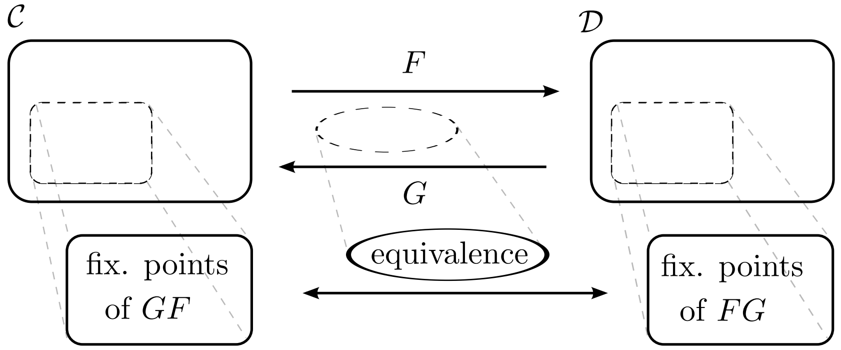

Figure 6.2: An adjunction between two functors F and G, and the induced equivalence.

the mediation establishes the correspondence between *radical ideals* of the polynomial ring and *algebraic* subsets of the affine space (Hilbert's Nullstellensatz); and in the case of the adjunction $pt \vdash \mathcal{O}$, the mediation is precisely the correspondence between *sober spaces* and *spatial locales*.

Now we proceed to approach the sign *sounds/scores/gestures* by means of pairs of adjoint functors between its terms. First, we enrich the initial diagram of Western performance (Section 6.1) as shown in Figure 6.3. This enrichment is mainly obtained by using Peirce's semeiosis as follows. First, we add suitable representation contexts or quasi-minds, namely body (producing gestures), instrument (producing sounds), and (quasi-)mind[8] (producing formulas); then we add the functors *hear* and *compression*, which render the inner triangle and the outer triangle of the diagram commutative. We can understand each functor as part of a sign whose object is *musical ideas*, and whose representamen is the domain of the functor (the term in the respective box), which is transformed into a new interpretant in the quasi-mind associated with the codomain of the functor (a box). Moreover, each functor can be interpreted as a categorical *action*, in metaphorical sense: scores-formulas act on the body to produce gestures, which act on the instrument to produce sounds, this entire process being the action of formulas on the instrument or *performance*; on the other hand, sounds act on the body to produce gestures, which act on a (quasi-)mind to take the abstract form of formulas, this entire process being the action of sounds on the (quasi-)mind or *compression*.

The modeling by categorical actions that we propose may be insufficient because in what concerns the polarity *sounds/gestures* perception is strongly involved (touch and hearing) and perception, following Merleau-Ponty, is related to a chiasm (which we will study from the perspective of adjunctions) rather than to the mere action of a thing on the body. Nevertheless, categorical actions have features that are far from classical approaches. As we have observed before, we call our functors actions given the analogy between categories and monoids and because an action from a monoid to a set is just a functor from the category of the monoid[9] to the category of sets. With this analogy, we access the realm of spatiality

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[8] Here, we stress the term quasi-mind since it could be a computer transforming sounds or gestures into codes or formulas.

[9] Given a monoid $(M, *)$, its category has a unique object, as morphisms the elements of M, and as composition the operation $*$.

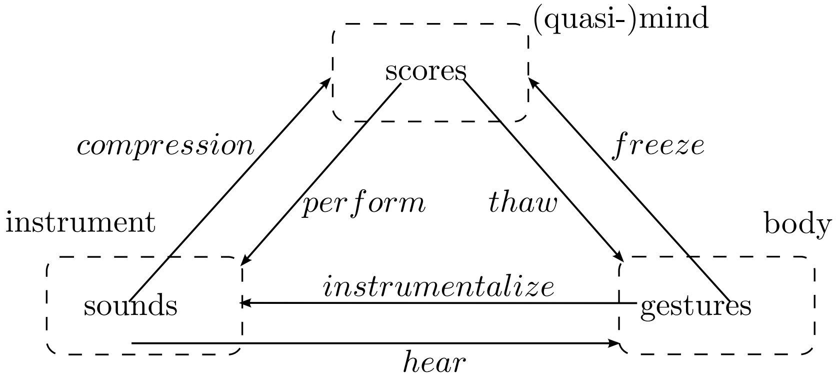

Figure 6.3: Enriched diagram of Western performance and its approach by pairs of adjoint actions.

as follows. Given a monoid M, its associated topos of monoid actions, which is the category of functors from the category of M to the category of sets, is useful to show how from the action it is possible to obtain space-time, in a remarkable inversion of usual prejudices. Zalamea's argument [37, §2] consists in showing that the intrinsic logic of this topos (that of the Heyting algebra $Sub(\Omega)$ of subobjects of the classifier[10] Ω) is classical, that is, that $Sub(\Omega)$ is a boolean algebra, if and only if M is a group. Thus, if M is a monoid that is not a group, then the intrinsic logic of the topos of actions is intuitionistic, which forces the development of time, through the associated Kripke models[11]. In a more direct way, note that the temporal aspect of monoid actions can be obtained by the fact that the operation $*$ on the monoid M defines a preorder[12] $\leq$ on M by decreeing

$$x \leq y \text{ if and only if there exists } a \in M \text{ such that } a * x = y,$$

so the action is related to an order that could express an internal variation of the monoid action through time. But there is a geometrical counterpoint to this logical point of view. The category of monoid actions, like any category of presheaves on a small category, is a *Grothendieck topos*, that is, a generalized space, or rather, a space in its own right. In particular, and so as to show the link with classical spatiality, recall that the set of subactions (subobjects) of a given action has the structure of a complete Heyting algebra, that is, that of a locale. Moreover, this kind of locale has enough points according to Proposition 2.8.1, so it is the lattice of opens of a certain topological space. Finally, under an Einsteinian perspective, spatiality involves time. The spatiality of Grothendieck topoi and locales will be commented in more detail in Section 6.9.

Now we give a further discussion on the pairs of functors.

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[10] In this particular case, the classifier Ω is a certain natural action of M on its left ideals.

[11] A Kripke model is a functor from the category of a fixed partially ordered set to **Set**, that is, a presheaf. A Kripke model can be regarded as a family of sets of truths varying through a time expressed by the underlying ordered set. The strong relation between Kripke models and intuitionistic logic relies on the fact that the former make up a complete semantics for the latter (see footnote 343 in [39, p. 283]).

[12] Reflexive and transitive relation.

First, as said before, the functors *freeze* and *thaw* between formulas and gestures resemble the relation between mathematics and music proposed by Mazzola. The functor *freeze* corresponds to a projection of gestures on formulas-scores. In this instance, we could think of a suitable functor from topological spaces to abelian groups and affine transformations between them, or to some algebraic category, which should be a morphism of gestural structures, abstracting the real movements to yield formulas. The functor *thaw* corresponds to an incarnation of formulas in terms of gestures, in the same category of gestural structures. A satisfactory mathematical formulation of the functors *freeze* and *thaw* has not been given yet. We know that, at bottom, musical formulas and gestures are the same thing (particular cases of the general notion of abstract gestures), but the ways of transformation into each other are not clear. However, some work has been done. On the one hand, we know a way of assigning a space to a group using algebraic topology because every group is the fundamental group of a space (fact), so we can regard the operation in the group as a (gestural) concatenation of curves in a space. In this case, we can recover gestural information from algebraic structures related to formulas[13]. On the other hand, given a sober space X, we know a way of assigning an algebraic structure to the space of gestures $\Gamma@X$, without loss of information, namely the locale of gestures $\Gamma@\mathcal{O}(X)$, because the space of points of $\Gamma@\mathcal{O}(X)$ is essentially $\Gamma@X$ (Corollary 2.4.3). Hence, we have a way (computational, by the way, since locales are geometric theories) of manipulating gestures on sober spaces, including Hausdorff and Euclidean ones, in algebraic terms; moreover the functor pt is a morphism of gestural structures[14]: the space $pt(\Gamma@L)$ is homeomorphic to $\Gamma@pt(L)$ (Section 3.8).

Second, we have the functor *hear*, which is the adjoint to the functor *instrumentalize*. Note that the latter is more related to *touch*, while the former is more related to *feel*; in fact, we have defined *hearing* as the reactive action of the sound on the body with a certain degree of consciousness. This action in the body produces gestures, which need not be related to any intellectual operation, they are like reflexes and reverberations of sound and envelopment. A further discussion on the emphasized terms in this paragraph will be given in Section 6.6 under the guidance of the late thinking of Merleau-Ponty.

Third, we have the functors *compression* and *perform*. The functor *compression* is the composition of the functors *freeze* and *hear* and is a process that resembles that of dictation-transcription of a musical piece; it is a process of abstraction of the substance of sounds, synthesis, and abstract representation in terms of symbols in their Peircean meaning. Regarding the functor *perform*, we would prefer the term *decompression*, since it seems restrictive to consider performance as the mere composition of the actions *thaw* and *instrumentalize*, omitting *hear* and *freeze*. As we have seen before, we consider interpretation as a mediation between the action of gestures on the instrument and their reaction in form of sounds. The levels of moderation can vary, but the importance of hearing in performance is unavoidable, as in the case of Western tradition. Also, the functor *freeze* may be important in performance because the performer may need to do auxiliary representations of the score in his mind (for

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[13] Here we are using the term formula in a vague way, relating it to both groups and locales.

[14] Certainly, our definition of *gestural structure*, was inspired by this example. However, we do not claim originality here: our notion of preservation of gestural structures is basically the notion of preservation of Kan extensions.

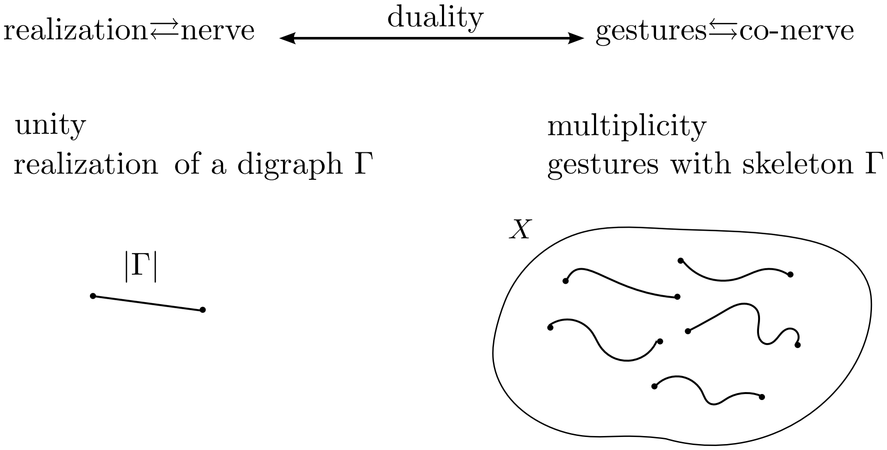

Figure 6.4: The duality realization/gestures. Here, the digraph Γ is a single arrow.

example harmonic or of form) during the performance. For that reason, performance could be more related to the interactions of sounds, gestures, and formulas.

6.5 Kan: extensions and concepts

Kan extensions [14, X.3] are particular examples of free objects, that is, they are fragments of adjunctions. They are useful to solve the problem of extension of the domain of a functor along another one, in an optimal and universal way. Surprisingly, this concept is a *transmutation* of the concept of adjunction because adjunctions can also be shown to be particular examples of Kan extensions; see Theorem [14, X.7.2]. Kan extensions are important in our discussion on gestures because the contravariant gesture functors (see Subsection 3.4.4 and Section 3.6) are suitable right Kan extensions. Moreover, *the concept of gestures is the dual of that of realization* (Subsection 3.4.3), the latter being a particular example of left Kan extension made up from tensor products. Also, since realization has an associated adjunction, namely the *realization/nerve* one (realization being the left adjoint, see 3.4.2), by duality, the gesture functor is a right adjoint; this situation is illustrated in Figure 6.4.

The duality *realization/gestures* deserves a further comment. In the preceding sections, we have based all our discussions on the concept of gestures; but according to the duality *realization/gestures*, this discussion should have a negative side. Consequently, one may ask for a philosophical definition of realization. To obtain an answer, we first note that, as suggested by the presentation of the space of gestures with skeleton Γ (digraph) and body in X as a function space $\mathbf{Top}(|\Gamma|, X)$ established in Theorem 1.5.4, and by the presentation of the category of gestures with skeleton Γ (digraph) and body in a small category $\mathcal{C}$ as a category of functors $\mathcal{C}^{Path(\Gamma)}$ made in Subsection 3.11.1, *realization corresponds to the abstract geometric form (unity) of which gestures (multiplicity) are all the possible incarnations in the space X (respectively, category $\mathcal{C}$).* Here it is worth stressing the duality *unity/multiplicity*: for example, the realization $|\Gamma|$ of a skeleton is *a topological space* with a marked unity since it is a free object, whereas the space of gestures $\mathbf{Top}(|\Gamma|, X)$ consists of a *multiplicity of* *gestures* (continuous functions), all the possible incarnations of the free object $|\Gamma|$ in the space. This leads us to think of music (and the other performing arts), which deals with gestures, multiplicities, incarnations, and *variations*, as a discipline located on the opposite position of geometry, which deals with abstract shapes and their *invariants*. On the other hand, regarding the Peircean categories, note that the concept of realization of a skeleton is marked by firstness since it is a free object, a potentiality, and a unity, in utter contrast to the object of gestures (not the concept of *a gesture*, discussed before as a form of thirdness), which is marked by secondness since it consists of all the reactions of the realization in a given space.

Then we ask whether there is a dual of the sign *sounds/scores/gestures*. As we have said repeatedly, both gestures and scores are different incarnations of the concept of abstract gestures and are associated with certain realizations, which are just the dual concepts behind gestures. In this way, we obtain that *geometric realization* is the dual term of topological *gestures* and that *scores-formulas* is the dual of some free construction, say the construction of *free categories*, given the analogy between diagrams and scores-formulas. Therefore, it remains to find the dual of *sounds*. Since topological gestures and formulas are representations of sound, the desired dual of *sounds* must have as representations the different realizations, including geometric realization and free categories, and hence the most natural candidate, in a mathematical sense, is the concept of digraph. In a wider conceptual sense, the concept of digraph is related to that of sketch, draft, and vision, but a first one with a few definite features. Moreover, in a musical sense, digraphs could also be related to *the sound spectrum*, which is a sort of sketch or projection of sound. In this way, the dual of the triad *sounds/scores/gestures* is *skeleta/free categories/geometric realization*; moreover, we can regard this sign as the beginning of a new semeiotic process: the different representations of skeleta and their relations (recall Gabriel's representation theory of digraphs). This sign suggests a relation with painting, though probably it is more related to the techniques thereof or the modes of representation rather than to the act of painting itself, which is mainly gestural. Probably, this triad is more related to the mental images of the painter and their transformations, and recalling Merleau-Ponty, with *spirit* (mind in some translations) or the non-visible counterpart of the body [33, 32]. The whole picture of our duality between triads corresponds to Figure 6.5.

There is another reason why Kan extensions are relevant in our discussion: the famous slogan '*All concepts are Kan extensions*' [14, X.7] of Mac Lane. If we adhere to Mac Lane, we have located gestures as a true concept of category theory that deserves to be studied and that has a mathematical interest regardless its musical origin, though, of course, the importance of the concept, in a wider philosophical sense, relies on its substance both mathematical and musical. Finally, the concept of gestures that we have been trying to elucidate has the following features (see the mathematical part of this thesis for details):

i) The gesture functors are *right Kan extensions*.

ii) The concept of *gestures* is the dual of that of *realization*, the case of *geometric realization* (case when the ground category is that of topological spaces) being a well-established concept in algebraic topology .

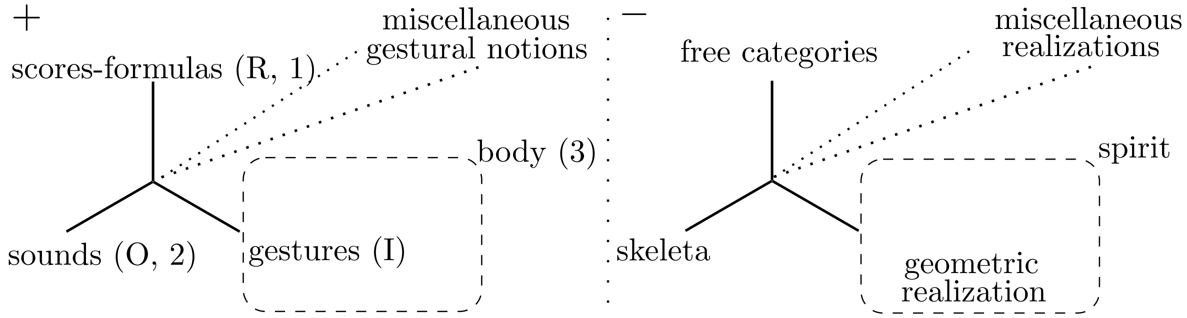

Figure 6.5: The duality between the signs *sounds/scores/gestures* and *skeleta/free categories/geometric realization.*

iii) Since the realization of a skeleton is an instance of *tensor product*, it has an associated adjunction *realization/nerve* (with realization being a *left adjoint*, and in fact, a *left Kan extension*), and therefore the object of gestures is a *co-tensor product* with an associated adjunction *gestures/co-nerve.*

iv) The abstract gestures contain as instances several concepts of mathematical music theory in an optimal way: diagrams in categories, musical formulas, gestures on topological spaces, gestures on locales, gestures on topological categories, and gestures on Grothendieck topoi. Thus, it is at the core of the fundamental tensions *gestures/formulas* and *continuous/discrete*, relating mathematics and music.

v) The gesture functors are sheaves on natural Grothendieck topologies on the category of skeleta (see Subsection 6.9.1).

6.6 Merleau-Ponty: the chiasmata of music and the raw being

In *The Visible and the Invisible* [33], Merleau-Ponty provokes us to criticize the deceitful extremes, but without falling in a vain relativism, proposing a mutual fertilization in which the identity of each extreme is only achieved by their transfiguration into the other. Merleau-Ponty intends to come back to things themselves, the flesh of the world, that is, to the visible, though taking into account the invisible, that is, mind, knowledge, truth. Moreover he intends to propose a true continuous bond between the visible and the invisible, the *chiasm*, rather than the classical detachments *subject/object*, *being (self)/world*, *body/mind*, *idealism/realism*, *essence/existence.* This should be the point of departure for a new construction of our concepts of body, flesh, perception, thing, world, being.

Certainly, the gestural approach to mathematics, and in particular to mathematical music theory; that is, the recovering of the gesture behind the phantom morphism, corresponds to Merleau-Ponty's effort to recover the experience of things, the flesh of the world, the movement of the body in it, and the contact, rather than structures or laws. To introduce gesturality, even in a mathematical way, is to introduce an attempt to go back to the world; and it is important not to forget that even the mathematical world bears an immense resemblance to our carnal world. If the mathematical world were totally isolated from the flesh of the world there would be no necessity of naturalness, expressiveness, and intuition in mathematics. For example, if topology were such an isolated piece, there would be no necessity of Grothendieck topologies, in which there is a passage from the idea of points-elements to a true reaffirmation of the concept of neighborhood, and there is a primacy of the concept of cover: the introduction of a notion of envelopment that resembles that of human body.

According to the *Oxford Dictionary*[15], *chiasm* or *chiasma* is 'the X-shaped structure formed at the point below the brain where the two optic nerves cross over each other'. The origin of this word is Greek, from the word *khiasma* 'crosspiece, cross-shaped mark', which comes from *khiazein* 'mark with the letter χ', that is, from a gesture. This definition is mainly related to vision, and thanks to Merleau-Ponty it gains an astonishing amplitude in its extension to the whole domain of perception: there is a chiasm of vision, a chiasm self/world, a chiasm of touch, and a *chiasm of phonation and of hearing* [33, p. 144]. Indeed, there is an *aural chiasm*, a place where the auditory nerves cross over each other, which is not as evident as the visual chiasm from an anatomical point of view, but of which we have enough evidence, as phenomenon, from the unity of the experience of hearing. For this reason, the definition above is not suitable for our discussion oriented towards music. However, our discussion would probably pose a false problem if we confine ourselves to a mere application of Merleau-Ponty's insights to hearing; rather we must pose the problem of a true reintegration of the different senses. As an evidence of this possibility of reintegration, it is worth recalling Peirce's words regarding qualities (firstness: colors, sounds) and the unity and the multiplicity of our senses [35, p. 77]:

> (...) it can hardly be doubted that the specializing effect of the evolutionary process which has made us what we are has been to blot the greater part of the senses and sensations which were once dimly felt, and to render bright, clear, and separate the rest. (...) The qualities merge into one another. They have no perfect identities, but only likenesses, or partial identities. Some of them, as the colours and the musical sounds, form well-understood systems. Probably, were our experience of them not so fragmentary, there would be no abrupt demarcations between them at all. Still, each one is what it is in itself without help from the others. They are single but partial determinations.

This suggests, since derivatives of a concept (in this case, perception) are as important as the pragmaticist integral of them, the possibility of a *synesthesia*, which could be regarded as a chiasm of higher order, now between senses and not inside a particular one; a gluing of senses that would allow a better experience of the world and a greater openness to it. There are physiological obstacles, but at least this corresponds to efforts to open up a dialogue among disciplines that once were conceived as part of the same body of knowledge but that modern specialization has separated. In this way, we could give a general philosophical definition of

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[15] `http://en.oxforddictionaries.com`.

chiasm as a gesture in which the modality is the crossing between two (or more) fibers or limbs. In fact, following Saint-Victor's definition, we thus have the movement of two fibers and a configuration of them that could be expressed as a certain X-shaped digraph, these two fibers point at a common objective as in the case of vision and hearing, and there is a fundamental modality: a crossing already expressed in the form of the digraph but that needs to be reaffirmed.

In the case of an adjunction, it is valid to give an interpretation of its fixed points in terms of a chiasm; in fact, if we are faithful to Mazzola's vision of recovering the gesture from the morphism (in particular, from a functor), then we can regard the functors of an adjunction as two gestures forming a χ, and moreover, we can regard the induced equivalence between subcategories of fixed points as the place where the two gestures cross. This is a very remarkable trait of adjunctions, at bottom, the underlying equivalence is expressing the contact surface of the chiasm that formed the pair of adjoint functors if they recovered their gestural essence. Thus, the concept of adjunction, far from being a bad dialectic of mere inversion or even of bijection or equivalence, could be interpreted, recalling the possibility of vision of the (in principle indiscernible) roots of a polynomial achieved by Galois with the field automorphisms and the associated Galois connection, as a fundamental tool for passing from the invisible to the visible by recovering the surface of contact of a chiasm, even if the original underlying continuity of the fibers remains a phantom (recall the phantom character of morphisms). Beyond Merleau-Ponty's criticism of the possibility of expressing dialectics and mediations between extremes in terms of thesis and statements, adjunctions offer a first approximation to catching dialectic in a precise way, though in a metaphorical and mathematical way, but with an astonishing effectiveness as testified by Galois theory, algebraic geometry, topology, and in general mathematics with a strong categorical flavor; this effectiveness should not be exclusively mathematical, but could also be extended to other branches of culture.

Now we can try to give an idea of what the fixed points of the adjunctions involved in the triad of musical Western performance are. However, we emphasize beforehand that the chiasm is not only a way to understand the unity of the triad of Western musical performance and the mediations involved between the pairs of adjoints composing this triad, in particular to understand the interpretant 'gestures' that composes the triad, but a gesture itself.

First, consider the case of the adjunction *instrumentalize/hear*, which is a piece of the more general dialectic *touching/touched*, which in turn can be regarded as a fragment of the more general dialectic *being/world*. The instrument, in our case, the piano, can be regarded as a metaphor, in musical practice, for the world, and of course, the performer is an incarnation of being. We are before a situation similar to that of 'man against the dolmen' recalled by Merleau-Ponty [33, p. 7] in which our movements allow our openness to the world and move our perception of it, but do not alter the fundamental solidity of the dolmen or the piano. The chiasm between *instrumentalize* and *hear* is therefore a fragment of the chiasm between *world* and *being* at which Merleau-Ponty's thought points. Yet, in the concrete case of the adjunction *instrumentalize/hear*, the place where instrument-sounds fuse with body-gestures in a non-trivial way is *improvisation*, but a raw one in which there is no mediation beyond the body of the performer and the flesh of the instrument. We thus are not far from free jazz.

The other two chiasmata or mediations are those between the poles *sounds/scores* and *gestures/scores*, that is, between materials for making music and their representations. From a mathematical point of view, since the functors between *sounds/scores* are appropriate composites of the other in the triangle, we can approximate the fixed points of the adjunction *compression/perform* via the fixed points of the other two adjunctions. First, recall the general fact on adjoint functors that given two pairs of adjoint functors (left adjoints on the left)

$$F' : \mathcal{E} \rightleftarrows \mathcal{D} : G' \text{ and } F : \mathcal{D} \rightleftarrows \mathcal{C} : G$$

with units and counits

$$\eta : id \longrightarrow GF,\ \epsilon : FG \longrightarrow id,\ \eta' : id \longrightarrow G'F',\text{ and } \epsilon' : F'G' \longrightarrow id,$$

we have an adjunction

$$FF' : \mathcal{E} \rightleftarrows \mathcal{C} : G'G,$$

where FF' is left adjoint to GG', with unit and counit given by the composites

$$id \xrightarrow{\eta'} G'F' \xrightarrow{G'\eta_{F'}} G'GFF' \text{ and } FF'G'G \xrightarrow{F\epsilon'_G} FG \xrightarrow{\epsilon} id.$$

This ensures that the fixed points of $G'GFF'$ *include at least* the fixed points of $G'F'$ such that their image under F' are fixed points of GF and that symmetrically, the fixed points of $FF'G'G$ include at least the fixed points of FG such that their image under G are fixed points of $F'G'$. In other words, this means that the mediation between the composites includes at least the inter-section of the individual mediations, which is a good approximation for our purposes—though note that the complete determination of the fixed points of the composition of adjunctions is more complex than that inter-section. Thus, to study the mediation *sounds/scores*, we can start by having a vision of the mediation between *scores/gestures*, and then search for common features between this mediation and improvisation.

A first candidate for the mediation between *scores/gestures* could be *gestural scores*, that is, scores with an internal movement. We could consider the case, a very important one to contemporary music (say from XX century onwards), of scores that include *structures in movement*; a feature that is not only present in twentieth-century music and that can also be found in the *mobiles* of sculpture, which in turn recall Miró's painting, or in mathematics, if category theory is regarded as an expression of movement (concrete categories) framed within the stability of particular collections of axioms. Actually, structures in movement belong to the more general dialectics *design/freedom* and *fixed/variable*, which surpass cultural instances and can be observed in[16] the constitution of universe (physics), the forms of government (politics), our beliefs and relation to God (theology), or our behavior (psychology). In music, our main concern together with mathematics, we have at least two ways to introduce movement in scores: we can introduce random variables (discrete or continuous) for pitch,

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[16] Some ideas in that direction can be found in [28].

onset, dynamics, order of the sound events, etc., or, on the other hand, it is possible to leave to the criterion of the performer an internal development as in the case of jazz standard.

After the big step towards atonality started by Schönberg in a sort of affirmation of the *relativity* of the concepts of consonance and dissonance, already announced by the works of late Romanticism, in which the modulations (sorts of localizations) led the composers to the dissolution of the empire of tonality, and as recalled by Merleau-Ponty, to a 'being in indivision' of musical substance[17]; after the Second World War, several avant-garde composers dealt with the edges of this new territory coming up with different solutions. Next we comment on some avant-garde composers that are directly related to structures in movement.

The case of Xenakis is very important since it is a paradigmatic case where there is a conscious use of mathematical tools. According to Xenakis, the problem of the uniform saturation of the sonic spectrum that serialist music poses for hearing, can be overcome, if instead of thinking internally (the detachment of the different voices), we manage the sounds as conglomerates or clouds by means of the external configuration of probability distributions from probability theory, thus having a strong movement in these distributions: chance.

On the other hand, one of the most important tensions in twentieth-century music, namely *choice/chance*, related to other ones as *European art/American art*, *Paris/New York*, was incarnated by Boulez and Cage. To a great extent, Boulez was one of the greatest continuators of the ideas of Schönberg, leading them, in his early career, to their boundaries, so in principle he could be related to choice and determination in music. On the other hand, Cage was probably the prominent figure of a group of composers including M. Feldman and E. Brown, established in New York, who according to [22, p. 344], 'began to make graphic scores that would provoke the human body into performance' as an inverse reaction to 'the fact that the canvases of the New York school bore so vividly the traces of the bodily actions', thus having a strong relation to abstract expressionism (Pollock, de Kooning), though this relation was more of affinity than of opposition. In these composers, it is recurrent the idea of graphic notations and, like in the paradigmatic case of Cage, there is a strong use of chance in their works. A very eloquent example of structures in movement is E. Brown's *Calder's Piece* (1966), a sort of piece conducted by a Calder's mobile (his *Chef d'orchestre*), which also serves as an instrument to be hit with hammers. But the place of Boulez is particularly important because, beyond its opposition to the obsession with chance, at the same time he reflects a mediation between chance and choice, freedom and design, particularly after his Third Piano Sonata (1957), where both the will of the composer and the renewal of the piece in each performance are taken into account. Of course, the influence of Brown and Calder are latent in this new stage of Boulez.

Also, at the core of the polarity *formulas/gestures*, which involves the very relation between mathematics and music (as we recalled in Section 6.1), and includes as a particular case the polarity *scores/gestures*, lies a very interesting mediation: Peirce's existential graphs. It is intriguing to see how the mathematical formulas, as defined by mathematical logic, more

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[17] According to Merleau-Ponty [33, p. 218]: 'atonal music = the equivalent of the philosophy of Being in indivision. Like paintings without identifiable things, without the skin of things, but giving their flesh'. On the other hand, the title 'The principle of ontology: being in indivision' in [33, p. 208] is very suggestive of the place that would have had the term 'Being in indivision' in Merleau's unfinished work.

specifically, classical propositional calculus[18], first-order logic, and some modal logics, can be presented by means of topological gestures in the assertion sheet (modeled, for instance, by the complex plane). This idea is certainly a true chiasm between mathematical formulas and topological gestures, where the deductions are performed as gestural movements (erasures, iterations, deiterations, homotopies) of geometric forms (ovals, pointed ovals, lines) and certain terms. To sum up, we are before a gestural game for expressing logical operations. A deeper discussion on existential graphs can be found in [38].

In this way, our proposal for the chiasm between *sounds/scores* is just the contact (Riemann) surface where the introduction of structures in movement in scores meets improvisation. There are two initial possibilities, which are related to two great currents of improvisation: jazz, and on the other hand, the improvisation linked to academic music; even if the detachment is odious. The dramatic feature here, which brings an important degree of elaboration to our concepts, is the use of scores. In the case of jazz, there is a certain use of scores, which can be avoided in the case of free jazz, though the term *scores* probably has a wider meaning than usual and could be interpreted at least in two ways: we can consider jazz standards as very open scores that are mainly represented in recordings and hence correspond to a sophisticated oral tradition, or we can consider written jazz, which is strongly intertwined with academic tradition. In the other case, that of academic tradition, we have thematic improvisation, as discussed in our first stages of modalization of the body's gestures with respect to the instrument. If the performer's instrumental playing is a valid metaphor for our encounter with the world in Merleau-Ponty's sense (man against dolmen), then improvisation should be a metaphor of our existence in the world, our overture to it, but in a way that is intensified by sounds, which at bottom are themselves flesh of the world; to sum up, improvisation is the perfect metaphor of the ecstasy in the world. As we said before, sound is not only a wave as described by their mathematical models (which could be reviewed under a richer conception, even mathematical), but it is the carnal extension of the instrument, of its carnal existence.

Now we focus on the problem of the chiasm of the entire triad *sounds/formulas/gestures*. As we have seen, from a mathematical point of view, gestures and formulas are just two different incarnations of the concept of abstract gestures, and at the same time, we can regard them as two particular ways of representing sounds. We thus have two manifestations of the idea of sound, each one related to a branch of mathematical music theory (formulas belong to the discrete one, gestures belong to the continuous one). Moreover, these branches are related to Thom's foundational antinomy of mathematics, namely the constant struggle between the continuous and the discrete. Yet, beyond the fragmentation of sound and music into these two branches, a reintegration of sound from formulas and gestures is needed, in a similar way that the partial visions of each eye are reintegrated in a whole Cyclopean vision by means of the chiasm (recalling [33, p. 141]). Thus, we have three initial possibilities for this desired reintegration.

We can regard the conjectural unification of the two branches of mathematical music theory, into the same category harmonizing the category of gestures and the category of

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[18] Though, existential graphs have been recast by A. Oostra so as to show that they are more naturally related to intuitionistic logic, that is, to topological spaces, locales, and Heyting algebras.

formulas and connected to them by means of pairs of adjoints, that is, Mazzola's diamond conjecture, as the chiasm of the triad *sounds/formulas/gestures*, or rather, the surface of contact between the two fibers-branches: the formulaic and the gestural.

On the other hand, our definition of abstract gestures (including the different incarnations in the categories of spaces, locales, Grothendieck topoi, topological categories, categories, and linear categories) can be regarded as another possibility of chiasm, though in this case it is an abstract feature, it belongs to non-visibility, since the gluing substance is not an object or category as in the case of the conjectural universe of the diamond conjecture, but *a concept*. At this point, it must be stressed that our definition of abstract gestures is not intended to be a solution to the diamond conjecture; rather we conceive the topic of abstract gestures as a possible mathematical framework where the conjecture could be formulated more appropriately. In this way, we have been more attentive to a suitable formulation of the problem than to giving a solution.

The third possibility for a chiasm is the introduction of Peircean pragmaticism in our situation. As we have seen, in our category of gestural structures we have those related to formulas and gestures, but we have much more: the different notions of gesturality, including those that come from realizations like the different incarnations of Mazzola's gestures. In fact, what we have is a differentiation of the concept of gesture view from all its perspectives. Then we could ask for a pragmaticist integration of gesture and sound, from all these diverse manifestations, but now regarding this integration as a truly continuous movement, that is, as a chiasm of fragmentary visions. In this case, the pragmaticist integration-differentiation is enriched by the underlying continuity associated with the junction of fibers. Of course, this is only a possibility, which could be explored in further research under the general orientation of Peirce's continuum.

6.6.1 Chiasm, exponential presentations, and painting

The theorems that express objects of hypergestures on digraphs as objects of gestures on skeleta of higher dimensions (Theorem 3.5.16: case topological spaces, Theorem 3.11.2: case small categories, iii) in Subsection 4.1.4: case of an elementary topos), and at the same time as exponentials, can be interpreted as instances of chiasmata of different visions. We next justify this assertion.

First, there is a relation between vision and the notation @ used by Mazzola. In Mazzola's works, the symbol @ is by no means arbitrary. The e-mail address of the author is `jsariasv1@gmail.com`, which means that `jsariasv1` *is located at* `gmail.com`, or that `jsariasv1` *is addressed at the site* `gmail.com`. Similarly, in category theory, if A and B are objects of a category $\mathcal{C}$, Mazzola uses the notation $A@B$ for the collection $\mathcal{C}(A, B)$ of morphisms from A to B, which means that we are looking at B from the perspective of A. Under this notation, it is said that a morphism $f : A \longrightarrow B$ is an A-addressed point[19] of B. This stresses the use of generalized points in category theory instead of atomic points, which are omnipresent in set-theory, and the introduction of relativity in category theory

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[19] In this way, in this case, the notation should be read from right to left, and not in the usual way of reading e-mail directions.

since the perspective of the observer is introduced. In this way, the Yoneda embedding (see (6) in [14, p. 62]), which asserts that each object C of the category can be identified with the representable functor $\mathcal{C}(\_, C)$, can be interpreted as the fact that we understand an object C by considering all the perspectives on it, by observing it from all possible perspectives (which can be expressed by the symbol $\_@C$ denoting the representable functor associated with C), or, in Zalamea's words, by capturing its aura. This viewpoint, which approaches objects externally and by their relations with the environment, contrasts with the usual way of approaching the objects in mathematics (and more generally in most sciences), in which we internally analyze and dissect them. Moreover, it is fully justified by the notions of space other than the usual of topological space: topoi and locales, whose essential ingredients are not points, but coverings and neighborhoods. In these notions of space, it is more natural to study generalized elements[20], for ordinary points can be absent in many non-trivial cases (see Proposition 2.5.4).

Second, there is a relation between vision and the object of Mazzola's gestures $\Gamma@C$ in a category (definition in Subsection 3.3.3). Recall from subsections 3.3.3 and 3.4.4 that the object $\Gamma@C$ of gestures with skeleton Γ and body in C is a cotensor product of functors, a limit, rather than a collection of morphisms, so it seems that the notation $\Gamma@C$ is not faithful to the initial insight of Mazzola. But, in the case of topological spaces, there is a correspondence between the elements of the space $\Gamma@X$ and the morphisms of digraphs from Γ to the digraph of paths in X. And moreover, the exponential presentations assert that, in some cases, $\Gamma@C$ is isomorphic to $C^{|\Gamma|}$, the latter exponential being an internal version of the collection of morphisms $\mathcal{C}(|\Gamma|, C)$. For these reasons we use the notation $\Gamma@C$ for the object of gestures—after all, we are looking at C from the perspective of the digraph Γ.

Third, under this visual interpretation of the symbol @, Mazzola's result that given two digraphs Γ_1 and Γ_2 there is an isomorphism between the objects of hypergestures $\Gamma_1@\Gamma_2@C$ and $\Gamma_2@\Gamma_1@C$, which Mazzola has christened *the Escher theorem* 'in view of the flip-flop nature of a number of Maurits Cornelis Escher's art work in the sense that one may view the same graphics from different perspectives or also convex versus concave' [27, p. 32], can be interpreted by saying that the result of the observation of C from two perspectives is independent of the order in which we perform the associated visions. Also, as we have shown (Section 3.10), this theorem is just a categorical generalization, given the analogy between ends and integrals, of Fubini's theorem[21], which asserts that a suitable double integral over a region of the plane can be computed as an iterated integral in either order. However, unlike the classical theorem of Fubini, where the original region of integration is rendered explicit, the Escher theorem, even in our proof, does not exhibit the analogue of the region of integration. In this way, the Escher theorem is more a theorem of differentiation, of dissection of a unified perspective into two partial ones, so one may ask whether it is possible to recover the unified vision.

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[20] Generalized points of Grothendieck topoi are well studied. An $\mathcal{E}$-addressed point of a topos of sheaves $\widetilde{\mathcal{C}}$, where $\mathcal{E}$ is an arbitrary topos, correspond to a continuous flat functor from $\mathcal{C}$ to $\mathcal{E}$, according to Corollary [15, VII.7.4].

[21] In fact, in the proof of the Escher theorem (Theorem 3.10.2), the central argument is Fubini's theorem for ends (Theorem and Corollary in [14, §8]).

Finally, the unified vision therefore is a chiasm between the two visions associated with the digraphs Γ_1 and Γ_2. Certainly, there is an object that represents the unified vision or region of integration. As we remarked at the beginning of this subsection, in certain representative cases, we have an isomorphism between the object of hypergestures $\Gamma_1@\Gamma_2@C$ and the object of gestures $(\Gamma_1 \times_g \Gamma_2)@C$, where $\Gamma_1 \times_g \Gamma_2$ is a semi-simplicial set possibly of higher dimension, which can be related to the chiasm. With this motivation at hand, the whole statement (in the cases cited, under the suitable hypotheses)

$$\Gamma_2@\Gamma_1@C \cong \Gamma_1@\Gamma_2@C \cong C^{|\Gamma_1|\times|\Gamma_2|} \cong C^{|\Gamma_1 \times_g \Gamma_2|} \cong (\Gamma_1 \times_g \Gamma_2)@C,$$

can be related in a more consistent way to Escher's works, where different perspectives fuse in a true hallucinatory chiasm that could only happen in spaces of higher dimensions.

Further, this discussion is at the core of painting; in this respect it is useful to recall Merleau-Ponty's words in [32, pp. 21]:

> Un corps humain est là quand, entre voyant et visible, entre touchant et touché, entre un œil et l'autre, entre la main et la main se fait une sorte de recroisement, quand s'allume l'étincelle du sentant-sensible, quand prend ce feu qui ne cessera pas de brûler, jusqu'à ce que tel accident du corps défasse ce que nul accident n'aurait suffi à faire…
>
> Or, dès que cet étrange système d'échanges est donné, tous les problèmes de la peinture sont là. Ils illustrent l'énigme du corps et elle les justifie. Puisque les choses et mon corps sont faits de la même étoffe, il faut que sa vision se fasse de quelque manière en elles, ou encore que leur visibilité manifeste se double en lui d'une visibilité secrète: «la nature est à l'intérieur», dit Cézanne.

According to the last paragraph, the chiasm between the seer and the visible consists of two main traits. The first one, the fact that the vision of the thing takes place in some respect in the thing itself, can be related to the Yoneda Lemma, in the sense that the latter establishes a bijection between the interior of a presheaf P (the elements of any of its values), which we relate to the thing, and the visions thereof from a suitable representable functor, which we relate to the seer:

$$Nat(\mathcal{C}(\_, C), P) \cong P(C).$$

The second trait establishes that the visibility of the thing is duplicated in the seer. This could be related to the fact (sometimes called the co-Yoneda lemma) that every presheaf P (thing) can be expressed as a colimit, indexed by the (category of) elements of P, of representable functors (seers):

$$P = Colim\left(\int P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{\mathbf{y}} \widehat{C}\right).$$

For, in particular, since colimits of representable functors are computed pointwise, this (plus the Yoneda lemma) means that for each object D of $\mathcal{C}$, we have the identity

$$Nat(\mathcal{C}(\_, D), P) \cong P(D) = Colim\left(\int P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{\mathcal{C}(D,\_)} \mathbf{Set}\right),$$

which can be interpreted, when D runs over $\mathcal{C}$, as the fact that the visibility of P (left-hand term) is made up from pieces of the form $\mathcal{C}(D, C)$, which are (once again by Yoneda) isomorphic to $Nat(\mathcal{C}(\_, D), \mathcal{C}(\_, C))$, that is, to visibilities of the seers. In this way, the visibility of the thing has an echo in the visibility of the seer. This chiasm between the seer and the visible, together with the chiasmata between the touching and the touched, between the eyes (which can be exemplified with the presentation above of objects of hypergestures as gestures of higher dimension), and the hands, comprise the problems of painting, according to Merleau-Ponty.

6.7 Merleau-Ponty: music and meaning

Merleau-Ponty's philosophy is a revelation in what concerns meaning. If we embark on the problem of meaning and signification of music, Merleau-Ponty [33] teaches us that though such a search is utterly legitimate, what instead is illusory is the false hope that words and signification will replace music itself. This situation has a parallel in mathematics, because after Gödel's incompleteness theorems, and after Grothendieck's practice of relative mathematics beyond pretended absolute foundations, it makes little sense to think of mathematics as product of logic and language. Instead, mathematics, or rather, the mathematical gestures, give rise to logic; recall Zalamea's cite of Lautman in [39, p. 67]: 'logic requires a mathematics in order to exist'. In the same way, it is absurd to try to replace, or even to found, music by systems of significations, which are at bottom logical systems. Notations, symbols, and meanings are all facsimiles of music itself; according to Merleau-Ponty, they are used for ease of understanding, or rather for rational understanding, rather than for a real access to music. Here we recall the constant cite of Alunni's words in Mazzola's work, which are fully vindicated in our characterization of diagrams as true abstract gestures: 'Ce n'est pas la règle qui gouverne l'action diagrammatique, mais l'action qui fait émerger la règle.'

Nevertheless, our access to music is never total. This situation illustrates Merleau-Ponty's fundamental paradox-chiasm of perception: *we reveal things by veiling them*. In fact, according to Peirce's pragmaticist maxim, to which we adhere, our knowledge of an object (music in our case) consists of all its differentiations obtained by a sort of filtering through lenses, which are just veils, but constantly seeking a reintegration of the whole thing from its veiled manifestations, by means of an endless process. Thus, paradoxically we pin all our hopes on languages and systems of signs (like musical notation), trying to capture the essence of things. This is as if truth, the light of things, were so intense, like the light of sun, which scatters through the electromagnetic spectrum, that we only access this light by the most diverse processes of filtration with a set of lenses that capture the diverse frequencies, even those that are obscure and invisible to our eyes. Here we recall from Merleau-Ponty [32] that every work of art is almost by definition unfinished, always open, so are also its meaning and understanding.

In the same way that language, meaning, and signification are not able to replace our experience of the world since the *character of being* of meaning locates it in a position similar to that of the world (and hence meaning solves the problem of the access to the world by that of the access to itself), the same thing could be said of the relation between meaning and music, which is first and foremost flesh and world (recall Merleau-Ponty's words on music [32, p. 14], cited in the introduction of this thesis). Language and signification are themselves an enigma, so the access to music by means of them only replaces a problem by another that is far from being clear—as if language were finally more clear than music. And it is important to recall here that even the problem of language, sooner or later, necessarily refers to the problem of language as an operative phenomenon, and as a consequence of speech and communication, to summarize, of gestures, the converse alternative, that is, every intent to reduce communication to linguistics and grammar being vain. We first, or at least it should be the most natural order, get acquainted with the liveliness of language, with its gestures, its substance, and then we grasp the grammar and the notation, but by no means the latter process replaces the former. To pretend to access music by means of signification is to pretend a complete access to the world by means of words: the invitation of music is to install us, by means of our body, in the world of the experience of sounds, to sink us into musical experience.

6.8 Riemann: semeiotic ramifications, continuation of gestures

The idea of a Riemann surface can be sketched as follows. Given a holomorphic function

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

(think of $f(z) = z^n$ for instance), that is, a differentiable function under the usual definition, there are two ways to render it injective so as to solve the problem of its inversion. The first one is to make a restriction of the domain (as usual in real variable). But there is another alternative, which is more natural: just expand the codomain continuously so as to let the periodicity of the function flow. First, we perform a division of the domain $\mathbb{C}$ in appropriate maximal regions on which the function turns out to be injective—in the case of z^n, these regions can be taken to be n congruent angles whose union is the whole complex plane. Then we consider each image, under f, of a region as lying in a different copy of the codomain, and paste all these different images continuously in the natural way suggested by the function. The resulting figure is a *Riemann surface*, and the different images of regions that form it are its *sheets*. The points around which different sheets are pasted are the *branch points* of the surface. For example, the Riemann surface of z^n has n branches or sheets connected by a single branch point (O), and the Riemann surface of e^z, a *helicoid*, has infinitely many sheets pasted around a single point (O). In this way, Riemann surfaces are natural objects for capturing ramification, continuous gluing, and periodicity.

Riemann surfaces are important for our discussion since the idea of a branch point around which the different sheets of the surface are glued together can be regarded as a natural codification of chiasm between different (parallel) worlds. Moreover, in the case of infinite semeiosis, the different stages of semeiosis, that is, its cycles made up from triads of the

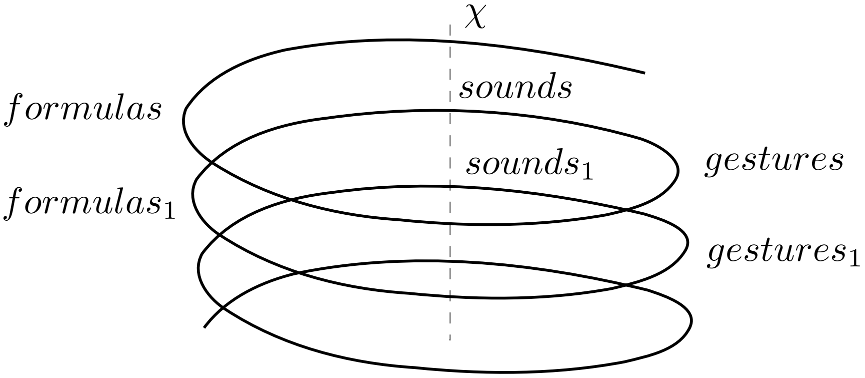

Figure 6.6: Zalamea's spiral of endless semeiosis spinning around chiasm and inscribed in a helicoid.

form object/representamen/interpretant (Section 6.3), can be placed in different sheets of a helicoid. In this way, we cipher the possibility, both upwards and downwards, of Peirce's semeiosis (see Figure 6.6), the existence of an ideal gluing feature (branch point-chiasm) of the triads involved, and the continuity of the passage from a term to another (which would express a gesturality in semeiosis, close to Peirce's continuum). Then, in our spatial unfolding (helicoid) of the Riemann surface of the exponential (Figure 6.6) we project the branch point vertically to form an axis, the chiasm axis χ, around which semeiosis takes place.

Thus, with these perspectives on the chiasm of the triad of Western performance at hand, we have Zalamea's spiral of endless semeiosis around the chiasm axis (Figure 6.6), in which the different transmutations of scores into gestures, of gestures into sounds and so on, occur through space-time. Moreover, this spiral can be recast as a Riemann surface, namely a helicoid, where the chiasm, from which the different ramifications unfold and which is at the same time surface of contact between the axis (the unified vision) and the different terms of the triad. This model has the virtue that grasps the continuity of the fibers or branches better than a spiral that is separated from its center.

On the other hand, the concept of Riemann surface of a holomorphic function is strongly related to that of *sheaf.* In fact, we can extend holomorphic functions defined locally to obtain new ones in larger domains by *analytic continuation.* This process allows the definition of the sheaf of holomorphic functions, whose global analytic functions can be identified with their respective Riemann surfaces. As we will see in the next section, this idea is taken up by Grothendieck as the main tool for unifying the notions of number and continuous magnitude. Moreover, analytic continuation, in the general language of Grothendieck (sheaves on sites), is useful for discussing gestures, which are, of course, coherently glued to produce new gestures and reverberate through time in a continuous overture.

A deeper philosophical discussion on Riemann and his legacy can be found in [36]. A mathematical introduction to Riemann Surfaces, analytic continuation, and the sheaf of holomorphic functions, on which the author partially based the previous discussion can be found in Ahlfors' *Complex Analysis.*

6.9 Grothendieck: topoi of gestures, listening to things

Topos theory was developed by Grothendieck, and his school, as a generalized topological framework, from which it was possible to formulate the appropriate cohomological theory for the solution of the Weil conjectures. This new topology was based on the notion of *topos*, which was an extension of the notion of topological space. The appropriate cohomology turns out to be the étale cohomology, obtained by means of the general treatment of cohomology made in the *Tôhoku* paper, and associated with the étale topos of a scheme. The latter is constructed as the topos of sheaves on the étale topology of a scheme and can be regarded as a refinement of the Zariski topology on the prime spectrum of a commutative and unitary ring, which tends to have few opens.

The notion of topos, inspired by that of topological space as the similarity in terminology suggests, can be sketched as follows. We start with a category $\mathcal{C}$ instead of a set. Roughly speaking, a Grothendieck topology on $\mathcal{C}$ is obtained by assigning to each object C of $\mathcal{C}$ a collection of coverings, each covering being formed from arrows with codomain C, in such a way that three requirements are satisfied: T1) the restriction (or rather, the pullback) of a covering along a morphism is a covering again, T2) if the restriction of a collection of arrows along each morphism of a given covering is a covering, then the collection is a covering (in other words, covering of coverings is covering), and T3) all the morphisms with codomain C form a covering. The fundamental analogy with topological spaces is that each topological space can be regarded as a category, namely the category of the poset of its opens, and that there is a Grothendieck topology whose coverings of a given open U in the space (or equivalently, of an object of the associated category) are related to the families of open subsets of U whose union is U. Given a category with a Grothendieck topology, that is, a site, the crucial point is to construct the associated topos of all sheaves on this site. There are two initial reasons to do this: first, the site is arbitrary so it need not have desirable properties as limits, colimits, and exponentials, which hold in the topos of sheaves; second, in the case of topological spaces, we can recover the original space from its category of sheaves, so both presentations are interchangeable.

Certainly, the profound notion of *sheaf* is the core of all this new world. It embraces the notions of *number* and *magnitude*, and therefore is a true chiasm between the discrete and the continuous, and at the same time it unifies the visions of Galois and Riemann: on the one hand, the notion of scheme, a suitable gluing of affine schemes (sheaves on the prime spectrum of a commutative and unitary ring whose fibers are local rings), can be regarded as a generalization of that of algebraic variety (number) in the line of Galois-Hilbert-Zariski; and on the other hand, the notion of topos, which is made up of sheaves that resemble the sheaf of germs of holomorphic functions in complex analysis, can be regarded as a generalization of that of topological space (magnitude) in the line of Riemann-Leray-Cartan. The great discovery of Grothendieck is that the notion of sheaf is not inherent to topological spaces but it can be formulated in terms of the abstract notion of coverings given above. Moreover, as pointed out by Johnstone, this point is more dramatic: essentially, we only need the base change axiom T1 to give the definition of sheaf. For example, on this observation relies the definition of sites for locales, and in particular, the construction of the product of two locales;

see [9].

6.9.1 Sheaves of gestures

In Zalamea's words, sheaves are the simplest mathematical structures that allow the transit from the local to the global. Given a site, a sheaf is (informally) a correspondence assigning to each object C of the category a set, whose elements are called sections over C, and to each morphism a restriction function, in such a way that coherent families of sections on the domains of morphisms of a covering of the Grothendieck topology glue together to form a unique section. The main application of sheaves (in Grothendieck's sense) to our discussion on gestures is that gestures can be glued together to produce new ones, thus forming sheaves. The precise definition of sheaves that we needed was that in Section 3.12, which allows the construction of sheaves with values in a suitable category rather than in **Set**. There, we observed that the definition of sheaves can be given in terms of limits that are very similar to those used to define the object of gestures. Probably, this could express that in a very abstract sense, at bottom, a sheaf is made up from gestures: in fact, for each object of the category C, each covering of it, and each coherent family, a sheaf expresses a sort of movement of individual sections (limbs), with the configuration given by the covering involved, an objective (to form a section), and a modality (uniqueness of the section). In this way, solidarity and will expressed in sheaves constitute authentic gestures, related to the respective (abstract) human gestures of solidarity and will, which oppose chaos.

The fact that the gesture functor

$$\_@S : \left(\widehat{G}\right)^{op} \longrightarrow \mathcal{C}$$

is a sheaf could be considered as the central fact of all our discussion on gestures for several reasons. From a purely technical point of view, it is based on the fundamental adjunction for gestures, which is obtained by using the categorical duality realization/gestures and the existence of a fundamental adjunction for realization; these results of duality and adjunction being the foundations of the theory of gestures presented in this monograph. Moreover, this fact shows that it is worth regarding the category of skeleta $\widehat{G}$ as a generalized space, where the Grothendieck topology has as coverings epimorphic families of morphisms of skeleta; and this is clearer if we replace the category of semi-simplicial sets $\widehat{G}$ by the category of *simplicial sets*[22] $\widehat{\Delta}$, the latter consisting of combinatorial models for spaces. But perhaps the importance of Theorem 3.12.1 is mainly philosophical because it gives a precise mathematical proof of a previous intuition due to Châtelet and Cavaillès: the importance and possibility of multiplication of gestures, their solidarity, and their *analytic continuation* in Riemann's sense, as basis of the understanding. Recalling Châtelet [29, pp. 9-10]:

> The gesture inaugurates a family of gestures, whereas the rule only enunciates 'instructions', a protocol for decomposing the action into endlessly repeatable acts.

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[22] This is absolutely feasible and has advantages since simplicial sets have a better behavior than semi-simplicial ones, for example, regarding realizations. See the discussions in Section 3.5.

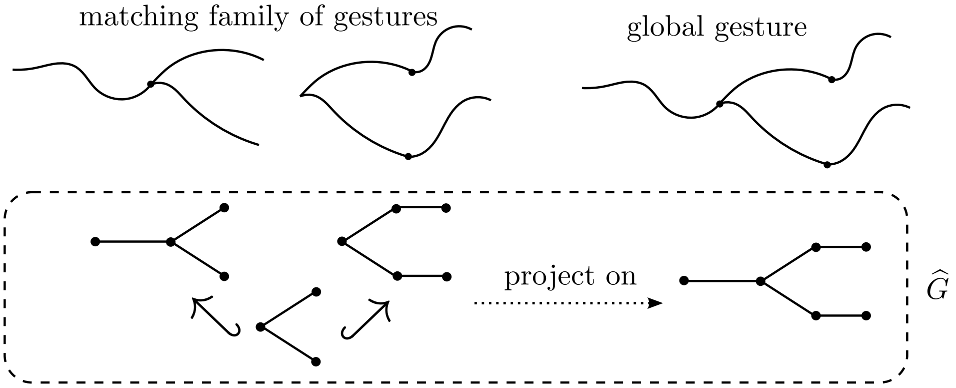

Figure 6.7: The sheaf of gestures: local gestures that are compatible extend to global ones.

> — a gesture awakens other gestures: it is able to store up all allusion's provocative virtualities, without debasing it into abbreviation.

Moreover, recalling the recurrent reference to Cavaillès in Mazzola's works [26]:

> Comprendre est attraper le geste et pouvoir continuer.

Though the theorem is given in a very abstract language because the values of the sheaves are in a category $\mathcal{C}$ instead of the category of sets and $\widehat{G}$ is not a topological space, the essential idea of gluing the parts to form a whole remains; what we need is to interpret the result in terms of generalized elements since the objects of $\mathcal{C}$, as in the case of locales, need not be characterized by their points (there may be no points at all). The interpretation is that if we have a cover of a skeleton by skeleta that project exactly on it and a coherent family (indexed by the skeleta of the cover) of packs of gestures (generalized elements) varying over a common parameter (we no longer think of families of elements-points), then the family yields a unique pack of gestures, varying over the same parameter as before, built from the packs in the initial family. The interesting feature of this process is that it extends throughout the giant topos of skeleta $\widehat{G}$, this is the endless possibility of extension, solidarity, and multiplication of gestures. Though in this general case we have not a presentation of the sheaf of gestures as an *étalé* space ($\widehat{G}$ is not a topological space), we can give a similar representation of the sheaf of gestures by means of Figure 6.7. It is also interesting to observe that this behavior of sheaves of gestures, which is marked by plasticity and flow, contrast with the internal behavior of the objects of gestures involved, whose structure can be very complicated. For example, recall the diverse possibilities for the object of gestures $\Gamma@\mathbf{3}$ with body in the Sierpiński locale: when we left the relative stability of the exponential presentation, $\Gamma@\mathbf{3}$ can be spatial or non-spatial depending on Γ (see Section 2.7). Moreover, the computation of the object of gestures as a limit in the category of locales and in the 2-category of Grothendieck topoi has not a presentation as simple as in the case of topological spaces (where spaces of gestures are simply function spaces), for the presentations of fibered products is not elementary—by the way, this fact explains the difficulty of making explicit calculations of the object of gestures in these universes.

6.9.2 Topoi of gestures

As we have pointed out through this thesis, there is a natural way of comparing Grothendieck topoi, namely through geometric morphisms, which are the analogues of continuous functions between topological spaces; but moreover, since topoi are categories, there is an additional notion of comparison between parallel geometric morphisms: (geometric) natural transformations between them. Thus, as stressed by Grothendieck himself [1, p. 331], there is a fundamental difference between topos theory and topology: the structure of the universe of all Grothendieck topoi is that of a 2-category and not simply that of a plain category, like the category of topological spaces. As we will see, it is necessary to take into account this additional structure, at least for our purposes.

The definition of gestures on Grothendieck topoi, regarded as a natural extension of that of gestures on topological spaces, which was one of the main aims of this thesis, deserves several remarks.

First, after the categorical characterization of the space of gestures in terms of limits, the most natural step towards the desired generalization is to carry over this definition into the 2-category of Grothendieck topoi. However, since there is no way of formulating ordinary limits in the category of Grothendieck topoi, we necessarily enter the world of *bilimits* in the 2-category of Grothendieck topoi, and this requires a high technology, since more information (that of geometric natural transformations, that is, 2-cells) and a more active role of categorical equivalences are involved. We are more demanding and this is reflected in the fact that only the existence of bilimits of finite type can be ensured in our 2-category, and moreover, in the difficult computation of the fibered biproduct of Grothendieck topoi. In consequence, we can only ensure the existence of the topos of gestures in the case when the skeleton (which rules the character, finite or infinite, of the limit involved) is finite, and the construction of explicit examples of the topos of gestures is a very difficult enterprise and hardly illustrative[23]. This is an important obstruction, which poses the problem of a simplification of the notion of gestures in such abstract universes, or the search for new notions of space that yield categories with better properties. Regarding the simplification, it is remarkable that the characterization of the objects of gestures as *weighted bilimits* in 2-categories (Subsection 4.2.4), being the 2-categorical analogue of the fundamental adjunction for gestures, provides a way of giving explicit computations of their generalized elements, namely, as pseudonatural transformations; in particular, the points of the Grothendieck topos of Mazzola's gestures have a simple presentation.

Second, the importance of the definition of gestures on Grothendieck topoi is that since topoi offer a synthesis of topology and algebra, which, as pointed out before, is framed in the dialect continuous/discrete, the same dialectic in which the diamond conjecture can be located, we can explore the perspectives that topos theory brings to that conjecture. The conjectural universe could be related to some topos. Nevertheless, in this thesis, regarding this possible topos, we only reached the definition of gestures on Grothendieck topoi, and

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[23] At this point, the problem is more dramatic because an explicit construction of the topos of Mazzola's gestures by bilimits would require the explicit construction of topoi of paths, which are exponentials (in a suitable 2-categorical sense): another extremely involved explicit construction; see [12].

an attempt to construct the conjectured universe could be based on our definition, though it seems a very hard enterprise. In any case, this enterprise could also take into account the unification of the concepts of diagram, formulas, and gestures on different notions of space, achieved by means of our definition of abstract gestures.

Third, points. . . Look at the following discussion.

6.9.3 Locales and points

Locales are halfway between topological spaces and Grothendieck topoi[24]. They correspond to the effort to take seriously the fact that the opens of a topological space form a lattice. In fact, we try to abandon points, and we are more concerned with the notion of open as a primitive one. This change of point of view, also present in Grothendieck's vision of topoi[25], is one of the most important features in our discussion on gestures, since it seems more natural to work with lattices of opens instead of spaces made up of points, which in several cases (especially in algebra: prime ideals, prime elements) are related to the use of the axiom of choice or Zorn's lemma (see Proposition 2.8.1), and hence are non-constructive—a feature that should be examined carefully in relation to performance and corporal movements, more related to naturalness and intuition. But this is more profound. As stressed by Isbell, products of locales have a better behavior than products of spaces [10]: the locale of opens of the product of two spaces is a sublocale of their localic product, which shows that, in general, there are more relations that hold between the opens of the product space than between the 'opens' of the localic product, and certainly these relations are artificial (see example/proposition [9, II.2.14]). Of course, there are more properties that behave better: unlike spaces, connected locally connected locales[26] are path connected; see [17] and Subsection 7.2.1 for a counterexample in point-set topology that is no longer a counterexample in the realm of locales—it can be tamed.

Thus, we are before a world that is between the artifices of point-set topology (often regarded by beginners as anti-intuitive mathematics), and Grothendieck's insights of a moderate topology where these artifices are softened. And, in fact, this world is strongly related to (if not a consequence of) Grothendieck topoi. Locales are just residues, or fragments, of Grothendieck topoi: the localic reflection of a Grothendieck topos, that is, the lattice of subobject of the final object, or rather the lattice of opens of the topos involved according to Grothendieck [1, IV.8], is a locale, and this procedure defines a left (bi)adjoint (Subsection

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[24] In this way, since locales are also generalized spaces, the terms *spatial locale* and *non-spatial locale*, used in this thesis so as to be coherent with standard texts, can be conceptually confusing. To say that a locale is non-spatial does not mean that it is not a (generalized) space. For that reason, it seems more appropriate the terms *has enough points* and *has not enough points*, respectively.

[25] It is to be studied to what extent Grothendieck's works on topoi influenced the birth of locales; in particular the works by J. Isbell and A. Joyal. Anyway, the actual treatment of locales is regarded as a part of topos theory, and several techniques of topos theory have been central in the development of locales: for example, sites, which have been introduced in locales, though only preserving the essential axiom on base change; and on the other hand, nuclei, which are just the analogues of Lawvere-Tierney topologies for locales.

[26] See the footnote 4 in Subsection 7.2.1 below for the definition of these concepts.

4.3.1) to the pseudofunctor Sh (from locales to Grothendieck topoi); moreover, every locale L, can be recovered as the lattice of opens of the topos $Sh(L)$.

Paradoxically, a more natural and intuitive behavior is achieved at the cost of a greater abstraction (recall the same paradox between the extreme concreteness of the Weil conjectures and the considerable abstraction in Grothendieck's language of schemes and topoi), and moreover, at the cost of difficult explicit presentations for exponentials and fibered products of locales and Grothendieck topoi. In particular, this happens in the case of objects of gestures, which involve limits and exponentials—though these constructions are manageable, with smoothness, from a synthetic point of view, by means of their universal properties, so that we can give the definitions of gestures in very arbitrary universes.

Now we reflect on the mathematical definition of gesture and its effectiveness as a model. In Mazzola's articles, which have been focused on the concepts of gestures on topological spaces and gestures on topological categories, and even this monograph, where to a great extent we follow Mazzola's ideas, the definition of gestures is based on the *unit interval* I, which is a subset of the real numbers. Essentially, with this interval we model time, and the space where the movements take place is a topological space (Mazzola), topological category (Mazzola), locale, or a Grothendieck topos (see Figure 3.1 for the different transmutations of this interval in each category). But this interval is separated and atomic in essence: it is a Hausdorff topological space, and the locale of opens in I has enough points, so the locale is completely characterized by its space of points. This yields a profound question: is I a suitable model of time in our context of performance? More generally one may ask whether the euclidean space $\mathbb{R}^n$ (and even any topological space) is a suitable model of space-time. Our conviction is a negative answer for both questions, because performance is a process related to envelopes and indecomposable movements that occur through non-atomic neighborhoods of space-time, as in the case of the human body (absolutely indecomposable in terms of points) or the pianist's hand[27]. As to the interval I, we have tried to broaden the notion of interval object to a more flexible notion of (semi-)simplicial object, which in principle changes the interval for any object in the background category (see Section 3.1). Concerning the notion of space we have tried to release it from its set-theoretic prison, and to allow more flexible notions where the neighborhoods (locales) and the coverings (Grothendieck topoi) are primary and the points are secondary, though it looks more abstract and difficult.

This automatically refers us to the mathematical problem of recovering the movement or continuity from the morphisms of a category (gestural embedding). We have to take into account that these conceptualizations exist so as to give an answer to the problem of the gestural embedding and to the diamond conjecture, and here we enter a new problem. Mazzola's first insight consists in recovering the implicit movement in a morphism of a topological category by means of a sort of infinite factorization of it (see [27]), which in turn is used to construct the bicategory of gestures. This solution is not definitive since an embedding into the bicategory of gestures is not guaranteed. Besides, there is a feature of topological categories that is a subject of further examination: the topology on a topological

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[27] I borrowed this idea from Octavio Agustín-Aquino.

category, which is simply a topology on the set of morphisms of the category, is not intrinsic, in the sense that its nature may be different from that of the morphisms, it is external. In contrast, in a Grothendieck topos, the topology is given by coverings of a given object of the category by means of morphisms whose codomain is that object. This suggest an intrinsic relation between the morphisms of the category and the Grothendieck topology on the category.

6.9.4 Hearing and sight

There is a parallel between Merleau-Ponty's conception of *vision* [33] and Grothendieck's conception of *hearing* [30]. As we mentioned before, the classical conception of perception is that something acts on our senses to produce feelings in ourselves, and hence it is intended as a sort of centripetal movement of perception. In the case of Merleau-Ponty, the chiasm of vision (not only in the sense of the junction of the optic nerves but as gluing *self/world*) allows the mediation between two aspects of vision: on the one hand, with our vision we touch and envelope things and, on the other hand, we are touched by things. Moreover, there is a constant aural reference, though in a negative sense, in Merleau-Ponty's phrases: the access to the world by means of the expression of things themselves from their *silence*. Similarly, though in a more mathematical and spiritual sense, Grothendieck proposes a hearing of the essence of things by means of a strong will. But once again, there is a chiasm between a meditative attitude or *yin* in which we let things talk, and a strong will that separates myself and the thing from chaos and noise so as to hear the thing.

Thus, there is an approach (perception and vision in Merleau-Ponty, perception and will in Grothendieck) and at the same time a distance (enveloping and thickness of vision and the body in Merleau-Ponty, detachment from chaos and silence in Grothendieck). And, as noted by Merleau-Ponty in [33, p. 135] 'this distance is not the contrary of this proximity, it is deeply consonant with it, it is synonymous with it.' Moreover, as in the case of vision, the thickness of the ears (whether internal or external) is perhaps a more eloquent way of enveloping the world (recall the human gesture of enveloping the ears, which is a gesture of enveloping the sound and things, so as to hear better), this thickness is our way of detaching things from the noise of the world, and hence is our distancing from the world. But at the same time, we try to hear, we try to access things, we perceive. Thus, perception is no longer an action of the external to the internal, but mutual fecundation between who perceives and what is perceived, fecundation that is only possible when self and world are made from the same flesh, identification between passivity and activity. In Merleau-Ponty's words [33, p. 135]: 'as many painters have said, I feel myself looked at by the things, my activity is equally passivity'.

Now, by stressing hearing, which is a physical sense, Grothendieck probably intends to render explicit the gesturality of the act of hearing (gesturality is the essence of understanding according to Cavaillès, as we noted before), a process more direct than a further mediation of written language (once again comfortable for those, like the author, that have not ears to hear), indiscriminate freezing, which needs an additional and often painful decoding. The virtue of music is to present itself stripped before us, there is no special system of significations required to access it, and this directness is probably what is intended. Therefore, if we are faithful to our hope of removing the demarcations between the senses (if according to Merleau-Ponty vision is a remarkable variant of touch, then, in a similar way, hearing is), we automatically come back to the importance of vision too, and in the particular case of mathematics, the importance of category theory as a sort of mathematical drawing. Categories are essentially digraphs (though usually placed in universes larger than the universe of sets) with some particular properties. In this way, categories are fully expressed as diagrams, which as we have seen are just abstract gestures, or skeletal gestures, discrete gestures, recalling Châtelet [29, p. 10]: "Diagrams are in a degree the accomplices of poetic metaphor". Therefore, there is no surprise in the fact that vision and hearing are usually very effective ways of learning.

6.9.5 Gesture and intuition

The Mazzolian-Châteletian mathematical gesture, as we said early, corresponds to the underlying substance, movement, and intuition behind the functional-morphic correspondences. Unlike a supposed inaccessible abstraction in Grothendieck's mathematics, and the contemptuous description of category theory as abstract nonsense, Grothendieck was characterized by a preference for motivations, examples, and a fecund reflection on its own mathematical work, which is now very unusual in our times of cult of ultra-specialization; also, he stresses the importance of hearing and vision, as we have observed, because the intuition and the gesture, the liveliness of mathematical thinking, are more directly grasped through the senses than through mathematical reading, which is an experience too elaborated and at the same inarticulate.

Probably, our experience as mathematicians before the explosive, extensive, and ultra-specialized corpus of mathematical papers is not far from our disorientation before some currents of contemporary music: serialism, stochastic music, electroacoustic music, etc. The problem of communication may be related to the fact that the extreme position is usually assumed, forgetting the intermediate process. Of course, there is also a musical intuition, though it may sometimes be biased by our cultural or educational environment, and we would like to be more independent of it, as in the case of the tonal paradigm, or if we extrapolate this situation to mathematics, as in the case of set theory, of which there are serious reasons (uncertainty of consistence, inclination to analysis and dissection of the mathematical objects, artificiality) to be more independent. This happens when we shock with category theory and realize the degree to which our mathematical thinking has been conditioned by the use of sets, or when we realize the perils of the atomic character of classical points-elements in our exploration of topos theory and the theory of locales—which does not mean that category theory be a sort of cloud of intelligibility that is the ultimate solution for mathematics. In the same way as it is very difficult to understand topos theory without the underlying intuition of classical topology (recall Grothendieck's emphasis upon this point), it is difficult to get acquainted with serialism without the suitable references to the common practice period of Western musical tradition. This is what we mean by musical intuition: a continuous reference to early stages of knowledge, which is linked to a certain gesturality, a continuous movement of thinking, rather than to abrupt negations.

The composers of the Second Viennese School (Schönberg, Webern, and Berg), founders of musical serialism, were well aware of this situation, and we cannot forget that one of the main concerns of avant-garde composers is the problem of communication; see [22]. Of course, there are very interesting explicit references to tradition in Second Viennese School's work (Bach orchestrations and, in a freer way, Schönberg's Cello Concerto based on a baroque concert for clavicembalo by Monn). Moreover, there is an entire musical current that reflects the necessity of intuition and reference: the moderate composers of the middle of the twentieth century, who correspond to a true mediation between the Western tradition (the common practice period) and the avant-garde. Probably, the distance between the public and the works of Shostakovich and Britten (regarded as the greatest composers of the moderate mainstream in [22, Ch. 14]) shortens. This is precisely because they preserve a strong bond with tonality but profiting from serial techniques, which they are well aware of. Therefore, we could regard these composers as those that are able to exemplify beyond mere abstraction, and that reconcile theorizing and the intentions of 'what must be', the artistic idealizations (analogue to the idealizations of mathematical practice, as Bourbaki's one, later nuanced by Grothendieck, or Hilbert's one, later nuanced by Gödel), with 'what is', the listener, traditions, established intuitions. Thus, our inclination is that the problem of contemporary music is not a problem of capacity of decoding of the difficulties posed by avant-garde, problem that would negate the possibility of intelligibility of a superior music for superior minds. Instead, we think that it is a problem of reference, intuition, gesturality, and orientation. The exposition of music to the amateur is usually a better test than the exposition to the expert, since the fire of gesturality and intuition is livelier in the amateur, who has not been biased by the traps of reason and excessive specialization.

Chapter 7

Notes

In this chapter we record some auxiliary results used through this thesis, which are not explicitly stated in the bibliography but are likely well known, add some final remarks, and point out some problems for further research.

The following fact, which is usually an assignment, deserves to be included here by its importance and because it is not stated in this form in our references. It is a general categorical version of the fundamental theorem of Galois theory, was actively used both mathematically and conceptually through this monograph, and its richness is comparable to that of the Yoneda lemma. This fact shows that each categorical adjunction restricts to an equivalence between the respective subcategories of fixed points (cf. [15, II.6.4]).

Proposition 7.0.1. *Let*

$$F : \mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$$

be an adjunction (G right adjoint to F) with unit η and counit ϵ. Define $Fix(GF)$ to be the full subcategory of $\mathcal{C}$ whose objects are those C for which η_C is an isomorphism. The subcategory $Fix(FG)$ is defined dually. This adjunction restricts to an equivalence of categories

$$Fix(GF) \underset{F}{\overset{G}{\leftrightarrows}} Fix(FG).$$

Moreover, if the morphisms

$$\eta_{GD} : GD \longrightarrow GFGD, \quad \epsilon_{FC} : FGFC \longrightarrow FC$$

are isomorphisms for each object D of $\mathcal{D}$ and each object C of $\mathcal{C}$, then $Fix(GF)$ consists of all the objects C of $\mathcal{C}$ that are isomorphic to GD for some D in $\mathcal{D}$, and $Fix(FG)$ consists of all the objects D in $\mathcal{D}$ that are isomorphic to FC for some C in $\mathcal{C}$.

Proof. Contemplation of the triangle identities. □

7.1 Chapter 1

Remark

Given a digraph Γ corresponding to the tuple (A, V, t, h), its realization $|\Gamma|$ was claimed, in [26, p. 34], to be a subspace of the product $\mathbb{R}^V \times \mathbb{C}^A$ (usual topology), and in particular, a metrizable space. This claim is wrong. The reason is that, since $|\Gamma|$ is a CW-complex, by Theorem [7, 1.5.17], the space $|\Gamma|$ is metrizable if and only if it is locally finite (that is, if Γ is locally finite[1]). This implies that if Γ is not locally finite, then it cannot be a subspace of $\mathbb{R}^V \times \mathbb{C}^A$.

Moreover, the representation fails even if Γ is locally finite. Consider the case when Γ only consists of a set of vertices V. In that case, the embedding proposed in [26, p. 34] identifies the vertex $x \in V$ with the canonical vector e_v in $\mathbb{R}^V$, so it characterizes $|\Gamma|$ as a set with the cofinite topology. But according to the construction of $|\Gamma|$ as a quotient space (Section 1.3), it is just the set V with the discrete topology, which coincides with the cofinite topology if and only if V is finite.

A suitable statement of the metrizability of $|\Gamma|$ is given by theorems 1.5.16 and 1.5.18 of [7], which assert that, since $|\Gamma|$ has dimension less than or equal to 1, it can be embedded in $\mathbb{R}^3$ if and only if $|\Gamma|$ is locally finite and countable[2].

7.2 Chapter 2

Open question

The following problem is inspired by Theorem 2.7.1. We say that a digraph Γ is *locally countable* if for each vertex x of Γ the set of arrows whose tail or head is x is countable. It is an open question whether $\Gamma@\mathbf{3}$ is spatial whenever Γ is not locally finite and locally countable, and whether $\Gamma@\mathbf{3}$ is non-spatial whenever Γ is not locally countable, that is, if there is a vertex in Γ that is the tail or head of an uncountable infinity of arrows.

7.2.1 A non-spatiality criterion for locales of paths on spatial locales

Consider the lexicographic order on the set $\omega_1 \times [0, 1)$ where ω_1 is the first uncountable ordinal. *The long line* $\mathbb{L}$ is the set $\omega_1 \times [0, 1)$ endowed with the order topology. *The long segment* $\mathbb{L}^*$ is obtained by adding a greatest element Ω to the order on $\mathbb{L}$, so it can be equipped with the order topology. Also, the long segment is the Alexandroff compactification of the long line and hence is a compact Hausdorff space, in particular, it is locally compact sober and $\mathbb{L}^* \times \mathbb{L}^*$ (Hausdorff) is a T_D-space[3].

---

[1] For the definition of locally finite digraph, see Section 1.6. Also, see Corollary 3.5.10.

[2] Recall that $|\Gamma|$ is *countable* if it has countably many cells, or equivalently, since 0-cells and 1-cells of $|\Gamma|$ are in correspondence with vertices and arrows respectively (Section 1.3), if both A and V are countable.

[3] A topological space X is T_D if for each $x \in X$, $\{x\}$ is open in its closure.

The following proposition shows that the locale of paths in the order topology on the long segment is non-spatial. Though this proposition seems to be well known, we include it since we used it to give a non-trivial example of a non-spatial locale of gestures on a spatial locale, and since there is no explicit reference to it in our bibliography.

Proposition 7.2.1. *Let X be a connected locally connected topological space that is not path connected. If X is locally compact sober and $X \times X$ is a T_D-space, then the locale of paths $\mathcal{O}(X)^I$ is non-spatial.*

Proof. Since X is connected and locally connected, its associated locale $\mathcal{O}(X)$ is connected and locally connected[4]. Thus, by the theorem in [17], which is also valid for locales, $\mathcal{O}(X)$ is a path connected locale. That is, the morphism $\langle e_0, e_1\rangle : \mathcal{O}(X)^I \longrightarrow \mathcal{O}(X) \times_l \mathcal{O}(X)$ is an epimorphism of locales. Also, there is a commutative square

$$\begin{tikzcd}
\mathcal{O}(X)^I \arrow[r, "\langle e_0, e_1\rangle"] & \mathcal{O}(X) \times_l \mathcal{O}(X) \\
\mathcal{O}(X^I) \arrow[u, "i"] \arrow[r, "\mathcal{O}(\langle e_0, e_1\rangle)"] & \mathcal{O}(X \times X) \arrow[u, "j"']
\end{tikzcd},$$

where i and j are the natural inclusions from the spatializations of $\mathcal{O}(X)^I$ and $\mathcal{O}(X) \times_l \mathcal{O}(X)$ respectively. The commutativity of this diagram is given by the naturality of the counit $\mathcal{O}pt \longrightarrow id$ of the adjunction $\mathcal{O} \dashv pt$. Moreover, note that the morphism j is an isomorphism. In fact, $\mathcal{O}(X) \times_l \mathcal{O}(X)$ is spatial since X is locally compact (Proposition [9, II.2.13]). Thus, if $\mathcal{O}(X)^I$ is spatial, that is, if i is an isomorphism, then $\mathcal{O}(\langle e_0, e_1\rangle)$ is an epimorphism and hence $\langle e_0, e_1\rangle : X^I \longrightarrow X \times X$ is an epimorphism since $X \times X$ is T_D (Exercise [9, II.2.1]), a contradiction. Therefore, $\mathcal{O}(X)^I$ is non-spatial. □

7.3 Chapter 3

Open question

In the case of Mazzola's gestures on locales derived from the standard simplex functor in **Loc** (see Example 3.3.2), the author does not know whether the general exponential presentation holds for an arbitrary locally finite (each simplex is the face of only finitely many simplices) semi-simplicial set Γ. We know that it holds whenever Γ is a digraph (Theorem 2.6.10). Also, as established in Theorem 3.5.14, the general exponential presentation holds for Mazzola's gestures derived from the standard simplex in **Top**. However, the strategy used in the case of locales is different from that used for topological spaces and the generalization of the former does not seem immediate.

---

[4] Recall that a locale L is *connected* if for each pair of elements $a, b \in L$, the conditions $a \wedge b = \mathbf{0}$ and $a \vee b = \mathbf{1}$ imply $a = \mathbf{0}$ or $b = \mathbf{0}$. Also, L is *locally connected* if for each element $a \in L$ there exists $S \subseteq L$, with $\bigvee S = a$, such that the locale $\downarrow(b)$ is connected for each $b \in S$. Note that in the case when L is the locale of opens of a space we recover the classical definitions.

7.4 Chapter 4

Open question

The problem of the presentation of the Grothendieck topos of Mazzola's gestures $\Gamma@\mathcal{E}$ (Definition 4.2.1) as the exponential $\mathcal{E}^{\Gamma\otimes_G\mathcal{T}}$, where $\mathcal{T}$ is the standard simplex pseudofunctor in the 2-category of Grothendieck topoi (Subsection 4.2.2) and $\Gamma \otimes_G \mathcal{T}$ is as in Subsection 4.2.4, does not seem easy because the tools used in the case of locales (Γ digraph) have no straightforward generalizations; for example, the analogue of the adjunction $\mathcal{O} \dashv pt$ for Grothendieck topoi is not clear. On the other hand, a constructive and explicit presentation of the Grothendieck topos of gestures (in terms of sites) is desirable but difficult.

7.5 Chapter 5

Open question

The problem of the presentation of the topological category of Mazzola's gestures $\Gamma@\mathbb{K}$ (Subsection 5.3.1) as an exponential should be addressed, starting by studying the suitable notion of realization of digraphs and semi-simplicial sets in $\mathbf{Cat}(\mathbf{Top})$. Probably, this problem is more manageable than its analogue for Grothendieck topoi and the tools exposed in this thesis could be helpful. Also, it could be explored whether the formulas of reduction of hypergestures to gestures for semi-simplicial sets (Theorem 3.5.16) can be generalized to topological categories.

Notation

$\mathcal{O}(X)$	The lattice of opens of the topological space X
$A \subseteq_{fin} B$	A is a finite subset of B
$\widehat{\mathcal{C}}$	The category $\mathbf{Set}^{\mathcal{C}^{op}}$ of presheaves on the small category $\mathcal{C}$
$\widetilde{\mathcal{C}}$, $Sh(\mathcal{C}, J)$	The category of sheaves on the site $(\mathcal{C}, J)$
$L \times_l M$	The product of L and M in the category of locales
$\Gamma_1 \times_g \Gamma_2$	The geometric product of the semi-simplicial sets Γ_1 and Γ_2
$\Sigma(A)$	The Scott topology on the lattice A
$\mathbb{S}$	The Sierpiński space
$\mathbf{2}$	The final object in **Loc**
$\mathbf{3}$	The locale of opens of the Sierpiński space
$\uparrow(a)$	the set $\{b \in L \mid a \leq b\}$ in a poset L
$\twoheaduparrow(a)$	the set of all elements b such that $a \ll b$ in a locale L [9, VII.2.2]
Loc	The category of locales
Top	The category of topological spaces
Sob	The category of sober spaces
CGHaus	The category of compactly generated Hausdorff spaces
Cat	The 2-category of small categories
$\mathfrak{Cat}$	The 2-category of categories
$\mathfrak{Loc}$	The 2-category of locales
$\mathfrak{BTop}$	The 2-category of Grothendieck topoi (bounded over **Set**)
$\mathfrak{LTop}$	The 2-category of localic topoi
$\mathbf{Cat}(\mathcal{C})$	The category of internal categories in $\mathcal{C}$
G	The semi-simplicial category
Δ	The simplicial category
$\mathbf{y}$	The Yoneda embedding
$\mathbf{a}$	The associated sheaf functor
$\diamond$	End of example
$\square$	End of proof

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