Algebraic and analytic properties associated with spectral transformations of matrix orthogonal polynomials

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Title in English

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Abstract: The main purpose of this work is the study of algebraic and analytic properties associated with spectral transformations of sequences of matrix orthogonal polynomials with respect to a matrix measure supported either on the unit circle or on the real line. In both frameworks, we study some properties related with a perturbation of a sequence of matrix moments. We extend to the matrix case some algebraic and analytic properties of matrix orthogonal polynomials on the unit circle, that are known in the scalar case, associated with the Uvarov and Christoffel matrix transformations of a Hermitian matrix measure supported on the unit circle. We also study properties of block Hessenberg and block CMV matrices associated with sequences of matrix orthonormal polynomials, under certain transformations of the corresponding matrix measure. In addition, we extend to the matrix case some properties of the Szegő transformation and then use these properties to analyze its effect on some spectral transformations, focusing on the relationship between the matrix-valued Stieltjes and Carathéodory functions.

Keywords: Matrix measure, spectral transformation, matrix orthogonal polynomials on the real line, matrix orthogonal polynomials on the unit circle, matrix moments, Szegő matrix transformation, Christoffel and Uvarov matrix transformations, relative asymptotics, block Hessenberg matrices, block CMV matrices, matrix-valued Stieltjes function, matrix-valued Carathéodory function.

Mathematics Subject Classification MSC (2010): 33C45; 33D45; 42C05.

Título en español

Propiedades algebraicas y analíticas asociadas con transformaciones espectrales de polinomios ortogonales matriciales

Resumen: El objetivo principal de este trabajo es estudiar propiedades algebraicas y analíticas asociadas a transformaciones espectrales de polinomios ortogonales matriciales con respecto a una medida matricial con soporte ya sea sobre la circunferencia unidad o sobre la recta real. En ambos casos, se estudian algunas propiedades relacionadas con una perturbación de una sucesión de momentos matriciales. Extendemos al caso matricial algunas propiedades algebraicas y analíticas de polinomios ortogonales matriciales sobre la circunferencia unidad, que son conocidas en el caso escalar, asociadas con las transformaciones matriciales de Uvarov y Christoffel de una medida matricial Hermitiana con soporte sobre la circunferencia unidad. También estudiamos propiedades de matrices Hessenberg y CMV por bloques, asociadas con sucesiones de polinomios ortonormales matriciales, bajo ciertas transformaciones de la medida matricial correspondiente. Además, se extienden al caso matricial algunas propiedades de la transformación Szegő y se utilizan estas propiedades para analizar su efecto sobre algunas transformaciones espectrales, enfocándose en la relación para las funciones a valor matricial de Stieltjes y Carathéodory.

Palabras clave: Medida matricial, transformación espectral, polinomios ortogonales matriciales en la recta real, polinomios ortogonales matriciales en la circunferencia unidad, momentos matriciales, transformación matricial de Szegő, transformación matricial de Christoffel y Uvarov, asintótica relativa, matrices de Hessenberg, matrices CMV, función a valor matricial de Stieltjes, función a valor matricial de Carathéodory.

Clasificación de asignaturas matemáticas MSC (2010): 33C45; 33D45; 42C05.

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To the memory of my dear brother,
Omar Yesid Quintero Fuentes 1988–2018

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Nomenclature

The following nomenclature will be used throughout the document. Some symbols take different meanings depending on the chapter, and therefore we do not include them in the following list.

Roman Symbols

A^T	—	Transpose of A.
A^H	—	Conjugate transpose of A.
c_n	—	Moment of order $n-$th associated to the matrix measure σ.
$\mathbb{C}$	—	The set of complex numbers.
$\mathcal{C}$	—	Block CMV matrix.
det	—	Determinant.
deg	—	Degree of a polynomial.
$\mathbb{D}$	—	The unit disc $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$.
F	—	Matrix-valued Carathéodory function.
H_n	—	Block (finite) Hankel matrix.
$\mathbb{H}_{\varphi^L}$	—	Block (left) Hessenberg matrix.
$\mathbb{H}_{\varphi^R}$	—	Block (right) Hessenberg matrix.
$\mathbf{I}$	—	$l \times l$ identity matrix.
$\mathfrak{Im}(\cdot)$	—	$\frac{1}{2i}(\cdot - \cdot^H)$ for $\cdot \in \mathcal{M}_l$.
$\mathbf{J}$	—	Block Jacobi matrix.
K_n^R	—	Right kernel matrix.
K_n^L	—	Left kernel matrix.
$\mathcal{M}_{k\times l}$	—	Set of $k \times l$ matrices with complex entries.
$\mathcal{M}_l$	—	Ring of $l \times l$ matrices with complex entries.
MOPUC	—	Matrix orthogonal polynomials on the unit circle.
MOPRL	—	Matrix orthogonal polynomials on the real line.
$n!!$	—	Double factorial of number n.
$\mathbb{N}$	—	Set of natural numbers.
$\mathcal{N}$	—	Nevai class of matrix measures.
p_n^R	—	Right matrix orthonormal polynomial on the real line.
p_n^L	—	Left matrix orthonormal polynomial on the real line.
P_n^R	—	Right monic matrix orthogonal polynomial on the real line.
P_n^L	—	Left monic matrix orthogonal polynomial on the real line.
$\mathbb{P}^l[z]$	—	Set of matrix polynomials in $z \in \mathbb{C}$.
$\mathbb{P}^l_n[z]$	—	Set of matrix polynomials of degree less than or equal to n in $z \in \mathbb{C}$.
$\mathbb{R}$	—	Real line.
$\mathfrak{Re}(\cdot)$	—	$\frac{1}{2}(\cdot + \cdot^H)$ for $\cdot \in \mathcal{M}_l$.

s_n	—	$\langle P_n^R, P_n^R\rangle_{R,\mu} = \langle P_n^L, P_n^L\rangle_{L,\mu}$.
S	—	Matrix-valued Stieltjes function.
$\mathcal{S}$	—	Szegő class of matrix measures.
S_n^R	—	$\langle \Phi_n^R, \Phi_n^R\rangle_{R,\sigma}$.
S_n^L	—	$\langle \Phi_n^L, \Phi_n^L\rangle_{L,\sigma}$.
Tr	—	Trace.
$\mathbb{T}$	—	The unit circle $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$\mathcal{W}$	—	Radon-Nikodym derivative of the absolutely continuous part of σ.

Greek Symbols

α_n	—	Verblunsky coefficients associated with the matrix measure σ.
δ	—	Dirac matrix measure.
$\delta_{n,k}$	—	Kronecker delta.
κ_n^R	—	Leading coefficient of φ_n^R.
κ_n^L	—	Leading coefficient of φ_n^L.
μ	—	Matrix measure supported on the real line.
μ_n	—	Moment $n-$th associated to the matrix measure μ.
ρ_n^R	—	$(\kappa_{n+1}^R)^{-1}\kappa_n^R = \sqrt{\mathbf{I} - \alpha_n\alpha_n^H}$.
ρ_n^L	—	$\kappa_n^L(\kappa_{n+1}^L)^{-1} = \sqrt{\mathbf{I} - \alpha_n^H\alpha_n}$.
$\varrho(\cdot)$	—	Spectrum of $\cdot \in \mathcal{M}_l$.
σ	—	Matrix measure support on unit circle.
φ_n^R	—	Right matrix orthonormal polynomial on the unit circle.
φ_n^L	—	Left matrix orthonormal polynomial on the unit circle.
ω	—	Radon-Nikodym derivative of the absolutely continuous part of μ.
$\Pi(\cdot)$	—	Spectral radius of $\cdot \in \mathcal{M}_l$.
Φ_n^R	—	Right monic matrix orthogonal polynomial on the unit circle.
Φ_n^L	—	Left monic matrix orthogonal polynomial on the unit circle.
$\Phi_n^{R,*}$	—	Reverse matrix polynomial $z^n\Phi_n^R(1/\bar{z})^H$.

Other Symbols

$\lVert\cdot\rVert_2$	—	Euclidean matrix norm.
$\lVert\cdot\rVert_F$	—	Frobenius spectral matrix norm.

Introduction

"Mathematics is not a prudent journey on a clear
highway, but a journey to a wild and strange
terrain, in which Explorers are often lost"
W.S. Anglin

Throughout history, the theory of orthogonal polynomials has been developed in two closely related directions:

1. Algebraic aspects (such as explicit expressions, recurrence relations, zeros localization, among others).
2. Analytic aspects (such as distribution of zeros, asymptotic behavior, associated differential operators, convergence of Fourier-type developments, among others).

The first direction is closely related with the theory of special functions, combinatorial and linear algebra, and is mainly dedicated to the study of concrete orthogonal systems or hierarchies of such systems, such as the Jacobi polynomials, Hahn, Askey-Wilson, etc. The theory of discrete orthogonal polynomials and the q-polynomials are also found in this direction, as well as many of the current advances in the study of orthogonal polynomials of several variables.

The second direction is characterized by the use of methods of mathematical analysis or methods related to it. The general properties of systems of orthogonal polynomials comprise a small part of these analytic aspects, while two extremely rich branches cover the larger part:

1. The theory of orthogonal polynomials on the real line.
2. The theory of orthogonal polynomials on the unit circle.

The theory of orthogonal polynomials on the real line originated in the work of A. M. Legendre, who was interested in solving the equations of motion of celestial mechanics. Thus A. M. Legendre built in 1785 the first family of orthogonal polynomials on the real line. These polynomials were also considered by P. S. Laplace relating them to spherical functions in the context of planetary motion. Subsequently, in 1821 Rodrigues managed to express these polynomials in terms of what we know today as the Rodrigues' formula. Later on, K. G. Jacobi in 1826 introduced the first family of polynomials which nowadays is known as classic, the Jacobi polynomials, that generalized those initially introduced by

Legendre. Then, Hermite polynomials appeared in 1864, although its weight function had already been studied by P. S. Laplace in 1810 as an application to probability theory. The last family (of the so-called classics) to appear were the Laguerre polynomials, introduced in 1879 when he was looking for a solution for the integral $\int_x^\infty e^{-t}t^{-1}dt$ by developing it in continued fractions, although they were already partially known by N.H Abel and J. L. Lagrange. Later on, T. S. Stieltjes introduced the so-called moment problem [164, 165] and P. L. Chebyshev introduced the polynomials that bear his name while considering the problem of finding the best uniform polynomial approximation of a continuous function [37, 38]. All of the above and other researchers helped the theory of orthogonal polynomials on the real line to be fully constituted. In 1939, the monograph *Orthogonal Polynomials* by G. Szegő [166] appears, in which all the results up to that moment are unified.

Similarly, the study of the theory orthogonal polynomials with respect to a nontrivial measure supported on the unit circle was initiated by G. Szegő in a series of articles published between 1915 and 1925, and later extended to a more general context of orthogonality respect to linear functionals by Ya. L. Geronimus. The two volumes of B. Simon's monograph [156, 157] appeared in the beginning of the 21st century, and they constitute the most exhaustive description of the state of art in the theory of orthogonal polynomials on the unit circle to this date.

Thanks to the origin of orthogonal polynomials in modern science, their study has been performed from many different points of view. That's why orthogonal polynomials both on the real line and on the unit circle have very useful properties in the solution of mathematical and physical problems. Their relations with moment problems [113, 155], rational approximation [21, 147], operator theory [114, 119], interpolation, Gaussian quadrature formulas [39, 89, 99, 166], electrostatics [111], statistical quantum mechanics [159], special functions [9], number theory [11] (irrationality [13] and transcendence [46]), graph theory [23], combinatorics, random matrices [47], stochastic process [153], data sorting and compression [72], and their role in the spectral theory of linear differential operators and Sturm-Liouville problems [145], as well as their applications in the theory of integrable systems [73, 98, 143, 144] constitute some illustrative samples of their impact.

In the last decades some generalizations of the theory of orthogonal polynomials, on both the real line and the unit circle, have been developed in the literature. One of these generalizations appears when a matrix of orthogonality measures (or simply, a matrix measure) is considered. In this case, two sequences of orthogonal matrix polynomials (polynomials with matrix coefficients) are obtained due to the non-commutativity of the matrices and were considered first by M. G. Krein [121, 122] in 1949. Later on, several researchers have made contributions in this theory until today. In the last 30 years, several known properties of orthogonal polynomials in the scalar case have been extended to the matrix case, such as algebraic aspects related to their zeros, recurrence relations, Favard type theorems and Christofell-Darboux formulas, Jacobi, Hessenberg and CMV block matrices, matrix-valued Stieltjes and Carathéodory functions, among many others. A nice summary, as well as many references, can be found in [42]. The extensions to the matrix case are usually developed by using procedures that are analogous to the ones used in the scalar case, with the added complications that result from the non commutativity of the matrices. Due to this non commutativity, two different (although related) orthogonal sequences, left and right, are usually considered.

As in the scalar case, matrix orthogonal polynomials on the real line (**MOPRL**) have proved to be a useful tool in the analysis of many problems of mathematics such as

rational approximation theory [83], spectral theory of Jacobi matrices [10, 146], analysis of polynomial sequences satisfying higher order recurrence relations [54, 71], quantum random walks [24] and Gaussian quadrature formulas [59, 69, 160], among many others. The problem of **MOPRL** whose derivatives also orthogonal was considered in [28]. Other differential properties, such as the role of **MOPRL** as solutions of first and second order differential equations has been analyzed in [27, 35, 36, 55, 71, 60, 61, 62, 63, 64, 102, 142], among many others.

The theory of matrix orthogonal polynomials with respect to a matrix measure supported on unit circle (**MOPUC**) is relatively recent and was indirectly developed as a consequence of the study of problems related to prediction theory [106, 107, 118, 120, 171]. More recently, a connection between **MOPUC** and quantum random walks has been explored in [24]. Some of the applications of **MOPUC** are found in the analysis of sequences of polynomials orthogonal with respect to a scalar measure supported on the complex plane [138], the analysis of time series in relation with the estimation of the frequencies of a stationary harmonic process [163], circuits and systems theory [48], linear algebra (in particular, the factorization of positive definite block-Toeplitz matrices [177]), spectral operator theory [95] and, more recently, Toda type integrable systems [6]. Moreover, they constitute a basic tool in Gaussian matrix quadrature formulas [21, 161, 163].

Most of the results contained in this document are related with spectral transformations of matrix measures supported on unit circle. Let us clarify what we mean by spectral transformation. Let $\mathcal{P}_l$ be the set of normalized positive semi-definite $l \times l$ matrix measures supported on unit circle, and let $\mathbf{D}_l^\infty$ be the set of matrix sequences $(\alpha_n)_{n\geq 0}$ with $\|\alpha_n\|_2 < 1$ (where $\|\cdot\|_2$ is the Euclidean matrix norm [109]). Moreover, let $\mathbf{F}_l$ be the set of matrix-valued Carathéodory functions (see Subsection 1.4.4) associated with positive semi-definite matrix measures supported on unit circle. Let us consider an injective mapping

$$\Delta : \mathbf{D}_l^\infty \to \mathbf{D}_l^\infty.$$

By Verblunsky's theorem for **MOPUC** (see Theorem 1.4.2), there exists a bijective mapping $\mathcal{S} : \mathcal{P}_l \to \mathbf{D}_l^\infty$. Δ induces a mapping

$$\Gamma : \mathcal{P}_l \to \mathcal{P}_l$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{P}_l & \xrightarrow{\Gamma} & \mathcal{P}_l \\ {\scriptstyle\mathcal{S}}\downarrow & & {\scriptstyle\mathcal{S}}\downarrow \\ \mathbf{D}_l^\infty & \xrightarrow[\Delta]{} & \mathbf{D}_l^\infty \end{array}$$

is commutative, i.e., $\Delta \circ \mathcal{S} = \mathcal{S} \circ \Gamma$. We will call the mapping Γ a **spectral transformation** of a matrix measure supported on unit circle. Similarly, let us consider an injective mapping

$$\mathcal{F} : \mathbf{F}_l \to \mathbf{F}_l.$$

By Riesz-Herglotz Theorem for **MOPUC** (see Theorem 1.4.8), there exists a bijective mapping $\mathcal{S}' : \mathcal{P}_l \to \mathbf{F}_l$. $\mathcal{F}$ induces a mapping

$$\Gamma : \mathcal{P}_l \to \mathcal{P}_l,$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{P}_l & \xrightarrow{\Gamma} & \mathcal{P}_l \\ {\scriptstyle \mathcal{S}'}\downarrow & & {\scriptstyle \mathcal{S}'}\downarrow \\ \mathbf{F}_l & \xrightarrow[\mathcal{F}]{} & \mathbf{F}_l \end{array}$$

is commutative, i.e., $\mathcal{F} \circ \mathcal{S}' = \mathcal{S}' \circ \Gamma$. The mapping Γ will also be called **spectral transformation** of a matrix measure supported on unit circle.

The theory of orthogonal polynomials associated to spectral transformations of measures supported on the unit circle was developed in the 1990s (see [1, 17, 93, 94, 112, 127, 136, 137, 148] among many others). It is also worth mentioning an interpretation based on the matrix representation of the multiplication operator with respect to an orthonormal basis, when the corresponding measures are supported on an arc of the unit circle ([96, 97]). Spectral transformations find applications in the integrable systems theory [169], fourth order linear differential equations with polynomial coefficients [103] and the bispectral problem [105], among others.

The theory of spectral transformations of matrix measures has not received a lot of attention in the literature, although there are several contributions in this direction. For instance, for **MOPRL**, connection formulas and asymptotic properties associated with the Uvarov matrix transformations are deduced in [174, 175, 176]. In [2, 3, 84, 85], some generalizations of the Christoffel and Geronimus matrix transformations are studied, obtaining connection formulas for the associated **MOPRL** that are expressed in terms of quasi-determinants. Spectral transformations of matrix measures supported on unit circle are much less studied, although the Uvarov matrix transformation, consisting in the addition of a Dirac matrix measure, has been studied in [173], where the authors studied asymptotic relative properties of the corresponding orthogonal sequences. As in the classical theory of orthogonal polynomials on the unit circle, the asymptotic behavior of sequences of Sobolev type (in particular, of Uvarov type) in the matrix case plays a central role in questions related to their application in approximation processes, Gaussian matrix quadrature formulas and Fourier expansions (see [21, 161, 163]). Recently, spectral transformations associated with perturbations of matrix moments have been studied in [40, 41], in connection with the so-called matrix moment problem.

As stated above, matrix orthogonal polynomials both on the real line and on the unit circle have been substantially studied in the last years not only because its intrinsic mathematical interest, but also because of its many applications. This constitutes a motivation to develop this work. Furthermore, in [48], the authors present a general and self-contained theory of **MOPUC**, connecting topics as diverse as factorization of positive definite block-Toeplitz matrices, Fourier expansion of positive definite periodic function matrices and spectral factorization of such matrices, least-squares approximation on the unit circle, and characterization of positive function matrices in the unit disk. Thus, properties of **MOPUC**, both algebraic and analytic, can be used as a tool in order to obtain results for the scalar case. In a similar way, it is possible that by analyzing properties associated with spectral transformations of **MOPUC**, one can deduce properties associated with spectral transformations of orthogonal polynomials on the scalar case, which is also a motivation of this work.

Most of the results of this work are oriented in the study of some algebraic and analytic properties associated with spectral transformations of a Hermitian matrix measure

supported on the unit circle. Those results consist in extending to the matrix case several properties known in the scalar case, related to recurrence formulas, connection formulas, relative asymptotics, block Hessenberg and CMV matrix factorizations, among others.

Finally, there exists an interesting relation between (scalar) measures supported on $[-1, 1]$ and measures supported on the unit circle, known in the literature as the Szegő transformation. In [156, 157], the author shows not only how these measures are related, but also the relation between the corresponding families of orthogonal polynomials, as well as the relation between the sequences of parameters of the recurrence relation on the real line and the sequence of Verblunsky coefficients on the unit circle, known as Geronimus relation. The interest of the Szegő transformation is not only from a theoretical perspective, but also from a computational point of view, since it has been used in a systematic way in the study of Szegő's quadrature formulas on the unit circle and its application to quadrature formulas in intervals of the real line [12, 43, 44].

The matrix version of Szegő transformation has been studied in [42, 51, 172]. In [172], the authors show that when two positive definite matrix measures supported on the real line and on the unit circle, respectively, are related through of the Szegő matrix transformation, there exists a relation between the corresponding sequences of orthogonal matrix polynomials. The relation between the Verblunsky matrix coefficients and the coefficients of the three-term recurrence relation known as Geronimus matrix relations is shown in [51, 116]. In this work, we extend to the matrix case some properties of the direct and inverse Szegő transformation and then we use these properties to analyze its effect on some spectral transformations, focusing on the relationship between the matrix-valued Stieltjes and Carathéodory functions.

Contributions and structure of the thesis

The contributions of this thesis are included in seven articles. Among these we have

1. On a finite moment perturbation of linear functionals and the inverse Szegő transformation [78]. (Published)
2. CMV block matrices for symmetric measures on the unit circle [76]. (In press)
3. Matrix moment perturbations and the inverse Szegő matrix transformation [79]. (In press)

and the following are currently submitted

1. Block Hessenberg matrices and orthonormal matrix polynomials on the unit circle [75].
2. On a Christoffel transformation for matrix measures supported on the unit circle [80].
3. On the Szegő transformation for some spectral perturbations of matrix measures [81].
4. Matrix Uvarov transformation on the unit circle: Asymptotic properties [82].

On the other hand, this work is divided into chapters whose content is distributed as follows:

Chapter 1 is meant for non-experts and therefore it contains some introductory and background material. In the first section some matrix analysis results are recalled. Then, we state some properties of matrix orthogonal polynomials both on the real line and on the unit circle that will be used in the sequel. There are many contributions in the literature where algebraic and analytic properties of scalar orthogonal polynomials are generalized to the matrix case. An outstanding summary and references on the subject can be found in [42, 48] among others. In the last section, we define the Szegő matrix transformation and state some of its properties, including a relation between the matrix-valued Stieltjes function associated with a matrix measure supported on $[-1, 1]$ and the matrix-valued Carathéodory function associated with a matrix measure supported on the unit circle.

In Chapter 2, we study a problem associated with a perturbation of the sequence of moments of a scalar measure. Given a perturbation of a measure supported on unit circle, we analyze the perturbation obtained on the corresponding measure on the real line,

when both measures are related through the inverse Szegő transformation. Moreover, using the connection formulas for the corresponding sequences of orthogonal polynomials, we deduce several properties such as relations between the corresponding *norms*. This analysis is presented on Section 1 and is later generalized to the matrix case in Section 2. The results of this chapter are contained in [78, 79].

In Chapter 3, we extend to the matrix case some algebraic and analytic properties of **MOPUC**, that are known in the scalar case, associated with a generalized Uvarov matrix transformation of a Hermitian matrix measure supported on the unit circle. We study properties such as connection formulas between the corresponding sequences of **MOPUC**, relative asymptotics for the matrix polynomials, as well as for their leading principal (matrix) coefficients, and properties of the zeros for the Uvarov matrix polynomials by using the associated (truncated) block Hessenberg matrices. Similar properties are deduced for the case of the Christoffel matrix transformation. The results for the Uvarov matrix transformation are contained in [82] and the Christoffel case appears in [80].

In the first section of Chapter 4, we study properties of block Hessenberg matrices associated with **MOPUC**. We also consider the Uvarov and Christoffel spectral matrix transformations of the orthogonality measure, and obtain some relations between the associated Hessenberg matrices. The results are contained in [75]. In the second section, we present some properties of symmetric positive definite matrix measures supported on unit circle, and relate the CMV block matrices associated with two symmetric matrix measures. Moreover, we present some sequences of Verblunsky matrix coefficients associated with certain transformations of matrix measures. The results can be found in [76].

In Chapter 5, given a normalized positive definite matrix measure μ supported on $[-1, 1]$, we obtain necessary conditions such that some spectral perturbations applied to μ are preserved under the Szegő matrix transformation. The results are included in [81].

Finally, in Chapter 6 first we present our conclusions, and we formulate some open problems which have arisen during our research. They constitute research problems that will be addressed in the near future.

CHAPTER 1

Matrix orthogonal polynomials and the Szegő matrix transformation

1.1 Matrix analysis

In the remaining of the document, we will use the following notation. Let $\mathbb{C}$ be the set of complex numbers and denote by $\mathcal{M}_{k\times l} = \mathcal{M}_{k\times l}(\mathbb{C})$ the set of $k \times l$ matrices with complex entries. In particular $\mathcal{M}_l = \mathcal{M}_{l\times l}(\mathbb{C})$ is the ring of $l \times l$ matrices and $\mathbf{I}$, $\mathbf{0}$ will denote the identity and the zero matrix, respectively, in $\mathcal{M}_l$. Let $\varrho(\mathbf{A})$ be the spectrum of $\mathbf{A} \in \mathcal{M}_l$, i.e. the set of its eigenvalues (an eigenvalue of $\mathbf{A}$ is a number $\lambda \in \mathbb{C}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v} \in \mathcal{M}_{l\times 1}$ called eigenvector).

Theorem 1.1.1. [19]. *Let* $\mathbf{A} \in \mathcal{M}_l$.

1. *The matrix* $\mathbf{A}$ *has* l *eigenvalues.*
2. *The sum of the* l *eigenvalues of* $\mathbf{A}$ *(including algebraic multiplicities) is the same as the trace of* $\mathbf{A}$ *(that is, the sum of the diagonal elements of* $\mathbf{A}$*).*
3. *The product of the* l *eigenvalues of* $\mathbf{A}$ *is the same as the determinant of* $\mathbf{A}$.

Let $\mathbf{A}^H$ be the conjugate transpose of $\mathbf{A} \in \mathcal{M}_l$. A matrix satisfying $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$ is called normal. There are two kinds of normal matrices of particular interest:

1. Hermitian matrices, satisfying $\mathbf{A}^H = \mathbf{A}$.
2. Unitary matrices, satisfying $\mathbf{A}^H\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^H$.

Hermitian matrices have real eigenvalues and unitary matrices have eigenvalues on the unit circle. The set of Hermitian matrices is a real vector space.

A matrix $\mathbf{A} \in \mathcal{M}_l$ is called positive definite (resp. semi-definite) if it is Hermitian and $u^H\mathbf{A}u > 0$ (resp. $u^H\mathbf{A}u \geq 0$) for every nonzero vector $u \in \mathcal{M}_{l\times 1}$. The notation $\mathbf{A} > \mathbf{B}$ (resp. $\mathbf{A} \geq \mathbf{B}$) for Hermitian matrices $\mathbf{A}, \mathbf{B} \in \mathcal{M}_l$ means that $\mathbf{A} - \mathbf{B}$ is positive definite (resp. semi-definite). The next theorem establishes some results on definite positive matrices whose proof can be found in [53, 109, 124].

Theorem 1.1.2. *Let $\mathbf{A} \in \mathcal{M}_l$ be a Hermitian matrix.*

1. *The matrix $\mathbf{A}$ is positive definite (resp. semi-definite) if for every $\lambda \in \varrho(\mathbf{A})$, $\lambda > 0$ (resp. $\lambda \geq 0$).*
2. *The matrix $\mathbf{A}$ is positive definite (resp. semi-definite) if and only if all principal minors of $\mathbf{A}$ are positive (resp. non-negative).*
3. *If $\mathbf{B} \in \mathcal{M}_l$ is a definite positive matrix, then every $\lambda \in \varrho(\mathbf{BA})$ is a real number. If both $\mathbf{A}$ and $\mathbf{B}$ are definite positive, then every $\lambda \in \varrho(\mathbf{BA})$ is positive.*
4. (Cholesky factorization). *The matrix $\mathbf{A}$ is positive definite if and only if there exists a unique lower triangle matrix $\mathbf{L} \in \mathcal{M}_l$ with positive entries on the main diagonal and such that $\mathbf{A} = \mathbf{L}\mathbf{L}^H$.*

A matrix $\mathbf{A} \in \mathcal{M}_l$ is unitarily similar to $\mathbf{B} \in \mathcal{M}_l$ if there exists a unitary matrix $U \in \mathcal{M}_l$ such that $\mathbf{A} = U\mathbf{B}U^H$. $\mathbf{A}$ is unitarily diagonalizable if it is unitarily similar to a diagonal matrix. It is easy to verify that unitary similarity is an equivalence relation, and the canonic form for complex square matrices is given in [110]. The Spectral Theorem for normal matrices establishes a bijection between unitarily diagonalizable matrices and normal matrices.

Theorem 1.1.3. (Spectral Theorem [109]). *A matrix is normal if and only if it is unitarily diagonalizable.*

A square root of a matrix $\mathbf{A} \in \mathcal{M}_l$ is any matrix $\mathbf{B} \in \mathcal{M}_l$ such that $\mathbf{B}^2 = \mathbf{A}$. We also write $\mathbf{B} := \mathbf{A}^{1/2} = \sqrt{\mathbf{A}}$, and $\mathbf{A}^{-1/2}$ denotes the inverse of the square root. The following results guarantee the existence and uniqueness of square roots for non-singular matrices.

Theorem 1.1.4. [15]. *If $\mathbf{A} \in \mathcal{M}_l$ is a non-singular matrix, then there exists a non-singular matrix $\mathbf{B} \in \mathcal{M}_l$ such that $\mathbf{B} = \mathbf{A}^{1/2}$.*

Theorem 1.1.5. [109]. *If $\mathbf{A} \in \mathcal{M}_l$ is a positive definite (resp. semi-definite) matrix, then there exists a unique positive definite (resp. semi-definite) matrix $\mathbf{B} \in \mathcal{M}_l$ such that $\mathbf{B} = \mathbf{A}^{1/2}$. Furthermore, $\mathbf{A}^{1/2}$ commutes with $\mathbf{A}$.*

Let $\|\cdot\|_2$ and $\|\cdot\|_F$ be, respectively, the Euclidean and Frobenius spectral matrix norms (see [109]) defined by

$$\|\mathbf{A}\|_2^2 = \Pi(\mathbf{A}^H\mathbf{A}) \text{ and } \|\mathbf{A}\|_F^2 = \sum_{i,j=1}^{l} |a_{i,j}|^2,$$

where $\Pi(\mathbf{A})$ is the spectral radius of $\mathbf{A} \in \mathcal{M}_l$, i.e. $\Pi(\mathbf{A}) = \max\{|\lambda| : \lambda \in \varrho(\mathbf{A})\}$. These norms satisfy

1. $\|\mathbf{A}\|_F^2 = Tr(\mathbf{A}^H\mathbf{A})$,
2. $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F$,

where $Tr(\mathbf{A})$ denotes the trace of $\mathbf{A}$.

Given a block matrix $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{M}_l$, its inverse can be computed by using the Schur complement (see [178])

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix}. \tag{1.1}$$

1.2 Matrix polynomials space and matrix-valued measures

Let $\mathbf{R}$ be a ring, then a right module over $\mathbf{R}$ is a set $\mathbf{M}$ together with two operations

$$+ : \mathbf{M} \times \mathbf{M} \to \mathbf{M} \text{ and } \cdot : \mathbf{M} \times \mathbf{R} \to \mathbf{M}$$

such that for $v, w \in \mathbf{M}$ and $\alpha, \beta \in \mathbf{R}$ we have

1. $(\mathbf{M}, +)$ is an Abelian group.
2. $v \cdot (\alpha + \beta) = v \cdot \alpha + v \cdot \beta$.
3. $(v + w) \cdot \alpha = v \cdot \alpha + w \cdot \alpha$.
4. $v \cdot (\alpha \cdot \beta) = (v \cdot \alpha) \cdot \beta$.

In a similar way, one defines a left module over $\mathbf{R}$. If $\mathbf{M}$ is a right and left module over $\mathbf{R}$, then $\mathbf{M}$ is said to be a bi-module [125, 152]. $\mathbf{M}$ is said to be a free right (or left) module over $\mathbf{R}$ if $\mathbf{M}$ admits a basis, that is, there exists a subset $\mathbf{S}$ of $\mathbf{M}$ such that $\mathbf{S}$ is not empty, $\mathbf{S}$ generates $\mathbf{M}$, ($\mathbf{M} = \langle \mathbf{S} \rangle = span(\mathbf{S})$) and $\mathbf{S}$ is linearly independent.

Recall that for any matrices $\mathbf{A}_k \in \mathcal{M}_l$, $0 \leq k \leq n$, with $\mathbf{A}_n \neq \mathbf{0}$, the matrix $\Phi(z) = \sum_{k=0}^{n} \mathbf{A}_k z^k$ is said to be a matrix polynomial of degree n. In particular, if $\mathbf{A}_n = \mathbf{I}$, then the matrix polynomial is said to be monic. Notice that, when $\mathbf{A}_n$ is non-singular, it is possible to turn Φ into a monic matrix polynomial. The set of matrix polynomials in $z \in \mathbb{C}$ (resp. $x \in \mathbb{R}$) with coefficients in $\mathcal{M}_l$ will be denoted by $\mathbb{P}^l[z]$ (resp. $\mathbb{P}^l[x]$). The set $\mathbb{P}^l[z]$ can be considered as a free bi-module (and, in particular, a right module) over the ring $\mathcal{M}_l$ with basis $(\mathbf{I}z^n)_{n\geq 0}$. An important submodule of $\mathbb{P}^l[z]$ (resp. $\mathbb{P}^l[x]$) is the set $\mathbb{P}^l_n[z]$ (resp. $\mathbb{P}^l_n[x]$) of matrix polynomials of degree less than or equal to n with the basis $(\mathbf{I}z^j)_{j=0}^n$ (resp. $(\mathbf{I}x^j)_{j=0}^n$).

The fundamental aspect in the study of the theory of matrix orthogonal polynomials is the shift from a scalar to a matrix measure. A detailed study on matrix measures can be found in [42, 151].

An $l \times l$ matrix-valued measure (or simply, matrix measure) $\sigma = (\sigma_{i,j})_{i,j=1}^l$ supported on $E \subseteq \mathbb{R}$ (in particular on the unit circle with $E = (-\pi, \pi]$) with finite moments of any order is said to be Hermitian (resp. positive semi-definite, positive definite), if for every Borel subset $\mathcal{B}$ on E, the matrix $\sigma(\mathcal{B})$ is a Hermitian (resp. positive semi-definite, positive definite) matrix. Notice that the assumption of positive definiteness implies the Hermitian character of the measure. The matrix measure σ is said to be normalized if $\sigma(E) = \mathbf{I}$ (i.e. a probability matrix measure).

1.3 Matrix orthogonal polynomials on the real line (MOPRL)

Let $\mu = (\mu_{ij})_{i,j=1}^{l}$ be a $l \times l$ Hermitian matrix measure supported on a compact subset $E \subseteq \mathbb{R}$. If the scalar measures $\mu_{i,j}$, $i, j = 1, \ldots, l$ are absolutely continuous with respect to the scalar Lebesgue measure dx, then according to the Radon-Nikodym Theorem, they can always be expressed as $d\mu_{i,j}(x) = \omega_{i,j}(x)dx$. Since μ is a Hermitian matrix measure, then the matrix $\omega(x) = (\omega_{i,j}(x))_{i,j=1}^{l}$ is also a Hermitian matrix measure, and thus

$$d\mu(x) = \omega(x)dx. \tag{1.2}$$

A matrix measure μ induces right and left sesquilinear forms, defined respectively as

$$\langle f, g\rangle_{R,\mu} = \int_E f(x)^H d\mu(x) g(x) \in \mathcal{M}_l, \quad f, g \in \mathbb{P}^l[x], \tag{1.3}$$

$$\langle f, g\rangle_{L,\mu} = \int_E f(x) d\mu(x) g(x)^H \in \mathcal{M}_l, \quad f, g \in \mathbb{P}^l[x]. \tag{1.4}$$

The sesquilinearity of these forms means that for every $f, g, h \in \mathbb{P}^l[x]$, and $\mathbf{A}, \mathbf{B} \in \mathcal{M}_l$, they satisfy

1. $\langle f, g\mathbf{A} + h\mathbf{B}\rangle_{R,\mu} = \langle f, g\rangle_{R,\mu}\mathbf{A} + \langle f, h\rangle_{R,\mu}\mathbf{B}$ and $\langle g\mathbf{A} + h\mathbf{B}, f\rangle_{R,\mu} = \mathbf{A}^H\langle g, f\rangle_{R,\mu} + \mathbf{B}^H\langle h, f\rangle_{R,\mu}$.
2. $\langle f, \mathbf{A}g + \mathbf{B}h\rangle_{L,\mu} = \langle f, g\rangle_{L,\mu}\mathbf{A}^H + \langle f, h\rangle_{L,\mu}\mathbf{B}^H$ and $\langle \mathbf{A}g + \mathbf{B}h, f\rangle_{L,\mu} = \mathbf{A}\langle g, f\rangle_{L,\mu} + \mathbf{B}\langle h, f\rangle_{L,\mu}$.

Directly from the definition for every $f, g \in \mathbb{P}^l[x]$, we have

1. $\langle f, g\rangle_{R,\mu} = \langle g, f\rangle_{R,\mu}^H, \quad \langle f, g\rangle_{L,\mu} = \langle g, f\rangle_{L,\mu}^H$.
2. $\langle f, g\rangle_{R,\mu} = \langle f^H, g^H\rangle_{L,\mu}, \quad \langle f, g\rangle_{L,\mu} = \langle f^H, g^H\rangle_{R,\mu}$.

Proposition 1.3.1. *([104], Proposition 2.2.) Let $f(x) = \sum_{k=0}^{n} \mathbf{A}_k x^k \in \mathbb{P}[x]$ be a matrix polynomial of degree n. If $\mathbf{A}_k$ is non-singular for some $0 \leq k \leq n$, then $\langle f, f\rangle_{R,\mu}$ and $\langle f, f\rangle_{L,\mu}$ are non-singular matrices. Moreover, $\langle f, f\rangle_{R,\mu} = \mathbf{0}$ (resp. $\langle f, f\rangle_{L,\mu} = \mathbf{0}$) if and only if $f = \mathbf{0}$.*

As a consequence of the above proposition, for any monic matrix polynomial $f \in \mathbb{P}[x]$, $\langle f, f\rangle_{R,\mu}$ and $\langle f, f\rangle_{L,\mu}$ are non-singular matrices. Therefore, it is possible to construct a sequence of monic matrix polynomials $(P_n^R(x))_{n\geq 0}$ (resp. $(P_n^L(x))_{n\geq 0}$) orthogonal with respect to (1.3) (resp. (1.4)) by applying the Gram-Schmidt orthogonalization process to the basis $(x^n\mathbf{I})_{n\geq 0}$. This is, P_n^R and P_n^L are the unique monic matrix polynomials satisfying

$$\langle x^k\mathbf{I}, P_n^R\rangle_{R,\mu} = \langle P_n^L, x^k\mathbf{I}\rangle_{L,\mu} = \mathbf{0}, \quad k = 0, 1, \ldots, n-1, \tag{1.5}$$

the polynomials $(P_n^R(x))_{n\geq 0}$ and $(P_n^L(x))_{n\geq 0}$ are called, respectively, the right and left-orthogonal matrix polynomials sequences with respect to the matrix measure μ.

From the Lemma 1.3 and (1.5) it follows that $P_n^R(x) = P_n^L(x)^H$ for all $x \in \mathbb{R}$ and from the properties of sesquilinear forms we have, for every $n \in \mathbb{N}$

$$\langle P_n^R, P_n^R\rangle_{R,\mu} = \langle P_n^L, P_n^L\rangle_{L,\mu} = s_n, \tag{1.6}$$

for some invertible matrix $s_n \in \mathcal{M}_l$. Furthermore, $(P_k^R(x))_{k=0}^n$ is a right module basis for $\mathbb{P}_n^l[x]$; indeed, any $f \in \mathbb{P}_n^l[x]$ has a unique expansion,

$$f(x) = \sum_{k=0}^{n} P_k^R(x) f_k^R, \quad f_k^R = s_k^{-1} \langle P_k^R, f \rangle_{R,\mu}. \tag{1.7}$$

Similarly for $(P_k^L(x))_{k=0}^n$

$$f(x) = \sum_{k=0}^{n} f_k^L P_k^L(x), \quad f_k^L = \langle f, P_k^L \rangle_{L,\mu} s_k^{-1}.$$

Moreover, if μ is a positive semi-definite matrix measure, there exist sequences of matrix polynomials $(p_n^R(x))_{n\geq 0}$ and $(p_n^L(x))_{n\geq 0}$, with

$$p_n^R(x) = \zeta_n^R x^n + \text{ lower order terms },$$
$$p_n^L(x) = \zeta_n^L x^n + \text{ lower order terms },$$

for the right and left cases, respectively, with non-singular matrices ζ_n^R and ζ_n^L. They are orthonormal with respect to (1.3) and (1.4), respectively. This is,

$$\langle p_k^R, p_n^R, \rangle_{R,\mu} = \langle p_n^L, p_k^L, \rangle_{L,\mu} = \mathbf{I}\delta_{n,k}, \quad k = 0, 1, \ldots, n.$$

Clearly, any orthonormal matrix polynomial has the form

$$p_n^R(x) = P_n^R(x)\zeta_n^R \nu_n, \text{ and } p_n^L(x) = \tau_n \zeta_n^L P_n^L(x),$$

where $\nu_n, \tau_n \in \mathcal{M}_l$ are unitary matrices. This is, matrix orthonormal polynomials on the real line are unique up to multiplication by unitary matrices, since

$$\langle p_n^R \nu_n^H, p_n^R \nu_n^H \rangle_{R,\mu} = \nu_n \langle p_n^R, p_n^R \rangle_{R,\mu} \nu_n^H = \nu_n \nu_n^H = \mathbf{I},$$

$$\langle \tau_n^H p_n^L, \tau_n^H p_n^L \rangle_{L,\mu} = \tau_n^H \langle p_n^L, p_n^L \rangle_{L,\mu} \tau_n = \tau_n^H \tau_n = \mathbf{I}.$$

By (1.7), the sequence of polynomials $(p_n^R(x))_{n\geq 0}$ also form a right orthonormal module basis in $\mathbb{P}^l[x]$. So for any $f \in \mathbb{P}^l[x]$, the expansion

$$f(x) = \sum_{k=0}^{\infty} p_k^R(x) f_k^R, \quad f_k^R = \langle p_k^R, f \rangle_{R,\mu} \tag{1.8}$$

Similarly, the sequence of polynomials $(p_n^L(x))_{n\geq 0}$ is a left module basis for $\mathbb{P}^l[x]$, and the expansion

$$f(x) = \sum_{k=0}^{\infty} f_k^L p_k^L(x), \quad f_k^L = \langle f, p_k^L \rangle_{L,\mu} \tag{1.9}$$

is unique.

If the Hermitian matrix measure is positive definite, we can derive the corresponding scalar products

$$\langle f, g \rangle_{\mu_{esc},R} = Tr\left[\langle f, g \rangle_{R,\mu}\right], \ \langle f, g \rangle_{\mu_{esc},L} = Tr\left[\langle f, g \rangle_{L,\mu}\right], \quad f, g \in \mathbb{P}^l[x],$$

and the corresponding Hilbert space $\mathcal{H}^R$, $\mathcal{H}^L$, with norms of Frobenius type given by

$$\|f\|_R = \sqrt{\langle f, f\rangle_{\mu_{esc},R}} \text{ and } \|f\|_L = \sqrt{\langle f, f\rangle_{\mu_{esc},L}}.$$

Moreover, from (1.8) and (1.9) the Parseval identity

$$\sum_{k=0}^{\infty} Tr(f_k^{R,H} f_k^R) = \|f\|_R^2 \text{ and } \sum_{k=0}^{\infty} Tr(f_k^L f_k^{L,H}) = \|f\|_L^2 \tag{1.10}$$

holds true. Obviously, since f is a polynomial, there are only finitely many non-zero terms in (1.8), (1.9) and (1.10).

1.3.1 Recurrence relations for MOPRL

Let $\mathbf{J}$ be the block Hessenberg matrix representing the multiplication by x operator with respect to the basis $\{p_n^R\}_{n\geq 0}$. As in the scalar case, using the orthogonality properties of p_n^R, we get that $\mathbf{J}_{nm} = \mathbf{0}$ if $|n-m| > 1$. Denote

$$B_n = \mathbf{J}_{nn} = \langle p_{n-1}^R, x p_{n-1}^R\rangle_{R,\mu} \text{ and } A_n = \mathbf{J}_{n,n+1} = \mathbf{J}_{n+1,n}^H = \langle p_{n-1}^R, x p_n^R\rangle_{R,\mu}.$$

Then we have

$$\mathbf{J} = \begin{bmatrix} B_1 & A_1 & \mathbf{0} & \cdots \\ A_1^H & B_2 & A_2 & \cdots \\ \mathbf{0} & A_2^H & B_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.11}$$

Applying (1.8) to $f(x) = x p_n^R(x)$, we get the three-term recurrence relation

$$x p_n^R(x) = p_{n+1}^R(x) A_{n+1}^H + p_n^R(x) B_{n+1} + p_{n-1}^R(x) A_n, \ n \geq 1. \tag{1.12}$$

If we set $p_{-1}^R(x) = \mathbf{0}$ and $A_0 = \mathbf{I}$, the relation (1.12) also holds for $n = 0$. Since for x real we have $p_n^L(x)^H = p_n^R(x)$, by conjugating (1.12), we get

$$x p_n^L(x) = A_{n+1} p_{n+1}^L(x) + B_{n+1} p_n^L(x) + A_n^H p_{n-1}^L(x), \ n \geq 1. \tag{1.13}$$

The matrices $B_n \in \mathcal{M}_l$ are Hermitian and the matrices $A_n \in \mathcal{M}_l$ are non-singular for $n \geq 0$. Any block matrix of the form (1.11) with $B_n = B_n^H$ and $\det A_n \neq 0$ for all n will be called a block Jacobi matrix corresponding to the Jacobi parameters A_n and B_n. It is well known (see Favard's Theorem [42], Section 2.2.4) that there exits a bijection between Hermitian matrix measures with compact support and block Jacobi matrices.

1.3.2 Zeros

A zero of a matrix polynomial f is $z_0 \in \mathbb{C}$ such that $\det f(z_0) = 0$. In [42], it is shown that:

1. All the zeros of $P_n^R(x)$ are real. Moreover, $\det(P_n^R(x)) = \det(P_n^L(x))$.

2. Let $(x_{n,j})_{j=1}^{nl}$ be the zeros of $P_n^R(x)$, counting multiplicities, ordered by

$$x_{n,1} \leq x_{n,2} \leq \cdots \leq x_{n,nl},$$

then

$$x_{n+1,j} \leq x_{n,j} \leq x_{n+1,j+l}.$$

Example 1.3.2. *Consider the 2×2 Hermitian matrix-valued measure supported on $(-1,1)$, defined by*

$$d\mu(x) = \frac{1}{\sqrt{1-x^2}} \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} dx. \tag{1.14}$$

The $n-$th degree monic **MOPRL** *(see [35]) is given by*

$$P_n^R(x) = P_n^L(x) = \frac{1}{2^n} \begin{bmatrix} U_n(x) & -U_{n-1}(x) \\ -U_{n-1}(x) & U_n(x) \end{bmatrix}, \quad n \geq 1,$$

where $(U_n(x))_{n\geq 0}$ is the sequence of Chebyshev polynomials of the second kind on $\mathbb{R}$ (see [39]). We have that

$$\langle P_k^R, P_n^R \rangle_{R,\mu} = \frac{1}{2^n} \int_{-1}^{1} x^n T_n(x) \frac{dx}{\sqrt{1-x^2}} \mathbf{I}_2 \delta_{nk} = \frac{\pi}{2^{2n}} \mathbf{I}_2 \delta_{nk}, \ k = 0, \cdots, n,$$

where $(T_n(x))_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind on $\mathbb{R}$ (see [39]) and $\mathbf{I}_2 \in \mathcal{M}_2$ is the 2×2 identity matrix. The coefficients of the recurrence relation (1.12) are:

$$B_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_n = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ n \geq 2 \text{ and } A_n = \frac{1}{4} \mathbf{I}_2, \ n \geq 1.$$

Furthermore, the zeros of $P_n^R(x)$ are given by

$$x_{n,k} = \pm \cos\left(\frac{2k\pi}{2n+1}\right), \quad k = 1, \ldots, n,$$

notice that $P_n^R(x)$ has $2n$ real and distinct zeros.

1.3.3 Kernel matrix polynomial and the Christoffel-Darboux formula

The kernel matrix polynomial of degree n for $x, y \in \mathbb{R}$ is defined as

$$K_n^R(x,y) = \sum_{k=0}^{n} p_k^R(x) p_k^R(y)^H = \sum_{k=0}^{n} p_k^L(x)^H p_k^L(y) = K_n^L(x,y). \tag{1.15}$$

In [42] it is shown that they satisfy the Christoffel-Darboux formulas

$$K_n^R(x,y) = \frac{p_{n+1}^R(x) A_{n+1}^H p_n^R(y)^H - p_n^R(x) A_{n+1} p_{n+1}^R(y)^H}{x-y}, \ n \geq 1 \tag{1.16}$$

and

$$K_n^L(x,y) = \frac{p_{n+1}^L(x)^H A_{n+1}^H p_n^L(y) - p_n^L(x)^H A_{n+1} p_{n+1}^L(y)}{x-y}, \ n \geq 1 \tag{1.17}$$

for any $x, y \in \mathbb{R}$ and $x \neq y$.

Furthermore, denoting $K_n(x,y) = K_n^R(x,y) = K_n^L(x,y)$, for any matrix polynomial $\pi \in \mathbb{P}_n^l[x]$, the kernel matrix polynomial have the reproducing properties ([42]), i.e., for any $x, y, z \in \mathbb{R}$

1. $\int_E K_n(y,x)d\mu(x)\pi(x) = \pi(y)$.
2. $\int_E \pi(x)d\mu(x)K_n(x,y) = \pi(y)$.
3. $\int_E K_n(y,x)d\mu(x)K_n(x,z) = K_n(y,z)$.

1.3.4 The matrix-valued Stieltjes function

The matrix-valued Stieltjes function (or Cauchy transformation) associated with the normalized matrix measure μ is defined by

$$S(x) = \int_{-1}^{1} \frac{1}{x-t} d\mu(t), \quad x \in \mathbb{C}\backslash[-1,1]. \tag{1.18}$$

It is of great interest in the theory of **MOPRL** (see [91, 92]), and admits the following series expansion

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \mu_n, \tag{1.19}$$

where μ_n are the moments associated with μ, i.e.,

$$\mu_n = \langle \mathbf{I}, x^n\mathbf{I}\rangle_{R,\mu} = \langle x^n\mathbf{I}, \mathbf{I}\rangle_{L,\mu} = \int_{-1}^{1} x^n d\mu(x), \ n \geq 0. \tag{1.20}$$

Notice that every moment is a Hermitian matrix, i.e., $\mu_n^H = \mu_n$ for $n \geq 0$, since μ is Hermitian.

By a transformation of the Stieltjes function $S(x)$, we mean a new Stieltjes function associated with a measure $\tilde{\mu}$, a spectral transformation of the original measure μ.

Example 1.3.3. *The double factorial of number n (denoted by $n!!$) is the product of all the integers from 1 up to n that have the same parity (odd or even) as n. That is,*

$$n!! = \begin{cases} \prod_{k=1}^{n/2}(2k) = n(n-2)(n-4)\cdots(4)(2), & \text{if } n \text{ is even,} \\ \prod_{k=1}^{(n+1)/2}(2k-1) = n(n-2)(n-4)\cdots(3)(1), & \text{if } n \text{ is odd.} \end{cases} \tag{1.21}$$

On the other hand, since

$$\int_{-1}^{1} \frac{x^n}{\sqrt{1-x^2}} dx = 2\int_0^{\pi/2} \sin^n t dt = \pi \frac{(n-1)!!}{n!!} \frac{1+(-1)^n}{2},$$

thus, the n-th moment matrix associated with the normalized matrix measure (1.14) *is given by*

$$\mu_n = \begin{bmatrix} \frac{(n-1)!!}{n!!}\frac{1+(-1)^n}{2} & \frac{n!!}{(n+1)!!}\frac{1-(-1)^n}{2} \\ \frac{n!!}{(n+1)!!}\frac{1-(-1)^n}{2} & \frac{(n-1)!!}{n!!}\frac{1+(-1)^n}{2} \end{bmatrix},$$

with $(-1)!! = 1$. *Hence the matrix-valued Stieltjes function is given by*

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \begin{bmatrix} \frac{(n-1)!!}{n!!}\frac{1+(-1)^n}{2} & \frac{n!!}{(n+1)!!}\frac{1-(-1)^n}{2} \\ \frac{n!!}{(n+1)!!}\frac{1-(-1)^n}{2} & \frac{(n-1)!!}{n!!}\frac{1+(-1)^n}{2} \end{bmatrix}.$$

1.4 Matrix orthogonal polynomials on the unit circle (MOPUC)

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle and unit disc, respectively, where $z \in \mathbb{T}$ is parametrized as $z = e^{i\theta}$ with $\theta \in (-\pi, \pi]$. In what follows, $\sigma = (\sigma_{i,j})_{i,j=1}^{l}$ will be an $l \times l$ Hermitian matrix measure supported on $\mathbb{T}$ (it can be supported on a finite subset of $\mathbb{T}$). Notice that such a measure can be decomposed as

$$d\sigma(\theta) = \mathcal{W}(\theta)\frac{d\theta}{2\pi} + d\sigma_s(\theta), \tag{1.22}$$

where $\mathcal{W} \in \mathcal{M}_l$ denotes the Radon-Nikodym derivative of the absolutely continuous part of σ with respect to the scalar Lebesgue's measure, and σ_s represents the singular part. If, in addition, the matrix measure σ is Hermitian and positive definite, then the matrix $\mathcal{W}(\theta)$ is a Hermitian and positive definite matrix.

As in the real line case, a matrix measure σ induces right and left sesquilinear forms, defined respectively as

$$\langle p, q\rangle_{R,\sigma} = \int_{\mathbb{T}} p(z)^H \frac{d\sigma(z)}{iz} q(z) = \int_{-\pi}^{\pi} p(e^{i\theta})^H d\sigma(\theta) q(e^{i\theta}) \in \mathcal{M}_l, \quad p, q \in \mathbb{P}^l[z], \tag{1.23}$$

$$\langle p, q\rangle_{L,\sigma} = \int_{\mathbb{T}} p(z) \frac{d\sigma(z)}{iz} q(z)^H = \int_{-\pi}^{\pi} p(e^{i\theta}) d\sigma(\theta) q(e^{i\theta})^H \in \mathcal{M}_l, \quad p, q \in \mathbb{P}^l[z]. \tag{1.24}$$

These sesquilinear forms satisfy the same properties that their real line counterparts. By applying the Gram-Schmidt orthogonalization process to the basis $(z^n\mathbf{I})_{n\geq 0}$, it is possible to construct a sequence of monic matrix polynomials $(\Phi_n^R(z))_{n\geq 0}$ (resp. $(\Phi_n^L(z))_{n\geq 0}$) orthogonal with respect to (1.23) (resp. (1.24)). This is, $\Phi_n^R(z)$ (resp. $\Phi_n^L(z)$) is the unique monic matrix polynomial satisfying

$$\langle z^k\mathbf{I}, \Phi_n^R\rangle_{R,\sigma} = \langle \Phi_n^L, z^k\mathbf{I}\rangle_{L,\sigma} = \mathbf{0}, \quad k = 0, 1, \ldots, n-1. \tag{1.25}$$

The sequences $(\Phi_n^R(z))_{n\geq 0}$ and $(\Phi_n^L(z))_{n\geq 0}$ are called, respectively, the monic right and left-orthogonal matrix polynomials sequences with respect to the matrix measure σ. It is important to notice that, unlike the real line case, on the unit circle there is not a direct relation between the right and left orthogonal polynomials. This is a consequence of the fact that $\langle zf, g\rangle_{R,\sigma} = \langle f, \bar{z}g\rangle_{R,\sigma}$. On the other hand, we denote

$$S_n^R = \langle \Phi_n^R, \Phi_n^R\rangle_{R,\sigma}, \text{ and } S_n^L = \langle \Phi_n^L, \Phi_n^L\rangle_{L,\sigma},$$

with $S_n^R \in \mathcal{M}_l$ and $S_n^L \in \mathcal{M}_l$ non-singular for $n \geq 0$. As in the real line case, we have that $(\Phi_k^R(z))_{k=0}^n$ is a right module basis for $\mathbb{P}_n^l[z]$; indeed, any $\pi \in \mathbb{P}_n^l[z]$ has a unique expansion,

$$\pi(z) = \sum_{k=0}^{n} \Phi_k^R(z)\pi_k^R, \quad \pi_k^R = S_k^{-R}\langle \Phi_k^R, \pi\rangle_{R,\sigma}. \tag{1.26}$$

Similarly, $(\Phi_k^L(z))_{k=0}^n$ is a left module basis for $\mathbb{P}_n^l[z]$, and

$$\pi(z) = \sum_{k=0}^{n} \pi_k^L \Phi_k^L(z), \quad \pi_k^L = \langle \pi, \Phi_k^L\rangle_{L,\sigma} S_k^{-L}.$$

Moreover, if σ is a positive semi-definite matrix measure, there exist sequences of matrix polynomials $(\varphi_n^R(z))_{n\geq 0}$, $(\varphi_n^L(z))_{n\geq 0}$, with

$$\begin{aligned} \varphi_n^R(z) &= \kappa_n^R z^n + \text{ lower order terms}, \\ \varphi_n^L(z) &= \kappa_n^L z^n + \text{ lower order terms}, \end{aligned}$$

for the right and left cases, respectively, with non-singular matrices κ_n^R and κ_n^L. They are orthonormal with respect to the sesquilinear forms (1.23) and (1.24) respectively. This is,

$$\langle \varphi_k^R, \varphi_n^R\rangle_{R,\sigma} = \langle \varphi_n^L, \varphi_k^L\rangle_{L,\sigma} = \delta_{n,k}\mathbf{I}, \ k = 0, 1, \ldots, n.$$

Clearly, any matrix orthonormal polynomial has the form

$$\varphi_n^R(z) = \Phi_n^R(z)\kappa_n^R\nu_n, \quad \varphi_n^L(z) = \tau_n\kappa_n^L\Phi_n^L(z), \tag{1.27}$$

where $\nu_n, \tau_n \in \mathcal{M}_l$ are unitary matrices. This is, as in the real case, matrix orthonormal polynomial are unique up to multiplication by unitary matrices. If we restrict κ_n^R (resp. κ_n^L) to be positive definite, then $\nu_n = \mathbf{I}$ (resp. $\tau_n = \mathbf{I}$) and the orthonormal sequence is uniquely determined (see [5]) with

$$\kappa_n^R = S_n^{-R/2} \text{ and } \kappa_n^L = S_n^{-L/2}. \tag{1.28}$$

Furthermore, κ_n^R (resp. κ_n^L) can always be uniquely chosen to satisfy

$$\left(\kappa_n^R\right)^{-1}\kappa_{n+1}^R > \mathbf{0} \text{ and } \kappa_{n+1}^L\left(\kappa_n^L\right)^{-1} > \mathbf{0}. \tag{1.29}$$

By (1.26), the sequence of polynomials $(\varphi_n^R(z))_{n\geq 0}$ forms a right module basis for $\mathbb{P}^l[z]$; indeed, any $\pi \in \mathbb{P}^l[z]$, has a unique expansion,

$$\pi(z) = \sum_{k=0}^{\infty} \varphi_k^R(z)\pi_k^R, \quad \pi_k^R = \langle \varphi_k^R, \pi\rangle_{R,\sigma}. \tag{1.30}$$

Similarly, the sequence of polynomials $(\varphi_n^L(z))_{n\geq 0}$ is a left module basis for $\mathbb{P}^l[z]$, and the expansion

$$\pi(z) = \sum_{k=0}^{\infty} \pi_k^L \varphi_k^L(z), \quad \pi_k^L = \langle \pi, \varphi_k^L\rangle_{L,\sigma} \tag{1.31}$$

is unique.

If the Hermitian matrix measure is positive definite, we can derive the corresponding scalar products

$$\langle \pi, q\rangle_{\sigma_{esc},R} = Tr\left[\langle \pi, q\rangle_{R,\sigma}\right],\ \langle \pi, q\rangle_{\sigma_{esc},L} = Tr\left[\langle \pi, q\rangle_{L,\sigma}\right],\quad \pi, q \in \mathbb{P}^l[z],$$

and their corresponding Hilbert spaces $\mathcal{H}^R$, $\mathcal{H}^L$, with norms of Frobenius type given by

$$\|\pi\|_R = \sqrt{\langle \pi, \pi\rangle_{\sigma_{esc},R}} \text{ and } \|\pi\|_L = \sqrt{\langle \pi, \pi\rangle_{\sigma_{esc},L}}.$$

Moreover, from (1.30) and (1.31) the Parseval identity

$$\sum_{k=0}^{\infty} Tr(\pi_k^{R,H}\pi_k^R) = \|\pi\|_R^2 \text{ and } \sum_{k=0}^{\infty} Tr(\pi_k^L\pi_k^{L,H}) = \|\pi\|_L^2 \tag{1.32}$$

holds true. Obviously, since π is a polynomial, there are only finitely many non-zero terms in (1.30), (1.31) and (1.32).

1.4.1 The Szegő recursion and further results

Let $(\varphi_n^R(z))_{n\geq 0}$ and $(\varphi_n^L(z))_{n\geq 0}$ be the right and left-orthonormal matrix polynomial sequences, respectively, with respect to the Hermitian matrix measure σ along with the condition (1.29).

For a matrix polynomial $\pi \in \mathbb{P}_n^l[z]$ of degree n, we define the reversed polynomial π^* by

$$\pi^*(z) = z^n \pi\left(\frac{1}{\bar{z}}\right)^H.$$

Moreover, clearly we have for $\pi, q \in \mathbb{P}_n^l[z]$ of degree n and for any $\mathbf{A} \in \mathcal{M}_l$

1. $(\pi^*(z))^* = \pi(z)$.
2. $(\mathbf{A}\pi(z))^* = \pi^*(z)\mathbf{A}^H,\ (\pi(z)\mathbf{A})^* = \mathbf{A}^H\pi^*(z)$.
3. $\langle \pi^*, q^*\rangle_{R,\sigma} = \langle \pi, q\rangle_{L,\sigma},\ \langle \pi^*, q^*\rangle_{L,\sigma} = \langle \pi, q\rangle_{R,\sigma}$.

Now define

$$\rho_n^R := (\kappa_{n+1}^R)^{-1}\kappa_n^R \text{ and } \rho_n^L := \kappa_n^L(\kappa_{n+1}^L)^{-1}. \tag{1.33}$$

Notice that we have $\rho_n^R > \mathbf{0}$ and $\rho_n^L > \mathbf{0}$.

Theorem 1.4.1. (Szegő recursion [42]).

1. *For suitable matrices α_n, one has*

$$z\varphi_n^R(z) = \varphi_{n+1}^R(z)\rho_n^R + \varphi_n^{L,*}(z)\alpha_n^H, \tag{1.34}$$
$$z\varphi_n^L(z) = \rho_n^L\varphi_{n+1}^L(z) + \alpha_n^H\varphi_n^{R,*}(z). \tag{1.35}$$

   These recurrence relations are known in the literature as forward Szegő recurrence relations.

2. $$\rho_n^R = \sqrt{\mathbf{I} - \alpha_n\alpha_n^H} \text{ and } \rho_n^L = \sqrt{\mathbf{I} - \alpha_n^H\alpha_n}. \tag{1.36}$$

The matrices α_n will henceforth be called the Verblunsky matrix coefficients associated with the matrix measure σ. Since ρ_n^R is positive definite and therefore non-singular, we have

$$\|\alpha_n\|_2 < 1, \quad \text{for all } n \geq 0. \tag{1.37}$$

In fact, the following converse result holds.

Theorem 1.4.2. (Verblunsky's Theorem [42]). *Any sequence $(\alpha_n)_{n\geq 0}$ satisfying (1.37) is the sequence of Verblunsky matrix coefficients of a unique positive semi-definite matrix measure supported on $\mathbb{T}$.*

Notice that, if the support of σ contains only finitely many points, say n points, then we can define a truncated sequence $(\Phi_k(z))_{k=0}^n$, and the recurrence relations terminate after that. This situation is closely related to the development of Gaussian quadrature formulas for matrix-valued functions (see [21, 161, 162]).

From (1.34) and (1.35), it can be deduced that

$$\alpha_n = -\left(\kappa_n^R\right)^H \Phi_{n+1}^R(0)^H \left(\kappa_n^L\right)^{-1}, \tag{1.38}$$

$$\alpha_n = -\left(\kappa_n^R\right)^{-1} \Phi_{n+1}^L(0)^H \left(\kappa_n^L\right)^H. \tag{1.39}$$

and from (1.33)

$$\kappa_n^L = (\rho_0^L \cdots \rho_{n-1}^L)^{-1} \text{ and } \kappa_n^R = (\rho_{n-1}^R \cdots \rho_0^R)^{-1}. \tag{1.40}$$

Notice that the Szegő recursion for the monic **MOPUC** reads as

$$\Phi_{n+1}^R(z) = z\Phi_n^R(z) + \Phi_n^{L,*}(z)\Phi_{n+1}^R(0), \tag{1.41}$$

$$\Phi_{n+1}^L(z) = z\Phi_n^L(z) + \Phi_{n+1}^L(0)\Phi_n^{R,*}(z), \tag{1.42}$$

and, as a consequence,

$$\Phi_{n+1}^{R,*}(z) = \Phi_n^{R,*}(z) + z\Phi_{n+1}^{R,H}(0)\Phi_n^L(z), \tag{1.43}$$

$$\Phi_{n+1}^{L,*}(z) = \Phi_n^{L,*}(z) + z\Phi_n^R(z)\Phi_{n+1}^{L,H}(0). \tag{1.44}$$

Proposition 1.4.3. *For $n \geq 1$, we have*

$$\Phi_n^{R,*}(z) = S_n^R \left(\sum_{k=0}^{n} S_k^{-R} \Phi_k^{R,H}(0) \Phi_k^L(z) \right), \tag{1.45}$$

$$\Phi_n^{L,*}(z) = \left(\sum_{k=0}^{n} \Phi_k^R(z) \Phi_k^{L,H}(0) S_k^{-L} \right) S_n^L. \tag{1.46}$$

Proof. (1.41) and (1.44) can be written in matrix form as

$$[\Phi_{n+1}^R(z), \Phi_{n+1}^{L,*}(z)] = [\Phi_n^R(z), \Phi_n^{L,*}(z)] \begin{bmatrix} z\mathbf{I} & z\Phi_{n+1}^{L,H}(0) \\ \Phi_{n+1}^R(0) & \mathbf{I} \end{bmatrix}. \tag{1.47}$$

and, by using (1.1) to determine the inverse of the block matrix, we get

$$\begin{bmatrix} z\mathbf{I} & z\Phi_{n+1}^{L,H}(0) \\ \Phi_{n+1}^{R}(0) & \mathbf{I} \end{bmatrix}^{-1} = \\ \begin{bmatrix} \frac{1}{z}\left(\mathbf{I}-\Phi_{n+1}^{L,H}(0)\Phi_{n+1}^{R}(0)\right)^{-1} & -\Phi_{n+1}^{L,H}(0)\left(\mathbf{I}-\Phi_{n+1}^{R}(0)\Phi_{n+1}^{L,H}(0)\right)^{-1} \\ -\frac{1}{z}\Phi_{n+1}^{R}(0)\left(\mathbf{I}-\Phi_{n+1}^{L,H}(0)\Phi_{n+1}^{R}(0)\right)^{-1} & \left(\mathbf{I}-\Phi_{n+1}^{R}(0)\Phi_{n+1}^{L,H}(0)\right)^{-1} \end{bmatrix}.$$

As a consequence, (1.47) becomes

$$z\Phi_n^R(z)\left(\mathbf{I}-\Phi_{n+1}^{L,H}(0)\Phi_{n+1}^{R}(0)\right) = \Phi_{n+1}^R(z) - \Phi_{n+1}^{L,*}(z)\Phi_{n+1}^R(0),$$
$$\Phi_n^{L,*}(z)\left(\mathbf{I}-\Phi_{n+1}^{R}(0)\Phi_{n+1}^{L,H}(0)\right) = \Phi_{n+1}^{L,*}(z) - \Phi_{n+1}^R(z)\Phi_{n+1}^{L,H}(0).$$

These recurrence relations are known in the literature as backward Szegő recurrence relations. Since $\mathbf{I}-\Phi_{n+1}^{L,H}(0)\Phi_{n+1}^R(0) = S_n^{-R}S_{n+1}^R$ and $\mathbf{I}-\Phi_{n+1}^R(0)\Phi_{n+1}^{L,H}(0) = S_n^{-L}S_{n+1}^L$ (see [141], Proposition 4.1), we have

$$z\Phi_n^R(z)S_n^{-R}S_{n+1}^R = \Phi_{n+1}^R(z) - \Phi_{n+1}^{L,*}(z)\Phi_{n+1}^R(0), \tag{1.48}$$
$$\Phi_n^{L,*}(z)S_n^{-L}S_{n+1}^L = \Phi_{n+1}^{L,*}(z) - \Phi_{n+1}^R(z)\Phi_{n+1}^{L,H}(0). \tag{1.49}$$

On the other hand, from (1.42) and (1.43) and taking into account

$$\Phi_{n+1}^R(0)\left(\mathbf{I}-\Phi_{n+1}^{L,H}(0)\Phi_{n+1}^R(0)\right)^{-1} = \left(\mathbf{I}-\Phi_{n+1}^R(0)\Phi_{n+1}^{L,H}(0)\right)^{-1}\Phi_{n+1}^R(0),$$

by using a similar procedure we get

$$zS_{n+1}^L S_n^{-L}\Phi_n^L(z) = \Phi_{n+1}^L(z) - \Phi_{n+1}^L(0)\Phi_{n+1}^{R,*}(z), \tag{1.50}$$
$$S_{n+1}^R S_n^{-R}\Phi_n^{R,*}(z) = \Phi_{n+1}^{R,*}(z) - \Phi_{n+1}^{R,H}(0)\Phi_{n+1}^L(z). \tag{1.51}$$

From (1.51), recursively, we obtain $S_n^{-R}\Phi_n^{R,*}(z) = S_0^{-R}\Phi_0^{R,*}(z) + \sum_{k=1}^n S_k^{-R}\Phi_k^{R,H}(0)\Phi_k^L(z)$, and since $\Phi_0^{R,*}(z) = \Phi_0^{R,H}(0)\Phi_0^L(0)$, so (1.45) follows. Similarly, from (1.49) we obtain (1.46) □

Proposition 1.4.4. *Let* $\mathbf{A},\mathbf{B}\in\mathcal{M}_l$.

If $\Phi_n^R(z)\mathbf{A} = \Phi_n^{L,*}(z)\mathbf{B}$ *for some* $n\geq 1$, *then* $\mathbf{A}=\mathbf{B}=\mathbf{0}$.

If $\mathbf{A}\Phi_n^L(z) = \mathbf{B}\Phi_n^{R,*}(z)$ *for some* $n\geq 1$, *then* $\mathbf{A}=\mathbf{B}=\mathbf{0}$.

Proof. Assume $\Phi_n^R(z)\mathbf{A} = \Phi_n^{L,*}(z)\mathbf{B}$. Then, $\mathbf{A}^H\Phi_n^{R,*}(z) = \mathbf{B}^H\Phi_n^L(z)$. Setting $z=0$, we have $\Phi_n^R(0)\mathbf{A}=\mathbf{B}$ and $\mathbf{A}^H = \mathbf{B}^H\Phi_n^L(0)$, since $\Phi_n^{R,*}(0)=\mathbf{I}$. As a consequence, we get

$$\mathbf{0} = \left(\mathbf{I}-\Phi_n^R(0)\Phi_n^{L,H}(0)\right)\mathbf{B}. \tag{1.52}$$

Using (1.38), (1.52) becomes

$$\mathbf{0} = \left(\mathbf{I}-(\kappa_{n-1}^L)^H\alpha_{n-1}^H\alpha_{n-1}(\kappa_{n-1}^{-L})^H\right)\mathbf{B} = (\kappa_{n-1}^L)^H\left(\mathbf{I}-\alpha_{n-1}^H\alpha_{n-1}\right)(\kappa_{n-1}^{-L})^H\mathbf{B},$$

and since κ_{n-1}^L and $\mathbf{I} - \alpha_{n-1}^H \alpha_{n-1}$ are non-singular for $n \geq 1$, we get $\mathbf{B} = \mathbf{0}$ and $\mathbf{A} = \mathbf{0}$. The other statement can be proved in a similar way. □

1.4.2 Zeros

A zero of a matrix polynomial $\pi \in \mathbb{P}^l[z]$ is a $z_0 \in \mathbb{C}$ such that $\det \pi(z_0) = 0$. In [42], it is shown that:

1. All the zeros of $\varphi_n^R(z)$ and $\varphi_n^L(z)$ lie in $\mathbb{D}$.
2. All the zeros of $\varphi_n^{R,*}(z)$ and $\varphi_n^{L,*}(z)$ lie in $\mathbb{C} \setminus \bar{\mathbb{D}}$.
3. For each $n \geq 0$,
$$\det\left(\varphi_n^R(z)\right) = \det\left(\varphi_n^L(z)\right).$$
4. For each $n \geq 0$,
$$\det\left(\kappa_n^R\right) = \det\left(\kappa_n^L\right).$$

As a consequence, it follows that for $n \geq 0$, both $(\varphi_n^R(z))^{-1}$ and $(\varphi_n^L(z))^{-1}$ exist for $|z| \geq 1$ and both $(\varphi_n^{R,*}(z))^{-1}$ and $(\varphi_n^{L,*}(z))^{-1}$ exist for $|z| < 1$. Moreover, since $\det\left(\kappa_n^R\right) = \det\left(\kappa_n^L\right)$ for $n \geq 1$, then
$$\det\left(\Phi_n^R(z)\right) = \det\left(\Phi_n^L(z)\right), \quad n \geq 0. \tag{1.53}$$

Example 1.4.5. *Consider the* 2×2 *Hermitian matrix-valued measure supported on* $\mathbb{T}$
$$d\sigma(\theta) = \mathcal{W}(\theta)\frac{d\theta}{2\pi} = \begin{bmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{bmatrix} \frac{d\theta}{2\pi}. \tag{1.54}$$

As a consequence, the $n-$*th degree monic* **MOPUC***, using the matrix Heine's formula* [141] *is*
$$\Phi_n^R(z) = \begin{bmatrix} \phi_n(z) & -\frac{n}{n+1} i \phi_{n-1}(z) \\ -\frac{n}{n+1} i \phi_{n-1}(z) & \phi_n(z) \end{bmatrix}, \ n \geq 0$$
where $\phi_{-1}(z) = 0$, $\phi_0(z) = 1$ *and for* $n \geq 1$
$$\phi_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} \left(\frac{1 + (-1)^{n-k}}{2} \right) (-1)^{\frac{n-k}{2}} (k+1) z^k.$$

Moreover
$$S_n^R = \frac{n+2}{2(n+1)} \mathbf{I}_2, \ n \geq 0.$$

k	$\det(\Phi_k^R(z))$	Zeros
$1-*$	$z^2+\frac{1}{4}$	$\pm 0.5i$
$2-*$	$z^4-\frac{2}{9}z^2+\frac{1}{9}$	$\pm 0.4714 \pm 0.33333i$
$3-*$	$z^6-\frac{7}{16}z^4-\frac{1}{8}z^2+\frac{1}{16}$	$\pm 0.63833 \pm 0.072085i$ $\pm 0.60583i$
$4-*$	$z^8-\frac{14}{25}z^6+\frac{3}{25}z^4-\frac{2}{25}z^2+\frac{1}{25}$	$\pm 0.67815 \pm 0.13783i$ $\pm 0.35828 \pm 0.53783i$
$5-*$	$z^{10}-\frac{23}{36}z^8+\frac{5}{18}z^6+\frac{1}{12}z^4-\frac{1}{18}z^2+\frac{1}{36}$	$\pm 0.66837 \pm 0.29419i$ $\pm 0.57018 \pm 0.37570i$ $\pm 0.67033i$
$6-*$	$z^{12}-\frac{34}{49}z^{10}+\frac{19}{49}z^8-\frac{4}{49}z^6+\frac{3}{49}z^4-\frac{2}{49}z^2+\frac{1}{49}$	$\pm 0.28765 \pm 0.63411i$ $\pm 0.63989 \pm 0.41068i$ $\pm 0.68380 \pm 0.20514i$

TABLE 1.1. Zeros of the first six **MOPUC**.

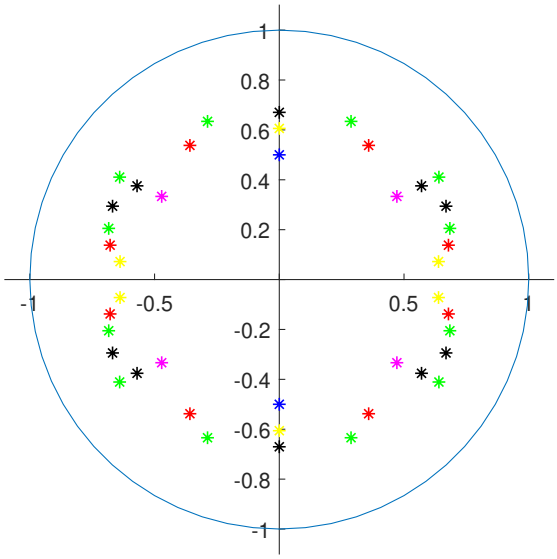

FIGURE 1.1. Zeros of the first six **MOPUC**.

Notice that all the zeros of $\Phi_k^R(z)$ *with* $k=1,\dots,6$ *lie in* $\mathbb{D}$. *In the same way, the reversed matrix polynomial of* $\Phi_n^R(z)$ *is*

$$\Phi_n^{R,*}(z)=\begin{bmatrix} \phi_n^{R,*}(z) & \frac{n}{n+1}iz\phi_{n-1}^*(z) \\ \frac{n}{n+1}iz\phi_{n-1}^*(z) & \phi_n^*(z) \end{bmatrix},\ n\geq 0,$$

where $\phi_n^*(z)=\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1+(-1)^k}{2}\right)(-)^{\frac{k}{2}}(n+1-k)z^k$, $\phi_0^*(z)=1$ *and* $\phi_{-1}^*(z)=0$.

k	$\det(\Phi_k^{R,*}(z))$	Zeros
1 − ∗	$1+\frac{1}{4}z^2$	$\pm 2.0i$
2 − ∗	$1-\frac{2}{9}z^2+\frac{1}{9}z^4$	$\pm 1.4142 \pm 1.0i$
3 − ∗	$1-\frac{7}{16}z^2-\frac{1}{8}z^4+\frac{1}{16}z^6$	$\pm 1.5469 \pm 0.17469i$ $\pm 1.6506i$
4 − ∗	$1-\frac{14}{25}z^2+\frac{3}{25}z^4-\frac{2}{25}z^6+\frac{1}{25}z^8$	$\pm 1.4161 \pm 0.28782i$ $\pm 0.8579 \pm 1.2878i$
5 − ∗	$1-\frac{23}{36}z^2+\frac{5}{18}z^4+\frac{1}{12}z^6-\frac{1}{18}z^8+\frac{1}{36}z^{10}$	$\pm 1.2533 \pm 0.55169i$ $\pm 1.2229 \pm 0.80579i$ $\pm 1.4918i$
6 − ∗	$1-\frac{34}{49}z^2+\frac{19}{49}z^4-\frac{4}{49}z^6+\frac{3}{49}z^8-\frac{2}{49}z^{10}+\frac{1}{49}z^{12}$	$\pm 0.59329 \pm 1.3079i$ $\pm 1.1068 \pm 0.71038i$ $\pm 1.3417 \pm 0.40251i$

TABLE 1.2. Zeros of the first six reversed **MOPUC**.

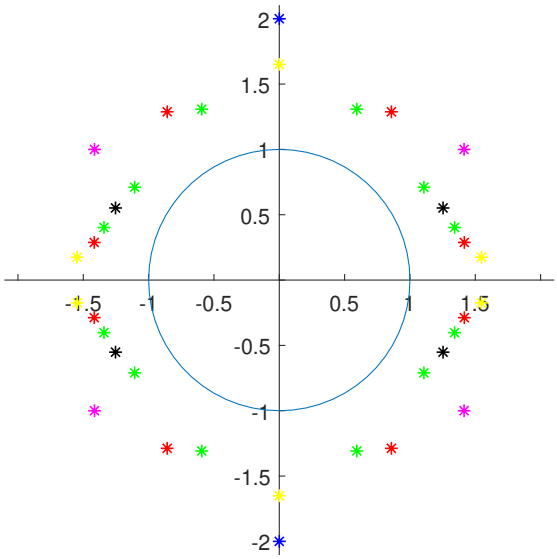

FIGURE 1.2. Zeros of the first six reversed **MOPUC**.

Notice that all the zeros of $\Phi_k^{R,*}(z)$ *with* $k=1,\ldots,6$ *lie in* $\mathbb{C}\setminus\bar{\mathbb{D}}$.

1.4.3 Kernel matrix polynomial and the Christoffel-Darboux formula

The kernel matrix polynomial of degree n for $z, \zeta \in \mathbb{C}$ is defined as

$$K_n^R(\zeta, z) = \sum_{k=0}^{n} \varphi_k^R(z)\varphi_k^R(\zeta)^H \text{ and } K_n^L(z, \zeta) = \sum_{k=0}^{n} \varphi_k^L(\zeta)^H \varphi_k^L(z), \tag{1.55}$$

for the right and left cases, respectively. It is shown in [42] that for $n \geq 1$ they satisfy the Christoffel-Darboux formulas

$$K_n^R(\zeta, z) = \frac{\varphi_{n+1}^{L,*}(z)\varphi_{n+1}^{L,*}(\zeta)^H - \varphi_{n+1}^R(z)\varphi_{n+1}^R(\zeta)^H}{1 - z\bar{\zeta}}, \tag{1.56}$$

$$K_n^L(z, \zeta) = \frac{\varphi_{n+1}^{R,*}(\zeta)^H\varphi_{n+1}^{R,*}(z) - \varphi_{n+1}^L(\zeta)^H\varphi_{n+1}^L(z)}{1 - z\bar{\zeta}}, \tag{1.57}$$

$$K_n^R(\zeta, z) = \frac{\varphi_n^{L,*}(z)\varphi_n^{L,*}(\zeta)^H - z\bar{\zeta}\varphi_n^R(z)\varphi_n^R(\zeta)^H}{1 - z\bar{\zeta}}, \tag{1.58}$$

$$K_n^L(z, \zeta) = \frac{\varphi_n^{R,*}(\zeta)^H\varphi_n^{R,*}(z) - z\bar{\zeta}\varphi_n^L(\zeta)^H\varphi_n^L(z)}{1 - z\bar{\zeta}}, \tag{1.59}$$

for $z\bar{\zeta} \neq 1$.

Proposition 1.4.6. *For $n \geq 1$, we have,*

1.
$$\varphi_n^{R,*}(z) = (\kappa_n^R)^{-1}K_n^L(z, 0) \text{ and } \varphi_n^{L,*}(z) = K_n^R(0, z)(\kappa_n^L)^{-1}. \tag{1.60}$$

2.
$$K_n^L(0, 0) = \kappa_n^R(\kappa_n^R)^H = S_n^{-R} \text{ and } K_n^R(0, 0) = (\kappa_n^L)^H\kappa_n^L = S_n^{-L}. \tag{1.61}$$

3.
$$(\kappa_n^{-L})^H\varphi_n^R(0) = \varphi_n^L(0)(\kappa_n^{-R})^H \text{ and } \varphi_n^{R,H}(0)\kappa_n^{-L} = \kappa_n^{-R}\varphi_n^{L,H}(0). \tag{1.62}$$

4.
$$\begin{aligned}\left(\kappa_{n-1}^L\kappa_n^{-L}\right)^H \kappa_{n-1}^L\kappa_n^{-L} &= \mathbf{I} - \varphi_n^L(0)(\kappa_n^{-R})^H\kappa_n^{-R}\varphi_n^{L,H}(0) \\ &= \mathbf{I} - \varphi_n^L(0)K_n^L(0, 0)^{-1}\varphi_n^{L,H}(0),\end{aligned} \tag{1.63}$$

and

$$\begin{aligned}\kappa_n^{-R}\kappa_{n-1}^R\left(\kappa_n^{-R}\kappa_{n-1}^R\right)^H &= \mathbf{I} - \varphi_n^{R,H}(0)\kappa_n^{-L}(\kappa_n^{-L})^H\varphi_n^R(0) \\ &= \mathbf{I} - \varphi_n^{R,H}(0)K_n^R(0, 0)^{-1}\varphi_n^R(0).\end{aligned} \tag{1.64}$$

5. *For any matrix polynomial $\pi \in \mathbb{P}_n^l[z]$ and $\zeta \in \mathbb{C}$, the kernel polynomials have the reproducing properties* ([42])
$$\langle \pi(z), K_n^R(\zeta, z)\rangle_{R,\sigma} = \pi(\zeta)^H, \text{ and } \langle K_n^L(z, \zeta), \pi(z)\rangle_{L,\sigma} = \pi(\zeta)^H. \tag{1.65}$$

Proof.

1. Taking into account that

$$\varphi_n^{R,*}(0)^H = \Phi_n^{R,*}(0)^H \kappa_n^R = \kappa_n^R \text{ and } \varphi_n^{L,*}(0)^H = \kappa_n^L \Phi_n^{L,*}(0)^H = \kappa_n^L, \tag{1.66}$$

from (1.58) , (1.59) we get (1.60).

2. The equality $K_n^L(0,0) = \kappa_n^R(\kappa_n^R)^H$ follows setting $z = 0$ in (1.60). On the other hand,

$$S_n^R = \langle \Phi_n^R, \Phi_n^R \rangle_{R,\sigma} = \langle \varphi_n^R \kappa_n^{-R}, \varphi_n^R \kappa_n^{-R} \rangle_{R,\sigma} = (\kappa_n^{-R})^H \kappa_n^{-R}.$$

3. Using (1.34) and (1.35) with the identity $\alpha_n \rho_n^L = \rho_n^R \alpha_n$ (see [42] Lemma 1.3.), one can verify

$$\varphi_n^{L,*}(z)\varphi_n^L(\zeta) - \varphi_n^R(z)\varphi_n^{R,*}(\zeta) = \zeta \varphi_{n-1}^{L,*}(z)\varphi_{n-1}^L(\zeta) - z\varphi_{n-1}^R(z)\varphi_{n-1}^{R,*}(\zeta),$$

and taking $z = \zeta$,

$$\begin{aligned} \varphi_n^{L,*}(z)\varphi_n^L(z) - \varphi_n^R(z)\varphi_n^{R,*}(z) &= z\left(\varphi_{n-1}^{L,*}(z)\varphi_{n-1}^L(z) - \varphi_{n-1}^R(z)\varphi_{n-1}^{R,*}(z)\right) \\ &\vdots \\ &= z^n\left(\varphi_0^{L,*}(z)\varphi_0^L(z) - \varphi_0^R(z)\varphi_0^{R,*}(z)\right), \end{aligned}$$

so that $\varphi_n^{L,*}(z)\varphi_n^L(z) = \varphi_n^R(z)\varphi_n^{R,*}(z)$. From the previous equation and (1.66) we get (1.62).

4. From (1.61), we have

$$(\kappa_n^L)^H \kappa_n^L - (\kappa_{n-1}^L)^H \kappa_{n-1}^L = K_n^R(0,0) - K_{n-1}^R(0,0) = \varphi_n^R(0)\varphi_n^{R,H}(0),$$

and therefore

$$\left(\kappa_{n-1}^L \kappa_n^{-L}\right)^H \kappa_{n-1}^L \kappa_n^{-L} = \mathbf{I} - (\kappa_n^{-L})^H \varphi_n^R(0)\varphi_n^{R,H}(0)\kappa_n^{-L}.$$

Then, (1.61) and (1.62) imply (1.63). (1.64) follows in a similar way. □

Notice that from (1.63) and (1.64) we deduce

$$\|\kappa_{n-1}^L \kappa_n^{-L}\|_2 \leq 1 \text{ and } \|\kappa_n^{-R} \kappa_{n-1}^R\|_2 \leq 1,$$

since $K_n^L(0,0)^{-1}$ and $K_n^R(0,0)^{-1}$ are positive semi-definite matrices.

Example 1.4.7. *Consider the matrix measure*

$$d\sigma(\theta) = \Omega \frac{d\theta}{2\pi}, \tag{1.67}$$

where $\Omega \in \mathcal{M}_l$ is a positive definite matrix. The corresponding $n-$th degree monic **MOPUC** *is*

$$\Phi_n(z) = \Phi_n^R(z) = \Phi_n^L(z) = z^n \mathbf{I},$$

Moreover

$$S_n^R = S_n^L = \langle \Phi_n, \Phi_n \rangle_{R,\sigma} = \int_{-\pi}^{\pi} e^{-in\theta} \Omega e^{in\theta} \frac{d\theta}{2\pi} = \Omega > \mathbf{0}.$$

The respective $n-$th reversed polynomial is

$$\Phi_n^*(z) = \Phi_n^{R,*}(z) = \Phi_n^{L,*}(z) = \mathbf{I}.$$

Thus, $\varphi_n^{L,*}(z) = \Phi_n^{L,*}(z) S_n^{-L/2} = \Omega^{-1/2}$, hence the kernel matrix polynomial is

$$K_n^R(\zeta, z) = \frac{\Omega^{-1} - z^{n+1}\Omega^{-1/2}(\zeta^{n+1}\Omega^{-1/2})^H}{1 - z\bar{\zeta}} = \Omega^{-1}\frac{1 - (z\bar{\zeta})^{n+1}}{1 - (z\bar{\zeta})} = \Omega^{-1}\sum_{k=0}^{n}(z\bar{\zeta})^k.$$

In particular, if $z = \zeta \in \mathbb{T}$

$$K_n^R(\zeta, \zeta) = (n+1)\Omega^{-1}.$$

Let $\pi(z) = z^k B_k + \cdots + zB_1 + B_0$ be a matrix polynomial of degree $0 \leq k \leq n$. For $\mathbf{A} \in \mathcal{M}_l$, we define

$$\pi(\mathbf{A}) = \mathbf{A}^k B_k + \cdots + \mathbf{A}B_1 + B_0, \text{ (in particular } \varphi_k^R(\mathbf{A}) = \mathbf{A}^k \kappa_k^R + \cdots)$$

when we use right matrix polynomials or the right sesquilinear form, and

$$\pi(\mathbf{A}) = B_k \mathbf{A}^k + \cdots + B_1\mathbf{A} + B_0, \text{ (in particular } \varphi_k^L(\mathbf{A}) = \kappa_k^L \mathbf{A}^k + \cdots)$$

for left matrix polynomials or the left sesquilinear form. With this notation, the reproducing property of the kernel matrix polynomial (1.65) can be extended to

$$\langle \pi(z), K_n^R(\mathbf{A}, z)\rangle_{R,\sigma} = \pi(\mathbf{A})^H \text{ and } \langle K_n^L(z, \mathbf{A}), \pi(z)\rangle_{L,\sigma} = \pi(\mathbf{A})^H. \tag{1.68}$$

Indeed, let $\pi(z) = \sum_{i=0}^k \varphi_i^R(z)\lambda_i$, with $0 \leq k \leq n$. Then, we have

$$\begin{aligned}
\langle \pi(z), K_n^R(\mathbf{A}, z)\rangle_{R,\sigma} &= \left\langle \sum_{i=0}^{k} \varphi_i^R(z)\lambda_i, \sum_{j=0}^{n} \varphi_j^R(z)\varphi_m^R(\mathbf{A})^H \right\rangle_{R,\sigma} \\
&= \sum_{i=0}^{k}\sum_{j=0}^{n} \lambda_i^H \langle \varphi_i^R, \varphi_j^R\rangle_{R,\sigma}\varphi_j^R(\mathbf{A})^H \\
&= \sum_{i=0}^{k} \lambda_i^H \varphi_i^R(\mathbf{A})^H = \pi(\mathbf{A})^H.
\end{aligned}$$

1.4.4 The matrix-valued Carathéodory function

An analytic matrix-valued function F defined on $\mathbb{D}$ is called a (matrix-valued) Carathéodory function if

$$F(0) = \mathbf{I} \text{ and } \mathfrak{Re}F(z) = \frac{1}{2}\left(F(z) + F(z)^H\right) \geq \mathbf{0} \text{ for all } z \in \mathbb{D}.$$

The following result can be found in [52] Theorem 2.2.2.

Theorem 1.4.8. (Riesz-Herglotz's Theorem). *If F is a matrix-valued Carathéodory function, then there exists a unique normalized positive semi-definite matrix measure σ such*

that

$$F(z) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta). \tag{1.69}$$

The matrix measure σ is given by the unique weak limit of the measures $d\sigma_r(\theta) = \mathfrak{Re} F(re^{i\theta}) \frac{d\theta}{2\pi}$ as $r \uparrow 1$. Moreover, we have

$$F(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n = \mathbf{I} + 2\sum_{n=1}^{\infty} c_{-n} z^n, \tag{1.70}$$

where c_n is the $n-th$ matrix moment associated with the measure σ, i.e.,

$$c_n = \langle \mathbf{I}, z^n \mathbf{I} \rangle_{R,\sigma} = \langle z^n \mathbf{I}, \mathbf{I} \rangle_{L,\sigma} = \int_{-\pi}^{\pi} e^{in\theta} d\sigma(\theta), \quad n \in \mathbb{Z}. \tag{1.71}$$

Conversely, if σ is a positive semi-definite matrix measure, then (1.69) defines a matrix-valued Carathéodory function.

Notice that every matrix moment satisfies $c_{-n}^H = c_n$ for $n \in \mathbb{Z}$, since σ is a Hermitian matrix measure.

Example 1.4.9. *Consider the positive semi-definite matrix measure* (1.54) *supported on* $\mathbb{T}$. *The sequence of matrix moments is*

$$c_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; c_1 = \frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; c_{-1} = -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$c_n = c_{-n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } n \geq 2.$$

Therefore, the respective matrix-valued Carathéodory function is

$$F(z) = \begin{bmatrix} 1 & -iz \\ -iz & 1 \end{bmatrix}.$$

The following result is a direct consequence of the matrix version of the Riesz-Herglotz's Theorem.

Proposition 1.4.10. *Let σ and $\tilde{\sigma}$ be normalized positive semi-definite matrix measures supported on $\mathbb{T}$, with associated matrix-valued Carathéordory functions F and $\tilde{F}$, respectively. If $F(z) = \tilde{F}(z)$ for $z \in \mathbb{D}$, then $\sigma = \tilde{\sigma}$.*

Proof. Let

$$F(z) = \mathbf{I} + 2\sum_{n=1}^{\infty} c_{-n} z^n \text{ and } \tilde{F}(z) = \mathbf{I} + 2\sum_{n=1}^{\infty} \tilde{c}_{-n} z^n,$$

be matrix-valued Carathéodory functions. Then, from the Riesz-Herglotz's Theorem, they have unique, positive semi-definite, associated matrix measures σ and $\tilde{\sigma}$. If $F(z) = \tilde{F}(z)$, and since the corresponding matrix measures are Hermitian, we get for every $n \in \mathbb{Z}$

$$c_n = \tilde{c}_n.$$

This means that for every $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{in\theta} d\sigma = \int_{-\pi}^{\pi} e^{in\theta} d\tilde{\sigma},$$

and, as a consequence, for arbitrary matrix polynomials $p, q \in \mathbb{P}^l[z]$,

$$\int_{-\pi}^{\pi} p(e^{i\theta}) d\sigma(\theta) q(e^{i\theta}) = \int_{-\pi}^{\pi} p(e^{i\theta}) d\tilde{\sigma}(\theta) q(e^{i\theta})$$

and therefore σ and $\tilde{\sigma}$ coincide. □

As in the real line case, by a transformation of the function F we mean a new Carathéodory function associated with a measure $\tilde{\sigma}$, which is a spectral transformation of the original measure σ.

1.4.5 Biorthogonal matrix polynomials

Let $\mathcal{X}(z) = [\mathbf{I}, \mathbf{I}z^{-1}, \mathbf{I}z, \mathbf{I}z^{-2}, \mathbf{I}z^2, \ldots]^T$ be a block vector, and denote the CMV left and right moment matrices associated with the matrix measure σ by

$$\mathbf{F}^R := \langle \mathcal{X}(z)^T, \mathcal{X}(z)^T \rangle_{R,\sigma} = \begin{bmatrix} c_0 & c_{-1} & c_1 & c_{-2} & \cdots \\ c_1 & c_0 & c_2 & c_{-1} & \cdots \\ c_{-1} & c_{-2} & c_0 & c_{-3} & \cdots \\ c_2 & c_1 & c_3 & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.72}$$

$$\mathbf{F}^L := \langle \mathcal{X}(z), \mathcal{X}(z) \rangle_{L,\sigma} = \begin{bmatrix} c_0 & c_1 & c_{-1} & c_2 & \cdots \\ c_{-1} & c_0 & c_{-2} & c_1 & \cdots \\ c_1 & c_2 & c_0 & c_3 & \cdots \\ c_{-2} & c_{-1} & c_{-3} & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.73}$$

respectively. Notice that if the matrix measure σ is Hermitian, then so are $\mathbf{F}^R$ and $\mathbf{F}^L$. Let $(\mathbf{F}^K)_n \in \mathcal{M}_{nl}$ be the truncated moment matrices with $K = R, L$. The matrix measure σ is said to be *quasi-definite* if $(\mathbf{F}^K)_n$ satisfies

$$\det(\mathbf{F}^K)_n \neq 0 \quad \text{for } K = R, L \text{ and } n \geq 1.$$

Definition 1.4.11. Given a quasi-definite matrix measure σ, the set of monic matrix polynomials $(\Phi^R_{i,n}(z))_{n\geq 0}$, $(\Phi^L_{i,n}(z))_{n\geq 0}$, $i = 1, 2$ are said to be biorthogonal with respect to σ if

1. for all $n \geq 0$, $\deg(\Phi^R_{i,n}) = \deg(\Phi^L_{i,n}) = n$ with $i = 1, 2$,
2. $\langle \Phi^R_{2,k}, \Phi^R_{1,n} \rangle_{R,\sigma} = \delta_{k,n} S^R_{2,1,n}$ and $\langle \Phi^L_{1,k}, \Phi^L_{2,n} \rangle_{L,\sigma} = \delta_{k,n} S^L_{1,2,n}$, with $S^R_{2,1,n} \neq \mathbf{0}$ and $S^L_{1,2,n} \neq \mathbf{0}$.

Proposition 1.4.12. ([6] Proposition 2) *Given a quasi-definite matrix measure σ, there exist sequences of monic biorthogonal matrix matrix polynomials $(\Phi^R_{i,n}(z))_{n\geq 0}$, $(\Phi^L_{i,n}(z))_{n\geq 0}$, $i = 1, 2$ and they are unique.*

Here, it is important to notice the order of the matrix polynomials in the sesquilinear form; if $k \neq n$, then $\langle \Phi_{1,k}^R, \Phi_{2,n}^R \rangle_{R,\sigma}$ and $\langle \Phi_{1,k}^L, \Phi_{2,n}^L \rangle_{L,\sigma}$ could be different from $\mathbf{0}$. Notice that this constitutes a more general framework to study matrix orthogonal polynomials. However, if σ is Hermitian,

$$\langle p, q \rangle_{R,\sigma} = \langle q, p \rangle_{R,\sigma}^H \text{ and } \langle p, q \rangle_{L,\sigma} = \langle q, p \rangle_{L,\sigma}^H, \ p, q \in \mathbb{P}^l[z].$$

Therefore, we conclude that the sequences $(\Phi_{i,n}^R(z))_{n\geq 0}$, and $(\Phi_{i,n}^L(z))_{n\geq 0}$, are orthogonal with respect to σ, i.e.

$$\langle \Phi_{i,k}^R, \Phi_{i,n}^R \rangle_{R,\sigma} = \delta_{k,n} S_{i,n}^R \text{ and } \langle \Phi_{i,n}^L, \Phi_{i,k}^L \rangle_{L,\sigma} = \delta_{k,n} S_{i,n}^L.$$

From the uniqueness of $\Phi_n^R(z)$ and $\Phi_n^L(z)$, we obtain that $\Phi_n^R(z) = \Phi_{1,n}^R(z) = \Phi_{2,n}^R(z)$ and $\Phi_n^L(z) = \Phi_{1,n}^L(z) = \Phi_{2,n}^L(z)$. In this document, we deal always with Hermitian matrix measures and, as a consequence, the more general case of biorthogonal sequences of matrix polynomials is not considered here.

1.4.6 Asymptotic properties

Let σ be a Hermitian matrix measure supported on $\mathbb{T}$ defined as in (1.22).

Definition 1.4.13. The Hermitian matrix measure σ belongs to the Nevai matrix class $\mathcal{N}$, if

$$\det \mathcal{W}(\theta) > 0$$

almost everywhere on unit circle.

Definition 1.4.14. The Hermitian matrix measure σ belongs to the Szegő matrix class $\mathcal{S}$, if

$$\log(\det \mathcal{W}(\theta))$$

is integrable over unit circle.

Necessary conditions for σ to belong to the Nevai or Szegő class can be given in terms of the Verblunsky matrix coefficients, as follows.

Theorem 1.4.15. (Rakhmanov's Theorem [168]). *If the Hermitian matrix measure $\sigma \in \mathcal{N}$, then*

$$\lim_{n\to\infty} \alpha_n = \mathbf{0}.$$

Theorem 1.4.16. [48]. *The Hermitian matrix measure $\sigma \in \mathcal{S}$ if and only if*

$$\sum_{n=1}^{\infty} \|\alpha_n\|_F < \infty.$$

Notice that $\mathcal{S} \subset \mathcal{N}$. For the scalar case, the Nevai and Szegő conditions were introduced in [166]. In the matrix case, the Szegő matrix condition was introduced in [48] and some well known results in the scalar case, such as the Bernstein-Szegő Theorem, were extended to the matrix case in [49].

When $z \in \mathbb{T}$ and $z = \zeta$ the Christoffel-Darboux formula (1.57) becomes

$$\varphi_n^L(z)^H \varphi_n^L(z) = \varphi_n^{R,*}(z)^H \varphi_n^{R,*}(z)$$

and rearranging the terms $\left(\varphi_n^L(z)\varphi_n^{R,*}(z)^{-1}\right)^H \left(\varphi_n^L(z)\varphi_n^{R,*}(z)^{-1}\right) = \mathbf{I}$, so that $\varphi_n^L(z)\varphi_n^{R,*}(z)^{-1}$ is a unitary matrix. For $z \in \bar{\mathbb{D}}$, let $\Delta(z) = \varphi_n^L(z)\varphi_n^{R,*}(z)^{-1}$. Then Δ is analytic in $\mathbb{D}$, continuous in $\bar{\mathbb{D}}$ and $\|\Delta(z)\|_2 = 1$ on $\mathbb{T}$ so by the maximum principle

$$\|\varphi_n^L(z)\varphi_n^{R,*}(z)^{-1}\|_2 \leq 1 \text{ for } |z| \leq 1,$$

equivalently

$$\left\|\varphi_n^R(z)^{-1}\varphi_n^{L,*}(z)\right\|_2 \leq 1 \text{ for } |z| \geq 1. \tag{1.74}$$

In the scalar case, it is very well known (see [150]) that if $\alpha_n \to 0$, then

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z, \quad |z| > 1, \text{ with } \varphi_n = \varphi_n^R = \varphi_n^L.$$

In [173], the authors extended the result for **MOPUC** associated with a positive definite matrix measure in the Szegő class. The following proposition generalizes the ratio asymptotics for Hermitian matrix measures in the Nevai class.

Proposition 1.4.17. *If $\sigma \in \mathcal{N}$, then*

$$\lim_{n\to\infty} \left(\varphi_n^R(z)\right)^{-1} \varphi_{n+1}^R(z) = \lim_{n\to\infty} \varphi_{n+1}^L(z) \left(\varphi_n^L(z)\right)^{-1} = z\mathbf{I}, \tag{1.75}$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$. Equivalently,

$$\lim_{n\to\infty} \varphi_{n+1}^{R,*}(z) \left(\varphi_n^{R,*}(z)\right)^{-1} = \lim_{n\to\infty} \left(\varphi_n^{L,*}(z)\right)^{-1} \varphi_{n+1}^{L,*}(z) = \mathbf{I}, \tag{1.76}$$

uniformly on every compact subset of $z \in \mathbb{D}$.

Proof. From (1.34), $\left(\varphi_n^R(z)\right)^{-1} \varphi_{n+1}^R(z)\rho_n^R - z\mathbf{I} = -\left(\varphi_n^R(z)\right)^{-1} \varphi_n^{L,*}(z)\alpha_n^H$ and thus

$$\left\|\left(\varphi_n^R(z)\right)^{-1} \varphi_{n+1}^R(z)\rho_n^R - z\mathbf{I}\right\|_2 \leq \|\alpha_n\|_2 \left\|\left(\varphi_n^R(z)\right)^{-1} \varphi_n^{L,*}(z)\right\|_2. \tag{1.77}$$

From (1.74), we have that (1.77) becomes

$$\left\|\left(\varphi_n^R(z)\right)^{-1} \varphi_{n+1}^R(z)\rho_n^R - z\mathbf{I}\right\|_2 \leq \|\alpha_n\|_2, \text{ for } |z| > 1, \tag{1.78}$$

and since $\alpha_n \to \mathbf{0}$ implies $\rho_n^R \to \mathbf{I}$, (1.78) implies that (1.75) holds if $\sigma \in \mathcal{N}$. On the other hand, (1.76) follows from the definition of the reversed polynomials. □

Proposition 1.4.18. *If $\sigma \in \mathcal{N}$, then*

$$\lim_{n\to\infty} \varphi_n^L(z) \left(\varphi_n^{R,*}(z)\right)^{-1} = \lim_{n\to\infty} \left(\varphi_n^{L,*}(z)\right)^{-1} \varphi_n^R(z) = \mathbf{0}, \tag{1.79}$$

uniformly on every compact subset of $z \in \mathbb{D}$. Equivalently,

$$\lim_{n\to\infty} \left(\varphi_n^R(z)\right)^{-1} \varphi_n^{L,*}(z) = \lim_{n\to\infty} \varphi_n^{R,*}(z) \left(\varphi_n^L(z)\right)^{-1} = \mathbf{0}, \tag{1.80}$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$.

Proof. For $|z|<1$, from (1.34)

$$\left(\varphi_n^{L,*}(z)\right)^{-1}\varphi_{n+1}^{L,*}(z)\left(\varphi_{n+1}^{L,*}(z)\right)^{-1}\varphi_{n+1}^{R}(z)\rho_n^R = z\left(\varphi_n^{L,*}(z)\right)^{-1}\varphi_n^R(z)-\alpha_n^H.$$

If $\sigma\in\mathcal{N}$, then $\rho_n^R\to\mathbf{I}$ and by (1.76) $\left(\varphi_n^{L,*}(z)\right)^{-1}\varphi_{n+1}^{L,*}(z)\to\mathbf{I}$, so we have

$$\lim_{n\to\infty}\left(\varphi_{n+1}^{L,*}(z)\right)^{-1}\varphi_{n+1}^{R}(z) = \lim_{n\to\infty} z\left(\varphi_n^{L,*}(z)\right)^{-1}\varphi_n^R(z) = \cdots = \lim_{n\to\infty} z^n\mathbf{I} = \mathbf{0}.$$

On the other hand, for $|z|>1$, since

$$\left(\varphi_n^R(z)\right)^{-1}\varphi_n^{L,*}(z) = \left(\varphi_n^L(1/\bar z)\left(\varphi_n^{R,*}(1/\bar z)\right)^{-1}\right)^H,$$

if $n\to\infty$, taking into account (1.79), we get (1.80). □

Also from (1.34), we have

$$\left(\varphi_n^R(z)\right)^{-1} = \rho_{n-1}^R\left(z\mathbf{I}-(\varphi_{n-1}^R(z))^{-1}\varphi_{n-1}^{L,*}(z)\alpha_{n-1}^H\right)^{-1}(\varphi_{n-1}^R(z))^{-1}.$$

If $n\to\infty$, using (1.80) we have the following proposition.

Proposition 1.4.19. *If $\sigma\in\mathcal{N}$, then*

$$\lim_{n\to\infty}\left(\varphi_n^R(z)\right)^{-1} = \lim_{n\to\infty}\left(\varphi_n^L(z)\right)^{-1} = \mathbf{0}, \tag{1.81}$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus\bar{\mathbb{D}}$.

Proposition 1.4.20. *If $\sigma\in\mathcal{N}$ and $\zeta\in\mathbb{C}\setminus\mathbb{D}$, then*

$$\lim_{n\to\infty}\left(\varphi_n^R(z)\right)^{-1}K_{n-1}^R(\zeta,z)\left(\varphi_n^{R,H}(\zeta)\right)^{-1} = \frac{1}{z\bar\zeta-1}\mathbf{I}, \tag{1.82}$$

$$\lim_{n\to\infty}\left(\varphi_n^{L,H}(\zeta)\right)^{-1}K_{n-1}^L(z,\zeta)\left(\varphi_n^{L}(z)\right)^{-1} = \frac{1}{z\bar\zeta-1}\mathbf{I},$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus\bar{\mathbb{D}}$.

Proof. For $z\in\mathbb{C}\setminus\bar{\mathbb{D}}$, the result follows from the previous Propositions and (1.56), (1.57). Let $z\in\mathbb{T}$. From (1.56), we have

$$(1-z\bar\zeta)\left(\varphi_n^R(z)\right)^{-1}K_{n-1}^R(\zeta,z)\left(\varphi_n^{R,H}(\zeta)\right)^{-1} = \left(\varphi_n^R(z)\right)^{-1}\varphi_n^{L,*}(z)\left(\left(\varphi_n^R(\zeta)\right)^{-1}\varphi_n^{L,*}(\zeta)\right)^H-\mathbf{I},$$

and thus from (1.74)

$$\left\|(1-z\bar\zeta)\left(\varphi_n^R(z)\right)^{-1}K_{n-1}^R(\zeta,z)\left(\varphi_n^{R,H}(\zeta)\right)^{-1}+\mathbf{I}\right\|_2^2 \le \left\|\left(\varphi_n^R(\zeta)\right)^{-1}\varphi_n^{L,*}(\zeta)\right\|_2^2.$$

As a consequence, from (1.80) we get (1.82). The result for the left kernel is obtained in a similar way. □

Corollary 1.4.21. *If $\sigma \in \mathcal{N}$ and $\zeta \in \mathbb{T}$, then*

$$\lim_{n\to\infty} \varphi_n^{R,H}(\zeta)\left(K_n^R(\zeta,\zeta)\right)^{-1}\varphi_n^R(\zeta) = \mathbf{0}, \tag{1.83}$$

$$\lim_{n\to\infty} \varphi_n^{L}(\zeta)\left(K_n^L(\zeta,\zeta)\right)^{-1}\varphi_n^{L,H}(\zeta) = \mathbf{0}.$$

The following result establishes bounds for the leading principal coefficients of the orthonormal matrix polynomials.

Proposition 1.4.22.

$$\sigma \in \mathcal{S} \text{ if and only if } \lim_{n\to\infty} \|\kappa_n^L\|_2 < \infty. \tag{1.84}$$

Similarly,

$$\sigma \in \mathcal{S} \text{ if and only if } \lim_{n\to\infty} \|\kappa_n^R\|_2 < \infty. \tag{1.85}$$

Proof. Let $A_n(0) = \left(\kappa_n^L\right)^H \kappa_n^L$, and notice that

$$\|\kappa_n^L\|_2^2 = \Pi\left(\left(\kappa_n^L\right)^H \kappa_n^L\right) = \Pi(A_n(0)). \tag{1.86}$$

If $\sigma \in \mathcal{S}$, from [48] (Theorem 17), $\lim_{n\to\infty}\Pi(A_n(0)) < \infty$ for every $n \in \mathbb{N}$, then from (1.86) $\lim_{n\to\infty}\|\kappa_n^L\|_2 < \infty$.

Conversely, if $\lim_{n\to\infty}\|\kappa_n^L\|_2 < \infty$, then from (1.86) $\Pi(A_n(0)) < \infty$ for every $n \in \mathbb{N}$, thus the numbers $\det(A_n(0))$ are bounded from above (since the product of the eigenvalues of a matrix equals its determinant). Therefore, from [48] (Theorem 19) and the Theorem 1.4.16 it follows that $\sigma \in \mathcal{S}$. In a similar way (1.85) can be proved. □

Now, we state an asymptotic property for the reproducing kernel.

Proposition 1.4.23. *If $\sigma \in \mathcal{S}$, then*

$$K_n^R(z,z) \text{ and } K_n^L(z,z),$$

converge uniformly on every compact subset of $z \in \mathbb{D}$ as $n \to \infty$.

Proof. If $\sigma \in \mathcal{S}$, from [48] the sequence $(A_n(z))_{n\geq 0}$ converges uniformly on every compact subset of $\mathbb{D}$, where $A_n(z) = U_n(z)\mathbf{T}_{n+1}^{-1}U_n(0)^T$ with $U_n(z) = [\mathbf{I}, \cdots, z^n\mathbf{I}]$ and $\mathbf{T}_{n+1} \in \mathcal{M}_{l(n+1)}$ is a block Toeplitz matrix. Moreover,

$$\varphi_n^L(z) = z^n(\kappa_n^{-L})^H A_n(1/\bar{z})^H \text{ and } A_n(0) = (\kappa_n^L)^H\kappa_n^L. \tag{1.87}$$

On the other hand, using the Christoffel-Darboux formula (1.56) with $z = \zeta \in \mathbb{D}$ and (1.87), we have

$$(1-|z|^2)K_n^R(z,z) \leq \varphi_{n+1}^{L,*}(z)\varphi_{n+1}^{L,*}(z)^H = A_{n+1}(z)A_{n+1}^{-1}(0)A_{n+1}^H(z),$$

so we conclude that $K_n^R(z,z)$ converges uniformly on every compact subset of $z \in \mathbb{D}$. □

Notice that in particular we have

$$\lim_{n\to\infty} \varphi_n^R(z)\varphi_n^R(z)^H = \lim_{n\to\infty} \varphi_n^L(z)^H\varphi_n^L(z) = \mathbf{0},$$

thus

$$\lim_{n\to\infty} \|\varphi_n^R(z)\|_2 = \lim_{n\to\infty} \|\varphi_n^L(z)\|_2 = 0,$$

uniformly on every compact subset of $z \in \mathbb{D}$.

1.5 The Szegő matrix transformation

In this section, we are interested in extending to the matrix case some properties from the Szegő transformation. The matrix version of the Szegő transformation has been studied in [42, 51, 172]. Properties such as the relation between the associated Stieltjes and Carathéodory functions in the scalar case are given in Theorem 13.1.2 of [157]. It will be the key tool to analyze the effect on some spectral transformations caused by the application of the matrix Szegő transformation in Chapters 2 and 5.

A matrix measure $\sigma(\theta)$ supported on $\mathbb{T}$ that is invariant under $\theta \mapsto -\theta$ (i.e., $z \mapsto \bar{z} = z^{-1}$) is called **symmetric** (in Section 4.2, we will study some properties of symmetric matrix measures, as well as the relation with block CMV matrices). If $\mu(x)$ is a normalized positive semi-definite matrix measure supported on $[-1, 1]$, then it is possible to define a normalized symmetric positive semi-definite matrix measure $\sigma(\theta)$ supported on $(-\pi, \pi]$ by

$$d\sigma(\theta) = \begin{cases} \frac{1}{2}d\mu(\cos\theta), & -\pi \le \theta \le 0, \\ \\ -\frac{1}{2}d\mu(\cos\theta), & 0 \le \theta \le \pi. \end{cases} \tag{1.88}$$

This is the so-called Szegő matrix transformation (see [42, 51]) that relates normalized matrix measures supported on $[-1, 1]$ with normalized (symmetric) matrix measures supported on $\mathbb{T}$. We will denote this transformation by $d\sigma = Sz(d\mu)$. These measures satisfy

$$\int_{-1}^{1} f(x)d\mu(x) = \int_{-\pi}^{\pi} f(\cos\theta)d\sigma(\theta) \tag{1.89}$$

for every integrable function f on $[-1, 1]$. Similarly, the inverse Szegő matrix transformation satisfies

$$\int_{-\pi}^{\pi} g(\theta)d\sigma(\theta) = \int_{-1}^{1} g(\arccos x)d\mu(x)$$

for every integrable function g on $\mathbb{T}$ with $g(\theta) = g(-\theta)$. We will denote this relation by $d\mu = Sz^{-1}(d\sigma)$.

In [172], the authors show that when the measures μ and $d\sigma = Sz(d\mu)$ are positive semi-definite, there exists a relation between the corresponding sequences of orthogonal matrix polynomials, as follows.

Theorem 1.5.1. [172]. *Let μ be a positive semi-definite matrix measure supported on $[-1, 1]$ and let $(\varphi_n^L(z))_{n\ge0}$, $(\varphi_n^R(z))_{n\ge0}$ be two sequences of orthonormal matrix polynomials with respect to the sesquilinear forms* (1.23) *and* (1.24)*, associated with $d\sigma = Sz(d\mu)$.*

Then, the matrix polynomials

$$p_n^L(x) = \frac{1}{2z^n}(\mathbf{I} - \alpha_{2n-1})^{-1/2}\left(\varphi_{2n}^L(z) + \varphi_{2n}^{R,*}(z)\right), \quad n \geq 1,$$

where $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$, *are orthonormal with respect to* μ.

In particular, if μ has the form (1.2), then (1.88) becomes

$$d\sigma(\theta) = \begin{cases} -\frac{1}{2}\omega(\cos\theta)\sin\theta d\theta, & -\pi \leq \theta \leq 0, \\ \\ \frac{1}{2}\omega(\cos\theta)\sin\theta d\theta, & 0 \leq \theta \leq \pi, \end{cases}$$

that can be written as

$$d\sigma(\theta) = \frac{1}{2}|\sin\theta|\omega(\cos\theta)d\theta. \tag{1.90}$$

Example 1.5.2. *Consider the normalized positive semi-definite matrix measure supported on* $[-1, 1]$

$$d\mu(x) = \omega(x)dx = \frac{1}{\pi\sqrt{1-x^2}}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} dx, \tag{1.91}$$

defined in (1.14). *Since* $\sqrt{1-\cos^2\theta} = |\sin\theta|$ *for* $\theta \in [-\pi, \pi]$, *then from* (1.90) *the normalized positive semi-definite matrix measure supported on* $\mathbb{T}$ *is*

$$d\sigma(\theta) = \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \frac{d\theta}{2\pi}. \tag{1.92}$$

As a consequence, the $n-$*th degree monic* **MOPUC**, *using the matrix Heine's formula* [141] *is*

$$\Phi_n^R(z) = \begin{bmatrix} \phi_n(z) & -\frac{n}{n+1}\phi_{n-1}(z) \\ -\frac{n}{n+1}\phi_{n-1}(z) & \phi_n(z) \end{bmatrix}, \quad n \geq 1,$$

where $\phi_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1+(-1)^{n-k}}{2}\right)(k+1)z^k$ *and* $\phi_0(z) = 1$. *Taking into account that*

$$\langle \Phi_n^R, \Phi_n^R \rangle_{R,\sigma} = \frac{n+2}{2(n+1)}\mathbf{I}_2,$$

we conclude that the $n-$*th degree orthonormal matrix polynomial is*

$$\varphi_n^R(z) = \varphi_n^L(z) = \sqrt{\frac{2(n+1)}{n+2}}\Phi_n^R(z).$$

The above example shows that

$$Sz\left(\frac{1}{\pi\sqrt{1-x^2}}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} dx\right) = \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \frac{d\theta}{2\pi}.$$

In general, if $d\mu(x) = \omega(x)dx$ and $d\sigma(\theta) = \mathcal{W}(\theta)\frac{d\theta}{2\pi}$, then the measures are related by the matrix Szegő transformation when

$$\mathcal{W}(\theta) = \pi|\sin\theta|\omega(\cos\theta), \quad \omega(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathcal{W}(\arccos x), \tag{1.93}$$

i.e.,

$$Sz\left(\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathcal{W}(\arccos x)dx\right)=\pi|\sin\theta|\omega(\cos\theta)\frac{d\theta}{2\pi}. \tag{1.94}$$

The proof is analogous to the one given in [157] Proposition 13.1.1. In particular, for $\mathcal{W}(\theta)=\cos^k\theta$, with $k\geq 0$, we have

$$Sz^{-1}\left(\cos^k\theta\frac{d\theta}{2\pi}\right)=\frac{1}{\pi}\frac{x^k}{\sqrt{1-x^2}}dx. \tag{1.95}$$

Notice that, if σ is a symmetric matrix measure, its associated sequence of moments are Hermitian matrices, since

$$c_n=\int_{-\pi}^{\pi}e^{in\theta}d\sigma(\theta)=\int_{-\pi}^{\pi}e^{-in\theta}d\sigma(\theta)=c_{-n}=c_n^H. \tag{1.96}$$

The following result establishes a relation between the Verblunsky matrix coefficients and the coefficients of the three-term recurrence relation.

Theorem 1.5.3. *Let μ be a normalized positive semi-definite matrix measure supported on $[-1,1]$, and let $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$, be the matrix recurrence coefficients appearing in* (1.12)*. Denote by $(\alpha_n)_{n\geq 0}$ be the sequence of Verblunsky matrix coefficients in* (1.34)-(1.35)*, associated with the symmetric normalized matrix measure $d\sigma=Sz(d\mu)$. Then, for $n\geq 1$,*

$$\begin{aligned}2A_{n+1}&=\sqrt{\mathbf{I}-\alpha_{2n-1}}\sqrt{\mathbf{I}-\alpha_{2n}^2}\sqrt{\mathbf{I}+\alpha_{2n+1}},\\ 2B_{n+1}&=\sqrt{\mathbf{I}-\alpha_{2n-1}}\alpha_{2n}\sqrt{\mathbf{I}-\alpha_{2n-1}}-\sqrt{\mathbf{I}+\alpha_{2n-1}}\alpha_{2n-2}\sqrt{\mathbf{I}+\alpha_{2n-1}},\end{aligned}$$

with $\alpha_{-1}=-\mathbf{I}$. Conversely, if $\Pi_n(x)=(p_{n+1}^L(x)^H)^{-1}p_n^L(x)^HA_{n+1}$, then for $n\geq 0$

$$\begin{aligned}2\alpha_{2n+1}&=\Pi_n(1)+\Pi_n(-1)-\mathbf{I},\\ 2\alpha_{2n}&=2\left(\Pi_n(1)-\Pi_n(-1)\right)^{-1/2}\Pi_n(1)\left(\Pi_n(1)-\Pi_n(-1)\right)^{-1/2}-\mathbf{I}.\end{aligned}$$

The above equations are known as Geronimus relations, and have been proved in [51, 172].

There is also a relation between the matrix-valued Stieltjes function and the matrix-valued Carathéodory function associated with μ and σ, respectively, as follows.

Theorem 1.5.4. *Let μ be a normalized positive semi-definite matrix measure supported on $[-1,1]$ and let $d\sigma=Sz(d\mu)$ be the normalized symmetric positive semi-definite matrix measure supported on $\mathbb{T}$ defined by the Szegő matrix transformation. Let S and F be the corresponding matrix-valued Stieltjes and Carathéodory functions defined in* (1.18) *and* (1.69) *respectively. Then*

$$F(z)=\frac{1-z^2}{2z}S(x), \tag{1.97}$$

with $x=\frac{1}{2}(z+\frac{1}{z})$, $z=x+\sqrt{x^2-1}$.

Proof. Notice that

$$\frac{1}{2}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}+\frac{e^{-i\theta}+z}{e^{-i\theta}-z}\right)=\frac{1-z^2}{z}\frac{1}{z+z^{-1}-2\cos\theta}.$$

Since σ is symmetric and $d\sigma = Sz(d\mu)$, we have

$$\begin{aligned}
F(z) &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\sigma(\theta)+\frac{1}{2}\int_{-\pi}^{\pi}\frac{e^{-i\theta}+z}{e^{-i\theta}-z}d\sigma(\theta)\\
&= \frac{1}{4}\int_{-\pi}^{0}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}+\frac{e^{-i\theta}+z}{e^{-i\theta}-z}\right)d\mu(\cos\theta)-\frac{1}{4}\int_{0}^{\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}+\frac{e^{-i\theta}+z}{e^{-i\theta}-z}\right)d\mu(\cos\theta)\\
&= \frac{1-z^2}{2z}\left(\int_{-\pi}^{0}\frac{1}{z+z^{-1}-2\cos\theta}d\mu(\cos\theta)-\int_{0}^{\pi}\frac{1}{z+z^{-1}-2\cos\theta}d\mu(\cos\theta)\right),
\end{aligned}$$

setting $t=\cos\theta$ and $2x = z+z^{-1}$, we get

$$\begin{aligned}
F(z) &= \frac{1-z^2}{2z}\left(\frac{1}{2}\int_{-1}^{1}\frac{1}{x-t}d\mu(t)-\frac{1}{2}\int_{1}^{-1}\frac{1}{x-t}d\mu(t)\right)\\
&= \frac{1-z^2}{2z}\int_{-1}^{1}\frac{1}{x-t}d\mu(t)
\end{aligned}$$

so that (1.97) follows. □

CHAPTER 2

Analysis of perturbation of moments

Summary

The structure of the chapter is as follows. In the first section, given a sequence of moments (complex numbers) associated with a Hermitian linear functional $\mathcal{L}$ defined in the space of Laurent polynomials, we study a new functional $\mathcal{L}_\Omega$ which is a perturbation of $\mathcal{L}$ in such a way that a finite number of moments are perturbed. Necessary and sufficient conditions are given for the regularity of $\mathcal{L}_\Omega$, and a connection formula between the corresponding families of orthogonal polynomials is obtained. On the other hand, assuming $\mathcal{L}_\Omega$ is positive definite, the perturbation is analyzed through the inverse Szegő transformation. We point out that this is the only part of the document where we are restricted to the case $l = 1$ (scalar case). Of course, these results can be obtained from Section 2.2 by setting $l = 1$, but we decided to include them in a separate section since the formulas are slightly different, because the use of scalar objects allows several simplifications. Moreover, in this case we deal with the more general case of a linear functional. Later on, some of the results are extended to the matrix case. First, we introduce a perturbation on a matrix measure supported on the unit circle that results in a perturbation of the corresponding sequence of matrix moments, and analyze the resulting perturbation on the associated matrix measure on the real line, when both measures are related through the Szegő matrix transformation. This is the main contribution of this chapter. Then, we deal with connection formulas and some other properties associated with the corresponding matrix orthogonal polynomials sequences on the real line and on the unit circle, respectively. Connection formulas for perturbations of matrix moments have been studied previously in [40, 41]. Finally, we illustrate the obtained results with an example.

2.1 On a finite moment perturbation of linear functionals and the inverse Szegő transformation (scalar case)

In the scalar case ($l = 1$), $\Phi_n^R(z) = \Phi_n^L(z)$ and $\varphi_n^R(z) = \varphi_n^L(z)$, for all $n \geq 0$ and $z \in \mathbb{C}$. Therefore, in this section it suffices to consider $\Phi_n(z) = \Phi_n^R(z)$ and $\varphi_n(z) = \varphi_n^R(z)$. Moreover, $S_n = S_n^R = S_n^L$ for all $n \geq 0$.

2.1.1 Introduction

Consider a linear functional $\mathcal{L}$ (more information about linear functionals see [39]) defined in the linear space of Laurent polynomials $\Lambda = span\{z^n\}_{n\in\mathbb{Z}}$ such that $\mathcal{L}$ is Hermitian, i.e.,

$$c_n = \langle \mathcal{L}, z^n\rangle = \overline{\langle \mathcal{L}, z^{-n}\rangle} = \overline{c}_{-n}, \quad n \in \mathbb{Z}.$$

Then, a bilinear functional can be defined in the linear space $\mathbb{P}^1[z] = span\{z^n\}_{n\geq 0}$ of polynomials with complex coefficients by

$$\langle p, q\rangle_{\mathcal{L}} = \langle \mathcal{L}, p(z)\bar{q}(z^{-1})\rangle, \quad p, q \in \mathbb{P}^1[z].$$

The sequence of complex numbers $(c_n)_{n\in\mathbb{Z}}$ is called the sequence of moments associated with $\mathcal{L}$. On the other hand, a sequence of monic polynomials $(\Phi_n(z))_{n\geq 0}$, with $deg\,(\Phi_n) = n$, is said to be orthogonal with respect to $\mathcal{L}$ if the condition

$$\langle \Phi_n, \Phi_m\rangle_{\mathcal{L}} = \delta_{n,m} S_n,$$

where $S_n \neq 0$, holds for every $n, m \geq 0$. The necessary and sufficient conditions for the existence of such a sequence can be expressed in terms of the Toeplitz matrix $\mathbf{T} = \lim_{n\to\infty} \mathbf{T}_n$, where $\mathbf{T}_n$ is defined by

$$\mathbf{T}_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_0 \end{bmatrix}.$$

The sequence $(\Phi_n(z))_{n\geq 0}$ satisfies the orthogonality condition if and only if $\mathbf{T}_n$ is non-singular for every $n \geq 0$. In such a case, $\mathcal{L}$ is said to be quasi-definite (or regular). On the other hand, if $\det \mathbf{T}_n > 0$ for every $n \geq 1$, then $\mathcal{L}$ is said to be positive definite and it has the integral representation

$$\langle p, q\rangle_\sigma = \langle p, q\rangle_{\mathcal{L}} = \int_{\mathbb{T}} p(z)\overline{q(z)}\frac{d\sigma(z)}{iz} = \int_{-\pi}^{\pi} p(e^{i\theta})\overline{q(e^{i\theta})}d\sigma(\theta), \quad p, q \in \mathbb{P}^1[z], \tag{2.1}$$

where σ is a nontrivial positive Borel measure supported on the unit circle $\mathbb{T}$. In such a case, there exists a (unique) family of polynomials $(\varphi_n(z))_{n\geq 0}$, with $deg\,\varphi_n = n$ and positive leading coefficient, such that

$$\int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)}d\sigma(z) = \delta_{m,n}.$$

Notice that (2.1) is a particular case of the sesquilinear forms (1.23) and (1.24).

In [30], the authors studied the perturbation associated with the linear functional

$$\langle p, q\rangle_{\tilde{\mathcal{L}}} := \langle p, q\rangle_{\mathcal{L}} + m\int_{\mathbb{T}} p(z)\overline{q(z)}\frac{dz}{iz}, \quad p, q \in \mathbb{P}^1[z],$$

where $m \in \mathbb{R}$ and $\mathcal{L}$ is (at least) a quasi-definite Hermitian linear functional defined in the linear space of Laurent polynomials. Notice that all moments associated with $\tilde{\mathcal{L}}$ are equal to the moments associated with $\mathcal{L}$, except for the first moment, which is $\tilde{c}_0 = c_0 + m$. The corresponding Toeplitz matrix $\tilde{\mathbf{T}}$ is the result of adding m to the main diagonal of $\mathbf{T}$.

Later on, the linear functional

$$\langle p,q\rangle_{\mathcal{L}_j} = \langle p,q\rangle_{\mathcal{L}} + m\langle z^j p,q\rangle_{\mathcal{L}_\theta} + \bar{m}\langle p,z^j q\rangle_{\mathcal{L}_\theta}, \tag{2.2}$$

where $j\in\mathbb{N}$ is fixed and $\langle\cdot,\cdot\rangle_{\mathcal{L}_\theta}$ is the bilinear functional associated with the normalized Lebesgue measure on the unit circle was studied in [31]. It is easily seen that the moments associated with $\mathcal{L}_j$ are equal to those of $\mathcal{L}$, except for the moments of order j and $-j$, which are perturbed by adding m and $\bar{m}$, respectively. In other words, the corresponding Toeplitz matrix is perturbed on the j-th and $-j$-th subdiagonals. In both cases, the authors obtained the regularity conditions for such a linear functional and deduced connection formulas between the corresponding orthogonal sequences.

Assuming that both $\mathcal{L}$ and $\mathcal{L}_j$ are positive definite, the perturbation (2.2) can be expressed in terms of the corresponding measures as

$$d\tilde{\sigma}(\theta) = d\sigma(\theta) + mz^j\frac{d\theta}{2\pi} + \bar{m}z^{-j}\frac{d\theta}{2\pi}. \tag{2.3}$$

On the other hand, the connection between the measure (2.3) and its corresponding measure supported in $[-1,1]$ via the inverse Szegő transformation was analyzed in [77], and it is deduced that the perturbed moments on the real line depend on the Chebyshev polynomials of the first kind.

In this section, we will extend those results to the case where a perturbation of a finite number of moments is introduced in (2.2). In next subsection, for the positive definite case, the study of the perturbation through the inverse Szegő transformation will be analyzed. In Subsection 2.1.3 necessary and sufficient conditions for the regularity of the perturbed functional are obtained, as well as a connection formula that relates the corresponding families of monic orthogonal polynomials. An illustrative example will be presented in the last subsection.

2.1.2 Finite moments perturbation through the inverse Szegő transformation

Let $\mathcal{L}$ be a quasi-definite linear functional on the linear space of Laurent polynomials, and let $(c_n)_{n\in\mathbb{Z}}$ be its associated sequence of moments.

Definition 2.1.1. Let Ω be a finite set of non negative integers. The linear functional $\mathcal{L}_\Omega$ is defined such that the associated bilinear functional satisfies

$$\langle p,q\rangle_{\mathcal{L}_\Omega} = \langle p,q\rangle_{\mathcal{L}} + \sum_{r\in\Omega}\left(m_r\langle z^r p,q\rangle_{\mathcal{L}_\theta} + \bar{m}_r\langle p,z^r q\rangle_{\mathcal{L}_\theta}\right), \quad p,q\in\mathbb{P}^1[z]. \tag{2.4}$$

where $m_r\in\mathbb{C}$ and

$$\langle p,q\rangle_{\mathcal{L}_\theta} = \int_{-\pi}^{\pi} p(e^{i\theta})\overline{q(e^{i\theta})}\frac{d\theta}{2\pi}, \quad p,q\in\mathbb{P}^1[z],$$

is the bilinear functional associated with the normalized Lebesgue measure in the unit circle.

Notice that, from (2.4), one easily sees that

$$\tilde{c}_n = \langle z^n, 1\rangle_{\mathcal{L}_\Omega} = \begin{cases} c_n, & \text{if} \quad n \notin \Omega, \\ c_{-n} + m_{-n}, & \text{if} \quad -n \in \Omega \text{ and } n \in \mathbb{Z}^-, \\ c_n + \bar{m}_n, & \text{if} \quad n \in \Omega \text{ and } n \notin \mathbb{Z}^-. \end{cases}$$

In other words, $\mathcal{L}_\Omega$ represents an additive perturbation of the moments c_r and c_{-r} of $\mathcal{L}$, with $r \in \Omega$. The rest of the moments remain unchanged. This is, the Toeplitz matrix associated with $\mathcal{L}_\Omega$ is

$$\widetilde{\mathbf{T}} = \mathbf{T} + \sum_{r\in\Omega} \underbrace{\begin{bmatrix} 0 & \cdots & \bar{m}_r & 0 & \cdots \\ \vdots & 0 & \cdots & \bar{m}_r & \cdots \\ m_r & \vdots & \ddots & \vdots & \ddots \\ 0 & m_r & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}}_{m_r \text{ is on the } r-\text{th subdiagonal}},$$

and therefore $\mathcal{L}_\Omega$ is also Hermitian. Moreover, if $\mathcal{L}$ is a positive definite functional, then the above perturbation can be expressed in terms of the corresponding measures as

$$d\tilde{\sigma}_\Omega(\theta) = d\sigma(\theta) + \sum_{r\in\Omega}\left(m_r z^r \frac{d\theta}{2\pi} + \bar{m}_r z^{-r}\frac{d\theta}{2\pi}\right) = d\sigma(\theta) + 2\sum_{r\in\Omega}\mathfrak{Re}(m_r z^r)\frac{d\theta}{2\pi}. \tag{2.5}$$

On the other hand, if $F_\Omega(z)$ is the Carathéodory function associated with $\mathcal{L}_\Omega$, then from (1.70)

$$F_\Omega(z) = F(z) + 2\sum_{r\in\Omega} m_r z^r,$$

which is a linear spectral transformation of $F(z)$.

Let σ be a (scalar) positive measure supported on the unit circle such that its corresponding moments $(c_n)_{n\in\mathbb{Z}}$ are real, i.e., σ is a symmetric measure. Assume also that the perturbed measure $\tilde{\sigma}_\Omega$, defined by (2.5), is also positive and that m_r with $r \in \Omega$ is real, so that the moments associated with $\tilde{\sigma}_\Omega$ are also real. Our goal in this subsection is to determine the relation between the positive Borel measures μ and $\tilde{\mu}_\Omega$, supported in $[-1, 1]$, which are associated with σ and $\tilde{\sigma}_\Omega$, respectively, via the inverse Szegő transformation defined in the Section 1.5. This relation will be stated in terms of the corresponding measure and their sequences of moments.

Proposition 2.1.2. *Let σ be a positive nontrivial Borel measure with real moments supported on the unit circle, let μ be its corresponding measure supported in $[-1, 1]$, such that $d\mu = Sz^{-1}(d\sigma)$ and define $\tilde{\sigma}_\Omega$ as in (2.5). Then, the measure $\tilde{\mu}_\Omega = Sz^{-1}(\tilde{\sigma}_\Omega)$ is given by*

$$d\tilde{\mu}_\Omega(x) = d\mu(x) + 2\sum_{r\in\Omega} m_r T_r(x)\frac{dx}{\pi\sqrt{1-x^2}}, \tag{2.6}$$

where $T_r(x) := \cos(r\theta)$ is the r-th degree Chebyshev polynomial of the first kind on $\mathbb{R}$ (see [39]) and m_r real.

Proof. Notice that, setting $z = e^{i\theta} = \cos\theta + i\sin\theta$, $x = \cos\theta$, $T_r(x) = \sum_{k=0}^{r} a_k x^k$ and m_r real. Then, from (2.5)

$$\begin{aligned} d\tilde{\sigma}_\Omega(\theta) &= d\sigma(\theta) + \sum_{r\in\Omega}\left(m_r(\cos\theta + i\sin\theta)^r\frac{d\theta}{2\pi} + m_r(\cos\theta + i\sin\theta)^{-r}\frac{d\theta}{2\pi}\right) \\ &= d\sigma(\theta) + 2\sum_{r\in\Omega} T_r(x)m_r\frac{d\theta}{2\pi} \\ &= d\sigma(\theta) + 2\sum_{r\in\Omega} m_r\sum_{k=0}^{r} a_k\cos^k\theta\frac{d\theta}{2\pi}. \end{aligned}$$

Since $\tilde{\sigma}_\Omega$ is a symmetric positive measure, by applying the inverse Szegő transformation and taking into account (1.95), we have

$$\begin{aligned} d\tilde{\mu}_\Omega(x) &= d\mu(x) + 2\sum_{r\in\Omega} m_r\sum_{k=0}^{r} a_k x^k\frac{dx}{\pi\sqrt{1-x^2}} \\ &= d\mu(x) + 2\sum_{r\in\Omega} m_r T_r(x)\frac{dx}{\pi\sqrt{1-x^2}} \end{aligned}$$

which is (2.6). □

The next proposition expresses the considered perturbation in terms of the moments associated with the matrix measure (2.6). Recall that $n!!$ is the double factorial or semi-factorial of a number $n \in \mathbb{N}$ defined as in (1.21).

Proposition 2.1.3. *Let μ be a positive measure supported in $[-1,1]$ and $\tilde{\mu}_\Omega$ defined as in (2.6) and $(\mu_n)_{n\geq 0}$, $(\tilde{\mu}_n)_{n\geq 0}$ be their corresponding sequences of moments. Then*

$$\tilde{\mu}_n = \begin{cases} \mu_n, & \text{if} \quad 0 \leq n < r_m, \\ \mu_n + \frac{2}{\pi}\sum_{r\in\Omega} m_r B(n,r), & \text{if} \quad r_m \leq n, \end{cases} \tag{2.7}$$

with $r_m = \min\{r : r \in \Omega\}$ and

- $B(n,r) = \frac{\pi r}{2}\sum_{k=0}^{[r/2]}\left(\frac{(-1)^k(r-k-1)!(2)^{r-2k}}{k!(r-2k)!}\right), \quad$ *if* $\quad r+n-2k=0,$
- $B(n,r) = 0, \quad$ *if $r+n$ is odd,*
- $B(n,r) = \frac{\pi r}{2}\sum_{k=0}^{[r/2]}\left(\frac{(-1)^k(r-k-1)!(2)^{r-2k}}{k!(r-2k)!}\frac{(r+n-2k-1)!!}{(r+n-2k)!!}\right)$, *if $r+n$ is even.*

Proof. The moments associated with $\tilde{\mu}_\Omega$ are given by

$$\tilde{\mu}_n = \int_{-1}^{1} x^n d\tilde{\mu}_\Omega(x) = \mu_n + \frac{2}{\pi}\sum_{r\in\Omega} m_r\int_{-1}^{1} x^n\frac{T_r(x)dx}{\sqrt{1-x^2}}.$$

As a consequence, by the orthogonality of $T_r(x)$ with respect to $\frac{1}{\sqrt{1-x^2}}$, we obtain for the $n-$th moments with $n \notin \Omega$

$$\tilde{\mu}_n = \begin{cases} \mu_n, & \text{if} \quad 0 \leq n < r_m, \\ \mu_n + \frac{2}{\pi}\sum_{r\in\Omega} m_r\int_{-1}^{1} x^n\frac{T_r(x)dx}{\sqrt{1-x^2}}, & \text{if} \quad r_m \leq n. \end{cases} \tag{2.8}$$

Furthermore (see [123]), we have

$$T_r(x) = \frac{r}{2}\sum_{k=0}^{[r/2]} \frac{(-1)^k (r-k-1)!(2x)^{r-2k}}{k!(r-2k)!}, \quad r \geq 1,$$

where $[r/2] = r/2$ if r is even and $[r/2] = (r-1)/2$ if r is odd. Therefore,

$$\begin{aligned}\int_{-1}^{1} x^n T_r(x) \frac{dx}{\sqrt{1-x^2}} &= \frac{r}{2}\int_{-1}^{1} x^n \sum_{k=0}^{[r/2]} \frac{(-1)^k (r-k-1)!(2x)^{r-2k}}{k!(r-2k)!} \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{r}{2}\sum_{k=0}^{[r/2]} \frac{(-1)^k (r-k-1)!(2)^{r-2k}}{k!(r-2k)!} \int_{-1}^{1} x^{r+n-2k} \frac{dx}{\sqrt{1-x^2}},\end{aligned}$$

and, since

$$\int_{-1}^{1} x^n \frac{dx}{\sqrt{1-x^2}} = \pi \frac{(n-1)!!}{n!!} \frac{1+(-1)^n}{2}, \text{ with } (-1)!! = 1$$

(2.8) becomes (2.7). □

From the previous proposition we can conclude that a perturbation of the moments c_r and c_{-r} with $r \in \Omega$, associated with a measure σ supported in the unit circle, results in a perturbation, defined by (2.7), of the moments μ_n, $n \geq r_m$, associated with a measure μ supported in $[-1, 1]$, when both measures are related through the inverse Szegő transformation.

2.1.3 A perturbation on a finite number of moments associated with a linear functional $\mathcal{L}$

Let $\mathcal{L}$ be a quasi-definite linear functional on the linear space of Laurent polynomials.

The following notation will be used hereinafter:

1. $\mathbf{A}_{(s_1,s_2;l_1,l_2;r)}$ will denote a $(s_2 - s_1 + 1) \times (l_2 - l_1 + 1)$ matrix whose entries are $a_{(s,l)_r}$, where $s_1 \leq s \leq s_2$ and $l_1 \leq l \leq l_2$,. For instance,

$$\mathbf{A}_{(2,3;4,5;6)} = \begin{bmatrix} a_{(2,4)_6} & a_{(2,5)_6} \\ a_{(3,4)_6} & a_{(3,5)_6} \end{bmatrix}.$$

2. $\Upsilon_n^{n-1}(0) = [\Phi_n^{(0)}(0), \cdots, \Phi_n^{(n-1)}(0)]^T \in \mathcal{M}_{n\times 1}$.
3. $\tilde{\Upsilon}_n^{n-1}(0) = [\tilde{\Phi}_n^{(0)}(0), \cdots, \tilde{\Phi}_n^{(n-1)}(0)]^T \in \mathcal{M}_{n\times 1}$.
4. Derivatives of negative order are defined as zero. For instance, $K_n^{(0,-2)}(z, y) \equiv 0$.

Necessary and sufficient conditions for the regularity of $\mathcal{L}_\Omega$, as well as the relation between the corresponding sequences of orthogonal polynomials, are given in the next result.

Proposition 2.1.4. *Let $\mathcal{L}$ be a quasi-definite linear functional and let $(\Phi_n(z))_{n\geq 0}$ be its associated monic orthogonal polynomials sequence. The following statements are equivalent*

1. $\mathcal{L}_\Omega$ *defined as in* (2.4) *is a quasi-definite linear functional.*

2. *The matrix* $\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{L}_n^r$ *is nonsingular, and*

$$\tilde{S}_n = S_n + (\boldsymbol{Q}_n \boldsymbol{W}_n)^T \sum_{r\in\Omega} \boldsymbol{Y}_n^r + \sum_{r\in\Omega} \bar{m}_r \overline{\frac{\Phi_n^{(n-r)}(0)}{(n-r)!}} \neq 0,\ n \geq 0,$$

with

$$\boldsymbol{Y}_n^r = \begin{bmatrix} m_r \overline{\frac{\Phi_n^{(r)}(0)}{0!r!}} \\ \vdots \\ m_r \overline{\frac{\Phi_n^{(2r-1)}(0)}{(r-1)!(2r-1)!}} \\ m_r \overline{\frac{\Phi_n^{(2r)}(0)}{r!(2r)!}} + \bar{m}_r \overline{\frac{\Phi_n^{(0)}(0)}{r!0!}} \\ \vdots \\ m_r \overline{\frac{\Phi_n^{(n)}(0)}{(n-r)!(n)!}} + \bar{m}_r \overline{\frac{\Phi_n^{(n-2r)}(0)}{(n-r)!(n-2r)!}} \\ \bar{m}_r \overline{\frac{\Phi_n^{(n-2r+1)}(0)}{(n-r+1)!(n-2r+1)!}} \\ \vdots \\ \bar{m}_r \overline{\frac{\Phi_n^{(n-r-1)}(0)}{(n-1)!(n-r-1)!}} \end{bmatrix}, \tag{2.9}$$

$$\begin{aligned} \boldsymbol{Q}_n &= (\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{L}_n^r)^{-1}, \\ \boldsymbol{W}_n &= \Phi_n(0) - \sum_{r\in\Omega} \bar{m}_r n! \boldsymbol{C}_{(0,n-1;n,n;r)}, \end{aligned}$$

$$\mathbb{K}_{n-1}^r(z,0) = \begin{bmatrix} m_r \frac{K_{n-1}^{(0,r)}(z,0)}{0!r!} \\ \vdots \\ m_r \frac{K_{n-1}^{(0,2r-1)}(z,0)}{(r-1)!(2r-1)!} \\ m_r \frac{K_{n-1}^{(0,2r)}(z,0)}{r!(2r)!} + \bar{m}_r \frac{K_{n-1}^{(0,0)}(z,0)}{r!0!} \\ \vdots \\ m_r \frac{K_{n-1}^{(0,n-1)}(z,0)}{(n-r-1)!(n-1)!} + \bar{m}_r \frac{K_{n-1}^{(0,n-2r-1)}(z,0)}{(n-r-1)!(n-2r-1)!} \\ \bar{m}_r \frac{K_{n-1}^{(0,n-2r)}(z,0)}{(n-r)!(n-2r)!} \\ \vdots \\ \bar{m}_r \frac{K_{n-1}^{(0,n-r-1)}(z,0)}{(n-1)!(n-r-1)!}. \end{bmatrix},$$

and

$$\mathbf{L}_n^r = \begin{bmatrix} m_r \boldsymbol{A}_{(0,r-1;0,r-1;r)} & \boldsymbol{B}_{(0,r-1;r,n-r-1;r)} & \bar{m}_r \boldsymbol{C}_{(0,r-1;n-r,n-1;r)} \\ m_r \boldsymbol{A}_{(r,n-r-1;0,r-1;r)} & \boldsymbol{B}_{(r,n-r-1;r,n-r-1;r)} & \bar{m}_r \boldsymbol{C}_{(r,n-r-1;n-r,n-1;r)} \\ m_r \boldsymbol{A}_{(n-r,n-1;0,r-1;r)} & \boldsymbol{B}_{(n-r,n-1;r,n-r-1;r)} & \bar{m}_r \boldsymbol{C}_{(n-r,n-1;n-r,n-1;r)} \end{bmatrix},$$

where the entries of the matrices $\boldsymbol{A}$*,* $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are given by*

$$a_{(s,l)_r} = \frac{K_{n-1}^{(s,l+r)}(0,0)}{l!(l+r)!}, \tag{2.10}$$

$$b_{(s,l)_r} = m_r \frac{K_{n-1}^{(s,l+r)}(0,0)}{l!(l+r)!} + \bar{m}_r \frac{K_{n-1}^{(s,l-r)}(0,0)}{l!(l-r)!}, \tag{2.11}$$

$$c_{(s,l)_r} = \frac{K_{n-1}^{(s,l-r)}(0,0)}{l!(l-r)!}. \tag{2.12}$$

Furthermore, if $(\tilde{\Phi}_n(z))_{n\geq 0}$ *denotes the monic orthogonal polynomials sequence associated with* $\mathcal{L}_\Omega$*, then*

$$\tilde{\Phi}_n(z) = \Phi_n(z) - (\boldsymbol{Q}_n \boldsymbol{W}_n)^T \sum_{r\in\Omega} \mathbb{K}_{n-1}^r(z,0) - \sum_{r\in\Omega} \bar{m}_r \frac{K_{n-1}^{(0,n-r)}(z,0)}{(n-r)!}, \tag{2.13}$$

for every $n \geq 1$*.*

Remark 2.1.5. Notice that $\mathbf{Q}_n$ and $\mathbf{L}_n^r$ are $n \times n$ matrices, whereas $\mathbf{Y}_n^r, \mathbf{W}_n$ and $\mathbb{K}_{n-1}^r(z,0)$ are n-th dimensional column vectors.

Proof. Assume $\mathcal{L}_\Omega$ is a quasi-definite linear functional, and denote by $(\tilde{\Phi}_n(z))_{n\geq 0}$ its associated monic orthogonal polynomials sequence. Let us write

$$\tilde{\Phi}_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k} \Phi_k(z), \tag{2.14}$$

where, for $0 \leq k \leq n-1$,

$$\begin{aligned} \lambda_{n,k} &= \frac{\langle \tilde{\Phi}_n, \Phi_k \rangle_{\mathcal{L}}}{S_n} \\ &= \frac{\langle \tilde{\Phi}_n, \Phi_k \rangle_{\mathcal{L}_\Omega} - \sum_{r\in\Omega} m_r \int_{\mathbb{T}} y^r \tilde{\Phi}_n(y) \overline{\Phi_k(y)} \frac{dy}{2\pi i y} - \sum_{r\in\Omega} \bar{m}_r \int_{\mathbb{T}} y^{-r} \tilde{\Phi}_n(y) \overline{\Phi_k(y)} \frac{dy}{2\pi i y}}{S_k} \\ &= -\sum_{r\in\Omega} \frac{m_r}{S_k} \int_{\mathbb{T}} y^r \tilde{\Phi}_n(y) \overline{\Phi_k(y)} \frac{dy}{2\pi i y} - \sum_{r\in\Omega} \frac{\bar{m}_r}{S_k} \int_{\mathbb{T}} y^{-r} \tilde{\Phi}_n(y) \overline{\Phi_k(y)} \frac{dy}{2\pi i y}, \end{aligned}$$

and notice that $\langle \tilde{\Phi}_n, \Phi_k \rangle_{\mathcal{L}} \neq 0$ (in general), and $\langle \tilde{\Phi}_n, \Phi_k \rangle_{\mathcal{L}_\Omega} = 0$ for $n > k$.

Substituting in (2.14) and using (1.56), we get

$$\begin{aligned}\tilde{\Phi}_n(z) = \Phi_n(z) &- \sum_{r\in\Omega}\left(m_r\int_{\mathbb{T}} y^r\tilde{\Phi}_n(y)\sum_{k=0}^{n-1}\frac{\Phi_k(z)\overline{\Phi_k(y)}}{S_k}\frac{dy}{2\pi iy}\right)\\ &- \sum_{r\in\Omega}\left(\bar{m}_r\int_{\mathbb{T}} y^{-r}\tilde{\Phi}_n(y)\sum_{k=0}^{n-1}\frac{\Phi_k(z)\overline{\Phi_k(y)}}{S_k}\frac{dy}{2\pi iy}\right)\\ = \Phi_n(z) &- \sum_{r\in\Omega}\left(m_r\int_{\mathbb{T}} y^r\tilde{\Phi}_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} - \bar{m}_r\int_{\mathbb{T}} y^{-r}\tilde{\Phi}_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}\right).\end{aligned}$$

From the power series expansion of $\tilde{\Phi}_n(y)$ and $K_{n-1}(z,y)$, we have

$$y^r\tilde{\Phi}_n(y) = y^r\sum_{l=0}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}y^l = \sum_{l=r}^{n+r}\frac{\tilde{\Phi}_n^{(l-r)}(0)}{(l-r)!}y^l,$$

and for $|y|=1$,

$$K_{n-1}(z,y) = \sum_{l=0}^{n-1}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}\frac{1}{y^l},$$

and since $\int_{\mathbb{T}} y^{r-t}\frac{dy}{2\pi iy} = 1$ if $r=t$ and zero otherwise, we arrive at

$$\begin{aligned}\int_{\mathbb{T}} y^r\tilde{\Phi}_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} &= \int_{\mathbb{T}}\sum_{l=r}^{n+r}\frac{\tilde{\Phi}_n^{(l-r)}(0)}{(l-r)!}y^l\sum_{l=0}^{n-1}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}\frac{1}{y^l}\frac{dy}{2\pi iy}\\ &= \sum_{l=0}^{n-r-1}\frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}.\end{aligned}$$

Similarly,

$$\int_{\mathbb{T}} y^{-r}\tilde{\Phi}_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} = \sum_{l=0}^{n-r}\frac{\tilde{\Phi}_n^{(l+r)}(0)}{(l+r)!}\frac{K_{n-1}^{(0,l)}(z,0)}{l!} = \sum_{l=r}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}.$$

As a consequence, we get

$$\tilde{\Phi}_n(z) = \Phi_n(z) - \sum_{r\in\Omega}\left(m_r\sum_{l=0}^{n-r-1}\frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!} - \bar{m}_r\sum_{l=r}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}\right), \tag{2.15}$$

which after a reorganization of the terms becomes

$$\begin{aligned}\tilde{\Phi}_n(z) \;=\; \Phi_n(z) &- \sum_{r\in\Omega}\left(m_r\sum_{l=0}^{r-1}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!} + \bar{m}_r\sum_{l=n-r}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}\right)\\ &- \sum_{r\in\Omega}\left(\sum_{l=r}^{n-r-1}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}\left(m_r\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!} + \bar{m}_r\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}\right)\right).\end{aligned}$$

In order to find the constant values $\tilde{\Phi}_n^{(l)}(0)$, we take s derivatives, $0 \leq s \leq n$, with respect to the variable z and evaluate at $z = 0$ to obtain the $(n+1) \times (n+1)$ linear system

$$\begin{aligned}\tilde{\Phi}_n^{(s)}(0) &= \Phi_n^{(s)}(0) - \sum_{r\in\Omega}\left(m_r \sum_{l=0}^{r-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(s,l+r)}(0,0)}{(l+r)!} + \bar{m}_r \sum_{l=n-r}^{n} \frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!} \frac{K_{n-1}^{(s,l-r)}(0,0)}{(l-r)!} \right) \\ &\quad - \sum_{r\in\Omega}\left(\sum_{l=r}^{n-r-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \left(m_r \frac{K_{n-1}^{(s,l+r)}(0,0)}{(l+r)!} + \bar{m}_r \frac{K_{n-1}^{(s,l-r)}(0,0)}{(l-r)!} \right) \right).\end{aligned}$$

If we use (2.10), (2.11) and (2.12), then the linear system becomes

$$\tilde{\Phi}_n^{(s)}(0) = \Phi_n^{(s)}(0) - \sum_{r\in\Omega}\left(m_r \sum_{l=0}^{r-1} a_{(s,l)_r} \tilde{\Phi}_n^{(l)}(0) + \sum_{l=r}^{n-r-1} b_{(s,l)_r} \tilde{\Phi}_n^{(l)}(0) + \bar{m}_r \sum_{l=n-r}^{n} c_{(s,l)_r} \tilde{\Phi}_n^{(l)}(0) \right).$$

Notice that the last equation (i.e., when $s = n$) gives no information, since $a_{(n,l)_r} = b_{(n,l)_r} = c_{(n,l)_r} = 0$ and $\tilde{\Phi}_n^{(n)}(0) = \Phi_n^{(n)}(0) = n!$. As a consequence, the $(n+1) \times (n+1)$ linear system can be reduced to an $n \times n$ linear system that can be expressed in matrix form as

$$\left(\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{L}_n^r \right) \tilde{\Upsilon}_n^{n-1} = \Upsilon_n^{n-1} - \sum_{r\in\Omega} \bar{m}_r n! \mathbf{C}_{(0,n-1;n,n;r)} = \mathbf{W}_n.$$

Since $\mathcal{L}_\Omega$ is quasi-definite, it has a unique monic orthogonal polynomial sequence and therefore the linear system has a unique solution. As a consequence, the matrix $\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{L}_n^r$ is nonsingular and

$$\tilde{\Upsilon}_n^{n-1} = \left(\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{L}_n^r \right)^{-1} \mathbf{W}_n. \tag{2.16}$$

This is, we have

$$\tilde{\Phi}_n(z) = \Phi_n(z) - \sum_{r\in\Omega} \mathbf{W}_n^T \left(\left(\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{L}_n^r \right)^{-1} \right)^T \mathbb{K}_{n-1}^r(z,0) - \sum_{r\in\Omega} \bar{m}_r \frac{K_{n-1}^{(0,n-r)}(z,0)}{(n-r)!},$$

which is (2.13).

On the other hand, for $n \geq 0$,

$$\begin{aligned}
\tilde{S}_n &= \langle \tilde{\Phi}_n, \tilde{\Phi}_n \rangle_{\mathcal{L}_\Omega} = \langle \tilde{\Phi}_n, \Phi_n \rangle_{\mathcal{L}_\Omega} = \langle \tilde{\Phi}_n, \Phi_n \rangle_{\mathcal{L}} + \sum_{r \in \Omega} \left(m_r \langle z^r \tilde{\Phi}_n, \Phi_n \rangle_{\mathcal{L}_\theta} + \bar{m}_r \langle \tilde{\Phi}_n, z^r \Phi_n \rangle_{\mathcal{L}_\theta} \right) \\
&= S_n + \sum_{r \in \Omega} \left(m_r \int_{\mathbb{T}} \sum_{l=0}^{n} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} z^{l+r} \sum_{l=0}^{n} \overline{\frac{\Phi_n^{(l)}(0)}{l!}} z^{-l} \frac{dz}{2\pi i z} \right) \\
&\quad + \sum_{r \in \Omega} \left(\bar{m}_r \int_{\mathbb{T}} \sum_{l=0}^{n} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} z^{l-r} \sum_{l=0}^{n} \overline{\frac{\Phi_n^{(l)}(0)}{l!}} z^{-l} \frac{dz}{2\pi i z} \right) \\
&= S_n + \sum_{r \in \Omega} \left(m_r \sum_{l=0}^{r-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \overline{\frac{\Phi_n^{(l+r)}(0)}{(l+r)!}} + \sum_{l=r}^{n-r} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \left(m_r \overline{\frac{\Phi_n^{(l+r)}(0)}{(l+r)!}} + \bar{m}_r \overline{\frac{\Phi_n^{(l-r)}(0)}{(l-r)!}} \right) \right) \\
&\quad + \sum_{r \in \Omega} \left(\bar{m}_r \sum_{l=n-r+1}^{n-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \overline{\frac{\Phi_n^{(l-r)}(0)}{(l-r)!}} \right) + \sum_{r \in \Omega} \bar{m}_r \overline{\frac{\Phi_n^{(n-r)}(0)}{(n-r)!}}.
\end{aligned}$$

Using (2.9) and (2.16), we get

$$\tilde{S}_n = S_n + \sum_{r \in \Omega} \mathbf{W}_n^T \left(\left(\mathbf{I}_n + \sum_{r \in \Omega} \mathbf{L}_n^r \right)^{-1} \right)^T \mathbf{Y}_n^r + \sum_{r \in \Omega} \bar{m}_r \overline{\frac{\Phi_n^{(n-r)}(0)}{(n-r)!}}.$$

Conversely, assume $\mathbf{I}_n + \sum_{r \in \Omega} \mathbf{L}_n^r$ is nonsingular for every $n \geq 1$ and define $(\tilde{\Phi}_n(z))_{n \geq 0}$ as in (2.13). For $0 \leq k \leq n-1$, we have

$$\begin{aligned}
\left\langle \sum_{l=0}^{n-r-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}, \Phi_k(z) \right\rangle_{\mathcal{L}} &= \sum_{l=0}^{n-r-1} \left\langle \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}, \Phi_k(z) \right\rangle_{\mathcal{L}} \\
&= \sum_{l=0}^{n-r-1} \left\langle \frac{\tilde{\Phi}_n^{(l)}(0)}{l!(l+r)!} \sum_{t=0}^{n-1} \frac{\Phi_t(z) \overline{\Phi_t^{(l+r)}(0)}}{S_t}, \Phi_k(z) \right\rangle_{\mathcal{L}} \\
&= \sum_{l=0}^{n-r-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \overline{\frac{\Phi_k^{(l+r)}(0)}{(l+r)!}},
\end{aligned}$$

and, similarly, $\left\langle \sum_{l=j}^{n} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}, \Phi_k(z) \right\rangle_{\mathcal{L}} = \sum_{l=r}^{n} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \overline{\frac{\Phi_k^{(l-r)}(0)}{(l-r)!}}$. In addition,

$$\begin{aligned}
\langle z^r \tilde{\Phi}_n, \Phi_k \rangle_{\mathcal{L}_\theta} &= \int_{\mathbb{T}} z^r \tilde{\Phi}_n(z) \overline{\Phi_k(z)} \frac{dz}{2\pi i z} \\
&= \int_{\mathbb{T}} z^r \left(\sum_{l=0}^{n} \frac{\tilde{\Phi}_n^{(l)}(0) z^l}{l!} \right) \left(\sum_{l=0}^{k} \overline{\frac{\Phi_n^{(l)}(0)}{l!}} z^{-l} \right) \frac{dz}{2\pi i z} = \sum_{l=0}^{n-r-1} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!} \overline{\frac{\Phi_k^{(l+r)}(0)}{(l+r)!}},
\end{aligned}$$

and also $\langle\tilde{\Phi}_n, z^r\Phi_k\rangle_{\mathcal{L}_\theta} = \sum_{l=r}^{n} \frac{\tilde{\Phi}_n^{(l)}(0)}{l!}\overline{\frac{\Phi_k^{(l-r)}(0)}{(l-r)!}}$. Thus, for $0 \leq k \leq n-1$, and taking into account (2.15) and the previous equations, we have

$$\begin{aligned}
\langle\tilde{\Phi}_n, \Phi_k\rangle_{\mathcal{L}_\Omega} &= \langle\tilde{\Phi}_n, \Phi_k\rangle_{\mathcal{L}} + \sum_{r\in\Omega}\Big(m_r\langle z^r\tilde{\Phi}_n, \Phi_k\rangle_{\mathcal{L}_\theta} + \bar{m}_r\langle\tilde{\Phi}_n, z^r\Phi_k\rangle_{\mathcal{L}_\theta}\Big)\\
&= \langle\Phi_n, \Phi_k\rangle_{\mathcal{L}} - \sum_{r\in\Omega} m_r\left\langle\sum_{l=0}^{n-r-1}\frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}, \Phi_k(z)\right\rangle_{\mathcal{L}}\\
&\qquad - \sum_{r\in\Omega}\bar{m}_r\left\langle\sum_{l=r}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}, \Phi_k(z)\right\rangle_{\mathcal{L}}\\
&\quad + \sum_{r\in\Omega} m_r\left(\sum_{l=0}^{n-r-1}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}\overline{\frac{\Phi_k^{(l+r)}(0)}{(l+r)!}}\right) + \sum_{r\in\Omega}\bar{m}_r\left(\sum_{l=r}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}\overline{\frac{\Phi_k^{(l-r)}(0)}{(l-r)!}}\right) = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{S}_n = \langle\tilde{\Phi}_n, \Phi_n\rangle_{\mathcal{L}_\Omega} &= \langle\tilde{\Phi}_n, \Phi_n\rangle_{\mathcal{L}} + \sum_{r\in\Omega}\Big(m_r\langle z^r\tilde{\Phi}_n, \Phi_n\rangle_{\mathcal{L}_\theta} + \bar{m}_r\langle\tilde{\Phi}_n, z^r\Phi_n\rangle_{\mathcal{L}_\theta}\Big)\\
&= S_n + \sum_{r\in\Omega}\left(m_r\int_{\mathbb{T}}\sum_{l=0}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}z^{l+r}\sum_{l=0}^{n}\overline{\frac{\Phi_n^{(l)}(0)}{l!}}z^{-l}\frac{dz}{2\pi iz}\right)\\
&\qquad + \sum_{r\in\Omega}\left(\bar{m}_r\int_{\mathbb{T}}\sum_{l=0}^{n}\frac{\tilde{\Phi}_n^{(l)}(0)}{l!}z^{l-r}\sum_{l=0}^{n}\overline{\frac{\Phi_n^{(l)}(0)}{l!}}z^{-l}\frac{dz}{2\pi iz}\right)\\
&= S_n + \sum_{r\in\Omega}\left(\tilde{\Upsilon}_n^{n-1}\right)^T(0)\mathbf{Y}_n^r(0) + \sum_{r\in\Omega}\bar{m}_r\overline{\frac{\Phi_n^{(n-r)}(0)}{(n-r)!}},
\end{aligned}$$

which is different from 0 by assumption. Therefore, $\mathcal{L}_\Omega$ is quasi-definite. □

Notice that if $r_m = \min\{r : r \in \Omega\}$, then from Proposition 2.1.4 we conclude that if $n < r_m$ then $\mathbb{K}_{n-1}^{r_m}(z,0)$ is the zero vector, $K_{n-1}^{(0,n-r_m)}(z,0) = 0$ and according to (2.13) we have $\tilde{\Phi}_n(z) = \Phi_n(z)$. This means that the only affected polynomials are those with degree $n \geq r_m$.

The above proposition generalizes the case when Ω has a single element $k \neq 0$ (see [31]) and the case $\Omega = \{0\}$ (see [30]).

2.1.4 Example

Let $\mathcal{L}$ be the Christoffel transformation of the normalized Lebesgue measure with parameter $\xi = 1$, i.e.,

$$\langle p, q\rangle_{\mathcal{L}} = \langle(z-1)p, (z-1)q\rangle_{\mathcal{L}_\theta},$$

and let $\Omega = \{1, 2\}$. Then,

$$\langle p, q\rangle_{\mathcal{L}_{\{1,2\}}} = \langle p, q\rangle_{\mathcal{L}} + m_1\langle zp, q\rangle_{\mathcal{L}_\theta} + \bar{m}_1\langle p, zq\rangle_{\mathcal{L}_\theta} + m_2\langle z^2p, q\rangle_{\mathcal{L}_\theta} + \bar{m}_2\langle p, z^2q\rangle_{\mathcal{L}_\theta}, \quad (2.17)$$

i.e., the moments of order 1 and 2 are perturbed. Since the sequence $(z^n)_{n\geq 0}$ is orthogonal with respect to $\mathcal{L}_\theta$, the monic orthogonal polynomials sequence with $\langle (z-1)p, (z-1)q\rangle_{\mathcal{L}_\theta}$ is given by (see [20])

$$\Phi_n(z) = \frac{1}{z-1}\left(z^{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} z^j\right), \quad n \geq 1,$$

or, equivalently,

$$\Phi_n(z) = z^n + \frac{n}{n+1}\Phi_{n-1}(z), \quad n \geq 1,$$

and its corresponding reversed polynomial is

$$\Phi_n^*(z) = \frac{1}{1-z}\left(1 - \frac{1}{n+1}\sum_{j=0}^{n} z^{j+1}\right), \quad n \geq 1.$$

Furthermore, if $0 \leq s \leq n$, we have

$$\Phi_n^{(s)}(0) = \frac{(s+1)!}{(n+1)}, \ \Phi_n^{*(s)}(0) = \frac{s!(n+1-s)}{(n+1)}, \ S_n = \frac{n+2}{n+1},$$

and if $0 \leq t, s \leq n-1$,

$$K_{n-1}^{(0,t)}(z,0) = \sum_{p=t}^{n-1} \frac{(t+1)!}{p+2}\Phi_p(z),$$

$$K_{n-1}^{(s,t)}(0,0) = \sum_{p=\max\{s,t\}}^{n-1} \frac{(t+1)!(s+1)!}{(p+1)(p+2)}.$$

As a consequence, we have

$$a_{(s,l)_r} = \frac{(s+1)!(l+r+1)}{l!} \sum_{p=\max\{s,l+r\}}^{n-1} \frac{1}{(p+1)(p+2)},$$

$$c_{(s,l)_r} = \frac{(s+1)!(l-r+1)}{l!} \sum_{p=\max\{s,l-r\}}^{n-1} \frac{1}{(p+1)(p+2)},$$

$$b_{(s,l)_r} = \frac{(s+1)!}{l!}\left(m_r \sum_{p=\max\{s,l+r\}}^{n-1} \frac{l+r+1}{(p+1)(p+2)} + \bar{m}_r \sum_{p=\max\{s,l-r\}}^{n-1} \frac{l-r+1}{(p+1)(p+2)}\right).$$

We now proceed to obtain the monic orthogonal polynomial sequence associated with $\mathcal{L}_{\{1,2\}}$, denoted by $(\tilde{\Phi}_n(z))_{n\geq 0}$. Notice that we have

$$c_{(s,n)_1} = \frac{(s+1)!}{n!(n+1)}, \ c_{(s,n)_2} = \frac{2(s+1)!}{n!(n+1)} \ \text{ if } 0 \leq s \leq n-2$$

and

$$c_{(n-1,n)_2} = \frac{n-1}{n(n+1)},$$

and thus

$$\mathbf{W}_n = \begin{bmatrix} \Phi_n^{(0)}(0) \\ \Phi_n^{(1)}(0) \\ \vdots \\ \Phi_n^{(n-2)}(0) \\ \Phi_n^{(n-1)}(0) \end{bmatrix} - \bar{m}_1 n! \begin{bmatrix} c_{(0,n)_1} \\ c_{(1,n)_1} \\ \vdots \\ c_{(n-2,n)_1} \\ c_{(n-1,n)_1} \end{bmatrix} - \bar{m}_2 n! \begin{bmatrix} c_{(0,n)_2} \\ c_{(1,n)_2} \\ \vdots \\ c_{(n-2,n)_2} \\ c_{(n-1,n)_2} \end{bmatrix}$$
$$= \frac{1-\bar{m}_1}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n! \end{bmatrix} - \frac{2\bar{m}_2}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n!\frac{n-1}{2n} \end{bmatrix}.$$

On the other hand, for $n \geq 2$ we have

$$\mathbb{K}_{n-1}^1(z,0) = \begin{bmatrix} 2m_1 \sum_{p=1}^{n-1} \frac{\Phi_p(z)}{p+2} \\ 3m_1 \sum_{p=2}^{n-1} \frac{\Phi_p(z)}{p+2} + \bar{m}_1 \sum_{p=0}^{n-1} \frac{\Phi_p(z)}{p+2} \\ \vdots \\ \frac{nm_1}{(n-2)!} \sum_{p=n-1}^{n-1} \frac{\Phi_p(z)}{p+2} + \frac{\overline{m_1}}{(n-3)!} \sum_{p=n-3}^{n-1} \frac{\Phi_p(z)}{p+2} \\ \frac{\bar{m}_1}{(n-2)!} \sum_{p=n-2}^{n-1} \frac{\Phi_p(z)}{p+2} \end{bmatrix},$$

$$\mathbf{L}_n^1 = \begin{bmatrix} m_1 a_{(0,0)_1} & b_{(0,1)_1} & \cdots & b_{(0,n-2)_1} & \bar{m}_1 c_{(0,n-1)_1} \\ m_1 a_{(1,0)_1} & b_{(1,1)_1} & \cdots & b_{(1,n-2)_1} & \bar{m}_1 c_{(1,n-1)_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1 a_{(n-2,0)_1} & b_{(n-2,1)_1} & \cdots & b_{(n-2,n-2)_1} & \bar{m}_1 c_{(n-2,n-1)_1} \\ m_1 a_{(n-1,0)_1} & b_{(n-1,1)_1} & \cdots & b_{(n-1,n-2)_1} & \bar{m}_1 c_{(n-1,n-1)_1} \end{bmatrix},$$

and for $n \geq 3$,

$$\mathbb{K}_{n-1}^2(z,0) = \begin{bmatrix} 3m_2 \sum_{p=2}^{n-1} \frac{\Phi_p(z)}{p+2} \\ 4m_2 \sum_{p=3}^{n-1} \frac{\Phi_p(z)}{p+2} \\ \frac{5m_2}{2} \sum_{p=4}^{n-1} \frac{\Phi_p(z)}{p+2} + \frac{\bar{m}_2}{2} \sum_{p=0}^{n-1} \frac{\Phi_p(z)}{p+2} \\ \vdots \\ \frac{nm_2}{(n-3)!} \sum_{p=n-1}^{n-1} \frac{\Phi_p(z)}{p+2} + \frac{\bar{m}_2}{(n-3)(n-5)!} \sum_{p=n-5}^{n-1} \frac{\Phi_p(z)}{p+2} \\ \frac{\bar{m}_2}{(n-2)(n-4)!} \sum_{p=n-4}^{n-1} \frac{\Phi_p(z)}{p+2} \\ \frac{\bar{m}_2}{(n-1)(n-3)!} \sum_{p=n-3}^{n-1} \frac{\Phi_p(z)}{p+2} \end{bmatrix},$$

$$\mathbf{L}_n^2 =$$

$$\begin{bmatrix} m_2a_{(0,0)_2} & m_2a_{(0,1)_2} & b_{(0,2)_2} & \cdots & b_{(0,n-3)_2} & \bar{m}_2c_{(0,n-2)_2} & \bar{m}_2c_{(0,n-1)_2} \\ m_2a_{(1,0)_2} & m_2a_{(1,1)_2} & b_{(1,2)_2} & \cdots & b_{(1,n-3)_2} & \bar{m}_2c_{(1,n-2)_2} & \bar{m}_2c_{(1,n-1)_2} \\ m_2a_{(2,0)_2} & m_2a_{(2,1)_2} & b_{(2,2)_2} & \cdots & b_{(2,n-3)_2} & \bar{m}_2c_{(2,n-2)_2} & \bar{m}_2c_{(2,n-1)_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m_2a_{(n-3,0)_2} & m_2a_{(n-3,1)_2} & b_{(n-3,2)_2} & \cdots & b_{(n-3,n-3)_2} & \bar{m}_2c_{(n-3,n-2)_2} & \bar{m}_2c_{(n-3,n-1)_2} \\ m_2a_{(n-2,0)_2} & m_2a_{(n-2,1)_2} & b_{(n-2,2)_2} & \cdots & b_{(n-2,n-3)_2} & \bar{m}_2c_{(n-2,n-2)_2} & \bar{m}_2c_{(n-2,n-1)_2} \\ m_2a_{(n-1,0)_2} & m_2a_{(n-1,1)_2} & b_{(n-1,2)_2} & \cdots & b_{(n-1,n-3)_2} & \bar{m}_2c_{(n-1,n-2)_2} & \bar{m}_2c_{(n-1,n-1)_2} \end{bmatrix},$$

For illustrative purposes, we compute the first polynomials of the sequence:

1. Degree one:

$$\tilde{\Phi}_1(z) = \Phi_1(z) - \frac{\bar{m}_1 K_0^{(0,0)}(z,0)}{0!} = z + \frac{(1-\bar{m}_1)}{2}\Phi_0(z),$$

since

$$\mathbb{K}_0^1(z,0) = 0,\ \mathbb{K}_0^2(z,0) = 0$$

and

$$\Phi_1(z) = z + \frac{1}{2}\Phi_0(z).$$

2. Degree two:

$$\begin{aligned}\tilde{\Phi}_2(z) = \Phi_2(z) &- \left(\frac{1-\bar{m}_1}{3}\begin{bmatrix} 1! \\ 2! \end{bmatrix}^T - \frac{2\bar{m}_2}{3}\begin{bmatrix} 1! \\ 1/2 \end{bmatrix}^T\right)\left(\begin{bmatrix} \frac{m_1}{3}+1 & \frac{2\bar{m}_1}{3} \\ \frac{2m_1}{3} & \frac{\bar{m}_1}{3}+1 \end{bmatrix}^{-1}\right)^T \\ &\begin{bmatrix} \frac{2m_1}{3}\Phi_1(z) \\ \bar{m}_1\left(\frac{\Phi_0(z)}{2}+\frac{\Phi_1(z)}{3}\right) \end{bmatrix} - \frac{2\bar{m}_1}{3}\Phi_1(z) - \bar{m}_2\left(\frac{\Phi_0(z)}{2}+\frac{\Phi_1(z)}{3}\right) \\ &= z^2 + \frac{1-\bar{m}_1}{3}\left(2\Phi_1(z) - \mathbf{K}\begin{bmatrix} 1! \\ 2! \end{bmatrix}^T\begin{bmatrix} \frac{\bar{m}_1}{3}+1 & -\frac{2m_1}{3} \\ -\frac{2\bar{m}_1}{3} & \frac{m_1}{3}+1 \end{bmatrix}\begin{bmatrix} \frac{2m_1}{3}\Phi_1(z) \\ \bar{m}_1\left(\frac{\Phi_0(z)}{2}+\frac{\Phi_1(z)}{3}\right) \end{bmatrix}\right) \\ &- \bar{m}_2\left(\frac{\Phi_0(z)}{2}+\frac{\Phi_1(z)}{3} - \frac{2}{3}\mathbf{K}\begin{bmatrix} 1! \\ 1/2 \end{bmatrix}^T\begin{bmatrix} \frac{\bar{m}_1}{3}+1 & -\frac{2m_1}{3} \\ -\frac{2\bar{m}_1}{3} & \frac{m_1}{3}+1 \end{bmatrix}\begin{bmatrix} \frac{2m_1}{3}\Phi_1(z) \\ \bar{m}_1\left(\frac{\Phi_0(z)}{2}+\frac{\Phi_1(z)}{3}\right) \end{bmatrix}\right),\end{aligned}$$

since

$$\mathbb{K}_1^2(z,0) = 0,\ \Phi_2(z) = z^2 + \frac{2}{3}\Phi_1(z)$$

and

$$\mathbf{K} = \frac{1}{\det(\mathbf{I}_2 + \mathbf{L}_2^1 + \mathbf{L}_2^2)} = \frac{9}{|m_1+3|^2 - 4|m_1|^2}.$$

3. In general, the n-th degree polynomial is

$$\tilde{\Phi}_n(z) = \Phi_n(z) - \left(\frac{1-\bar{m}_1}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n! \end{bmatrix}^T - \frac{2\bar{m}_2}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n!(n-1)/2n \end{bmatrix}^T \right) \left[\left(\mathbf{I}_n + \sum_{r=1}^{2} \mathbf{L}_n^r \right)^{-1} \right]^T$$
$$\times \left(\sum_{r=1}^{2} \mathbb{K}_{n-1}^r(z,0) \right) - \frac{n\bar{m}_1}{n+1}\Phi_{n-1}(z) - \bar{m}_2(n-1)\left(\frac{\Phi_{n-2}(z)}{n} + \frac{\Phi_{n-1}(z)}{n+1} \right)$$
$$= z^n + \frac{1-\bar{m}_1}{n+1} \left(n\Phi_{n-1}(z) - \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n! \end{bmatrix}^T \left[(\mathbf{I}_n + \sum_{r=1}^{2} \mathbf{L}_n^r)^{-1} \right]^T \left(\sum_{r=1}^{2} \mathbb{K}_{n-1}^r(z,0) \right) \right)$$
$$- \bar{m}_2(n-1)\left(\frac{\Phi_{n-1}(z)}{n+1} + \frac{\Phi_{n-2}(z)}{n} \right) + \frac{2\bar{m}_2}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n!\frac{n-1}{2n} \end{bmatrix}^T$$
$$\left[\left(\mathbf{I}_n + \sum_{r=1}^{2} \mathbf{L}_n^r \right)^{-1} \right]^T \left(\sum_{r=1}^{2} \mathbb{K}_{n-1}^r(z,0) \right),$$

since $\Phi_n(z) = z^n + \frac{n}{n+1}\Phi_{n-1}(z)$.

On the other hand, assuming that the linear functional (2.17) is positive definite, the associated measure is

$$d\sigma(\theta) = |z-1|^2 \frac{d\theta}{2\pi} + 2\mathfrak{Re}(m_1 z)\frac{d\theta}{2\pi} + 2\mathfrak{Re}(m_2 z^2)\frac{d\theta}{2\pi},$$

and the corresponding moments are given by

$$\tilde{c}_n = \begin{cases} 2, & \text{if} \quad n = 0, \\ -1 + \bar{m}_1, & \text{if} \quad n = 1, \\ -1 + m_1, & \text{if} \quad n = -1, \\ \bar{m}_2, & \text{if} \quad n = 2, \\ m_2, & \text{if} \quad n = -2, \\ 0, & \text{in other case.} \end{cases}$$

Thus, the perturbed Toeplitz matrix is

$$\widetilde{\mathbf{T}} = \begin{bmatrix} 2 & -1+\bar{m}_1 & \bar{m}_2 & 0 & 0 & \cdots \\ -1+m_1 & 2 & -1+\bar{m}_1 & \bar{m}_2 & 0 & \cdots \\ m_2 & -1+m_1 & 2 & -1+\bar{m}_1 & \bar{m}_2 & \cdots \\ 0 & m_2 & -1+m_1 & 2 & -1+\bar{m}_1 & \cdots \\ 0 & 0 & m_2 & -1+m_1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

i.e., the first and second subdiagonals are perturbed. Furthermore, from (1.94)

$$Sz\left(\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}dx\right)=|z-1|^2\frac{d\theta}{2\pi},$$

then according to (2.6) the measure supported in $[-1,1]$ is

$$d\tilde{\mu}(x)=\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}dx+\frac{2}{\pi}m_1T_1(x)\frac{1}{\sqrt{1-x^2}}dx+\frac{2}{\pi}m_2T_2(x)\frac{1}{\sqrt{1-x^2}}dx. \tag{2.18}$$

Thus, from (2.7), the perturbed moments associated with the measure (2.18) are

$$\tilde{\mu}_n=\begin{cases}\mu_n, & \text{if} \quad n=0,\\ \mu_n+2m_1\frac{n!!}{(n+1)!!}, & \text{if} \quad n \text{ is odd},\\ \mu_n+2m_2\left(\frac{2(n+1)!!}{(n+2)!!}-\frac{(n-1)!!}{n!!}\right), & \text{if} \quad n \text{ is even},\end{cases}$$

where $(\mu_n)_{n\geq 0}$ are the moments associated with the measure $d\mu(x)=\frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}dx$, and

$$\mu_n=\frac{2}{\pi}\int_{-1}^{1}x^n\sqrt{\frac{1-x}{1+x}}dx=\frac{2}{\pi}\int_{-1}^{1}\frac{x^n-x^{n+1}}{\sqrt{1-x^2}}dx=\begin{cases}2, & \text{if} \quad n=0,\\ -2\frac{n!!}{(n+1)!!}, & \text{if} \quad n \text{ is odd},\\ 2\frac{(n-1)!!}{n!!}, & \text{if} \quad n \text{ is even}.\end{cases}$$

2.2 Matrix moment perturbations and the inverse Szegő matrix transformation

2.2.1 Introduction

The term *moment problem* was used for the first time in T. J. Stieltjes' classic memoir [165] (published posthumously between 1894 and 1895) dedicated to the study of continued fractions. The moment problem is a question in classical analysis that has produced a rich theory in applied and pure mathematics. This problem is beautifully connected to the theory of orthogonal polynomials, spectral representation of operators, matrix factorization problems, probability, statistics, prediction of stochastic processes, polynomial optimization, inverse problems in financial mathematics and function theory, among many other areas. In the matrix case, M. Krein was the first to consider this problem in [121], and later on the study of density questions related to the matrix moment problem were studied in [66, 67, 130, 131]. Recently, the theory of matrix moment problem is used in [50] for the analysis of random matrix-valued measures. In this section, we are interested in the study of some properties related with a perturbation of a sequence of matrix moments, within the framework of the theory of matrix orthogonal polynomials both on the real line and on the unit circle.

2.2.2 Matrix moments on the real line and the unit circle related through the inverse Szegő transformation

Let $\langle \cdot, \cdot \rangle_{R,\theta}$ be a right sesquilinear form defined by

$$\langle p, q \rangle_{R,\theta} = \int_{-\pi}^{\pi} p(e^{i\theta})^H q(e^{i\theta}) \frac{d\theta}{2\pi}, \quad p, q \in \mathbb{P}^l[z]. \tag{2.19}$$

Given a positive semi-definite matrix measure σ supported on $\mathbb{T}$, we can consider the perturbation

$$d\sigma_j(\theta) = d\sigma(\theta) + z^j m_j \frac{d\theta}{2\pi} + z^{-j} m_j^H \frac{d\theta}{2\pi}, \quad z = e^{i\theta}, \tag{2.20}$$

for a fixed j and $m_j \in \mathcal{M}_l$ such that σ_j is positive semi-definite. In particular, if σ is symmetric and m_j is a Hermitian matrix, then (2.20) becomes

$$d\sigma_j(\theta) = d\sigma(\theta) + 2\cos(j\theta) m_j \frac{d\theta}{2\pi}, \tag{2.21}$$

and thus σ_j is a symmetric, positive semi-definite matrix measure supported on unit circle.

Notice that the right sesquilinear form associated with (2.20) is

$$\langle p, q \rangle_{R,\sigma_j} = \langle p, q \rangle_{R,\sigma} + \langle m_j^H p, z^j q \rangle_{R,\theta} + \langle z^j m_j p, q. \rangle_{R,\theta}, \quad p, q \in \mathbb{P}^l[z]. \tag{2.22}$$

Notice that (2.20) and (2.22) are extensions to the matrix case of (2.3) and (2.2), respectively, with $\Omega = \{j\}$. From (2.22), one easily sees that the corresponding matrix moment $\tilde{c}_k$ satisfies

$$\tilde{c}_k = \begin{cases} c_k, & \text{if} \quad k \notin \{j, -j\}, \\ c_j + m_j^H, & \text{if} \quad k = j, \\ c_{-j} + m_j, & \text{if} \quad k = -j. \end{cases}$$

In other words, σ_j represents an additive perturbation of the matrix moments c_j and c_{-j} of σ.

The following result shows the matrix measure supported on $[-1, 1]$ that is related with (2.20) through the inverse Szegő matrix transformation. This result constitutes a generalization of the corresponding result on the scalar case studied in [77, 78]. The proof is analogous to the one given in Proposition 2.1.2 for matrix measure, with $\Omega = \{j\}$ and $m_j \in \mathcal{M}_l$ a Hermitian matrix.

Proposition 2.2.1. *Let σ be a positive semi-definite matrix measure supported on $\mathbb{T}$, and let $m_j \in \mathcal{M}_l$ be a Hermitian matrix such that σ_j defined as in (2.20) is positive semi-definite. Assume $d\mu = Sz^{-1}(d\sigma)$. Then, $d\mu_j = Sz^{-1}(d\sigma_j)$ is given by*

$$d\mu_j(x) = d\mu(x) + 2T_j(x) m_j \frac{dx}{\pi\sqrt{1-x^2}}, \quad j \geq 0, \tag{2.23}$$

where $T_j(x) := \cos(j\theta)$ is the $j-$th degree Chebyshev polynomial of the first kind on $\mathbb{R}$ (see [39]).

The next proposition is the matrix version of Proposition 2.1.3 with $\Omega = \{j\}$. We also omit the proof, since it is similar to the one of the scalar case. Recall that $n!!$ is the double factorial or semi-factorial of a number $n \in \mathbb{N}$ defined as in (1.21).

Proposition 2.2.2. *Let μ be a positive semi-definite matrix measure supported on $[-1,1]$, μ_j defined as in (2.23) and $(\mu_n)_{n\geq 0}$, $(\tilde{\mu}_n)_{n\geq 0}$ be their corresponding sequences of matrix moments. Then,*

$$\tilde{\mu}_n = \begin{cases} \mu_n + m_j \mathbb{B}(n,j), & \text{if } \ j \leq n \ \text{ and } \ n+j \text{ is even,} \\ \mu_n, & \text{otherwise,} \end{cases} \tag{2.24}$$

where

$$\mathbb{B}(n,j) = j \sum_{k=0}^{[j/2]} (-1)^k \frac{(j-k-1)!(2)^{j-2k}}{k!(j-2k)!} \frac{(j+n-2k-1)!!}{(j+n-2k)!!}.$$

This means that the perturbation of the matrix moments c_j and c_{-j} of a matrix measure σ supported on the unit circle results in a perturbation, defined by (2.24), of the moments μ_n associated with a measure μ supported in $[-1,1]$, when both measures are related through the inverse matrix Szegő transformation.

2.2.3 Connection formulas and further results

In the remaining of the section, we will use the following notation. For $0 \leq k \leq j-1$, let

$$\chi_k^{(j)} = [\mathbf{0} \cdots \mathbf{0}\ \mathbf{I}\ \mathbf{0} \cdots \mathbf{0}]^H \in \mathcal{M}_{jl\times l},$$

where $\mathbf{I}$ is in the k-th position, and define $\chi_k^{(j)} = [\mathbf{0}, \cdots, \mathbf{0}]^H$ for $j \geq k$ or $k < 0$. We also define

$$\Lambda_{n,m}^{(j+1)} := [\chi_n^{(j+1)}\ \chi_{n+1}^{(j+1)}\ \cdots\ \chi_m^{(j+1)}] \in \mathcal{M}_{(j+1)l\times(m-n+1)l},$$

with $n \leq m$, for $n, m \in \mathbb{Z}$.

2.2.3.1 Connection formulas on the real line and further results

Let $(\mu_j)_{j\geq 0}$ be a sequence of matrix moments defined as in (1.20)where μ is a Hermitian matrix measure and we denote by $H_j \in \mathcal{M}_{lj}$ and $F_j \in \mathcal{M}_{l(j+1)}$ the block matrices

$$H_j = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{j-1} \\ \mu_1 & \mu_2 & \cdots & \mu_j \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{j-1} & \mu_j & \cdots & \mu_{2j-2} \end{bmatrix},\ F_j = \begin{bmatrix} & & & \mu_j \\ & H_j & & \vdots \\ & & & \mu_{2j-1} \\ \mathbf{I} & x\mathbf{I} & \cdots & x^j\mathbf{I} \end{bmatrix},$$

with $x \in \mathbb{R}$ and $j \geq 1$, and the block vector

$$v_{j,2j-1} = [\mu_j\ \mu_{j+1}\ \cdots\ \mu_{2j-1}]^H \in \mathcal{M}_{lj\times l}.$$

Notice that H_j is a Hankel invertible and Hermitian matrix for $j \geq 1$. With these matrices, it is shown in [140] that the sequence of (right) matrix polynomials $(P_j^R(x))_{j\geq 0}$ defined by

$$P_j^R(x) = x^j\mathbf{I} - [\mathbf{I}\ x\mathbf{I}\ \cdots\ x^{j-1}\mathbf{I}]H_j^{-1}v_{j,2j-1},\ \text{wiht } P_0^R(x) = \mathbf{I}, \tag{2.25}$$

i.e. each P_j^R is defined as the Schur complement of $x^j\mathbf{I}$ in the matrix F_j, is orthogonal with respect the sesquilinear form (1.3), this is

$$\langle P_m^R, P_j^R\rangle_{R,\mu} = \delta_{m,j}s_j, \quad m = 0,\dots,j,$$

where s_j is the Schur complement of H_{j+1} with respect to μ_{2j}, i.e.

$$s_j = \mu_{2j} - v_{j,2j-1}^H H_j^{-1} v_{j,2j-1}, \text{ with } s_0 = \mu_0. \tag{2.26}$$

Conversely, any sequence of matrix moments $(\mu_j)_{j\geq 0}$ satisfying that the corresponding Hankel matrices are invertible and Hermitian defines a Hermitian matrix measure. In this situation, we will say $(\mu_j)_{j\geq 0}$ is a Hermitian sequence.

In a similar way, a sequence of left matrix polynomials $(P_n^L(x))_{n\geq 0}$ can be defined such that it is orthogonal with respect to sesquilinear form (1.4). Since $P_j^L(x) = P_j^R(x)^H$ for all $x \in \mathbb{R}$, it suffices to consider $P_j(x) = P_j^R(x)$.

Let $P_j^{(k)}(x)$ be the $k-$th derivative of $P_j(x)$. From (2.25) we can deduce

$$\frac{P_j^{(k)}(0)}{k!} = \begin{cases} -(\chi_k^{(j)})^H H_j^{-1} v_{j,2j-1}, & \text{if } 0 \leq k \leq j-1, \\ \mathbf{I}, & \text{if } k = j. \end{cases} \tag{2.27}$$

The kernel matrix defined as (1.15) can be expressed also in terms of the moments by (see [140])

$$K_j(y,x) = [\mathbf{I}\ x\mathbf{I}\ \cdots\ x^j\mathbf{I}]H_{j+1}^{-1}\begin{bmatrix} \mathbf{I} \\ y\mathbf{I} \\ \vdots \\ y^j\mathbf{I} \end{bmatrix}. \tag{2.28}$$

Denoting $K_j^{(i,r)}(y,x)$ the $i-$th derivative (resp. $r-$th) $K_j(y,x)$ with respect to the variable y (resp. x), we have from (2.28), for $0 \leq i \leq j$ and $0 \leq r \leq j$,

$$\begin{aligned} \frac{K_j^{(i,0)}(0,x)}{i!} &= [\mathbf{I}\ x\mathbf{I}\ \cdots\ x^j\mathbf{I}]H_{j+1}^{-1}\chi_i^{(j+1)}, \\ \frac{K_j^{(0,r)}(y,0)}{r!} &= (\chi_r^{(j+1)})^H H_{j+1}^{-1}\begin{bmatrix} \mathbf{I} \\ y\mathbf{I} \\ \vdots \\ y^j\mathbf{I} \end{bmatrix}, \\ \frac{K_j^{(i,r)}(0,0)}{i!r!} &= (\chi_r^{(j+1)})^H H_{j+1}^{-1}\chi_i^{(j+1)}. \end{aligned}$$

Let μ be a Hermitian matrix measure supported on $E \subseteq \mathbb{R}$ with an associated sequence of monic matrix orthogonal polynomials $(P_n(x))_{n\geq 0}$, and a Hermitian sequence of matrix moments $(\mu_n)_{n\geq 0}$. Define a new matrix moments sequence $(\tilde{\mu}_n)_{n\geq 0}$ by $\tilde{\mu}_n = \mu_n + \mathbf{M}_n \in \mathcal{M}_l$ with $\mathbf{M}_n$ Hermitian, in such a way that $(\tilde{\mu}_n)_{n\geq 0}$ is also a Hermitian sequence. Then, there exists an associated measure $\tilde{\mu}$ that is Hermitian, with a corresponding sequence of matrix orthogonal polynomials $(\tilde{P}_n(x))_{n\geq 0}$. Also, define

$$s_j = \langle P_j, P_j\rangle_{R,\mu} \text{ and } \tilde{s}_j = \langle \tilde{P}_j, \tilde{P}_j\rangle_{R,\tilde{\mu}}.$$

Now, we will show the relation between the polynomials matrix $\tilde{P}_j$ and P_j, as well as the relation between $\tilde{s}_j$ and s_j, in terms of the sequence of matrix moments $(\mu_n)_{n\geq 0}$. Notice that the perturbation of a matrix measure μ supported on $E=[-1,1]$ defined by (2.23) constitutes a particular case with

$$\mathbf{M}_n=\begin{cases} m_j\mathbb{B}(n,j), & \text{if} \quad j\leq n \ \text{ and } \ n+j \text{ is even},\\ \mathbf{0}, & \text{otherwise.}\end{cases}$$

We will use the following notation

$$\begin{aligned}
\mathbf{K}_{j-1}^{k}(0,x) &:= \left[K_{j-1}^{(k,0)}(0,x)\ K_{j-1}^{(k-1,0)}(0,x)\ \ldots\ K_{j-1}^{(0,0)}(0,x)\right]\in\mathcal{M}_{l\times(k+1)l},\\
\mathbf{P}_{j,k}(0) &:= [P_j^{(0)}(0)^T\ P_j^{(1)}(0)^T\ \ldots\ P_j^{(k)}(0)^T]^T\in\mathcal{M}_{(k+1)l\times l},\\
\tilde{\mathbf{P}}_{j,k}(0) &:= [\tilde{P}_j^{(0)}(0)^T\ \tilde{P}_j^{(1)}(0)^T\ \ldots\ \tilde{P}_j^{(k)}(0)^T]^T\in\mathcal{M}_{(k+1)l\times l},\\
\mathbf{D}_{0,k} &:= diag\{0!\mathbf{I}\ 1!\mathbf{I}\ \ldots\ k!\mathbf{I}\}\in\mathcal{M}_{(k+1)l},\\
\mathbf{D}_{k,0} &:= diag\{k!\mathbf{I}\ (k-1)!\mathbf{I}\ \ldots\ 0!\mathbf{I}\}\in\mathcal{M}_{(k+1)l},\\
\mathbf{D}_{\mathbf{M}_k} &:= diag\{\mathbf{M}_k\ \ldots\ \mathbf{M}_k\}\in\mathcal{M}_{(k+1)l}.
\end{aligned}\tag{2.29}$$

For a fixed k, with $0\leq k\leq 2j-1$, the authors proved in [40] that

$$\tilde{P}_j(x)=P_j(x)-\sum_{p=0}^{k}\frac{K_{j-1}^{(k-p,0)}(0,x)}{(k-p)!}\mathbf{M}_k\frac{\tilde{P}_j^{(p)}(0)}{p!}=P_j(x)-\mathbf{K}_{j-1}^{k}(0,x)\mathbf{D}_{k,0}^{-1}\mathbf{D}_{\mathbf{M}_k}\mathbf{D}_{0,k}^{-1}\tilde{\mathbf{P}}_{j,k}(0)$$

with

$$\tilde{\mathbf{P}}_{j,k}(0)=\left[\mathbf{I}_{(k+1)l}+\mathbf{D}_{0,k}(\Lambda_{0,k}^{(j)})^H H_j^{-1}\Lambda_{0,k}^{(j)}\mathbf{D}_{k,0}^{-1}\mathbf{D}_{\mathbf{M}_k}\Lambda_{k,0}^{(k+1)}\right]^{-1}\mathbf{P}_{j,k}(0).\tag{2.30}$$

As a consequence we have the next result.

Theorem 2.2.3. [40]. *Let μ and $\tilde{\mu}$ be Hermitian matrix measures associated with the Hermitian sequences $(\mu_k)_{k=0}^{2n-1}$ and $(\tilde{\mu}_k)_{k=0}^{2n-1}$, where $\tilde{\mu}_k=\mu_k+\mathbf{M}_k$, with $0\leq k\leq 2n-1$. For $1\leq j\leq n$, $(\tilde{P}_j(x))_{j=0}^{n}$ is orthogonal with respect to $\tilde{\mu}$ if*

$$\tilde{P}_j(x)=P_j(x)-\sum_{k=0}^{2n-1}\mathbf{K}_{j-1}^{k}(0,x)\mathbb{M}(k,j)\mathbf{P}_{j,k}(0),\ \textit{with}\ \tilde{P}_0(x)=P_0(x)=\mathbf{I},$$

where

$$\mathbb{M}(k,j)=\mathbf{D}_{k,0}^{-1}\mathbf{D}_{\mathbf{M}_k}\mathbf{D}_{0,k}^{-1}\left[\mathbf{I}_{(k+1)l}+\mathbf{D}_{0,k}(\Lambda_{0,k}^{(j)})^H H_j^{-1}\Lambda_{0,k}^{(j)}\mathbf{D}_{k,0}^{-1}\mathbf{D}_{\mathbf{M}_k}\Lambda_{k,0}^{(k+1)}\right]^{-1}.\tag{2.31}$$

Now, we will deduce a relation for the corresponding *norms*. We will need the following auxiliary results. For $0 \leq j \leq n$, let $\tilde{\mathbf{M}}_j \in \mathcal{M}_{jl}$ be defined as

$$\tilde{\mathbf{M}}_j := \begin{bmatrix} \mathbf{M}_0 & \cdots & \mathbf{M}_k & \cdots & \mathbf{M}_{j-1} \\ \vdots & \mathbf{M}_k & & & \vdots \\ \mathbf{M}_k & & \vdots & & \mathbf{M}_{2j-2-k} \\ \vdots & & & \mathbf{M}_{2j-2-k} & \vdots \\ \mathbf{M}_{j-1} & \cdots & \mathbf{M}_{2j-2-k} & \cdots & \mathbf{M}_{2j-2} \end{bmatrix} = \sum_{k=0}^{j-1} \mathbf{M}_{j,k} + \sum_{k=j}^{2j-2} \mathbf{M}_{j,k}^t,$$

where for $k = 0, \ldots, j-1$, (resp. $k = j, \ldots, 2j-2$,) the matrices $\mathbf{M}_{j,k} \in \mathcal{M}_{jl}$ (resp. $\mathbf{M}_{j,k}^t \in \mathcal{M}_{jl}$) are given by

$$\mathbf{M}_{j,k} := \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{M}_k & \cdots & \mathbf{0} \\ \vdots & \mathbf{M}_k & & & \vdots \\ \mathbf{M}_k & & \vdots & & \vdots \\ \vdots & & & \ddots & \\ \mathbf{0} & \cdots & & \cdots & \mathbf{0} \end{bmatrix}, \ \mathbf{M}_{j,k}^t := \begin{bmatrix} \mathbf{0} & \cdots & & \cdots & \mathbf{0} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \vdots & & \mathbf{M}_k \\ \vdots & & & \mathbf{M}_k & \vdots \\ \mathbf{0} & \cdots & \mathbf{M}_k & \cdots & \mathbf{0} \end{bmatrix},$$

and for $k \geq 2j-1$, $\mathbf{M}_{j,k} = \mathbf{M}_{j,k}^t = \mathbf{0} \in \mathcal{M}_{jl}$.

Lemma 2.2.4. *Let*

$$\tilde{H}_{j,k} = \begin{cases} H_j + \mathbf{M}_{j,k} & \text{if} \quad k = 0, \ldots, j-1, \\ H_j + \mathbf{M}_{j,k}^t & \text{if} \quad k = j, \ldots, 2j-2. \end{cases} \tag{2.32}$$

1. *If $k = 0, \ldots, j-1$, then*

$$\tilde{H}_{j,k}^{-1} - H_j^{-1} = -\sum_{p=0}^{k} H_j^{-1} \chi_{k-p}^{(j)} \mathbf{M}_k (\chi_p^{(j)})^H \tilde{H}_{j,k}^{-1}. \tag{2.33}$$

2. *If $k = j, \ldots, 2j-2$, then*

$$\tilde{H}_{j,k}^{-1} - H_j^{-1} = -\sum_{p=k-(j-1)}^{j-1} H_j^{-1} \chi_{k-p}^{(j)} \mathbf{M}_k (\chi_p^{(j)})^H \tilde{H}_{j,k}^{-1}. \tag{2.34}$$

Proof. For $k = 0, \ldots, j-1$, we have

$$\tilde{H}_{j,k}^{-1} - H_j^{-1} = H_j^{-1}(H_j - \tilde{H}_{j,k})\tilde{H}_{j,k}^{-1} = -H_j^{-1}\mathbf{M}_{j,k}\tilde{H}_{j,k}^{-1}, \tag{2.35}$$

and thus

$$\mathbf{M}_{j,k} = \chi_k^{(j)} \mathbf{M}_k (\chi_0^{(j)})^H + \cdots + \chi_0^{(j)} \mathbf{M}_k (\chi_k^{(j)})^H = \sum_{p=0}^{k} \chi_{k-p}^{(j)} \mathbf{M}_k (\chi_p^{(j)})^H.$$

Substituting $\mathbf{M}_{j,k}$ in (2.35), (2.33) follows. A similar procedure can be used to prove (2.34) □

Let $\tilde{H}_j \in \mathcal{M}_{lj}$ be the block Hankel matrix associated with the perturbed matrix moments sequence, i.e.

$$\tilde{H}_j = \begin{bmatrix} \tilde{\mu_0} & \tilde{\mu_1} & \cdots & \tilde{\mu_{j-1}} \\ \tilde{\mu_1} & \tilde{\mu_2} & \cdots & \tilde{\mu_j} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\mu_{j-1}} & \tilde{\mu_j} & \cdots & \tilde{\mu_{2j-2}} \end{bmatrix} = H_j + \tilde{\mathbf{M}}_j.$$

Thus, from (2.32) for $1 \leq j \leq n$,

$$\tilde{H}_j = \tilde{H}_{j,0} + \cdots + \tilde{H}_{j,2j-2}. \tag{2.36}$$

Theorem 2.2.5. *Let μ be a Hermitian matrix measure with matrix moments $(\mu_k)_{k=0}^{2n-1}$. Define a new Hermitian sequence of matrix moments $(\tilde{\mu}_k)_{k=0}^{2n-1}$ by $\tilde{\mu}_k = \mu_k + \mathbf{M}_k$ with $0 \leq k \leq 2n-1$, and denote by $\tilde{\mu}$ its corresponding Hermitian matrix measure. Then, for $1 \leq j \leq n$, we have*

$$\tilde{s}_j = s_j + \sum_{k=0}^{2n-1} \mathbf{P}_{j,k}(0)^H \Lambda_{k,0}^{(k+1)} \mathbb{M}(k,j) \mathbf{P}_{j,k}(0), \tag{2.37}$$

with $\tilde{s}_0 = \tilde{\mu}_0 = \mu_0 + \mathbf{M}_0$ and $\mathbb{M}(k,j)$ as in (2.31).

Proof. Let us assume first that only the $k-$th moment is perturbed. From (2.26) we have that if $2j < k$, then $\tilde{s}_j = s_j$. We will show that if $k \leq 2j$, we have

$$\tilde{s}_j = s_j + \sum_{p=0}^{k} \frac{(P_j^{(k-p)}(0))^H}{(k-p)!} \mathbf{M}_k \frac{\tilde{P}_j^{(p)}(0)}{p!}. \tag{2.38}$$

For $0 \leq k \leq j-1$, notice that $\tilde{\mu}_{2j} = \mu_{2j}$ and $\tilde{v}_{j,2j-1} = v_{j,2j-1}$, and thus from (2.26) we get

$$\tilde{s}_j = s_j - \mu_{2j} + v_{j,2j-1}^H H_j^{-1} v_{j,2j-1} + \tilde{\mu}_{2j} - \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1} \tilde{v}_{j,2j-1} = s_j - v_{j,2j-1}^H (\tilde{H}_{j,k}^{-1} - H_j^{-1}) \tilde{v}_{j,2j-1}.$$

From (2.33) and using (2.27), we deduce (2.38).
On the other hand, for $j \leq k \leq 2j-1$ we have $\tilde{\mu}_{2j} = \mu_{2j}$ and $\tilde{v}_{j,2j-1} = v_{j,2j-1} + \chi_{k-j}^{(j)} \mathbf{M}_k$. From (2.26) and (2.27), proceeding as above, we get

$$\begin{aligned} \tilde{s}_j &= s_j - \mu_{2j} + v_{j,2j-1}^H H_j^{-1} v_{j,2j-1} + \tilde{\mu}_{2j} - \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1} \tilde{v}_{j,2j-1} \\ &= s_j - v_{j,2j-1}^H (\tilde{H}_{j,k}^{-1} - H_j^{-1}) \tilde{v}_{j,2j-1} + \frac{(P_j^{(k-j)}(0))^H}{(k-j)!} \mathbf{M}_k + \mathbf{M}_k \frac{\tilde{P}_j^{(k-j)}(0)}{(k-j)!}, \end{aligned}$$

and from (2.34) and using (2.27)

$$\tilde{s}_j = s_j + \sum_{p=k-(j-1)}^{j-1} \frac{(P_j^{(k-p)}(0))^H}{(k-p)!}\mathbf{M}_k\frac{\tilde{P}_j^{(p)}(0)}{p!} + \frac{(P_j^{(k-j)}(0))^H}{(k-j)!}\mathbf{M}_k\frac{\tilde{P}_j^{(j)}(0)}{j!} + \frac{(P_j^{(j)}(0))^H}{j!}\mathbf{M}_k\frac{\tilde{P}_j^{(k-j)}(0)}{(k-j)!}$$
$$= s_j + \sum_{p=k-j}^{j} \frac{(P_j^{(k-p)}(0))^H}{(k-p)!}\mathbf{M}_k\frac{\tilde{P}_j^{(p)}(0)}{p!}.$$

Since

$$\begin{cases} \tilde{P}_j^{(p)}(0) = \mathbf{0}, & \text{if } j < p, \\ P_j^{(k-p)}(0) = \mathbf{0}, & \text{if } p < k-j, \end{cases}$$

we get (2.38).
Finally, for $k = 2j$ (notice that if $j = n$ the result is also valid with $\mathbf{M}_{2j} = \mathbf{0}$),

$$\tilde{\mu}_{2j} = \mu_{2j} + \mathbf{M}_{2j} \quad \text{and} \quad \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1}\tilde{v}_{j,2j-1} = v_{j,2j-1}^H H_j^{-1} v_{j,2j-1},$$

and thus from (2.26)

$$\begin{aligned} \tilde{s}_j &= s_j - \mu_{2j} + v_{j,2j-1}^H H_j^{-1} v_{j,2j-1} + \tilde{\mu}_{2j} - \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1}\tilde{v}_{j,2j-1} \\ &= s_j - \mu_{2j} + \tilde{\mu}_{2j} = s_j + \mathbf{M}_{2j} = s_j + \sum_{p=0}^{k} \frac{(P_j^{(k-p)}(0))^H}{(k-p)!}\mathbf{M}_k\frac{\tilde{P}_j^{(p)}(0)}{p!}. \end{aligned}$$

On the other hand, using the notation given in (2.29), (2.38) can be written as

$$\tilde{s}_j = s_j + [P_j^{(k)}(0)^H \; P_j^{(k-1)}(0)^H \; \cdots \; P_j^{(0)}(0)^H]\mathbf{D}_{k,0}^{-1}\mathbf{D}_{\mathbf{M}_k}\mathbf{D}_{0,k}^{-1}\tilde{\mathbf{P}}_{j,k}(0),$$

that, using (2.30) and (2.31), becomes

$$\tilde{s}_j = s_j + \mathbf{P}_{j,k}(0)^H \Lambda_{k,0}^{(k+1)} \mathbb{M}(k,j)\mathbf{P}_{j,k}(0).$$

The general case, i.e. when all moments are perturbed ($k = 0, \ldots, 2n-1$) is easy to deduce, since we have (2.36) and

$$\tilde{v}_{j,2j-1} = v_{j,2j-1} + \sum_{k=0}^{2j-1} \chi_{k-j}^{(j)}\mathbf{M}_k,$$

so (2.37) follows. □

2.2.3.2 Connection formulas for the unit circle and further results

Let $(c_j)_{j\geq 0}$ be a sequence of matrix moments defined as in (1.71) where σ is a Hermitian matrix measure with $c_0 + c_0^H := \int_{-\pi}^{\pi} d\sigma(\theta)$ and let $T_j \in \mathcal{M}_{lj}$ and $G_j \in \mathcal{M}_{l(j+1)}$ be defined

as

$$T_j = \begin{bmatrix} c_0^H + c_0 & c_1 & \cdots & c_{j-1} \\ c_1^H & c_0^H + c_0 & \cdots & c_{j-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{j-1}^H & c_{j-2}^H & \cdots & c_0^H + c_0 \end{bmatrix}, \; G_j = \begin{bmatrix} & & & c_j \\ & T_j & & \vdots \\ & & & c_1 \\ \mathbf{I} & z\mathbf{I} & \cdots & z^j\mathbf{I} \end{bmatrix},$$

and the vector block

$$\nu_j = \begin{bmatrix} c_j^H & c_{j-1}^H & \cdots & c_1^H \end{bmatrix}^H \in \mathcal{M}_{lj\times l}.$$

Notice that T_j is a Toeplitz invertible and Hermitian matrix for $j \geq 1$. As in the real line case, any sequence $(c_j)_{j\in\mathbb{Z}}$ such that T_j is invertible and Hermitian for $j \geq 1$ will be called a Hermitian sequence, and has an associated Hermitian matrix measure σ supported on $\mathbb{T}$. The sequence of monic matrix polynomials $(\Phi_j^R(z))_{j\geq 0}$ defined by (see [141])

$$\Phi_j^R(z) = z^j\mathbf{I} - [\mathbf{I} \; z\mathbf{I} \; \cdots \; z^{j-1}\mathbf{I}]T_j^{-1}\nu_j, \text{ with } \Phi_0^R(z) = \mathbf{I}, \tag{2.39}$$

i.e. each Φ_j^R is defined as the Schur complement of $z^j\mathbf{I}$ in the matrix G_j, is orthogonal with respect to the sesquilinear form (1.23) and satisfies

$$\langle \Phi_m^R, \Phi_j^R \rangle_{R,\sigma} = \delta_{m,j} S_j^R, \quad m = 0, \ldots, j,$$

where the Hermitian matrix S_j^R is the Schur complement of T_{j+1} with respect to $c_0 + c_0^H$, i.e.,

$$S_j^R = c_0^H + c_0 - \nu_j^H T_j^{-1}\nu_j, \text{ with } S_0^R = c_0 + c_0^H. \tag{2.40}$$

In a similar way, a sequence $(\Phi_n^L(z))_{n\geq 0}$ can be constructed such that it is orthogonal with respect to the sesquilinear form (1.24). Henceforth we will consider only the right matrix polynomials, denoted by $\Phi_j = \Phi_j^R$ and the corresponding *norms* $S_j = S_j^R$ for $j \geq 0$, since the results for the left polynomials are analogous.

Let $\Phi_j^{(k)}(x)$ be the $k-$th derivative of $\Phi_j(x)$, from (2.39) we can deduce

$$\frac{\Phi_j^{(k)}(0)}{k!} = \begin{cases} -(\chi_k^{(j)})^H T_j^{-1}\nu_j, & \text{if } 0 \leq k \leq j-1, \\ \mathbf{I}, & \text{if } k = j. \end{cases} \tag{2.41}$$

The kernel matrix polynomial defined as (1.55) can be expressed also in terms of the moments by (see [141])

$$K_j(w,z) = [\mathbf{I} \; z\mathbf{I} \; \cdots \; z^j\mathbf{I}]T_{j+1}^{-1} \begin{bmatrix} \mathbf{I} \\ \bar{w}\mathbf{I} \\ \vdots \\ \bar{w}^j\mathbf{I} \end{bmatrix}. \tag{2.42}$$

As in the real case, $K_j^{(i,r)}(w,z)$ is the $i-$th derivative (resp. $r-$th) $K_j(w,z)$ with respect to the variable w (resp. z). Thus, from (2.42), for $0 \leq i \leq j$ and $0 \leq r \leq j$, we have

$$\begin{aligned}
\frac{K_j^{(i,0)}(0,z)}{i!} &= [\mathbf{I}\ x\mathbf{I}\ \cdots\ z^j\mathbf{I}]T_{j+1}^{-1}\chi_i^{(j+1)}, \\
\frac{K_j^{(0,r)}(w,0)}{r!} &= (\chi_r^{(j+1)})^H T_{j+1}^{-1}\begin{bmatrix}\mathbf{I}\\ \bar{w}\mathbf{I}\\ \vdots \\ \bar{w}^j\mathbf{I}\end{bmatrix}, \\
\frac{K_j^{(i,r)}(0,0)}{i!r!} &= (\chi_r^{(j+1)})^H T_{j+1}^{-1}\chi_i^{(j+1)}.
\end{aligned}$$

Let σ be a Hermitian matrix measure supported on $\mathbb{T}$, with associated monic **MOPUC** $(\Phi_n(z))_{n\geq 0}$, and matrix moments $(c_n)_{n\geq 0}$. We define a new Hermitian sequence of matrix moments by $(\tilde{c}_n)_{n\geq 0}$ where $\tilde{c}_n = c_n + m_n \in \mathcal{M}_l$ for $n \geq 0$, and denote its corresponding Hermitian matrix measure and monic **MOPUC** by $\tilde{\sigma}$ and $(\tilde{\Phi}_n(z))_{n\geq 0}$, respectively. Furthermore, we define

$$S_j = \langle \Phi_j, \Phi_j\rangle_{R,\sigma} \text{ and } \tilde{S}_j = \langle \tilde{\Phi}_j, \tilde{\Phi}_j\rangle_{R,\tilde{\sigma}}.$$

In this subsection, in a similar way as in the real case, we will show the relation between the matrix polynomials $\tilde{\Phi}_j$ and Φ_j, as well as the relation between $\tilde{S}_j$ and S_j, in terms of the matrix moments $(c_n)_{n\geq 0}$. Notice that (2.20) constitutes a particular case, when only one of the moments is perturbed. Let us define the following notation

$$\begin{aligned}
\mathbb{K}_{j-1}^k(0,z) &:= [K_{j-1}^{(0,0)}(0,z)\ K_{j-1}^{(1,0)}(0,z)\ \cdots\ K_{j-1}^{(k,0)}(0,z)] \in \mathcal{M}_{l\times(k+1)l}, \\
\Upsilon_j^k(0) &:= [\Phi_j^{(0)}(0)^T\ \Phi_j^{(1)}(0)^T\ \cdots\ \Phi_j^{(k)}(0)^T]^T \in \mathcal{M}_{(k+1)l\times l}, \\
\tilde{\Upsilon}_j^k(0) &:= [\tilde{\Phi}_j^{(0)}(0)^T\ \tilde{\Phi}_j^{(1)}(0)^T\ \cdots\ \tilde{\Phi}_j^{(k)}(0)^T]^T \in \mathcal{M}_{(k+1)l\times l}, \\
\mathbf{D}_{m_k}^{j+1} &:= diag\{m_k\ \cdots\ m_k\} \in \mathcal{M}_{(j+1)l}, \\
\mathbf{D}_{k,j+k} &:= diag\{k!\mathbf{I}\ \cdots\ (j+k)!\mathbf{I}\} \in \mathcal{M}_{(j+1)l}.
\end{aligned} \tag{2.43}$$

As in the real line case, for a fixed k, with $0 \leq k \leq j-1$, in [41] the authors proved that

$$\begin{aligned}
\tilde{\Phi}_j(z) &= \Phi_j(z) - \sum_{p=0}^{j-k}\frac{K_{j-1}^{(p,0)}(0,z)}{p!}m_k\frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} - \sum_{p=0}^{j-k-1}\frac{K_{j-1}^{(p+k,0)}(0,z)}{(p+k)!}m_k^H\frac{\tilde{\Phi}_j^{(p)}(0)}{p!} \\
&= \Phi_j(z) - \mathbb{K}_{j-1}^j(0,z)\left(\mathbf{D}_{0,j}^{-1}\mathbf{D}_{k,j+k}^{-1}\Lambda_{-k,j-k}^{(j+1)}\mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)}\mathbf{D}_{0,j}^{-1}\mathbf{D}_{k,j+k}^{-1}\mathbf{D}_{m_k^H}^{j+1}\right)\tilde{\Upsilon}_j^j(0)
\end{aligned}$$

with

$$\tilde{\Upsilon}_j^j(0) = \left(\mathbf{I}_{(j+1)l} + \mathbf{D}_{0,j}(\Lambda_{0,j}^{(j)})^H T_j^{-1}\Lambda_{0,j}^{(j)}\left(\mathbf{D}_{k,j+k}^{-1}\Lambda_{-k,j-k}^{(j+1)}\mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)}\mathbf{D}_{0,j}^{-1}\mathbf{D}_{m_k^H}^{j+1}\right)\right)^{-1}\Upsilon_j^j(0) \tag{2.44}$$

As a consequence, we have the following result.

Theorem 2.2.6. [41]. *Let σ be a Hermitian matrix measure supported on $\mathbb{T}$ with matrix moments $(c_k)_{k=0}^n$. Define a new Hermitian sequence of matrix moments $(\tilde{c}_k)_{k=0}^n$ by $\tilde{c}_k = c_k + m_k$, for $0 \leq k \leq n$. Then, if $(\Phi_j(z))_{j=0}^n$ and $(\tilde{\Phi}_j(z))_{j=0}^n$ are the corresponding monic*

MOPUC, *we have*

$$\tilde{\Phi}_j(z) = \Phi_j(z) - \sum_{k=0}^{j} \mathbb{K}_{j-1}^{j}(0,z)\mathbb{N}(k,j)\Upsilon_j^j(0), \text{ with } \tilde{\Phi}_0(x) = \Phi_0(x) = \mathbf{I},$$

where

$$\begin{aligned}\mathbb{N}(k,j) = &\left(\mathbf{D}_{0,j}^{-1}\mathbf{D}_{k,j+k}^{-1}\Lambda_{-k,j-k}^{(j+1)}\mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)}\mathbf{D}_{0,j}^{-1}\mathbf{D}_{k,j+k}^{-1}\mathbf{D}_{m_k^H}^{j+1}\right)\\ &\left(\mathbf{I}_{(j+1)l} + \mathbf{D}_{0,j}(\Lambda_{0,j}^{(j)})^H T_j^{-1}\Lambda_{0,j}^{(j)}\left(\mathbf{D}_{k,j+k}^{-1}\Lambda_{-k,j-k}^{(j+1)}\mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)}\mathbf{D}_{0,j}^{-1}\mathbf{D}_{m_k^H}^{j+1}\right)\right)^{-1}.\end{aligned} \tag{2.45}$$

The above theorem is a generalization to the matrix case of what was studied in [31, 78]. The particular case of a perturbation of a single matrix moment given by (2.20) is shown in the following corollary.

Corollary 2.2.7. *Let $k \in \mathbb{N}$ (fixed) with $0 \leq k \leq n$, and let σ be a Hermitian matrix measure supported on $\mathbb{T}$. Define σ_k as a perturbation of σ given by (2.20). Then,*

$$\begin{aligned}\tilde{\Phi}_j(z) &= \Phi_j(z), \quad 0 \leq j < k,\\ \tilde{\Phi}_j(z) &= \Phi_j(z) - \mathbb{K}_{j-1}^{j}(0,z)\mathbb{N}(k,j)\Upsilon_j^j(0), \quad k \leq j \leq n,\end{aligned}$$

with $\mathbb{N}(k,j)$ as in (2.45).

On the other hand, for $1 \leq j \leq n$, let $\tilde{\mathbf{N}}_j \in \mathcal{M}_{jl}$ be defined as

$$\tilde{\mathbf{N}}_j := \begin{bmatrix} m_0^H + m_0 & \cdots & m_k & \cdots & m_{j-1} \\ \vdots & \ddots & & \ddots & \\ m_k^H & & m_0^H + m_0 & & m_k \\ \vdots & \ddots & & \ddots & \\ m_{j-1}^H & \cdots & m_k^H & \cdots & m_0^H + m_0 \end{bmatrix} = \mathbf{N}_{j,0} + \cdots + \mathbf{N}_{j,j-1},$$

where for $k = 0, ..., j-1$, we have

$$\mathbf{N}_{j,k} := \begin{bmatrix} \mathbf{0} & \cdots & m_k & \cdots & \mathbf{0} \\ \vdots & \ddots & & \ddots & \vdots \\ \mathbf{0} & & \vdots & & m_k \\ \vdots & & & \ddots & \\ \mathbf{0} & \cdots & & \cdots & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & & & \vdots \\ m_k^H & & \vdots & & \vdots \\ \vdots & \ddots & & \ddots & \\ \mathbf{0} & \cdots & m_k^H & \cdots & \mathbf{0} \end{bmatrix},$$

and for $k \geq j$, $\mathbf{N}_{j,k} = \mathbf{0} \in \mathcal{M}_{lj}$.

Lemma 2.2.8. *For $1 \leq j \leq n$ and $0 \leq k \leq j-1$, define*

$$\tilde{T}_{j,k} = T_j + \mathbf{N}_{j,k}. \tag{2.46}$$

Then,

$$T_j^{-1} - \tilde{T}_{j,k}^{-1} = \sum_{p=0}^{j-k-1} T_j^{-1}\chi_p^{(j)} m_k (\chi_{p+k}^{(j)})^H \tilde{T}_{j,k}^{-1} + \sum_{p=0}^{j-k-1} T_j^{-1}\chi_{p+k}^{(j)} m_k^H (\chi_p^{(j)})^H \tilde{T}_{j,k}^{-1}. \tag{2.47}$$

Proof. We have

$$\tilde{T}_{j,k}^{-1} - T_j^{-1} = T_j^{-1}(T_j - \tilde{T}_{j,k})\tilde{T}_{j,k}^{-1} = -T_j^{-1}\mathbf{N}_{j,k}\tilde{T}_{j,k}^{-1}. \tag{2.48}$$

Notice that

$$\mathbf{N}_{j,k} = \sum_{p=0}^{j-k-1} \chi_p^{(j)} m_k (\chi_{p+k}^{(j)})^H + \sum_{p=0}^{j-k-1} \chi_{p+k}^{(j)} m_k^H (\chi_p^{(j)})^H,$$

and thus substituting $\mathbf{N}_{j,k}$ in (2.48), (2.47) follows. □

Let $\tilde{T}_j \in \mathcal{M}_{lj}$ be the block matrix Toeplitz with every moment perturbed, i.e.

$$\tilde{T}_j = \begin{bmatrix} \tilde{c}_0^H + \tilde{c}_0 & \tilde{c}_1 & \cdots & \tilde{c}_{j-1} \\ \tilde{c}_1^H & \tilde{c}_0^H + \tilde{c}_0 & \cdots & \tilde{c}_{j-2} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{c}_{j-1}^H & \tilde{c}_{j-2}^H & \cdots & \tilde{c}_0^H + \tilde{c}_0 \end{bmatrix} = T_j + \tilde{\mathbf{N}}_j,$$

then, from (2.46), for $1 \leq j \leq n$, we get

$$\tilde{T}_j = \tilde{T}_{j,0} + \cdots + \tilde{T}_{j,j-1}. \tag{2.49}$$

Theorem 2.2.9. *Let σ be a Hermitian matrix measure supported on $\mathbb{T}$ with matrix moments $(c_k)_{k=0}^n$. Define a Hermitian sequence $(\tilde{c}_k)_{k=0}^n$ with $\tilde{c}_k = c_k + m_k$, for $0 \leq k \leq n$, and denote by $\tilde{\sigma}$ the associated Hermitian matrix measure. Then, for $0 \leq j \leq n$, we have*

$$\tilde{S}_j = S_j + \sum_{k=0}^{j} \Upsilon_j^j(0)^H \mathbb{N}(k,j) \Upsilon_j^j(0), \tag{2.50}$$

with $\tilde{S}_0 = \tilde{c}_0^H + \tilde{c}_0 = c_0 + m_0 + (c_0 + m_0)^H$ and $\mathbb{N}(k,j)$ as in (2.45).

Proof. Let us first assume that only a single k-th matrix moment is perturbed. From (2.40), we have that if $j < k$, then $\tilde{S}_j = S_j$. Let us show that for $0 \leq k \leq j$,

$$\tilde{S}_j = S_j + \sum_{p=0}^{j-k} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} + \sum_{p=0}^{j-k} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!}. \tag{2.51}$$

For $k = 0$, $\tilde{\nu}_j = \nu_j$, and from (2.40) we get

$$\begin{aligned} \tilde{S}_j &= \tilde{c}_0^H + \tilde{c}_0 - \tilde{\nu}_j^H \tilde{T}_{j,k}^{-1} \tilde{\nu}_j = S_j - c_0^H - c_0 + \nu_j^H T_j^{-1} \nu_j + \tilde{c}_0^H + \tilde{c}_0 - \tilde{\nu}_j^H \tilde{T}_{j,k}^{-1} \tilde{\nu}_j \\ &= S_j + m_0^H + m_0 - \nu_j (\tilde{T}_{j,k}^{-1} - T_j^{-1}) \tilde{\nu}_j. \end{aligned}$$

By using (2.41) and (2.47), we obtain

$$\tilde{S}_j = S_j + m_0^H + m_0 + \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} + \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!},$$

and since for $k=0$ we have

$$m_0 = \frac{(\Phi_j^{(j)}(0))^H}{j!} m_k \frac{\tilde{\Phi}_{j-k+k}^{(j)}(0)}{(j-k+k)!}, \quad m_0^H = \frac{(\Phi_j^{(j-k+k)}(0))^H}{(j-k+k)!} m_k^H \frac{\tilde{\Phi}_j^{(j)}(0)}{j!},$$

we deduce (2.51).
For $1 \le k \le j$, $\tilde{\nu}_j = \nu_j + \chi_{j-k}^{(j)} m_k$, and proceeding as above we get

$$\tilde{S}_j = S_j - \nu_j^H (\tilde{T}_{j,k}^{-1} - T_j^{-1}) \tilde{\nu}_j - \nu_j^H T_j^{-1} \chi_{j-k}^{(j)} m_k - m_k^H (\chi_{j-k}^{(j)})^H \tilde{T}_{j,k}^{-1} \tilde{\nu}_j.$$

By using (2.41) and (2.47), we obtain

$$\begin{aligned}\tilde{S}_j = S_j &+ \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} + \frac{(\Phi_j^{(j-k)}(0))^H}{(j-k)!} m_k \frac{\tilde{\Phi}_j^{(j)}(0)}{j!} \\ &+ \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} + \frac{(\Phi_j^{(j)}(0))^H}{j!} m_k^H \frac{\tilde{\Phi}_j^{(j-k)}(0)}{(j-k)!},\end{aligned}$$

so (2.51) follows.

On the other hand, using (2.43), the first sum of (2.51) can be written as

$$\begin{aligned}\sum_{p=0}^{j-k} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} &= \sum_{p=k}^{j} \frac{(\Phi_j^{(p-k)}(0))^H}{(p-k)!} m_k \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} = \sum_{p=0}^{j} \frac{(\Phi_j^{(p-k)}(0))^H}{(p-k)!} m_k \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} \\ &= \Upsilon_j^j(0)^H \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} \tilde{\Upsilon}_j^j(0),\end{aligned}$$

and in a similar way, the second sum of (2.51) becomes

$$\sum_{p=0}^{j-k} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} = \Upsilon_j^j(0)^H \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \mathbf{D}_{m_k^H}^{j+1} \tilde{\Upsilon}_j^j(0).$$

As a consequence, (2.51) becomes

$$\tilde{S}_j = S_j + \Upsilon_j^j(0)^H \left(\mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \mathbf{D}_{m_k^H}^{j+1} \right) \tilde{\Upsilon}_j^j(0),$$

that, using (2.44) and (2.45), can be written as

$$\tilde{S}_j = S_j + \Upsilon_j^j(0)^H \mathbb{N}(k,j) \Upsilon_j^j(0).$$

The general case, when all moments are perturbed, follows immediately taking into account (2.49) and

$$\tilde{\nu}_j = \nu_j + \sum_{k=1}^{j} \chi_{j-k}^{(j)} m_k,$$

so (2.50) follows. □

For the particular case of (2.20), we have

Corollary 2.2.10. *Let $k \in \mathbb{N}$ (fixed) with $0 \leq k \leq n$, and let σ be a Hermitian matrix measure supported on $\mathbb{T}$. Let σ_k be a perturbation of σ defined by (2.20). Then, for $0 \leq j \leq n$,*

$$\begin{aligned} \tilde{S}_j &= S_j, \quad 0 \leq j < k, \\ \tilde{S}_j &= S_j + \Upsilon_j^j(0)^H \mathbb{N}(k,j) \Upsilon_j^j(0), \quad k \leq j \leq n, \end{aligned}$$

with $\mathbb{N}(k,j)$ as in (2.45).

2.2.4 Example

As an illustrative example of the results of the above subsections, consider the matrix measure that appears in [24], Section 8,

$$d\sigma(\theta) = \frac{1}{\sqrt{\cos 2\theta}} \begin{bmatrix} \sqrt{1+\cos 2\theta} & \pm 1 \\ \pm 1 & \sqrt{1+\cos 2\theta} \end{bmatrix} \frac{d\theta}{2\pi}, \text{ if } \begin{cases} \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right], \\ \\ \theta \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right], \end{cases} \tag{2.52}$$

and their corresponding matrix moments are given by

$$c_0 = \mathbf{I}_2, \ c_{4m} = c_{4m+2} = \frac{\alpha_m}{2}\mathbf{I}_2, \ c_{4m+1} = \frac{\alpha_m}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ c_{4m+3} = \mathbf{0}_2, \tag{2.53}$$

with

$$\alpha_0 = 1, \ \alpha_n = (-1)^n \prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right), \ n \geq 1.$$

Now, we consider the problem of determining the matrix measure associated if some of the moments given by (2.53) change. For instance, let us introduce a perturbation on the matrix moment c_{4m+1}, for some $m \in \mathbb{Z}$, given by

$$\tilde{c}_{4m+1} = \frac{\tilde{\alpha}_m}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\alpha_m + \epsilon}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = c_{4m+1} + \frac{\epsilon}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

i.e. $m_{4m+1} = \frac{\epsilon}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The perturbed matrix measure, obtained by using (2.21) ($j = 4m+1$), is

$$d\sigma_{4m+1}(\theta) = d\sigma(\theta) + \sqrt{2}\epsilon \begin{bmatrix} 0 & \cos(4m+1)\theta \\ \cos(4m+1)\theta & 0 \end{bmatrix} \frac{d\theta}{2\pi}, \tag{2.54}$$

where σ is as in (2.52). Notice that $\sqrt{2}\epsilon \begin{bmatrix} 0 & \cos(4m+1)\theta \\ \cos(4m+1)\theta & 0 \end{bmatrix}$ is a positive semi-definite matrix if $\epsilon < 0$. Hence, since σ is also positive semi-definite, σ_{4m+1} is positive semi-definite if $\epsilon < 0$. Furthermore, the matrix measure (2.52) and its perturbation (2.54) are Hermitian and thus there exists an associated sequence of orthogonal matrix polynomials, as well as their norms, given by the Theorems 2.2.6 and 2.2.9.

On the other hand, to illustrate the results in Section 2.2.2, by using (1.93), (1.94) and taking into account that $\cos 2\theta = 2\cos^2\theta - 1$, we get

$$d\mu(x) = Sz^{-1}(d\sigma(\theta)) = \frac{1}{\pi}\frac{1}{\sqrt{2x^2-1}}\begin{bmatrix} \sqrt{2}|x| & \pm 1 \\ \pm 1 & \sqrt{2}|x| \end{bmatrix}\frac{dx}{\sqrt{1-x^2}}, \tag{2.55}$$

with $x \in [-1, -\frac{1}{\sqrt{2}}] \cup [\frac{1}{\sqrt{2}}, 1]$. (2.55) is the measure associated with the measure (2.52), connected through the inverse Szegő matrix transformation. Furthermore, from Proposition 2.2.1 with $\epsilon < 0$ we get

$$d\mu_{4m+1}(x) = d\mu(x) + \frac{\sqrt{2}\epsilon}{\pi}\begin{bmatrix} 0 & T_{4m+1}(x) \\ T_{4m+1}(x) & 0 \end{bmatrix}\frac{dx}{\sqrt{1-x^2}}, \tag{2.56}$$

where μ is as in (2.55). Notice that the matrix measure (2.55) and its perturbation (2.56) are positive semi-definite, so there exists a corresponding sequence of matrix orthogonal polynomials on the real line and their norms, that can be obtained from Theorems 2.2.3 and 2.2.5.

CHAPTER 3

On the Uvarov and Christoffel spectral matrix transformations

Summary
The aim of this chapter is to study algebraic and analytic properties of the **MOPUC** associated with the Uvarov and Christoffel spectral transformations of a Hermitian matrix measure σ supported on the unit circle, defined respectively by

$$\begin{aligned} d\sigma_{u_m}(z) &= d\sigma(z) + \sum_{j=1}^{m} \mathbf{M}_j \delta(z-\zeta_j), \\ d\sigma_{c_m}(z) &= W_m(z)^H d\sigma(z) W_m(z), \end{aligned}$$

where $\zeta_j \in \mathbb{C}$, $\mathbf{M}_j$ is a positive definite matrix, δ is the Dirac matrix measure and $W_m(z) = \prod_{j=1}^{m}(z\mathbf{I} - A_j)$ with A_j a square matrix for $j = 1, \ldots, m$. These properties include connection formulas between the corresponding sequences of **MOPUC**, relative asymptotics for the **MOPUC**, as well as for their leading principal (matrix) coefficients (that depend on the location of the mass points) as well as properties of the zeros for the Uvarov matrix polynomials by using the associated (truncated) block Hessenberg matrices.

3.1 On the Uvarov transformation for matrix measure with support on the unit circle

3.1.1 Introduction

In this section, we consider the Uvarov matrix transformation with $m > 0$ masses of a Hermitian matrix measure σ supported on $\mathbb{T}$, defined by

$$d\sigma_{u_m}(z) = d\sigma(z) + \sum_{j=1}^{m} \mathbf{M}_j \delta(z-\zeta_j), \tag{3.1}$$

where $\zeta_i \neq \zeta_j$, $\zeta_i \in \mathbb{C}$, $\mathbf{M}_j$ is an $l \times l$ positive definite matrix for $j = 1, ..., m$, and δ is the Dirac matrix measure. These properties include connection formulas, ratio and

relative asymptotics when $\sigma \in \mathcal{N}$, and properties of their zeros, by using truncated block Hessenberg matrices. Notice that σ_{u_k} is the Uvarov matrix transformation of the measure $\sigma_{u_{k-1}}$ with a single mass point, i.e.,

$$d\sigma_{u_k}(z) = d\sigma_{u_{k-1}}(z) + \mathbf{M}_k \delta(z-\zeta_k), \quad k = 1, \ldots, m, \tag{3.2}$$

with $\sigma_{u_0} := \sigma$. The right and left sesquilinear forms associated with the matrix measure (3.1) are

$$\langle p, q\rangle_{R,\sigma_{u_m}} = \langle p, q\rangle_{R,\sigma} + \sum_{j=1}^{m} p(\zeta_j)^H \mathbf{M}_j q(\zeta_j), \quad p, q \in \mathbb{P}^l[z], \tag{3.3}$$

$$\langle p, q\rangle_{L,\sigma_{u_m}} = \langle p, q\rangle_{L,\sigma} + \sum_{j=1}^{m} p(\zeta_j) \mathbf{M}_j q(\zeta_j)^H, \quad p, q \in \mathbb{P}^l[z]. \tag{3.4}$$

If we define $p(\zeta) := \left[p(\zeta_1)^T, ..., p(\zeta_m)^T\right]^T \in \mathcal{M}_{ml\times l}$, $q(\zeta) := \left[q(\zeta_1)^T, ..., q(\zeta_m)^T\right]^T \in \mathcal{M}_{ml\times l}$, $\hat{p}(\zeta) := [p(\zeta_1), ..., p(\zeta_m)] \in \mathcal{M}_{l\times ml}$, $\hat{q}(\zeta) := [q(\zeta_1), ..., q(\zeta_m)] \in \mathcal{M}_{l\times ml}$ and the block diagonal matrix

$$\mathbb{M} = \begin{bmatrix} \mathbf{M}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{M}_m \end{bmatrix} \in \mathcal{M}_{lm},$$

then (3.3) and (3.4) can be written as

$$\langle p, q\rangle_{R,\sigma_{u_m}} = \langle p, q\rangle_{R,\sigma} + p(\zeta)^H \mathbb{M} q(\zeta), \quad p, q \in \mathbb{P}^l[z], \tag{3.5}$$

$$\langle p, q\rangle_{L,\sigma_{u_m}} = \langle p, q\rangle_{L,\sigma} + \hat{p}(\zeta) \mathbb{M} \hat{q}(\zeta)^H, \quad p, q \in \mathbb{P}^l[z], \tag{3.6}$$

respectively. The $n-$th matrix moment associated to the matrix measure (3.1) is $\tilde{c}_{m,n} = \langle \mathbf{I}, z^n \mathbf{I}\rangle_{R,\sigma_{u_m}} = c_n + \sum_{j=1}^{m} \zeta_j^n \mathbf{M}_j$ for $n \in \mathbb{Z}$, and the respective CMV left and right moment matrices defined as in (1.72) and (1.73), associated with of the matrix measure (3.1) are

$$\tilde{\mathbf{F}}_m^R = \mathbf{F}^R + \sum_{j=1}^{m} \begin{bmatrix} \mathbf{M}_j & \zeta_j^{-1}\mathbf{M}_j & \zeta_j\mathbf{M}_j & \zeta_j^{-2}\mathbf{M}_j & \cdots \\ \zeta_j\mathbf{M}_j & \mathbf{M}_j & \zeta_j^{2}\mathbf{M}_j & \zeta_j^{-1}\mathbf{M}_j & \cdots \\ \zeta_j^{-1}\mathbf{M}_j & \zeta_j^{-2}\mathbf{M}_j & \mathbf{M}_j & \zeta_j^{-3}\mathbf{M}_j & \cdots \\ \zeta_j^{2}\mathbf{M}_j & \zeta_j\mathbf{M}_j & \zeta_j^{3}\mathbf{M}_j & \mathbf{M}_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\tilde{\mathbf{F}}_m^L = \mathbf{F}^L + \sum_{j=1}^{m} \begin{bmatrix} \mathbf{M}_j & \zeta_j\mathbf{M}_j & \zeta_j^{-1}\mathbf{M}_j & \zeta_j^{2}\mathbf{M}_j & \cdots \\ \zeta_j^{-1}\mathbf{M}_j & \mathbf{M}_j & \zeta_j^{-2}\mathbf{M}_j & \zeta_j\mathbf{M}_j & \cdots \\ \zeta_j\mathbf{M}_j & \zeta_j^{2}\mathbf{M}_j & \mathbf{M}_j & \zeta_j^{3}\mathbf{M}_j & \cdots \\ \zeta_j^{-2}\mathbf{M}_j & \zeta_j^{-1}\mathbf{M}_j & \zeta_j^{-3}\mathbf{M}_j & \mathbf{M}_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since σ_{u_m} is an $l \times l$ Hermitian matrix measure on $\mathbb{T}$, there exist sequences $(\mathcal{U}_{m,n}^R(z))_{n\geq 0}$ (right) and $(\mathcal{U}_{m,n}^L(z))_{n\geq 0}$ (left), which are orthonormal with respect to (3.5) and (3.6), respectively. In the remaining of this section, we will consider only the right sequences $(\varphi_n^R(z))_{n\geq 0}$ and $(\mathcal{U}_{m,n}^R(z))_{n\geq 0}$, since all the results can be obtained for the left sequences in a similar way.

3.1.2 Connection formula and further properties

In this subsection, we find the connection formula between the corresponding sequences of **MOPUC** and further properties.

Lemma 3.1.1. *For $n \geq 0$ and $\zeta_i \neq \zeta_j$ $i,j = 1,\ldots,m$, the block matrix $\mathcal{K}_n^R(\zeta)$ defined by*

$$\mathcal{K}_n^R(\zeta) = \begin{bmatrix} K_n^R(\zeta_1,\zeta_1) & \cdots & K_n^R(\zeta_m,\zeta_1) \\ \vdots & \ddots & \vdots \\ K_n^R(\zeta_1,\zeta_m) & \cdots & K_n^R(\zeta_m,\zeta_m) \end{bmatrix} \in \mathcal{M}_{lm},$$

is positive definite.

Proof. First notice that $\mathcal{K}_n^R(\zeta)$ is Hermitian because $\left(K_n^R(\zeta_i,\zeta_j)\right)^H = K_n^R(\zeta_j,\zeta_i)$ with $i,j = 1,\ldots m$, and nonsingular because $\zeta_i \neq \zeta_j$ $i,j = 1,\ldots,m$. Now, let us consider the block matrix

$$G_n^R(\zeta) = \begin{bmatrix} \varphi_0^R(\zeta_1) & \cdots & \varphi_n^R(\zeta_1) \\ \vdots & \ddots & \vdots \\ \varphi_0^R(\zeta_m) & \cdots & \varphi_n^R(\zeta_m) \end{bmatrix} \in \mathcal{M}_{lm\times l(n+1)}.$$

Notice that $\mathcal{K}_n^R(\zeta) = G_n^R(\zeta)G_n^R(\zeta)^H$, and for any nonzero $u \in \mathcal{M}_{1\times lm}$ we have

$$u\mathcal{K}_n^R(\zeta)u^H = uG_n^R(\zeta)\left(uG_n^R(\zeta)\right)^H \geq 0.$$

Since $G_n^R(\zeta)$ is nonzero for every $n \geq 0$, the inequality is strict. As a consequence, $\mathcal{K}_n^R(\zeta)$ is a positive definite matrix for $n \geq 0$. □

Proposition 3.1.2. *The sequence of monic matrix polynomials $(U_{m,n}^R(z))_{n\geq 0}$ is orthogonal with respect to the sesquilinear form* (3.5) *if and only if the matrix $\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}$ is nonsingular for $n \geq 1$. Furthermore,*

$$U_{m,n}^R(z) = \Phi_n^R(z) - \mathbb{K}_{n-1}^R(\zeta,z)\mathbb{M}\left[\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}\right]^{-1}\mathbb{F}_n^R(\zeta), \tag{3.7}$$

where

$$\mathbb{K}_{n-1}^R(\zeta,z) := \left[K_{n-1}^R(\zeta_1,z),\ldots,K_{n-1}^R(\zeta_m,z)\right] \in \mathcal{M}_{l\times ml},$$
$$\mathbb{F}_n^R(\zeta) := \left[\Phi_n^R(\zeta_1)^T,\ldots,\Phi_n^R(\zeta_m)^T\right]^T \in \mathcal{M}_{ml\times l}.$$

Moreover,

$$\left(\tilde{\kappa}_{m,n}^{-R}\right)^H \tilde{\kappa}_{m,n}^{-R} = \tilde{S}_{m,n}^R = S_n^R + \mathbb{F}_n^{R,H}(\zeta)\mathbb{M}\left[\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}\right]^{-1}\mathbb{F}_n^R(\zeta), \tag{3.8}$$

is positive definite for $n \geq 1$.

Proof. Assume that $(U_{m,n}^R(z))_{n\geq 0}$ is the sequence of monic **MOPUC** with respect to (3.5). We can write

$$U_{m,n}^R(z) = \Phi_n^R(z) + \sum_{k=0}^{n-1}\Phi_k^R(z)\lambda_{n,k}^R, \tag{3.9}$$

where the coefficients $\lambda_{n,k}^R$ are given by

$$\lambda_{n,k}^R = -S_k^{-R}\sum_{j=1}^{m}\Phi_k^{R,H}(\zeta_j)\mathbf{M}_j U_{m,n}^R(\zeta_j), \quad 0\leq k\leq n-1.$$

Substituting this expression in (3.9), we obtain

$$U_{m,n}^R(z) = \Phi_n^R(z) - \sum_{j=0}^{m} K_{n-1}^R(\zeta_j, z)\mathbf{M}_j U_{m,n}^R(\zeta_j), \tag{3.10}$$

and writing $\mathbb{U}_{m,n}^R(\zeta) := \left[U_{m,n}^R(\zeta_1)^T, \ldots, U_{m,n}^R(\zeta_m)^T\right]^T \in \mathcal{M}_{l\times ml}$, (3.10) becomes

$$U_{m,n}^R(z) = \Phi_n^R(z) - \mathbb{K}_{n-1}^R(\zeta, z)\mathbb{M}\mathbb{U}_{m,n}^R(\zeta). \tag{3.11}$$

Setting $z = \zeta_1, \ldots, = \zeta_m$ in (3.11), we get the linear system $\mathbb{U}_{m,n}^R(\zeta) = \mathbb{F}_n^R(\zeta) - \mathcal{K}_{n-1}^R\mathbb{M}\mathbb{U}_{m,n}^R(\zeta)$, which is equivalent to $\left[\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}\right]\mathbb{U}_{m,n}^R(\zeta) = \mathbb{F}_n^R(\zeta)$. The uniqueness of $(U_{m,n}^R(z))_{n\geq 0}$ and $(\Phi_n^R(z))_{n\geq 0}$ implies that $\mathbf{I}_{lm}+\mathcal{K}_{n-1}^R(\zeta)\mathbb{M}$ is a non-singular matrix, and

$$\mathbb{U}_{m,n}^R(\zeta) = \left[\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}\right]^{-1}\mathbb{F}_n^R(\zeta), \tag{3.12}$$

so (3.11) becomes (3.7).

Conversely, assume that $\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}$ is non-singular and define $(U_{m,n}^R(z))_{n\geq 0}$ as in (3.7). To prove orthogonality, let $0\leq k\leq n$. From (3.3), we have

$$\left\langle z^k\mathbf{I}, U_{m,n}^R\right\rangle_{R,\sigma_{u_m}} = \left\langle z^k\mathbf{I}, U_{m,n}^R\right\rangle_{R,\sigma} + \sum_{j=1}^{m}\left(\zeta_j^k\mathbf{I}\right)^H\mathbf{M}_j U_{m,n}^R(\zeta_j). \tag{3.13}$$

Since $K_{n-1}^R(\zeta_j, z) = K_n^R(\zeta_j, z) - \varphi_n^R(z)\varphi_n^{R,H}(\zeta_j)$, from (1.65) and (3.10) we have

$$\begin{aligned}
\left\langle z^k\mathbf{I}, U_{m,n}^R\right\rangle_{R,\sigma} &= \left\langle z^k\mathbf{I}, \Phi_n^R\right\rangle_{R,\sigma} - \sum_{j=1}^{m}\left\langle z^k\mathbf{I}, K_{n-1}^R(\zeta_j, z)\right\rangle_{R,\sigma}\mathbf{M}_j U_{m,n}^R(\zeta_j)\\
&= \delta_{n,k}S_n^R + \sum_{j=1}^{m}\left(-\left\langle z^k\mathbf{I}, K_n^R(\zeta_j, z)\right\rangle_{R,\sigma} + \left\langle z^k\mathbf{I}, \varphi_n^R\right\rangle_{R,\sigma}\varphi_n^{R,H}(\zeta_j)\right)\mathbf{M}_j U_{m,n}^R(\zeta_j)\\
&= \delta_{n,k}S_n^R - \sum_{j=1}^{m}\left(\zeta_j^k\mathbf{I}\right)^H\mathbf{M}_j U_{m,n}^R(\zeta_j) + \sum_{j=1}^{m}\left\langle z^k\mathbf{I}, \Phi_n^R\right\rangle_{R,\sigma}S_n^{-R}\Phi_n^{R,H}(\zeta_j)\mathbf{M}_j U_{m,n}^R(\zeta_j)\\
&= \delta_{n,k}S_n^R - \sum_{j=1}^{m}\left(\zeta_j^k\mathbf{I}\right)^H\mathbf{M}_j U_{m,n}^R(\zeta_j) + \delta_{n,k}\sum_{j=1}^{m}\Phi_n^{R,H}(\zeta_j)\mathbf{M}_j U_{m,n}^R(\zeta_j).
\end{aligned}$$

Substituting on (3.13)

$$\left\langle z^k\mathbf{I}, U_{m,n}^R\right\rangle_{R,\sigma_{u_m}} = \delta_{n,k}\left[S_n^R + \sum_{j=1}^{m}\Phi_n^{R,H}(\zeta_j)\mathbf{M}_j U_{m,n}^R(\zeta_j)\right] = \delta_{n,k}\left[S_n^R + \mathbb{F}_n^R(\zeta)^H\mathbb{M}\mathbb{U}_{m,n}^R(\zeta)\right]$$

so that $(U^R_{m,n}(z))_{n\geq 0}$ is orthogonal with respect to (3.5). From (3.12) we obtain (3.8) for $k=n$. On the other hand, since $\mathbb{M}$ is positive definite,

$$\mathbb{M}\left[\mathbf{I}_{lm}+\mathcal{K}^R_{n-1}(\zeta)\mathbb{M}\right]^{-1}=\left[\mathbb{M}^{-1}+\mathcal{K}^{-R}_{n-1}(\zeta)\right]^{-1}$$

is positive definite, then $\tilde{S}^R_{m,n}$ is positive definite for $n\geq 1$. □

Notice that the $n-$th degree matrix orthonormal polynomial has the form

$$\mathcal{U}^R_{m,n}(z)=U^R_{m,n}(z)\tilde{\kappa}^R_{m,n}=\tilde{\kappa}^R_{m,n}z^n+\cdots,$$

and thus from (3.7) we have

$$\mathcal{U}^R_{m,n}(z)=\left[\varphi^R_n(z)-\mathbb{K}^R_{n-1}(\zeta,z)\mathbb{M}\left(\mathbf{I}_{lm}+\mathcal{K}^R_{n-1}(\zeta)\mathbb{M}\right)^{-1}\mathbb{L}^R_n(\zeta)\right]\kappa_n^{-R}\tilde{\kappa}^R_{m,n},\tag{3.14}$$

where $\mathbb{L}^R_n(\zeta):=\left[\varphi^R_n(\zeta_1)^T,\ldots,\varphi^R_n(\zeta_m)^T\right]^T\in\mathcal{M}_{ml\times l}$.

Example 3.1.3. *Consider the Hermitian matrix measure*

$$d\sigma(\theta)=\frac{1}{2}\begin{bmatrix}2 & e^{i\theta}\\ e^{-i\theta} & 2\end{bmatrix}\frac{d\theta}{2\pi}.\tag{3.15}$$

We have $\log(\det\omega(\theta))=\log(3/4)$ *is integrable over unit circle, then* $\sigma\in\mathcal{S}\subset\mathcal{N}$. *Moreover, the matrix moments are* $c_0=\mathbf{I}_2$, $c_1=c_{-1}^H=\begin{bmatrix}0&0\\1/2&0\end{bmatrix}$ *and* $c_n=\mathbf{0}_2$ *for* $|n|\geq 2$. *Therefore,*

$$\mathbf{F}^R=\begin{bmatrix}\mathbf{I}_2 & c_{-1} & c_1 & & \\ c_1 & \mathbf{I}_2 & \mathbf{0}_2 & c_{-1} & \\ c_{-1} & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \ddots\\ & c_1 & \mathbf{0}_2 & \mathbf{I}_2 & \ddots\\ & & \ddots & \ddots & \ddots\end{bmatrix},$$

where $\det(\mathbf{F}^R)_n\neq 0$ *for* $n\geq 1$, *so* σ *is also quasi-definite. The corresponding monic* **MOPUC** *are*

$$\Phi^R_0(z)=\Phi^L_0(z)=\mathbf{I}_2,\quad \Phi^R_n(z)=\Phi^L_n(z)=\begin{bmatrix}z^n & 0\\ -\frac{1}{2}z^{n-1} & z^n\end{bmatrix},\ n\geq 1,$$

and we have $S^R_0=S^L_0=\mathbf{I}_2$, $S^R_n=\begin{bmatrix}\frac{3}{4}&0\\0&1\end{bmatrix}$, $S^L_n=\begin{bmatrix}1&0\\0&\frac{3}{4}\end{bmatrix}$, $n\geq 1$. *Therefore, the matrix orthonormal polynomials are given by*

$$\varphi^R_0(z)=\mathbf{I},\ \varphi^R_n(z)=\begin{bmatrix}\frac{2}{\sqrt{3}}z^n & 0\\ -\frac{1}{\sqrt{3}}z^{n-1} & z^n\end{bmatrix}\ \text{and}\ \varphi^{L,*}_n(z)=\begin{bmatrix}1 & -\frac{1}{\sqrt{3}}z\\ 0 & \frac{2}{\sqrt{3}}\end{bmatrix}\ n\geq 1.$$

Thus, from (1.55) with $z\bar{\zeta}\neq 1$, *we get*

$$K^R_{n-1}(\zeta,z)=\mathbf{I}+\sum_{k=1}^{n-1}(z\bar{\zeta})^k\begin{bmatrix}\frac{4}{3} & -\frac{2}{3\bar{\zeta}}\\ -\frac{2}{3z} & \frac{1}{3\bar{\zeta}z}+1\end{bmatrix}=\mathbf{I}+\frac{z\bar{\zeta}-(z\bar{\zeta})^n}{3(1-z\bar{\zeta})}\begin{bmatrix}4 & -\frac{2}{\bar{\zeta}}\\ -\frac{2}{z} & \frac{1}{\bar{\zeta}z}+3\end{bmatrix}.$$

Now, let us consider the Uvarov transformation of (3.15) with one mass point at $\zeta_1 = 2$ *and* $\mathbf{M}_1 = \mathbf{I}_2$*, i.e.*

$$d\sigma_{u_1}(\theta) = d\sigma(\theta) + \mathbf{I}_2\delta(z-2).$$

Notice that for $n \geq 2$ *we have*

$$\mathbf{I}_2 + K_{n-1}^R(2,2) = 2\mathbf{I}_2 + \frac{4^n-4}{9}\begin{bmatrix} 4 & -1 \\ -1 & \frac{13}{4} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4^{n+1}+2 & 4-4^n \\ 4-4^n & \frac{13}{4}4^n+5 \end{bmatrix},$$

and denoting $\Xi_{n-1} = \det\left(\mathbf{I}_2 + K_{n-1}^R(2,2)\right) = \frac{4}{27}(16)^n + \frac{23}{54}(4^n) - \frac{2}{27}$*, we get*

$$\left(\mathbf{I}_2 + K_{n-1}^R(2,2)\right)^{-1} = \frac{1}{9\Xi_{n-1}}\begin{bmatrix} \frac{13}{4}4^n+5 & 4^n-4 \\ 4^n-4 & 4^{n+1}+2 \end{bmatrix},$$
$$\left(\mathbf{I}_2 + K_{n-1}^R(2,2)\right)^{-1}\Phi_n^R(2) = \frac{2^n}{9\Xi_{n-1}}\begin{bmatrix} 3(4^n)+6 & 4^n-4 \\ -\frac{9}{2} & 4^{n+1}+2 \end{bmatrix},$$

and

$$\begin{aligned} &K_{n-1}^R(2,z)\left(\mathbf{I}_2 + K_{n-1}^R(2,2)\right)^{-1}\Phi_n^R(2) \\ &= \frac{2^n}{9\Xi_{n-1}}\left(\frac{2z-(2z)^n}{1-2z}\begin{bmatrix} 4^{n+1}+\frac{19}{2} & -6 \\ -\frac{1}{4z}(18z+2^{2n+3}+19) & \frac{1}{z}(2z+4^{n+1}z+3) \end{bmatrix}\right. \\ &\qquad\left. + \begin{bmatrix} 3\times 4^n+16 & 4^n-4 \\ -\frac{9}{2} & 4^{n+1}+2 \end{bmatrix}\right). \end{aligned}$$

As a consequence, from (3.7)*, we get*

$$U_{1,1}^R(z) = \begin{bmatrix} z-1 & 0 \\ -\frac{1}{4} & z-1 \end{bmatrix},\ U_{1,n}^R(z) = \begin{bmatrix} \phi_{1,1}(z,n) & \phi_{1,2}(z,n) \\ \phi_{2,1}(z,n) & \phi_{2,2}(z,n) \end{bmatrix},\ \textit{for } n \geq 2,$$

where

$$\begin{aligned} \phi_{1,1}(z,n) &= z^n - \frac{2^n}{9\Xi_{n-1}}\left((4^{n+1}+\frac{19}{2})\sum_{k=1}^{n-1}(2z)^k + 3(4^n) + 16\right) \\ \phi_{1,2}(z,n) &= \frac{2^n}{9\Xi_{n-1}}\left(6\sum_{k=1}^{n-1}(2z)^k - 4^n + 4\right) \\ \phi_{2,1}(z,n) &= -\frac{1}{2}z^{n-1} + \frac{2^n}{9\Xi_{n-1}}\left(\frac{18z+2^{2n+3}+19}{4}\sum_{k=1}^{n-1}2^k z^{k-1} + \frac{9}{2}\right) \\ \phi_{2,2}(z,n) &= z^n - \frac{2^n}{9\Xi_{n-1}}\left((2z+4^{n+1}z+3)\sum_{k=1}^{n-1}2^k z^{k-1} + 4^{n+1} + 2\right). \end{aligned}$$

Now, let us consider the Uvarov transformation with two masses of (3.15), with mass points $\zeta_1 = 2$ *and* $\zeta_2 = 3$ *and* $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{I}_2$*, i.e.*

$$d\sigma_{u_2}(\theta) = d\sigma(\theta) + \mathbf{I}_2\delta(z-2) + \mathbf{I}_2\delta(z-3). \tag{3.16}$$

Proceeding as above, we have, for $n \geq 2$,

$$\left[\mathbf{I}_4 + \mathcal{K}_{n-1}^R(\zeta)\right]^{-1} = \begin{bmatrix} \frac{4^{n+1}+2}{9} & \frac{4-4^n}{9} & \frac{24(6^n)-54}{90} & \frac{24-4(6^n)}{90} \\ \frac{4-4^n}{9} & \frac{13(4^{n-1})+5}{9} & \frac{36-6^{n+1}}{90} & \frac{19(6^n)-24}{90} \\ \frac{24(6^n)-54}{90} & \frac{36-6^{n+1}}{90} & \frac{9^{n+1}+27}{54} & \frac{27-3(9^n)}{108} \\ \frac{24-4(6^n)}{90} & \frac{19(6^n)-24}{90} & \frac{27-3(9^n)}{108} & \frac{7(9^n)+45}{108} \end{bmatrix}^{-1},$$

$$\mathbb{K}_{n-1}^R(\zeta, z) = \left[\mathbf{I}_2 + \frac{2z-(2z)^n}{3(1-2z)} \begin{bmatrix} 4 & -1 \\ -\frac{2}{z} & \frac{1}{2z}+3 \end{bmatrix}, \mathbf{I}_2 + \frac{3z-(3z)^n}{3(1-3z)} \begin{bmatrix} 4 & -\frac{2}{3} \\ -\frac{2}{z} & \frac{1}{3z}+3 \end{bmatrix}\right],$$

$$\mathbb{F}_n^R(\zeta) = \begin{bmatrix} 2^n & -2^{n-2} & 3^n & -\frac{1}{2}3^{n-1} \\ 0 & 2^n & 0 & 3^n \end{bmatrix}.$$

Thus, $U_{2,1}^R(z) = \begin{bmatrix} z-\frac{5}{2} & 0 \\ 0 & z-\frac{5}{2} \end{bmatrix}$, and from (3.7) we can compute the sequence of monic matrix polynomials orthogonal with respect to (3.16) for $n \geq 2$.

In the following example, we consider a matrix measure with $l = 3$.

Example 3.1.4. *Consider the Hermitian matrix measure*

$$d\sigma(\theta) = \frac{1}{2} \begin{bmatrix} 2 & e^{i\theta} & 0 \\ e^{-i\theta} & 2 & e^{i\theta} \\ 0 & e^{-i\theta} & 2 \end{bmatrix} \frac{d\theta}{2\pi}. \tag{3.17}$$

We have $\log(\det \omega(\theta)) = \log(1/2)$ is integrable over unit circle, then $\sigma \in \mathcal{S} \subset \mathcal{N}$. Using the matrix Heine's formula [141]*, we obtain*

$$\Phi_0^R(z) = S_0^R = \mathbf{I}_3,\ \Phi_1^R(z) = \begin{bmatrix} z & 0 & 0 \\ -\frac{1}{2} & z & 0 \\ 0 & -\frac{1}{2} & z \end{bmatrix},\ S_1^R = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Phi_0^L(z) = S_0^L = \mathbf{I}_3,\ \Phi_1^L(z) = \begin{bmatrix} z & 0 & 0 \\ -\frac{1}{2} & z & 0 \\ 0 & -\frac{1}{2} & z \end{bmatrix},\ S_1^L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} \end{bmatrix},$$

and for $n \geq 2$

$$\Phi_n^R(z) = \begin{bmatrix} z^n & 0 & 0 \\ -\frac{2}{3}z^{n-1} & z^n & 0 \\ \frac{1}{3}z^{n-2} & -\frac{1}{2}z^{n-1} & z^n \end{bmatrix},\ S_n^R = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\Phi_n^L(z) = \begin{bmatrix} z^n & 0 & 0 \\ -\frac{1}{2}z^{n-1} & z^n & 0 \\ \frac{1}{3}z^{n-2} & -\frac{2}{3}z^{n-1} & z^n \end{bmatrix},\ S_n^L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix},$$

Now, let us consider the Uvarov transformation of (3.17) *with one mass point at $\zeta = \zeta_1 \in \mathbb{T}$ and $\mathbf{M} = \mathbf{M}_1 \in \mathcal{M}_3$, i.e.*

$$d\sigma_{u_1}(\theta) = d\sigma(\theta) + \mathbf{M}\delta(z-\zeta).$$

From (3.7) *and* (3.8) *with* $m = 1$, *we obtain for* $n \geq 1$

$$\begin{aligned} U_{1,n}^R &= \Phi_n^R(z) - K_{n-1}^R(\zeta, z)\mathbf{M}\left[\mathbf{I}_3 + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\Phi_n^R(\zeta), \\ \tilde{S}_{1,n}^R &= S_n^R(z) + \Phi_n^{R,H}(\zeta)\mathbf{M}\left[\mathbf{I}_3 + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\Phi_n^R(\zeta). \end{aligned} \tag{3.18}$$

Notice that $K_0^R(\zeta, z) = \mathbf{I}_3$,

$$K_1^R(\zeta,z) = \mathbf{I}_3 + \varphi_1^R(z)\varphi_1^{R,H}(\zeta) = \mathbf{I}_3 + \Phi_1^R(z)S_1^{-R}\Phi_1^{R,H}(\zeta) = \begin{bmatrix} \frac{4}{3}\bar{\zeta}z + 1 & -\frac{2}{3}z & 0 \\ -\frac{2}{3}\bar{\zeta} & \frac{4}{3}\bar{\zeta}z + \frac{4}{3} & -\frac{2}{3}z \\ 0 & -\frac{2}{3}\bar{\zeta} & \bar{\zeta}z + \frac{4}{3} \end{bmatrix},$$

and for $k > 2$, *it is possible to compute* $K_{n-1}^R(\zeta, z)$ *using* (1.56) *with*

$$\Phi_n^{L,*}(z) = \begin{bmatrix} 1 & -\frac{1}{2}z & \frac{1}{3}z^2 \\ 0 & 1 & -\frac{2}{3}z \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the Nevai condition only depends on the absolutely continuous component of the matrix measure, and the Uvarov matrix transformation only modifies the singular part of the measure, the following result is straightforward.

Proposition 3.1.5. *If* $\sigma \in \mathcal{N}$, *then* $\sigma_{u_m} \in \mathcal{N}$.

Lemma 3.1.6. *For* $n \geq 1$,

1. *If* $|\zeta| > 1$ *and* $\mathbf{M} \in \mathcal{M}_l$ *is a positive definite matrix, we have*

$$\lim_{n\to\infty} \varphi_n^{R,H}(\zeta)\mathbf{M}\left[\mathbf{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\varphi_n^R(\zeta) = (|\zeta|^2 - 1)\mathbf{I}. \tag{3.19}$$

2. *If* $|\zeta| = 1$, *we have*

$$\lim_{n\to\infty} \varphi_n^{R,H}(\zeta)\mathbf{M}\left[\mathbf{I} + K_n^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\varphi_n^R(\zeta) = \mathbf{0}. \tag{3.20}$$

Proof. Let $|\zeta| > 1$. Since

$$\begin{aligned} &\varphi_n^{R,H}(\zeta)\mathbf{M}\left[\mathbf{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\varphi_n^R(\zeta) = \\ &\qquad \left[\left(\varphi_n^R(\zeta)\right)^{-1}\mathbf{M}^{-1}\left(\varphi_n^{R,H}(\zeta)\right)^{-1} + \left(\varphi_n^R(\zeta)\right)^{-1}K_{n-1}^R(\zeta,\zeta)\left(\varphi_n^{R,H}(\zeta)\right)^{-1}\right]^{-1}, \end{aligned}$$

from (1.81) and (1.82) with $z = \zeta$ we obtain (3.19). On the other hand, let $|\zeta| = 1$. Notice that we have

$$K_n^R(\zeta,\zeta) \leq \left[\mathbf{I} + K_n^R(\zeta,\zeta)\mathbf{M}\right]\mathbf{M}^{-1} = \mathbf{M}^{-1} + K_n^R(\zeta,\zeta),$$

and thus $\mathbf{M}\left[\mathbf{I} + K_n^R(\zeta,\zeta)\mathbf{M}\right]^{-1} \leq K_n^R(\zeta,\zeta)^{-1}$. Therefore

$$\left\|\varphi_n^{R,H}(\zeta)\mathbf{M}\left[\mathbf{I} + K_n^R(\zeta,\zeta)\right]^{-1}\varphi_n^R(\zeta)\right\|_2 \leq \left\|\varphi_n^{R,H}(\zeta)K_n^R(\zeta,\zeta)^{-1}\varphi_n^R(\zeta)\right\|_2.$$

Then, using (1.83), we get (3.20). □

Notice that, if we denote $\mathbb{T}_{m,n}^R := \left[\mathcal{U}_{m,n}^R(\zeta_1)^T, \ldots, \mathcal{U}_{m,n}^R(\zeta_m)^T\right]^T \in \mathcal{M}_{ml\times l}$, then (3.12) becomes

$$\mathbb{T}_{m,n}^R(\zeta) = \left[\mathbf{I}_{lm} + \mathcal{K}_{n-1}^R(\zeta)\mathbb{M}\right]^{-1} \mathbb{L}_n^R(\zeta)\kappa_n^{-R}\tilde{\kappa}_{m,n}^R. \tag{3.21}$$

This expression will allow us to obtain asymptotics for the principal leading matrix coefficients.

Proposition 3.1.7. *For $n \geq 0$, we have*

$$0 < \left\| \left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{m,n}^R \right\|_2^2 \leq 1. \tag{3.22}$$

Proof. From (3.3), we have

$$\begin{aligned} \left\langle \mathcal{U}_{m,n}^R, \varphi_n^R \right\rangle_{R,\sigma_{u_m}} &= \left\langle \mathcal{U}_{m,n}^R, \varphi_n^R \right\rangle_{R,\sigma} + \sum_{j=1}^{m} \left(\mathcal{U}_{m,n}^R(\zeta_j)\right)^H \mathbf{M}_j \varphi_n^R(\zeta_j) \\ &= \left\langle \mathcal{U}_{m,n}^R, \varphi_n^R \right\rangle_{R,\sigma} + \mathbb{T}_{m,n}^R(\zeta)^H \mathbb{M}\mathbb{L}_n^R(\zeta). \end{aligned} \tag{3.23}$$

Considering $\langle \mathcal{U}_{m,n}^R, \varphi_n^R\rangle_{R,\sigma_{u_m}} = \tilde{\kappa}_{m,n}^{-R}\kappa_n^R$, $\langle \mathcal{U}_{m,n}^R, \varphi_n^R\rangle_{R,\sigma} = (\kappa_n^{-R}\tilde{\kappa}_{m,n}^R)^H$ and taking into account (3.21), (3.23) becomes

$$\tilde{\kappa}_{m,n}^{-R}\kappa_n^R = \left(\kappa_n^{-R}\tilde{\kappa}_{m,n}^R\right)^H + \left(\kappa_n^{-R}\tilde{\kappa}_{m,n}^R\right)^H \mathbb{L}_n^R(\zeta)^H \left[\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^R(\zeta)\right]^{-1} \mathbb{M}\mathbb{L}_n^R(\zeta).$$

Multiplying on the right by $\kappa_n^{-R}\tilde{\kappa}_{m,n}^R$, we get

$$\mathbf{I} - \left(\kappa_n^{-R}\tilde{\kappa}_{m,n}^R\right)^H \kappa_n^{-R}\tilde{\kappa}_{m,n}^R = \left(\mathbb{L}_n^R(\zeta)\kappa_n^{-R}\tilde{\kappa}_{m,n}^R\right)^H \left[\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^R(\zeta)\right]^{-1} \mathbb{M} \left(\mathbb{L}_n^R(\zeta)\kappa_n^{-R}\tilde{\kappa}_{m,n}^R\right).$$

Since the right-hand side is positive definite because $\mathbb{M}$ and $\mathcal{K}_{n-1}^R(\zeta)$ are both positive definite, we have $\left(\kappa_n^{-R}\tilde{\kappa}_{m,n}^R\right)^H \kappa_n^{-R}\tilde{\kappa}_{m,n}^R \leq \mathbf{I}$, and therefore (3.22) follows. □

The following result was proved in [173] for the particular case $m = 1$. Since the proof for an arbitrary value of m is analogous, we do not include it here.

Lemma 3.1.8. *For $n \geq 0$, the matrix $\left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{m,n}^R$ can be assumed to be lower triangular matrices with positive entries on the main diagonal.*

3.1.3 Relative Asymptotics

In this subsection, we deal with relative asymptotics. Asymptotic properties of matrix orthogonal polynomials, both on the real line and on the unit circle, have been studied in [48, 56, 57, 58, 68], among others. In particular, asymptotic properties have been deduced for **MOPRL** whose corresponding matrix measure belongs to the Nevai class. In addition, some asymptotic properties of matrix orthogonal polynomials associated with the Uvarov transformation of a matrix measure on the real line have been studied in the literature in [18, 176, 174, 175]. The case of the unit circle, however, has not drawn as much attention, although we can find some results in [173]. In the latter, the authors studied relative asymptotics of the sequences of **MOPUC** associated with the Uvarov matrix transformation with one point mass of a matrix measure satisfying a matrix Szegő condition. For the scalar case, relative asymptotics for sequences of orthogonal polynomials associated with Uvarov type perturbations of scalar measures in $\mathcal{N}$ supported on the unit

circle have been widely studied in the literature (see [29, 33, 74, 126, 135] among many others). Some applications in this direction are asymptotics of orthogonal polynomials on unbounded intervals [129, 149, 167], and (one-point) Padé [128] and Hermite-Padé approximations [22]. Moreover, in [101] the author studies Padé approximation to Markov functions to which a rational function is added, which is equivalent to adding mass points to a given measure, i.e. an Uvarov transformation with several masses.

As expected, the asymptotic behavior of the polynomials will depend on the location of the mass points.

3.1.3.1 Case $\zeta_j \in \mathbb{C} \setminus \bar{\mathbb{D}}$

Lemma 3.1.9. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{C} \setminus \bar{\mathbb{D}}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{m,n}^R = \prod_{j=1}^{m} \frac{1}{|\zeta_j|}\mathbf{I}. \tag{3.24}$$

Proof. We use induction on m. For $m = 1$ the result was proved in [173] Theorem 3.4, i.e,

$$\lim_{n\to\infty} \left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{1,n}^R = \frac{1}{|\zeta_1|}\mathbf{I}. \tag{3.25}$$

For $m > 1$, from Lemma 3.1.5 we have that σ_{u_k} belongs to $\mathcal{N}$ for $k = 2, \ldots, m$, and σ_{u_k} satisfies (3.2). Then, we can apply the previous argument to $U_{k,n}^R(z) = z^n \tilde{\kappa}_{k,n}^R + \cdots$ in terms of $U_{k-1,n}^R(z) = z^n \tilde{\kappa}_{k-1,n}^R + \cdots$ to obtain

$$\lim_{n\to\infty} \left(\tilde{\kappa}_{k-1,n}^R\right)^{-1} \tilde{\kappa}_{k,n}^R = \frac{1}{|\zeta_k|}\mathbf{I} \text{ with } k = 2, \ldots, m. \tag{3.26}$$

Since $\left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{m,n}^R = \left(\left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{1,n}^R\right) \left(\left(\tilde{\kappa}_{1,n}^R\right)^{-1} \tilde{\kappa}_{2,n}^R\right) \cdots \left(\left(\tilde{\kappa}_{m-1,n}^R\right)^{-1} \tilde{\kappa}_{m,n}^R\right)$, from (3.25) and (3.26) we easily get (3.24). □

Proposition 3.1.10. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{C} \setminus \bar{\mathbb{D}}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \left(\varphi_n^R(z)\right)^{-1} \mathcal{U}_{m,n}^R(z) = \prod_{j=1}^{m} \frac{\bar{\zeta}_j}{|\zeta_j|} \frac{z - \zeta_j}{z\bar{\zeta}_j - 1}\mathbf{I}, \tag{3.27}$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \mathbb{D}$.

Proof. We use induction on m. For $m = 1$, the result was proved in [173] Theorem 4.1, i.e,

$$\lim_{n\to\infty} \left(\varphi_n^R(z)\right)^{-1} \mathcal{U}_{1,n}^R(z) = \frac{\bar{\zeta}_1}{|\zeta_1|} \frac{z - \zeta_1}{z\bar{\zeta}_1 - 1}\mathbf{I}, \tag{3.28}$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \mathbb{D}$. For $m \geq 1$, from Lemma 3.1.5 we know σ_{u_k} with $k = 2, \ldots, m$ belongs to $\mathcal{N}$, and taking into account (3.2), we have

$$\lim_{n\to\infty} \left(\mathcal{U}_{k-1,n}^R(z)\right)^{-1} \mathcal{U}_{k,n}^R(z) = \frac{\bar{\zeta}_k}{|\zeta_k|} \frac{z - \zeta_k}{z\bar{\zeta}_k - 1}\mathbf{I}, \quad \text{with} \quad k = 2, \ldots, m. \tag{3.29}$$

Since

$$\left(\varphi_n^R(z)\right)^{-1}\mathcal{U}_{m,n}^R(z)=\left(\varphi_n^R(z)\right)^{-1}\mathcal{U}_{1,n}^R(z)\left(\mathcal{U}_{1,n}^R(z)\right)^{-1}\mathcal{U}_{2,n}^R(z)\cdots\left(\mathcal{U}_{m-1,n}^R(z)\right)^{-1}\mathcal{U}_{m,n}^R(z),$$

from (3.28) and (3.29) for $|z|>1$, (3.27) follows. □

Since $\mathcal{U}_{m,n}^{R,*}(z)\left(\varphi_n^{R,*}(z)\right)^{-1}=\left(\varphi_n^R(1/\bar{z})^{-1}\mathcal{U}_{m,n}^R(1/\bar{z})\right)^H$, from the previous proposition we get the following result.

Corollary 3.1.11. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{C}\setminus\bar{\mathbb{D}}$ for $j=1,\ldots,m$, then*

$$\lim_{n\to\infty}\mathcal{U}_{m,n}^{R,*}(z)\left(\varphi_n^{R,*}(z)\right)^{-1}=\lim_{n\to\infty}\left(\varphi_n^{L,*}(z)\right)^{-1}\mathcal{U}_{m,n}^{L,*}(z)=\prod_{j=1}^{m}\frac{\zeta_j}{|\zeta_j|}\frac{z\bar{\zeta}_j-1}{z-\zeta_j}\mathbf{I},\tag{3.30}$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Moreover, since

$$\left(\mathcal{U}_{m,n}^R(z)\right)^{-1}\mathcal{U}_{m,n+1}^R(z)=\left(\mathcal{U}_{m,n}^R(z)\right)^{-1}\varphi_n^R(z)\left(\varphi_n^R(z)\right)^{-1}\varphi_{n+1}^R(z)\left(\varphi_{n+1}^R(z)\right)^{-1}\mathcal{U}_{m,n+1}^R(z),$$

another immediate consequence of the above proposition and (1.75) is the following.

Corollary 3.1.12. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{C}\setminus\bar{\mathbb{D}}$ for $j=1,\ldots,m$, then*

$$\lim_{n\to\infty}\left(\mathcal{U}_{m,n}^R(z)\right)^{-1}\mathcal{U}_{m,n+1}^R(z)=z\mathbf{I},$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus(\bar{\mathbb{D}}\cup\{\zeta_j\}_{j=1}^m)$.

Proposition 3.1.13. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{C}\setminus\bar{\mathbb{D}}$ for $j=1,\ldots,m$, then*

$$\lim_{n\to\infty}\left(\varphi_n^{L,*}(z)\right)^{-1}\mathcal{U}_{m,n}^R(z)=\mathbf{0},$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Proof. For $m=1$, from (3.14) with $m=1$ and multiplying on the left by $(\varphi_n^{L,*}(z))^{-1}$, we get

$$\begin{aligned}(\varphi_n^{L,*}(z))^{-1}\mathcal{U}_{1,n}^R(z)=\big[&(\varphi_n^{L,*}(z))^{-1}\varphi_n^R(z)\\&-(\varphi_n^{L,*}(z))^{-1}K_{n-1}^R(\zeta_1,z)\mathbf{M}_1\left[\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1)\Big]\kappa_n^{-R}\tilde{\kappa}_{1,n}^R.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\varphi_n^{L,*}(z))^{-1}K_{n-1}^R(\zeta_1,z)\mathbf{M}_1\left[\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1)\\&\qquad=(\varphi_n^{L,*}(z))^{-1}K_{n-1}^R(\zeta_1,z)\varphi_n^{R,H}(\zeta_1)^{-1}\mathbf{B}\\&\qquad=\left(\frac{(\varphi_n^R(\zeta_1)^{-1}\varphi_n^{L,*}(\zeta_1))^H-\varphi_n^{L,*}(z)^{-1}\varphi_n^R(z)}{1-z\bar{\zeta}_1}\right)\mathbf{B},\end{aligned}$$

with $\mathbf{B} = \varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left[\mathbf{I} + K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1)$, so from Proposition 1.4.18 and Lemma 3.1.6, we obtain

$$\lim_{n\to\infty}(\varphi_n^{L,*}(z))^{-1}K_{n-1}^R(\zeta_1,z)\mathbf{M}_1\left[\mathbf{I} + K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1) = \mathbf{0}.$$

Therefore, from (1.79) and (3.24) with $m = 1$, we get

$$\lim_{n\to\infty}(\varphi_n^{L,*}(z))^{-1}\mathcal{U}_{1,n}^R(z) = \Big[\lim_{n\to\infty}(\varphi_n^{L,*}(z))^{-1}\varphi_n^R(z) + \mathbf{0}\Big]\prod_{j=1}^{m}\frac{1}{|\zeta_j|}\mathbf{I} = \mathbf{0}.$$

Since $\sigma_{u_k} \in \mathcal{N}$ for $k = 2,\ldots,m$, we have

$$\lim_{n\to\infty}(\mathcal{U}_{m-1,n}^{L,*}(z))^{-1}\mathcal{U}_{m,n}^R(z) = \mathbf{0}. \tag{3.31}$$

For $m > 1$, from (3.30) and (3.31), we obtain

$$\begin{aligned}\lim_{n\to\infty}\left(\varphi_n^{L,*}(z)\right)^{-1}\mathcal{U}_{m,n}^R(z) &= \lim_{n\to\infty}\left(\varphi_n^{L,*}(z)\right)^{-1}\mathcal{U}_{m-1,n}^{L,*}(z)\left(\mathcal{U}_{m-1,n}^{L,*}(z)\right)^{-1}\mathcal{U}_{m,n}^R(z)\\ &= \left(\prod_{j=1}^{m-1}\frac{\zeta_j}{|\zeta_j|}\frac{z\bar{\zeta}_j - 1}{z-\zeta_j}\mathbf{I}\right)\times\mathbf{0} = \mathbf{0}.\end{aligned}$$

□

Corollary 3.1.14. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{C}\setminus\bar{\mathbb{D}}$ for $j = 1,\ldots,m$, then*

$$\lim_{n\to\infty}\mathcal{U}_{m,n}^{R,*}(z)\left(\varphi_n^L(z)\right)^{-1} = \mathbf{0},$$

uniformly on every compact subset of $z \in \mathbb{C}\setminus\bar{\mathbb{D}}$.

3.1.3.2 Case $\zeta_j \in \mathbb{T}$

Lemma 3.1.15. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{T}$ for $j = 1,\ldots,m$, then*

$$\lim_{n\to\infty}\left(\kappa_n^R\right)^{-1}\tilde{\kappa}_{m,n}^R = \mathbf{I}. \tag{3.32}$$

Proof. From (3.8) with $m = 1$, we have

$$\langle U_{1,n}^R, U_{1,n}^R\rangle_{R,\sigma_{u_1}} = \langle\Phi_n^R,\Phi_n^R\rangle_{R,\sigma} + \left(\Phi_n^R(\zeta_1)\right)^H\mathbf{M}_1\left[\mathbf{I} + K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\Phi_n^R(\zeta_1),$$

and since $\langle U_{1,n}^R, U_{1,n}^R\rangle_{R,\sigma_{u_1}} = (\tilde{\kappa}_{1,n}^{-R})^H\tilde{\kappa}_{1,n}^{-R}$ and $\langle\Phi_n^R,\Phi_n^R\rangle_{R,\sigma} = (\kappa_n^{-R})^H\kappa_n^{-R}$, we get

$$\begin{aligned}(\tilde{\kappa}_{1,n}^{-R})^H\tilde{\kappa}_{1,n}^{-R} &= (\kappa_n^{-R})^H\kappa_n^{-R} + (\kappa_n^{-R})^H\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left[\mathbf{I} + K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1)\kappa_n^{-R}\\ &= (\kappa_n^{-R})^H\left[\mathbf{I} + \varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left[\mathbf{I} + K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1)\right]\kappa_n^{-R}.\end{aligned}$$

As a consequence,

$$(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H = \left[\mathbf{I} + \varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left(\mathbf{I} + K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right)^{-1}\varphi_n^R(\zeta_1)\right]^{-1}.$$

Notice that

$$\mathbf{I}+\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right)^{-1}\varphi_n^R(\zeta_1)$$
$$=\varphi_n^R(\zeta_1)^{-1}(\mathbf{I}+K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1)(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1)^{-1}\varphi_n^R(\zeta_1).$$

Thus,

$$\begin{aligned}&(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H\\&\quad=\varphi_n^R(\zeta_1)^{-1}(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1)(\mathbf{I}+K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1)^{-1}\varphi_n^R(\zeta_1).\end{aligned}\tag{3.33}$$

Furthermore,

$$\mathbf{I}-\varphi_n^R(\zeta_1)^{-1}(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1)(\mathbf{I}+K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1)^{-1}\varphi_n^R(\zeta_1)$$
$$=\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1(\mathbf{I}+K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1)^{-1}\varphi_n^R(\zeta_1),$$

so we obtain

$$(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H=\mathbf{I}-\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1(\mathbf{I}+K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1)^{-1}\varphi_n^R(\zeta_1),\tag{3.34}$$

and from (3.20), we get

$$\lim_{n\to\infty}(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H=\mathbf{I}.\tag{3.35}$$

Moreover, from (3.22) with $m=1$

$$\left\|\kappa_n^{-R}\tilde{\kappa}_{1,n}^R\right\|_F^2=Tr((\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)\leq l\left\|\kappa_n^{-R}\tilde{\kappa}_{1,n}^R\right\|_2^2\leq l,$$

and thus we necessarily have $\lim_{n\to\infty}\left(\kappa_n^R\right)^{-1}\tilde{\kappa}_{1,n}^R=\mathbf{I}$, since we can consider a subsequence $(n_v)_{v\geq 0}$ of positive integers such that $\lim_{v\to\infty}\left(\kappa_{n_v}^R\right)^{-1}\tilde{\kappa}_{1,n_v}^R=\mathbf{L}<\infty$, and then from (3.35) we have

$$\mathbf{L}\mathbf{L}^H=\mathbf{I}.\tag{3.36}$$

Since $\mathbf{I}$ is positive definite and $\mathbf{L}$ has positive entries on the main diagonal from Lemma 3.1.8, then the Cholesky factorization (see Theorem 1.1.2 item 4) of (3.36) is unique and we deduce $(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)_{n\geq 1}$ does not have subsequences converging to a limit different than $\mathbf{I}$. For $m>1$, proceeding as in the Lemma 3.1.9, we get (3.32). □

Proposition 3.1.16. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\ldots,m$, then*

$$\lim_{n\to\infty}\left(\varphi_n^R(z)\right)^{-1}\mathcal{U}_{m,n}^R(z)=\mathbf{I},\tag{3.37}$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus\bar{\mathbb{D}}$.

Proof. Let $m=1$. From (3.14) with $m=1$ and multiplying on the left by $(\varphi_n^R(z))^{-1}$ and on the right by $(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H$, we get

$$\left(\varphi_n^R(z)\right)^{-1}\mathcal{U}_{1,n}^R(z)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H=\Big[\mathbf{I}-\left(\varphi_n^R(z)\right)^{-1}K_{n-1}^R(\zeta_1,z)\left(\varphi_n^{R,H}(\zeta_1)\right)^{-1}$$
$$\times\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right)^{-1}\varphi_n^R(\zeta_1)\Big]\kappa_n^{-R}\tilde{\kappa}_{1,n}^R(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H.$$

Using (3.33), notice that

$$\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right)^{-1}\varphi_n^R(\zeta_1)\kappa_n^{-R}\tilde{\kappa}_{1,n}^R(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H \\ =\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left(\mathbf{I}+K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1\right)^{-1}\varphi_n^R(\zeta_1),$$

so from (3.20)

$$\lim_{n\to\infty}\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left(\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right)^{-1}\varphi_n^R(\zeta_1)\kappa_n^{-R}\tilde{\kappa}_{1,n}^R(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H=\mathbf{0}. \tag{3.38}$$

Therefore, using the previous lemma and (1.82) with $\zeta_1\in\mathbb{T}$ and $|z|>1$, we get

$$\lim_{n\to\infty}\left(\varphi_n^R(z)\right)^{-1}\mathcal{U}_{1,n}^R(z)=\lim_{n\to\infty}\left(\varphi_n^R(z)\right)^{-1}\mathcal{U}_{1,n}^R(z)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H \\ =\lim_{n\to\infty}\left(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H-\frac{1}{z\bar{\zeta}_1-1}\mathbf{I}\times\mathbf{0}\right)=\mathbf{I}.$$

For $m>1$, proceeding as in the proof of Proposition 3.1.10, we get (3.37). □

Corollary 3.1.17. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\ldots,m$, then*

$$\lim_{n\to\infty}\mathcal{U}_{m,n}^{R,*}(z)\left(\varphi_n^{R,*}(z)\right)^{-1}=\mathbf{I},$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Proposition 3.1.18. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\ldots,m$, then*

$$\lim_{n\to\infty}\left(\varphi_n^{L,*}(z)\right)^{-1}\mathcal{U}_{m,n}^R(z)=\mathbf{0},$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Proof. For $m=1$, from (3.14) with $m=1$ and multiplying on the left by $(\varphi_n^{L,*}(z))^{-1}$ and on the right by $(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H$, we get

$$(\varphi_n^{L,*}(z))^{-1}\mathcal{U}_{1,n}^R(z)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H=\big[(\varphi_n^{L,*}(z))^{-1}\varphi_n^R(z)-(\varphi_n^{L,*}(z))^{-1}K_{n-1}^R(\zeta_1,z) \\ \varphi_n^{R,H}(\zeta_1)^{-1}\varphi_n^{R,H}(\zeta_1)\mathbf{M}_1\left[\mathbf{I}+K_{n-1}^R(\zeta_1,\zeta_1)\mathbf{M}_1\right]^{-1}\varphi_n^R(\zeta_1)\Big]\kappa_n^{-R}\tilde{\kappa}_{1,n}^R(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H.$$

Since $\varphi_n^R(\zeta_1)^{-1}\varphi_n^{L,*}(\zeta_1)$ is an unitary matrix when $\zeta_1\in\mathbb{T}$, from (1.74) we get

$$\left\|(\varphi_n^{L,*}(z))^{-1}K_{n-1}^R(\zeta_1,z)\varphi_n^{R,H}(\zeta_1)^{-1}\right\|_2= \\ \left\|\frac{(\varphi_n^R(\zeta_1)^{-1}\varphi_n^{L,*}(\zeta_1))^H-\varphi_n^{L,*}(z)^{-1}\varphi_n^R(z)}{1-z\bar{\zeta}_1}\right\|_2\leq\frac{1+\|\varphi_n^{L,*}(z)^{-1}\varphi_n^R(z)\|_2}{|1-z\bar{\zeta}_1|}, \tag{3.39}$$

and from (1.79) it follows that (3.39) is bounded when n tends to infinity. Therefore, using (1.79), (3.32) and (3.38), we obtain

$$\lim_{n\to\infty}(\varphi_n^{L,*}(z)\mathcal{U}_{1,n}^R(z))^{-1}=\lim_{n\to\infty}(\varphi_n^{L,*}(z)\mathcal{U}_{1,n}^R(z))^{-1}(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H \\ =\lim_{n\to\infty}(\varphi_n^{L,*}(z)\varphi_n^R(z))^{-1}\kappa_n^{-R}\tilde{\kappa}_{1,n}^R(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H=\mathbf{0}.$$

For $m>1$, we proceed as in the proof of Proposition 3.1.13 using Corollary 3.1.17. □

Corollary 3.1.19. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{T}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \mathcal{U}_{m,n}^{R,*}(z) \left(\varphi_n^L(z)\right)^{-1} = \mathbf{0},$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$.

3.1.3.3 Case $\zeta_j \in \mathbb{D}$

Let $\zeta_j \in \mathbb{D}$ for $j, 1 \ldots, m$, and assume that ζ_j is not a zero of φ_n^R for all $n > 0$.

Lemma 3.1.20. *If $\sigma \in \mathcal{S}$ and $\zeta_j \in \mathbb{D}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \left(\kappa_n^R\right)^{-1} \tilde{\kappa}_{m,n}^R = \mathbf{I}. \tag{3.40}$$

Proof. For $m = 1$. If $\sigma \in \mathcal{S}$, from (3.34)

$$\left\|(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H - \mathbf{I}\right\|_2 \leq \|\varphi_n^R(\zeta_1)\|_2^2 \|\mathbf{M}_1\|_2 \left\|(\mathbf{I} + K_n^R(\zeta_1,\zeta_1)\mathbf{M}_1)^{-1}\right\|_2.$$

Since $K_n^R(z,z)$, converges uniformly on every compact subset of $z \in \mathbb{D}$ (see Proposition 1.4.23), thus $\lim_{n\to\infty} \|\varphi_n^R(\zeta_1)\|_2 = 0$, therefore

$$\lim_{n\to\infty} \left\|(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_n^{-R}\tilde{\kappa}_{1,n}^R)^H - \mathbf{I}\right\|_2 = 0,$$

so that (3.40) follows. The case $m > 1$ follows immediately. □

3.1.4 Zeros

Here, we study the zeros of the **MOPUC** associated with an Uvarov perturbation. It is worth noticing that zeros of orthogonal polynomials on the unit circle play an important role in topics ranging from mathematics to physics (see [86]).

Let us consider the left monic sequence $(\Phi_n^L(z))_{n\geq 0}$. Expand $z\Phi_j^L(z)$ into a Fourier series using the left **MOPUC**

$$z\Phi_j^L(z) = \sum_{k=0}^{j+1} H_{j,k}^L \Phi_k^L(z), \quad j = 0, \ldots, n-1. \tag{3.41}$$

Using the Szegő recurrence relation (1.42) and (1.45), we get

$$z\Phi_j^L(z) = \Phi_{j+1}^L(z) - \Phi_{j+1}^L(0) S_j^R \left(\sum_{k=0}^{j} S_k^{-R} \Phi_k^{R,H}(0) \Phi_k^L(z)\right), \quad j = 0, \ldots, n-1, \tag{3.42}$$

and comparing (3.41) and (3.42), we deduce

$$H_{j,k}^L = \begin{cases} \mathbf{I}, & \text{if } k = j+1, \\ -\Phi_{j+1}^L(0) S_j^R S_k^{-R} \Phi_k^{R,H}(0), & \text{if } 0 \leq k \leq j, \\ \mathbf{0}, & \text{if } k > j+1. \end{cases} \tag{3.43}$$

In matrix form, we have

$$z\begin{bmatrix}\Phi_0^L(z)\\ \vdots\\ \Phi_{n-1}^L(z)\end{bmatrix}=(\mathbb{H}_{\Phi^L})_{n-1}\begin{bmatrix}\Phi_0^L(z)\\ \vdots\\ \Phi_{n-1}^L(z)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \vdots\\ \mathbf{I}\end{bmatrix}\Phi_n^L(z), \tag{3.44}$$

where

$$(\mathbb{H}_{\Phi^L})_{n-1}=\begin{bmatrix} H_{0,0}^L & \mathbf{I} & & & \mathbf{0}\\ H_{1,0}^L & H_{1,1}^L & \mathbf{I} & & \\ \vdots & \vdots & \ddots & \ddots & \\ H_{n-2,0}^L & H_{n-2,1}^L & \cdots & H_{n-2,n-2}^L & \mathbf{I}\\ H_{n-1,0}^L & H_{n-1,1}^L & \cdots & H_{n-1,n-2}^L & H_{n-1,n-1}^L \end{bmatrix}\in\mathcal{M}_{nl},$$

is the truncated block Hessenberg matrix. Block Hessenberg matrices are very useful in multivariate time series analysis, multichannel signal processing and Gaussian quadrature of matrix-valued functions (see [161, 163]).

Notice that if $\xi\in\mathbb{D}$ is a zero of $\Phi_n^L(z)$, then one can find a nonzero $\alpha(\xi)\in\mathcal{M}_{l\times 1}$, such that $\Phi_n^L(\xi)\alpha(\xi)=\mathbf{0}_{l\times 1}$. Thus, from (3.44), we conclude that the zeros of $\Phi_n^L(z)$ are the eigenvalues of $(\mathbb{H}_{\Phi^L})_{n-1}$ with eigenvector

$$[(\Phi_0^L(\xi)\alpha(\xi))^T,\dots,(\Phi_{n-1}^L(\xi)\alpha(\xi))^T]^T\in\mathcal{M}_{nl\times 1}.$$

Example 3.1.21. *Consider the Hermitian matrix measure σ defined as in* (3.17)*. Notice that Φ_n^L and Φ_n^R have a zero of multiplicity $3n$ in $z=0$. On the other hand, from* (3.43) *for this particular case, we get*

$$(\mathbb{H}_{\Phi^L})_1=\begin{bmatrix} H_{0,0}^L & \mathbf{I}_3\\ H_{1,0}^L & H_{1,1}^L\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0 & 1 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{6} & 0\end{bmatrix}.$$

It is easy to see that the only eigenvalue of the block Hessenberg matrix $(\mathbb{H}_{\Phi^L})_1$ is $z=0$. In general form, the block Hessenberg matrix

$$(\mathbb{H}_{\Phi^L})_{n-1}=\begin{bmatrix} H_{0,0}^L & \mathbf{I}_3 & & & \mathbf{0}_3\\ H_{1,0}^L & H_{1,1}^L & \mathbf{I}_3 & & \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \mathbf{I}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3\end{bmatrix}, \tag{3.45}$$

has only a eigenvalue in $z=0$.

On the other hand, we also have

$$z\begin{bmatrix} U_{m,0}^L(z) \\ \vdots \\ U_{m,n-1}^L(z) \end{bmatrix} = (\mathbb{H}_{U^L})_{n-1}\begin{bmatrix} U_{m,0}^L(z) \\ \vdots \\ U_{m,n-1}^L(z) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix} U_{m,n}^L(z). \tag{3.46}$$

Proceeding as in the Proposition 3.1.2 the relation between $(\Phi_n^L(z))_{n\geq 0}$ and $(U_{m,n}^L(z))_{n\geq 0}$ is

$$\begin{aligned} U_{m,n}^L(z) &= \Phi_n^L(z) - \mathbb{F}_n^L(\zeta)\left[\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right]^{-1}\mathbb{M}\mathbb{K}_{n-1}^L(z,\zeta) \\ &= \Phi_n^L(z) - \sum_{k=0}^{n-1}\mathbb{F}_n^L(\zeta)\left[\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right]^{-1}\mathbb{M}\mathbb{F}_k^{L,H}(\zeta)S_k^{-L}\Phi_k^L(z), \end{aligned} \tag{3.47}$$

where $\mathbb{K}_{n-1}^L(z,\zeta) := \left[K_{n-1}^L(z,\zeta_1)^T,\ldots,K_{n-1}^L(z,\zeta_m)^T\right]^T \in \mathcal{M}_{ml\times l}$ and $\mathbb{F}_n^L(\zeta) := \left[\Phi_n^L(\zeta_1),\ldots,\Phi_n^L(\zeta_m)\right] \in \mathcal{M}_{l\times ml}$.

Let $(\mathbf{L}_{U\Phi}^L)_{n-1}$ be the change of basis matrix such that

$$\begin{bmatrix} U_{m,0}^L(z) \\ \vdots \\ U_{m,n-1}^L(z) \end{bmatrix} = (\mathbf{L}_{U\Phi}^L)_{n-1}\begin{bmatrix} \Phi_0^L(z) \\ \vdots \\ \Phi_{n-1}^L(z) \end{bmatrix}. \tag{3.48}$$

From (3.47), $(\mathbf{L}_{U\Phi}^L)_{n-1}$ is a lower triangular block matrix, where $(\mathbf{L}_{U\Phi}^L)_{n-1} = (l_{j,k})_{j,k=0}^{n-1}$ with

$$l_{j,k} = \begin{cases} \mathbf{0}, & \text{if } j<k, \\ \mathbf{I}, & \text{if } j=k, \\ -\mathbb{F}_j^L(\zeta)\left[\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{j-1}^L(\zeta)\right]^{-1}\mathbb{M}\mathbb{F}_k^{L,H}(\zeta)S_k^{-L}, & \text{if } k<j. \end{cases} \tag{3.49}$$

Substituting (3.48) in (3.46)

$$\begin{aligned} z(\mathbf{L}_{U\Phi}^L)_{n-1}\begin{bmatrix} \Phi_0^L(z) \\ \vdots \\ \Phi_{n-1}^L(z) \end{bmatrix} &= (\mathbb{H}_{U^L})_{n-1}(\mathbf{L}_{U\Phi}^L)_{n-1}\begin{bmatrix} \Phi_0^L(z) \\ \vdots \\ \Phi_{n-1}^L(z) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix}\Phi_n^L(z) \\ &\quad + \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix}\sum_{k=0}^{n-1} l_{n,k}\Phi_k^L(z). \end{aligned} \tag{3.50}$$

Notice that

$$\begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix}\sum_{k=0}^{n-1} l_{n,k}\Phi_k^L(z) = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ l_{n,0} & \cdots & l_{n,n-1} \end{bmatrix}\begin{bmatrix} \Phi_0^L(z) \\ \vdots \\ \Phi_{n-1}^L(z) \end{bmatrix} = \mathbb{A}_n\begin{bmatrix} \Phi_0^L(z) \\ \vdots \\ \Phi_{n-1}^L(z) \end{bmatrix}.$$

Moreover, since $(\mathbf{L}_{U\Phi}^L)_{n-1}^{-1}$ is a lower triangular block matrix with $\mathbf{I}$ in the main diagonal, we have

$$(\mathbf{L}_{U\Phi}^L)_{n-1}^{-1}\begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{I} \end{bmatrix} \text{ and } (\mathbf{L}_{U\Phi}^L)_{n-1}^{-1}\mathbb{A}_n = \mathbb{A}_n.$$

As a consequence, (3.50) becomes

$$z\begin{bmatrix}\Phi_0^L(z)\\ \vdots\\ \Phi_{n-1}^L(z)\end{bmatrix}=\left[(\mathbf{L}_{U\Phi}^L)_{n-1}^{-1}(\mathbb{H}_{U^L})_{n-1}(\mathbf{L}_{U\Phi}^L)_{n-1}+\mathbb{A}_n\right]\begin{bmatrix}\Phi_0^L(z)\\ \vdots\\ \Phi_{n-1}^L(z)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \vdots\\ \mathbf{I}\end{bmatrix}\Phi_n^L(z).$$

Comparing (3.44) with the above equation, we get

$$(\mathbb{H}_{U^L})_{n-1}=(\mathbf{L}_{U\Phi}^L)_{n-1}\left[(\mathbb{H}_{\Phi^L})_{n-1}-\mathbb{A}_n\right](\mathbf{L}_{U\Phi}^L)_{n-1}^{-1}.$$

Therefore, we have proved the following proposition. Notice that we have followed the ideas on [32], where the scalar case is considered.

Proposition 3.1.22. *The zeros of the matrix polynomial $U_{m,n}^L$ are the eigenvalues of the block matrix*

$$(\mathbb{H}_{\Phi^L})_{n-1}-\mathbb{A}_n.$$

Example 3.1.23. *We consider the Uvarov transformation with one mass point of the Hermitian matrix measure σ defined as in* (3.17)

$$d\sigma_{u_1}(\theta)=d\sigma(\theta)+\mathbf{M}\delta(z-\zeta),$$

with $\mathbf{M}\in\mathcal{M}_3$ and $\zeta\in\mathbb{T}$. As consequence of the above proposition the zeros of the matrix polynomial

$$U_{1,n}^L(z)=\Phi_n^L(z)-\Phi_n^L(\zeta)\left[\mathbf{I}_3+\mathbf{M}K_{n-1}^L(\zeta,\zeta)\right]^{-1}\mathbf{M}K_{n-1}^L(z,\zeta)\tag{3.51}$$

are the eigenvalues of the block matrix

$$(\mathbb{H}_{\Phi^L})_0-\mathbb{A}_1=H_{0,0}^L-l_{1,0},\ \textit{for } n=1,$$

$$(\mathbb{H}_{\Phi^L})_1-\mathbb{A}_2=\begin{bmatrix}H_{0,0}^L & \mathbf{I}_3\\ H_{1,0}^L & H_{1,1}^L\end{bmatrix}-\begin{bmatrix}\mathbf{0}_3 & \mathbf{0}_3\\ l_{2,0} & l_{2,1}\end{bmatrix},\ \textit{for } n=2,$$

and

$$(\mathbb{H}_{\Phi^L})_{n-1}-\mathbb{A}_n=\begin{bmatrix}H_{0,0}^L & \mathbf{I}_3 & & & \mathbf{0}_3\\ H_{1,0}^L & H_{1,1}^L & \mathbf{I}_3 & & \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \mathbf{I}_3\\ -l_{n,0} & -l_{n,1} & -l_{n,2} & \cdots & -l_{n,n-1}\end{bmatrix}\in\mathcal{M}_{3n},\ \textit{for } n>2,$$

with $(\mathbb{H}_{\Phi^L})_{n-1}$ as in (3.45) *and from* (3.49) *for $k=0,\dots,n-1$*

$$l_{n,k}=-\Phi_n^L(\zeta)\left[\mathbf{I}_3+\mathbf{M}K_{n-1}^L(\zeta,\zeta)\right]^{-1}\mathbf{M}\Phi_k^{L,H}(\zeta)S_k^{-L}.\tag{3.52}$$

In particular, if $\mathbf{M}=\mathbf{I}_3$ and $\zeta=1$, for $n=1$, from (3.51) *we have*

$$U_{1,1}^L(z)=\begin{bmatrix}z-\frac{1}{2} & 0 & 0\\ -\frac{1}{4} & z-\frac{1}{2} & 0\\ 0 & -\frac{1}{4} & z-\frac{1}{2}\end{bmatrix}.$$

Notice that the zeros of $U_{1,1}^L(z)$ *are* $z=\frac{1}{2}$ *with multiplicity* 3*, coinciding with the eigenvalues of*

$$H_{0,0}^L - l_{1,0} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

For $n=2$*, from* (3.51)*, we have*

$$U_{1,2}^L(z) = \begin{bmatrix} z^2 - \frac{53}{170}z - \frac{32}{85} & \frac{11}{85}z - \frac{6}{85} & \frac{2}{85}z - \frac{1}{85} \\ \frac{26}{85} - \frac{137}{340}z & z^2 - \frac{71}{170}z - \frac{27}{85} & \frac{9}{85}z - \frac{9}{170} \\ \frac{13}{85} - \frac{13}{170}z & \frac{29}{85} - \frac{39}{85}z & z^2 - \frac{38}{85}z - \frac{47}{170} \end{bmatrix},$$

with

$$\det(U_{1,2}^L(z)) = z^6 - \frac{20}{17}z^5 - \frac{7}{17}z^4 + \frac{46}{85}z^3 + \frac{73}{340}z^2 - \frac{8}{85}z - \frac{4}{85}.$$

On the other hand,

$$(\mathbb{H}_{\Phi^L})_1 - \mathbb{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{6} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{53}{170} & -\frac{1}{17} & -\frac{1}{85} & -\frac{53}{170} & \frac{11}{85} & \frac{2}{85} \\ \frac{33}{340} & -\frac{9}{34} & -\frac{9}{170} & \frac{33}{340} & -\frac{71}{170} & \frac{9}{85} \\ -\frac{13}{170} & \frac{2}{17} & -\frac{47}{170} & -\frac{13}{170} & \frac{53}{255} & -\frac{38}{85} \end{bmatrix},$$

and the corresponding characteristic polynomial is

$$z^6 - \frac{20}{17}z^5 - \frac{7}{17}z^4 + \frac{46}{85}z^3 + \frac{73}{340}z^2 - \frac{8}{85}z - \frac{4}{85}.$$

Their zeros are plotted the following figure

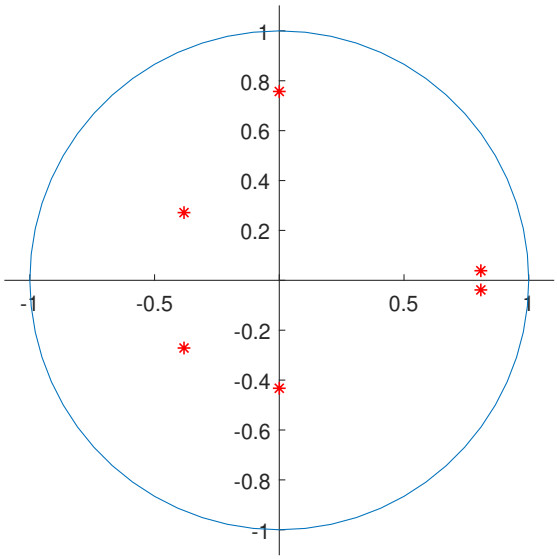

FIGURE 3.1. Zeros of $U_{1,2}^L(z)$ with $\zeta = 1$ and $\mathbf{M} = \mathbf{I}_3$.

For $n>2$, from (3.52)

$$l_{n,0}=-\begin{bmatrix} 1 & 0 & 0\\ -\frac{1}{2} & 1 & 0\\ \frac{1}{3} & -\frac{2}{3} & 1\end{bmatrix}\begin{bmatrix} \frac{3}{2}n+\frac{1}{3} & \frac{4}{3}-n & \frac{1}{2}n-1\\ \frac{4}{3}-n & 2n-\frac{1}{3} & \frac{4}{3}-n\\ \frac{1}{2}n-1 & \frac{4}{3}-n & \frac{3}{2}n+\frac{1}{3}\end{bmatrix}^{-1},\ l_{n,1}=\frac{1}{3}l_{n,0}\begin{bmatrix} 3 & -2 & 0\\ 0 & 4 & -2\\ 0 & 0 & 4\end{bmatrix},$$

and for $k=2,\dots,n-1$

$$l_{n,k}=l_{n,0}\begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{2}\\ 0 & \frac{4}{3} & -1\\ 0 & 0 & \frac{3}{2}\end{bmatrix}.$$

For instance, if we consider $n=5$

$$l_{5,0}=-\begin{bmatrix} \frac{59}{380} & \frac{11}{190} & -\frac{1}{380}\\ -\frac{3}{152} & \frac{9}{76} & \frac{9}{152}\\ \frac{1}{95} & -\frac{2}{95} & \frac{11}{95}\end{bmatrix},\ l_{5,1}=\begin{bmatrix} -\frac{59}{380} & \frac{1}{38} & \frac{4}{95}\\ \frac{3}{152} & -\frac{13}{76} & 0\\ -\frac{1}{95} & \frac{2}{57} & -\frac{16}{25}\end{bmatrix},$$

and for $k=2,3,4$,

$$l_{5,k}=\begin{bmatrix} -\frac{59}{380} & \frac{1}{38} & -\frac{3}{190}\\ \frac{3}{152} & -\frac{13}{76} & \frac{3}{76}\\ -\frac{1}{95} & \frac{2}{57} & -\frac{1}{5}\end{bmatrix}.$$

Thus, the zeros of $U_{1,5}^{L}(z)$, or the eigenvalues of $(\mathbb{H}_{\Phi^L})_4-\mathbb{A}_5$, are plotted in the following figure

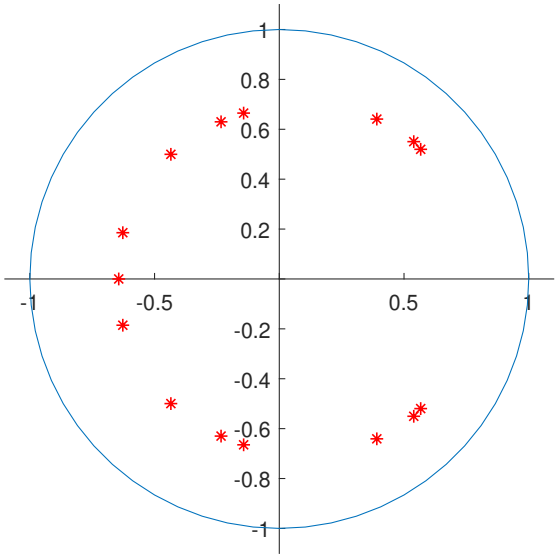

FIGURE 3.2. Zeros of $U_{1,5}^{L}(z)$ with $\zeta=1$ and $\mathbf{M}=\mathbf{I}_3$.

For $n=10$, we have

$$l_{10,0}=-\begin{bmatrix} \frac{1019}{11730} & \frac{13}{345} & -\frac{8}{5865}\\ -\frac{9}{1564} & \frac{3}{46} & \frac{15}{391}\\ \frac{29}{11730} & -\frac{2}{345} & \frac{719}{11730}\end{bmatrix},\ l_{10,1}=\begin{bmatrix} -\frac{1019}{11730} & \frac{3}{391} & \frac{158}{5865}\\ \frac{9}{1564} & -\frac{71}{782} & -\frac{3}{391}\\ -\frac{29}{11730} & \frac{11}{1173} & -\frac{502}{5865}\end{bmatrix},$$

and for $k = 2, \cdots 9$,

$$l_{10,k} = \begin{bmatrix} -\frac{1019}{11730} & \frac{3}{391} & -\frac{29}{7820} \\ \frac{9}{1564} & -\frac{71}{782} & \frac{33}{3128} \\ -\frac{29}{11730} & \frac{11}{1173} & -\frac{387}{3910} \end{bmatrix}.$$

The zeros are in the following figure

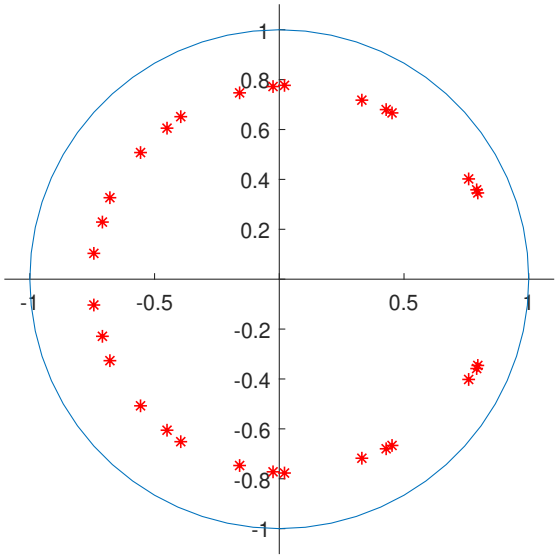

FIGURE 3.3. Zeros of $U^L_{1,10}(z)$ with $\zeta = 1$ and $\mathbf{M} = \mathbf{I}_3$.

3.2 On a Christoffel transformation for a matrix measure with support on unit circle

3.2.1 Introduction

The study of the Christoffel matrix transformation for matrix measures has not received much attention in the literature. There are, however, some contributions in this direction. For instance, for transformations of matrix measures supported on the real line, in [2, 3] the authors consider the sesquilinear form defined by

$$\langle f, g \rangle_\mu = \int_E f(x) w(x) d\mu(x) g(x)^T, \quad f, g \in \mathbb{P}^l[x],$$

where μ is a positive definite matrix measure supported in some compact subset $E \subseteq \mathbb{R}$, f, g are real matrix polynomials, and w is a fixed matrix polynomial. This can be seen as a generalization of the Christoffel transformation for scalar orthogonal polynomials on the real line, and the authors obtain, among other results, connection formulas for the corresponding orthogonal matrix polynomials, which are given in terms of certain quasi-determinants. Transformations of matrix measures supported on $\mathbb{T}$ are much less studied, although the Uvarov matrix transformation, consisting in the addition of a Dirac matrix measure, has been studied in [173] and the above chapter. These contributions are focused in the study of the relative asymptotics between both sequences of orthogonal matrix polynomials. To the best of our knowledge, asymptotic properties associated with the Christoffel transformation on the unit circle have not been studied in the literature.

The results on this subsection are oriented in that direction. We will consider certain Christoffel transformation of a matrix measure supported on the unit circle and our aim is to obtain several algebraic and analytic properties associated with it.

In the scalar case, the Christoffel transformation of a nontrivial positive Borel measure λ supported on $\mathbb{T}$ is defined by (see [88, 133])

$$d\tilde{\lambda}(z) = \overline{(z-\zeta)}d\lambda(z)(z-\zeta) = |z-\zeta|^2 d\lambda(z),\ \zeta \in \mathbb{C}, \tag{3.53}$$

and the associated inner product

$$\langle p, q\rangle_{\tilde{\lambda}} = \langle (z-\zeta)p, (z-\zeta)q\rangle_{\lambda} = \int_{-\pi}^{\pi} \overline{p(e^{i\theta})}q(e^{i\theta})|e^{i\theta}-\zeta|^2 d\lambda(\theta), \quad p, q \in \mathbb{P}^1[z].$$

Notice that $\tilde{\lambda}$ is also positive definite. In this section, we consider the Christoffel transformation of a Hermitian matrix measure σ supported on $\mathbb{T}$, defined by

$$d\sigma_{c_m}(z) = W_m(z)^H d\sigma(z) W_m(z), \tag{3.54}$$

where

$$W_m(z) = \prod_{j=1}^{m}(z\mathbf{I} - A_j) \text{ with } (A_j)_{j=1}^{m} \in \mathcal{M}_l, \quad m > 0,$$

with $A_j \neq A_i$, for $i, j = 1, \ldots, m$. Notice that (3.54) is a natural extension to the matrix case of (3.53). Moreover, the matrix measure σ_{c_k} is the Christoffel transformation of the measure $\sigma_{c_{k-1}}$

$$d\sigma_{c_k}(z) = (z\mathbf{I} - A_k)^H d\sigma_{c_{k-1}}(z)(z\mathbf{I} - A_k), \quad k = 1, \ldots, m, \tag{3.55}$$

with $\sigma_{c_0} := \sigma$. The corresponding right and left sesquilinear forms associated with the measure (3.54) are

$$\langle p, q\rangle_{R,\sigma_{c_m}} = \langle W_m(z)p, W_m(z)q\rangle_{R,\sigma}, \quad p, q \in \mathbb{P}^l[z], \tag{3.56}$$

$$\langle p, q\rangle_{L,\sigma_{c_m}} = \langle pW_m(z), qW_m(z)\rangle_{L,\sigma}, \quad p, q \in \mathbb{P}^l[z]. \tag{3.57}$$

Notice that σ_{c_m} is an $l \times l$ Hermitian matrix measure, and thus in this case there exist sequences $(\mathcal{C}^R_{m,n}(z))_{n\geq 0}$ (right) and $(\mathcal{C}^L_{m,n}(z))_{n\geq 0}$ (left) which are orthonormal with respect to (3.56) and (3.57) respectively.

3.2.2 Connection formula and further properties

In this section, we state several relations between the sequences of **MOPUC** $(\mathcal{C}^R_{m,n}(z))_{n\geq 0}$ and $(\varphi^R_{n+m}(z))_{n\geq 0}$. They will be used in the next subsections to derive relative asymptotic properties. In the sequel, we will only consider the right sequence $(\mathcal{C}^R_{m,n}(z))_{n\geq 0}$ since all the results can be obtained for the left sequence in a similar way. We first need a lemma.

Lemma 3.2.1. *For $n+m \geq 1$ and $A_i \neq A_j$ $i,j = 1,\ldots,m$, the block matrix $\mathcal{K}^R_{n+m-1}(A)$ defined by*

$$\mathcal{K}^R_{n+m-1}(A) = \begin{bmatrix} K^R_{n+m-1}(A_1,A_1) & \cdots & K^R_{n+m-1}(A_m,A_1) \\ \vdots & \ddots & \vdots \\ K^R_{n+m-1}(A_1,A_m) & \cdots & K^R_{n+m-1}(A_m,A_m) \end{bmatrix} \in \mathcal{M}_{lm},$$

is a positive definite matrix.

Proof. First notice that the block matrix $\mathcal{K}^R_{n+m-1}(A)$ is Hermitian because $\left(K^R_{n+m-1}(A_i,A_j)\right)^H = K^R_{n+m-1}(A_j,A_i)$ with $i,j = 1,\ldots m$. Now let us consider the block matrix

$$G^R_{n+m-1}(A) = \begin{bmatrix} \varphi^R_0(A_1) & \cdots & \varphi^R_{n+m-1}(A_1) \\ \vdots & \ddots & \vdots \\ \varphi^R_0(A_m) & \cdots & \varphi^R_{n+m-1}(A_m) \end{bmatrix} \in \mathcal{M}_{lm\times l(n+m)}.$$

Notice that

$$\mathcal{K}^R_{n+m-1}(A) = G^R_{n+m-1}(A)G^R_{n+m-1}(A)^H,$$

and for any nonzero $u \in \mathbb{C}^{lm}$ we have

$$u\mathcal{K}^R_{n+m-1}(A)u^H = uG^R_{n+m-1}(A)\left(uG^R_{n+m-1}(A)\right)^H \geq 0.$$

Since $A_i \neq A_j$ $i,j = 1,\ldots,m$, the inequality is strict. As a consequence, $\mathcal{K}^R_{n+m-1}(A)$ is a positive definite matrix for $n+m \geq 1$. □

Proposition 3.2.2. *Let z be such that $\mathbf{I}z - A_j$ is non-singular for $j = 1,\ldots,m$. The sequence of monic matrix polynomials $(C^R_{m,n}(z))_{n\geq 0}$ defined by*

$$C^R_{m,n}(z) = W_m(z)^{-1}\left[\Phi^R_{n+m}(z) - \mathbb{K}^R_{n+m-1}(A,z)\left(\mathcal{K}^R_{n+m-1}(A)\right)^{-1}\mathbb{F}^R_{n+m}(A)\right], \tag{3.58}$$

where

$$\begin{aligned} \mathbb{K}^R_{n+m-1}(A,z) &:= \left[K^R_{n+m-1}(A_1,z),\ldots,K^R_{n+m-1}(A_m,z)\right] \in \mathcal{M}_{l\times ml}, \\ \mathbb{F}^R_{n+m}(A) &:= \left[\Phi^R_{n+m}(A_1)^T,\ldots,\Phi^R_{n+m}(A_m)^T\right]^T \in \mathcal{M}_{ml\times l}, \end{aligned} \tag{3.59}$$

is orthogonal with respect to the sesquilinear form (3.56). Furthermore,

$$\left(\tilde{\kappa}^{-R}_{m,n}\right)^H \tilde{\kappa}^{-R}_{m,n} = \tilde{S}^R_{m,n} = S^R_{n+m} + \mathbb{F}^{R,H}_{n+m}(A)\left(\mathcal{K}^R_{n+m-1}(A)\right)^{-1}\mathbb{F}^R_{n+m}(A) \tag{3.60}$$

is a positive definite matrix for $n \geq 1$.

Proof. Let us write

$$\Delta^R_{n+m} = -\left(\mathcal{K}^R_{n+m-1}(A)\right)^{-1}\mathbb{F}^R_{n+m}(A). \tag{3.61}$$

Let $0 \leq k \leq n$. Then, from (3.58)

$$\begin{aligned}\langle z^k\mathbf{I}, C^R_{m,n}\rangle_{R,\sigma_{c_m}} &= \left\langle W_m(z)z^k, W_m(z)C^R_{m,n}\right\rangle_{R,\sigma}\\ &= \left\langle W_m(z)z^k, \Phi^R_{n+m}\right\rangle_{R,\sigma} + \left\langle W_m(z)z^k, \mathbb{K}^R_{n+m-1}(A,z)\Delta^R_{n+m}\right\rangle_{R,\sigma}\\ &= S^R_{n+m}\delta_{n,k} + \sum_{j=1}^{m}\left\langle W_m(z)z^k, K^R_{n+m-1}(A_j,z)\right\rangle_{R,\sigma}\lambda^R_{n+m,j}. \qquad (3.62)\end{aligned}$$

Since $K^R_{n+m-1}(A_j,z) = K^R_{n+m}(A_j,z) - \varphi^R_{n+m}(z)\varphi^{R,H}_{n+m}(A_j)$, we have from (1.68)

$$\left\langle W_m(z)z^k, K^R_{n+m-1}(A_j,z)\right\rangle_{R,\sigma} = -\left\langle W_m(z)z^k, \varphi^R_{n+m}(z)\varphi^{R,H}_{n+m}(A_j)\right\rangle_{R,\sigma} = -\Phi^{R,H}_{n+m}(A_j)\delta_{n,k},$$

and, substituting in (3.62),

$$\langle z^k\mathbf{I}, C^R_{m,n}\rangle_{R,\sigma_{c_m}} = \left(S^R_{n+m} - \sum_{j=1}^{m}\Phi^{R,H}_{n+m}(A_j)\lambda^R_{n+m,j}\right)\delta_{n,k} = \left(S^R_{n+m} - \mathbb{F}^{R,H}_{n+m}(A)\Delta^R_{n+m}\right)\delta_{n,k},$$

so that $(C^R_{m,n}(z))_{n\geq 0}$ is orthogonal with respect to (3.56). For $k = n$ and from (3.61) we obtain (3.60). On the other hand, since S^R_{n+m} and $(\mathcal{K}^R_{n+m-1}(A))^{-1}$ are positive definite matrices, then $\tilde{S}^R_{m,n}$ is a positive definite matrix for $n \geq 1$. □

Example 3.2.3. *Consider the matrix measure $d\sigma(\theta) = \Omega\frac{d\theta}{2\pi} \in \mathcal{S} \subset \mathcal{N}$, where $\Omega \in \mathcal{M}_l$ is a non-singular Hermitian matrix. The corresponding n-th degree monic polynomial is*

$$\Phi_n(z) = \Phi^R_n(z) = \Phi^L_n(z) = z^n\mathbf{I}.$$

The Christoffel matrix transformation of σ with $m = 1$ is then defined by

$$d\sigma_{c_1}(z) = (z\mathbf{I} - A_1)^H\Omega\frac{d\theta}{2\pi}(z\mathbf{I} - A_1), \quad z = e^{i\theta}.$$

From (1.55), we have

$$K^R_n(A_1,z) = \Omega^{-1}\sum_{k=0}^{n}\left(zA_1^H\right)^k, \quad K^R_n(A_1,A_1) = \sum_{k=0}^{n}A_1^k\Omega^{-1}(A_1^k)^H.$$

Thus, from (3.58) and (3.60) *with $m = 1$, we obtain for $n \geq 1$*

$$(z\mathbf{I} - A_1)C^R_{1,n}(z) = z^{n+1}\mathbf{I} - \Omega^{-1}\sum_{k=0}^{n}\left(zA_1^H\right)^k\left(\sum_{k=0}^{n}A_1^k\Omega^{-1}(A_1^k)^H\right)^{-1}A_1^{n+1},$$

$$\tilde{S}^R_{1,n} = \Omega + (A^{n+1})^H\left(\sum_{k=0}^{n}A_1^k\Omega^{-1}(A_1^k)^H\right)^{-1}A_1^{n+1}.$$

In particular, if $A_1 = \zeta_1\mathbf{I}$ with $\zeta_1 \in \mathbb{C}$ we have two cases:

1. $\zeta_1 \notin \mathbb{T}$, *since* $\varphi_n^R(z) = z^n\Omega^{-1/2}$ *we have from (1.55) with* $\zeta = \zeta_1$

$$K_n^R(\zeta_1, z) = \Omega^{-1}\sum_{k=0}^{n}(\bar{\zeta}_1 z)^k = \Omega^{-1}\frac{1-(\bar{\zeta}_1 z)^{n+1}}{1-\bar{\zeta}_1 z},$$

and therefore for $z = \zeta_1$, *we get*

$$(K_n^R(\zeta_1, \zeta_1))^{-1} = \Omega\frac{1-|\zeta_1|^2}{1-|\zeta_1|^{2(n+1)}}.$$

Thus, from (3.58) *with* $m = 1$, *for* $n \geq 1$, *we obtain in this diagonal case*

$$\begin{aligned} C_{1,n}^R(z) &= \frac{1}{z-\zeta_1}\left(z^{n+1} - \zeta_1^{n+1}\frac{1-|\zeta_1|^2}{1-|\zeta_1|^{2(n+1)}}\sum_{k=0}^{n}(\bar{\zeta}_1 z)^k\right)\mathbf{I} \\ &= \frac{1}{z-\zeta_1}\left(z^{n+1} - \zeta_1^{n+1}\frac{1-|\zeta_1|^2}{1-|\zeta_1|^{2(n+1)}}\frac{1-(\bar{\zeta}_1 z)^{n+1}}{1-\bar{\zeta}_1 z}\right)\mathbf{I}, \end{aligned}$$

and from (3.60) *with* $m = 1$, *since* $S_n^R = \Omega$ *for* $n \geq 0$,

$$\tilde{S}_{1,n}^R = \Omega\left(1 + |\zeta_1|^{2(n+1)}\frac{1-|\zeta_1|^2}{1-|\zeta_1|^{2(n+1)}}\right).$$

2. $\zeta_1 \in \mathbb{T}$, *from (1.55) with* $z = \zeta = \zeta_1$ *we have*

$$(K_n(\zeta_1, \zeta_1))^{-1} = \left(\Omega^{-1}\sum_{k=0}^{n}1\right)^{-1} = \frac{1}{n+1}\Omega$$

and thus from 3.58 with $m = 1$, *we obtain for* $n \geq 1$

$$(z-\zeta_1)C_{1,n}^R(z) = \left(z^{n+1} - \frac{\zeta_1^{n+1}}{n+1}\sum_{k=0}^{n}(\bar{\zeta}_1 z)^k\right)\mathbf{I},$$

or, equivalently,

$$C_{1,n}^R(z) = \left(z^n + \zeta_1\frac{n}{n+1}C_{1,n-1}^R(z)\right)\mathbf{I}, \quad n \geq 1.$$

Furthermore, from (3.60) *with* $m = 1$, $\tilde{S}_{1,n}^R = \left(1 + \frac{1}{n+1}\right)\Omega$.

Notice that this example generalizes to the matrix case the Christoffel transformation of the normalized Lebesgue measure studied in [20].

Now, we will show that σ_{c_m} belongs to the Szegő class by using a different approach to the one used in the scalar case.

Lemma 3.2.4. *If* σ *belongs to* $\mathcal{S}$, *then* σ_{c_m} *belongs to* $\mathcal{S}$.

Proof. From 3.60, since $\mathbb{F}_{n+m}^{R,H}(A)^H \left(\mathcal{K}_{n+m-1}^R(A)\right)^{-1} \mathbb{F}_{n+m}^R(A)$ is a positive semi-definite matrix, then for all $n \geq 1$ we have $S_{n+m}^R \leq \tilde{S}_{m,n}^R$. From [48], we get

$$S_{n+m}^R = \left\langle \Phi_{n+m}^R, \Phi_{n+m}^R \right\rangle_{R,\sigma} = \left(\kappa_{n+m}^R\right)^{-H} \kappa_{n+m}^{-R} = \left(B_{n+m}(0)\right)^{-1},$$
$$\tilde{S}_{m,n}^R = \left\langle U_{m,n}^R, U_{m,n}^R \right\rangle_{R,\sigma_{c_m}} = \left(\tilde{\kappa}_{m,n}^R\right)^{-H} \tilde{\kappa}_{m,n}^{-R} = \left(\tilde{B}_{m,n}(0)\right)^{-1},$$

so that $\tilde{B}_{m,n}(0) \leq B_{n+m}(0)$. Notice that $\left(\det B_{n+m}(0)\right)_{n\geq 0}$ is bounded from above, since σ belongs to $\mathcal{S}$, and thus $\left(\det \tilde{B}_{m,n}(0)\right)_{n\geq 0}$ is bounded from above. From [48] Theorem 19, it follows that σ_{c_m} belongs to $\mathcal{S}$. □

Since the $n-$th orthonormal matrix polynomial has the form

$$\mathcal{C}_{m,n}^R(z) = C_{m,n}^R(z)\tilde{\kappa}_{m,n}^R = \tilde{\kappa}_{m,n}^R z^n + \cdots,$$

then from (3.58) we have

$$W_m(z)\mathcal{C}_{m,n}^R(z) = \left[\varphi_{n+m}^R(z) - \mathbb{K}_{n+m-1}^R(A,z)\left(\mathcal{K}_{n+m-1}^R(A)\right)^{-1}\mathbb{L}_{n+m}^R(A)\right]\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R, \tag{3.63}$$

where $\mathbb{L}_{n+m}^R(A) := \left[\varphi_{n+m}^R(A_1)^T, \ldots, \varphi_{n+m}^R(A_m)^T\right]^T \in \mathcal{M}_{ml\times l}$. Furthermore, taking into account that $S_{n+m}^R = (\kappa_{n+m}^{-R})^H \kappa_{n+m}^{-R}$ from (3.60) we can deduce

$$\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R(\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R)^H = \left[\mathbf{I} + \mathbb{L}_{n+m}^{R,H}(A)\left(\mathcal{K}_{n+m-1}^R(A)\right)^{-1}\mathbb{L}_{n+m}^R(A)\right]^{-1}. \tag{3.64}$$

Lemma 3.2.5. *For $n \geq 0$, we have*

$$0 < \left\|\left(\kappa_{n+m}^R\right)^{-1}\tilde{\kappa}_{m,n}^R\right\|_2^2 \leq 1 \text{ and } 0 < \left\|\tilde{\kappa}_{m,n}^L\left(\kappa_{n+m}^L\right)^{-1}\right\|_2^2 \leq 1. \tag{3.65}$$

Proof. From (3.56) and (3.63), we have

$$\begin{aligned}\mathbf{I} = \langle \mathcal{C}_{m,n}^R, \mathcal{C}_{m,n}^R\rangle_{R,\sigma_{c_m}} &= \left\langle W_m(z)\mathcal{C}_{m,n}^R, \varphi_{n+m}^R\right\rangle_{R,\sigma}\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R \\ &- \left\langle W_m(z)\mathcal{C}_{m,n}^R, \mathbb{K}_{n+m-1}^R(A,z)\right\rangle_{R,\sigma}\left(\mathcal{K}_{n+m-1}^R(A)\right)^{-1}\mathbb{L}_{n+m}^R(A)\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R.\end{aligned} \tag{3.66}$$

On the one hand, we have

$$\left\langle W_m(z)\mathcal{C}_{m,n}^R, \varphi_{n+m}^R\right\rangle_{R,\sigma} = \left(\tilde{\kappa}_{m,n}^R\right)^H\left(\kappa_{n+m}^{-R}\right)^H\langle\varphi_{n+m}^R, \varphi_{n+m}^R\rangle_{R,\sigma} = \left(\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right)^H, \tag{3.67}$$

Moreover, since $\mathbb{K}_{n+m-1}^R(A,z)$ is the block vector defined in 3.59, we have for the j-th block entry

$$K_{n+m-1}^R(A_j,z) = K_{n+m}^R(A_j,z) - \varphi_{n+m}^R(z)\varphi_{n+m}^{R,H}(A_j), \quad j = 1,\ldots,m,$$

then from (1.68) for each entry we have

$$\begin{aligned}&\left\langle W_m(z)\mathcal{C}_{m,n}^R, K_{n+m-1}^R(A_j,z)\right\rangle_{R,\sigma}\\ &\quad=\left\langle W_m(z)\mathcal{C}_{m,n}^R, K_{n+m}^R(A_j,z)\right\rangle_{R,\sigma}-\left\langle W_m(z)\mathcal{C}_{m,n}^R,\varphi_{n+m}^R\right\rangle_{R,\sigma}\varphi_{n+m}^{R,H}(A_j)\\ &\quad=-\left(\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right)^H\varphi_{n+m}^{R,H}(A_j),\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\left\langle W_m(z)\mathcal{C}_{m,n}^R, \mathbb{K}_{n+m-1}^R(A,z)\right\rangle_{R,\sigma}&=-\left(\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right)^H\mathbb{L}_{n+m}^{R,H}(A)\\ &=-\left(\mathbb{L}_{n+m}^R(A)\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right)^H.\end{aligned}\tag{3.68}$$

Thus, substituting (3.67) and (3.68) in (3.66), we get

$$\begin{aligned}&\mathbf{I}-\left(\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right)^H\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\\ &\qquad=\left(\mathbb{L}_{n+m}^R(A)\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right)^H\left(\mathcal{K}_{n+m-1}^R(A)\right)^{-1}\left(\mathbb{L}_{n+m}^R(A)\kappa_{n+m}^{-R}\tilde{\kappa}_{m,n}^R\right).\end{aligned}$$

Since the right-hand member is positive semi-definite because $\mathcal{K}_{n+m-1}^R(A)$ is positive definite, then

$$\left(\left(\kappa_{n+m}^R\right)^{-1}\tilde{\kappa}_{m,n}^R\right)^H\left(\kappa_{n+m}^R\right)^{-1}\tilde{\kappa}_{m,n}^R\leq\mathbf{I}$$

so that (3.65) follows. □

Lemma 3.2.6. *For $n\geq 0$, the matrices $\left(\kappa_{m+n}^R\right)^{-1}\tilde{\kappa}_{m,n}^R$ and $\left(\tilde{\kappa}_{m,n}^L\left(\kappa_{m+n}^L\right)^{-1}\right)^H$ can be assumed to be lower triangular with positive entries on the main diagonal.*

Proof. If $(\mathcal{C}_{m,n}^R(z))_{n\geq 0}$ is such that the matrices $\left(\left(\kappa_{n+m}^R\right)^{-1}\tilde{\kappa}_{m,n}^R\right)^H$, $n\geq 0$, are not lower triangular matrices with positive entries in the main diagonal, then we can find a sequence of unitary matrices $(Q_n)_{n\geq 0}$ such that

$$\mathcal{C}_{m,n}^R(z)Q_n=\tilde{\lambda}_{m,n}^R z^n+\cdots,$$

is a sequence of orthonormal matrix polynomials such that $\left(\left(\kappa_{n+m}^R\right)^{-1}\tilde{\lambda}_{m,n}^R\right)^H$ are upper triangular matrices with positive entries on the main diagonal. To show this, notice that the sequences of orthonormal matrix polynomials associated with the measures σ and σ_{c_m} have the form

$$(\varphi_n^R(z)\nu_n)_{n\geq 0}\ \text{ and }(\mathcal{C}_{m,n}^R(z)\tau_n)_{n\geq 0},$$

where ν_n and τ_n are unitary matrices. As a consequence, their coefficients are related by $\lambda_n^R=\kappa_n^R\nu_n$ and $\tilde{\lambda}_{m,n}^R=\tilde{\kappa}_{m,n}^R\tau_n$, and therefore

$$\left(\left(\lambda_{n+m}^R\right)^{-1}\tilde{\lambda}_{m,n}^R\right)^H=\left(\left(\kappa_{n+m}^R\nu_{n+m}\right)^{-1}\tilde{\kappa}_{m,n}^R\tau_n\right)^H=\tau_n^H\left(\left(\kappa_{n+m}^R\right)^{-1}\tilde{\kappa}_{m,n}^R\right)^H\nu_{n+m}.$$

Taking $\nu_{n+m} = \mathbf{I}$, we get $\kappa_{n+m} > \mathbf{0}$ (see [5]), and thus

$$\left(\left(\lambda^R_{n+m}\right)^{-1}\tilde{\lambda}^R_{m,n}\right)^H = \tau_n^H\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{m,n}\right)^H. \tag{3.69}$$

Now, consider the sequence $(\mathcal{C}^R_{m,n}(z)Q^H_n)_{n\geq 0}$, where $(Q^H_n)_{n\geq 0}$ are the unitary matrices given by the Francis and Kublanovskaja's QR factorization (see [108]) of $\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{m,n}\right)_{n\geq 0}$, i.e.

$$\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{m,n} = S_nQ_n,$$

where $(S_n)_{n\geq 0}$ are lower triangular matrices. Thus, $\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{m,n}\right)^H$ can be expressed as

$$\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{m,n}\right)^H = \tilde{Q}_n\tilde{S}_n,$$

where $\tilde{Q}_n$ is a unitary matrix and $\tilde{S}_n$ is a nonsingular upper triangular matrix. Therefore, taking

$$S_n = \tilde{S}^H_nJ_n \text{ and } Q_n = J^H_n\tilde{Q}^H_n,$$

where

$$(J_n)_{i,j} = \left\{\frac{(\tilde{S}_n)_{i,i}}{|(\tilde{S}_n)_{i,i}|} \text{ if } i=j \text{ and } 0 \text{ otherwise } \right\},$$

we have $S_nQ_n = \tilde{S}^H_nJ^H_nJ_n\tilde{Q}^H_n = (\tilde{Q}_n\tilde{S}_n)^H$. Then, from (3.69),

$$\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\lambda}^R_{m,n}\right)^H = Q_n\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{m,n}\right)^H = Q_n\tilde{Q}_n\tilde{S}_n = J^H_n\tilde{S}_n = S^H_n$$

and thus $\left(\left(\kappa^R_{n+m}\right)^{-1}\tilde{\lambda}^R_{m,n}\right)^H$ is an upper triangular matrix with positive entries on the main diagonal, since

$$diag(S^H_n) = (|(\tilde{S}_n)_{1,1}|, |(\tilde{S}_n)_{2,2}|, \cdots, |(\tilde{S}_n)_{l,l}|).$$

□

3.2.3 Relative Asymptotics

In this subsection, we deal with relative asymptotics for the Christoffel matrix transformation in the diagonal case, i.e. $A_j = \zeta_j\mathbf{I}$, where $\zeta_j \in \mathbb{C}$ for $j = 1, \ldots, m$. Therefore, here we consider $W_m(z) = \prod_{j=1}^m(z-\zeta_j)\mathbf{I}$. For the scalar case, relative asymptotics for sequences of orthogonal polynomials associated with Christoffel type perturbations of scalar measures in $\mathcal{N}$ supported on the unit circle have been widely studied in the literature (see, among others, [93, 133, 135]).

First, we will show that the matrix measure (3.54) belongs to $\mathcal{N}$. This will allow us to prove several asymptotic properties in the remaining of the chapter.

Lemma 3.2.7. *If $\zeta \in \mathbb{C}\setminus\mathbb{D}$ and $\sigma \in \mathcal{N}$, then $\sigma_{c_m} \in \mathcal{N}$.*

Proof. Taking into account that $\varphi^{L,*}_{n+1}(0) = (\kappa^L_{n+1})^H$, we obtain from the Christoffel-Darboux formula (1.56) $K^R_n(\zeta_1, 0) = (\kappa^L_{n+1})^H\varphi^{L,*}_{n+1}(\zeta_1)^H - \varphi^R_{n+1}(0)\varphi^{R,H}_{n+1}(\zeta_1)$, so evaluating

$z=0$ on (3.58) with $m=1$, we get

$$\begin{aligned}-\zeta_1 C_{1,n}^R(0) = \Phi_{n+1}^R(0) - (\kappa_{n+1}^L)^H \varphi_{n+1}^{L,*}(\zeta_1)^H K_n^{-R}(\zeta_1,\zeta_1)\Phi_{n+1}^R(\zeta_1) \\ + \varphi_{n+1}^R(0)\varphi_{n+1}^{R,H}(\zeta_1)K_n^{-R}(\zeta_1,\zeta_1)\Phi_{n+1}^R(\zeta_1).\end{aligned} \tag{3.70}$$

From (1.38), $\alpha_n^H = -(\kappa_n^{-L})^H \Phi_{n+1}^R(0)\kappa_n^R$, multiplying on the left by $(\kappa_n^{-L})^H$ and on the right by κ_n^R the equation (3.70), we get

$$\begin{aligned}\zeta_1(\kappa_n^{-L})^H C_{1,n}^R(0)\kappa_n^R =& \alpha_n^H + (\kappa_{n+1}^L\kappa_n^{-L})^H \varphi_{n+1}^{L,*}(\zeta_1)^H K_n^{-R}(\zeta_1,\zeta_1)\varphi_{n+1}^R(\zeta_1)\kappa_{n+1}^{-R}\kappa_n^R \\ & - (\kappa_n^{-L})^H \varphi_{n+1}^R(0)\varphi_{n+1}^{R,H}(\zeta_1)K_n^{-R}(\zeta_1,\zeta_1)\varphi_{n+1}^R(\zeta_1)\kappa_{n+1}^{-R}\kappa_n^R.\end{aligned}$$

On the one hand, since $\sigma\in\mathcal{N}$ from (1.33) and (1.36) we get that $\kappa_{n+1}^L\kappa_n^{-L}\to\mathbf{I}$, $\kappa_{n+1}^{-R}\kappa_n^R\to\mathbf{I}$, and

$$\varphi_{n+1}^{L,*}(\zeta_1)^H K_n^{-R}(\zeta_1,\zeta_1)\varphi_{n+1}^R(\zeta_1) = (\varphi_{n+1}^R(\zeta_1)^{-1}\varphi_{n+1}^{L,*}(\zeta_1))^H \varphi_{n+1}^{R,H}(\zeta_1)K_n^{-R}(\zeta_1,\zeta_1)\varphi_{n+1}^R(\zeta_1),$$

so from (1.80), (1.82) and (1.83) we obtain

$$\lim_{n\to\infty}(\kappa_{n+1}^L\kappa_n^{-L})^H \varphi_{n+1}^{L,*}(\zeta_1)^H K_n^{-R}(\zeta_1,\zeta_1)\varphi_{n+1}^R(\zeta_1)\kappa_{n+1}^{-R}\kappa_n^R = \mathbf{0}.$$

On the other hand, since $\alpha_n^H = -(\kappa_n^{-L})^H\Phi_{n+1}^R(0)\kappa_n^R = -(\kappa_n^{-L})^H\varphi_{n+1}^R(0)\kappa_{n+1}^{-R}\kappa_n^R$, and $\kappa_{n+1}^{-R}\kappa_n^R\to\mathbf{I}$, therefore

$$\lim_{n\to\infty}(\kappa_n^{-L})^H\varphi_{n+1}^R(0) = \mathbf{0},$$

so from (1.82) and (1.83), we get

$$\lim_{n\to\infty}(\kappa_n^{-L})^H\varphi_{n+1}^R(0)\varphi_{n+1}^{R,H}(\zeta_1)K_n^{-R}(\zeta_1,\zeta_1)\varphi_{n+1}^R(\zeta_1)\kappa_{n+1}^{-R}\kappa_n^R = \mathbf{0}.$$

In this way

$$\lim_{n\to\infty}(\kappa_n^{-L})^H C_{1,n}^R(0)\kappa_n^R = \mathbf{0}. \tag{3.71}$$

Let $\tilde{\alpha}_{1,n}^H = -(\tilde{\kappa}_{1,n-1}^{-L})^H C_{1,n}^R(0)\tilde{\kappa}_{1,n-1}^R \Rightarrow C_{1,n}^R(0) = -(\tilde{\kappa}_{1,n-1}^L)^H\tilde{\alpha}_{1,n}^H\tilde{\kappa}_{1,n-1}^{-R}$, substituting in (3.71)

$$\lim_{n\to\infty}\left(\tilde{\kappa}_{1,n-1}^L\kappa_n^{-L}\right)^H\tilde{\alpha}_{1,n}^H\left(\tilde{\kappa}_{1,n-1}^{-R}\kappa_n^R\right) = \mathbf{0},$$

from (3.65) it follows $\lim_{n\to\infty}\tilde{\alpha}_{1,n}^H = \mathbf{0}$. Therefore, we conclude that $\sigma_{c_1}\in\mathcal{N}$.

For $m>1$, taking into account (3.55) and applying the previous argument to $C_{k,n}^R(z)$ in terms of $C_{k-1,n}^R(z)$, the lemma is proved. □

As expected, the asymptotic behavior of the polynomials will depend on the location of the mass points.

3.2.3.1 Case $\zeta_j\in\mathbb{C}\setminus\bar{\mathbb{D}}$

Lemma 3.2.8. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{C}\setminus\bar{\mathbb{D}}$ for $j=1,\dots,m$ then*

$$\lim_{n\to\infty}\left(\kappa_{n+m}^R\right)^{-1}\tilde{\kappa}_{m,n}^R = \prod_{j=1}^m\frac{1}{|\zeta_j|}\mathbf{I}. \tag{3.72}$$

Proof. We use induction on m. From (3.64) with $m = 1$, we get

$$(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R})(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R})^{H} = \left[\mathbf{I} + \varphi_{n+1}^{R,H}(\zeta_1)\left(K_n^R(\zeta_1,\zeta_1)\right)^{-1}\varphi_{n+1}^{R}(\zeta_1)\right]^{-1}. \tag{3.73}$$

Therefore, from (1.82) with $z = \zeta = \zeta_1$

$$\lim_{n\to\infty}(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R})(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R})^{H} = \left(\mathbf{I} + \left(\frac{1}{|\zeta_1|^2-1}\right)^{-1}\mathbf{I}\right)^{-1} = \frac{1}{|\zeta_1|^2}\mathbf{I}. \tag{3.74}$$

Moreover, from (3.65) with $m = 1$

$$\left\|\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R}\right\|_F^2 = Tr((\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R})^{H}\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R}) \leq l\left\|\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R}\right\|_2^2 \leq l,$$

then we necessarily have

$$\lim_{n\to\infty}\left(\kappa_{n+1}^{R}\right)^{-1}\tilde{\kappa}_{1,n}^{R} = \frac{1}{|\zeta_1|}\mathbf{I}, \tag{3.75}$$

since otherwise we can consider a subsequence $(n_v)_{v\geq 0}$ of positive integers such that

$$\lim_{v\to\infty}\left(\kappa_{n_v}^{R}\right)^{-1}\tilde{\kappa}_{1,n_v}^{R} = \mathbf{D} < \infty,$$

and then from (3.74) we have

$$\mathbf{D}\mathbf{D}^{H} = \frac{1}{|\zeta_1|^2}\mathbf{I}. \tag{3.76}$$

Since $\frac{1}{|\zeta_1|^2}\mathbf{I}$ is positive definite and $\mathbf{D} \in \mathcal{M}_l$ has positive entries on the main diagonal from Lemma 3.2.6, then the Cholesky factorization (see 1.1.2 item 4) of (3.76) is unique and we deduce $(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^{R})_{n\geq 1}$ does not have subsequences converging to a limit other than $\frac{1}{|\zeta_1|}\mathbf{I}$.

Furthermore, from Lemma 3.2.7, σ_{c_k} belongs to $\mathcal{N}$ for $k = 2,\ldots,m$, and each σ_{c_k} satisfies (3.55), then we can apply the previous argument to $C_{k,n}^{R}(z) = z^n\tilde{\kappa}_{k,n}^{R} + \cdots$ in terms of $C_{k-1,n+1}^{R}(z) = z^n\tilde{\kappa}_{k-1,n+1}^{R} + \cdots$ to obtain

$$\lim_{n\to\infty}\left(\tilde{\kappa}_{k-1,n+1}^{R}\right)^{-1}\tilde{\kappa}_{k,n}^{R} = \frac{1}{|\zeta_k|}\mathbf{I} \text{ with } k = 2,\ldots,m. \tag{3.77}$$

Since

$$\begin{aligned}\left(\kappa_{n+m}^{R}\right)^{-1}\tilde{\kappa}_{m,n}^{R} \quad &= \left(\kappa_{n+m}^{R}\right)^{-1}\tilde{\kappa}_{1,n+m-1}^{R}\left(\tilde{\kappa}_{1,n+m-1}^{R}\right)^{-1}\tilde{\kappa}_{2,n+m-2}^{R}\cdots \\ &\quad \cdots\left(\tilde{\kappa}_{m-2,n+2}^{R}\right)^{-1}\tilde{\kappa}_{m-1,n+1}^{R}\left(\tilde{\kappa}_{m-1,n+1}^{R}\right)^{-1}\tilde{\kappa}_{m,n}^{R}\end{aligned} \tag{3.78}$$

from (3.75) and (3.77) we deduce

$$\begin{aligned}
\lim_{n\to\infty}\left(\kappa^R_{n+m}\right)^{-1}\tilde{\kappa}^R_{1,n+m-1} &= \lim_{n\to\infty}\left(\kappa^R_{n+1}\right)^{-1}\tilde{\kappa}^R_{1,n} = \frac{1}{|\zeta_1|}\mathbf{I},\\
\lim_{n\to\infty}\left(\tilde{\kappa}^R_{1,n+m-1}\right)^{-1}\tilde{\kappa}^R_{2,n+m-2} &= \lim_{n\to\infty}\left(\tilde{\kappa}^R_{1,n+1}\right)^{-1}\tilde{\kappa}^R_{2,n} = \frac{1}{|\zeta_2|}\mathbf{I},\\
&\vdots\\
\lim_{n\to\infty}\left(\tilde{\kappa}^R_{m-2,n+2}\right)^{-1}\tilde{\kappa}^R_{m-1,n+1} &= \lim_{n\to\infty}\left(\tilde{\kappa}^R_{m-2,n+1}\right)^{-1}\tilde{\kappa}^R_{m-1,n} = \frac{1}{|\zeta_{m-1}|}\mathbf{I},\\
\lim_{n\to\infty}\left(\tilde{\kappa}^R_{m-1,n+1}\right)^{-1}\tilde{\kappa}^R_{m,n} &= \frac{1}{|\zeta_m|}\mathbf{I},
\end{aligned}$$

This together with (3.78) implies (3.72). □

Proposition 3.2.9. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{C}\setminus\bar{\mathbb{D}}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\left(\varphi^R_{n+m}(z)\right)^{-1}\mathcal{C}^R_{m,n}(z) = \prod_{j=1}^{m}\frac{\bar{\zeta}_j}{|\zeta_j|}\frac{1}{z\bar{\zeta}_j-1}\mathbf{I}, \tag{3.79}$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus\mathbb{D}$.

Proof. Let $|z|\geq 1$. From (3.63) with $m=1$ and multiplying on the left by $(\varphi^R_{n+1}(z))^{-1}$, we get

$$\left(\varphi^R_{n+1}(z)\right)^{-1}\left((z-\zeta_1)\mathcal{C}^R_{1,n}(z)\right) = (\mathbf{I}-B_n(z,\zeta_1))\kappa^{-R}_{n+1}\tilde{\kappa}^R_{1,n}, \tag{3.80}$$

where

$$B_n(z,\zeta_1) = \varphi^R_{n+1}(z)^{-1}K^R_n(\zeta_1,z)\varphi^{R,H}_{n+1}(\zeta_1)^{-1}\varphi^{R,H}_{n+1}(\zeta_1)\left(K^R_n(\zeta_1,\zeta_1)\right)^{-1}\varphi^R_{n+1}(\zeta_1),$$

and thus using (1.82) (with $\zeta=\zeta_1$ for the first factor and $z=\zeta=\zeta_1$ for the second factor)

$$\lim_{n\to\infty}B_n(z,\zeta_1) = \left(\frac{1}{z\bar{\zeta}_1-1}\right)\left(\frac{1}{|\zeta_1|^2-1}\right)^{-1}.$$

Therefore, taking limit when n tends to infinity in (3.80) and using (3.72) with $m=1$, we obtain

$$\lim_{n\to\infty}\left(\varphi^R_{n+1}(z)\right)^{-1}\left((z-\zeta_1)\mathcal{C}^R_{1,n}(z)\right) = \frac{\bar{\zeta}_1}{|\zeta_1|}\frac{z-\zeta_1}{z\bar{\zeta}_1-1}\mathbf{I},$$

and thus

$$\lim_{n\to\infty}\left(\varphi^R_{n+1}(z)\right)^{-1}\mathcal{C}^R_{1,n}(z) = \frac{\bar{\zeta}_1}{|\zeta_1|}\frac{1}{z\bar{\zeta}_1-1}\mathbf{I}. \tag{3.81}$$

On the other hand, since all σ_{c_k} with $k=2,\dots,m$ belong to $\mathcal{N}$, and taking into account (3.55), we can apply the previous argument to $\mathcal{C}^R_{k,n}(z)$ in terms of $\mathcal{C}^R_{k-1,n+1}(z)$, to obtain

$$\lim_{n\to\infty}\left(\mathcal{C}^R_{k-1,n+1}(z)\right)^{-1}\mathcal{C}^R_{k,n}(z) = \frac{\bar{\zeta}_k}{|\zeta_k|}\frac{1}{z\bar{\zeta}_k-1}\mathbf{I},\quad k=2,\dots,m. \tag{3.82}$$

Since

$$\varphi_{n+m}^R(z)^{-1}\mathcal{C}_{m,n}^R(z) = \left(\varphi_{n+m}^R(z)^{-1}\mathcal{C}_{1,n+m-1}^R(z)\right)\left(\mathcal{C}_{1,n+m-1}^R(z)^{-1}\mathcal{C}_{2,n+m-2}^R(z)\right) \cdots \left(\mathcal{C}_{m-2,n+2}^R(z)^{-1}\mathcal{C}_{m-1,n+1}^R(z)\right)\left(\mathcal{C}_{m-1,n+1}^R(z)^{-1}\mathcal{C}_{m,n}^R(z)\right), \tag{3.83}$$

then from (3.81) and (3.82) we get

$$\begin{aligned}
\lim_{n\to\infty} \varphi_{n+m}^R(z)^{-1}\mathcal{C}_{1,n+m-1}^R(z) &= \lim_{n\to\infty} \varphi_{n+1}^R(z)^{-1}\mathcal{C}_{1,n}^R(z) = \frac{\bar{\zeta}_1}{|\zeta_1|}\frac{1}{z\bar{\zeta}_1 - 1}\mathbf{I},\\
\lim_{n\to\infty} \mathcal{C}_{1,n+m-1}^R(z)^{-1}\mathcal{C}_{2,n+m-2}^R(z) &= \lim_{n\to\infty} \mathcal{C}_{1,n+1}^R(z)^{-1}\mathcal{C}_{2,n}^R(z) = \frac{\bar{\zeta}_2}{|\zeta_2|}\frac{1}{z\bar{\zeta}_2 - 1}\mathbf{I},\\
&\vdots\\
\lim_{n\to\infty} \mathcal{C}_{m-2,n+2}^R(z)^{-1}\mathcal{C}_{m-1,n+1}^R(z) &= \lim_{n\to\infty} \mathcal{C}_{m-2,n+1}^R(z)^{-1}\mathcal{C}_{m-1,n}^R(z)\\
&= \frac{\bar{\zeta}_{m-1}}{|\zeta_{m-1}|}\frac{1}{z\bar{\zeta}_{m-1} - 1}\mathbf{I},\\
\lim_{n\to\infty} \mathcal{C}_{m-1,n+1}^R(z)^{-1}\mathcal{C}_{m,n}^R(z) &= \frac{\bar{\zeta}_m}{|\zeta_m|}\frac{1}{z\bar{\zeta}_m - 1}\mathbf{I}.
\end{aligned}$$

As a consequence, from (3.83) we conclude (3.79) for $|z| \geq 1$. □

Corollary 3.2.10. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{C}\setminus\bar{\mathbb{D}}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \left(\varphi_{n+m}^R(z)\right)^{-1}\left(W_m(z)\mathcal{C}_{m,n}^R(z)\right) = \prod_{j=1}^{m} \frac{\bar{\zeta}_j}{|\zeta_j|}\frac{z-\zeta_j}{z\bar{\zeta}_j - 1}\mathbf{I},$$

uniformly on every compact subset of $z \in \mathbb{C}\setminus\mathbb{D}$.

Since $\left(W_m(z)\mathcal{C}_{m,n}^R(z)\right)^*\left(\varphi_{n+m}^{R,*}(z)\right)^{-1} = \left(\varphi_{n+m}^R(1/\bar{z})^{-1}(W_m(z)\mathcal{C}_{m,n}(1/\bar{z}))\right)^H$ from the previous proposition we get the following result.

Corollary 3.2.11. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{C}\setminus\bar{\mathbb{D}}$ for $j = 1, \ldots, m$, then*

$$\begin{aligned}
\prod_{j=1}^{m} \frac{\zeta_j}{|\zeta_j|}\frac{z\bar{\zeta}_j - 1}{z-\zeta_j}\mathbf{I} &= \lim_{n\to\infty} \left(W_m(z)\mathcal{C}_{m,n}^R(z)\right)^*\left(\varphi_{n+m}^{R,*}(z)\right)^{-1}\\
&= \lim_{n\to\infty} \left(\varphi_{n+m}^{L,*}(z)\right)^{-1}\left(W_m(z)\mathcal{C}_{m,n}^L(z)\right)^*,
\end{aligned} \tag{3.84}$$

uniformly on every compact subset of $z \in \mathbb{D}$.

Moreover, since

$$\begin{aligned}
(\mathcal{C}_{m,n}^R(z))^{-1}\mathcal{C}_{m,n+1}^R(z) = &\left[\varphi_{n+m}^R(z)^{-1}(W_m(z)\mathcal{C}_{m,n}^R(z))\right]^{-1}\\
&\times \varphi_{n+m}^R(z)^{-1}\varphi_{n+m+1}^R(z)\left[\varphi_{n+m+1}^R(z)^{-1}(W_m(z)\mathcal{C}_{m,n+1}^R(z))\right],
\end{aligned}$$

another immediate consequence of the above proposition and (1.75) is the following.

Corollary 3.2.12. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{C}\setminus\bar{\mathbb{D}}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} (\mathcal{C}_{m,n}^R(z))^{-1}\mathcal{C}_{m,n+1}^R(z) = z\mathbf{I},$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus (\bar{\mathbb{D}} \cup \{\zeta_j\}_{j=1}^m)$.

Proposition 3.2.13. *If* $\sigma \in \mathcal{N}$ *and* $\zeta_j \in \mathbb{C} \setminus \bar{\mathbb{D}}$ *for* $j = 1, \ldots, m$, *then*

$$\lim_{n\to\infty} \left(\varphi_{n+m}^{L,*}(z)\right)^{-1} \mathcal{C}_{m,n}^R(z) = \mathbf{0}, \tag{3.85}$$

uniformly on every compact subset of $z \in \mathbb{D}$.

Proof. For $m = 1$, from (3.63) with $m = 1$ and multiplying on the left by $(\varphi_{n+1}^{L,*}(z))^{-1}$, we get

$$\varphi_{n+1}^{L,*}(z)^{-1} \left(W_1(z)\mathcal{C}_{1,n}^R(z)\right) = \left[\varphi_{n+1}^{L,*}(z)^{-1}\varphi_{n+1}^R(z) - B_n(z,\zeta_1)\right] \kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R \tag{3.86}$$

where

$$\begin{aligned} B_n(z,\zeta_1) &= \varphi_{n+1}^{L,*}(z)^{-1} K_n^R(\zeta_1, z)(K_n^R(\zeta_1,\zeta_1))^{-1}\varphi_{n+1}^R(\zeta_1) \\ &= \varphi_{n+1}^{L,*}(z)^{-1} K_n^R(\zeta_1, z)\varphi_{n+1}^{R,H}(\zeta_1)^{-1}\varphi_{n+1}^{R,H}(\zeta_1)(K_n^R(\zeta_1,\zeta_1))^{-1}\varphi_{n+1}^R(\zeta_1) \\ &= \left[\frac{(\varphi_{n+1}^R(\zeta_1)^{-1}\varphi_{n+1}^{L,*}(\zeta_1))^H - \varphi_{n+1}^{L,*}(z)^{-1}\varphi_{n+1}^R(z)}{1 - z\zeta_1}\right] \\ &\quad \times \left[\varphi_{n+1}^R(\zeta_1)^{-1} K_n^R(\zeta_1,\zeta_1)\varphi_{n+1}^{R,H}(\zeta_1)^{-1}\right]^{-1} \end{aligned}$$

so from Proposition 1.4.17 and (1.82), we obtain that $\lim_{n\to\infty} B_n(z,\zeta_1) = \mathbf{0}$. Therefore, from (3.86)

$$\lim_{n\to\infty} \varphi_{n+1}^{L,*}(z)^{-1} \left((z-\zeta_1)\mathcal{C}_{1,n}^R(z)\right) = \mathbf{0} \Rightarrow \lim_{n\to\infty} \varphi_{n+1}^{L,*}(z)^{-1}\mathcal{C}_{1,n}^R(z) = \mathbf{0},$$

for $|z| < 1$. Moreover, since $\sigma_{c_k} \in \mathcal{N}$ with $k = 1, \ldots, m$ we have

$$\lim_{n\to\infty} \mathcal{C}_{m-1,n+1}^{L,*}(z)^{-1}\mathcal{C}_{m,n}^R(z) = \mathbf{0}. \tag{3.87}$$

For $m > 1$, since

$$(\varphi_{n+m}^{L,*}(z))^{-1}\mathcal{C}_{m,n}^R(z) = (\varphi_{n+m}^{L,*}(z))^{-1}\mathcal{C}_{m-1,n+1}^{L,*}(z)(\mathcal{C}_{m-1,n+1}^{L,*}(z))^{-1}\mathcal{C}_{m,n}^R(z)$$

Therefore, from (3.84) and (3.87) and the above equation we get (3.85). □

Corollary 3.2.14. *If* $\sigma \in \mathcal{N}$ *and* $\zeta_j \in \mathbb{C} \setminus \bar{\mathbb{D}}$ *for* $j = 1, \ldots, m$, *then*

$$\lim_{n\to\infty} \left(\varphi_{n+m}^{L,*}(z)\right)^{-1} (W_m(z)\mathcal{C}_{m,n}^R(z)) = \mathbf{0},$$

uniformly on every compact subset of $z \in \mathbb{D}$.

Corollary 3.2.15. *If* $\sigma \in \mathcal{N}$ *and* $\zeta_j \in \mathbb{C} \setminus \bar{\mathbb{D}}$ *for* $j = 1, \ldots, m$, *then*

$$\lim_{n\to\infty} \mathcal{C}_{m,n}^{R,*}(z) \left(\varphi_{n+m}^L(z)\right)^{-1} = \mathbf{0},$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$.

3.2.3.2 Case $\zeta_j \in \mathbb{T}$

Lemma 3.2.16. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{T}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \left(\kappa_{n+m}^R\right)^{-1} \tilde{\kappa}_{m,n}^R = \mathbf{I}. \tag{3.88}$$

Proof. We use induction on m. For $m = 1$, since

$$\mathbf{I} + \varphi_{n+1}^{R,H}(\zeta_1) \left(K_n^R(\zeta_1,\zeta_1)\right)^{-1} \varphi_{n+1}^R(\zeta_1) = \varphi_{n+1}^R(\zeta_1)^{-1} K_{n+1}^R(\zeta_1,\zeta_1) K_n^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1).$$

From (3.73)

$$(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H = \varphi_{n+1}^R(\zeta_1)^{-1} K_n^R(\zeta_1,\zeta_1) K_{n+1}^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1). \tag{3.89}$$

Furthermore,

$$\mathbf{I} - \varphi_{n+1}^R(\zeta_1)^{-1} K_n^R(\zeta_1,\zeta_1) K_{n+1}^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1) = \varphi_{n+1}^{R,H}(\zeta_1) K_{n+1}^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1),$$

so we obtain

$$(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H = \mathbf{I} - \varphi_{n+1}^{R,H}(\zeta_1) K_{n+1}^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1), \tag{3.90}$$

and from (1.83), we get

$$\lim_{n\to\infty} (\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H = \mathbf{I}.$$

Then proceeding as in Lemma 3.2.8, we get (3.88). □

Proposition 3.2.17. *If $\sigma \in \mathcal{N}$ and $\zeta_j \in \mathbb{T}$ for $j = 1, \ldots, m$, then*

$$\lim_{n\to\infty} \left(\varphi_{n+m}^R(z)\right)^{-1} \mathcal{C}_{m,n}^R(z) = W_m(z)^{-1}, \tag{3.91}$$

uniformly on every compact subset of $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$.

Proof. We use induction on m. From 3.63 with $m = 1$, multiplying on the left by $(\varphi_{n+1}^R(z))^{-1}$ and on the right by $(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H$, we get

$$(\varphi_{n+1}^R(z))^{-1} \left(W_1(z)\mathcal{C}_{1,n}^R(z)\right) (\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H = (\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H - B_n(z,\zeta_1), \tag{3.92}$$

where

$$\begin{aligned} B_n(z,\zeta_1) =& \varphi_{n+1}^R(z)^{-1} K_n^R(\zeta_1,z) \varphi_{n+1}^{R,H}(\zeta_1)^{-1} \varphi_{n+1}^{R,H}(\zeta_1) K_n^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1) \\ & (\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H. \end{aligned}$$

Using (3.89)

$$B_n(z,\zeta_1) = \varphi_{n+1}^R(z)^{-1} K_n^R(\zeta_1,z) \varphi_{n+1}^{R,H}(\zeta_1)^{-1} \varphi_{n+1}^{R,H}(\zeta_1) K_{n+1}^R(\zeta_1,\zeta_1)^{-1} \varphi_{n+1}^R(\zeta_1),$$

therefore from (1.82) and (1.83), $B_n(z,\zeta_1) \to \mathbf{0}$. Thus taking limit when n tends to infinity in (3.92) and using (3.88) we obtain that

$$\lim_{n\to\infty} (\varphi_{n+1}^R(z))^{-1} \left((z-\zeta_1)\mathcal{C}_{1,n}^R(z)\right) = \mathbf{I},$$

and as consequence

$$\lim_{n\to\infty}(\varphi_{n+1}^R(z))^{-1}\left(\mathcal{C}_{1,n}^R(z)\right)=\frac{1}{z-\zeta_1}\mathbf{I}.$$

For $m>1$, proceeding as in the Proposition 3.2.9, we get (3.91). □

Corollary 3.2.18. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\left(\varphi_{n+m}^R(z)\right)^{-1}\left(W_m(z)\mathcal{C}_{m,n}^R(z)\right)=\mathbf{I},$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus\bar{\mathbb{D}}$.

Corollary 3.2.19. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\left(W_m(z)\mathcal{C}_{m,n}^R(z)\right)^*\left(\varphi_{n+m}^{R,*}(z)\right)^{-1}=\mathbf{I},$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Proposition 3.2.20. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\left(\varphi_{n+m}^{L,*}(z)\right)^{-1}\mathcal{C}_{m,n}^R(z)=\mathbf{0},$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Proof. For $m=1$, from 3.63 and multiplying on the left by $(\varphi_{n+1}^{L,*}(z))^{-1}$ and on the right by $(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H$, we get

$$\begin{aligned}\varphi_{n+1}^{L,*}(z)^{-1}&\left((z-\zeta_1)\mathcal{C}_{1,n}^R(z)\right)(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H\\&=\varphi_{n+1}^{L,*}(z)^{-1}\varphi_{n+1}^R(z)\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H-B_n(z,\zeta_1),\end{aligned}\tag{3.93}$$

where

$$\begin{aligned}B_n(z,\zeta_1)&=\varphi_{n+1}^{L,*}(z)^{-1}K_n^R(\zeta_1,z)\varphi_{n+1}^{R,H}(\zeta_1)^{-1}\\&\quad\times\varphi_{n+1}^{R,H}(\zeta_1)K_n^R(\zeta_1,\zeta_1)^{-1}\varphi_{n+1}^R(\zeta_1)\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R(\kappa_{n+1}^{-R}\tilde{\kappa}_{1,n}^R)^H.\end{aligned}$$

Using (3.89)

$$B_n(z,\zeta_1)=\varphi_{n+1}^{L,*}(z)^{-1}K_n^R(\zeta_1,z)\varphi_{n+1}^{R,H}(\zeta_1)^{-1}\times\varphi_{n+1}^{R,H}(\zeta_1)K_{n+1}^R(\zeta_1,\zeta_1)^{-1}\varphi_{n+1}^R(\zeta_1).$$

Since $\varphi_{n+1}^R(\zeta_1)^{-1}\varphi_{n+1}^{L,*}(\zeta_1)$ is a unitary matrix when $\zeta_1\in\mathbb{T}$, from (1.74), we get

$$\begin{aligned}&\left\|(\varphi_{n+1}^{L,*}(z))^{-1}K_n^R(\zeta_1,z)\varphi_{n+1}^{R,H}(\zeta_1)^{-1}\right\|_2=\\&\left\|\frac{(\varphi_{n+1}^R(\zeta_1)^{-1}\varphi_{n+1}^{L,*}(\zeta_1))^H-\varphi_{n+1}^{L,*}(z)^{-1}\varphi_{n+1}^R(z)}{1-z\bar{\zeta_1}}\right\|_2\leq\frac{1+\|\varphi_{n+1}^{L,*}(z)^{-1}\varphi_{n+1}^R(z)\|_2}{|1-z\bar{\zeta_1}|}\end{aligned}\tag{3.94}$$

and from (1.79) it follows that (3.94) is bounded when n tends to infinity. Therefore, using (1.83), we have $\lim_{n\to\infty}B_n(z,\zeta_1)=\mathbf{0}$. Therefore, taking limit in (3.93), using (1.79) and (3.88) with $m=1$, we obtain

$$\lim_{n\to\infty}\left(\varphi_{n+1}^{L,*}(z)\right)^{-1}\left((z-\zeta_1)\mathcal{C}_{1,n}^R(z)\right)=\lim_{n\to\infty}\left(\varphi_{n+1}^{L,*}(z)\right)^{-1}\varphi_{n+1}^R(z)=\mathbf{0}.$$

Thus

$$\lim_{n\to\infty}\left(\varphi_{n+1}^{L,*}(z)\right)^{-1}\left(\mathcal{C}_{1,n}^{R}(z)\right)=\mathbf{0}.$$

For $m>1$, we proceed as in the Proposition 3.2.13 using the Corollary 3.2.19. □

Corollary 3.2.21. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\left(\varphi_{n+m}^{L,*}(z)\right)^{-1}\left(W_m(z)\mathcal{C}_{m,n}^{R}(z)\right)=\mathbf{0},$$

uniformly on every compact subset of $z\in\mathbb{D}$.

Corollary 3.2.22. *If $\sigma\in\mathcal{N}$ and $\zeta_j\in\mathbb{T}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\mathcal{C}_{m,n}^{R,*}(z)\left(\varphi_{n+m}^{L}(z)\right)^{-1}=\mathbf{0},$$

uniformly on every compact subset of $z\in\mathbb{C}\setminus\bar{\mathbb{D}}$.

3.2.3.3 Case $\zeta_j\in\mathbb{D}$

Let $\zeta_j\in\mathbb{D}$ for $j=1\dots,m$, and assume that ζ_j is not a zero of φ_n^R for all $n>0$.

Lemma 3.2.23. *If $\sigma\in\mathcal{S}$ and $\zeta_j\in\mathbb{D}$ for $j=1,\dots,m$, then*

$$\lim_{n\to\infty}\left(\kappa_{n+m}^{R}\right)^{-1}\tilde{\kappa}_{m,n}^{R}=\mathbf{I}. \tag{3.95}$$

Proof. Let $m=1$. If $\sigma\in\mathcal{S}$, then $K_n^R(z,z)$ converges uniformly on every compact subset of $z\in\mathbb{D}$ (see Proposition 1.4.23) so we have $\lim_{n\to\infty}\|\varphi_n^R(z)\|_2=0$. Therefore, from (3.90)

$$\left\|\left(\kappa_{n+1}^{R}\right)^{-1}\tilde{\kappa}_{1,n}^{R}\left(\left(\kappa_{n+1}^{R}\right)^{-1}\tilde{\kappa}_{1,n}^{R}\right)^{H}-\mathbf{I}\right\|_2\leq\|\varphi_{n+1}^{R}(\zeta_1)\|_2^2\left\|K_{n+1}^{-R}(\zeta_1,\zeta_1)\right\|_2,$$

when n tends to infinity, we conclude that

$$\lim_{n\to\infty}\left(\kappa_{n+1}^{R}\right)^{-1}\tilde{\kappa}_{1,n}^{R}\left(\left(\kappa_{n+1}^{R}\right)^{-1}\tilde{\kappa}_{1,n}^{R}\right)^{H}=\mathbf{I}.$$

Proceeding as in the Lemma 3.2.8, we get (3.95). □

CHAPTER 4

Block matrix representations and spectral transformations

Summary
The structure of the chapter is as follows. In the first section, some properties of the block Hessenberg matrices associated with orthonormal matrix polynomials on the unit circle are deduced. In particular, we show that the block Hessenberg matrix associated with a matrix measure supported on the unit circle is unitary if and only if the measure does not belong to the Szegő matrix class. Then, we deduce explicit relations between the Uvarov matrix transformation and the corresponding block Hessenberg matrices, and similar results for the Christoffel matrix transformation. Finally, we show a factorization of the block Hessenberg matrix. As an application, we provide another proof of the fact that all the zeros of an orthonormal matrix polynomial are located in the unit disk. In Section 2, we deal with properties of symmetric matrix measures supported on the unit circle. In particular, given two symmetric matrix measures σ and $\tilde{\sigma}$, we establish a relationship between the corresponding CMV matrices $\mathcal{C}$ and $\tilde{\mathcal{C}}$, based in a well-known factorization in terms of certain $2l \times 2l$ blocks, which are defined in terms of the Verblunsky matrix coefficients. Then, we consider sequences of Verblunsky matrix coefficients associated with certain transformations of the measure σ, which constitute a generalization to the matrix case of the results obtained in [88]. Finally, in the last subsection we analyze properties of a CMV matrix associated with a new sequence of Verblunsky matrix coefficients that is constructed from two given sequences.

4.1 Block Hessenberg matrices and orthonormal matrix polynomials on the unit circle

4.1.1 Introduction

As in the scalar case, there are many similarities between **MOPRL** and **MOPUC**. One of those similarities is related to the matrix representation of the multiplication operator with respect to the basis of orthogonal polynomials. In the **MOPRL** case, we have the so-called block Jacobi matrix (defined as in (1.11)). Block Jacobi matrices give information about properties of a sequence of **MOPRL**, such as the distribution of the zeros, among

others (see [65, 154, 160, 163]). For **MOPUC**, the representation of the multiplication operator in terms of the orthogonal matrix polynomials yields a block Hessenberg matrix (see [76, 163]) that is known in the literature as GGT matrix (see[156] Chapter 4) in the scalar case. The block Hessenberg matrix also gives information about properties such as the distribution of the zeros ([163]). Furthermore, block Hessenberg matrices are very useful in multivariate time series analysis, multichannel signal processing and Gaussian quadrature formulas for matrix-valued functions (see [21, 161, 163]).

4.1.2 Block Hessenberg matrices for MOPUC

In this section, we deal with properties of the block Hessenberg matrix that represents the multiplication by z operator with respect to a sequence of orthonormal matrix polynomials on the unit circle. Let us express $z\varphi_n^L(z)$ in terms of $(\varphi_k^L(z))_{k=0}^{n+1}$, i.e.,

$$z\varphi_n^L(z) = \sum_{k=0}^{n+1} h_{n,k}^L \varphi_k^L(z), \quad h_{n,k}^L = \langle z\varphi_n^L, \varphi_k^L \rangle_{L,\sigma} \in \mathcal{M}_l. \tag{4.1}$$

Using the Szegő recurrence relation (1.35) and (1.60), we get

$$z\varphi_n^L(z) = \rho_n^L \varphi_{n+1}^L(z) + \alpha_n^H (\kappa_n^R)^{-1} \sum_{k=0}^{n} \varphi_k^L(0)^H \varphi_k^L(z), \tag{4.2}$$

and comparing (4.1) and (4.2), we deduce

$$h_{n,k}^L = \begin{cases} \rho_n^L = \kappa_n^L (\kappa_{n+1}^L)^{-1}, & \text{if } k = n+1, \\ \alpha_n^H \kappa_n^{-R} \varphi_k^{L,H}(0) = -(\kappa_n^{-L})^H \Phi_{n+1}^R(0) \Phi_k^{L,H}(0) (\kappa_k^L)^H, & \text{if } 0 \le k \le n, \\ \mathbf{0}, & \text{if } k > n+1. \end{cases} \tag{4.3}$$

In matrix form, we have

$$z\varphi^L(z) = \mathbb{H}_{\varphi^L} \varphi^L(z), \tag{4.4}$$

where $\varphi^L(z) = [\varphi_0^L(z)^T, \varphi_1^L(z)^T, \ldots, \varphi_n^L(z)^T, \ldots]^T$ and $\mathbb{H}_{\varphi^L}$ is the lower block Hessenberg matrix

$$\mathbb{H}_{\varphi^L} = \begin{bmatrix} h_{0,0}^L & h_{0,1}^L & \mathbf{0} & \mathbf{0} & \cdots \\ h_{1,0}^L & h_{1,1}^L & h_{1,2}^L & \mathbf{0} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.5}$$

$\mathbb{H}_{\varphi^L}$ will henceforth be called *block Hessenberg matrix* associated with the sequence $(\varphi_n^L(z))_{n\ge 0}$. Notice that from (1.39) $\varphi_k^{L,H}(0) = -\kappa_{k-1}^R \alpha_{k-1} \rho_{k-1}^{-L}$, and since $\alpha_{k-1}\rho_{k-1}^{-L} = \rho_{k-1}^{-R}\alpha_{k-1}$ (see [42] Lemma 1.3.), we get $\varphi_k^{L,H}(0) = -\kappa_k^R \alpha_{k-1}$, with $\alpha_{-1} = -\mathbf{I}$. Moreover, from (1.33) $\kappa_n^{-R}\kappa_k^R = \rho_{n-1}^R \cdots \rho_k^R$. Therefore (4.3) becomes

$$h_{n,k}^L = \begin{cases} \rho_n^L = \kappa_n^L (\kappa_{n+1}^L)^{-1}, & \text{if } k = n+1, \\ -\alpha_n^H \kappa_n^{-R} \kappa_k^R \alpha_{k-1} = -\alpha_n^H \rho_{n-1}^R \cdots \rho_k^R \alpha_{k-1}, & \text{if } 0 \le k \le n, \\ \mathbf{0}, & \text{if } k > n+1. \end{cases}$$

The previous expression appears in [158] for right orthonormal matrix polynomials.

Example 4.1.1. *Consider the matrix of measures*

$$d\sigma(\theta) = \frac{1}{2}\begin{bmatrix} 2 & e^{i\theta} \\ e^{-i\theta} & 2 \end{bmatrix}\frac{d\theta}{2\pi}.$$

Then, it is not difficult to show that the corresponding monic **MOPUC** *are*

$$\Phi_0^R(z) = \Phi_0^L(z) = \mathbf{I}_2, \quad \Phi_n^R(z) = \Phi_n^L(z) = \begin{bmatrix} z^n & 0 \\ -\frac{1}{2}z^{n-1} & z^n \end{bmatrix}, \ n \geq 1,$$

and we have

$$S_0^R = S_0^L = \mathbf{I}_2, \quad S_n^R = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{bmatrix}, \ S_n^L = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}, \qquad n \geq 1.$$

Thus,

$$\kappa_0^L = \mathbf{I}_2, \quad \kappa_n^L = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{\sqrt{3}} \end{bmatrix}, \ (\kappa_n^L)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}, \qquad n \geq 1.$$

From (4.3)*, we deduce that the entries of* $\mathbb{H}_{\varphi^L}$ *are given by*

$$h_{n,k}^L = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3}/2 \end{bmatrix}, & \text{if } k = 1 \text{ and } n = 0, \\ \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}, & \text{if } k = n = 0, \\ \mathbf{I}_2, & \text{if } k = n+1 \text{ and } n \geq 1, \\ \mathbf{0}_2, & \text{otherwise.} \end{cases}$$

On the other hand, for any $n \geq 0$ and $\zeta \in \mathbb{C}$, we define

$$\lambda_n^L(\zeta) = K_n^{-L}(\zeta,\zeta) \text{ and } \lambda_n^R(\zeta) = K_n^{-R}(\zeta,\zeta),$$
$$\lambda_\infty^L(\zeta) = \lim_{n\to\infty} \lambda_n^L(\zeta) \text{ and } \lambda_\infty^R(\zeta) = \lim_{n\to\infty} \lambda_n^R(\zeta).$$

In particular, from (1.61)

$$\lambda_\infty^L(0) = \lim_{n\to\infty} K_n^{-L}(0,0) = \lim_{n\to\infty} \left(\kappa_n^{-R}\right)^H \kappa_n^{-R}, \tag{4.6}$$

and

$$\lambda_\infty^R(0) = \lim_{n\to\infty} K_n^{-R}(0,0) = \lim_{n\to\infty} \kappa_n^{-L}\left(\kappa_n^{-L}\right)^H.$$

Also notice that

$$\left(\prod_{k=0}^{\infty}\sqrt{\mathbf{I}-\alpha_k\alpha_k^H}\right)^H\left(\prod_{k=0}^{\infty}\sqrt{\mathbf{I}-\alpha_k\alpha_k^H}\right) = \left(\prod_{k=0}^{\infty}\rho_k^R\right)^H\left(\prod_{k=0}^{\infty}\rho_k^R\right) = \lambda_\infty^L(0). \tag{4.7}$$

Indeed, from (1.40), we have

$$\left(\prod_{k=0}^{n}\rho_k^R\right)^H\left(\prod_{k=0}^{n}\rho_k^R\right)=(\kappa_{n+1}^{-R})^H\kappa_{n+1}^{-R},$$

and thus, taking limits when $n\to\infty$ and using (4.6), we obtain (4.7). In the scalar case, $\lambda_n(\zeta)=\lambda_n^L(\zeta)=\lambda_n^R(\zeta)$ is the well-known Christoffel function (see [156]). The matrix version of the Christoffel function for a matrix measure supported on the real line was defined in [70]. The following result states that, as in the scalar case, $\mathbb{H}_{\varphi^L}$ is *almost* unitary.

Proposition 4.1.2. *The block Hessenberg matrix $\mathbb{H}_{\varphi^L}$ satisfies*

$$\mathbb{H}_{\varphi^L}\mathbb{H}_{\varphi^L}^H = \mathbf{I}_\infty, \tag{4.8}$$
$$\mathbb{H}_{\varphi^L}^H\mathbb{H}_{\varphi^L} = \mathbf{I}_\infty-\varphi^L(0)\lambda_\infty^L(0)\varphi^{L,H}(0), \tag{4.9}$$

where $\mathbf{I}_\infty$ is the infinite dimensional identity matrix.

Proof. For the first statement, notice that $\mathbf{I}_\infty=\langle\varphi^L,\varphi^L\rangle_{L,\sigma}$, and we have $\langle zp,zq\rangle_{L,\sigma}=\langle p,q\rangle_{L,\sigma}$ for every $p,q\in\mathbb{P}^l[z]$. Thus, from (4.4) we get

$$\begin{aligned}\mathbb{H}_{\varphi^L}\mathbb{H}_{\varphi^L}^H&=\mathbb{H}_{\varphi^L}\langle\varphi^L,\varphi^L\rangle_{L,\sigma}\mathbb{H}_{\varphi^L}^H\\&=\langle\mathbb{H}_{\varphi^L}\varphi^L,\mathbb{H}_{\varphi^L}\varphi^L\rangle_{L,\sigma}=\langle z\varphi^L,z\varphi^L\rangle_{L,\sigma}=\langle\varphi^L,\varphi^L\rangle_{L,\sigma}=\mathbf{I}_\infty.\end{aligned}$$

Notice that this is the proof given in [163] for the truncated case. On the other hand, let $A(k,n)=\left(\alpha_k^H\kappa_k^{-R}\varphi_n^{L,H}(0)\right)^H\alpha_k^H\kappa_k^{-R}\varphi_n^{L,H}(0)$. For $n\geq 0$, we have

$$\begin{aligned}&\left(\mathbb{H}_{\varphi^L}^H\right)_{(n)}\left(\mathbb{H}_{\varphi^L}\right)^{(n)}\\&\quad=\left(\kappa_{n-1}^L\kappa_n^{-L}\right)^H\kappa_{n-1}^L\kappa_n^{-L}+\left(\alpha_n^H\kappa_n^{-R}\varphi_n^{L,H}(0)\right)^H\alpha_n^H\kappa_n^{-R}\varphi_n^{L,H}(0)+\sum_{k=n+1}^{\infty}A(k,n),\end{aligned}$$

where $(\mathbb{H})_{(n)}$ and $(\mathbb{H})^{(n)}$ denote the $n-$th block row and the $n-$th block column, respectively, of the block matrix $\mathbb{H}$. Using (1.63), we have

$$\left(\mathbb{H}_{\varphi^L}^H\right)_{(n)}\left(\mathbb{H}_{\varphi^L}\right)^{(n)}=\mathbf{I}-\varphi_n^L(0)(\kappa_n^{-R})^H(\mathbf{I}-\alpha_n\alpha_n^H)\kappa_n^{-R}\varphi_n^{L,H}(0)+\sum_{k=n+1}^{\infty}A(k,n),$$

and since $\mathbf{I}-\alpha_n\alpha_n^H=\left(\kappa_{n+1}^{-R}\kappa_n^R\right)^2=\left(\kappa_{n+1}^{-R}\kappa_n^R\right)^H\kappa_{n+1}^{-R}\kappa_n^R=(\kappa_n^R)^H(\kappa_{n+1}^{-R})^H\kappa_{n+1}^{-R}\kappa_n^R$, we get

$$\left(\mathbb{H}_{\varphi^L}^H\right)_{(n)}\left(\mathbb{H}_{\varphi^L}\right)^{(n)}=\mathbf{I}-\varphi_n^L(0)(\kappa_{n+1}^{-R})^H\kappa_{n+1}^{-R}\varphi_n^{L,H}(0)+\sum_{k=n+1}^{\infty}A(k,n),$$

that, recursively, becomes

$$\begin{aligned}\left(\mathbb{H}^H_{\varphi^L}\right)_{(n)}\left(\mathbb{H}_{\varphi^L}\right)^{(n)} &= \mathbf{I} - \varphi_n^L(0)(\kappa_{n+2}^{-R})^H\kappa_{n+2}^{-R}\varphi_n^{L,H}(0) + \sum_{k=n+2}^{\infty} A(k,n)\\ &\vdots\\ &= \mathbf{I} - \varphi_n^L(0)\left(\lim_{k\to\infty}(\kappa_k^{-R})^H\kappa_k^{-R}\right)\varphi_n^{L,H}(0)\\ &= \mathbf{I} - \varphi_n^L(0)\lambda_\infty^L(0)\varphi_n^{L,H}(0),\end{aligned}$$

where we have used (4.6). On the other hand, for $0 \le j < n$, using (1.63) and (4.6)

$$\begin{aligned}\left(\mathbb{H}^H_{\varphi^L}\right)_{(n)}\left(\mathbb{H}_{\varphi^L}\right)^{(j)} &= \varphi_n^L(0)\left(-(\kappa_n^{-R})^H\kappa_n^{-R} + \sum_{k=n}^{\infty}\left(\alpha_k^H\kappa_k^{-R}\right)^H\alpha_k^H\kappa_k^{-R}\right)\varphi_j^{L,H}(0)\\ &= \varphi_n^L(0)\left(-(\kappa_{n+1}^{-R})^H\kappa_{n+1}^{-R} + \sum_{k=n+1}^{\infty}\left(\alpha_k^H\kappa_k^{-R}\right)^H\alpha_k^H\kappa_k^{-R}\right)\varphi_j^{L,H}(0)\\ &\vdots\\ &= -\varphi_n^L(0)\left(\lim_{k\to\infty}(\kappa_k^{-R})^H\kappa_k^{-R}\right)\varphi_j^{L,H}(0) = -\varphi_n^L(0)\lambda_\infty^L(0)\varphi_j^{L,H}(0).\end{aligned}$$

□

Theorem 4.1.3. *$\mathbb{H}_{\varphi^L}$ is a unitary matrix if and only if $\sigma \notin \mathcal{S}$.*

Proof. Assume $\mathbb{H}_{\varphi^L}$ is a unitary matrix. Then, from (4.9), $\lambda_\infty^L(0) = \mathbf{0}$. As a consequence, from (4.6) we have

$$\mathbf{0} = \lambda_\infty^L(0) = \lim_{n\to\infty}(\kappa_n^{-R})^H\kappa_n^{-R} \Rightarrow \lim_{n\to\infty}\kappa_n^{-R} = \mathbf{0} \Rightarrow \lim_{n\to\infty}\|\kappa_n^{-R}\|_2 = 0.$$

Since $0 \le \|\kappa_n^R\|_2^{-1} \le \|\kappa_n^{-R}\|_2$, we get

$$\lim_{n\to\infty}\|\kappa_n^R\|_2^{-1} = 0 \Rightarrow \lim_{n\to\infty}\|\kappa_n^R\|_2 = \infty,$$

and from (1.85) it follows that $\sigma \notin \mathcal{S}$.

Conversely, we proceed by contrapositive. Assume that $\mathbb{H}_{\varphi^L}$ is not a unitary matrix, then from (4.9)

$$\lambda_\infty^L(0) = \lim_{n\to\infty}\lambda_n^L(0) > \mathbf{0}.$$

Therefore, we consider the following cases:

1. The sequence $(\lambda_n^L(0))_{n\ge0}$ does not converge, i.e., $\lim_{n\to\infty}\|\lambda_n^L(0)\|_2 = \infty$. Since $\alpha_n^H\alpha_n < \mathbf{I}$, we have $\rho_n^L \le \mathbf{I}$ and thus $\|\rho_n^L\|_2 < 1$ for every $n \in \mathbb{N}$. From (1.40) and (4.6) we get

   $$\|\lambda_n^L(0)\|_2 = \left\|(\kappa_n^{-R})^H\kappa_n^{-R}\right\|_2 \le \|\kappa_n^{-R}\|_2^2 = \|\rho_0^L\cdots\rho_{n-1}^L\|_2^2 \le 1,$$

   it follows that $\lim_{n\to\infty}\|\lambda_n^L(0)\|_2 < \infty$. Therefore, this case is not possible.

2. The sequence $(\lambda_n^L(0))_{n\geq 0}$ converges to $\mathbf{L} \in \mathcal{M}_l$, $\mathbf{L} \neq \mathbf{0}$. In this case the sequence $(\lambda_n^L(0)^{-1})_{n\geq 0}$ also converges to some $\mathbf{V} \in \mathcal{M}_l$, $\mathbf{V} \neq \mathbf{0}$. Thus,

$$0 < \lim_{n\to\infty} \|\lambda_n^L(0)^{-1}\|_2 < \infty. \tag{4.10}$$

Since $B_n(0) = \kappa_n^R(\kappa_n^R)^H = \lambda_n^L(0)^{-1}$, from (4.10) we see that $\det B_n(0)$ is bounded from above. As a consequence, from [48] (Theorem 19) and (1.84), it follows that $\sigma \in \mathcal{S}$.

□

We point out that the block Hessenberg matrix associated with $(\varphi_n^R(z))_{n\geq 0}$ has a similar expression, and the same properties. In the scalar case, the previous theorem can be found in [156].

4.1.3 Block Hessenberg matrices for perturbations of MOPUC

In this section, we consider some transformations of matrix measures supported on the unit circle and we study the relation between the corresponding block Hessenberg matrices. Furthermore, we give conditions for the perturbed matrix measures to belong to the Szegő class. In the scalar case, Hessenberg matrices associated with spectral transformations of orthogonal polynomials on the unit circle have been studied in [20, 30, 45], and results related with connection formulas between the perturbed and unperturbed sequences of polynomials, as well as conditions to remain in the Szegő class, can be found in [88].

In the remaining of the document, we will consider only left **MOPUC**, since the results for the right polynomials are analogous.

4.1.3.1 Uvarov matrix transformation with m mass points

In this subsection, we consider the Uvarov matrix transformation with $m > 0$ mass points of the matrix measure σ supported on $\mathbb{T}$, defined as in (3.1) by

$$d\sigma_{u_m}(z) = d\sigma(z) + \sum_{j=1}^{m} \mathbf{M}_j \delta(z - \zeta_j), \tag{4.11}$$

where $\zeta_i \neq \zeta_j$ for $i, j = 1, \ldots, m$, $\mathbf{M}_j \in \mathcal{M}_l$ is a positive definite matrix for $j = 1, ..., m$, and δ is the Dirac matrix measure.

Let $\mathbb{H}_{\mathcal{U}^L}$ be the block Hessenberg matrix such that

$$z\mathcal{U}^L(z) = \mathbb{H}_{\mathcal{U}^L}\mathcal{U}^L(z), \tag{4.12}$$

where $\mathcal{U}^L(z) = [\mathcal{U}_{m,0}^L(z)^T, \mathcal{U}_{m,1}^L(z)^T, \ldots, \mathcal{U}_{m,n}^L(z)^T, \ldots]^T$. Clearly, if $\mathbf{L}_{\mathcal{U}\varphi}^L$ is the change of basis matrix such that $\mathcal{U}^L(z) = \mathbf{L}_{\mathcal{U}\varphi}^L \varphi^L(z)$, we have

$$\mathbb{H}_{\mathcal{U}^L} = \mathbf{L}_{\mathcal{U}\varphi}^L \mathbb{H}_{\varphi^L} \mathbf{L}_{\mathcal{U}\varphi}^{-L}. \tag{4.13}$$

In order to compute $\mathbf{L}_{\mathcal{U}\varphi}^L$, proceeding as in the Proposition 3.1.2, the relation between $(\varphi_n^L(z))_{n\geq 0}$ and $(\mathcal{U}_{m,n}^L(z))_{n\geq 0}$ is given in the following proposition

Proposition 4.1.4. *The sequence* $(\mathcal{U}_{m,n}^L(z) = \tilde{\kappa}_{m,n}^L z^n + \cdots)_{n\geq 0}$ *is orthonormal with respect to* (3.5) *if and only if the matrix* $\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)$ *is nonsingular for* $n \geq 1$. *Furthermore,*

$$\mathcal{U}_{m,n}^L(z) = \tilde{\kappa}_{m,n}^L \kappa_n^{-L} \left[\varphi_n^L(z) - \mathbb{L}_n^L(\zeta) \left(\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right)^{-1} \mathbb{M}\mathbb{K}_{n-1}^L(z,\zeta)\right], \tag{4.14}$$

and

$$\tilde{\kappa}_{m,n}^{-L} \left(\tilde{\kappa}_{m,n}^{-L}\right)^H = \tilde{S}_{m,n}^L = S_n^L + \mathbb{F}_n^L(\zeta) \left[\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right]^{-1} \mathbb{M}\mathbb{F}_n^{L,H}(\zeta), \tag{4.15}$$

is a positive definite matrix for $n \geq 1$. *Where*

$$\begin{aligned}
\mathbb{L}_n^L(\zeta) &:= \left[\varphi_n^L(\zeta_1), \ldots, \varphi_n^L(\zeta_m)\right] \in \mathcal{M}_{l\times ml},\ \mathbb{F}_n^L(\zeta) := \left[\Phi_n^L(\zeta_1), \ldots, \Phi_n^L(\zeta_m)\right] \in \mathcal{M}_{l\times ml},\\
\mathbb{K}_{n-1}^L(z,\zeta) &:= \left[K_{n-1}^L(z,\zeta_1)^T, \ldots, K_{n-1}^L(z,\zeta_m)^T\right]^T \in \mathcal{M}_{ml\times l},\\
\mathcal{K}_{n-1}^L(\zeta) &:= \begin{bmatrix} K_{n-1}^L(\zeta_1,\zeta_1) & \cdots & K_{n-1}^L(\zeta_m,\zeta_1) \\ \vdots & \ddots & \vdots \\ K_{n-1}^L(\zeta_1,\zeta_m) & \cdots & K_{n-1}^L(\zeta_m,\zeta_m) \end{bmatrix} \in \mathcal{M}_{lm},\\
\mathbb{M} &:= \begin{bmatrix} \mathbf{M}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{M}_m \end{bmatrix} \in \mathcal{M}_{lm}.
\end{aligned}$$

Then, from (4.14)

$$\mathcal{U}_{m,n}^L(z) = \tilde{\kappa}_{m,n}^L \kappa_n^{-L} \left[\varphi_n^L(z) - \sum_{k=0}^{n-1} \mathbb{L}_n^L(\zeta) \left(\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right)^{-1} \mathbb{M}\mathbb{L}_k^{L,H}(\zeta)\varphi_k^L(z)\right].$$

Thus, the entries of $\mathbf{L}_{\mathcal{U}\varphi}^L$ are

$$\left(\mathbf{L}_{\mathcal{U}\varphi}^L\right)_{n,k} = \begin{cases} \tilde{\kappa}_{m,n}^L \kappa_n^{-L}, & \text{if } k = n,\\ -\tilde{\kappa}_{m,n}^L \kappa_n^{-L} \mathbb{L}_n^L(\zeta) \left(\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right)^{-1} \mathbb{M}\mathbb{L}_k^{L,H}(\zeta), & \text{if } 0 \leq k < n,\\ \mathbf{0}, & \text{if } n < k. \end{cases}$$

Lemma 4.1.5. *For* $n \geq 1$,

$$(\tilde{\kappa}_{m,n}^L \kappa_n^{-L})^H \tilde{\kappa}_{m,n}^L \kappa_n^{-L} = \left[\mathbf{I} + \mathbb{L}_n^L(\zeta) \left(\mathbf{I} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right)^{-1} \mathbb{M}\mathbb{L}_n^{L,H}(\zeta)\right]^{-1}. \tag{4.16}$$

Proof. From (1.61) and (4.15), we have

$$\begin{aligned}
\kappa_n^L \tilde{\kappa}_{m,n}^{-L} (\kappa_n^L \tilde{\kappa}_{m,n}^{-L})^H &= \kappa_n^L \left[S_n^L + \mathbb{F}_n^L(\zeta) \left(\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right)^{-1} \mathbb{M}\mathbb{F}_n^{L,H}(\zeta)\right] (\kappa_n^L)^H\\
&= \mathbf{I} + \mathbb{L}_n^L(\zeta) \left(\mathbf{I}_{lm} + \mathbb{M}\mathcal{K}_{n-1}^L(\zeta)\right)^{-1} \mathbb{M}\mathbb{L}_n^{L,H}(\zeta),
\end{aligned}$$

so we obtain (4.16). □

To compute $\mathbf{L}_{\mathcal{U}\varphi}^{-L}$, let us write

$$\Phi_n^L(z) = U_{m,n}^L(z) + \sum_{k=0}^{n-1} \lambda_{nk}^L U_{m,k}^L(z),$$

where, for $0 \leq k \leq n-1$,

$$\lambda_{nk}^L = \langle \Phi_n^L, U_{m,k}^L \rangle_{L,\sigma_{u_m}} \tilde{S}_{m,k}^{-L} = \left(\langle \Phi_n^L, U_{m,k}^L \rangle_{L,\sigma} + \sum_{j=1}^{m} \Phi_n^L(\zeta_j) \mathbf{M}_j U_{m,k}^{L,H}(\zeta_j) \right) \tilde{S}_{m,k}^{-L}$$
$$= \mathbb{F}_n^L(\zeta) \mathbb{M} [U_{m,k}^L(\zeta_1), \ldots, U_{m,k}^L(\zeta_m)]^H \tilde{S}_{m,k}^{-L}.$$

Thus,

$$\Phi_n^L(z) = U_{m,n}^L(z) + \sum_{k=0}^{n-1} \mathbb{F}_n^L(\zeta) \mathbb{M} \mathbb{T}_{m,k}^{L,H}(\zeta) \mathcal{U}_k^L(z), \tag{4.17}$$

where $\mathbb{T}_{m,k}^L(\zeta) = [\mathcal{U}_{m,k}(\zeta_1), \ldots, \mathcal{U}_{m,k}(\zeta_m)]$. From (4.14) with $z = \zeta_j$, $j = 1, \ldots, m$, it follows that $\mathbb{T}_{m,k}^{L,H}(\zeta) = (\mathbf{I}_{lm} + \mathcal{K}_{n-1}^L(\zeta)\mathbb{M})^{-1} \mathbb{L}_k^{L,H}(\zeta)(\tilde{\kappa}_{m,k}^L \kappa_k^{-L})^H$, and substituting in (4.17)

$$\varphi_n^L(z) = \kappa_n^L \tilde{\kappa}_{m,n}^{-L} \mathcal{U}_{m,n}^L(z) + \sum_{k=0}^{n-1} \mathbb{L}_n^L(\zeta) \mathbb{M} (\mathbf{I}_{lm} + \mathcal{K}_{k-1}^L(\zeta)\mathbb{M})^{-1} \mathbb{L}_k^{L,H}(\zeta)(\tilde{\kappa}_{m,k}^L \kappa_k^{-L})^H \mathcal{U}_k^L(z).$$

As a consequence,

$$\left(\mathbf{L}_{\mathcal{U}\varphi}^{-L}\right)_{n,k} = \begin{cases} \kappa_n^L \tilde{\kappa}_{m,n}^{-L}, & \text{if } k = n, \\ \mathbb{L}_n^L(\zeta) \mathbb{M} (\mathbf{I}_{lm} + \mathcal{K}_{k-1}^L(\zeta)\mathbb{M})^{-1} \mathbb{L}_k^{L,H}(\zeta)(\tilde{\kappa}_{m,k}^L \kappa_k^{-L})^H, & \text{if } 0 \leq k < n, \\ \mathbf{0}, & \text{if } n < k. \end{cases}$$

On the other hand, since the matrix Szegő condition only depends on the absolutely continuous component of the matrix measure, and the Uvarov matrix transformation only affects the singular part, the following result is immediate.

Proposition 4.1.6. *If $\sigma \in \mathcal{S}$, then $\sigma_{u_m} \in \mathcal{S}$.*

A straightforward consequence of Theorem 4.1.3 and the above proposition is that if $\mathbb{H}_{\varphi^L}^L$ is not a unitary matrix, neither is $\mathbb{H}_{\mathcal{U}^L}^L$.

4.1.3.2 Christoffel matrix transformation

Now, we consider the Christoffel matrix transformation defined as (3.54) with $m = 1$

$$d\sigma_c(z) = W(z) d\sigma(z) W(z)^H,$$

where $W(z) = z\mathbf{I} - A$ with $A \in \mathcal{M}_l$. The corresponding left sesquilinear form is

$$\langle p, q \rangle_{L,\sigma_c} = \langle p(z)W(z), q(z)W(z) \rangle_{L,\sigma}, \quad p, q \in \mathbb{P}^l[z]. \tag{4.18}$$

Proceeding as in the Proposition 3.2.2, we get the following proposition that establishes a connection formula for these matrix polynomials.

Proposition 4.1.7. *For all z such that $W(z) = \mathbf{I}z - A$ is non-singular, the sequence of matrix polynomials $(\mathcal{C}_n^L(z) = \tilde{\kappa}_n^L z^n + \cdots)_{n \geq 0}$ defined by*

$$\mathcal{C}_n^L(z) W(z) = \tilde{\kappa}_n^L \kappa_{n+1}^{-L} \left[\varphi_{n+1}^L(z) - \varphi_{n+1}^L(A) K_n^{-L}(A, A) K_n^L(z, A) \right] \tag{4.19}$$

is orthonormal with respect to the sesquilinear form (4.18). Furthermore,

$$\tilde{\kappa}_n^{-L}\left(\tilde{\kappa}_n^{-L}\right)^H = \tilde{S}_n^L = S_{n+1}^L + \Phi_{n+1}^L(A)K_n^{-L}(A,A)\Phi_{n+1}^{L,H}(A), \tag{4.20}$$

is a positive definite matrix for $n \geq 1$.

From (4.19), we have

$$\mathcal{C}_n^L(z)W(z) = \tilde{\kappa}_n^L\kappa_{n+1}^{-L}\left[\varphi_{n+1}^L(z) - \sum_{k=0}^{n}\varphi_{n+1}^L(A)K_n^{-L}(A,A)\varphi_k^{L,H}(A)\varphi_k^L(z)\right],$$

that can be written as

$$\mathcal{C}^L(z)W(z) = \mathfrak{C}^L\varphi^L(z), \tag{4.21}$$

where $\mathcal{C}^L(z) = [\mathcal{C}_0^L(z)^T, \mathcal{C}_1^L(z)^T, \ldots, \mathcal{C}_n^L(z)^T, \ldots]^T$, and

$$\left(\mathfrak{C}^L\right)_{n,k} = \begin{cases} \tilde{\kappa}_n^L(\kappa_{n+1}^L)^{-1}, & \text{if } k = n+1, \\ -\tilde{\kappa}_n^L(\kappa_{n+1}^L)^{-1}\varphi_{n+1}^L(A)K_n^{-L}(A,A)\varphi_k^{L,H}(A), & \text{if } 0 \leq k \leq n, \\ \mathbf{0}, & \text{if } k > n+1. \end{cases}$$

Lemma 4.1.8. *For* $n \geq 1$,

$$\left(\tilde{\kappa}_n^L\kappa_{n+1}^{-L}\right)^H\tilde{\kappa}_n^L\kappa_{n+1}^{-L} = \left[\mathbf{I} + \varphi_{n+1}^L(A)K_n^{-L}(A,A)\varphi_{n+1}^{L,H}(A)\right]^{-1}. \tag{4.22}$$

In particular, if $\varphi_{n+1}^L(A)$ *is non-singular*

$$\left(\tilde{\kappa}_n^L\kappa_{n+1}^{-L}\right)^H\tilde{\kappa}_n^L\kappa_{n+1}^{-L} = \varphi_{n+1}^L(A)K_{n+1}^{-L}(A,A)K_n^L(A,A)\varphi_{n+1}^{-L}(A). \tag{4.23}$$

Proof. From (1.61) and (4.20), we have

$$\begin{aligned}\kappa_{n+1}^L\tilde{\kappa}_n^{-L}\left(\kappa_{n+1}^L\tilde{\kappa}_n^{-L}\right)^H &= \kappa_{n+1}^L(S_{n+1}^L + \Phi_{n+1}^L(A)K_n^{-L}(A,A)\Phi_{n+1}^{L,H}(A))\left(\kappa_{n+1}^L\right)^H \\ &= \mathbf{I} + \varphi_{n+1}^L(A)K_n^{-L}(A,A)\varphi_{n+1}^{L,H}(A),\end{aligned}$$

which is equivalent to (4.22). In particular, from (4.22) we obtain

$$\begin{aligned}\left(\tilde{\kappa}_n^L\kappa_{n+1}^{-L}\right)^H\tilde{\kappa}_n^L\kappa_{n+1}^{-L} &= \left[\mathbf{I} + \varphi_{n+1}^L(A)K_n^{-L}(A,A)\varphi_{n+1}^{L,H}(A)\right]^{-1} \\ &= \left[\varphi_{n+1}^L(A)K_n^{-L}(A,A)\left(K_n^L(A,A) + \varphi_{n+1}^{L,H}(A)\varphi_{n+1}^L(A)\right)\varphi_{n+1}^{-L}(A)\right]^{-1} \\ &= \varphi_{n+1}^L(A)K_{n+1}^{-L}(A,A)K_n^L(A,A)\varphi_{n+1}^{-L}(A)\end{aligned}$$

which is (4.23). □

Proposition 4.1.9. *The block Hessenberg matrix* $\mathfrak{C}^L$ *satisfies* $\mathfrak{C}^L\mathfrak{C}^{L,H} = \mathbf{I}_\infty$, *and*

$$\left(\mathfrak{C}^{L,H}\mathfrak{C}^L\right)_{n,j} = \mathbf{I}\delta_{n,j} - \left[\left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\varphi_n^L(A)K_{n-1}^{-L}(A,A) - \varphi_n^L(A)\Lambda_n(A)\right]\varphi_j^{L,H}(A), \tag{4.24}$$

where

$$\Lambda_n(A) = \sum_{k=n}^{\infty} K_k^{-L}(A,A)\varphi_{k+1}^{L,H}(A)(\tilde{\kappa}_k^L\kappa_{k+1}^{-L})^H\tilde{\kappa}_k^L\kappa_{k+1}^{-L}\varphi_{k+1}^L(A)K_k^{-L}(A,A).$$

In particular, if $\varphi_n^L(A)$ is non-singular for $n \geq 0$, then

$$\mathfrak{C}^{L,H}\mathfrak{C}^L = \mathbf{I}_\infty - \varphi^L(A)\lambda_\infty^L(A)\varphi^{L,H}(A). \tag{4.25}$$

Proof. The first statement can be proved as in Proposition 4.1.2. On the other hand, for $n \geq 0$,

$$(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^L)^{(n)} = \left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L} + \varphi_n^L(A)\Lambda_n(A)\varphi_n^{L,H}(A). \tag{4.26}$$

From (4.22), $\mathbf{0} = \mathbf{I} - \left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\left(\mathbf{I} + \varphi_n^L(A)K_{n-1}^{-L}(A,A)\varphi_n^{L,H}(A)\right)$, and thus

$$\left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L} = \mathbf{I} - \left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\varphi_n^L(A)K_{n-1}^{-L}(A,A)\varphi_n^{L,H}(A).$$

Substituting in (4.26) and rearranging,

$$(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^L)^{(n)} = \mathbf{I} - \left(\left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\varphi_n^L(A)K_{n-1}^{-L}(A,A) - \varphi_n^L(A)\Lambda_n(A)\right)\varphi_n^{L,H}(A).$$

Moreover, for $0 \leq j < n$,

$$(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^L)^{(j)} = -\left(\left(\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\right)^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\varphi_n^L(A)K_{n-1}^{-L}(A,A) - \varphi_n^L(A)\Lambda_n(A),\right)\varphi_j^{L,H}(A).$$

Thus, we have proved (4.24). If $\varphi_n^L(A)$ is non-singular for $n \geq 0$, from (4.23) and (4.26) we get

$$\begin{aligned}(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^L)^{(n)} &= \varphi_n^L(A)K_n^{-L}(A,A)K_{n-1}^L(A,A)\varphi_n^{-L}(A)\\ &\quad + \varphi_n^L(A)\left(\sum_{k=n}^{\infty}K_k^{-L}(A,A)\varphi_{k+1}^{L,H}(A)\varphi_{k+1}^L(A)K_{k+1}^{-L}(A,A)\right)\varphi_n^{L,H}(A).\end{aligned}$$

Since

$$\begin{aligned}K_k^{-L}(A,A) - K_{k+1}^{-L}(A,A) &= K_k^{-L}(A,A)(K_{k+1}^L(A,A) - K_k^L(A,A))K_{k+1}^{-L}(A,A)\\ &= K_k^{-L}(A,A)\varphi_{k+1}^{L,H}(A)\varphi_{k+1}^L(A)K_{k+1}^{-L}(A,A),\end{aligned} \tag{4.27}$$

we obtain

$$\begin{aligned}(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^L)^{(n)} &= \varphi_n^L(A)K_n^{-L}(A,A)K_{n-1}^L(A,A)\varphi_n^{-L}(A)\\ &\quad + \varphi_n^L(A)\left(\sum_{k=n}^{\infty}K_k^{-L}(A,A) - K_{k+1}^{-L}(A,A)\right)\varphi_n^{L,H}(A).\end{aligned}$$

Notice that

$$\begin{aligned}&\mathbf{I} - \varphi_n^L(A)K_{n+1}^{-L}(A,A)\varphi_n^{L,H}(A) =\\ &\varphi_n^L(A)K_n^{-L}(A,A)K_{n-1}^L(A,A)\varphi_n^{-L}(A) + \varphi_n^L(A)\left(K_n^{-L}(A,A) - K_{n+1}^{-L}(A,A)\right)\varphi_n^{L,H}(A)\end{aligned}$$

and thus

$$\begin{aligned}(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^{L})^{(n)} &= \mathbf{I} - \varphi_n^L(A)K_{n+1}^{-L}(A,A)\varphi_n^{L,H}(A) \\ &\quad + \varphi_n^L(A)\left(\sum_{k=n+1}^{\infty} K_k^{-L}(A,A) - K_{k+1}^{-L}(A,A)\right)\varphi_n^{L,H}(A) \\ &\vdots \\ &= \mathbf{I} - \varphi_n^L(A)\lambda_\infty^L(A)\varphi_n^{L,H}(A).\end{aligned}$$

Moreover, for $0 \leq j < n$,

$$(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^{L})^{(j)} = -(\tilde{\kappa}_{n-1}^L\kappa_n^{-L})^H\tilde{\kappa}_{n-1}^L\kappa_n^{-L}\varphi_n(A)K_{n-1}^{-L}(A,A)\varphi_j^{L,H}(A)+\varphi_n^L(A)\Lambda_n(A)\varphi_j^{L,H}(A).$$

From (4.23) and (4.27) we get

$$\begin{aligned}(\mathfrak{C}^{L,H})_{(n)}(\mathfrak{C}^{L})^{(j)} &= \varphi_n^L(A)\left(-K_n^{-L}(A,A) + \sum_{k=n}^{\infty} K_k^{-L}(A,A) - K_{k+1}^{-L}(A,A)\right)\varphi_j^{L,H}(A) \\ &\vdots \\ &= -\varphi_n^L(A)\lambda_\infty^L(A)\varphi_j^{L,H}(A),\end{aligned}$$

so (4.25) follows. □

Additional properties of $\mathfrak{C}^L$ can be easily deduced in the case when A is a diagonal (constant) matrix.

Corollary 4.1.10. *If $A = \zeta\mathbf{I}$ with $|\zeta| \geq 1$, then*

$$\mathfrak{C}^L\mathfrak{C}^{L,H} = \mathbf{I}_\infty \text{ and } \mathfrak{C}^{L,H}\mathfrak{C}^L = \mathbf{I}_\infty - \varphi^L(\zeta)\lambda_\infty^L(\zeta)\varphi^{L,H}(\zeta).$$

The proof of the next result is analogous to the one given in [156] (Theorem 2.2.1), taking $l(z,\zeta) = \frac{1}{2}(1 + z\zeta^{-1})\mathbf{I}$.

Proposition 4.1.11. *Let σ be an $l \times l$ Hermitian matrix measure supported on the unit circle.*

1. *If $|\zeta| > 1$, then $\lambda_\infty^L(\zeta) = \mathbf{0}$.*
2. *If $|\zeta| = 1$, then $\lambda_\infty^L(\zeta) = \sigma(\{\zeta\})$.*

As a consequence, we conclude that if $|\zeta| > 1$, then $\mathfrak{C}^L$ is a unitary matrix. Moreover, if $|\zeta| = 1$ and $\sigma(\{\zeta\}) = \mathbf{0}$, then $\mathfrak{C}^L$ is also a unitary matrix. Notice that in the general case $A \in \mathcal{M}_l$, to determine whether $\mathfrak{C}^L$ is a unitary matrix or not becomes considerably more difficult. This case will be analyzed in a future contribution.

Let $\mathbf{L}_{\varphi\mathcal{C}}$ and $\mathbb{H}_{\mathcal{C}^L}$ be, respectively, the change of basis matrix and the block Hessenberg matrix such that

$$\varphi^L(z) = \mathbf{L}_{\varphi\mathcal{C}}^L\mathcal{C}^L(z), \tag{4.28}$$

$$z\mathcal{C}^L(z) = \mathbb{H}_{\mathcal{C}^L}\mathcal{C}^L(z). \tag{4.29}$$

Moreover, if A commutes with φ_n^L for all $n \geq 0$, we have that $\varphi^L(z)A = \mathbb{A}\varphi^L(z)$, where $\mathbb{A}$ is a block diagonal matrix with A in the diagonal.

Proposition 4.1.12. *If A commutes with φ_n^L for all $n \geq 0$, then*

$$\mathbf{L}_{\varphi\mathcal{C}}^{L} = (\mathbb{H}_{\varphi^L} - \mathbb{A})\mathfrak{C}^{L,H}, \tag{4.30}$$

and

$$\mathbb{H}_{\mathcal{C}^L} - \mathbb{A} = \mathfrak{C}^{L}\mathbf{L}_{\varphi\mathcal{C}}^{L}. \tag{4.31}$$

Proof. From (4.28), we have $\mathcal{C}^L(z) = \mathbf{L}_{\varphi\mathcal{C}}^{-L}\varphi^L(z)$. Multiplying on the right by $z\mathbf{I}-A$, taking into account that A commutes with $\varphi^L(z)$, and using (4.4) and (4.21), we get

$$\mathfrak{C}^{L}\varphi^L(z) = \mathbf{L}_{\varphi\mathcal{C}}^{-L}(\mathbb{H}_{\varphi^L} - \mathbb{A})\varphi^L(z).$$

As a consequence, $\mathfrak{C}^L = \mathbf{L}_{\varphi\mathcal{C}}^{-L}(\mathbb{H}_{\varphi^L} - \mathbb{A})$, and using $\mathfrak{C}^L\mathfrak{C}^{L,H} = \mathbf{I}_\infty$, we obtain (4.30). On the other hand, substituting (4.28) and (4.29) in (4.21) we obtain

$$(\mathbb{H}_{\mathcal{C}^L} - \mathbb{A})\mathcal{C}^L(z) = \mathfrak{C}^{L}\mathbf{L}_{\varphi\mathcal{C}}^{L}\mathcal{C}^L(z),$$

and thus (4.31) follows. □

Corollary 4.1.13. *If $A = \zeta\mathbf{I}$ with $\zeta \in \mathbb{C}$, then*

$$\mathbf{L}_{\varphi\mathcal{C}}^{L} = (\mathbb{H}_{\varphi^L} - \zeta\mathbf{I}_\infty)\mathfrak{C}^{L,H}, \text{ and } \mathbb{H}_{\mathcal{C}^L} - \zeta\mathbf{I}_\infty = \mathfrak{C}^{L}\mathbf{L}_{\varphi\mathcal{C}}^{L}.$$

Note that the previous proposition extends to the matrix case the result given in [45]. Finally, from Lemma 3.2.4 we deduce that if $\sigma \in \mathcal{S}$, then $\sigma_c \in \mathcal{S}$.

An interesting open problem is to consider the relation between the CMV block matrices associated with these transformations. We will analyze this problem in a future contribution. In the scalar case, this has been studied in [25].

4.1.4 A block factorization for block Hessenberg matrices

In this section, we use the ideas in [4] (for the scalar case) to obtain a factorization of $(\mathbb{H}_{\varphi^L})_n \in \mathcal{M}_{(n+1)l}$, the truncated block Hessenberg matrix defined as in (4.5). Denote by $\Theta \in \mathcal{M}_{2l}$ the unitary matrix (see [158], Section 4)

$$\Theta(\alpha) = \begin{bmatrix} \alpha^H & \rho^L \\ \rho^R & -\alpha \end{bmatrix}, \text{ with } \alpha \in \mathcal{M}_l,\ \rho^L = \sqrt{\mathbf{I} - \alpha^H\alpha} \text{ and } \rho^R = \sqrt{\mathbf{I} - \alpha\alpha^H}. \tag{4.32}$$

Set $\hat{\Theta}_n(\alpha_k) = \mathbf{I}_{kl} \oplus \Theta(\alpha_k) \oplus \mathbf{I}_{(n-k-1)l}$, where $\oplus$ is the direct sum, and define $\mathbf{I}_0 \oplus \mathbf{B} = \mathbf{B} \oplus \mathbf{I}_0 = \mathbf{B}$ for any matrix $\mathbf{B} \in \mathcal{M}_l$. Notice that

$$\begin{aligned}
(\mathbb{H}_{\varphi^L})_0 &= \alpha_0^H = \mathbf{I}_0 \oplus \alpha_0^H, \\
(\mathbb{H}_{\varphi^L})_1 &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \alpha_1^H \end{bmatrix} \Theta(\alpha_0) = [\mathbf{I}_l \oplus \alpha_1^H][\mathbf{I}_0 \oplus \Theta(\alpha_0) \oplus \mathbf{I}_0] = [\mathbf{I}_l \oplus \alpha_1^H]\hat{\Theta}_1(\alpha_0), \\
(\mathbb{H}_{\varphi^L})_2 &= \begin{bmatrix} \mathbf{I}_{2l} & \\ & \alpha_2^H \end{bmatrix} \begin{bmatrix} \mathbf{I} & \\ & \Theta(\alpha_1) \end{bmatrix} \begin{bmatrix} \Theta(\alpha_0) & \\ & \mathbf{I} \end{bmatrix} \\
&= [\mathbf{I}_{2l} \oplus \alpha_2^H][\mathbf{I}_l \oplus \Theta(\alpha_1) \oplus \mathbf{I}_0][\mathbf{I}_0 \oplus \Theta(\alpha_0) \oplus \mathbf{I}_l] = [\mathbf{I}_{2l} \oplus \alpha_2^H]\hat{\Theta}_2(\alpha_1)\hat{\Theta}_2(\alpha_0).
\end{aligned}$$

Thus, the following proposition follows using induction on n.

Proposition 4.1.14. *For $n \geq 0$, we have*

$$(\mathbb{H}_{\varphi^L})_n = [\mathbf{I}_{nl} \oplus \alpha_n^H] \prod_{k=0}^{n-1} \hat{\Theta}_n(\alpha_{n-1-k}). \tag{4.33}$$

Furthermore, when $n \to \infty$,

$$\mathbb{H}_{\varphi^L} = \lim_{n\to\infty} \prod_{k=0}^{n} \hat{\Theta}_{n+1}(\alpha_{n-k}). \tag{4.34}$$

In the scalar case, the above factorization is known as the AGR factorization of the GGT matrices, due to Ammar, Gragg, and Reichel (see [4]). Two additional proofs appear in [158] (Section 10). Moreover, as in the scalar case, the above proposition can be used to show that every unitary block matrix has a factorization without recourse to orthogonal polynomials (see [158] Theorem 10.3). Another immediate consequence of the above proposition is the following well-known result of the theory of orthogonal matrix polynomials.

Corollary 4.1.15. *All the zeros of $\varphi_n^L(z)$ lie in $\mathbb{D}$.*

Proof. It is well-known that the zeros of φ_n^L are the eigenvalues of $(\mathbb{H}_{\varphi^L})_{n-1}$ (see Subsection 3.1.4) Therefore, from (4.33) and taking into account that $\Theta(\alpha)$ is a unitary matrix, we have

$$(\mathbb{H}_{\varphi^L})_{n-1}(\mathbb{H}_{\varphi^L}^H)_{n-1} = \mathbf{I}_{(n-1)l} \oplus \alpha_{n-1}^H \alpha_{n-1}.$$

As a consequence, we can write

$$(\mathbb{H}_{\varphi^L})_{n-1}(\mathbb{H}_{\varphi^L}^H)_{n-1} - \mathbf{I}_{nl} = \begin{bmatrix} \mathbf{0}_{(n-1)l} & \\ & \alpha_{n-1}^H \alpha_{n-1} - \mathbf{I} \end{bmatrix}. \tag{4.35}$$

Since $\alpha_{n-1}^H \alpha_{n-1} < \mathbf{I}$, we conclude that all the eigenvalues of $(\mathbb{H}_{\varphi^L})_{n-1}$ lie in $\mathbb{D}$. □

Notice that taking $n \to \infty$ in (4.35), we obtain another proof of (4.8). On the other hand, from (4.13) and (4.34) we obtain the following factorization of the block Hessenberg matrix associated with the Uvarov matrix transformation

$$\mathbb{H}_{\mathcal{U}^L} = \mathbf{L}_{\mathcal{U}\varphi}^L \left[\lim_{n\to\infty} \prod_{k=0}^{n} \hat{\Theta}_{n+1}(\alpha_{n-k}) \right] \mathbf{L}_{\mathcal{U}\varphi}^{-L}.$$

In a similar way, for the Christoffel matrix transformation , from (4.30), (4.31) and (4.34), we get

$$\mathbb{H}_{\mathcal{C}^L} - \mathbb{A} = \mathfrak{C}^L \left[\lim_{n\to\infty} \prod_{k=0}^{n} \hat{\Theta}_{n+1}(\alpha_{n-k}) - \mathbb{A} \right] \mathfrak{C}^{L,H}.$$

Notice that if $\mathbb{A} = \zeta \mathbf{I}_\infty$ with $\zeta \in \mathbb{C}$ and taking into account that $\mathfrak{C}^L \mathfrak{C}^{L,H} = \mathbf{I}_\infty$ we obtain

$$\mathbb{H}_{\mathcal{C}^L} = \mathfrak{C}^L \left[\lim_{n\to\infty} \prod_{k=0}^{n} \hat{\Theta}_{n+1}(\alpha_{n-k}) \right] \mathfrak{C}^{L,H}.$$

4.2 CMV block matrices for symmetric matrix measures on the unit circle

4.2.1 Introduction

The block Hessenberg matrix representation has two limitations. In particular, the matrix is not always unitary. Indeed, it is unitary if and only if the associated measure does not belong to the Szegő matrix class (see [48, 76]). Also, it has the disadvantage of having rows with an infinite number non-zero block entries. This represents an important constraint in the applications. Moreover, its complicated structure yields some difficulties in the study of spectral theory [100]. A more convenient representation is given by the so-called block CMV matrix. This matrix is always unitary and their entries are given in terms of the Verblunsky matrix coefficients. The block CMV matrix comes from the representation of the multiplication operator in the linear space of Laurent polynomials, when a suitable orthonormal basis related to the matrix orthogonal polynomials is chosen. These matrices play a similar role among unitary matrices than block Jacobi matrices among all Hermitian matrices (see [116]). Moreover, they give a spectral interpretation for the zeros of orthogonal polynomials that is much simpler than the one given by the block Hessenberg matrix, according to its special structure (see [42], Theorem 3.10). The name comes from the initials CMV (Cantero, Moral and Velázquez, [26]), although it had appeared before in several places (see the discussion in [158, 170]). Recently, CMV block matrices have been used in the study of Toda type integrable systems [6] and quantum random walks [24]. In the later, the authors show how the theory of CMV block matrices gives a natural tool to study these processes and give results that are analogous to those that Karlin and McGregor developed in the study of (classical) birth-and-death processes using orthogonal polynomials on the real line [115]. CMV block matrices have also been used to describe the solutions to the Schur interpolation problem of certain operator-valued functions in the unit disc [8], in the study of conservative discrete time-invariant systems [7], and in the development of algorithms for Hessenberg reduction of structured matrices [90], by using a block Lanczos-type procedure. This indicates that, as in the scalar case, CMV block matrices are strongly related with the study of block Krylov methods. For a recent treatment of these methods for Hermitian positive-definite matrices, we refer the reader to [14].

4.2.2 CMV block matrices

For $k \geq 0$ we define the CMV basis by

$$\begin{aligned} \chi_{2k}(z) &= z^{-k}\varphi_{2k}^{L,*}(z), \quad \chi_{2k-1}(z) = z^{-k+1}\varphi_{2k-1}^{R}(z), \\ x_{2k}(z) &= z^{-k}\varphi_{2k}^{R}(z), \quad x_{2k-1}(z) = z^{-k}\varphi_{2k-1}^{L,*}(z). \end{aligned}$$

The sequences $(\chi_k(z))_{k\geq 0}$ and $(x_k(z))_{k\geq 0}$ are orthonormal (see [42], Proposition 3.20), i.e.

$$\langle \chi_k, \chi_n \rangle_{R,\sigma} = \langle x_k, x_n \rangle_{R,\sigma} = \delta_{k,n}\mathbf{I}.$$

By using the above basis, the matrix representation of the multiplication operator $f(z) \mapsto zf(z)$ is given by a five-diagonal block matrix called the CMV matrix (see [42], Section 3.11). It is given in terms of the Verblunsky matrix coefficients, and has the following

factorization

$$\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots) = \mathcal{L}\mathcal{M},$$

where $\mathcal{L} = \Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \Theta(\alpha_4) \oplus \cdots$ and $\mathcal{M} = \mathbf{I} \oplus \Theta(\alpha_1) \oplus \Theta(\alpha_3) \oplus \cdots$, are tridiagonal block matrices. In other words, the CMV matrix is given by

$$\mathcal{C} = \mathcal{L}\mathcal{M} = \begin{bmatrix} \alpha_0^H & \rho_0^L\alpha_1^H & \rho_0^L\rho_1^L & \mathbf{0} & \mathbf{0} & \cdots \\ \rho_0^R & -\alpha_0\alpha_1^H & -\alpha_0\rho_1^L & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \alpha_2^H\rho_1^R & -\alpha_2^H\alpha_1 & \rho_2^L\alpha_3^H & \rho_2^L\rho_3^L & \cdots \\ \mathbf{0} & \rho_2^R\rho_1^R & -\rho_2^R\alpha_1 & -\alpha_2\alpha_3^H & -\alpha_2\rho_3^L & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha_4^H\rho_3^R & -\alpha_4^H\alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Notice that $\mathcal{C}$ is unitary since $\Theta(\alpha_n)$ is unitary (see [158], Section 4.) for every $n \geq 0$.

4.2.3 CMV Matrices for symmetric matrix measures

Let σ be a **symmetric** matrix measure on the unit circle as the Section 1.5. Symmetric matrix measures supported on the unit circle frequently appear in the study of quantum random walks. For instance, the matrix measure associated with the well-known Hadamard quantum random walk with constant coin $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and local transition is

$$d\sigma(\theta) = \frac{1}{\sqrt{\cos 2\theta}} \begin{bmatrix} \sqrt{1+\cos 2\theta} & \pm 1 \\ \pm 1 & \sqrt{1+\cos 2\theta} \end{bmatrix} \frac{d\theta}{2\pi},$$

supported in $[-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}]$ (see [24], Section 8).

Symmetric matrix measures can be characterized in terms of the corresponding Verblunsky matrix coefficients as follows.

Theorem 4.2.1. ([42], Lemma 4.1) *The Hermitian matrix measure σ is symmetric if and only if all Verblunsky matrix coefficients α_n are Hermitian matrices.*

When the Verblunsky coefficients are Hermitian matrices (real numbers in the scalar case) as noted in Section 1.5 there exists an observation due to Szegő called the matrix Szegő transformation. Moreover, Theorem 1.5.3 showed a relation between the CMV matrix and the Jacobi matrix $\mathbf{J}$.

In what follows, we assume σ and $\tilde{\sigma}$ are positive definite symmetric matrix measures supported on $\mathbb{T}$. Therefore, the corresponding Verblunsky matrix coefficients $(\alpha_n)_{n\geq 0}$ and $(\tilde{\alpha}_n)_{n\geq 0}$ are Hermitian matrices. When σ is a symmetric matrix measure, the associated **MOPUC** satisfy some interesting properties. The following lemma states some of them.

Lemma 4.2.2. *Assume that κ_n^R and κ_n^L are positive definite. Then, for all $n \geq 0$,*

1. $\kappa_n^R = (\kappa_n^L)^H$ *and* $S_n^R = S_n^L$.
2. $(\varphi_n^R(\bar{z}))^H = \varphi_n^L(z)$.
3. $(\Phi_n^R(0))^H = \Phi_n^L(0)$.

Proof. For $0 \leq k \leq n$, there exist matrices $\lambda_{k,j} \in \mathcal{M}_l$, such that $\varphi_k^R(z) = \kappa_k^R z^k + \sum_{j=0}^{k-1} \lambda_{k,j} z^j$.

1. From the orthogonality, we have

$$\begin{aligned}\mathbf{I}\delta_{n,k} = \langle \varphi_k^R, \varphi_n^R \rangle_{R,\sigma} &= \int_{-\pi}^{\pi} \left(\kappa_k^R e^{ik\theta} + \sum_{j=0}^{k-1} \lambda_{k,j} e^{ij\theta} \right)^H d\sigma(\theta) \varphi_n^R(e^{i\theta}) \\ &= \int_{-\pi}^{\pi} \left((\kappa_k^R)^H e^{-ik\theta} + \sum_{j=0}^{k-1} \lambda_{k,j}^H e^{-ij\theta} \right) d\sigma(\theta) \varphi_n^R \left(\kappa_n^R e^{in\theta} + \sum_{p=0}^{n-1} \lambda_{n,p} e^{ip\theta} \right).\end{aligned}$$

On the other hand, setting $\theta \mapsto -\theta$, we get

$$\begin{aligned}\mathbf{I}\delta_{n,k} &= \int_{-\pi}^{\pi} \left((\kappa_k^R)^H e^{ik\theta} + \sum_{j=0}^{k-1} \lambda_{k,j}^H e^{ij\theta} \right) d\sigma(\theta) \left(\kappa_n^R e^{-in\theta} + \sum_{p=0}^{n-1} \lambda_{n,p} e^{-ip\theta} \right) \\ &= \int_{-\pi}^{\pi} \left((\kappa_k^R)^H e^{ik\theta} + \sum_{j=0}^{k-1} \lambda_{k,j}^H e^{ij\theta} \right) d\sigma(\theta) \left((\kappa_n^R)^H e^{in\theta} + \sum_{p=0}^{n-1} \lambda_{n,p}^H e^{ip\theta} \right)^H \\ &= \left\langle (\kappa_k^R)^H z^k + \sum_{j=0}^{k-1} \lambda_{kj}^H z^j, (\kappa_n^R)^H z^k + \sum_{p=0}^{n-1} \lambda_{np}^H z^p \right\rangle_{L,\sigma} = \langle \psi_k^L, \psi_n^L \rangle_{L,\sigma}\end{aligned}$$

where $\psi_k^L(z) = (\kappa_k^R)^H z^k + \sum_{j=0}^{k-1} \lambda_{kj}^H z^j$. Since $\langle \varphi_k^L, \varphi_n^L \rangle_{L,\sigma} = \mathbf{I}\delta_{n,k}$ and $\kappa_n^L > \mathbf{0}$, there is uniqueness and therefore $\psi_n^L(z) = \varphi_n^L(z)$. Comparing coefficients, we get $\kappa_n^R = (\kappa_n^L)^H$. Moreover, from (1.61) we obtain $S_n^R = S_n^L$.

2. We have $(\varphi_n^R(\bar{z}))^H = \psi_n^L(z)$ and since we know $\psi_n^L(z) = \varphi_n^L(z)$ from the previous item, the result follows.

3. Since σ is symmetric, we have $\alpha_n^H = \alpha_n$. The result follows from (1.38) and (1.39). □

In the remaining of this section, we will show that the matrices $\Theta(\alpha_n)$ and $\Theta(\tilde{\alpha}_n)$ are unitarily similar whenever the corresponding measure matrices are symmetric. Moreover, we show an explicit formula for the unitary matrices of the similarity relation. Analogous results for the scalar case have been studied in [139].

Proposition 4.2.3. *For $n \geq 0$, there exist unitary matrices $\Xi_n \in \mathcal{M}_{2l}$ such that*

$$\Theta(\alpha_n) = \Xi_n \Theta(\tilde{\alpha}_n) \Xi_n^H. \tag{4.36}$$

Proof. Since σ is symmetric, we have $\alpha_n^H = \alpha_n$ and thus $\Theta(\alpha_n)$ is an Hermitian and unitary matrix. As a consequence, its eigenvalues are in $\mathbb{T}$ and are real. This is, they are necessarily ± 1. Moreover, since the trace of $\Theta(\alpha_n)$ is zero, we deduce that each eigenvalue has algebraic multiplicity l, since the sum of the eigenvalues equals the trace of matrix (see [19], Section 4.3). On the other hand, since $\Theta(\alpha_n)$ is a normal matrix, then by the Spectral Theorem for normal matrices (see [109], Theorem 2.5.3) there exists a unitary matrix $U_{\Theta_n} \in \mathcal{M}_{2l}$, and a diagonal matrix $D_{\Theta_n} \in \mathcal{M}_{2l}$, containing the eigenvalues of $\Theta(\alpha_n)$, such that

$$\Theta(\alpha_n) = U_{\Theta_n} D_{\Theta_n} U_{\Theta_n}^H, \tag{4.37}$$

where we have arranged the eigenvalues $\Theta(\alpha_n)$ in such a way that $D_{\Theta_n} = [\mathbf{I} \oplus (-\mathbf{I})]$. Similarly, for $\tilde{\sigma}$ we have

$$\Theta(\tilde{\alpha}_n) = U_{\tilde{\Theta}_n} D_{\tilde{\Theta}_n} U_{\tilde{\Theta}_n}^H = U_{\tilde{\Theta}_n} [\mathbf{I} \oplus (-\mathbf{I})] U_{\tilde{\Theta}_n}^H. \tag{4.38}$$

Since $D_{\Theta_n} = D_{\tilde{\Theta}_n}$, from (4.37) and (4.38) we have $U_{\Theta_n}^H \Theta(\alpha_n) U_{\Theta_n} = U_{\tilde{\Theta}_n}^H \Theta(\tilde{\alpha}_n) U_{\tilde{\Theta}_n}$, and, as a consequence, $\Theta(\alpha_n) = U_{\Theta_n} U_{\tilde{\Theta}_n}^H \Theta(\tilde{\alpha}_n) (U_{\Theta_n} U_{\tilde{\Theta}_n}^H)^H$. Therefore, taking

$$\Xi_n = U_{\Theta_n} U_{\tilde{\Theta}_n}^H, \tag{4.39}$$

the result follows. □

Notice that (4.39) is a change of basis matrix between two orthonormal basis. On the other hand, since the matrices $(\alpha_n)_{n\geq 0}$ are Hermitian, then for all $n \geq 0$, we have $\mathbf{0} < \alpha_n^2 = \alpha_n^H \alpha_n < \mathbf{I}$, and therefore $\mathbf{0} < \mathbf{I} + \alpha_n < 2\mathbf{I}$. Also, we have $\mathbf{0} < \mathbf{I} - \alpha_n$. Since $\mathbf{I} + \alpha_n$ and $\mathbf{I} - \alpha_n$ are positive definite matrices, they have unique square roots (see [109], Theorem 7.2.6) $\sqrt{\mathbf{I} + \alpha_n}$ and $\sqrt{\mathbf{I} - \alpha_n}$, respectively, that are also positive definite. Thus, we can define the unitary block matrices

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\mathbf{I} + \alpha_n} & \sqrt{\mathbf{I} - \alpha_n} \\ \sqrt{\mathbf{I} - \alpha_n} & -\sqrt{\mathbf{I} + \alpha_n} \end{bmatrix} \in \mathcal{M}_{2l}, \ \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\mathbf{I} + \tilde{\alpha}_n} & \sqrt{\mathbf{I} - \tilde{\alpha}_n} \\ \sqrt{\mathbf{I} - \tilde{\alpha}_n} & -\sqrt{\mathbf{I} + \tilde{\alpha}_n} \end{bmatrix} \in \mathcal{M}_{2l}.$$

Proposition 4.2.4. *The unitary matrix Ξ_n defined in (4.39) has the form*

$$\Xi_n(\alpha_n, \tilde{\alpha}_n) = \frac{1}{2} \begin{bmatrix} \pi_{n,0,0} + \pi_{n,1,1} & \pi_{n,0,1} - \pi_{n,1,0} \\ -(\pi_{n,0,1} - \pi_{n,1,0}) & \pi_{n,0,0} + \pi_{n,1,1} \end{bmatrix} \in \mathcal{M}_{2l}, \tag{4.40}$$

with

$$\pi_{n,i,j} = \sqrt{\mathbf{I} + (-1)^i \alpha_n} \sqrt{\mathbf{I} + (-1)^j \tilde{\alpha}_n} \in \mathcal{M}_l, \quad i,j = 0,1. \tag{4.41}$$

Proof. Notice that $\Theta(\alpha_n)$ can be written as

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\mathbf{I} + \alpha_n} & \sqrt{\mathbf{I} - \alpha_n} \\ \sqrt{\mathbf{I} - \alpha_n} & -\sqrt{\mathbf{I} + \alpha_n} \end{bmatrix} [\mathbf{I} \oplus (-\mathbf{I})] \left(\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\mathbf{I} + \alpha_n} & \sqrt{\mathbf{I} - \alpha_n} \\ \sqrt{\mathbf{I} - \alpha_n} & -\sqrt{\mathbf{I} + \alpha_n} \end{bmatrix} \right)^H$$

and then from (4.37) we can take

$$U_{\Theta_n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\mathbf{I} + \alpha_n} & \sqrt{\mathbf{I} - \alpha_n} \\ \sqrt{\mathbf{I} - \alpha_n} & -\sqrt{\mathbf{I} + \alpha_n} \end{bmatrix}.$$

In a similar way, from (4.38) we get

$$U_{\tilde{\Theta}_n} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\mathbf{I} + \tilde{\alpha}_n} & \sqrt{\mathbf{I} - \tilde{\alpha}_n} \\ \sqrt{\mathbf{I} - \tilde{\alpha}_n} & -\sqrt{\mathbf{I} + \tilde{\alpha}_n} \end{bmatrix}.$$

Thus, according to (4.39) we have

$$\Xi_n(\alpha_n, \tilde{\alpha}_n) = \frac{1}{2} \begin{bmatrix} \sqrt{\mathbf{I} + \alpha_n} & \sqrt{\mathbf{I} - \alpha_n} \\ \sqrt{\mathbf{I} - \alpha_n} & -\sqrt{\mathbf{I} + \alpha_n} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{I} + \tilde{\alpha}_n} & \sqrt{\mathbf{I} - \tilde{\alpha}_n} \\ \sqrt{\mathbf{I} - \tilde{\alpha}_n} & -\sqrt{\mathbf{I} + \tilde{\alpha}_n} \end{bmatrix}^H,$$

and multiplying and using (4.41), (4.40) follows. □

Since we know the relation between $\Theta(\alpha_n)$ and $\Theta(\tilde{\alpha}_n)$, it is possible to deduce the relation between the corresponding CMV block matrices $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

Theorem 4.2.5. *Let $\mathcal{C}(\alpha_0, \alpha_1, ...) = \mathcal{L}\mathcal{M}$ and $\tilde{\mathcal{C}}(\tilde{\alpha}_0, \tilde{\alpha}_1, ...) = \tilde{\mathcal{L}}\tilde{\mathcal{M}}$. Then, there exist unitary matrices W and V such that*

$$\tilde{\mathcal{C}} = W\mathcal{C}V, \tag{4.42}$$

where

$$W = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \cdots, \ V = \mathbf{I} \oplus V_0 \oplus V_1 \oplus V_2 \oplus \cdots, \tag{4.43}$$

with $W_n, V_n \in \mathcal{M}_{2l}$ given by

$$W_n = \Xi_{2n}^H \Theta(\alpha_{2n}) \Xi_{2n} \Theta(\alpha_{2n})^H, \ V_n = \Theta(\alpha_{2n+1})^H \Xi_{2n+1}^H \Theta(\alpha_{2n+1}) \Xi_{2n+1}. \tag{4.44}$$

Proof. Let $\tilde{\mathcal{C}} = \tilde{\mathcal{L}}\tilde{\mathcal{M}} = \tilde{\mathcal{L}}\mathcal{L}^{-1}\mathcal{C}\mathcal{M}^{-1}\tilde{\mathcal{M}}$. Defining $W = \tilde{\mathcal{L}}\mathcal{L}^{-1}$ and $V = \mathcal{M}^{-1}\tilde{\mathcal{M}}$, we have

$$\begin{aligned} W &= (\Theta(\tilde{\alpha}_0) \oplus \Theta(\tilde{\alpha}_2) \oplus \Theta(\tilde{\alpha}_4) \oplus \cdots)(\Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \Theta(\alpha_4) \oplus \cdots)^{-1} \\ &= \Theta(\tilde{\alpha}_0)\Theta(\alpha_0)^{-1} \oplus \Theta(\tilde{\alpha}_2)\Theta(\alpha_2)^{-1} \oplus \Theta(\tilde{\alpha}_4)\Theta(\alpha_4)^{-1} \oplus \cdots, \end{aligned}$$

and thus, defining $W_n = \Theta(\tilde{\alpha}_{2n})\Theta(\alpha_{2n})^{-1} = \Theta(\tilde{\alpha}_{2n})\Theta(\alpha_{2n})^H$ for $n \geq 0$, from (4.36) we get $\Xi_n^H \Theta(\alpha_n) \Xi_n = \Theta(\tilde{\alpha}_n)$, and substituting in W_n we obtain the first equation of (4.44). In a similar way,

$$\begin{aligned} V &= (\mathbf{I} \oplus \Theta(\alpha_1) \oplus \Theta(\alpha_3) \oplus \cdots)^{-1}(\mathbf{I} \oplus \Theta(\tilde{\alpha}_1) \oplus \Theta(\tilde{\alpha}_3) \oplus \cdots) \\ &= \mathbf{I} \oplus \Theta(\alpha_1)^{-1}\Theta(\tilde{\alpha}_1) \oplus \Theta(\alpha_3)^{-1}\Theta(\tilde{\alpha}_3) \oplus \cdots, \end{aligned}$$

and defining $V_n = \Theta(\alpha_{2n+1})^{-1}\Theta(\tilde{\alpha}_{2n+1}) = \Theta(\alpha_{2n+1})^H\Theta(\tilde{\alpha}_{2n+1})$ for $n \geq 0$, again from (4.36) we get the second equation of (4.44). □

Notice that $\tilde{\mathcal{C}} = W\mathcal{C}V$ defines an equivalence relation on the set $\mathcal{A}$ of CMV block matrices associated with symmetric matrix measures.

4.2.4 Verblunsky matrix coefficients associated with transformations of the matrix measure

In this subsection, we will consider the effect that some transformations on the matrix measure have on the corresponding Verblunsky matrix coefficients. For all $p, q \in \mathbb{P}^l[z]$, we define the following (right) sesquilinear forms:

1. Christoffel transformation

$$\langle p, q \rangle_{R,\sigma_c} = \langle W(z)p, W(z)q \rangle_{R,\sigma}, \tag{4.45}$$

   where $W(z) = (z\mathbf{I} - A) \in \mathcal{M}_l$ and z is not an eigenvalue of A.

2. Uvarov transformation with one mass point

$$\langle p, q \rangle_{R,\sigma_u} = \langle p, q \rangle_{R,\sigma} + p(\zeta)^H \mathbf{M} q(\zeta), \tag{4.46}$$

   where $\mathbf{M} \in \mathcal{M}_l$ is a positive definite matrix and $\zeta \in \mathbb{T}$.

3. Uvarov transformation with two mass points

$$\langle p,q\rangle_{R,\sigma_v} = \langle p,q\rangle_{R,\sigma} + p(\bar{\zeta}^{-1})^H \mathbf{M} q(\zeta) + p(\zeta)^H \mathbf{M}^H q(\bar{\zeta}^{-1}), \tag{4.47}$$

where $\mathbf{M} \in \mathcal{M}_l$ and $\zeta \in \mathbb{C} \setminus \{\mathbb{T}, 0\}$.

Notice that, since σ is a Hermitian symmetric matrix measure, it follows that the sesquilinear forms associated with both Uvarov transformations are also symmetric. The same is true for the sesquilinear form associated with the Christoffel transformation if we take $W(\theta) = \cos\theta \mathbf{I} - A$.

Let $(C_n^R(z) = \kappa_{c,n} z^n + \cdots)_{n\geq 0}$, $(U_n^R(z) = \kappa_{u,n} z^n + \cdots)_{n\geq 0}$ and $(V_n^R(z) = \kappa_{v,n} z^n + \cdots)_{n\geq 0}$ be, respectively, the sequences of monic **MOPUC** associated with the sesquilineal forms (4.45), (4.46) and (4.47).

For the Christoffel matrix transformation, from the Proposition 3.2.2 with $m = 1$, we have that

$$C_n^R(z) = W(z)^{-1} \left[\Phi_{n+1}^R(z) - K_n^R(A,z) \left[K_n^R(A,A) \right]^{-1} \Phi_{n+1}^R(A) \right], \tag{4.48}$$

and

$$S_{c,n}^R = (\kappa_{c,n}^{-R})^H \kappa_{c,n}^{-R} = S_{n+1}^R + \Phi_{n+1}^R(A)^H \left[K_n^R(A,A) \right]^{-1} \Phi_{n+1}^R(A), \tag{4.49}$$

For the Uvarov transformation with one mass point, from the Proposition 3.1.2 with $m = 1$, we have that

$$U_n^R(z) = \Phi_n^R(z) - K_{n-1}^R(\zeta, z)\mathbf{M} \left[\boldsymbol{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M} \right]^{-1} \Phi_n^R(\zeta), \tag{4.50}$$

and

$$S_{u,n}^R = (\kappa_{u,n}^{-R})^H \kappa_{u,n}^{-R} = S_n^R + \Phi_n^{R,H}(\zeta)\mathbf{M} \left[\boldsymbol{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M} \right]^{-1} \Phi_n^R(\zeta),$$

is a positive definite matrix for $n \geq 1$.

Finally, we show the connection formula for the Uvarov transformation with two mass points. We define $\mathbb{K}_{n-1}^R(\vec{\zeta}, z) := \left[K_{n-1}^R(\zeta, z), K_{n-1}^R(\bar{\zeta}^{-1}, z) \right]$, $\mathbb{F}_n^R(\vec{\zeta}) := \left[\Phi_n^R(\zeta)^T, \Phi_n^R(\bar{\zeta}^{-1})^T \right]^T$ and $\mathbb{M} := \begin{bmatrix} \mathbf{0} & \mathbf{M}^H \\ \mathbf{M} & \mathbf{0} \end{bmatrix}$.

Proposition 4.2.6. *The sequence of monic matrix polynomials $(V_n^R(z))_{n\geq 0}$ is orthogonal with respect to the sesquilinear form* (4.47) *if and only if the matrix*

$$\Lambda_{n-1}^R(\vec{\zeta}) = \begin{bmatrix} \mathbf{I} + K_{n-1}^R(\bar{\zeta}^{-1}, \zeta)\mathbf{M} & K_{n-1}^R(\zeta,\zeta)\mathbf{M}^H \\ K_{n-1}^R(\bar{\zeta}^{-1}, \bar{\zeta}^{-1})\mathbf{M} & \mathbf{I} + K_{n-1}^R(\zeta, \bar{\zeta}^{-1})\mathbf{M}^H \end{bmatrix},$$

is non-singular for $n \geq 1$. Furthermore,

$$V_n^R(z) = \Phi_n^R(z) - \mathbb{K}_{n-1}^R(\vec{\zeta}, z)\mathbb{M} \left[\Lambda_{n-1}^R(\vec{\zeta}) \right]^{-1} \mathbb{F}_n^R(\vec{\zeta}), \tag{4.51}$$

and

$$S_{v,n}^R = (\kappa_{v,n}^{-R})^H \kappa_{v,n}^{-R} = S_n^R + \mathbb{F}_n^{R,H}(\vec{\zeta})\mathbb{M} \left[\Lambda_{n-1}^R(\vec{\zeta}) \right]^{-1} \mathbb{F}_n^R(\vec{\zeta}). \tag{4.52}$$

Proof. Assume that $(V_n^R(z))_{n\geq 0}$ is the sequence of monic **MOPUC** with respect to (4.47). We can write

$$V_n^R(z) = \Phi_n^R(z) + \sum_{k=0}^{n-1} \Phi_k^R(z)\lambda_{nk}^R, \tag{4.53}$$

where the coefficients λ_{nk}^R are given, for $0 \leq k \leq n-1$, by

$$\lambda_{nk}^R = -S_k^{-R}\left(\Phi_k^{R,H}(\bar{\zeta}^{-1})\mathbf{M}V_n^R(\zeta) + \Phi_k^{R,H}(\zeta)\mathbf{M}^H V_n^R(\bar{\zeta}^{-1})\right).$$

Substituting this expression in (4.53), we obtain

$$V_n^R(z) = \Phi_n^R(z) - \mathbb{K}_{n-1}^R(\vec{\zeta}, z)\mathbb{M}\mathbb{V}_n^R(\vec{\zeta}), \tag{4.54}$$

where $\mathbb{V}_n^R(\vec{\zeta}) := \left[V_n^R(\zeta)^T, V_n^R(\bar{\zeta}^{-1})^T\right]^T$. Setting $z = \zeta$ and $z = \bar{\zeta}^{-1}$ in (4.54) and rearranging, we get $\Lambda_{n-1}^R(\vec{\zeta})\mathbb{V}_n^R(\vec{\zeta}) = \mathbb{F}_n^R(\vec{\zeta})$. The uniqueness of $V_n^R(z)$ and $\Phi_n^R(z)$ implies that $\Lambda_{n-1}^R(\vec{\zeta})$ is a nonsingular matrix for $n \geq 1$. Thus $\mathbb{V}_n^R(\vec{\zeta}) = \left(\Lambda_{n-1}^R(\vec{\zeta})\right)^{-1}\mathbb{F}_n^R(\vec{\zeta})$, and therefore from (4.54) we get (4.51).

Conversely, define $(V_n^R(z))_{n\geq 0}$ as in (4.51). Let $0 \leq k \leq n$, and notice that from (4.47), we have

$$\langle (z-\zeta)^k\mathbf{I}, V_n^R\rangle_{R,\sigma_v} = \langle (z-\zeta)^k\mathbf{I}, V_n^R\rangle_{R,\sigma} + \left[\left((\bar{\zeta}^{-1}-\zeta)^k\right)^H\mathbf{M}, \mathbf{0}\right]\mathbb{V}_n^R(\vec{\zeta}). \tag{4.55}$$

First, for $0 \leq k \leq n-1$, using (1.65) and (4.54) we get

$$\langle (z-\zeta)^k\mathbf{I}, V_n^R\rangle_{R,\sigma} = -\left[\left((\bar{\zeta}^{-1}-\zeta)^k\right)^H\mathbf{M}, \mathbf{0}\right]\mathbb{V}_n^R(\vec{\zeta}),$$

and thus, substituting on (4.55), we get $\langle (z-\zeta)^k\mathbf{I}, V_n^R(z)\rangle_{R,\sigma_v} = \mathbf{0}$, so that $(V_n^R(z))_{n\geq 0}$ is orthogonal with respect to (4.47). For $k = n$, using (4.54),

$$\langle (z-\zeta)^n\mathbf{I}, V_n^R\rangle_{R,\sigma} = S_n^R - \left\langle (z-\zeta)^n\mathbf{I}, \mathbb{K}_{n-1}^R(\vec{\zeta}, z)\right\rangle_{R,\sigma}\mathbb{M}\mathbb{V}_n^R(\vec{\zeta}). \tag{4.56}$$

Since $K_{n-1}^R(\zeta, z) = K_n^R(\zeta, z) - \varphi_n^R(z)\varphi_n^{R,H}(\zeta)$ and $K_{n-1}^R(\bar{\zeta}^{-1}, z) = K_n^R(\bar{\zeta}^{-1}, z) - \varphi_n^R(z)\varphi_n^{R,H}(\bar{\zeta}^{-1})$ and using (1.65) we get

$$\left\langle (z-\zeta)^n\mathbf{I}, \mathbb{K}_{n-1}^R(\vec{\zeta}, z)\right\rangle_{R,\sigma} = \left[\mathbf{0}, \left((\bar{\zeta}^{-1}-\zeta)^n\right)^H\right] - \mathbb{F}_n^{R,H}(\vec{\zeta}),$$

and, as a consequence, from (4.56)

$$\langle (z-\zeta)^n\mathbf{I}, V_n^R\rangle_{R,\sigma} = S_n^R - \left[\left((\bar{\zeta}^{-1}-\zeta)^n\right)^H\mathbf{M}, \mathbf{0}\right]\mathbb{V}_n^R(\vec{\zeta}) + \mathbb{F}_n^{R,H}(\vec{\zeta})\mathbb{M}\mathbb{V}_n^R(\vec{\zeta}).$$

Substituting this expression in (4.55) with $k = n$, we get (4.52). □

By using (1.1), the inverse of $\Lambda_{n-1}^R(\vec{\zeta})$, can be computed with $\mathbf{A} = \mathbf{I} + K_{n-1}^R(\bar{\zeta}^{-1}, \zeta)\mathbf{M}$, $\mathbf{B} = K_{n-1}^R(\zeta, \zeta)\mathbf{M}^H$, $\mathbf{C} = K_{n-1}^R(\bar{\zeta}^{-1}, \bar{\zeta}^{-1})\mathbf{M}$ and $\mathbf{D} = \mathbf{I} + K_{n-1}^R(\zeta, \bar{\zeta}^{-1})\mathbf{M}^H$.

As a consequence, the perturbed matrix polynomials evaluated at $z = 0$ are deduced in the following proposition. An analogous result for the scalar case can be found in [88].

Proposition 4.2.7. *For $n \geq 1$, we have:*

$$1.\ C_n^R(0) \quad = \mathbb{A}_{c,n}^R(A)\Phi_{n+1}^R(0) + \mathbb{B}_{c,n}^R(A), \tag{4.57}$$

$$2.\ U_n^R(0) \quad = \mathbb{A}_{u,n-1}^R(\zeta)\Phi_n^R(0) + \mathbb{B}_{u,n-1}^R(\zeta), \tag{4.58}$$

$$3.\ V_n^R(0) \quad = \mathbb{A}_{v,n-1}^R(\zeta)\Phi_n^R(0) + \mathbb{B}_{v,n-1}^R(\zeta), \tag{4.59}$$

with

$$\begin{aligned} \mathbb{A}_{c,n}^R(A) &= A^{-1}\left[(S_n^L)^{-1}\Phi_n^{L,*}(A)^H\left[K_n^R(A,A)\right]^{-1}\Phi_n^{L,*}(A) - \mathbf{I}\right], \\ \mathbb{B}_{c,n}^R(A) &= A^{-1}(S_n^L)^{-1}\Phi_n^{L,*}(A)^H\left[K_n^R(A,A)\right]^{-1}A\Phi_n^R(A), \end{aligned} \tag{4.60}$$

$$\begin{aligned} \mathbb{A}_{u,n-1}^R(\zeta) &= \mathbf{I} - (S_{n-1}^L)^{-1}\Phi_{n-1}^{L,*}(\zeta)^H\mathbf{M}\left[\mathbf{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\Phi_{n-1}^{L,*}(\zeta), \\ \mathbb{B}_{u,n-1}^R(\zeta) &= -\zeta(S_{n-1}^L)^{-1}\Phi_{n-1}^{L,*}(\zeta)^H\mathbf{M}\left[\mathbf{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\Phi_{n-1}^R(\zeta), \end{aligned} \tag{4.61}$$

$$\begin{aligned} \mathbb{A}_{v,n-1}^R(\zeta) &= \mathbf{I} - (S_{n-1}^L)^{-1}\mathbb{F}_{n-1}^{L,*}(\vec{\zeta})^H\mathbb{M}\left[\Lambda_{n-1}^R(\vec{\zeta})\right]^{-1}\mathbb{F}_{n-1}^{L,*}(\vec{\zeta}), \\ \mathbb{B}_{v,n-1}^R(\zeta) &= -(S_{n-1}^L)^{-1}\mathbb{F}_{n-1}^{L,*}(\vec{\zeta})^H\mathbb{M}\left[\Lambda_{n-1}^R(\vec{\zeta})\right]^{-1}[\mathbf{I}\zeta \oplus \bar{\zeta}^{-1}\mathbf{I}]\mathbb{F}_{n-1}^R(\vec{\zeta}). \end{aligned} \tag{4.62}$$

Proof.

1. From (1.58) $K_n^R(A,0) = (S_n^L)^{-1}\Phi_n^{L,*}(A)^H$, and setting $z = 0$ in (4.48), we have $C_n^R(0) = -A^{-1}\left[\Phi_{n+1}^R(0) - (S_n^L)^{-1}\Phi_n^{L,*}(A)^H\left[K_n^R(A,A)\right]^{-1}\Phi_{n+1}^R(A)\right]$. Since using (1.41) we have $\Phi_{n+1}^R(A) = A\Phi_n^R(A) + \Phi_n^{L,*}(A)\Phi_{n+1}^R(0)$, substituting and considering (4.60), we get (4.57).

2. Since, from (1.58), we have

$$K_{n-1}^R(\zeta,0) = (S_{n-1}^L)^{-1}\Phi_{n-1}^{L,*}(\zeta)^H, \tag{4.63}$$

setting $z = 0$ in (4.50) we get,

$$U_n^R(0) = \Phi_n^R(0) - (S_{n-1}^L)^{-1}\Phi_{n-1}^{L,*}(\zeta)^H\mathbf{M}\left[\mathbf{I} + K_{n-1}^R(\zeta,\zeta)\mathbf{M}\right]^{-1}\Phi_n^R(\zeta). \tag{4.64}$$

From (1.41), $\Phi_n^R(\zeta) = \zeta\Phi_{n-1}^R(\zeta) + \Phi_{n-1}^{L,*}(\zeta)\Phi_n^R(0)$, and substituting in (4.64) and taking into account (4.61), (4.58) follows.

3. From (4.63), we have $\mathbb{K}_{n-1}^R(\vec{\zeta},0) = (S_{n-1}^L)^{-1}\mathbb{F}_{n-1}^{L,*}(\vec{\zeta})^H$. Thus, setting $z = 0$ in (4.51), we get $V_n^R(0) = \Phi_n^R(0) - (S_{n-1}^L)^{-1}\mathbb{F}_{n-1}^{L,*}(\vec{\zeta})^H\mathbb{M}\left[\Lambda_{n-1}^L(\vec{\zeta})\right]^{-1}\mathbb{F}_n^R(\vec{\zeta})$. From (1.41), $\mathbb{F}_n^R(\vec{\zeta}) = [\zeta\mathbf{I}\oplus\bar{\zeta}^{-1}\mathbf{I}]\mathbb{F}_{n-1}^R(\vec{\zeta}) + \mathbb{F}_{n-1}^{L,*}(\vec{\zeta})\Phi_n^R(0)$, so that substituting and using (4.62), we get (4.59). □

According to (1.38), the sequences of Verblunsky matrix coefficients associated with the previous transformations will be denoted by $(\alpha_{c,n})_{n\geq 0}$, $(\alpha_{u,n})_{n\geq 0}$ and $(\alpha_{v,n})_{n\geq 0}$, where $\alpha_{c,n} = -(\kappa_{c,n}^R)^H C_{n+1}^R(0)^H(\kappa_{c,n}^L)^{-1}$, $\alpha_{u,n} = -(\kappa_{u,n}^R)^H U_{n+1}^R(0)^H \quad (\kappa_{u,n}^L)^{-1}$ and $\alpha_{v,n} = -(\kappa_{v,n}^R)^H V_{n+1}^R(0)^H(\kappa_{v,n}^L)^{-1}$. If we assume that these matrices are Hermitian, then it is possible to give expressions for the entries of the corresponding unitary matrices $\Xi_n(\alpha_n, \tilde{\alpha}_n)$ defined in (4.40) with $\tilde{\alpha}_n = \alpha_{k,n}$ $(k = c, u, v)$. In this way, we can relate the matrices $\Theta(\alpha_n)$ and $\Theta(\alpha_{k,n})$ using (4.36) and obtain the relation between the corresponding CMV block matrices.

4.2.5 A CMV block matrix associated with two sequences of Verblunsky coefficients

In this section, from two known sequences of Verblunsky matrix coefficients, we construct a new one, and we show the relation between the corresponding CMV block matrices. Verblunsky's Theorem 1.4.2 guarantees the existence of the corresponding matrix measure, that can be approximated by using the Bernstein-Szegő approximation theorem (Theorem 3.11 of [42]).

Let $(\alpha_n)_{n\geq 0}$, $(\hat{\alpha}_n)_{n\geq 0}$ be two sequences of Verblunsky matrix coefficients associated with the positive definite matrix measures σ, $\hat{\sigma}$, respectively, supported on $\mathbb{T}$, with corresponding CMV block matrices $\mathcal{C} = \mathcal{L}\mathcal{M}$, $\hat{\mathcal{C}} = \hat{\mathcal{L}}\hat{\mathcal{M}}$.

We define a new sequence of Verblunsky matrix coefficients $(\alpha_{k,n})_{n\geq 0}$ by combining $k \geq 1$ elements of $(\alpha_n)_{n\geq 0}$ and k elements of $(\hat{\alpha}_n)_{n\geq 0}$, in the following way

$$\alpha_{k,n} = \begin{cases} \alpha_n, & n \in \{0, \cdots, k-1, 2k, \cdots, 3k-1, 4k, \cdots, 5k-1, \cdots\}, \\ \hat{\alpha}_n, & n \in \{k, \cdots, 2k-1, 3k, \cdots, 4k-1, 5k, \cdots, 6k-1, \cdots\}. \end{cases}$$

As a consequence, the matrices Θ are given by

$$\Theta(\alpha_{k,n}) = \begin{cases} \Theta(\alpha_n), & n \in \{0, \cdots, k-1, 2k, \cdots, 3k-1, 4k, \cdots, 5k-1, \cdots\}, \\ \Theta(\hat{\alpha}_n), & n \in \{k, \cdots, 2k-1, 3k, \cdots, 4k-1, 5k, \cdots, 6k-1, \cdots\}. \end{cases}$$

For instance, for $k = 1$, the Verblunsky matrix coefficients and the Θ matrices are

$$\alpha_{1,n} = \begin{cases} \alpha_n, & n \text{ is even}, \\ \hat{\alpha}_n, & n \text{ is odd}, \end{cases} ; \Theta(\alpha_{1,n}) = \begin{cases} \Theta(\alpha_n), & n \text{ is even}, \\ \Theta(\hat{\alpha}_n), & n \text{ is odd}. \end{cases}$$

Let $\mathcal{C}_k = \mathcal{C}_k(\alpha_{k,0}, \alpha_{k,1}, \ldots) = \mathcal{L}_k\mathcal{M}_k$ be the associated CMV block matrix. The case $k = 1$ becomes

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{L}_1\mathcal{M}_1 = (\Theta(\alpha_{1,0}) \oplus \Theta(\alpha_{1,2}) \oplus \Theta(\alpha_{1,4}) \oplus \cdots)(\mathbf{I} \oplus \Theta(\alpha_{1,1}) \oplus \Theta(\alpha_{1,3}) \oplus \cdots) \\ &= (\Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \Theta(\alpha_4) \oplus \cdots)(\mathbf{I} \oplus \Theta(\hat{\alpha}_1) \oplus \Theta(\hat{\alpha}_3) \oplus \cdots) \\ &= \mathcal{L}\hat{\mathcal{M}} = \mathcal{C}(\hat{\mathcal{L}}\mathcal{M})^{-1}\hat{\mathcal{C}}. \end{aligned}$$

In particular, if $\hat{\alpha}_n = \mathbf{0}$ for every $n \geq 1$, we have

$$\mathcal{C}_1 = (\Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \Theta(\alpha_4) \oplus \cdots) \left(\mathbf{I} \oplus \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \oplus \cdots\right).$$

In general, the expression for the CMV block matrix $\mathcal{C}_k$ is given in the following result.

Proposition 4.2.8. *The* CMV *block matrix $\mathcal{C}_k$ for $k > 1$ is given by*

1. *If k is a positive even number*

$$\mathcal{C}_k = \mathcal{L}P^kQ^k\hat{\mathcal{M}}, \tag{4.65}$$

where $P^k = P_0 \oplus P_2 \oplus P_4 \oplus \cdots$, *and* $Q^k = \mathbf{I} \oplus Q_1 \oplus Q_3 \cdots$, *with unitary matrices*

$$P_n = \begin{cases} \Theta(\alpha_n)^H \Theta(\hat{\alpha}_n), & n \in \{k, k+2, \ldots, 2k-2\} + 2mk,\ m \geq 0, \\ \mathbf{I}_{2l}, & n \in \{0, 2, \ldots, k-2\} + 2mk,\ m \geq 0. \end{cases}$$

$$Q_n = \begin{cases} \Theta(\alpha_n)\Theta(\hat{\alpha}_n)^H, & n \in \{1, 3, \ldots, k-1\} + 2mk,\ m \geq 0, \\ \mathbf{I}_{2l}, & n \in \{k+1, k+3, \ldots, 2k-1\} + 2mk,\ m \geq 0. \end{cases}$$

2. *If* $k > 1$ *is a positive odd number,* $\mathcal{C}_k$ *is as in (4.65) with unitary matrices*

$$P_n = \begin{cases} \Theta(\alpha_n)^H \Theta(\hat{\alpha}_n), & n \in \{k+1, k+3, \ldots, 2k-2\} + 2mk,\ m \geq 0, \\ \mathbf{I}_{2l}, & n \in \{0, 2, \ldots, k-1\} + 2mk,\ m \geq 0. \end{cases}$$

$$Q_n = \begin{cases} \Theta(\alpha_n)\Theta(\hat{\alpha}_n)^H, & n \in \{1, 3, \ldots, k-2\} + 2mk,\ m \geq 0, \\ \mathbf{I}_{2l}, & n \in \{k, k+2, \ldots, 2k-1\} + 2mk,\ m \geq 0. \end{cases}$$

Proof. Assume k is a positive even number. Then, we have

$$\begin{aligned} \mathcal{C}_k &= \mathcal{L}_k \mathcal{M}_k = (\Theta(\alpha_{k,0}) \oplus \Theta(\alpha_{k,2}) \oplus \Theta(\alpha_{k,4}) \oplus \cdots)(\mathbf{I} \oplus \Theta(\alpha_{k,1}) \oplus \Theta(\alpha_{k,3}) \oplus \cdots) \\ &= (\Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \cdots \oplus \Theta(\alpha_{k-2}) \oplus \Theta(\hat{\alpha}_k) \oplus \cdots)(\mathbf{I} \oplus \Theta(\alpha_1) \oplus \Theta(\alpha_3) \oplus \cdots \\ &\quad \oplus \Theta(\alpha_{k-1}) \oplus \Theta(\hat{\alpha}_{k+1}) \oplus \cdots) \\ &= \mathcal{L}(\mathbf{I}_{2l} \oplus \mathbf{I}_{2l} \oplus \cdots \oplus \mathbf{I}_{2l} \oplus \Theta(\alpha_k)^{-1}\Theta(\hat{\alpha}_k) \oplus \Theta(\alpha_{k+2})^{-1}\Theta(\hat{\alpha}_{k+2}) \oplus \cdots \oplus \\ &\quad \Theta(\alpha_{2k-2})^{-1}\Theta(\hat{\alpha}_{2k-2}) \oplus \mathbf{I}_{2l} \oplus \cdots)(\mathbf{I} \oplus \Theta(\alpha_1)\Theta(\hat{\alpha}_1)^{-1} \oplus \cdots \oplus \Theta(\alpha_{k-1}) \\ &\quad \Theta(\hat{\alpha}_{k-1})^{-1} \oplus \mathbf{I}_{2l} \oplus \cdots \oplus \mathbf{I}_{2l} \oplus \Theta(\alpha_{2k+1})\Theta(\hat{\alpha}_{2k+1})^{-1} \oplus \cdots)\hat{\mathcal{M}}, \end{aligned}$$

which is (4.65). The odd case can be proved in a similar way. P^k and Q^k are unitary because $\Theta(\alpha_n)$ and $\Theta(\hat{\alpha}_n)$ are unitary matrices for $n \geq 0$. □

Notice that $\lim_{k\to\infty} \mathcal{C}_k = \mathcal{L}\mathcal{M}$. Furthermore, if $(\alpha_n)_{n\geq 0}$ and $(\hat{\alpha}_n)_{n\geq 0}$ are Hermitian Verblunsky matrix coefficients, then $(\alpha_{k,n})_{n\geq 0}$ has also Hermitian elements for all $k \geq 1$. According to Theorem 4.2.5, there exist unitary matrices W_k, $\hat{W}_k$, V_k and $\hat{V}_k$ such that $\mathcal{C}_k = W_k \mathcal{C} V_k$, and $\mathcal{C}_k = \hat{W}_k \hat{\mathcal{C}} \hat{V}_k$. Therefore we have $\hat{\mathcal{C}} = W\mathcal{C}V$, where $W = \hat{W}_k^H W_k$ and $V = V_k \hat{V}_k^H$.

CHAPTER 5

On the Szegő transformation for some spectral transformations of matrix measures

Summary
In this chapter, we deduce conditions such that the Uvarov, Christoffel and Geronimus matrix transformations applied to a matrix measure supported on the real line, are preserved under the matrix Szegő transformation. In the scalar case, these transformations have been studied (in both the real line and unit circle cases) on [20, 45, 88, 132, 139, 134, 179] among others, and the analysis under the Szegő transformation has been developed in [87, 148].

5.1 Introduction

We aim to study some spectral transformations of matrix measures on the real line under the effect of the Szegő transformation. Namely, consider a normalized positive semi-definite matrix measure μ supported on $[-1,1]$. As noted in the Section 1.5, μ has an associated (symmetric, positive semi-definite) matrix measure σ, supported on the unit circle, which is the result of applying the matrix Szegő transformation to μ. The application of the Uvarov, Christoffel and Geronimus spectral transformations to μ yields another matrix measure, say $\tilde{\mu}$, supported on $[-1,1]$. Of course, the application of the Szegő transformation to $\tilde{\mu}$ will result in a matrix measure $\tilde{\sigma}$ supported on the unit circle. We will determine conditions on the parameters of the Uvarov, Christoffel, and Geronimus transformations in such a way that $\tilde{\sigma}$ is the corresponding spectral transformation of σ. The main tool will be the relation between the corresponding matrix-valued Stieltjes and Carathéodory functions given in Theorem 1.5.4. We will consider the aforementioned transformations because, in the scalar case (see [179]), a finite composition of the Christoffel and Geronimus transformations can be used to construct any perturbation of the Stieltjes function of the form

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)}, \tag{5.1}$$

where A, B, C are arbitrary polynomials such that $\widetilde{S}(x)$ is also a Stieltjes function. In other words, they *generate* perturbations of the form (5.1). In particular, the Uvarov transformation can be obtained by applying a Christoffel transformation followed by a Geronimus transformation.

In this chapter S and F will be the matrix-valued Stieltjes and Carathéodory functions defined in (1.18) and (1.69) associated with μ and σ, that are normalized positive semi-definite matrix measures supported on $[-1,1]$ and $\mathbb{T}$, respectively.

5.2 Uvarov matrix transformation

The Uvarov matrix transformations of μ and σ are defined as

$$\begin{aligned} d\mu_u(x) &= d\mu(x) + \mathbf{M}_r\delta(x-\beta), \\ d\sigma_u(z) &= d\sigma(z) + \mathbf{M}_c\delta(z-\alpha) + \mathbf{M}_c^H\delta(z-\bar{\alpha}^{-1}), \end{aligned} \tag{5.2}$$

with $\mathbf{M}_r \in \mathcal{M}_l$ an Hermitian matrix, $\mathbf{M}_c \in \mathcal{M}_l$, $\beta \in \mathbb{R}$, $\alpha \in \mathbb{C}\setminus\{0\}$ and δ is the Dirac matrix measure.

In order to relate both Uvarov transformations through the matrix Szegő transformation, we need them to be normalized and at least Hermitian. Assume $\mathbf{I}+\mathbf{M}_r \in \mathcal{M}_l$ and $\mathbf{I}+\mathbf{M}_c+\mathbf{M}_c^H \in \mathcal{M}_l$ are nonsingular. Then, Theorem 1.1.4 guarantees the existence of matrices $R_u \in \mathcal{M}_l$ and $T_u \in \mathcal{M}_l$ such that

$$R_u = \sqrt{\mathbf{I}+\mathbf{M}_r} \text{ and } T_u = \sqrt{\mathbf{I}+\mathbf{M}_c+\mathbf{M}_c^H}. \tag{5.3}$$

Moreover, since $\mathbf{I}+\mathbf{M}_r$ and $\mathbf{I}+\mathbf{M}_c+\mathbf{M}_c^H$ are Hermitian matrices, then R_u and T_u are also Hermitian matrices. Therefore, the normalized Uvarov transformations of the matrix measures μ and σ are defined as

$$\begin{aligned} d\mu_{u_N}(x) &= R_u^{-1}d\mu_u(x)R_u^{-1} = R_u^{-1}\left[d\mu(x) + \mathbf{M}_r\delta(x-\beta)\right]R_u^{-1}, \\ d\sigma_{u_N}(z) &= T_u^{-1}d\sigma_u(z)T_u^{-1} = T_u^{-1}\left[d\sigma(z) + \mathbf{M}_c\delta(z-\alpha) + \mathbf{M}_c^H\delta(z-\bar{\alpha}^{-1})\right]T_u^{-1}. \end{aligned} \tag{5.4}$$

Let $(\mu_n^u)_{n\geq 0}$ and $(c_n^u)_{n\in\mathbb{Z}}$ be the sequences of matrix moments associated with the normalized matrix measures (5.4) respectively. The matrix-valued Stieltjes and Carathéodory functions associated with the Uvarov transformation are deduced in the following result, that is a generalization from the scalar case given in [132].

Proposition 5.2.1. *Let S_u and F_u be the matrix-valued Stieltjes and Carathéodory functions associated with the normalized Hermitian matrix measures μ_{u_N} and σ_{u_N}, as defined in* (5.4). *Then,*

$$S_u(x) = R_u^{-1}\left[S(x) + \mathbf{M}_r(x-\beta)^{-1}\right]R_u^{-1}. \tag{5.5}$$

$$\begin{aligned} F_u(z) = T_u^{-1}\Bigg[F(z) + \frac{1}{(z-\alpha)(\bar{\alpha}z-1)}\Big(&\left(\mathbf{M}_c+\mathbf{M}_c^H\right)\alpha \\ &+(1-|\alpha|^2)\left(\mathbf{M}_c-\mathbf{M}_c^H\right)z - \bar{\alpha}\left(\mathbf{M}_c+\mathbf{M}_c^H\right)z^2\Big)\Bigg]T_u^{-1}. \end{aligned} \tag{5.6}$$

Proof. From (5.4), we get for $n \in \mathbb{N}$

$$\mu_n^u = \int_{-1}^{1} x^n R_u^{-1}[d\mu(x) + \mathbf{M}_r \delta(x-\beta)] R_u^{-1} = R_u^{-1}\left[\mu_n + \mathbf{M}_r \beta^n\right] R_u^{-1}.$$

Thus,

$$\begin{aligned} S_u(x) = \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \mu_n^u &= R_u^{-1}\left[\sum_{n=0}^{\infty} \frac{1}{x^{n+1}}\mu_n + \mathbf{M}_r \sum_{n=0}^{\infty} \frac{1}{x^{n+1}}\beta^n\right] R_u^{-1} \\ &= R_u^{-1}\left[S(x) + \mathbf{M}_r\left[\frac{1}{x}\sum_{n=0}^{\infty}\left(\frac{1}{x}\beta\right)^n\right]\right] R_u^{-1} \end{aligned}$$

with $|\beta| < |x|$ in order to guarantee the convergence of the series. As a consequence,

$$R_u S_u(x) R_u = S(x) + \mathbf{M}_r\left[\frac{1}{x}\left(1 - \frac{1}{x}\beta\right)^{-1}\right] = S(x) + \mathbf{M}_r (x-\beta)^{-1},$$

which is (5.5). On the other hand, from (5.4), we get for $n \in \mathbb{Z}$

$$c_{-n}^u = \int_{-\pi}^{\pi} e^{-in\theta} d\sigma_{u_N}(\theta) = T_u^{-1}\left[c_{-n} + \alpha^{-n}\mathbf{M}_c + \bar{\alpha}^n \mathbf{M}_c^H\right] T_u^{-1}.$$

Thus,

$$\begin{aligned} F_u(z) &= \mathbf{I} + 2\sum_{n=1}^{\infty} c_{-n}^u z^n = T_u^{-1}\left[\mathbf{I} + \mathbf{M}_c + \mathbf{M}_c^H\right] T_u^{-1} \\ &\quad + 2T_u^{-H}\left[\sum_{n=1}^{\infty}(c_{-n} + \alpha^{-n}\mathbf{M}_c + \bar{\alpha}^n \mathbf{M}_c^H) z^n\right] T_u^{-1}, \end{aligned}$$

that can be arranged as

$$\begin{aligned} T_u F_u(z) T_u &= \mathbf{I} + \mathbf{M}_c + \mathbf{M}_c^H + 2\sum_{n=1}^{\infty}[c_{-n} + \alpha^{-n}\mathbf{M}_c + \bar{\alpha}^n \mathbf{M}_c^H] z^n \\ &= F(z) + \left[\mathbf{I} + 2\sum_{n=1}^{\infty}(\alpha^{-1}z)^n\right]\mathbf{M}_c + \left[\mathbf{I} + 2\sum_{n=1}^{\infty}(\bar{\alpha}z)^n\right]\mathbf{M}_c^H. \end{aligned}$$

Notice that, for convergence purposes, we need $|\alpha^{-1}z| < 1$ and $|\alpha z| < 1$, so that if $|\alpha| < 1$ the region of convergence is $|z| < |\alpha|$. In this disc we have

$$T_u F_u(z) T_u = F(z) + \mathbf{M}_c \frac{\alpha + z}{\alpha - z} + \mathbf{M}_c^H \frac{1 + \bar{\alpha} z}{1 - \bar{\alpha} z},$$

which is (5.6). □

Now, let us determine the matrix-valued Carathéodory function F_1 associated with the normalized matrix measure $d\sigma_1 = Sz(d\mu_{u_N})$, i.e. the matrix Szegő transformation of

μ_{u_N}. From (1.97) and (5.5), we have

$$\frac{2z}{1-z^2}F_1(z) = S_u(x) = R_u^{-1}\left[\frac{2z}{1-z^2}F(z) + \mathbf{M}_r\left(\frac{z+z^{-1}}{2} - \beta\right)^{-1}\right]R_u^{-1}$$
$$= R_u^{-1}\left[\frac{2z}{1-z^2}F(z) + \mathbf{M}_r\left(\frac{2z}{z^2-2z\beta+1}\right)\right]R_u^{-1}.$$

Thus, we get

$$F_1(z) = R_u^{-1}\left[F(z) + \mathbf{M}_r\left(\frac{1-z^2}{z^2-2z\beta+1}\right)\right]R_u^{-1}. \tag{5.7}$$

If we assume $F_u(z) = F_1(z)$, then from Proposition 1.4.10 $\sigma_{u_N} = \sigma_1$ and $d\sigma_{u_N} = Sz(d\mu_{u_N})$. Moreover, comparing the coefficients of z in (5.6) and (5.7) we get

$$(1-|\alpha|^2)(\mathbf{M}_c - \mathbf{M}_c^H) = \mathbf{0}.$$

We can consider the following two cases:

1. If $|\alpha| \neq 1$, then $\mathbf{M}_c$ is a Hermitian matrix, and $F_u(z)$ becomes

$$F_u(z) = T_u^{-1}\left[F(z) + 2\mathbf{M}_c\left(\frac{\alpha - \bar{\alpha}z^2}{(z-\alpha)(\bar{\alpha}z-1)}\right)\right]T_u^{-1}.$$

   Comparing with $F_1(z)$, we get $T_u = R_u$ and therefore $\mathbf{M}_r = 2\mathbf{M}_c$. Moreover,

$$\frac{\alpha - \bar{\alpha}z^2}{(z-\alpha)(\bar{\alpha}z-1)} = \frac{1-z^2}{z^2-2z\beta+1}, \tag{5.8}$$

   so that $\frac{\alpha}{\bar{\alpha}} = 1$, and thus $\alpha \in \mathbb{R}$ and $\frac{1+\alpha^2}{\alpha} = 2\beta$. As a consequence,

$$\alpha = \beta \pm \sqrt{\beta^2-1}, \quad \text{for } |\beta| > 1.$$

2. If $|\alpha| = 1$, we get

$$F_u(z) = T_u^{-1}\left[F(z) + (\mathbf{M}_c + \mathbf{M}_c^H)\left(\frac{\alpha - \bar{\alpha}z^2}{(z-\alpha)(\bar{\alpha}z-1)}\right)\right]T_u^{-1}.$$

   Thus, $\mathbf{M}_r = \mathbf{M}_c + \mathbf{M}_c^H$ and from (5.8) we deduce $\alpha = \pm 1$ and $\beta = \pm 1$.

We summarize our findings in the following result.

Theorem 5.2.2. *Let μ be a positive semi-definite, and normalized matrix measure supported on $[-1,1]$ and let σ be its associated matrix Szegő transformation, i.e. $d\sigma = Sz(d\mu)$. Let μ_{u_N} and σ_{u_N} be their normalized Uvarov matrix transformations defined by (5.4). Then*

1. *For $|\beta| > 1$, $Sz(d\mu_{u_N}(x)) = d\sigma_{u_N}(z)$ if and only if*

$$\alpha = \beta \pm \sqrt{\beta^2-1} \text{ and } \mathbf{M}_c = \frac{1}{2}\mathbf{M}_r.$$

2. *For $\beta = \pm 1$, $Sz(d\mu_{u_N}(x)) = d\sigma_{u_N}(z)$ if and only if*

$$\alpha = \pm 1 \text{ and } \mathbf{M}_r = \mathbf{M}_c + \mathbf{M}_c^H.$$

Notice that this means that, under the stated conditions, the (matrix) Szegő transformation of the Uvarov transformation of μ coincides with the Uvarov transformation of the Szegő transformation of μ. The previous theorem constitutes a generalization of the results obtained in [87] for the scalar case.

Example 5.2.3. *Consider the Uvarov transformation of the matrix measure (1.91) with* $\beta = 1$ *and* $\mathbf{M}_r = \mathbf{I}_2$. *Thus,* $R_u = T_u = \sqrt{2}\mathbf{I}_2$ *and* $\alpha = 1$. *Then from* (1.94), *we have*

$$\begin{aligned} Sz(d\mu_{u_N}(x)) &= Sz\left(\frac{1}{2}\left(\frac{1}{\pi\sqrt{1-x^2}}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} dx + \mathbf{I}_2\delta(x-1)\right)\right) \\ &= \frac{1}{2}\left(\begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}\frac{d\theta}{2\pi} + \mathbf{I}_2\delta(z-1)\right). \end{aligned}$$

5.3 Christoffel matrix transformation

The Christoffel matrix transformation of μ can be defined as

$$d\mu_c(x) = d\mu(x)\,(x\mathbf{I} - B)\,,$$

where $B \in \mathcal{M}_l$ is a Hermitian matrix and $x \notin \varrho(B)$. Notice that, in order to apply the matrix Szegő transformation, we need μ_c to be an Hermitian, normalized matrix measure. As a consequence, we will require that the matrices B and μ commute. Also, we require $x\mathbf{I} - B$ to be a positive semi-definite matrix, so that μ_c is also positive semi-definite. Therefore, we will consider the normalized matrix measure

$$d\mu_{c_N}(x) = R_c^{-1} d\mu_c(x) R_c^{-1} = R_c^{-1}\left[d\mu(x)\,(x\mathbf{I} - B)\right] R_c^{-1}, \tag{5.9}$$

where

$$R_c^2 = \int_{-1}^{1} d\mu(x)(x\mathbf{I} - B) = \mu_1 - B, \tag{5.10}$$

and R_c is an Hermitian matrix.

Similarly, on the unit circle, the Christoffel matrix transformation of σ can be defined as

$$d\sigma_c(z) = (z\mathbf{I} - A)^H d\sigma(z)(z\mathbf{I} - A)$$

and the respective normalized matrix measure

$$d\sigma_{c_N}(z) = T_c^{-1} d\sigma_c(z) T_c^{-1} = T_c^{-1}\left[(z\mathbf{I} - A)^H d\sigma(z)(z\mathbf{I} - A)\right] T_c^{-1}, \tag{5.11}$$

with $A \in \mathcal{M}_l$, $z \in \mathbb{T}$, $z \notin \varrho(A)$ and

$$T_c^2 = \int_{-\pi}^{\pi} (e^{i\theta}I - A)^H d\sigma(\theta)(e^{i\theta}I - A) = \mathbf{I} - 2\mathfrak{Re}(c_{-1}A) + A^H A \geq 0. \tag{5.12}$$

Notice that if σ is positive semi-definite, then σ_{c_N} is also positive semi-definite.

If we denote by $(\mu_n^c)_{n\geq 0}$ and $(c_n^c)_{n\in\mathbb{Z}}$ the sequences of matrix moments associated with the matrix measure (5.9) and (5.11), respectively, then their Stieltjes and Carathéodory matrix-valued functions are given in the next result.

Proposition 5.3.1. *Let S_c and F_c be the matrix-valued Stieltjes and Carathéodory functions associated with the normalized Hermitian matrix measures μ_{c_N} and σ_{c_N}, as defined in (5.9) and (5.11) respectively. Then,*

$$S_c(x) = R_c^{-1}\left[S(x)(\mathbf{I}x - B) - \mathbf{I}\right] R_c^{-1}, \tag{5.13}$$

$$F_c(z) = T_c^{-1}\frac{1}{z}\left[\left(\mathbf{I} - zA^H\right) F(z)\left(z\mathbf{I} - A\right) - \left(z^2A^H - 2iz\mathfrak{Im}(c_{-1}A) - A\right)\right] T_c^{-1}. \tag{5.14}$$

Proof. From (5.9), for $n \in \mathbb{N}$ we have

$$\mu_n^c = \int_{-1}^{1} x^n R_c^{-1}\left(d\mu(x)\left(x\mathbf{I} - B\right)\right) R_c^{-1} = R_c^{-1}(\mu_{n+1} - \mu_n B)R_c^{-1}.$$

Therefore

$$\begin{aligned} S_c(x) = \sum_{n=0}^{\infty}\frac{1}{x^{n+1}}\mu_n^c &= R_c^{-1}\left[\sum_{n=0}^{\infty}\frac{1}{x^{n+1}}\mu_{n+1} - \sum_{n=0}^{\infty}\frac{1}{x^{n+1}}\mu_n B\right] R_c^{-1} \\ &= R_c^{-1}\left[-\mu_0 + x\sum_{n=0}^{\infty}\frac{1}{x^{n+1}}\mu_n - S(x)B\right] R_c^{-1} \\ &= R_c^{-1}\left[S(x)(\mathbf{I}x - B) - \mathbf{I}\right] R_c^{-1}, \end{aligned}$$

so that (5.13) follows. On the other hand, from (5.11), for $n \in \mathbb{Z}$

$$c_{-n}^c = T_c^{-1}\left(c_{-n} - A^H c_{-(n-1)} - c_{-(n+1)}A + A^H c_{-n}A\right) T_c^{-1}. \tag{5.15}$$

Since $F_c(z) = \mathbf{I} + 2\sum_{n=0}^{\infty} c_{-n}^c z^n$, we get

$$\begin{aligned} &T_c F_c(z) T_c \\ &= \mathbf{I} - c_{-1}A - A^H c_1 + A^H A + 2\sum_{n=1}^{\infty}(c_{-n} - A^H c_{-(n-1)} - c_{-(n+1)}A + A^H c_{-n}A)z^n \\ &= F(z) + A^H F(z)A - \left[c_{-1} + 2\sum_{n=1}^{\infty} c_{-(n+1)}z^n\right] A - A^H\left[c_1 + 2\sum_{n=1}^{\infty} c_{-(n-1)}z^n\right] \\ &= F(z) + A^H F(z)A - \left[c_{-1} + \frac{1}{z}\left(F(z) - \mathbf{I} - 2c_{-1}z\right)\right] A - A^H\left[c_1 + z(\mathbf{I} + F(z))\right] \\ &= \frac{1}{z}\left[-(zA^H - \mathbf{I})F(z)(z\mathbf{I} - A) + (-z^2A^H + zc_{-1}A - zA^H c_1 + A)\right]. \end{aligned}$$

Notice that $c_{-1}^H = c_1$, and thus $zc_{-1}A - zA^H c_1 = z(c_{-1}A - (c_{-1}A)^H) = 2iz\mathfrak{Im}(c_{-1}A)$, and (5.14) follows. □

Now, let us denote by F_2 the matrix-valued Carathéodory function associated with the matrix Szegő transformation $d\sigma_2 = Sz(\mu_{c_N})$. From (1.97) and (5.13), we get

$$\begin{aligned} \frac{2z}{1-z^2}F_2(z) = S_c(x) &= R_c^{-1}\left[S(x)(\mathbf{I}x - B) - \mathbf{I}\right] R_c^{-1} \\ &= R_c^{-1}\left[\frac{2z}{1-z^2}F(z)\left(\frac{z+z^{-1}}{2}\mathbf{I} - B\right) - \mathbf{I}\right] R_c^{-1}. \end{aligned}$$

Therefore,

$$F_2(z) = \frac{R_c^{-1}}{\sqrt{2}} \frac{1}{z} \left[F(z) \left((z^2+1)\mathbf{I} - 2zB \right) - (1-z^2)\mathbf{I} \right] \frac{R_c^{-1}}{\sqrt{2}}. \tag{5.16}$$

Notice that, in order to equate F_c and F_2, we will need to impose some conditions on the matrix A.

Lemma 5.3.2. *Let $A \in \mathcal{M}_l$ be a non-singular and Hermitian matrix. If $A^{1/2}$ commutes with T_c, then $A^{1/2}$ commutes with both F_c and F.*

Proof. It is clear that $A^{1/2}T_c = T_c A^{1/2}$ implies $A^{1/2}T_c^2 = T_c^2 A^{1/2}$, i.e.

$$A^{1/2} \int_{-\pi}^{\pi} d\sigma_c = \int_{-\pi}^{\pi} A^{1/2} d\sigma_c = \int_{-\pi}^{\pi} d\sigma_c A^{1/2}.$$

Therefore, we get, for $n \in \mathbb{Z}$,

$$A^{1/2} c_n^c = \int_{-\pi}^{\pi} A^{1/2} e^{in\theta} T_c^{-1} d\sigma_c T_c^{-1} = \int_{-\pi}^{\pi} e^{in\theta} T_c^{-1} d\sigma_c T_c^{-1} A^{1/2} = c_n^c A^{1/2},$$

and, as a consequence, $A^{1/2}$ commutes with F_c. On the other hand, since A is Hermitian then from (5.14) it follows that $A^{1/2}$ commutes with F. □

Let us assume that $A^{1/2}$ satisfies the conditions on the previous lemma. If $F_c = F_2$, then from Proposition 1.4.10 we have $\sigma_{c_N} = Sz(\mu_{c_N})$. Thus, from Lemma 5.3.2, (5.14) becomes

$$F_c(z) = T_c^{-1} \frac{1}{z} \left[-F(z) \left((z^2+1)A - (\mathbf{I}+A)z \right) - \left(z^2 - 1 \right) A \right] T_c^{-1}.$$

Comparing the terms not involving F, we have

$$\frac{R_c^{-1}}{\sqrt{2}} (z^2-1) \frac{R_c^{-1}}{\sqrt{2}} = -T_c^{-1}(z^2-1)AT_c^{-1} = iT_c^{-1}A^{1/2}(z^2-1)iA^{1/2}T_c^{-1}.$$

Therefore, since $A^{1/2}$ commutes with T_c, we have

$$\frac{R_c^{-1}}{\sqrt{2}} = iA^{1/2}T_c^{-1}. \tag{5.17}$$

As a consequence, from (5.10) and (5.12),

$$\mathbf{I} - 2c_{-1}A + A^2 = -2(\mu_1 - B)A. \tag{5.18}$$

Moreover, since $\sigma = Sz(\mu)$ is symmetric, we have $\int_{-\pi}^{\pi} \sin\theta d\sigma(\theta) = \mathbf{0}$. Then, from (1.89),

$$\mu_1 = \int_{-1}^{1} x dx = \int_{-\pi}^{\pi} \cos\theta d\sigma(\theta) - i \int_{-\pi}^{\pi} \sin\theta d\sigma(\theta) = \int_{-\pi}^{\pi} e^{-i\theta} d\sigma(\theta) = c_{-1}.$$

Substituting in (5.18), we get $A = B \pm \sqrt{B^2 - \mathbf{I}}$ for $B^2 \geq \mathbf{I}$. On the other hand, comparing the terms involving F, we obtain

$$-T_c^{-1}F(z)AT_c^{-1} = \frac{R_c^{-1}}{\sqrt{2}} F(z) \frac{R_c^{-1}}{\sqrt{2}},$$

which holds by (5.17). We have proved the following result.

Theorem 5.3.3. *Let μ be a positive semi-definite, and normalized matrix measure supported on $[-1,1]$ and let σ be its associated matrix Szegő transformation, i.e. $d\sigma = Sz(d\mu)$. Let $A \in \mathcal{M}_l$ be a nonsingular, Hermitian matrix such that $A^{1/2}$ commutes with T_c. If*

$$T_c = i\sqrt{2}R_c A^{1/2},\ A = B \pm \sqrt{B^2 - \mathbf{I}} \text{ with } B^2 \geq \mathbf{I},$$

and B commutes with μ, then $Sz(d\mu_{c_N}(x)) = d\sigma_{c_N}(z)$.

In other words, the (matrix) Szegő transformation of the Christoffel transformation of μ coincides with the Christoffel transformation of the Szegő transformation of μ.

The following examples show that we can find matrices satisfying the hypotheses of the previous result.

Example 5.3.4. *(Diagonal case) If we take $B = \beta\mathbf{I} \in \mathcal{M}_l$, with $|\beta| \geq 1$, then $A = \alpha\mathbf{I} = (\beta \pm \sqrt{\beta^2 - 1})\mathbf{I} \in \mathcal{M}_l$. For these matrices, the conditions of Theorem 5.3.3 are satisfied for any pair of matrix measures μ and σ related through the Szegő transformation. This is a generalization of the result given in [87] for the scalar case.*

Example 5.3.5. *Let $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Then,*

$$A = B + \sqrt{B^2 - \mathbf{I}_2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}^{1/2} = \begin{bmatrix} 2+\sqrt{2} & 1+\sqrt{2} \\ 1+\sqrt{2} & 2+\sqrt{2} \end{bmatrix},$$

where we have chosen the square root of $B - \mathbf{I}_2$ with positives entries. Notice that A is a Hermitian and non-singular matrix. Moreover,

$$A^{1/2} = \frac{1}{2}\begin{bmatrix} 2+\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{bmatrix}.$$

Consider the normalized semi-definite positive matrix measures (1.91) and (1.92) satisfying $d\sigma = Sz(d\mu)$, and notice that B commutes with (1.91)*. On the other hand, since $c_{-1} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$, we have*

$$T_c^2 = \mathbf{I}_2 - 2c_{-1}A + A^2 = \begin{bmatrix} 5\sqrt{2}+9 & 5\sqrt{2}+6 \\ 5\sqrt{2}+6 & 5\sqrt{2}+9 \end{bmatrix},$$

thus $T_c = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, where $a = \frac{1}{2}\sqrt{2}\sqrt{5\sqrt{2} + \sqrt{15} + \sqrt{30} + 9}$ and $b = \frac{5\sqrt{2}+6}{2a}$. It is easy to see that $A^{1/2}$ commutes with T_c. Furthermore, since $c_0 = \mathbf{I}_2$ and $c_{-n} = \mathbf{0}_2$ for $|n| \geq 2$, the corresponding matrix-valued Carathéodory function is

$$F(z) = \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}.$$

Notice that $A^{1/2}$ also commutes with F.

5.4 Geronimus matrix transformation

We say that the matrix measure μ_{G_N} is the normalized Geronimus matrix transformation of a matrix measure μ if it satisfies

$$d\mu(x) = R_G \left[d\mu_{G_N}(x) \, (x\mathbf{I} - B) \right] R_G, \tag{5.19}$$

where the Hermitian matrix

$$R_G^2 = \int_{-1}^{1} d\mu(x)(x\mathbf{I} - B)^{-1}$$

is the normalization factor, $B \in \mathcal{M}_l$ is an Hermitian matrix and $x \notin \varrho(B)$. As in the Christoffel case, we will require that $B\mu = \mu B$ and $x\mathbf{I} - B > 0$ in order for μ_{G_N} to be a positive definite (Hermitian) matrix measure. In a similar way, on the unit circle, a matrix measure σ_{G_N} is said to be the normalized Geronimus matrix transformation of σ if it satisfies

$$d\sigma(z) = T_G \left[(z\mathbf{I} - A)^H d\sigma_{G_N}(z)(z\mathbf{I} - A) \right] T_G, \tag{5.20}$$

where the Hermitian matrix

$$T_G^2 = \int_{-\pi}^{\pi} (e^{i\theta}\mathbf{I} - A)^{-H} d\sigma(z)(e^{i\theta}\mathbf{I} - A)^{-1} \geq \mathbf{0}$$

is the normalization factor, with $A \in \mathcal{M}_l$, $z \in \mathbb{T}$ and $z \notin \rho(A)$. Notice that if σ is semi-definite positive, then σ_{G_N} is also semi-definite positive.

We will denote by $(\mu_n^g)_{n\geq 0}$ and $(c_n^g)_{n\in\mathbb{Z}}$ the sequences of matrix moments associated with the matrix measures (5.19) and (5.20), respectively.

Proposition 5.4.1. *Let S_G and F_G be the matrix-valued Stieltjes and Carathéodory functions associated with the normalized Hermitian matrix measures μ_{G_N} and σ_{G_N} as defined in (5.19) and (5.20), respectively. Then,*

$$S_G(x) = \left[R_G^{-1} S(x) R_G^{-1} + \mathbf{I} \right] (\mathbf{I}x - B)^{-1}. \tag{5.21}$$

$$F_G(z) = (\mathbf{I} - zA^H)^{-1} \left[T_G^{-1} z F(z) T_G^{-1} + z^2 A^H - 2iz\mathfrak{Im}(c_{-1}^g A) - A \right] (z\mathbf{I} - A)^{-1}. \tag{5.22}$$

Proof. From (5.19), we get for $n \in \mathbb{N}$,

$$R_G^{-1} \mu_n R_G^{-1} = \mu_{n+1}^g - \mu_n^g B.$$

Proceeding as in the proof of Proposition 5.3.1, we get

$$R_G^{-1} S(x) R_G^{-1} = S_G(x)(x\mathbf{I} - B) - \mathbf{I},$$

which is (5.21). Similarly, from (5.20), we get for $n \in \mathbb{Z}$

$$T_G^{-1} c_{-n} T_G^{-1} = c_{-n}^g - A^H c_{-(n-1)}^g - c_{-(n+1)}^g A + A^H c_{-n}^g A, \tag{5.23}$$

and again proceeding as in the proof for the Christoffel transformation, we get

$$T_G^{-1} F(z) T_G^{-1} = \frac{1}{z} \left[(\mathbf{I} - zA^H) F_G(z)(z\mathbf{I} - A) + (-z^2 A^H + z c_{-1}^g A - z A^H c_1^g + A) \right].$$

Since σ_G is Hermitian, we have $c_1^g = (c_{-1}^g)^H$ and thus $A^H c_1^g - c_{-1}^g A = (c_{-1}^g A)^H - c_{-1}^g A = -2\mathfrak{Im}(c_{-1}^g A)$. As a consequence, we obtain (5.22). □

Now, let us compute F_3, the matrix-valued Carathéodory function associated with the matrix measure $d\sigma_3 = Sz(d\mu_{G_N})$. From (1.97) and (5.21),

$$\begin{aligned}\frac{2z}{1-z^2}F_3(z) &= S_G(x)\\ &= \left[R_G^{-1}\frac{2z}{1-z^2}F(z)R_G^{-1} + \mathbf{I}\right]\left(\frac{z+z^{-1}}{2}\mathbf{I} - B\right)^{-1}\\ &= \left[R_G^{-1}\frac{2z}{1-z^2}F(z)R_G^{-1} + \mathbf{I}\right]\left(\frac{(z^2+1)\mathbf{I} - 2zB}{2z}\right)^{-1}.\end{aligned}$$

Therefore,

$$F_3(z) = \left[R_G^{-1}2zF(z)R_G^{-1} + (1-z^2)\mathbf{I}\right]\left((z^2+1)\mathbf{I} - 2zB\right)^{-1}. \tag{5.24}$$

As in the Christoffel case, from (5.23) we can deduce the following auxiliary result.

Lemma 5.4.2. *Let $A \in \mathcal{M}_l$ be a non-singular and Hermitian matrix. If $A^{1/2}$ commutes with T_G, then $A^{1/2}$ commutes with both F_G and F.*

Let us assume that $A^{1/2}$ satisfies the conditions on the previous lemma. In this situation, F_G becomes

$$F_G(z) = (zA - \mathbf{I})^{-1}\left[-T_G^{-1}zF(z)T_G^{-1} + (1-z^2)A\right](z\mathbf{I} - A)^{-1}.$$

Now, as in the preceding subsection, we proceed to the comparison between F_G and F_3, to obtain

$$(zA - \mathbf{I})^{-1}A(z\mathbf{I} - A)^{-1} = \left((z^2+1)\mathbf{I} - 2zB\right)^{-1},$$

i.e.

$$(z\mathbf{I} - A)A^{-1}(zA - \mathbf{I}) = (z^2+1)\mathbf{I} - 2zB, \tag{5.25}$$

and thus $\mathbf{I} + A^2 = 2BA$, so that $A = B \pm \sqrt{B^2 - \mathbf{I}}$ for $B^2 \geq \mathbf{I}$. On the other hand, since $A^{1/2}$ commutes with T_G and with F by the Lemma 5.4.2, we must have

$$\left((z^2+1)\mathbf{I} - 2zB\right)\frac{R_G}{\sqrt{2}}(F(z))^{-1}\frac{R_G}{\sqrt{2}} = (z\mathbf{I} - A)A^{-1}(zA - \mathbf{I})\frac{T_G}{i}A^{1/2}(F(z))^{-1}\frac{T_G}{i}A^{1/2}.$$

and from (5.25) we deduce $\frac{1}{i}T_G A^{1/2} = \frac{1}{\sqrt{2}}R_G$. Thus, we have proved the following result, which constitutes the Geronimus transformation's version of Theorem 5.3.3.

Theorem 5.4.3. *Let μ be a positive semi-definite, and normalized matrix measure supported on $[-1,1]$ and let σ be its associated matrix Szegő transformation, i.e. $d\sigma = Sz(d\mu)$. Let $A \in \mathcal{M}_l$ be a non-singular, Hermitian matrix such that $A^{1/2}$ commutes T_G. If*

$$T_G = \frac{i}{\sqrt{2}}R_G A^{-1/2}, \quad A = B \pm \sqrt{B^2 - \mathbf{I}} \text{ for } B^2 \geq \mathbf{I},$$

where B commutes with μ, then $Sz(d\mu_{G_N}(x)) = d\sigma_{G_N}(z)$.

Notice that Example 5.3.5 and Example 5.3.4 also satisfy the hypotheses of Theorem 5.4.3. Furthermore, if we extend (1.18) for matrix values $X \in \mathcal{M}_l$, we have

$$S(X) = \int_{-1}^{1} d\mu(t)(X - t\mathbf{I})^{-1} = \sum_{n=0}^{\infty} \mu_n X^{-(n+1)}, \quad t \notin \varrho(X),$$

and, in this situation, $R_G^2 = -S(B)$.

CHAPTER 6

Conclusions and open problems

6.1 Conclusions

The results presented in this work have been focussed on the study of algebraic and analytic properties associated with spectral transformations of matrix measures supported either on the unit circle or on the real line. Next, we summarize the main contributions of this document. We point out that many of the results given in this work for matrix polynomials have their counterparts in the scalar case, and they were developed by using procedures and tools that are analogous to the ones used in the scalar case, with the added complications that result from the non-commutativity of the matrices. Despite the fact that two orthogonal sequences can be considered in the matrix framework, in most of the document we only show the results for one of the sequences, since the results can be obtained in a similar way for the other.

1. In Chapter 2, we show that the perturbation of the matrix moments c_j and c_{-j} of a positive definite matrix measure σ supported on the unit circle results in a perturbation, defined by (2.24), of the moments μ_n associated with a positive definite matrix measure μ supported on the interval $[-1,1]$, when both measures are related through the Szegő matrix transformation. Moreover

$$Sz^{-1}\left(d\sigma_j(\theta)\right) = d\mu(x) + 2T_j(x)m_j\frac{dx}{\pi\sqrt{1-x^2}} = d\mu_j(x),$$

   where $d\sigma_j$ and $d\mu_j$ are defined by (2.20) and (2.23) respectively and $T_j(x) = \cos(j\theta)$ is the $j-$th degree Chebyshev polynomial of the first kind on $\mathbb{R}$. The results are contained in [79], and the particular case $l = 1$ (scalar case) appears in [78].

2. In Chapter 3, we study the Uvarov and Christoffel matrix transformations of a Hermitian matrix measure σ defined by (3.1) and (3.54) respectively, and show that they have similar algebraic and analytic properties. Connection formulas between the corresponding orthogonal sequences, as well as for their leading coefficients, are obtained, and relative asymptotic properties were deduced. It is worth noticing that, when σ belongs to the Nevai matrix class $\mathcal{N}$, the relative asymptotic behavior

of both transformations is the same. These results can be found in [82] (Uvarov transformation) and [80] (Christoffel transformation).

3. In Chapter 4, Theorem 4.1.3 characterizes unitary block Hessenberg matrices associated with **MOPUC** in terms of the derivative of the absolutely continuous part of the associated matrix measure σ supported on the unit circle, that generalizes a well known result for the scalar case. On the other hand, it is shown that, as in the scalar case, the Hessenberg matrices associated with the Christoffel, Uvarov and Geronimus matrix transformations are related to factorizations of the original Hessenberg matrices. The corresponding contribution is [75].

4. Given two sequences of Verblunsky matrix coefficients $(\alpha_n)_{n\geq 0}$, $(\tilde{\alpha}_n)_{n\geq 0}$, where both α_n and $\tilde{\alpha}_n$ are Hermitian matrices for $n \in \mathbb{N}$, in Section 4.2 we show that the associated block matrices $\Theta(\alpha_n)$ and $\Theta(\tilde{\alpha}_n)$ are unitarily similar i.e., there exist unitary matrices $\Xi_n \in \mathcal{M}_{2l}$ such that

$$\Theta(\alpha_n) = \Xi_n \Theta(\tilde{\alpha}_n) \Xi_n^H, \quad n \geq 0,$$

where Ξ_n is defined as in (4.40). Therefore, the corresponding CMV matrices are also unitarily similar, since there exist unitary matrices W and V defined by (4.43), such that $\tilde{\mathcal{C}} = W\mathcal{C}V$. Moreover, the set $\mathcal{A}$ of CMV block matrices associated with symmetric matrix measures defines an equivalence relation given by (4.42). This is the problem considered in [76].

5. Under some conditions, in Chapter 5 we show that the Uvarov matrix transformation of a matrix measure μ supported on the interval $[-1, 1]$ coincides with the Uvarov matrix transformation of a matrix measure supported on unit circle, when both pair of measures (μ and σ, and their corresponding transformations) are related through the Szegő matrix transformation. The same occurs for the Christoffel and Geronimus matrix transformations. These results are contained in [81].

6.2 Open problems

In this section we formulate some open problems that have arisen during our research. We intend to address these questions in future contributions.

1. In the scalar case it is well known that the zeros of the orthogonal polynomials with respect to a measure supported on unit circle lie in $\mathbb{D}$, and the Lucas-Gauss Theorem (see [16]) guarantees that the zeros of the derivatives of the orthogonal polynomials also lie in $\mathbb{D}$. In the matrix case, the location of the zeros of the derivatives of matrix orthogonal polynomials, to the best of our knowledge, constitutes an open problem. This result would allow to study relative asymptotics properties of the sequences of derivatives of **MOPUC**.

2. In this document, we consider the case when the orthogonality measure is Hermitian. For more general measures, sequences of biorthogonal matrix polynomials appear. It would be interesting to extend the results in this document to the more general framework of biorthogonality.

3. Given a Hermitian matrix measure σ supported on unit circle, it is possible to obtain two sesquilinear forms of discrete Sobolev type defined by

$$\begin{aligned}\langle p,q\rangle_{R,S} &= \langle p,q\rangle_{R,\sigma} + p(\mathbf{Z})^H \mathbb{A} q(\mathbf{Z}), \quad p,q\in\mathbb{P}^l[z], &(6.1)\\ \langle p,q\rangle_{L,S} &= \langle p,q\rangle_{L,\sigma} + p(\mathbf{Z}) \mathbb{A} q(\mathbf{Z})^H, \quad p,q\in\mathbb{P}^l[z], &(6.2)\end{aligned}$$

where

$$\begin{aligned}p(\mathbf{Z}) &= (p(\zeta_1),\ldots,p^{(l_1)}(\zeta_1),\ldots,p(\zeta_m),\ldots,p^{(l_m)}(\zeta_m)),\\ q(\mathbf{Z}) &= (q(\zeta_1),\ldots,q^{(l_1)}(\zeta_1),\ldots,q(\zeta_m),\ldots,q^{(l_m)}(\zeta_m)),\end{aligned}$$

with $\zeta_i\in\mathbb{C}$, $i=1,2,\ldots,m$, $\mathbb{A}\in\mathcal{M}_M$ is a positive definite block matrix, $M = l(m+l_1+l_2+\cdots+l_m)$ and $q(\mathbf{Z})^H$ denotes the conjugate transpose of the vector $q(\mathbf{Z})$. These sesquilinear forms of Sobolev type are a natural extension to the matrix case of the inner product defined in the scalar case in [74]. Notice that the sesquilinear forms

$$\begin{aligned}\langle p,q\rangle_{R,S} &= \langle p,q\rangle_{R,\sigma} + \sum_{j=1}^{m} p(\zeta_j)^H \mathbf{M}_j q(\zeta_j), \quad p,q\in\mathbb{P}^l[z],\\ \langle p,q\rangle_{L,S} &= \langle p,q\rangle_{L,\sigma} + \sum_{j=1}^{m} p(\zeta_j) \mathbf{M}_j q(\zeta_j)^H, \quad p,q\in\mathbb{P}^l[z],\end{aligned}$$

considered in this document, where $\mathbf{M}_j\in\mathcal{M}_l$ is a positive definite matrix for $j=1,\ldots,m$, constitutes a particular case of (6.1) and (6.2) respectively, with $l_1=\cdots=l_m=0$. An interesting problem would be to analyze relative asymptotic properties for the **MOPUC** associated with the sesquilinear forms (6.1) and (6.2).

4. In sharp contrast with the scalar case, there exist matrix polynomials which do not have a unique factorization in terms of degree 1 factors, or it could happen that a factorization does not even exist. For example, the matrix polynomial

$$W(z) = \mathbf{I}_2 z^2 - \begin{bmatrix} 2 & 0\\ 0 & 2\end{bmatrix} z$$

can be written as

$$W(z) = \left(\mathbf{I}_2 z - \begin{bmatrix} 1 & -1\\ -1 & 1\end{bmatrix}\right)\left(\mathbf{I}_2 z - \begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix}\right) \text{ or } W(z) = \left(\mathbf{I}_2 z - \begin{bmatrix} 2 & 0\\ 0 & 2\end{bmatrix}\right)\mathbf{I}_2 z$$

but the matrix polynomial

$$W(z) = \mathbf{I}_2 z^2 - \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix}$$

can not be factorized with degree 1 matrix polynomials. In this work, we studied the Christoffel matrix transformation with $W(z)=\prod_{j=1}^{m}(z\mathbf{I}-A_j)$ with $A_j\in\mathcal{M}_l$. It would be an interesting problem to analyze the case when $W(z)$ is an arbitrary matrix polynomial. For **MOPRL**, a Christoffel matrix transformation has been studied in [2, 3] with $W(z)$ an arbitrary matrix polynomial.

5. In Chapter 4, the relation between the CMV matrices associated with two Verblunsky matrix sequences $(\alpha_n)_{n\geq 0}$, $(\tilde{\alpha}_n)_{n\geq 0}$ is studied, with α_n and $\tilde{\alpha}_n$ Hermitian matrices for all $n \geq 0$. The more general situation of non Hermitian matrices is an open problem. In the scalar case, this has been considered in [139].

6. Since that the determinant is a non-linear operator on the matrix space $\mathcal{M}_l$, it is not easy to study properties of the zeros associated with matrix orthogonal polynomials associated with Christoffel and Geronimus matrix transformations. It is an interesting problem to develop alternate techniques to study those zeros.

7. Algebraic and analytic properties for the Geronimus matrix transformation of **MOPUC** are, to the best of our knowledge, still not studied. On the real line, this transformation has been studied in [84, 85].

8. In Section 5.3, we assume that the matrix measure μ and the matrix B commute. In this case, it can be shown that μ is unitarily equivalent to a matrix measure consisting in two blocks (see [117]). It is an interesting open problem to deduce what kind of transformation is obtained in each block.

9. An interesting open problem is to give a consistent definition of linear spectral matrix transformation and find a system of generators for that set. For instance, in the scalar case, given a measure μ supported on the real line, the spectral transformations

   1. $d\tilde{\mu}_c = (x-\xi)d\mu$, $\xi \notin supp(\mu)$,
   2. $d\tilde{\mu}_u = d\mu + m_r\delta(x-\xi)$, $\xi \notin supp(\mu)$, $m_r \in \mathbb{C}$,
   3. $d\tilde{\mu}_g = \frac{d\mu}{x-\xi} + m_r\delta(x-\xi)$, $\xi \notin supp(\mu)$, $m_r \in \mathbb{C}$,

   are called Christoffel, Uvarov, and Geronimus transformations, respectively. In general, a linear spectral transformation of a Stieltjes function is another Stieltjes function $\tilde{S}(x)$ that has the form

   $$\tilde{S}(x) = \frac{A(x)S(x) + B(x)}{D(x)},$$

   where A, B and D are polynomials in x. The three transformations defined above are important due to the fact that any linear spectral transformation of a given Stieltjes function (i.e. for any polynomials A, B and D) can be obtained as a combination of Christoffel and Geronimus transformations (see [179]). A similar result holds for linear spectral transformations of Carathéodory functions, which are defined in a similar way (see [34]).

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