COUNTING ON GENERALIZED FIBONACCI OBJECTS

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Abstract

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Generalized Fibonacci numbers play a significant role in combinatorics, appearing in the enumeration of various combinatorial objects. This thesis investigates generalized Fibonacci numbers through the lens of the ECO (Enumeration of Combinatorial Objects) method, culminating in the introduction of a novel class of words enumerated by these numbers, referred to as generalized Fibonacci words. Using these words, we construct two additional families of objects: generalized Fibonacci bargraphs and generalized Fibonacci polytopes. The combinatorial richness of these families is thoroughly explored. For bargraphs, we present new results on their enumeration based on various statistics. For polytopes, we provide partial findings on their normalized volume, f-vector, and structure, shedding light on their intriguing geometric and combinatorial properties.

Key words

Generalized Fibonacci numbers, ECO method, bargraph, polytope, generating function.

Resumen

Conteo en Objetos Generalizados de Fibonacci

Los números de Fibonacci generalizados desempeñan un papel significativo en la combinatoria, apareciendo en la enumeración de diversos objetos combinatorios. Esta tesis investiga los números de Fibonacci generalizados a través del método ECO (Enumeración de Objetos Combinatorios), culminando en la introducción de una nueva clase de palabras enumeradas por estos números, denominadas palabras de Fibonacci generalizadas. Utilizando estas palabras, construimos dos familias adicionales de objetos: los gráficos de barras generalizados de Fibonacci y los politopos generalizados de Fibonacci. La riqueza combinatoria de estas familias se explora en detalle. Para los gráficos de barras, presentamos nuevos resultados sobre su enumeración respecto a diversas estadísticas. Para los politopos, proporcionamos hallazgos parciales sobre su volumen normalizado, su f-vector y su estructura, arrojando luz sobre sus interesantes propiedades geométricas y combinatorias.

Palabras clave
Números de Fibonacci generalizados, Método ECO, gráficos de barras, politopo, función generatriz.

Ackgnowlegdements

"No man is an island,
Entire of itself;
Every man is a piece of the continent,
A part of the main."
— John Donne

This thesis is a reflection of the many beings who have influenced and supported me along these wonderful years.

I am deeply grateful to Prof. José Luis Ramírez, whose mentorship, dedication, and trust have helped me grow both as a mathematician and as a person. I also extend my heartfelt thanks to Prof. Rafael González D'León for introducing me to the beauty of combinatorics and, through his passion and dedication, showing me the joy of embracing mathematics as a way of life.

At Universidad Nacional in Bogotá, I had the privilege of sharing this adventure with an incredible group of friends. Camilo, David, Ricardo, Victor, Emmanuel, Daniela, Alejandra, Luisa, and Brayan: thank you for your friendship, encouragement, and for the adventures we shared together. To the many others who have been part of this experience, even if not named here, please know that I deeply appreciate you and the role you've played in making these years so meaningful.

Beyond the academic world, I have been blessed with unwavering love and support. My mother, Nana, thank you for tirelessly working to give me opportunities you never had. Mamá Laura, your care and endless belief in me have been invaluable and I will work hard to honor your faith and resilience.

Among those who have supported me, Natalia holds a special place. Words cannot fully capture what your belief in me has meant, but I hope to show it in everything we build together. This achievement is as much yours as mine.

Finally, to Cande, Luffy, and Jack: life would not be the same without your presence. And to Sanji, I will always remember the time we spent together and your gentle and innocent way of seeing the world.

Contents

Introduction

Fibonacci numbers are the sequence of integers $(F_n)_{n\geq 0}$ defined by the initial values $F_0 = 0$, $F_1 = 1$, and the recursive relation $F_n = F_{n-1} + F_{n-2}$, for all integers $n \geq 2$. This sequence begins with the numbers:

$$0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \ldots$$

These numbers show a great importance within mathematics, permeating various fields. From the point of view of combinatorics, their relevance comes from their consistent appearance in different types of combinatorial objects. For instance, a *tiling* is a configuration of tiles that covers a given figure using a specified set of tiles, ensuring that there are no gaps or overlapping tiles. The Fibonacci number F_{n+1} counts the number of ways to tile a $2 \times n$ rectangle using dominoes (tiles of size 1×2 and 2×1).
Figure 1 illustrates some examples of such tilings. The appearance of Fibonacci numbers in this setting follows from the recursive structure of these tilings. Specifically, each tiling of a $2 \times n$ rectangle can be uniquely obtained in one of two ways: by appending two horizontal dominoes to a tiling of a $2 \times (n-2)$ rectangle or by appending a single vertical domino to a tiling of a $2 \times (n-1)$ rectangle. This recurrence directly corresponds to the Fibonacci sequence.

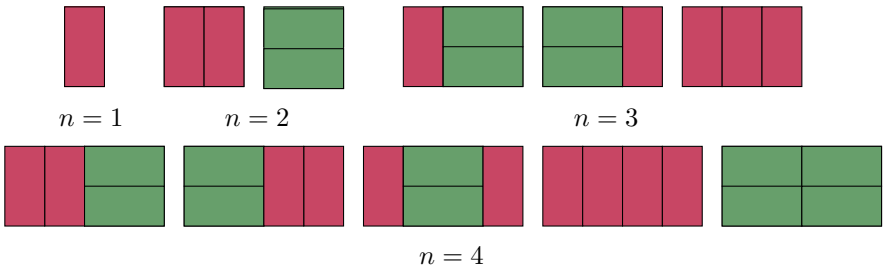


Figure 1: Tilings of a $2 \times n$ rectangle with domino tiles.

Other combinatorial objects counted by Fibonacci numbers are binary words avoiding consecutive occurrences of the digit 1, subsets of $[n]$ without consecutive integers, compositions of n with parts 1 and 2, and planar trees with $n+1$ vertices and leaves located at level 2,

among others. For more information about Fibonacci sequence, see the entry A000045 in the Online Encyclopedia of Integer Sequences (OEIS).

Fibonacci numbers serve as a foundation for constructing more general families of numbers, where the classical Fibonacci sequence appears as a particular case. In this context, the p-generalized Fibonacci numbers $(F_n^p)_{n\geq 0}$ are introduced; see [25, 5] for reference. These numbers are defined by initializing the sequence with $p-1$ zeroes followed by a single 1, with subsequent terms given by the sum of the preceding p terms. Formally, they satisfy the recurrence:

$$F_n^p = \begin{cases} 0, & \text{if } n = -p+2, -p+3, \ldots, 0, \\ 1, & \text{if } n = 1, \\ \sum_{i=1}^{p} F_{n-i}^p, & \text{if } n > 1. \end{cases}$$

This project aims to provide a comprehensive analysis of p-generalized Fibonacci numbers. We begin with a review of their fundamental properties and description within the ECO ("Enumeration of Combinatorial Objects") methodology. A key contribution of this work is the introduction of a novel family of p-ary words, which are shown to be enumerated by p-generalized Fibonacci numbers. These words serve as the foundation for generating other significant combinatorial objects. Firstly, we construct a family of bargraphs, obtaining new results regarding their enumeration with respect to the statistics of area, semiperimeter, number of internal points, and grade of points. Secondly, we construct a family of polytopes, where we explore their geometric and combinatorial properties. We also include significant coding related to each of these objects available at repository.

I am grateful to the Department of Mathematics at Universidad Nacional de Colombia in Bogotá for partially supporting my participation in various events, where I received valuable feedback and suggestions. Below are my presentations:

- **Posters:**
  - *p-Generalized Fibonacci Sequences*, ALTENCOA-9, Cali, Dec. 2023.
  - *Conteo en Políminos Generalizados de Fibonacci*, ECCO 2024, Popayán, June 2024.
- **Talks:**
  - *El Método ECO, una Herramienta para el Conteo de Estructuras Combinatorias*, RIIMAT, Virtual, July 2024.
- **Seminars:**
  - *Sucesiones Generalizadas de Fibonacci y sus Poliminós Asociados*, Seminario Discremath, UNAL Bogotá, May 2024.

Chapter 1

Foundations and Key Principles

This chapter aims to lay a foundational introduction to essential concepts and key objects that will play a central role throughout subsequent chapters.

1.1 An Overview of Enumerative Combinatorics

Enumerative combinatorics is a branch of mathematics focused on counting *combinatorial objects*: elements characterized by specific patterns, structures, or rules, within structured families. These families, known as *combinatorial classes*, arise from the need to classify combinatorial objects. Formally, a *combinatorial class* is a pair (C, ρ), where C is an enumerable set, and $\rho : C \to \mathbb{N}$ is a function, called *size function*, which assigns a nonnegative integer to each object. We require that for each $n \in \mathbb{N}$, the subset $C_n \coloneqq \{\gamma \in C \mid \rho(\gamma) = n\}$ consisting of all objects of size n, is finite. The enumerative goal relies on determining the sequence of numbers (c_n), defined by $c_n \coloneqq |C_n|$, known as *counting sequence* of the class.

Example 1. A *triangulation* of a regular polygon P_n with n vertices is a set of diagonals that subdivides the polygon into $n-2$ triangles, where no two diagonals intersect inside the polygon. Figure 1.1 exhibits various triangulations of regular polygons.

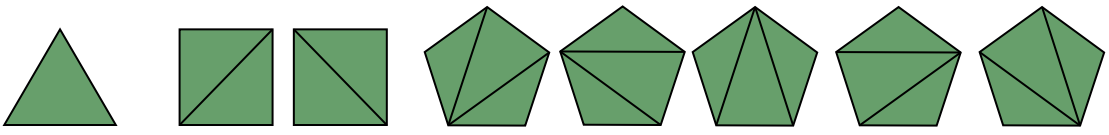


Figure 1.1: Triangulations of polygons.

We define the combinatorial class C, of all triangulations of regular polygons as follows:

- The elements of C are all the distinct triangulations of any regular polygon, i.e., the set of triangulations of P_n for each $n \geq 3$, where P_n is a regular polygon with n vertices.

- The size function $\rho : C \to \mathbb{N}$ assigns to each triangulation T of P_n the value $\rho(T) = n$, where n is the number of vertices of the polygon being triangulated.

For each fixed n, the number of distinct triangulations of a regular polygon with n vertices is given by the Catalan number C_{n-2}, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For a proof, see [36]. Therefore, the counting sequence for the class of triangulations of regular polygons is the sequence of Catalan numbers.

Example 2. An *integer partition* of a positive integer n is a way of writing n as a sum of positive integers, called the *parts* of the partition. Rearrangements of the same parts in different orders are considered identical. For instance, the partitions of 4 are $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. Let P_n denote the set of integer partitions of n. The counting sequence of partitions, denoted by (p_n), begins as follows:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, \ldots$$

See the sequence A000041 at OEIS for further details. Despite its significance, a closed formula for p_n, akin to that for triangulations, remains unknown. This is not an isolated case; many counting sequences lacking a closed formula have inspired diverse approaches for their study. One such approach is asymptotic analysis, which seeks to find a sequence (b_n) such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, where a_n is the counting sequence. This provides valuable insights into the sequence's growth for large values of n. In the case of integer partitions, Hardy and Ramanujan made a remarkable breakthrough in 1918 by providing the asymptotic approximation for p_n, given by:

$$p_n \approx \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}, \quad \text{as } n \to \infty.$$

The enumeration of combinatorial objects is closely tied to the study of their inherent properties. Certain characteristics, such as recursivity or symmetry, often provide crucial insights for deducing their counting sequences. Consequently, enumerative combinatorics focuses on developing methods to extract enumeration results based on the properties of the objects under consideration. For a deeper exploration of classical techniques, such as the addition, multiplication, and inclusion-exclusion principles, we recommend the books [32] and [29].
A common approach in combinatorics is to focus on subfamilies of combinatorial objects that satisfy specific properties. This strategy enables a more targeted exploration of particular characteristics and behaviors within a broader combinatorial structure, often yielding deeper

insights and more specialized results. For example, in the study of integer partitions, one might investigate subfamilies defined by particular constraints:

- *Partitions into distinct parts*: This subfamily includes partitions where each part is unique. For example, the partitions of 5 into distinct parts are $\{5\}$, $\{4, 1\}$, and $\{3, 2\}$. Studying this subfamily can reveal interesting properties related to uniqueness and combinatorial optimization.

- *Partitions into odd parts*: This subfamily consists of partitions where all parts are odd numbers. For example, the partitions of 7 into odd parts are $\{7\}$, $\{5, 2\}$, $\{3, 3, 1\}$, and $\{1, 1, 1, 1, 1, 1, 1\}$. Analyzing these partitions can provide insights into the distribution of odd numbers and their combinatorial significance.

- *Partitions into parts not exceeding a given number*: This subfamily includes partitions where no part is greater than a specified number. For example, the partitions of 6 with each part not exceeding 3 are $\{3, 3\}$, $\{3, 2, 1\}$, $\{3, 1, 1, 1\}$, $\{2, 2, 2\}$, $\{2, 2, 1, 1\}$, $\{2, 1, 1, 1, 1\}$, and $\{1, 1, 1, 1, 1, 1\}$. This subfamily can be useful in problems involving bounded resources or constraints.

- *Partitions with a fixed number of parts*: This subfamily consists of partitions with a specified number of parts. For example, the partitions of 7 into exactly three parts are $\{5, 1, 1\}$, $\{4, 2, 1\}$, $\{3, 3, 1\}$, and $\{3, 2, 2\}$. Studying these partitions can help understand the balance and distribution of parts within a fixed structure.

By focusing on these subfamilies, specialized techniques might be applied to derive results and properties that may not be evident in the broader context. It also facilitates the discovery of bijections, generating functions, and other combinatorial tools that can be used to study and solve problems related to these specific subsets.
In addition to these restrictions, it is also useful to study the distribution of combinatorial objects with respect to additional parameters. A *statistic* is a function that assigns to each object in the combinatorial class, a natural number.
Studying statistics can reveal intricate structures and patterns that are not apparent when only considering the size function. These statistics can measure various properties, such as height, width, number of components, or other specific attributes. For example, in the case of partitions, common statistics are:

- *Number of parts*: The number of summands in a partition. This statistic helps in analyzing the structure of partitions.

- *Largest part*: The largest integer in a partition. This provides a measure of the partition's spread.

- *Multiplicity of parts*: The frequency of each integer in a partition. This helps in understanding the distribution of summands.

In summary, enumerative combinatorics is a dynamic and expansive field that provides a robust framework for systematically counting and classifying combinatorial objects according to a wide range of parameters and properties. This discipline not only facilitates the organization and analysis of discrete structures but also fosters new insights into their combinatorial nature, essential for applications across mathematics, computer science, and beyond.

1.2 Foundational combinatorial structures

We offer a comprehensive journey through some of the most common combinatorial classes, each characterized by its intrinsic properties and significant contributions to the field.

Graphs

Graphs are objects that encode the relationships between elements of a given set. There are many types of graphs. Two remarkable types are the following: an *undirected graph* is a pair $G = (V, E)$, where V is a non-empty set of *vertices*, and E is a set of unordered pairs of vertices, called *edges*. In contrast, a *directed graph* (or digraph) is also defined as $G = (V, E)$, but here, E consists of ordered pairs of vertices, indicating directionality.
Graphs can be depicted (not uniquely) in the plane as collections of points (vertices) connected by lines (edges). Figure 1.2 illustrates examples of an undirected graph, $G_1 = (V, E_1)$, and a directed graph, $G_2 = (V, E_2)$, with the following sets:

$$
\begin{aligned}
V &= \{v_1, v_2, v_3, v_4, v_5, v_6\}, \\
E_1 &= \{\{v_1, v_1\}, \{v_1, v_2\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_5, v_6\}\}, \\
E_2 &= \{(v_1, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_5), (v_4, v_2), (v_5, v_3), (v_5, v_6)\}.
\end{aligned}
$$

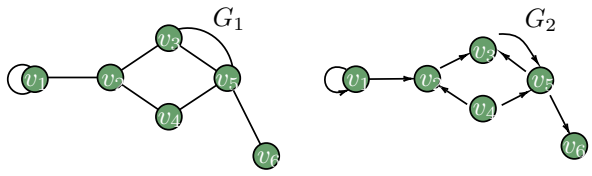


Figure 1.2: Examples of an undirected graph G_1 and a directed graph G_2.

The distinction between these two fundamental types of graphs is evident: in an undirected graph, edges represent mutual or bidirectional relationships, while in a directed graph, edges signify asymmetric or unidirectional relationships. On the other hand, a graph can have

various graphical representations. A graph is *planar* if it can be portrayed in the plane without any edge crossings. The complete graph on five vertices K_5, displayed in Figure 1.3 serves as an example of a non-planar graph.

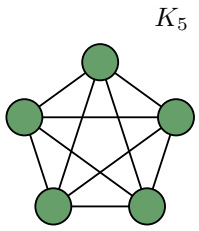


Figure 1.3: Non planar graph.

An undirected graph is *simple* if it lacks multiple edges between any pair of vertices and does not contain any loops (edges from a vertex to itself). Similarly, *simple directed graphs*, are characterized by the absence of loops and multiple edges with the same source and target vertices. Figure 1.4 provides visual examples of undirected and directed simple graphs.

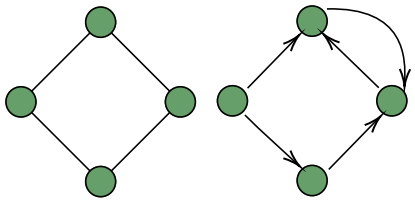


Figure 1.4: Simple graphs.

In some cases, it is important to distinguish between the vertices or edges of a graph. A graph $G = (V, E)$ is called *vertex-labeled* if it is equipped with an injective function $l_V : V \to A$, and *edge-labeled* if equipped with an injective function $l_E : E \to B$, where A and B are sets satisfying $|V| = |A|$ and $|E| = |B|$. Typically, A and B are subsets of the natural numbers, such as $A = \{1, \ldots, |V|\}$ and $B = \{1, \ldots, |E|\}$. Figure 1.5 illustrates a vertex-labeled graph and an edge-labeled graph derived from the same unlabeled graph.

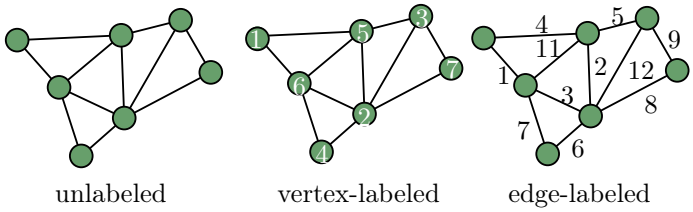


Figure 1.5: Different types of labeled graphs.

We introduce essential terminology and concepts pertinent to graphs: two vertices $u, v \in V$ are *adjacent* if there exists an edge $e \in E$ connecting u and v. The *degree* of a vertex v is

defined as the number of vertices adjacent to v. A *path* of length k between two vertices $u, v \in V$ is a subset $\{\{u, x_1\}, \{x_2, x_3\}, \cdots, \{x_k, v\}\} \subseteq E$, where $x_i \in V$. A graph is *connected* if there exists a path between any pair of vertices. Furthermore, a path that starts and ends at the same vertex is referred to as a *cycle*. The conditions for two given graphs to be considered as equal are encapsulated in the notion of isomorphism: two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are *isomorphic* if there exists a bijection $f : V_1 \to V_2$ such that $u, v \in V_1$ are adjacent if and only if $f(u), f(v) \in V_2$ are adjacent. The graphs G and H from Figure 1.6 are isomorphic via the function $f : V_1 \to V_2$, $f(u_i) = v_i$, for $1 \leq i \leq 8$.

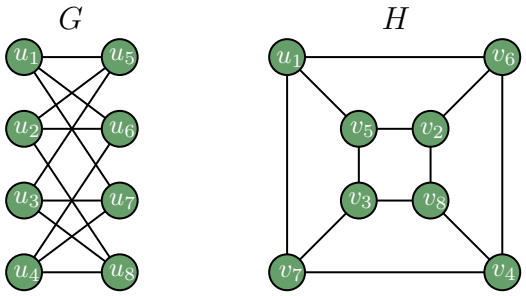


Figure 1.6: Isomorphic graphs.

Enumeration in graphs typically involves classifying them based on the number of vertices they contain. The number of labeled simple undirected graphs with n vertices is $2^{\frac{n(n-1)}{2}}$, while the number of labeled simple directed graphs with n vertices is $2^{n(n-1)}$. For unlabeled graphs, the enumeration problem becomes significantly more challenging. For a detailed discussion on the enumeration of distinct families of graphs, we recommend referring to [21, 34].

Trees

An important subfamily of graphs is the family of trees. A *tree* is an undirected simple graph without cycles, where any pair of vertices is connected by precisely one path. A tree becomes *rooted* when a vertex is designated as the root. Trees are particularly useful for describing hierarchical structures. The vertices within a tree are organized into levels: vertices reachable from the root via a path of distance k are located at level k. As a convention, the root itself is placed at level 0.

A vertex of a tree is said to be the *child* of the adjacent vertex located one level above it and the *parent* of the adjacent vertices positioned one level below it. When a vertex lacks children, it is termed a *leaf*. Conversely, if a vertex possesses one or more children, it is said to be an *internal node*. A *plane rooted tree* is a rooted tree where there is an ordering for the children of each vertex. For example, in Figure 1.8, the trees depicted are isomorphic when considered as rooted trees. However, they differ when viewed as plane rooted trees. This distinction arises because transforming one into the other necessitates a reflection, a operation that cannot be

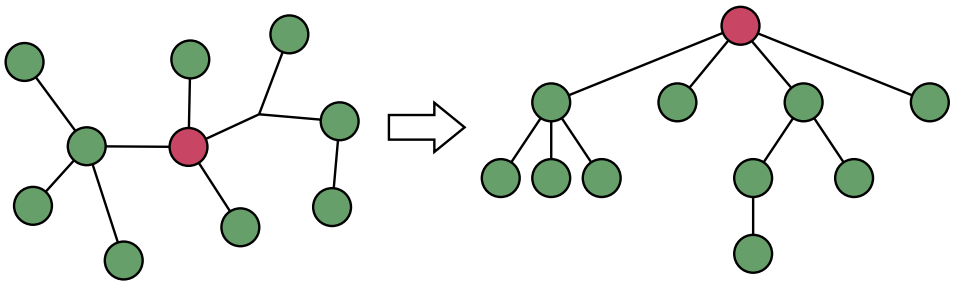


Figure 1.7: Hierarchy structure of a rooted tree.

performed on the plane without altering the structure.

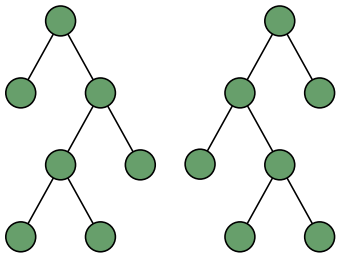


Figure 1.8: Non isomorphic plane trees.

Trees play a fundamental role in the development of ECO theory, as discussed in Chapter 2. This combinatorial framework facilitates the systematic generation and enumeration of various classes of combinatorial objects by leveraging the structure of rooted planar trees. These trees serve as a foundational tool, encoding the recursive construction rules and growth patterns of the objects under study.
The versatility of rooted planar trees in ECO theory lies in their ability to represent relationships and hierarchies in a visually intuitive and mathematically rigorous manner. By associating combinatorial objects with tree structures, one can derive enumeration formulas, explore structural properties, and analyze the behavior of these objects under different constraints. The centrality of trees in ECO theory underscores their importance in addressing a wide range of problems within enumerative combinatorics.

Words

A k-ary *word* is a finite string formed by digits from a set A comprising k elements, known as an *alphabet*. While the traditional understanding of words involves using alphabets made up of letters from a specific language, we often simplify by setting $A = [k] \coloneqq \{1, 2, \ldots, k\}$. Words serve as representations for arrangements, permutations or combinations. Moreover, exploring patterns within families of words is significant in both combinatorics and computer science, alongside the study of automata and formal languages. Additionally, words are also

studied from an algebraic perspective since they possess a monoid structure and serve as the basis for defining free objects within algebraic structures.

Words are typically classified by their length. The set of k-ary words is denoted by $\mathcal{W}^k$. It is also common to include the empty word, represented by ϵ. Since each position in a k-ary word can contain any of k possible digits, the multiplication principle implies that the number of words in $\mathcal{W}_n^k$, the subset of $\mathcal{W}^k$ consisting of words of length n, is k^n.

Interesting subfamilies of words arise when we consider words that avoid certain subwords. Let $W_{\leq n}^k$ denote the set of k-ary words of length less than or equal to n, and let $T \subseteq W_{\leq n}^k$ represent a subset of such words. The set of k-ary words of length n that avoid the subwords in T is denoted by $\mathcal{W}_n^k(T)$. For example, if $T = \{123, 12\}$, the set of 3-ary words of length 3 avoiding T is

$$\mathcal{W}_3^3(T) = \{111, 113, 131, 132, 133, 211, 213, 221, 222, 223, 311, 313, 321, 322, 333\}.$$

A more restrictive approach, which focuses on the relative growth and decline of digits within words, has gained considerable attention in recent research (see [23]). To formalize this, we define a word $w = w_1 w_2 \cdots w_n$ to be in its *reduced form* if it is rewritten as $w' = x_1 x_2 \cdots x_n$, where x_i corresponds to the position of w_i in the ordered list of distinct digits in w. Specifically, $x_i = l$ if w_i is the l-th smallest unique digit in w. For example, consider the word $w = 24352$. To reduce it, we first order its digits: $2 < 3 < 4 < 5$, and then replace each w_i with its position in this ordering. The reduced form of w is thus $w' = 13241$. A word w avoids a pattern $\tau = \tau_1 \tau_2 \cdots \tau_k$ if the reduced form of every subword of w of length k differs from τ. For instance, the word $w = 24352$ avoids the pattern $\tau = 321$ because its subwords of length 3 are $243, 435$, and 352, which reduce respectively to $132, 213$, and 231, none of which match τ. The family of k-ary words that avoid the pattern T is denoted by $\overline{\mathcal{W}}^k(T)$.

Counting both types of words presents significant challenges, and existing results are often confined to specific cases of subwords or patterns. For instance, while some combinatorial methods have provided valuable insights into counting certain classes of words (see, for example, [5]), many cases remain unresolved due to their complexity and the intricate nature of the restrictions involved. However, recent advancements in enumerative combinatorics have introduced promising techniques to address these challenges. One such approach is the ECO method, which has proven effective in handling a wide range of configurations for T, often yielding insights and solutions that were previously difficult to obtain.

Lattice paths

A *lattice* in the Euclidean space $\mathbb{R}^n$ is a discrete model of the space formed by an infinite set of vectors arranged in a particular pattern. Formally, a set $L \subset \mathbb{R}^n$ is a *lattice* if it satisfies the following conditions:

1. Any two points in L can be added or subtracted coordinate-wise, yielding another point within L.

2. For any two distinct points $x, y \in L$, the Euclidean distance $d(x, y)$ between them is always positive.

3. Each point in $\mathbb{R}^n$ lies within a finite distance from at least one point in L.

We will exclusively consider the *integer lattice* $\mathbb{Z}^d$ in the two-dimensional case ($d = 2$). A *lattice path* is a finite sequence of vectors $v_0, v_1, \ldots, v_k$, where for each $i = 1, 2, \ldots, k$, the vector $v_i - v_{i-1}$ belongs to a predefined set of allowable steps S. An example of a lattice path is the *Dyck path*, which is defined within the positive quadrant of the integer lattice $\mathbb{Z}^2$. Dyck paths are characterized by the set of steps $S = \{(1, 1), (1, -1)\}$, representing diagonal upward and downward movements. Furthermore, each Dyck path starts at the origin $(0, 0)$, ends on the horizontal axis, and stays weakly above the axis throughout its trajectory. The *length* of a path is the total number of steps it consists of. The number of Dyck paths with $2n$ steps corresponds to the Catalan number C_n. Figure 1.9 illustrates the Dyck paths of length 5.

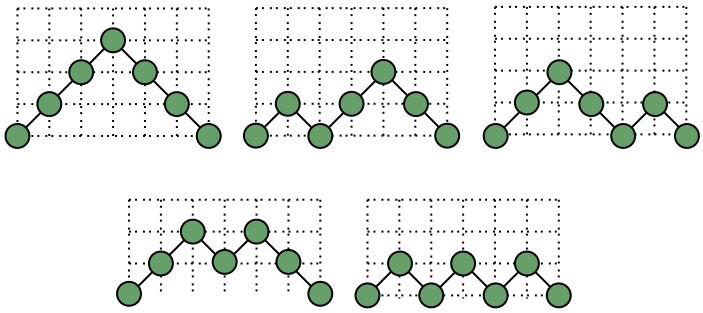


Figure 1.9: Dyck paths of length 5.

Polyominoes

A *cell* refers to the interior and boundary of a unit square with vertices lying on the integer lattice $\mathbb{Z}^2$. A *polyomino* is a finite collection of cells whose interior is connected. The collection P_1, depicted in Figure 1.10, forms a polyomino, whereas the collection P_2 fails to have a connected interior.

Polyominoes emerged in statistical physics (named as lattice animals). In mathematics they were introduced by Golomb in 1953 [19]. Let P(n) denote the number of polyominoes consisting of n cells. For instance, $P(4) = 19$ since there are 19 tetrominos (polyominoes formed with 4 cells), which correspond to the pieces used in Tetris game, see Figure 1.11.

Despite their widespread popularity, a closed formula for $P(n)$, remains elusive (see A001168). The most extensive enumeration to date, achieved in 2024 by Gill Barequet and Gil Ben-

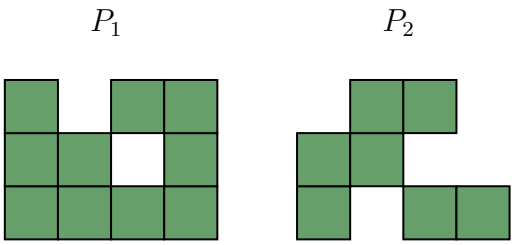


Figure 1.10: Cells collection.

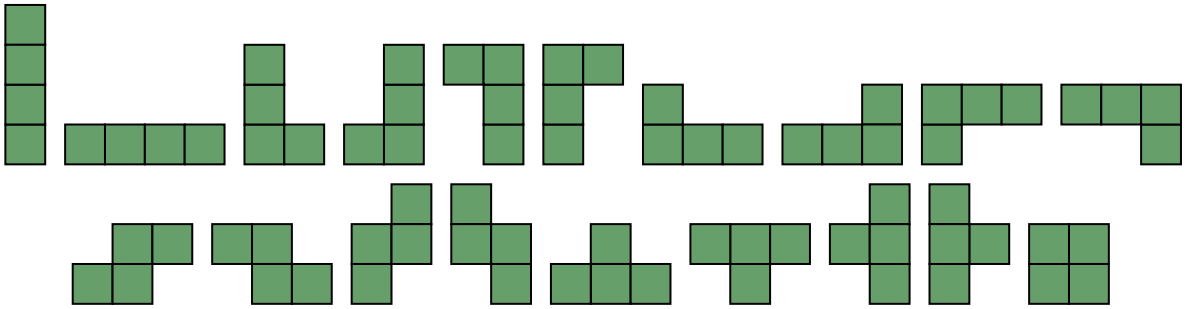


Figure 1.11: Examples of Tetris puzzles composed of tetrominos.

Shachar [4], corresponds to:

$$P(70) = 185007926458857112706528908119423434008 14.$$

In 1999, Madras [26] shed light on the asymptotic growth of polyominoes by showing the existence of Klarner's constant, defined as $\lambda = \lim_{n\to\infty} \frac{P(n+1)}{P(n)}$. However, the precise value of λ remains undetermined, though it is constrained within the range $3.98 < \lambda < 4.65$.
The difficulty in the enumeration of the family of polyominoes have led to the enumeration of restricted families. We highlight the family of column (respectively file) convex polyominoes, consisting of all polyominoes whose columns (resp. rows) are convex, and the family of convex polyominoes consisting of simultaneously file and column convex polyominoes. The Figure 1.12 illustrates the polyomino P_1 being convex by rows but not by columns, P_2 being column-convex but not row-convex and P_3, being both row and column-convex.

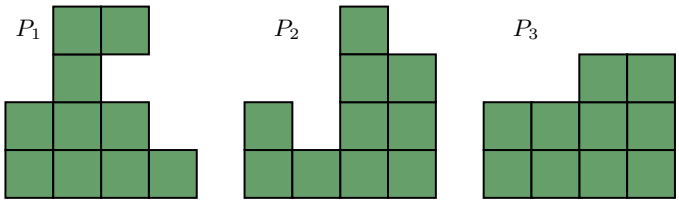


Figure 1.12: Convexity types in polyominoes.

Enumerative information about the different types of convex polyominoes might be found in

[8] and [9]. A relevant subfamily of column-convex polyominoes is the family of *bargraphs*, which are column-convex poloyminoes whose base file is also convex. The polyominoes P_2 and P_3 in Figure 1.12 are also bargraphs. An extensive study of statistics in bargraphs is available at [27]. In Chapter 4, we will study certain statistics in a special family of bargraphs denominated generalized Fibonacci bargraphs. As the name suggests, this family results to be enumerated (respectively to the number of columns) by the generalized Fibonacci numbers.

1.3 Generating Functions: A Tool for Counting

The theory of generating functions (see for example [33, 17]) is central in combinatorics for its effectiveness to solve enumeration problems. Generating functions compactly preserve the counting information of combinatorial objects. For a given counting sequence (c_n), the associated generating function is

$$F(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Generating functions are mostly treated as infinite polynomials to leverage its algebraic properties rather than as a traditional function. The core idea behind generating functions is finding a closed expression for $F(z)$, to then extract the coefficient $[z^n]F(z) = c_n$.
We exhibit below some of the basic operations of generating functions

- Sum of generating functions: If $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ are generating functions, then the sum is

$$A(z) + B(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n.$$

- Scalar multiplication: For a constant c,

$$cA(z) = \sum_{n=0}^{\infty} (ca_n) z^n.$$

- Cauchy product: The product of two generating functions $A(z)$ and $B(z)$ is

$$A(z) \cdot B(z) = \left(\sum_{n=0}^{\infty} a_n z^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

  This is also known as the convolution of the sequences (a_n) and (b_n).

- Shift property: Multiplying a generating function by z^k shifts the sequence by k places:

$$z^k A(z) = \sum_{n=0}^{\infty} a_n z^{n+k} = \sum_{n=k}^{\infty} a_{n-k} z^n.$$

- Derivative and integral:

$$A'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

$$\int A(z)\, dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}.$$

For instance, for Fibonacci numbers, the generating function is

$$F(z) = \sum_{i=0}^{\infty} F_i z^i,$$

where F_i is the i-th Fibonacci number. Thus, $F_0 = 0$, $F_1 = 1$. Applying the shifting property and the Fibonacci numbers recursion, it follows that

$$(1 - z - z^2)F(z) = 1.$$

Thus, a closed expression for the generating function of Fibonacci numbers is

$$F(z) = \frac{1}{1 - z - z^2}.$$

To obtain a closed expression for F_n, we perform partial fraction decomposition on this expression. The roots of the denominator are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. This means that we can rewrite the generating function as:

$$F(z) = \frac{z}{(1 - \alpha z)(1 - \beta z)}.$$

We want to find constants A and B such that:

$$\frac{z}{(1 - \alpha z)(1 - \beta z)} = \frac{A}{1 - \alpha z} + \frac{B}{1 - \beta z}.$$

These values are $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Using the geometrical series, we obtain that the series expansion of $F(z)$ is:

$$F(z) = \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} \alpha^n z^n - \sum_{n=0}^{\infty} \beta^n z^n \right).$$

Since $F_n = [z^n]F(z)$, it follows that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

In summary, generating functions are a versatile tool in combinatorics, enabling the encoding and manipulation of sequences. By transforming sequences into functions, we can apply operations to derive relationships and closed-form expressions. They are particularly useful for solving recurrence relations, counting combinatorial objects, and deriving explicit formulas for counting sequences.

Chapter 2

The ECO Method

The *Enumeration of Combinatorial Objects* (ECO) method targets the enumeration of combinatorial classes, focusing on the recursive and systematic generation of their combinatorial objects. By studying generative rules known as *production systems*, the ECO method enables the derivation of recurrence relations, generating functions, or direct enumerative formulas, illustrating how structural properties of production systems influence the enumeration of combinatorial objects. This technique originated from Chung's work on Baxter permutations in 1978 [10] and was formally developed into a comprehensive theory by Barcucci et al. in the 1990s [3]. Since then, it has proven effective in enumerating many combinatorial classes. In this chapter, we review the ECO theory from a historical perspective, first through generating trees and then via an alternative approach using matrices, with the aim of applying it to the context of generalized Fibonacci numbers in the following chapter.

2.1 Building Combinatorial classes Recursively

Several types of combinatorial classes contain objects of size 1 that serve as fundamental building blocks for systematically generating the entire class. For instance, trees are constructed by attaching a new node to a node of an existing tree. Polyominoes are formed by adding unit cells along the contour of a given polyomino, while Dyck paths are extended by appending the Dyck path UD (one step up, one step down) at a lattice point of an existing Dyck path. See figure 2.1 for reference.

Definition 3. A *parameter* for a combinatorial class $\mathcal{C}$ is a rule ϑ_n that produces objects of size $n+1$ from objects of size n. When a parameter acts on an object, we refer to that process as a *local expansion*. A *site* is a position in a combinatorial object where a local expansion can be performed. An *operator* is a family $(\vartheta_n)_{n\geq 0}$ of parameters, where each ϑ_n specifies a systematic rule to expand objects of size n into objects of size $n+1$. Given a fixed combinatorial class $\mathcal{C}$, different choices of operators correspond to different systematic ways

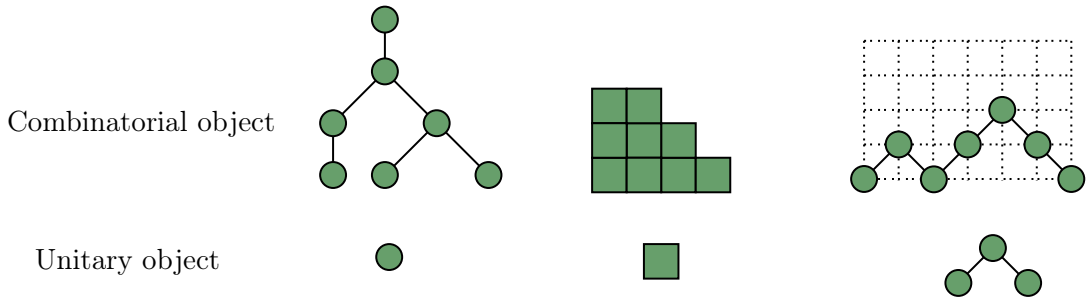


Figure 2.1: Ground objects for different combinatorial objects.

of generating larger objects from smaller ones. If every object in $\mathcal{C}$ can be generated through an operator, we say that the class admits a *recursive construction*.

Example 4. We illustrate the previous definitions within the context of two combinatorial classes:

- Let $\mathcal{W}^k$ be the class of k-ary words, where λ denotes the empty word. Fixing n, the parameter ϑ_n expands a word $\omega = \omega_1\omega_2\cdots\omega_n \in \mathcal{W}_n^k$ by inserting a digit $i \in \{1,\ldots,k\}$ into one of the $n+1$ possible sites: before ω_1, between two consecutive digits, or after ω_n. The operator, consisting of the family of parameters $(\vartheta_n)_{n\geq 0}$, provides a recursive construction for the class of k-ary words, since any word ω can be obtained from a smaller word ω' by deleting a single digit ω_i from ω.

- Let $\mathcal{T}$ be the class of planar rooted trees, with size function given by the number of nodes. The parameter ϑ_n generates trees in $\mathcal{T}_{n+1}$ by adding a child to a node in a tree $T \in \mathcal{T}_n$. Each node in T is a site. The operator $\vartheta = (\vartheta)_{n\geq 0}$ constructs T recursively: if a is a node of a given tree $T \in \mathcal{T}$, then T is generated from the tree $T \setminus \{a\}$, obtained by deleting the node a.

One of the key advantages of recursive constructions is their ability to systematically track the generation of objects within a combinatorial class, providing a clear hierarchical structure. Each operator naturally encodes the class as a planar tree $T_{\mathcal{C},\vartheta}$, which we denote simply by T when the context is clear. In T, objects correspond to nodes, and edges indicate that one object is derived from another of smaller size through a local expansion. This tree representation not only reflects the recursive nature of the construction but also serves as a powerful tool for enumeration and structural analysis.

Example 5. Figure 2.2 shows the resulting trees from the combinatorial classes and operators of example 4.

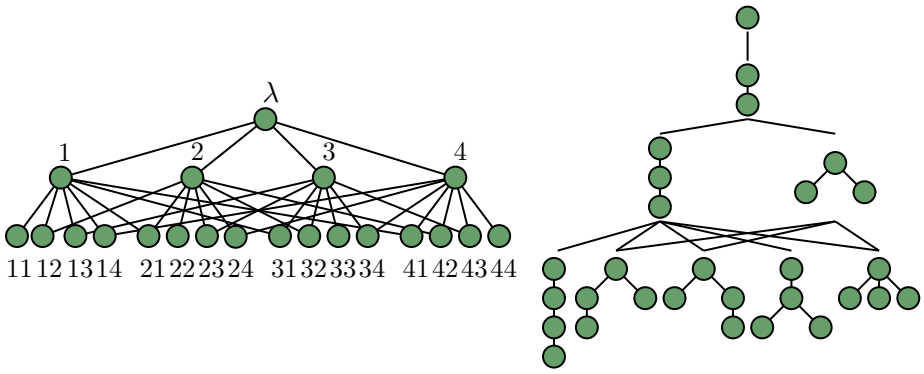


Figure 2.2: Tree structure for words and planar trees.

In the preceding example, the resulting trees exhibit irregular structures, with nodes potentially having multiple parents. In the late 1970s, Chung, Graham, Hoggatt, and Kleiman [10] explored these complexities while investigating Baxter permutations, identifying conditions in operators that could result in more regular and structured trees. Their work laid the foundation for the ECO (Enumeration of Combinatorial Objects) methodology, which was further formalized and expanded in the late 1990s by Barcucci, Del Lungo, Pergola, and Pinzani [3]. These authors developed a unified theoretical framework that was used to enumerate a wide range of combinatorial objects. They introduced techniques to codify these objects into generating trees, a specialized type of tree structure from which enumerative information could be extracted.

Building on this foundation, other researchers, such as Deutsch [11], have extended the ECO methodology by developing alternative approaches, such as codifying combinatorial objects into matrices, providing valuable new perspectives and tools for the analysis of these objects.

2.2 The ECO Method: A Generating Trees Perspective

We begin by introducing the version of the ECO method developed by Barcucci et al. in [3]. This method requires the combinatorial class to have a unique minimal object, ensuring that the resulting tree is rooted. This allows each element to be tracked by its distance from the root or, equivalently, by the number of expansions applied from the root. The following type of operator provides a more structured framework for handling expansions within this setting.

Definition 6. Let $\mathcal{C}$ be a combinatorial class and $n \in \mathbb{N}$. An *ECO operator* ϑ is an operator that satisfies the following conditions:

1. For each $x \in \mathcal{C}_{n+1}$, there exists an object $x' \in \mathcal{C}_n$ such that $x \in \vartheta(x')$.
2. For any two distinct objects $x, x' \in \mathcal{C}_n$, it holds that $\vartheta(x) \cap \vartheta(x') = \emptyset$.

The sites where an ECO operator performs local expansions are known as *active sites*.

We observe that the sets $\mathcal{F}_{n+1} = \{\vartheta(x) : x \in \mathcal{C}_n\}$ form a partition of $\mathcal{C}_{n+1}$, ensuring that each object in $\mathcal{C}_{n+1}$ is uniquely generated from an object in $\mathcal{C}_n$ through ϑ. The objects in $\mathcal{C}_n$ are positioned at level n of $T_{\mathcal{C},\vartheta}$, preserving the enumerative information of $\mathcal{C}$ within the tree structure. The ECO method provides techniques to extract the number of nodes at each level based on the structural properties of T. However, $T_{\mathcal{C},\vartheta}$ may lack the regularity needed for efficient enumeration. In such cases, generating trees offer a more structured approach: their systematic node generation provides a reliable framework for enumeration within the ECO method.

Definition 7. A *generating tree* is a planar, rooted, and labeled tree equipped with a *production system* of labels

$$\Omega = \begin{cases} (s_0) \\ (k) \rightsquigarrow (e_1(k)), (e_2(k)), \ldots, (e_k(k)). \end{cases}$$

The generating tree induced by the production system Ω comprises a root labeled with s_0, and each node labeled with k generates k new nodes labeled by $e_1(k), \ldots, e_k(k)$. The former condition is recognized as the *regularity principle* of generating trees.

The regularity principle implies that the counting sequences considered in the ECO method are non-decreasing. Consequently, while applicable in a wide variety of contexts, this approach may not be immediately applicable to many combinatorial classes due to the inherent constraints imposed by the regularity principle.

Example 8. The operator ϑ, consisting of the parameters $\vartheta_n, n \geq 0$, that append a digit $j \in \{1, \ldots, k\}$ to a word $\omega = \omega_1 \cdots \omega_n \in \mathcal{W}_n^{[k]}$ at the right position of the digit ω_n is an ECO operator:

- Given a word $\omega = \omega_1 \cdots \omega_n \in \mathcal{W}_n^k$, the word $\omega' = \omega_1 \cdots \omega_{n-1} \in \mathcal{W}_{n-1}^k$ is such that $\omega \in \vartheta(\omega')$.
- Given two different words ω and η in $\mathcal{W}_n^k$, there exists a position i for some $1 \leq i \leq n$, where the digits ω_i and η_i differ. Consequently, the digits at i-th position of any two words in $\vartheta(\omega)$ and $\vartheta(\eta)$ will also differ. Thus $\vartheta(\omega) \cap \vartheta(\eta) = \emptyset$.

Hence ϑ is an ECO operator. Since the empty word, denoted by λ, is the unique element of size 0, the ECO operator ϑ induces the planar tree $T_{\mathcal{W}^k,\vartheta}$, whose initial levels are shown in Figure 2.3, taking the value $k = 4$.

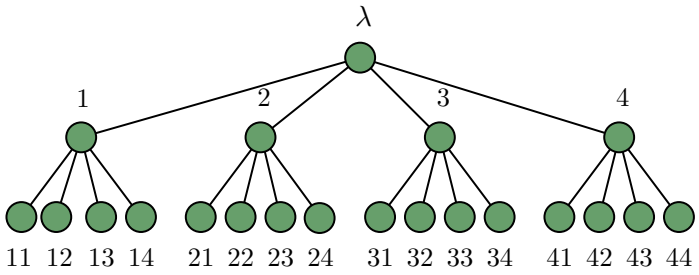


Figure 2.3: Hierarchy induced on $\mathcal{W}^4$ by the ECO operator ϑ.

Observe that $T_{\mathcal{W}^k,\vartheta}$ generates its nodes regularly, as each word produces k new words, and each of these, in turn, generates another k new words. Therefore, $T_{\mathcal{W}^k,\vartheta}$ is isomorphic to the generating tree defined by the production system:

$$\Omega = \begin{cases} (k) \\ (k) \rightsquigarrow (k)^k \,. \end{cases}$$

We use $(k)^n$ to denote the sequence of n labels $(k)\dots(k)$. As we have already discussed, the cardinality of $\mathcal{W}_n^k$ is k^n. However, the production system Ω has a property that enables the derivation of a closed formula for the generating function of $\mathcal{W}^k$. We will revisit and explore this example in greater detail once the necessary tools are introduced.

Having encoded the combinatorial class into a tree structure, we now outline the general procedure for enumeration using the ECO method:

1. Find an ECO operator for the combinatorial class.
2. Construct the tree $T_{\mathcal{C},\vartheta}$.
3. Find a generating tree isomorphic to $T_{\mathcal{C},\vartheta}$.
4. Use the production system of the generating tree to deduce the enumerative information.

In general, obtaining an explicit formula for the generating function from the production system remains an open problem, with complete solutions available only for specific cases. A key technique for handling the functional equations that arise in this context is the *kernel method*, which we introduce in the following example (see [13, pp. 13–14]).

Example 9 ([3])**.** Consider the class of planar trees $\mathcal{T}$ discussed in Example 4. In order to find a generating tree, we introduce the operator ϑ that takes a tree $T \in \mathcal{T}$ and produces a tree $T' \in \mathcal{T}$ by attaching a new node to one of the existing nodes in T. To ensure a structured expansion, the new node is always appended at the rightmost position. However, without imposing any restrictions on the set of nodes that can be expanded, ϑ does not qualify as an ECO operator, as some trees can be generated in multiple ways, as shown in Figure 2.4.

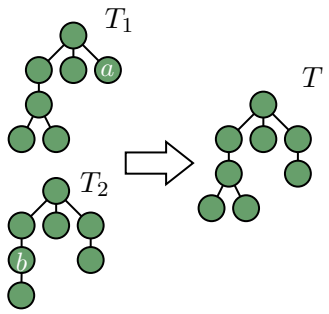


Figure 2.4: The expansions of ϑ in the nodes $a \in T_1$ and $b \in T_2$ generate the tree T.

To address this issue, we impose specific restrictions on the expansion process. Given a tree $T \in \mathcal{T}_n$, we first construct a tree T_1 that includes all possible expansions, where each node receives a new child. Next, we perform a preorder traversal of T_1, visiting each node from left to right along each branch. During this traversal, we identify the label l of the last node. We then select only as active sites the newly appended nodes whose labels exceed l. This selection process is illustrated in Figure 2.5.

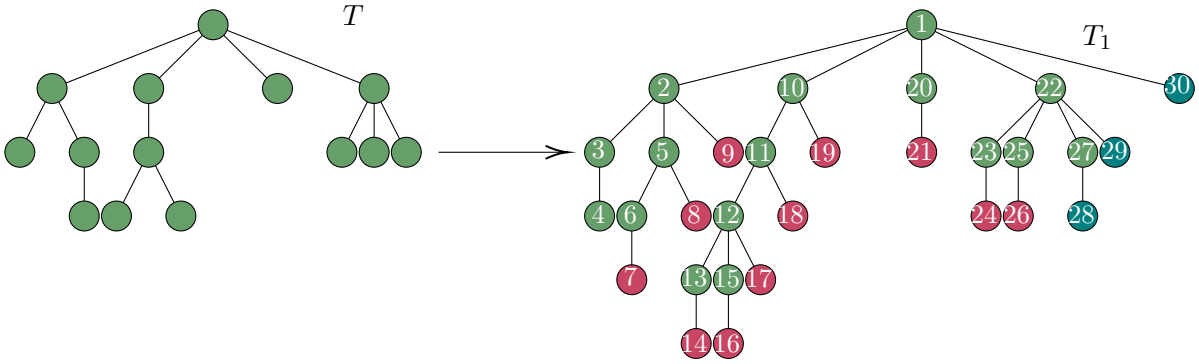


Figure 2.5: The nodes labeled 28, 29 and 30 are the local expansions of ϑ in T.

Each tree $T \in \mathcal{T}_n$ is now uniquely generated from the tree $T \setminus a$, where a is the last node in the pre-order traversal. This modification ensures that the second condition of ECO operators is met because ϑ does not alter the existing nodes but only adds new nodes to the rightmost branch. Furthermore, if γ is a node at level k on the rightmost branch of T, there is exactly one active node at each level above γ. Consequently, $\mathcal{T}$ is isomorphic to the generating tree with the following production system:

$$\Omega_{\mathcal{T}} = \begin{cases} (1) \\ (k) \rightsquigarrow (2), (3), \ldots, (k+1). \end{cases} \tag{2.1}$$

We aim to derive a functional equation for the ordinary generating function

$$F(x) = \sum_{n=0}^{\infty} t_n x^n,$$

where t_n represents the number of planar trees of size n. To achieve this, we define $f_{n,k}$ as the number of nodes at level n labeled k in the generating tree associated with Ω, and consider the bivariate generating function:

$$G(x,y) = \sum_{n\geq 0, k\geq 1} f_{n,k} x^n y^k.$$

Since each node at level n labeled k generates nodes labeled j for $j = 2, \ldots, k+1$, it follows that there are $f_{n,k}$ nodes labeled j at level $n+1$. Hence,

$$\begin{aligned} G(x,y) &= y + x \sum_{n\geq 0, k\geq 1} f_{n,k} x^n (y^2 + \cdots + y^{k+1}) \\ &= y + x \sum_{n\geq 0, k\geq 1} f_{n,k} x^n \frac{y^2 - y^{k+2}}{1-y}. \end{aligned}$$

Splitting the sum, we obtain:

$$G(x,y) = y + x \left(\sum_{n\geq 0, k\geq 1} f_{n,k} x^n \frac{y^2}{1-y} \right) - x \left(\sum_{n\geq 0, k\geq 1} f_{n,k} x^n \frac{y^{k+2}}{1-y} \right).$$

Thus, the functional equation is

$$G(x,y) \left(1 + x \frac{y^2}{1-y} \right) = y + \frac{y^2}{1-y} G(x,1).$$

To use the kernel method, we need to find y such that $1 + x\frac{y^2}{1-y} = 0$. The roots of this equation are:

$$y_0(x) = \frac{1 - \sqrt{1-4x}}{2x}, \quad y_1(x) = \frac{1 + \sqrt{1-4x}}{2x}.$$

We discard $y_1(x)$ because the coefficients in its power series expansion does not match with our problem. Substituting $y_0(x)$ into the right-hand side of the equation yields:

$$F(x) = G(x,1) = \frac{y_0(x) - 1}{x y_0(x)} = \frac{1 - \sqrt{1-4x}}{2x},$$

Consequently, the number of planar trees with n nodes is given by the coefficient of x^n, which is the Catalan number C_n.

An important aspect to consider is that different production systems, even when their generating trees are non-isomorphic, can still produce the same counting sequence; such systems are said to be *equivalent*. This highlights the flexibility and variety of approaches available in the enumeration of combinatorial objects, allowing multiple valid production systems to coexist for the same structure. Therefore, effective enumeration through the ECO methodology requires not only finding a production system but also identifying one that is most suitable for extracting the desired enumerative information.

Example 10. We will derive the production system (2.1) by defining an ECO operator on Dyck paths. Furthermore, we will introduce an alternative ECO operator to obtain an equivalent system. The class of Dyck paths has, as its fundamental building block, the unitary Dyck path UD. We define the operator ϑ_1, which expands each Dyck path by inserting a unitary Dyck path at one of the points $v_1, \ldots, v_k$ of the last descent (see Figure 2.6).

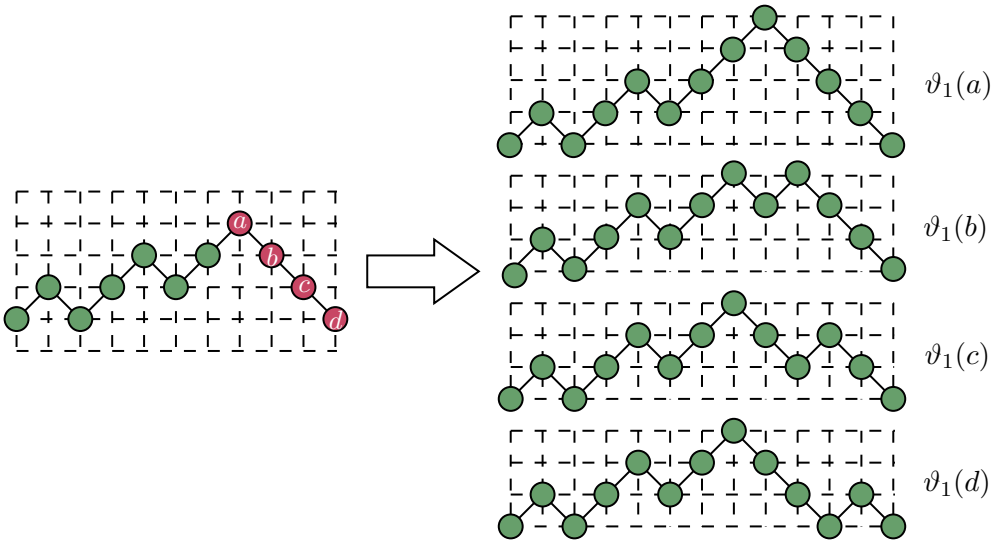


Figure 2.6: Image of a Dyck path via the ECO operator ϑ_1.

If there are k points in the last descent of a Dyck path, then k new Dyck paths (one for each point) are generated. Moreover, if the active site to be expanded is located at level i, the last descent of the resulting Dyck path will have a length of $i+1$. Consequently, the production system obtained is

$$\Omega = \begin{cases} (1), \\ (k) \rightsquigarrow (2)\,(3)\ldots(k+1)\,, \end{cases} \tag{2.2}$$

which corresponds to the system (2.1) derived from the class of planar trees in Example 10. Finally, we show that ϑ_1 is indeed an ECO operator. Let D be a Dyck path of length n. We identify and remove the rightmost peak of D, resulting in a path D' of length $n-1$, with $D \in \vartheta_1(D')$. On the other hand, consider two distinct paths of length n, say D_1 and D_2.

There must be a position i, $1 \leq i \leq 2n$, where D_1 and D_2 differ in their steps. Regardless of whether the endpoints of these positions are active sites, this difference persists in each of the generated paths. Thus, $\vartheta_1(D_1) \cap \vartheta_1(D_2) = \emptyset$.

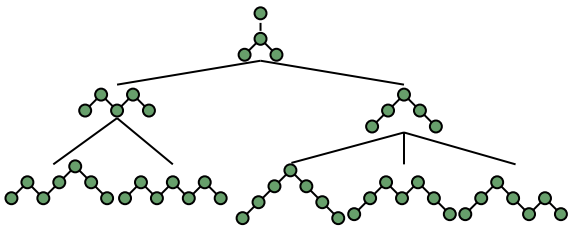


Figure 2.7: Initial levels of the planar tree induced by the ECO operator ϑ_1.

In addition to this construction, Duchi, in her doctoral thesis (see [13, pp. 25–26]), proposes an alternative ECO construction. She begins by observing that each Dyck path D can be uniquely decomposed into blocks of Dyck subpaths $D_1, D_2, \ldots, D_k$, where $1 \leq k \leq n$. These blocks are characterized by having their only points at the base level as their initial and ending points. This decomposition enables the classification of Dyck paths into two types: *even* type, when there is an even number of blocks, and *odd* type, when there is an odd number of blocks. Figure 2.8 illustrates an odd Dyck path and its corresponding decomposition.

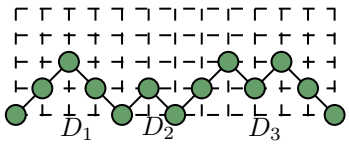


Figure 2.8: Odd Dyck path decomposition.

The set of active sites for even Dyck paths consists solely of the last point of the path, while for odd Dyck paths, it comprises the points in the last descent of the Dyck path, excluding the one at the base level. The operator ϑ_2 performs the following local expansions on even and odd Dyck paths:

- If the path is even, a unitary peak is attached at the end of the path.
- If the path is odd, each of the following expansions is considered:
    1. A unitary peak is attached at the active site.
    2. Let A be the (possibly empty) longest Dyck subpath such that the active point selected is its endpoint. A new Dyck path is generated by erasing A from the original path and then reattaching it, elevated by one level.

Figure 2.9 illustrates the expansions of several Dyck paths using the operator ϑ_2.

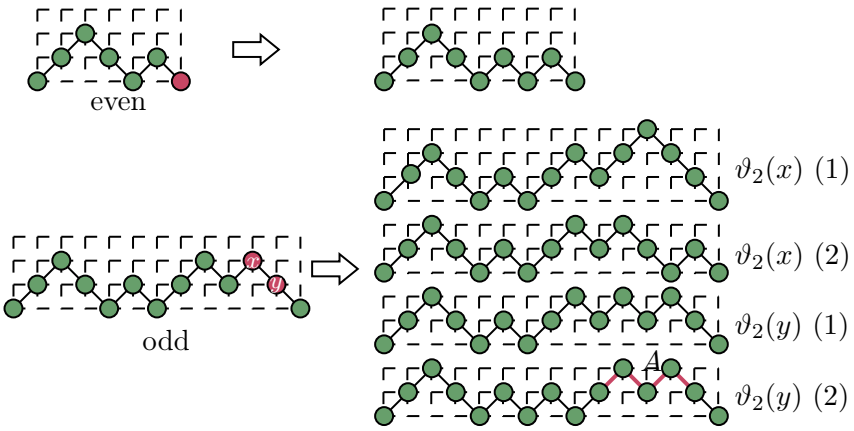


Figure 2.9: Dyck paths generation thought the operator ϑ_2.

We show that ϑ_2 is an ECO operator:

1. Let D be a Dyck path. If D ends in a unitary peak, it is generated from the path resulting from the removal of that unitary peak. On the other hand, if the last peak of D has height $i \geq 2$, then D is generated either from the path resulting from erasing the unitary peak at the top of the last peak, or from the path obtained by erasing the longest Dyck subpath whose endpoints are above the base level and reattaching it at a higher level.

2. Let D_1 and D_2 be different Dyck paths of length n. We split the proof into three cases:

   - If D_1 and D_2 are both even, it is easily seen that $\vartheta_2(D_1) \cap \vartheta_2(D_2) = \emptyset$.
   - If D_1 and D_2 are of different types, say D_1 is even and D_2 is odd, then each Dyck path $D_3 \in \vartheta_2(D_1)$ is odd and ends in a unitary peak, while the odd Dyck paths in $\vartheta_2(D_2)$ all have their last peak of height $i > 1$.
   - If D_1 and D_2 are both odd, we find the first peak where they differ. If that peak is located in the last position, then the expansions are clearly different. If it is located in any other position, then that difference is preserved in the resulting paths.

In summary, ϑ_2 adheres to the conditions required of an ECO operator by ensuring that each Dyck path is uniquely generated from a smaller path, and no two distinct paths generate overlapping sets of new paths. Figure 2.10 exhibits the generation of Dyck paths via the operator ϑ_2.

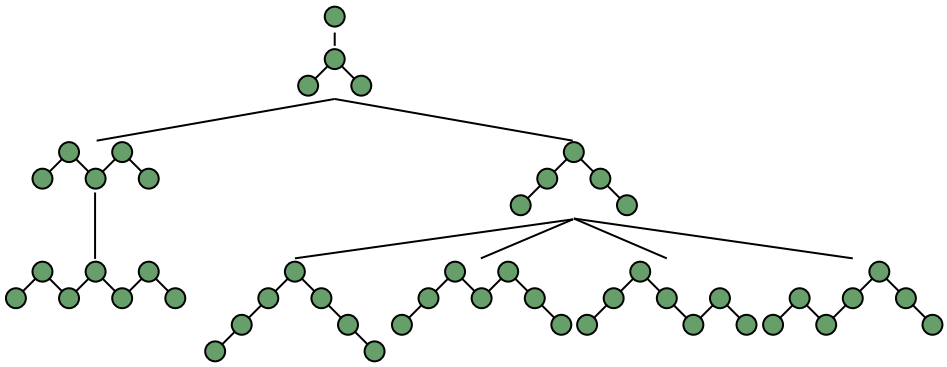


Figure 2.10: Generation of Dyck paths via the operator ϑ_2.

Since the even paths generate exactly one path by appending a unitary peak at the end, their corresponding nodes in the generating tree are labeled as (1), with the production rule $(1) \rightsquigarrow (2)$. On the other hand, each odd path with i active sites produces $2i$ Dyck paths: i of these paths, generated by rule 1), are odd, while the remaining i paths, generated by rule 2), are even. Furthermore, the resulting odd paths have $2, 3, \ldots, i+1$ active sites. Consequently, the corresponding nodes in the generating tree are labeled $(4)(6)\ldots(i+2)$, leading to the following production system obtained via the operator ϑ_2:

$$\Omega_{\vartheta_2} = \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2i) \rightsquigarrow (1)^i\,(4)\,(6)\cdots(2i+2). \end{cases} \tag{2.3}$$

Figure 2.11 displays the generating tree induced by this production system.

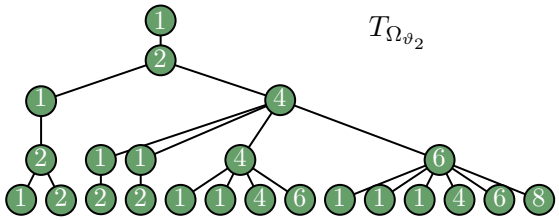


Figure 2.11: Initial levels of the generating tree induced by Ω_{ϑ_2}.

This example demonstrates that, while the ECO method allows us to define a production system, multiple production systems can exist for the same combinatorial class. These systems may exhibit different structures yet still produce the same generating function, underscoring the flexibility and diversity of approaches available in combinatorial enumeration.

2.3 The ECO method: A Matricial Approach.

Similarly to generating trees, production systems can also codify elements in a combinatorial class entries of a matrix. A key advantage of this approach, developed by E. Deutsch et al. in [11], is the ability to use algebraic tools for enumeration.

Definition 11. Given a production system

$$\Omega = \begin{cases} (s_0) \\ (k) \rightsquigarrow (e_1(k)), (e_2(k)), \ldots, (e_k(k)), \end{cases}$$

the *production matrix* determined by Ω is the matrix $M_\Omega = (a_{i,j})$ whose entry $a_{i,j}$ represents the number of times that the label e_j appears in the rule of the label e_i. In other words, according to our notation, the entry $a_{i,j}$ is the exponent of e_j in the rule for e_i, with the exponent being 0 when e_j does not appear. This matrix generally has infinite entries. However, if the production system is finite, the matrix M_Ω will have a finite number of non-zero rows and columns, so we consider instead the reduced matrix $M_{\Omega_{l\times l}}$, where l is the greatest label appearing in the production system Ω.

Example 12. Consider the rules of the production system that define the Fibonacci sequence[1]:

$$\Omega_2 = \begin{cases} (2) \\ (2) \to (2)(1) \\ (1) \to (2). \end{cases}$$

The generating tree of this system is exhibited in Figure 2.12. Note that the Fibonacci numbers count the number of nodes in each level of the tree.

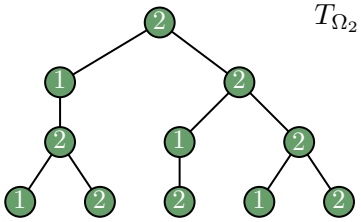


Figure 2.12: Generating tree of Fibonacci numbers.

[1] In the literature, it is also common to set the root to be labeled by 1, which results in a shifted Fibonacci sequence.

In exponential notation, these rules are given by:

$$(1) \rightsquigarrow (1)^0(2)^1(3)^0\cdots,$$
$$(2) \rightsquigarrow (1)^1(2)^1(3)^0\cdots.$$

Hence, the production matrix determined by Ω_2 is

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

However, the entries $a_{i,j}$ are zero for any $i \geq 3$ or $j \geq 3$. Hence, all the information of the production system Ω_2 is encoded in the 2×2 matrix:

$$M_{\Omega_2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

In the generating trees representation, objects of size n are encoded as nodes at level n of $T_{\mathcal{C},\vartheta}$. In the context of production matrices, this information can be reconstructed by iteratively applying matrix operations to the production matrix.

Definition 13. Let P be a production matrix and $\mathbf{u}^T$ be the row vector $(1, 0, 0, \dots)$. The *ECO* matrix determined by P is the matrix A_P given by:

$$A_P = \begin{pmatrix} \mathbf{u}^T \\ \mathbf{u}^T P \\ \mathbf{u}^T P^2 \\ \vdots \end{pmatrix}.$$

In this representation, the number of objects of size n, denoted as a_n, is encoded as the sum of the n-th row of A_P. This can be calculated using the formula $a_n = \mathbf{u}^T P^n \cdot e$, where $e = (1, 1, 1, \dots)^T$. In the case of a finite matrix, we adjust the dimensions of vectors $\mathbf{u}^T$ and e to ensure that the product $P\mathbf{u}^T$ is meaningful.

Example 14. The ECO matrix for the production system Ω_2 of Fibonacci numbers is given

by:

$$A_{\Omega_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 2 & 3 & 0 & 0 & \vdots \\ 3 & 5 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

Let a_n be the sum of the entries of the n-th row of A_{Ω_2}. Thus $a_1 = 1$ and $a_2 = 2$, whilst for $n \geq 3$, $a_n = a_{1,n-1} + a_{2,n-1}$. That is, a_n satisfies the Fibonacci recursion.

2.4 Properties of Production Systems

Banderier and his collaborators, as discussed in [2], extended the application of the kernel method to uncover relations between the structural properties of production systems and the algebraic characteristics of their corresponding generating functions. They categorize production systems into three types based on the nature of the generating function induced by the system:

- A production system is termed *rational* if its generating function is a rational function, meaning it is the quotient of two polynomials.
- An *algebraic* production system is one where the generating function satisfies an algebraic equation with polynomial coefficients. This implies that the generating function is a root of some polynomial equation with polynomial coefficients.
- A *transcendental* production system features a generating function that cannot be expressed as a solution to any polynomial equation with rational coefficients. Such generating functions are not algebraic and lie outside the scope of algebraic solutions.

This subsection follows the approach presented in [2], adapting its key ideas to our context.

Rational production systems

Let f_n be the number of nodes at level n in the generating tree induced by a production system Ω, and s_n denote the sum of the labels of those nodes. Considering the root at level 0, we have $f_0 = 1$. Moreover, each node labeled (k) has precisely k children, implying $s_n = f_{n+1}$. The number of nodes at level n with label k is denoted by $f_{n,k}$. We define the generating function in two variables:

$$F(z, u) = \sum_{n,k \geq 0} f_{n,k} z^n u^k,$$

which counts the distribution of nodes at n-th generation labeled k. We also consider the ordinary generating function

$$F_k(z) = \sum_{n \geq 0} f_{n,k} z^n.$$

These generating functions are related by the equation $F(z) = F(z,1) = \sum_{k \geq 1} F_k(z)$.

Theorem 15 ([2])**.** *The generating function $F_k(z)$ admits the following expression:*

$$F_k(z) = [\![k = s_0]\!] + z \sum_{j \geq 1} \pi_{j,k} F_j(z),$$

where

$$[\![P]\!] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise.} \end{cases}$$

and $\pi_{j,k} = |\{i \leq j : e_i(j) = k\}|$ counts the nodes labeled j that produce node labeled k.

Proof. If $k = s_0$, then $F_k(z)$ must count the initial node at level 0, which contributes exactly 1. Otherwise, each node of label j at level n generates $\pi_{j,k}$ nodes of label k at level $n+1$. Therefore, the number of nodes labeled k at level $n+1$ satisfies:

$$f_{n+1,k} = \sum_{j \geq 1} \pi_{j,k} f_{n,j}.$$

Summing both sides over all n and multiplying by z^n, we obtain:

$$\sum_{n \geq 0} f_{n+1,k} z^n = \sum_{j \geq 1} \pi_{j,k} \sum_{n \geq 0} f_{n,j} z^n.$$

Since the generating function satisfies:

$$F_k(z) = \sum_{n \geq 0} f_{n,k} z^n,$$

we rewrite the left-hand side as:

$$\frac{F_k(z) - [k = s_0]}{z} = \sum_{j \geq 1} \pi_{j,k} F_j(z).$$

Multiplying by z gives:

$$F_k(z) = [k = s_0] + z \sum_{j \geq 1} \pi_{j,k} F_j(z).$$

□

Theorem 16 ([2]). *If Ω is a production system with finite label rules, then $F(z)$ is rational.*

Proof. Since Ω has finite label rules, there are finite values of $k \in \mathbb{N}$, such that $F_k(z) \neq 0$. Hence $F(z)$ is described as a finite number of linear systems provided by description of theorem 15. Thus, $F(z)$ is rational. □

Example 17. In the production system for Fibonacci numbers

$$\Omega_2 = \begin{cases} (2) \\ (2) \to (2)(1) \\ (1) \to (2). \end{cases}$$

Each node labeled 1 produces a node labeled 2, and each node labeled 2 produces one node labeled 1 and one labeled 2. The ordinary generating function is given by:

$$F(z) = F_1(z) + F_2(z).$$

According to the rules for generation of labels, it follows that:

$$F_1(z) = zF_2(z),$$

$$F_2(z) = 1 + z(F_1(z) + F_2(z)).$$

Substituting $F_1(z)$ into $F_2(z)$ we find that

$$F_2(z) = \frac{1}{1 - z - z^2}.$$

Consequently,

$$F(z) = (1 + z)F_2(z) = \frac{1 + z}{1 - z - z^2},$$

which is rational and corresponds to the (shifted) generating function of Fibonacci numbers.

Definition 18. Let $f(x)$ be a function. We say that f is *affine* in the variable x if for every $x \in Dom(f)$, there exists constants α and β such that $f(x) = \alpha x + \beta$.

Theorem 19 ([2]). *Consider the function $\sigma(k) = e_1(k) + \cdots + e_k(k)$. If σ is affine, then $F(z)$ is rational and has the following closed formula:*

$$F(z) = \frac{1 + (s_0 - \alpha)z}{1 - \alpha z - \beta z^2}.$$

Proof. Let $k_1, \ldots, k_{f_n}$ be the labels of the f_n nodes at level n of the tree. Hence

$$f_{n+2} = s_{n+1} = (\alpha k_1 + \beta) + \cdots + (\alpha k_{f_n} + \beta) = \alpha s_n + \beta f_n = \alpha f_{n+1} + \beta f_n.$$

With the initial conditions $f_0 = 1$ y $f_1 = s_0$, thus

$$F(z)(1-\alpha z-\beta z^2) = (1+s_0 z+\cdots)-(\alpha z+\alpha s_0 z^2+\alpha f_2 z^3+\cdots)-(\beta z^2+\beta s_0 z^3+\beta f_3 z^5+\cdots).$$

Observe that according to the previous recurrence, for $n \geq 3$, it follows that

$$[z^n]F(z) = f_n - \alpha f_{n-1} - \beta f_{n-2} = 0.$$

Therefore, $F(z)(1 - \alpha z - \beta z^2) = 1 + (s_0 - \alpha)z$ and

$$F(z) = \frac{1 + (s_0 - \alpha)z}{1 - \alpha z - \beta z^2}.$$ □

Example 20. The production system of k-ary words

$$\Omega = \begin{cases} (1), \\ (k) \rightsquigarrow (k)^k \end{cases}$$

has the function $\sigma(k) = k^2$, which is affine with the values $\alpha = k$ and $\beta = 0$. Thus, the generating function is

$$F(z) = \frac{1}{1 - kz}. \tag{2.4}$$

This is a geometric series, so its Taylor expansion can be expressed as

$$F(z) = \sum_{k=0}^{\infty} k^n z^n.$$

Therefore, the counting sequence of k-ary words is $a_n = k^n$, for all $n \geq 0$.

We highlight that not every finite production system is affine. For instance, the production system

$$\Omega = \begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)\,(3) \\ (3) \rightsquigarrow (1)\,(2)\,(3)\,, \end{cases}$$

leads to the linear system

$$\begin{cases} \sigma(1) = 2 = \alpha + \beta, \\ \sigma(2) = 5 = 2\alpha + \beta, \\ \sigma(3) = 6 = 3\alpha + \beta. \end{cases}$$

which has no integer solutions. The presence of an affine function σ proves to be a sufficient yet not entirely necessary condition for the rationality of the generating function. The subsequent result delves into functions that exhibit near-affine characteristics.

Theorem 21 ([2])**.** *Consider a production system Ω in the form:*

$$\Omega = \begin{cases} (s_0)\,, & (k) \rightsquigarrow e_1^{[0]}(k), \ldots, e_k^{[0]}(k) \quad \text{if } k \text{ is even} \\ & (k) \rightsquigarrow e_1^{[1]}(k), \ldots, e_k^{[1]}(k) \quad \text{if } k \text{ is odd} \end{cases}$$

Additionally, if the following conditions hold:

i) The functions σ_0 and σ_1 are affine, both having the same leading coefficient α.

ii) There exist exactly m odd labels on the right-hand side of each rule, where $m \in \mathbb{N}$.

Then $F(z)$ is rational, and its form is given by:

$$F(z) = \frac{1 + (s_0 - \alpha)\, z + (s_1 - \alpha s_0 - \beta_0)\, z^2}{1 - \alpha z - \beta_0 z^2 - m\,(\beta_1 - \beta_0)\, z^3}.$$

It is worth noting that Theorem 19 is a special case of this result, with the value $\beta_0 = \beta_1$.

Proof. Without loss of generality, let us assume that the first $n - m$ nodes have even label. Thus:

$$\begin{aligned} f_{n+2} = s_{n+1} &= (\alpha k_1 + \beta_0) + \cdots + (\alpha k_{m-n} + \beta_0) + \alpha k_{n-m+1} + \beta_1 + \cdots + \alpha k_{f_n} + \beta_1 \\ &= \alpha s_n + \beta_0 (n - m) + \beta_1 m, \end{aligned}$$

where $f_0 = 1$ and $f_1 = s_0$. We expand the product $F(z)\left(1 - \alpha z - \beta_0 z^2 - m(\beta_1 - \beta_0) z^3\right)$ and notice that for $n \geq 3$, the coefficient of z^n is

$$[z^n] F(z) = f_n - \alpha f_{n-1} - \beta f_{n-2} - m(\beta_1 - \beta_0) f_{n-3} = 0.$$

Leading to the equation:

$$F(z)\left(1 - \alpha z - \beta_0 z^2 - m(\beta_1 - \beta_0) z^3\right) = 1 + s_0 z + f_2 z^2 - \alpha z + \alpha s_0 z^2 - \beta_0 z^2.$$

□

Example 22. The production system:

$$\Omega = \begin{cases} (2) \\ (k) \rightsquigarrow (2)^{k-2}(2 + (k \mod 2))(k+1) \end{cases}$$

is not affine, since the auxiliary function is given by:

$$\sigma(k) = \begin{cases} 3k-1, & \text{if } k \text{ is even,} \\ 3k, & \text{if } k \text{ is odd.} \end{cases}$$

See the generating tree in Figure 2.13. However, σ still fulfills the first condition of theorem 21 with the values $\alpha = 3$, $\beta_0 = -1$, and $\beta_2 = 0$. The second condition is also satisfied with $m = 1$, as each case generates only one odd label, consequently, Ω is rational, and its generating function is given by:

$$F(z) = \frac{1-z}{1-3z+z^2-z^3}.$$

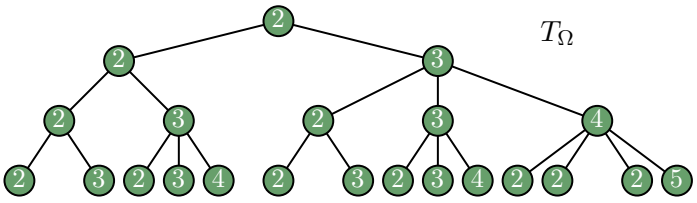


Figure 2.13: Initial levels of the generating tree induced by Ω.

Example 23. The production system

$$\Omega_1 = \begin{cases} (2) \\ (2) \rightsquigarrow (2)\,(3), \\ (3) \rightsquigarrow (2)(3)(4), \\ (4) \rightsquigarrow (2)\,(3)(4)^2. \end{cases}$$

is affine with $\alpha = 4$ y $\beta = -3$ and consequently its generating function is

$$F(z) = \frac{1-2z}{1-4z+3z^2}.$$

Algebraic production systems

In contrast to the rational case, deriving explicit formulas for algebraic generating functions directly from their production systems is more challenging. However, certain structured pro-

duction systems lead to algebraic generating functions. These systems follow specific recursive rules that constrain their growth, resulting in generating functions that satisfy algebraic equations. The main result is the following. For a detailed proof (in terms of factorial walks), see [2].

Theorem 24 ([2])**.** *Production systems of the form:*

$$\Omega = \begin{cases} (s_0), \\ (k) \rightsquigarrow (a)(a+1)\cdots(k-c-1)(k+d_1)\cdots(k+d_m), \quad k \geq a. \end{cases}$$

have an algebraic generating function.

In this system:

- The sequence of labels $(a)(a+1)\cdots(k-c-1)$ consists of consecutive ascending integers smaller than k, without repetition.
- The terms $(k+d_1)\cdots(k+d_m)$ allow for repetitions or jumps in the values.

Example 25. The production systems for Catalan numbers from Examples 9 and 10:

$$\Omega = \begin{cases} (1), \\ (k) \rightsquigarrow (2)\,(3)\ldots(k+1)\,, \end{cases}$$

and

$$\Omega' = \begin{cases} (1), \\ (1) \rightsquigarrow (2), \\ (2i) \rightsquigarrow (1)^i\,(4)\,(6)\cdots(2i+2) \end{cases}$$

are of the form given in Theorem 24, and their generating function is

$$F(z) = \frac{1-\sqrt{1-4z}}{2z},$$

which is algebraic.

Transcendental production systems

In the algebraic case, for each node in the production system labeled k, the number of nodes labeled with values greater than k is sufficiently restricted, ensuring that the generating function has a positive radius of convergence. In contrast, in the transcendental case, this set grows too rapidly, forcing the generating function to have a radius of convergence equal to zero.

Theorem 26 ([2]). *Let $b \geq 0$ be a number and define $m(k) \coloneqq |\{i : e_i(k) \geq k - b\}|$. If:*

- *For all k, there exists a forward jump.*
- *The sequence $(m(k))_k$ is non-decreasing and diverges to infinity.*

Then $F(z)$ has radius of convergence 0.

Proof. Consider the product

$$h_n \coloneqq m(s_0 + b)m(s_0 + 2b) \cdots m(s_0 + nb).$$

We will prove that there are at least h_n nodes at level $n(b+1)$ in the generating tree. At level nb, take a node v labeled k such that $k \geq s_0 + nb$. Then v has at least $m(k)$ children labeled at least $k - b$. Similarly, there are at least $m(s_0 + nb)$ children labeled at least $s_0 + (n-1)b$. Iterating this argument, we find that at level $nb + 1$, there are at least

$$m(s_0 + (n - i + 1)b) \cdots m(s_0 + nb)$$

nodes labeled at least $s_0 + (n - i)b$ for $0 \leq i \leq n$. When $i = n$, the level $n(b + 1)$ has exactly h_n nodes labeled at least s_0. Thus, we obtain

$$f_{n(b+1)} \geq h_n.$$

Since $h_n/h_{n-1} = m(s_0+nb)$ and $(m(k))_k$ is non-decreasing and unbounded, by the comparison test, the series $\sum_{n\geq 0} h_n z^n$ diverges, and so does $\sum_{n\geq 0} f_{n(b+1)}$. This implies that $F(z)$ has radius of convergence 0. □

Example 27. Consider the production system for permutations:

$$\Omega = \begin{cases} (2), \\ (k) \rightsquigarrow (k+1)^k \,. \end{cases}$$

Each node labeled k produces k children labeled $(k + 1)$, each of which represents a forward jump, as illustrated in Figure 2.14.

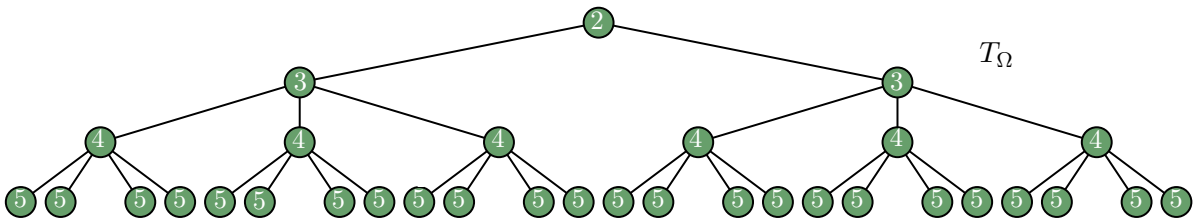


Figure 2.14: Initial levels of the generating tree induced by Ω.

If $b = 0$, then $m(k) = k$. Thus, $(m(k))_k$ is non-decreasing and unbounded as $k \to \infty$. Hence, Ω satisfies the conditions of Theorem 26, and its exponential generating function is

$$F(z) = \frac{1}{1-z},$$

which has radius of convergence 0.

Chapter 3

Generalized Fibonacci numbers

Expanding mathematical concepts into more abstract while retaining their core properties is a prevalent practice in mathematics. When it comes to the Fibonacci numbers, the generalization perspectives are multiple. Some authors such as V. Hogatt [24], S. Falcón ([15, 16]), and A. Horadam [25] have contributed to the exploration and characterization of generalized Fibonacci numbers, which stem from recursions of the form:

$$u(t) = Pu(t-1) + Qu(t-2), \;\; t \geq 2. \quad u(0) = a, u(1) = a + b,$$

where a, b, P and Q are natural numbers. Alternatively, other authors such as J. Baril and P. Do [5] and E. Egge and T. Mansour [14] have pursued a different path, focusing on sequences derived fixing a positive number p of initial terms and computing subsequent terms as the sum of the preceding p terms. In line with this alternative approach, we introduce the p-generalized Fibonacci numbers.

3.1 Generalized Fibonacci numbers

Definition 28. Let p be a positive number. The *p-generalized Fibonacci numbers*, denoted by $F_{p,k}$, are the numbers defined by the recursion:

$$F_{p,k} = \begin{cases} 0, & \text{if } k = -p+2, -p+1, \ldots, 0, \\ 1, & \text{if } k = 1, \\ \sum_{i=1}^{p} F_{p,k-i}, & \text{if } k > 1. \end{cases}$$

The inclusion of negative values for k is intended to allow the computation of the terms $F_{p,k}$ for $k = 2, 3, \ldots, p$. However, the terms $F_{p,k}, k \geq 0$, posses the relevant information about the series.

Illustratively, the Tribonacci sequence $(F_{3,k})$ (A000073), consists of the terms

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81 \ldots,$$

while the *Tetranacci* sequence $(F_{4,k})$ (A000078) consists of the numbers

$$0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56 \ldots.$$

Observe that for $1 < k \leq p$, the term $F_{p,k}$ is given by:

$$F_{p,k} = 1 + \sum_{i=0}^{k-2} 2^i = 2^{k-2}.$$

Theorem 29. *Let $f_p(z)$ be the ordinary generating function of the p-generalized Fibonacci sequence $(F_{p,k})$, then*

$$f_p(z) = \frac{1 + z + z^2 + z^3 + \cdots + z^{p-1}}{1 - z - z^2 - \cdots - z^p}. \tag{3.1}$$

Proof. We expand the product $f_p(z)(1 - z - z^2 - \cdots - z^p)$ as follows:

$$\begin{aligned} f_p(z)(1 - z - z^2 - \cdots - z^p) = {} & F_{p,0} + F_{p,1}z + F_{p,2}z^2 + \cdots + F_{p,n}z^n + \cdots \\ & - (F_{p,0}z + F_{p,1}z^2 + F_{p,2}z^3 + \cdots + F_{p,n}z^{n+1} + \cdots) \\ & - (F_{p,0}z^2 + F_{p,1}z^3 + F_{p,2}z^4 + \cdots + F_{p,n}z^{n+2} + \cdots) \\ & \qquad \cdots \\ & - (F_{p,0}z^p + F_{p,1}z^{p+1} + F_{p,2}z^{p+2} + \cdots + F_{p,n}z^{p+n} + \cdots) \end{aligned}$$

We observe that for $i \leq p - 1$, the coefficient $[z^i]f_p(z)$ is given by

$$[z^i]f_p(z) = F_{p,i} - F_{p,i-1} - \cdots - F_{p,0} = 2^{i-2} - 2^{i-3} - \cdots - 2^0 = 1.$$

On the other hand, when $i \geq p$, the coefficient $[z^i]f_p(z)$ is

$$[z^i]f_p(z) = F_{p,i} - \sum_{k=1}^{p} F_{p,n-k} = 0.$$

Thus, $f_p(z)(1 - z - z^2 - \cdots - z^p) = 1 + 1 + z^2 + \cdots + z^{p-1}$ and consequently:

$$f_p(z) = \frac{1 + z + z^2 + z^3 + \cdots + z^{p-1}}{1 - z - z^2 - \cdots - z^p}.$$

□

There exists an equivalent of Binet's formula for usual Fibonacci numbers.

Theorem 30. *([12]) The p-generalized Fibonacci numbers satisfy the closed formula:*

$$F_{p,n} = \sum_{i=1}^{p} \frac{\alpha_i - 1}{2 + (p+1)(\alpha_i - 2)} \alpha_i^{n-1},$$

where $\alpha_i, \ldots, \alpha_p$ *are the roots of* $z^p - z^{p-1} - \cdots - 1 = 0$.

The table below exhibits the roots of the polynomial for different values of p (the numbers are rounded up to the third decimal position):

p	Roots
2	$-0.618, 1.618$
3	$-0.420 - 0.606i, -0.420 + 0.606i, 1.839$
4	$-0.775, -0.076 - 0.815i, -0.076 + 0.815i, 1.928$
5	$-0.678 - 0.459i, -0.678 + 0.459i, 0.195 - 0.849i, 0.195 + 0.849i, 1.966$
6	$-0.840, -0.462 - 0.719i, -0.462 + 0.719i, 0.390 - 0.818i, 0.390 + 0.818i, 1.984$

The roots of the polynomial $z^p - z^{p-1} - \cdots - 1$ lie whitin the unit circle except for one root α, see for reference [12]. Thus, as n grows, the assymptotic behaviour of Binet's generalized formula is

$$F_{p,n} \sim \frac{\alpha - 1}{2 + (p+1)(\alpha - 2)} \alpha^{n-1}.$$

Theorem 31. *The generating function of p-generalized Fibonacci numbers converges to the generating function of binary words. That is* $\lim_{p\to\infty} f_p(z) = \frac{1}{1-2z}$.

Proof. Observe that

$$\begin{aligned}
\lim_{p\to\infty} f_p(z) &= \frac{\lim_{p\to\infty}(1 + z^2 + z^3 + \dots)}{\lim_{p\to\infty}(1 - z - z^2 - \dots)} \\
&= \frac{1}{1-z}\frac{1-z}{1-2z} \\
&= \frac{1}{1-2z}.
\end{aligned}$$

□

Additional combinatorial identities and properties regarding p-generalized Fibonacci numbers can be found in [18].

3.2 Production System for Generalized Fibonacci Numbers

We introduce previously the production system

$$\Omega_2 = \begin{cases} (2) \\ (1) \rightsquigarrow (2), \\ (2) \rightsquigarrow (1)(2) \end{cases}$$

which codifies the usual Fibonacci sequence. To establish the production system Ω_p for the general case $(F_{p,k})$, we include additional binary rules, which are distinguished by the sub-indices ranging from 1 to $p-1$.

Theorem 32. *([5]) The production system (Ω_p), given by*

$$\Omega_p = \begin{cases} (2_{p-1}) \\ (2_{p-1}) \to (2_{p-1})\,(2_{p-2}) \\ (2_{p-2}) \to (2_{p-1})\,(2_{p-3}) \\ \dots \\ (2_1) \to (2_{p-1})\,(1) \\ (1) \to (2_{p-1})\,. \end{cases} \tag{3.2}$$

codifies the generalized Fibonacci sequence $(F_{p,k})$.

As an example, we illustrate the generating tree obtained for $p = 3$ in Figure 3.1:

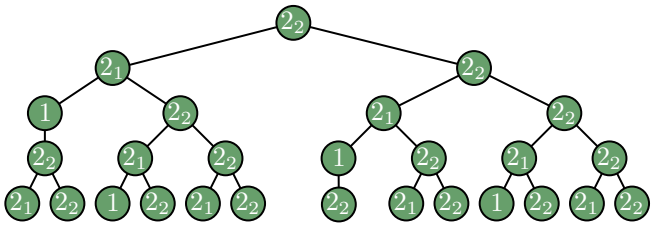


Figure 3.1: Generating tree induced by Ω_3.

Proof. The core idea lies in transforming the production system Ω_p into matrix notation, to use then the Cayley-Hamilton theorem to show that a_n adheres to the p-generalized Fibonacci recursion. The production matrix M_{Ω_p} is obtained by placing the exponents of the rule (1) at the first row and the exponents of the rule (2_i) at the $i+1$-th row, for $1 \leq i \leq p-1$.

That is,

$$M_{\Omega_p} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

Let x be a variable and d_p be the determinant of the matrix $A = (x \cdot \mathcal{I}_{p\times p}) - M_{\Omega_p}$. Thus:

$$\begin{aligned} d_p &= xd_{p-1} - (1)^{p-1} \det \begin{pmatrix} -1 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & x \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \\ &= xd_{p-1} - (1)^{p-1}(-1)^p = xd_{p-1} - 1. \end{aligned}$$

If we repeatedly replace the term $d_p - 1$, we find that $d_p = x^p - x^{p-1} - x^{p-2} - \cdots - x - 1$. By virtue of the Cayley-Hamilton theorem, we obtain that:

$$M^p_{\Omega_p} - M^p_{\Omega_p} - \cdots - M_{\Omega_p} - \mathcal{I} = 0.$$

Since $\det(M_{\Omega_P}) \neq 0$, it follows that $M - \Omega_p$ is invertible and then $M^n_{\Omega_p}$ can be expressed as

$$M^n_{\Omega_p} = \sum_{i=n-p}^{n-1} M^i_{\Omega_p}.$$

Since $a_n = \boldsymbol{u}^T M^n_{\Omega_p} \cdot e$, it follows that for all $n \geq p$:

$$a_n = \sum_{i=n-p}^{n-1} a_i.$$

Additionally, each node at level i, for $0 \leq i \leq p-2$, has exactly 2 successors. Hence $a_i = 2^i$, and

$$a_n = \begin{cases} 2^n, & \text{if} \quad 0 \leq n \leq p-1; \\ \sum_{i=n-p}^{n-1} a_i, & \text{if} \quad p \geq n. \end{cases}$$

□

3.3 Catalog of Objects Enumerated by Generalized Fibonacci Numbers

A comprehensive exploration of traditional Fibonacci numbers reveals their presence across various families of combinatorial objects, including tilings, sets, permutations, words, compositions, and more. By employing these objects as foundational examples, we derive families of objects governed by generalized Fibonacci sequences. We include combinatorial proofs of these relationships and establish bijections between the different objects.

Binary words

Let S^p be the class of binary words that avoid p consecutive occurrences of the digit 1. We refer to such binary words as generalized Fibonacci strings. As a particular example, we show below the words in S_4^3. That is, binary words of length 4 avoiding the word 111 :

$$\begin{array}{cccccccc} 0000 & 0001 & 0010 & 0100 & 1000 & 1001 & 1100 & 0101 \\ 0110 & 0011 & 1101 & 1011 & 1010. & & & \end{array}$$

Theorem 33. *The generalized Fibonacci strings are enumerated by the generalized Fibonacci sequences. More precisely:* $|S_n^p| = F_{p,n+2}$.

Proof.

- For $0 \leq n \leq p-1$, S_n^p is precisely the set of all binary strings of length n since each word is not long enough to have p consecutive occurrences of the digit 1. That is, $|S_n^p| = 2^n = F_{p,n+2}$.

- For $n \geq p$, we denote by A_n^p the words in S_n^p whose last occurrence is the digit 0, and by B_n^p the words in S_n^p whose last digit is 1. Each word in A_n^p is of the form $\omega' 0$, where ω' is a word of length $n-1$ that avoids p consecutive ocurrences of the digit 1. That is, $|A_n^p| = |S_{n-1}^p|$. On the other hand, each word in B_n^p consists of a word α_k in S_{n-k-1}^p followed by the digit 0 and k consecutive occurrences of the digit 1, for $k = 1, 2, \ldots, p-1$. Hence $|B_n^p| = \sum_{k=1}^{p-2} |S_{n-k}^p|$. Finally, since $S_n^p = A_n^p \cup B_n^p$ and $A_n^p \cap B_n^p = \emptyset$, it follows that

$$|S_n^p| = F_{p,n+2} + \sum_{i=1}^{p-2} F_{p,n-k} = F_{p,n+3}.$$

□

Theorem 34 ([5])**.** *The Fibonacci strings S^p are codified by the production system (Ω_p).*

Proof. We prove this statement by recursion over the levels of the generating tree induced by (Ω_p). For $n = 0$, the empty word λ is codified by the root (2_{p-1}). Assume that the strings

in S_i^p are codified as nodes at i-th level for each $i = 1, \ldots, k$. Let τ be the binary string of a node at level $k+1$ that arises from a binary string δ at level k.
If the last digit of τ is 0δ, then, since δ avoids p consecutive ocurrences of the digit 1, so do τ. If τ ends in the digit 1δ is codified by a binary rule (2_i), with $1 \leq i \leq p-1$. There are two options:

- If δ has the label (2_{p-1}), then δ is of the form $\delta = \delta'0$ and $\tau = \delta'01$ where δ' avoids p consecutive occurrences of the digit 1 according to the induction hypothesis. Consequently, τ also avoids p consecutive occurrences of the digit 1.

- If δ is labeled $(2i)$ where $1 \leq i \leq p-2$, its predecessor δ_1 is labeled $(2i+1)$. We repeat this process until we find a label $(2p-1)$. This results leads to a path $\delta_1, ..., \delta_{p-1-i}$, where δ_j is the successor of δ_{j+1} for $1 \leq j \leq p-2-i$, and δ_j is labeled $(2i+j)$. Particularly, δ_{p-1-i} is labeled $(2p-1)$. Thus, δ is of the form $\delta'01p-i-1$, and $\tau = \delta'01p-i$, where $\delta' \in F_{p,k-p+i}$ by the induction hypothesis. Since the length $p-1-i$ of the path verifies $p-1-i \leq p-2$, we conclude that τ belongs to $F_{p,k+1}$.

□

Observe that as p becomes very large, the p-generalized Fibonacci sequence starts to resemble the structure of binary words since such words are not long enough to have p occurrences of the digit 1.

p-ary words

The production system (Ω_p) of p-generalized Fibonacci numbers, allows to construct a new family of p-ary words enumerated by the p-generalized Fibonacci numbers: Let $\mathcal{W}^p$ be the subset of p-ary words, having the word consisting of the digit p as the unique word of size 1, and obtained through the following ECO operator ϑ, acting on the last position of the word. If $\omega = \omega_1 \cdots \omega_n$, then $\vartheta(\omega)$ produces new words by appending the digit ω_{n+1} according to the ECO local expansions:

$$\omega_{i+1} = \begin{cases} p, & \text{if } \omega_i = 1, \\ \omega_i - 1 \text{ or } p, & \text{if } 2 \leq \omega_i \leq p. \end{cases} \tag{3.3}$$

Observe that ϑ leads to the production system of p-generalized Fibonacci numbers (Ω_p). Figure 3.2 illustrates the generation of 3-generalized Fibonacci words, see the file *wordsgeneration.ipynb* in Github repository.

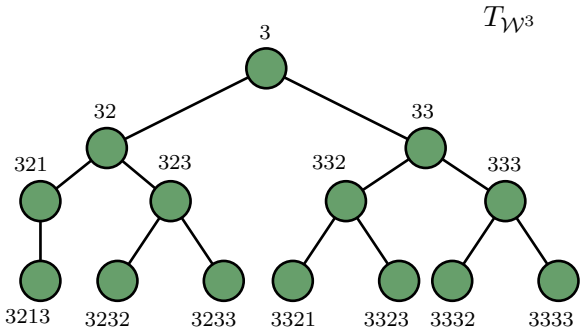


Figure 3.2: Initial levels of the planar tree associated to $\mathcal{W}^3$.

Theorem 35. *The p-generalized Fibonacci words of length n are in bijection with the binary words avoiding p consecutive 1s of length $n-1$.*

Proof.

- For a given generalized Fibonacci word $\omega = \omega_1\omega_2\cdots\omega_n$, we construct a binary string of length $n-1$ by examining the increases and decreases in ω. Reading the word from left to right, we assign 0 for increases or unchanged digits and 1 for decreases. For example, the word 32323 corresponds to the binary string 1010. This binary string avoids p consecutive occurrences of the digit 1, as each word has at most $p-1$ ascents. Furthermore, the string obtained is unique because the words are fully determined by the increases and decreases.

- Let α be a binary string of length $n-1$ that avoids p consecutive occurrences of the digit 1. Since each generalized Fibonacci word starts with an occurrence of the digit p, we construct the generalized Fibonacci word by appending the allowed descending digit at the right when there is a 0 and the allowed ascending or unchanged digit when there is a 1. For instance, the word 1011011 that avoids 3 consecutive occurrences of 1 generates the 3-generalized Fibonacci word of length 8 $\omega = 33233233$. By construction the word generated is a generalized Fibonacci word and is unique since it is fully determined by the increases and decreases once we set the first digit of ω to be the digit p.

□

Bargraphs

Each p-generalized Fibonacci word $\omega = \omega_1\cdots\omega_n$ induces a bargraph of the same length (with n columns) by taking the value of the digit ω_i as the number of cells in the bargraph at i-th column. Such family is said to be the class of p-generalized Fibonacci bargraphs. The Figure 3.3 shows some 3-generalized Fibonacci bargraphs, compare it with Figure 3.2. see for reference the file Bargraphsgeneration.ipynb in the repository.

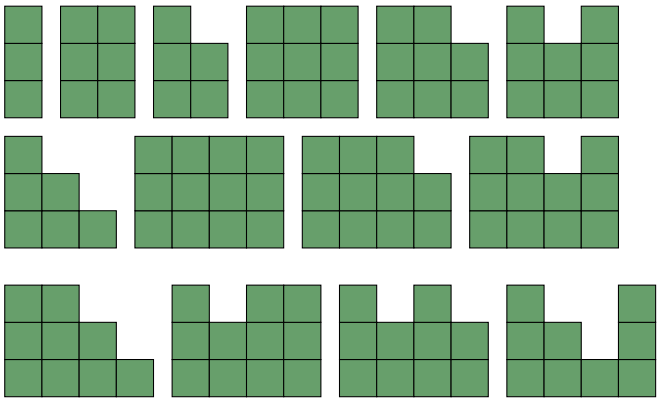


Figure 3.3: Bargraphs in $\mathcal{B}^p$.

By construction, p-generalized Fibonacci bargraphs are in bijection with p-generalized Fibonacci words, and consequently, enumerated by p-generalized Fibonacci numbers. It might seem that generalized Fibonacci bargraphs are just a reinterpretation of generalized Fibonacci words in the language of polyominoes. However, they allow the analysis of new statistics related to polyominoes, as we will explore in the next chapter.

Sets

Let $\mathcal{A}_n^p$ be the collection of subsets of $[n] \coloneqq \{1, \ldots, n\}$ without p consecutive numbers. As an illustrative example, the elements of $\mathcal{A}_4^3$ are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}.$$

Theorem 36. *The number of elements in $\mathcal{A}_n^p$ is $f_{p,n+2}$*

Proof. We establish a bijection between $\mathcal{A}_n^p$ and $\mathcal{S}_n^p$:

- Given a set A in $\mathcal{A}_n^p$, we obtain a binary word $\omega = \omega_1 \cdots \omega_n$ of length n by setting

$$\omega_i = \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

  Note that ω avoids p consecutive occurrences of the digit 1 since A avoids p consecutive numbers.

- Given a word $\omega = \omega_1\omega_2 \cdots \omega_n \in \mathcal{S}_n^p$, take the subset $A = \{i : w_i = 1, \text{ for } 1 \leq i \leq n\}$. It follows that both proceses are inverses of each other.

□

Tilings

For $n \geq 1$, let $\mathcal{R}_n^p$ be the set of tilings of the rectangle of size $1 \times n$ using tiles of size

$$1 \times 1, 1 \times 2, \ldots, 1 \times p.$$

Figure (3.4) exhibits such tilings for $p = 3$.

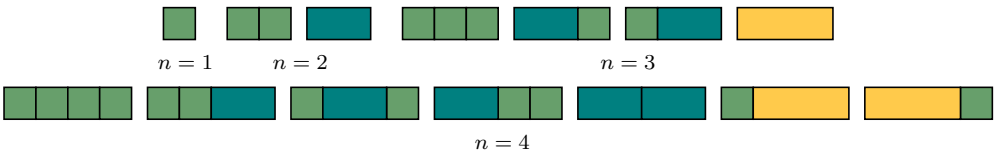


Figure 3.4: Tiling in $\mathcal{R}_n^3$ for $1 \leq n \leq 4$.

Theorem 37. *$|R_n|$ is given by the p-generalized Fibonacci number $F_{p,n}$.*

Proof. Let A_i be the subset of tilings in $\mathcal{R}_n$, $n > p$, ending in a tile of size $1 \times i$, for $i = 1, \ldots, p$. Each tiling $t \in A_i$ is formed by a tiling in R_{n-i} with the $1 \times i$ appended. Thus

$$|\mathcal{R}_n| = \sum_{i=1}^{p} |A_i| = \sum_{i=1}^{p} |R_{n-i}|.$$

Additionally, for a rectangle of size $1 \times n$, with $n = 1, \ldots, p$, there are $n - 1$ positions where a vertical bar, marking the possible contours of the tiles $1 \times 1, \ldots, 1 \times n$. Since each insertion or not of a vertical bar defines the tiling, there are 2^{n-1} tilings with the mentioned tiles. Since this family of tilings satisfies the p-generalized Fibonacci recursion and coincide with the $p - 1$ initial values, we conclude that $|R_n| = F_{p,n}$. □

We provide the bijection between $\mathcal{R}_n^p$ and the Fibonacci strings S_{n-1}^p: for each tiling A in $\mathcal{R}_n^p$, there are $n - 1$ positions in the interior of A with potential subdivisions (vertical lines). We construct a binary word traversing these positions from left to right and inserting 0 whenever there is a vertical line and 1, otherwise. Figure 3.5 displays the binary words obtained by this construction in the tilings of Figure 3.4.

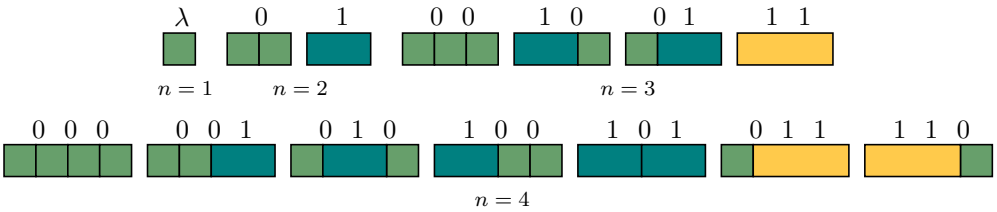


Figure 3.5: Binary words associated to 3-generalized Fibonacci tilings.

The resulting word avoids p consecutive occurrences of the digit 1 because there are only $p-1$ possible positions to subdivide A, and each word in S_{n-1}^{p} is generated through this process. Moreover, the positions of the digit 0 in the binary words uniquely determine the blocks of the tiling and their arrangement. Thus, this construction establishes a bijection.

Compositions of n having part smaller than $p+1$

Let M^p be the compositions of integers whose parts are less or equal than p. The number of such compositions of the integer $n-p$ is $F_{p,n}$. For instance, for $n=7, p=3$ those compositions are

$$4 = 1+1+1+1 = 1+3 = 3+1 = 2+1+1 = 1+2+1 = 1+1+2 = 2+2. \tag{3.4}$$

We represent each composition of n as a box of length $1\times n$ divided into sub-boxes whose length and order given by the parts of the composition. Figure 3.6 displays the box representation of the compositions of 4 with parts, while Figure 3.7 has the box representations of the compositions for the value $n=4$ displayed in (3.4).

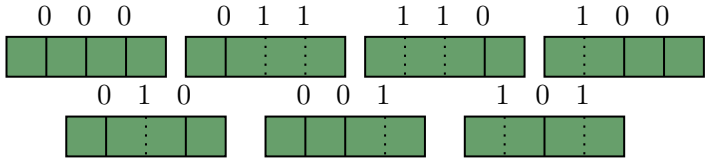


Figure 3.6: Binary words associated to compositions in $\mathcal{M}_4^3$.

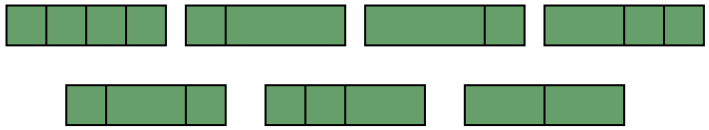


Figure 3.7: Representation into boxes of compositions of 4 with part 1,2 and 3.

Chapter 4

The Combinatorics of Generalized Fibonacci Bargraphs

In this chapter, we present new results concerning the enumeration of generalized Fibonacci bargraphs, obtaining closed formulas for the generating functions that count their distribution with respect to their area, semiperimeter, number of internal vertices, and the vertex degree.

4.1 A Decomposition of Generalized Fibonacci Bargraphs

The p-generalized Fibonacci bargraphs exhibit a systematic pattern in their initial and final columns. Let $\mathcal{B}^p$ denote the collection of all such bargraphs. This collection can be partitioned as $\mathcal{B}^p = \bigcup_{i=1}^{p} \mathcal{B}^{p,i}$, where $\mathcal{B}^{p,i}$ represents the subset of p-generalized Fibonacci bargraphs whose final column contains exactly i cells. Each bargraph in these subsets can be further decomposed as follows:

- For $1 \leq i \leq p-1$, the bargraphs in $\mathcal{B}^{p,i}$ begin with a column of p cells, followed by a p-generalized Fibonacci bargraph B[1], and conclude with two columns containing $i+1$ and i cells, respectively. Refer to Figure 4.1 for a visual representation of this decomposition.

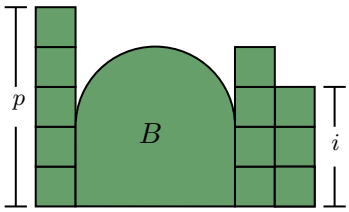


Figure 4.1: Decomposition of bargraphs in $\mathcal{B}^{p,i}$, $1 \leq i \leq p-1$.

[1]The structure of B depends on the value of i. However, it is not relevant for our enumeration purposes, and we will omit a precise description of it

- Bargraphs $\omega \in \mathcal{B}^{p,p}$ present several possibilities: ω either consists of a single column with p cells or begins with a column of p cells, followed by a p-generalized Fibonacci bargraph B, and concludes with columns with i and p cells, where $1 \leq i \leq p$. See Figure 4.2 for reference.

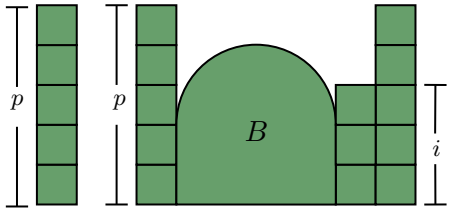


Figure 4.2: Structure of bargraphs in $\mathcal{B}^{p,p}$.

We leverage the interplay among the columns of bargraphs within the collections $\mathcal{B}^{p,i}$, to establish relations between the generating functions restricted to each $\mathcal{B}^{p,i}$. We are interested in finding expressions for multivariate generating functions of the form:

$$F_p(x; x_1, \ldots, x_k) \coloneqq \sum_{n \geq 0} x^n \sum_{\omega \in \mathcal{B}^p} x^{s_1(\omega)} x^{s_2(\omega)} \ldots x^{s_k(\omega)},$$

where $s_1(\omega), \ldots, s_k(\omega)$ are given statistics defined in $\mathcal{B}^p$. The coefficient $[x^n]F_p(x; x_1, \ldots, x_k)$ enumerates the number of p-generalized Fibonacci bargraphs with n columns, and the exponents of the variables $x_1, \ldots, x_k$ mark the distribution of the statistics $s_1, \ldots s_k$. The restriction of F to $\mathcal{B}^{p,i}$, for $1 \leq i \leq p$, is given by:

$$F_{p,i}(x; x_1, \ldots, x_k) \coloneqq \sum_{n \geq 0} x^n \sum_{\omega \in \mathcal{B}^{p,i}} x^{s_1(\omega)} x^{s_2(\omega)} \ldots x^{s_k(\omega)}.$$

Since $\mathcal{B}^p$ is partitioned into the sets $\mathcal{B}^{p,i}, 1 \leq i \leq p$, we obtain the following relation:

$$F_p(x; x_1, \ldots, x_k) = \sum_{i=1}^{p} F_{p,i}(x; x_1, \ldots, x_k).$$

4.2 Area and Semi-perimeter.

Let ω be a polyomino. The statistic *area*(ω) corresponds to the number cells comprising ω, while the statistic *sper*(ω) represents half of the edges in the contour outlining the polyomino. This definition arises from the observation that the perimeter of a polyomino is always even. Figure 4.3 illustrates a polyomino ω with area and semi-perimeter 9 (there are 18 edges in the countour of ω, colored red).

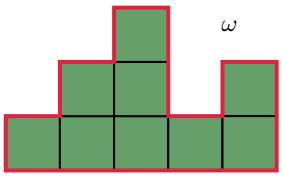


Figure 4.3: Area and semi-perimeter of ω.

We aim to find an expression for:

$$F_p(x;y,z) \coloneqq \sum_{n\geq 0} x^n \sum_{\omega\in\mathcal{B}^p} y^{\text{area}(\omega)} z^{\text{sper}(\omega)}, \tag{4.1}$$

which counts p-generalized Fibonacci bargraph with size n, area m, and semi-perimeter l through the coefficient $[x^n y^m z^l]F_p(x;y,z)$.

Example 38. Figure 4.4 displays the area and semiperimeter statistics of 3-generalized Fibonacci bargraphs up to size 3.

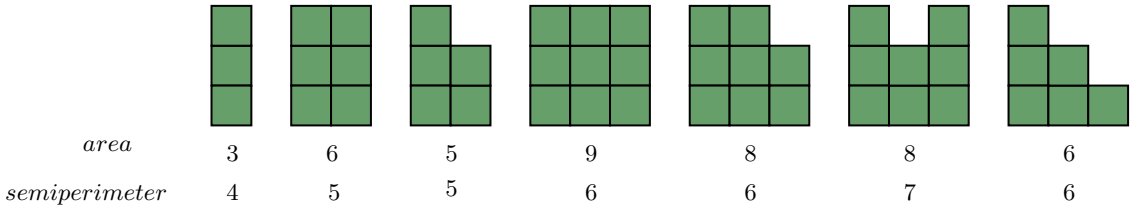


Figure 4.4: Area and semi-perimeter in 3-generalized Fibonacci bargraphs.

According to Figure 4.4, the multivariate generating function $F^3(x;y,z)$ has the initial terms:

$$F_3(x;y,z) = xy^3z^4 + x^2(y^6z^5 + y^5z^5) + x^3(y^9z^6 + y^8(z^6+z^7) + y^6z^6) + \mathcal{O}(x^4).$$

Theorem 39. *The multivariate generating function $F_p(x;y,z)$ is given by $\sum_{i=1}^{p} F_{p,i}(x;y,z)$, where $F_{p,i}(x;y,z)$, $1\leq i\leq p$, satisfies the system of equations:*

$$F_{p,i}(x;y,z) = xy^i z F_{p,i+1}(x;y,z), \quad 1\leq i\leq p-1,$$
$$F_{p,p}(x;y,z) = xy^p z^{p+1} + \sum_{i=1}^{p} xy^i z^{p-i+1} F_{p,i}(x;y,z).$$

Proof. We explore the relationship between the area and semiperimeter statistics and the local expansions that generate bargraphs in $\mathcal{B}^{p,i}$, for each $1\leq i\leq p$. These local expansions are based on the ECO construction rules for p-generalized Fibonacci words outlined in 3.3. Since adding an extra column with k cells to a bargraph increases its size by one unit and its area by k units, our analysis focuses primarily on the resulting changes to the semi-perimeter

statistic. This approach can be interpreted as backtracking the ECO expansions: rather than determining which bargraphs are generated from a given bargraph, we instead trace the origins of a given bargraph, identifying the expansions that led to its creation and the statistics alteration in this process.

- For $1 \leq i \leq p-1$, each bargraph in $\mathcal{B}^{p,i}$ is generated from a bargraph in $\mathcal{B}^{p,i+1}$ through a local expansion. Such expansion increases the semiperimeter of the original bargraph by one unit, reflecting the horizontal edges of the new column, exhibited in Figure 4.5.

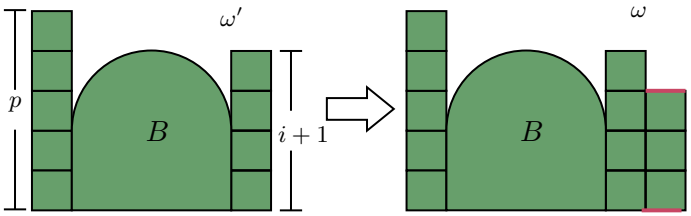


Figure 4.5: Backtracking of bargraphs in $\mathcal{B}^{p,i}$, for $1 \leq i \leq p-1$, showing changes to the semiperimeter.

These changes in the size, area and semiperimeter lead to the following identity between the generating functions:

$$F_{p,i}(x;y,z) = xy^i z F_{p,i+1}(x;y,z), \qquad 1 \leq i \leq p-1. \tag{4.2}$$

- For bargraphs ω in $\mathcal{B}^{p,p}$, we must consider the special case where ω consists of a single column with p cells. In all other cases, ω is generated from a bargraph $\omega' \in \mathcal{B}^{p,i}$, where $1 \leq i \leq p$. In each case, the expansion increases the semi-perimeter by $p-i+1$ units, reflecting the contribution of the horizontal edges of the appended column and the edges of the $p-i$ cells surpassing the penultimate column. These cases are illustrated in Figure 4.6.

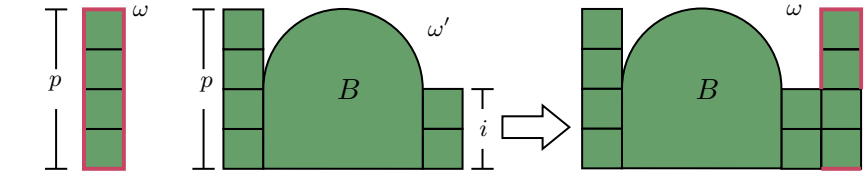


Figure 4.6: Backtracking of bargraphs in $\mathcal{B}^{p,p}$, showing changes to the semiperimeter.

As a result of these changes in the statistics, the following identity holds between the generating functions.

$$F_{p,p}(x;y,z) = xy^p z^{p+1} + \sum_{i=1}^{p} xy^i z^{p-i+1} F_{p,i}(x;y,z). \tag{4.3}$$

□

Theorem 40. *The generating function $F_p(x;y,z)$ is given by*

$$F_p(x;y,z) = \frac{xy^p z^{p+1}\left(\sum_{i=1}^{p-1} x^{p-i} y^{\frac{(p-i)(p+i-1)}{2}} z^{p-1}+1\right)}{1-\left(\sum_{i=1}^{p-1} x^{p-i+1} y^{\frac{p(p+1)-i(i-1)}{2}} z^{2(p-i)+1} + xy^p z\right)}.$$

Proof. Since $F_{p,i}(x;y,z) = xy^i z F_{p,i+1}(x;y,z)$ for $1 \le i \le p-1$, we can express each $F_{p,i}(x;y,z)$ in terms of $F_{p,p}(x;y,z)$ by iterating the recurrence relation:

$$\begin{aligned} F_{p,i}(x;y,z) &= (xy^i z)(xy^{i+1}z)\cdots(xy^{p-1}z)F_{p,p}(x;y,z) \\ &= x^{p-1} y^{\frac{(p-i)(p+i-1)}{2}} z^{p-1} F_{p,p}(x;y,z). \end{aligned}$$

Now substitute each $F_{p,i}$ into the equation for $F_{p,p}(x;y,z)$:

$$\begin{aligned} F_{p,p}(x;y,z) &= xy^p z^{p+1} + \sum_{i=1}^{p-1} xy^i z^{p-i+1} F_{p,i}(x;y,z) + xy^p z F_{p,p}(x;y,z) \\ &= xy^p z^{p+1} + \sum_{i=1}^{p-1} xy^i z^{p-i+1}\left(x^{p-1} y^{\frac{(p-i)(p+i-1)}{2}} z^{p-1} F_{p,p}(x;y,z)\right) + xy^p z F_{p,p}(x;y,z) \\ &= xy^p z^{p+1} + \sum_{i=1}^{p-1} x^{p-i+1} y^{\frac{p(p+1)-i(i-1)}{2}} z^{2(p-i)+1} F_{p,p}(x;y,z) + xy^p z F_{p,p}(x;y,z). \end{aligned}$$

To solve for $F_{p,p}(x;y,z)$, we rearrange:

$$F_{p,p}(x;y,z) = \frac{xy^p z^{p+1}}{1-\left(\sum_{i=1}^{p-1} x^{p-i+1} y^{\frac{p(p+1)-i(i-1)}{2}} z^{2(p-i)+1} + xy^p z\right)}.$$

Substituting this into the formula for $F_p(x;y,z)$:

$$\begin{aligned} F_p(x;y,z) &= \sum_{i=1}^{p} F_{p,i}(x;y,z) \\ &= F_{p,p}(x;y,z)\cdot\left(\sum_{i=1}^{p-1} x^{p-i}y^{\frac{(p-i)(p+i-1)}{2}}z^{p-i+1}\right) \\ &= \frac{xy^pz^{p+1}\left(\sum_{i=1}^{p-1} x^{p-i}y^{\frac{(p-i)(p+i-1)}{2}}z^{p-i}\right)}{1-\left(\sum_{i=1}^{p-1} x^{p-i+1}y^{\frac{p(p+1)-i(i-1)}{2}}z^{2(p-i)+1}+xy^pz\right)}. \end{aligned}$$

□

We have implemented a SageMath code to compute the generating function $F_p(x;y,z)$ and its series expansion for arbitrary values of p. The implementation is available in the file *areaandsemiperimeter.ipynb*, which can be found in the following repository: repository. This code allows for an efficient calculation of the generating functions and provides a detailed expansion of the series for various values of p.

In Table 4.1, we present the generating functions for $2 \leq p \leq 5$, along with the corresponding initial terms of their series expansions. This table offers valuable insight into the behavior of these functions as p varies, and it serves as a useful reference for understanding the underlying combinatorial structures.

p	$F_p(x;y,z)$	Series Expansion
2	$\frac{xy^2z^3(1+xyz)}{1-xy^2z-x^2y^3z^3}$	$xy^2z^3+x^2y^3(y+1)z^4+x^3y^4z^5(y^2+y+z)$ $+x^4y^5z^6(y^3+y^2+2yz+z)+\mathcal{O}(x^5)$
3	$\frac{xy^3z^4(1+xy^2z+x^2y^3z^2)}{1-xy^3z-x^2y^5z^3-x^3y^6z^5}$	$xy^3z^4+x^2y^5(y+1)z^5+x^3y^6z^6(y^3+y^2(z+1)+1)$ $+\mathcal{O}(x^4)$
4	$\frac{xy^4z^5(x^3y^6z^3+x^2y^5z^2+xy^3z+1)}{1-x^4y^{10}z^7-x^3y^9z^5-x^2y^7z^3-xy^4z}$	$y^4z^5x+x^2(y^8z^6+y^7z^6)$ $+x^3(y^{11}z^7+y^9z^7+(y^8z^2+y^7z^3)(y^4z^5))+\mathcal{O}(x^4)$
5	$\frac{xy^5z^6(x^4y^{10}z^4+x^3y^9z^3+x^2y^7z^2+xy^4z+1)}{1-x^5y^{15}z^9-x^4y^{14}z^7-x^3y^{12}z^5-x^2y^9z^3-xy^5z}$	$xy^5z^6+x^2(y^9z^7+y^{10}z^7)$ $+x^3(y^{12}z^8+y^{14}z^9+y^{14}z^8+y^{15}z^8)+\mathcal{O}(x^4)$

Table 4.1: Generating functions $F_p(x;y,z)$ for $2 \leq p \leq 5$.

Figure 4.7 displays Fibonacci bargraphs of sizes 1,2 and 3, and their distribution according to the number of columns, area and semiperimeter.

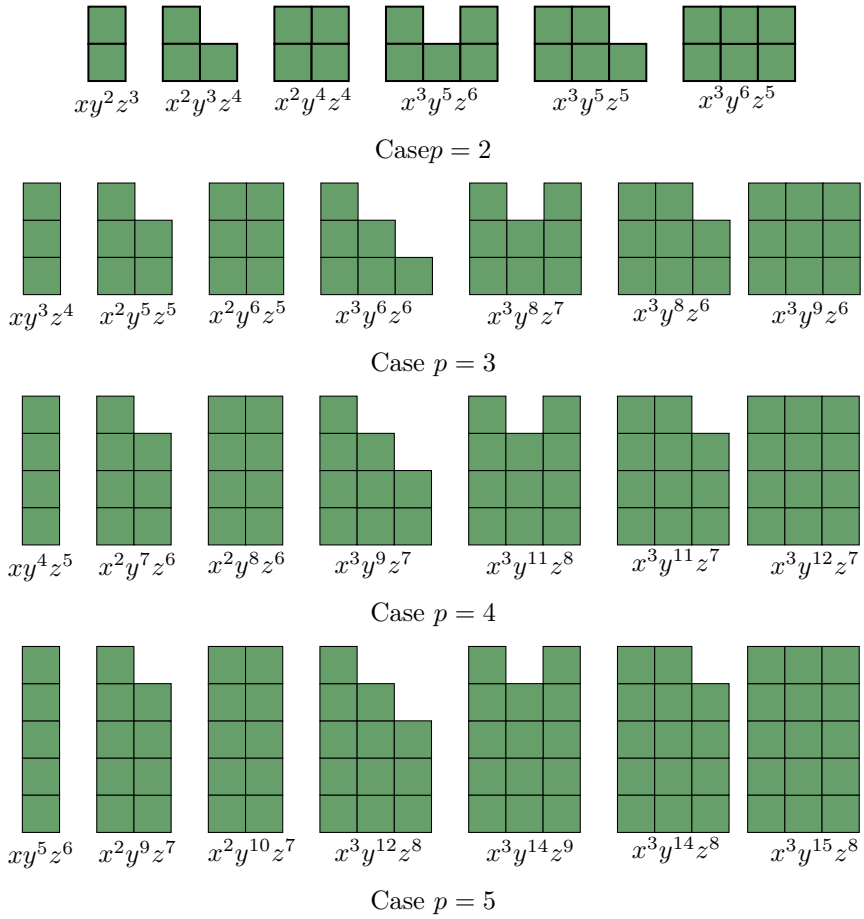


Figure 4.7: Bargraphs for cases $p = 2, 3, 4, 5$, showing their distributions based on the number of columns, area, and semiperimeter.

Notice that the multivariate generating function $F_p(x; y, z)$, allows us to obtain the generating functions with respect to the area and semiperimeter by taking the specializations $F_p(x; y, 1)$ and $F_p(x; 1, z)$.

Corollary 41. *The generating functions of p-generalized Fibonacci bargraphs with respect to the size and area, and size and perimeter are respectively:*

$$F_p(x; y, 1) = \frac{xy^p \left(\sum_{i=1}^{p-1} x^{p-i} y^{\frac{(p-i)(p+i-1)}{2}} \right)}{1 - \left(\sum_{i=1}^{p-1} x^{p-i+1} y^{\frac{p(p+1)-i(i-1)}{2}} + xy^p \right)}.$$

$$F_p(x; 1, z) = \frac{xz^{p+1} \left(\sum_{i=1}^{p-1} x^{p-i} z^{p-i} \right)}{1 - \left(\sum_{i=1}^{p-1} x^{p-i+1} z^{2(p-i)+1} + xz \right)}.$$

4.3 Number of Internal Vertices

An *internal vertex* of a polyomino is a vertex formed by the cells of the polyomino that do not lie on its boundary. The statistic $int(\omega)$ counts the number of internal vertices in the polyomino ω. Figure 4.8 illustrates the internal points in ω. Recent studies have focused on analyzing this statistic in various families of polyominoes. In particular, Baril and his collaborators have made significant contributions to understanding the internal vertices of Motzkin polyominoes, as detailed in [6]. Additionally, Toquica has explored this statistic, with a particular focus on bargraphs related to Catalan numbers, as presented in [30, 28].

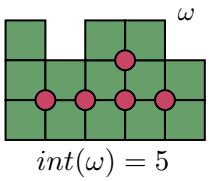


Figure 4.8: Internal vertices of ω.

We want to determine an expression for the multivariate generating function

$$G(x;q) \coloneqq \sum_{n\geq 0} x^n \sum_{\omega\in\mathcal{B}^p} q^{\mathrm{int}(\omega)}, \tag{4.4}$$

Similarly to the treatment of area and semiperimeter, we explore the multivariate generating functions associated with each set $\mathcal{B}^{p,i}$:

$$G_{p,i}(x;q) \coloneqq \sum_{n\geq 0} x^n \sum_{\omega\in\mathcal{B}^{p,i}} y^{\mathrm{int}(\omega)}, \quad 1 \leq i \leq p, \tag{4.5}$$

and analyze their interaction between the statistic of number of internal vertices and the ECO expansions for p-generalized Fibonacci bargraphs.

- The bargraphs $\omega \in \mathcal{B}^{p,i}$, for $1 \leq i \leq p-1$, are generated from a bargraph in $\mathcal{B}^{p,i+1}$. Each local expansion transforms the $i-1$ vertices, located between the first level and the i-th level of the last column of the original bargraph (located in the contour) into internal vertices in the expanded bargraph. For reference, see Figure 4.9.

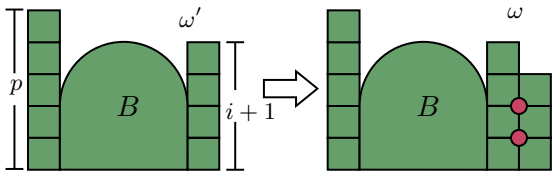


Figure 4.9: Backtracking of bargraphs in $\mathcal{B}^{p,i}$, for $1 \leq i \leq p-1$, showing changes to the number of internal vertices.

This relationship is expressed in terms of the generating functions by the equality:

$$G_{p,i}(x;q) = xq^{i-1}G_{p,i+1}(x;q), \quad 1 \leq i \leq p-1. \tag{4.6}$$

- The bargraphs in $\mathcal{B}^{p,p}$ consist either of a single column with p cells, which has no interior vertices, or are generated from an ECO expansion in a bargraph in $\mathcal{B}^{p,i}$, where $1 \leq i \leq p$. The cells of the appended column produce new internal vertices at levels $1, \ldots i-1$, as shown in Figure 4.10, we obtain the equation between generating functions:

$$G_{p,p}(x;q) = x + \sum_{i=1}^{p} xq^{i-1}G_{p,i}(x;q). \tag{4.7}$$

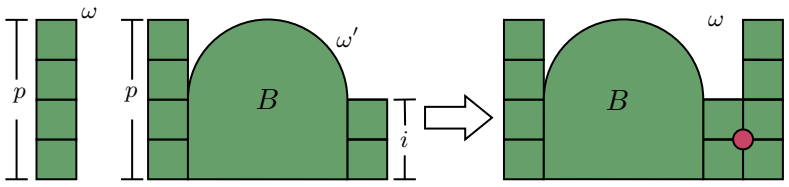


Figure 4.10: Backtracking of bargraphs in $\mathcal{B}^p$, showing changes to the number of internal vertices.

Theorem 42. *The generating function $G_p(x;q)$, given by $\sum_{i=1}^{p} G_{p,i}(x;q)$, where $G_{p,i}(x;q)$, for $1 \leq i \leq p$, satisfies the system of equations:*

$$G_{p,i}(x;q) = xq^{i-1}G_{p,i+1}(x;q), \quad 1 \leq i \leq p-1.$$
$$G_{p,p}(x;q) = x + \sum_{i=1}^{p} xq^{i-1}G_{p,i}(x;q).$$

Theorem 43. *The generating function that accounts for the distribution of p-generalized Fibonacci bargraphs according to size and the number of interior vertices is given by:*

$$G_p(x;q) = \frac{x\left(\sum_{i=1}^{p-1} x^{p-i} q^{\frac{(p-i)(p+i)-3(p-i)}{2}} + 1\right)}{1 - \left(\sum_{i=1}^{p-1} x^{p-i+1} q^{\frac{(p-i)(p+i)-3(p-i)+2(i-1)}{2}} + xq^{p-1}\right)}$$

Proof. We iterate the Equation (4.6) to express $G_{p,i}(x;q)$, for $1 \leq i \leq p-1$, as:

$$\begin{aligned} G_{p,i}(x;q) &= x^{p-i} q^{i+(i+1)+\cdots+(p-2)} G_{p,p}(x;q) \\ &= x^{p-i} q^{\frac{(p-i)(p+i)-3(p-i)}{2}} G_{p,p}(x;q). \end{aligned}$$

Hence, replacing in

$$G_{p,p}(x;q) = x + \sum_{i=1}^{p} xq^{i-1}G_{p,i}(x;q),$$

it follows that:

$$\begin{aligned} G_{p,p}(x;q) &= x + \sum_{i=1}^{p-1} xq^{i-1}x^{p-i}q^{\frac{(p-i)(p+i)-3(p-i)}{2}}G_{p,p}(x;q) + xq^{p-1}G_{p,p}(x;q) \\ &= x + \sum_{i=1}^{p-1} x^{p-i+1}q^{\frac{(p-i)(p+i)-3(p-i)+2(i-1)}{2}}G_{p,p}(x;q) + xq^{p-1}G_{p,p}(x;q). \end{aligned}$$

Thus,

$$G_{p,p}(x;q) = \frac{x}{1 - \left(\sum_{i=1}^{p-1} x^{p-i+1}q^{\frac{(p-i)(p+i)-3(p-i)+2(i-1)}{2}} + xq^{p-1}\right)}.$$

Now, since

$$\begin{aligned} G_p(x;q) &= \sum_{i=1}^{p} G_{p,i}(x;q) \\ &= \sum_{i=1}^{p-1} x^{p-i}q^{\frac{(p-i)(p+i)-3(p-i)}{2}}G_{p,p}(x;q) + G_{p,p}(x;q), \end{aligned}$$

we have:

$$G_p(x;q) = \frac{x\left(\sum_{i=1}^{p-1} x^{p-i}q^{\frac{(p-i)(p+i)-3(p-i)}{2}} + 1\right)}{1 - \left(\sum_{i=1}^{p-1} x^{p-i+1}q^{\frac{(p-i)(p+i)-3(p-i)+2(i-1)}{2}} + xq^{p-1}\right)}.$$

□

The expressions obtained for $G_p(x;q)$, with $2 \leq p \leq 5$ and their respective series expansions are listed in Table 4.2:

p	$G_p(x;q)$	Series Expansion
2	$\frac{x(1+x)}{1-qx-x^2}$	$x + (1+q)x^2 + (1+q+q^2)x^3 + (1+2q+q^2+q^3)x^4 + \mathcal{O}(x^5)$
3	$\frac{x(qx^2+qx+1)}{1-qx^3-q^2x^2-q^2x}$	$x + (q+q^2)x^2 + (q+q^2+q^3+q^4)x^3 + (q+2q^3+2q^4+q^5+q^6)x^4 + \mathcal{O}(x^5)$
4	$\frac{x(q^3x^3+q^3x^2+q^2x+1)}{1-q^3x^4-q^4x^3-q^4x^2-q^3}$	$x + q^2(q+1)x^2 + q^3(q^3+q^2+q+1)x^3 + q^4(q^5+q^4+2q^3+2q^2+2)x^4 + \mathcal{O}(x^5)$
5	$\frac{x(q^6x^4+q^6x^3+q^5x^2+q^3x+1)}{1-q^6x^5-q^7x^4-q^7x^3-q^6x^2-q^4x}$	$x + (q^3+q^4)x^2 + (q^5+q^6+q^7+q^8)x^3 + (q^6+q^7+2q^9+2q^{10}+q^{11}+q^{12})x^4 + \mathcal{O}(x^5)$

Table 4.2: Generating functions $G_p(x;y,z)$ for $2 \leq p \leq 5$.

Figure 4.11 displays Fibonacci bargraphs of sizes 1,2 and 3, and their distribution according to the number of columns and number of internal vertices.

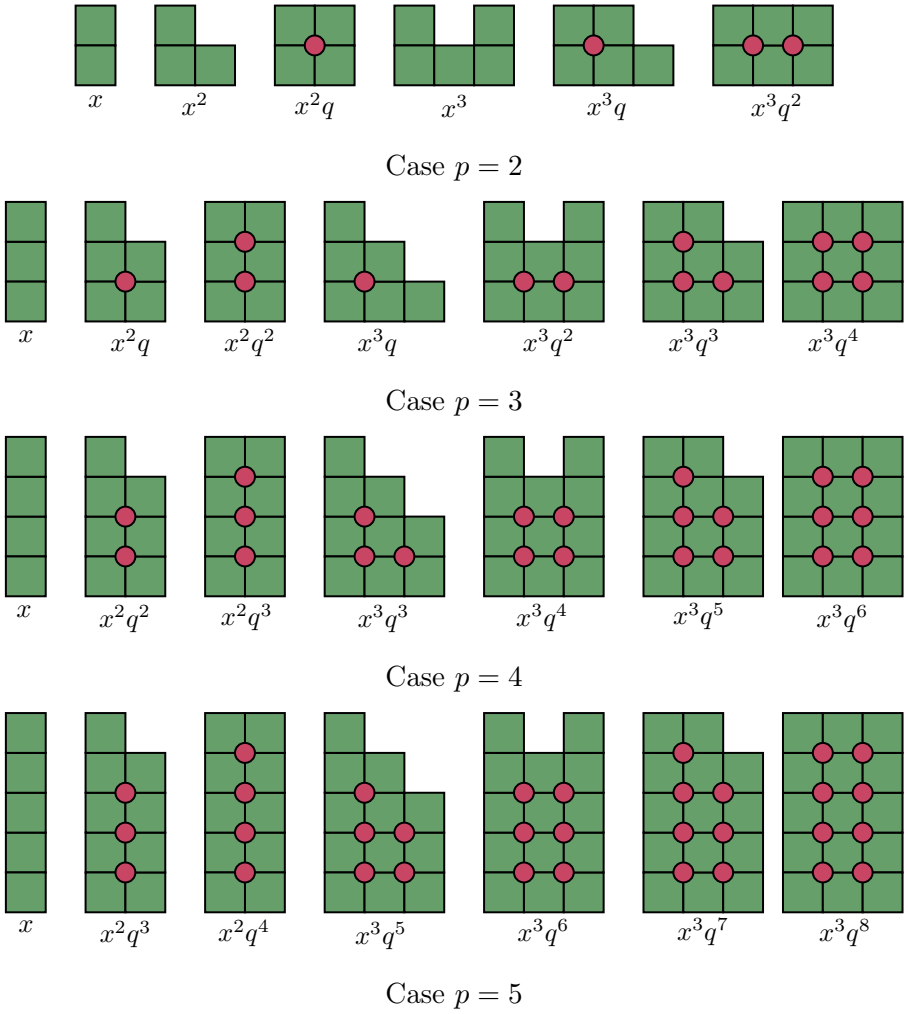


Figure 4.11: Bargraphs for cases $p = 1, 2, 3, 4$ showing their distributions based on the number of columns, and internal vertices.

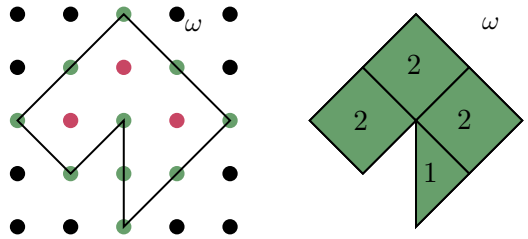
The area and the number of interior vertices in a generalized Fibonacci bargraph determine the number of *external vertices*. These are the vertices formed by the edges of the bargraph, with at least one of those edges lying on the contour. In other words, external vertices are all lattice points lying in the boundary of the bargraph.

Theorem 44 (Pick)**.** *Let P be a non-self-intersecting polygon without holes, with integer coordinates for its vertices. Then, the following relationship holds:*

$$area(P) = int(P) + \frac{ext(P)}{2} - 1,$$

where ext(P) represents the number of external vertices of P.

Example 45. Consider the polygon ω described in Figure (4.12). Pick's formula holds with int(ω)=3, ext(ω)=10 and area(ω)=7.



$int(\omega) = 3 \quad ext(\omega) = 10 \qquad area(\omega) = 7$

Figure 4.12: Interior, exterior vertices and area of the polygon ω.

4.4 Vertex Degree

In analogy to graphs, the grade of a vertex in a polyomino is the number of adjacent vertices connected to it. Let $grad_i(\omega)$, for $i = 2, 3, 4$ be the statistic that counts the number of vertices in ω having grade i. As an example, the polyomino depicted in Figure 4.13 comprises 11 (gray) vertices having grade 2, 2 (blue) vertices having grade 3, and 8 (red) vertices having grade 4.

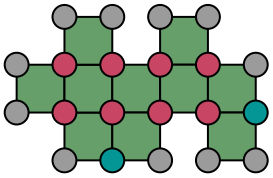


Figure 4.13: Vertices of a polyomino according to their grade

Our goal is to derive an expression for the generating function

$$H(x; r, s, t) = \sum_{n \geq 0} x^n \sum_{\omega \in \mathcal{B}^{p,i}} r^{grad_2(\omega)} s^{grad_3(\omega)} t^{grad_4(\omega)}$$

As in previous cases, we establish a system of equations for $H(x; r, s, t)$ by analyzing the behavior of the statistics under the ECO expansions (3.3). This translates into identities for the multivariate generating functions:

$$H_{p,i}(x; r, s, t) = \sum_{n \geq 0} x^n \sum_{\omega \in \mathcal{B}^{p,i}} r^{grad_2(\omega)} s^{grad_3(\omega)} t^{grad_4(\omega)}, \quad 1 \leq i \leq p$$

Theorem 46. *The multivariate generating function $H_p(x;r,s,t)$ is given by $\sum_{i=1}^{p} H_{p,i}(x;r,s,t)$, where $H_{p,i}(x;r,s,t), 1 \le i \le p$, satisfies the system of equations:*

$$H_{p,i}(x;r,s,t) = xrt^i H_{p,i+1}(x;r,s,t), \quad 1 \le i \le p-1.$$
$$H_{p,p}(x;r,s,t) = xr^4 s^{2(p-1)} + \sum_{i=1}^{p-1} xrs^{2(p-i)} t^i H_{p,i}(x;r,s,t) + xs^2 t^{p-1} H_{p,p}(x;r,s,t).$$

Proof. We identify the vertices of the bargraphs by its coordinates in the lattice, placing the bottom-left corner of the bargraph at the origin. Thus, the alteration of the grade of the vertices under ECO expansions is described as follows:

- The bargraphs in $\mathcal{B}_{n+1}^{p,i}$, for $1 \le i \le p-1$, are obtained through an ECO expansion from a bargraph ω' of in $\mathcal{B}_n^{p,i+1}$. The rightmost vertices of ω' are graded as follows: there are two vertices with grade 2, located at the lattice points $(n,0)$ and $(n,i+1)$. Additionally, there are i vertices with grade 3, located at coordinates (n,j), where $1 \le j \le i$. After appending the new column, all of these points (except for the one at level $i+1$) are connected to one of the vertices of the new column, increasing their grade by 1 unit. Moreover, the new vertices $(n+1,0)$ and $(n+1,i)$ have grade 2, while the vertices $(n+1,k)$, for $1 \le k \le i-1$, have grade 3. Refer to Figure (4.14) for an illustration. In summary, the vertices' grade transitions are as follows:

$$\mathrm{grad}_2(\omega') + 1 = \mathrm{grad}_2(\omega), \quad \mathrm{grad}_3(\omega') = \mathrm{grad}_3(\omega), \quad \text{and} \quad \mathrm{grad}_4(\omega') + i = \mathrm{grad}_4(\omega).$$

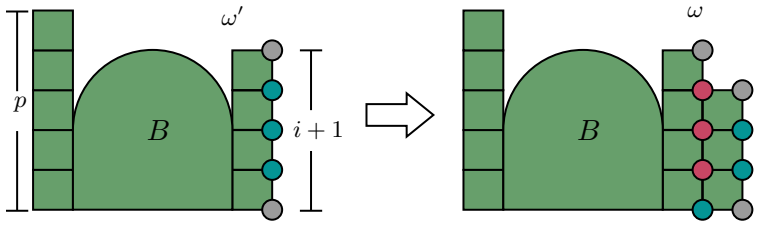


Figure 4.14: Backtracking of bargraphs in $\mathcal{B}^{p,i}$, for $1 \le i \le p-1$ showing changes to grade of the vertices.

Therefore, we obtain the following relation:

$$H_{p,i}(x;r,s,t) = xrt^i H_{p,i+1}(x;r,s,t), \quad 1 \le i \le p-1. \tag{4.8}$$

- The bargraphs $\omega \in \mathcal{B}_n^{p,p}$ can be described as follows: For $n=1$, ω is a single column with p cells. In this case, there are 4 vertices of grade 2 and $2(p-1)$ vertices of grade 3. For $n>1$, ω is obtained through an ECO expansion from a bargraph $\omega' \in \mathcal{B}_n^{p,i}$, where $1 \le i \le p$. See Figure (4.15) for reference.

1. Case: $1 \leq i \leq p-1$. In ω', there are 2 vertices of grade 2 located at $(n, 0)$ and (n, i), and $i-1$ vertices of grade 3 at coordinates (n, a) for $1 \leq a \leq i-1$. During the ECO expansion the vertices (n, j) connect to $(n+1, j)$ for $1 \leq j \leq i$ and the vertex (n, i) also connects to $(n, i+1)$. Furthermore, the newly added vertices $(n+1, 0)$, $(n+1, p)$, and (n, p) have grade 2, while the vertices $(n+1, k)$ for $1 \leq k \leq p-1$, and $(n, i+1), \ldots (n, p-1)$ have grade 3.

2. Case: $i = p$. In ω', the vertices are graded as follows: There are 2 vertices of grade 2 at $(n, 0)$ and (n, p) , and $p-1$ vertices of grade 3 at (n, a) for $1 \leq a \leq p-1$. During the ECO expansion, each vertex at (n, b), for $1 \leq b \leq p$ connects to the vertex $(n+1, b)$, increasing its grade by 1. The new vertices $(n+1, 0)$ and $(n+1, p)$ have grade 2, while the vertices $(n+1, a)$ for $1 \leq a \leq p-1$ have grade 3.

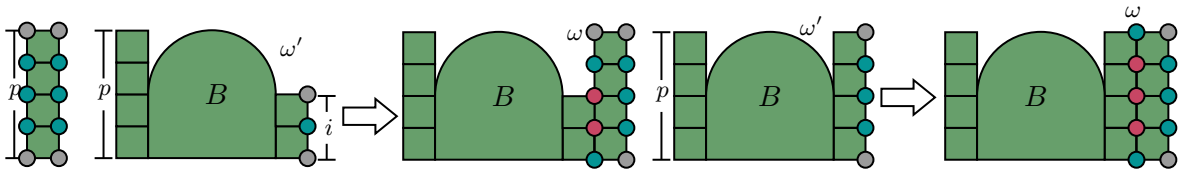


Figure 4.15: Backtracking of bargraphs in $\mathcal{B}^p$, showing changes to grade of the vertices.

In summary, the changes in the grade of the vertices under ECO expansions, when $\omega \in \mathcal{B}^{p,p}$ and $\omega' \in \mathcal{B}^{p,i}$ are expressed as:

$$\mathrm{grad}_2(\omega) = \begin{cases} \mathrm{grad}_2(\omega') + 1 & \text{for } 1 \leq i \leq p-1, \\ \mathrm{grad}_2(\omega') & \text{for } i = p, \end{cases}$$

$$\mathrm{grad}_3(\omega) = \begin{cases} \mathrm{grad}_3(\omega') + 2(p-i) & \text{for } 1 \leq i \leq p-1, \\ \mathrm{grad}_3(\omega') + 2 & \text{for } i = p, \end{cases}$$

and

$$\mathrm{grad}_4(\omega) = \begin{cases} \mathrm{grad}_4(\omega') + i & \text{for } 1 \leq i \leq p-1, \\ \mathrm{grad}_3(\omega') + p - 1 & \text{for } i = p. \end{cases}$$

Resulting in the identity between generating functions:

$$H_{p,p}(x; r, s, t) = xr^4 s^{2(p-1)} + \sum_{i=1}^{p-1} xrs^{2(p-i)} t^i H_{p,i}(x; r, s, t) + xs^2 t^{p-1} H_{p,p}(x; r, s, t).$$

□

Theorem 47. *The multivariate generating function $H_p(x;r,s,t)$ admits the closed formula:*

$$H_p(x;r,s,t) = \frac{xr^4s^{2(p-1)}\left(\sum_{i=1}^{p-1} x^{p-i}r^{p-i}t^{\frac{(p-i)(p+i-1)}{2}} + 1\right)}{1-\left(\sum_{i=1}^{p-1} x^{p-i+1}r^{p-i+1}s^{2(p-i)}t^{\frac{(p-i)(p+i)+3i-p}{2}} - xs^2t^{p-1}\right)}$$

Proof. By iterating in the formula for $H_{p,i}(x;r,s,t)$, we obtain for $1 \leq i \leq p-1$:

$$H_{p,i}(x;r,s,t) = x^{p-i}r^{p-i}t^{\frac{(p-i)(p+i)}{2}}H_{p,p}(x;r,s,t).$$

Thus, replacing in the formula for $H_{p,p}(x;r,s,t)$:

$$\begin{aligned}
H_{p,p}(x;r,s,t) &= xr^4s^{2(p-1)} + \sum_{i=1}^{p-1} xrs^{2(p-i)}t^i\left(x^{p-i}r^{p-i}t^{\frac{(p-i)(p+i-1)}{2}}H_{p,p}(x;r,s,t)\right) \\
&\quad + xs^2t^{p-1}H_{p,p}(x;r,s,t) \\
&= xr^4s^{2(p-1)} + \sum_{i=1}^{p-1} x^{p-i+1}r^{p-i+1}s^{2(p-i)}t^{\frac{(p-i)(p+i)+2i}{2}}H_{p,p}(x;r,s,t) \\
&\quad + xs^2t^{p-1}H_{p,p}(x;r,s,t).
\end{aligned}$$

Hence

$$H_{p,p}(x;r,s,t) = \frac{xr^4s^{2(p-1)}}{1-\left(\sum_{i=1}^{p-1} x^{p-i+1}r^{p-i+1}s^{2(p-i)}t^{\frac{(p-i)(p+i)+2i}{2}} + xs^2t^{p-1}\right)}.$$

Finally, we write the formula for $H_p(x;r,s,t)$ in terms of $H_{p,p}(x;r,s,t)$:

$$\begin{aligned}
H_p(x;r,s,t) &= \sum_{i=1}^{p} H_{p,i}(x;r,s,t) \\
&= H_{p,p}(x;r,s,t)\left(x^{p-i}r^{p-i}t^{\frac{(p-i)(p+i)}{2}} + 1\right) \\
&= \frac{xr^4s^{2(p-1)}\left(x^{p-i}r^{p-i}t^{\frac{(p-i)(p+i)}{2}} + 1\right)}{1-\left(\sum_{i=1}^{p-1} x^{p-i+1}r^{p-i+1}s^{2(p-i)}t^{\frac{(p-i)(p+i)+2i}{2}} + xs^2t^{p-1}\right)}.
\end{aligned}$$

□

Table 4.3 shows the multivariate generating functions $H_p(x;r,s,t)$ and their series expansion, for $2 \leq p \leq 5$. Figure 4.16 displays Fibonacci bargraphs of sizes 1,2 and 3, and their distribution according to the number of columns and grade of the vertices.

p	$H_p(x;r,s,t)$	Series Expansion
2	$\frac{r^4s^2x(1+rtx)}{1-r^2s^2t^2x^2-s^2tx}$	$r^4s^2x + (r^4s^4t + r^5s^4t^2)x^2 + \mathcal{O}(x^3)$
3	$\frac{r^4s^4x(1+rt^2x+r^2t^3x^2)}{1-r^3s^4t^4x^3-r^2s^2t^4x^2-s^2t^2x}$	$r^4s^4x + (r^4s^6t^2 + r^5s^2t^2)x^2 + \mathcal{O}(x^3)$
4	$\frac{r^4s^6x(r^2t^4x^2+rt^3x-rt^2x+1)(rt^2x+1)}{1-r^4s^6t^7x^4-r^3s^4t^7x^3-r^2s^2t^6x^2-s^2t^3x}$	$r^4s^6x + r^4s^6(rt^3 + s^2t^3)x^2 +$ $r^4s^6(r^2t^5 + rs^2t^6 + r^2s^2t^6 + s^4t^6)x^3 + \mathcal{O}(x^4)$
5	$\frac{r^4s^8x(r^4t^{10}x^4+r^3t^9x^3+r^2t^7x^2+rt^4x+1)}{1-r^5s^8t^{11}x^5-r^4s^6t^{11}x^4-r^3s^4t^{10}x^3-r^2s^2t^8x^2-s^2t^4x}$	$r^4s^8x + r^4s^8(rt^4 + s^2t^4)x^2 +$ $r^4s^8(r^2t^7 + rs^2t^8 + r^2s^2t^8 + s^4t^8)x^3 + \mathcal{O}(x^4)$

Table 4.3: Generating functions $H_p(x;r,s,t)$ for $2 \leq p \leq 5$.

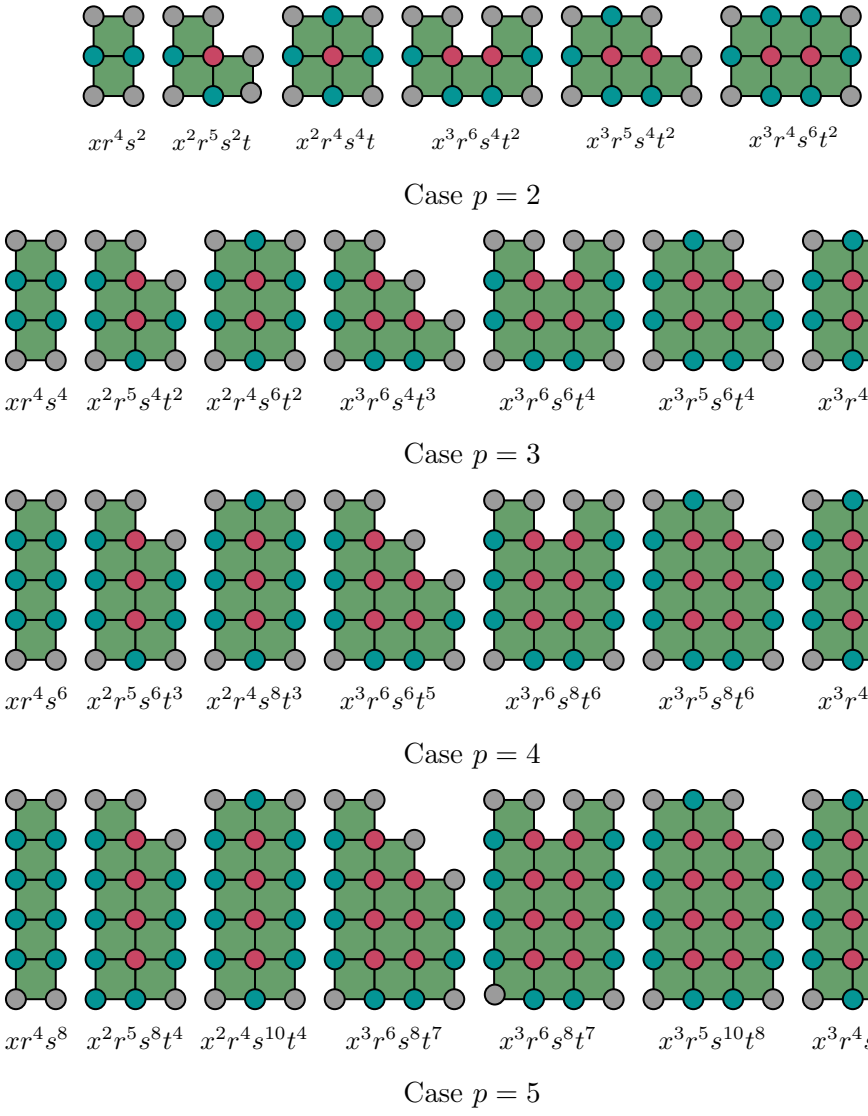


Figure 4.16: Bargraphs for cases $p = 1, 2, 3, 4$ showing their distributions based on the number of columns and vertex grades.

Chapter 5

Generalized Fibonacci Polytopes

In this chapter, we introduce a family of polytopes derived from generalized Fibonacci words, an examine their geometric and combinatorial properties. We include comprehensive SageMath documentation to support our exploration and computational analysis.

5.1 Polytopes

Polytopes are geometric objects that extend the concepts of polygons and polyhedra to higher-dimensional spaces. They are captivating not only for their intrinsic geometric and combinatorial properties but also for their remarkable ability to arise from diverse mathematical structures, such as graphs, posets, words, matroids, etc. This versatility underscores the profound interconnectedness of mathematics, as polytopes frequently act as a bridge between combinatorics, geometry, algebra, and optimization. For a deeper exploration of polytopes, we refer the reader to [7, 20].
In mathematics, it is common to understand a single concept from multiple perspectives. This is the case for polytopes, which can be interpreted both as bounded convex sets and as solution sets to specific systems of linear inequalities.

Definition 48. A set A is called *convex* if, for any two points $x, y \in A$, the line segment $s = \lambda x + (1 - \lambda)y$, where $\lambda \in [0, 1]$, lies entirely within A.

Example 49. In Figure 5.1, the set A is convex, while the set B is not.

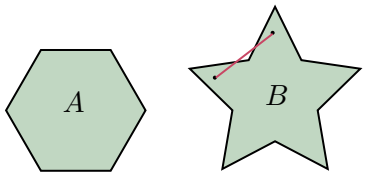


Figure 5.1: Convex and non-convex sets.

Definition 50. Let $x_1, \ldots, x_d \in \mathbb{R}^n$ be points in Euclidean space. The *convex hull* of these points, denoted by $conv(x_1, \ldots, x_d)$, is the smallest convex set that contains $x_1, \ldots, x_d$. In other words, if X is any convex set such that $x_1, \ldots, x_d \in X$, then $conv(x_1, \ldots, x_d) \subseteq X$. The convex hull can also by described by:

$$conv(x_1, \ldots, x_d) = \left\{ \sum_{i=1}^{d} \lambda_i x_i : \ \ \lambda_i \geq 0 \ \ and \ \ \sum_{i=1}^{d} \lambda_i = 1 \right\}$$

Definition 51. A set $P \subseteq \mathbb{R}^n$ is a *v-polytope* if it is the convex hull of a finite number of points $x_1, \ldots, x_d \in \mathbb{R}^n$.

Example 52. Figure 5.2 illustrates the regular hexagon with diameter 1 and center at the origin, in its v-polytope description, obtained as the convex hull of the points $(\pm 1, 0), (\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

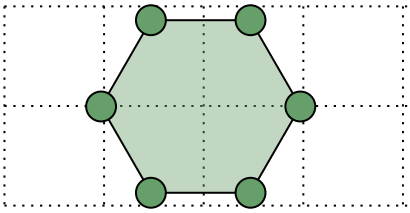


Figure 5.2: v-description of the hexagon.

While this approach emphasizes the geometric nature of polytopes, an alternative perspective arises from their algebraic characterization as solution sets of specific systems of linear inequalities.

Definition 53. A *half-space* in $\mathbb{R}^n$ is defined as the set of points that satisfy a linear inequality of the form

$$H^+ = \{x \in \mathbb{R}^n : a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b\},$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers, and not all of $a_1, a_2, \ldots, a_n$ are zero.

Definition 54. A *polyhedron* is the intersection of a finite number of half-spaces, each defined by a linear inequality. That is, a polyhedron P can be expressed as

$$P = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i\},$$

where each inequality defines a half-space in $\mathbb{R}^n$.

Definition 55. An *h-polytope* is a bounded polyhedron.

Example 56. The h-description of the hexagon 5.2 is given by the inequalities:

$$\sqrt{3}x + y \leq \sqrt{3}, -\sqrt{3}x + y \leq \sqrt{3}, \sqrt{3}x - y \leq \sqrt{3}, -\sqrt{3}x - y \leq \sqrt{3}, y \leq \frac{\sqrt{3}}{2}, -y \leq \frac{\sqrt{3}}{2}.$$

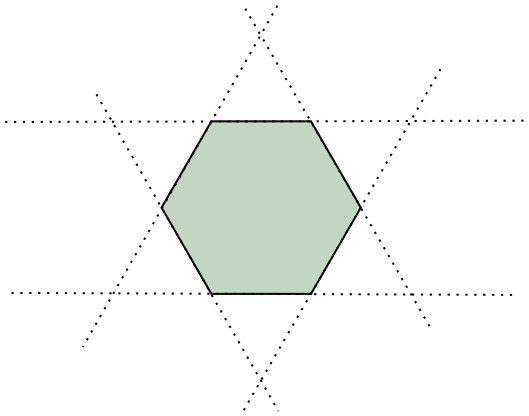


Figure 5.3: $h-$description of the hexagon.

A natural question to consider is whether every polytope defined as a convex hull can also be described as an intersection of hyperplanes and vice versa. This question, answered affirmatively, is fundamental in polytope theory. A proof is discussed in [35].

Theorem 57. *Every h-polytope is a v-polytope, and vice versa.*

Therefore, we refer to both v-polytopes (defined by vertices) and h-polytopes (defined by hyperplanes) simply as *polytopes*. We will introduce key geometric properties of polytopes that are essential for understanding their structure.

Definition 58. The *affine hull* of a set S is the minimal affine subspace (under inclusion), containing S. That is

$$\mathit{aff}(S) \coloneqq \left\{ \sum_{i=1}^{m} \alpha_i s_i \mid s_1, \ldots, s_m \in \mathbb{R}, \sum_{i=1}^{m} \alpha_i = 1. \right\}$$

Definition 59. The *dimension* of a polytope P in $\mathbb{R}^n$ is defined as the dimension of its affine hull, reflecting the number of independent directions in which the polytope extends.

Definition 60. A set $F \subset \mathbb{R}^n$ is a *face* of a polytope P in $\mathbb{R}^n$ if there exists a hyperplane

$$H = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i x_i = b\},$$

such that $F = P \cap H$ and P lies entirely within one of the closed half-spaces defined by H. In other words, F is a subset of P that is itself a polytope, and the intersection of P with a

hyperplane where the hyperplane is such that F does not extend into the exterior of P. Faces are categorized by their dimensions: 0-dimensional faces are called vertices, 1-dimensional faces are called edges, and the maximal faces (which are not the polytope itself) are called facets.

Example 61. Figure 5.4 shows supporting hyperplanes for a vertex and an edge in the hexagon.

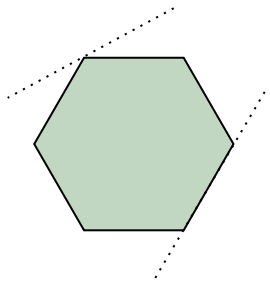


Figure 5.4: Faces of the hexagon.

Definition 62. The *f-vector* of a polytope P of dimension n is a vector that captures the number of faces of each dimension of P. Specifically, if P has f_i i-dimensional faces, then the f-vector is defined as

$$f(P) = (f_{-1}, f_0, f_1, \ldots, f_{n-1}, f_n).$$

The empty set is considered a face of dimension-1, while the polytope itself is considered an n-dimensional face, so $f_n = 1$.

Example 63. Let P be the regular hexagon. Then, the f-vector is given by $f(P) = (1, 6, 6, 1)$.

Definition 64. The *volume* of a polytope P is the euclidean volume of the region of space enclosed by P. A polytope whose vertices lie in the integer lattice $\mathbb{Z}^d$, for some $d > 0$, is said to be a *lattice polytope*. If P is a lattice polytope of dimension d, the *normalized volume* of P is defined by

$$NV(P) = \mathrm{Vol}(P) \cdot (n!)$$

The dimension, f-vector, and volume are fundamental in describing a polytope, but they do not fully capture its structure. For example, they do not reveal how the faces of the polytope interact with each other. The combinatorial information of a polytope, however, is independent of its geometric realization and provides an understanding of its intrinsic properties.

Definition 65. Let P be a polytope and $F(P)$ the set of all its faces. The set $F(P)$ is naturally endowed with a partially ordered set (POSET) structure under the inclusion relation $\subseteq$. This POSET, denoted by $(F(P), \subseteq)$, is known as the *face lattice* of P.

The face lattice of a polytope encodes the structure and relationships between the faces of the polytope independent of its geometric realization.

Definition 66. Two polytopes are said to be *combinatorially equivalent* if they face lattices are isomorphic.

Example 67. The square and the rhombus are combinatorially equivalent. Their face lattices are shown in Figure 5.5.

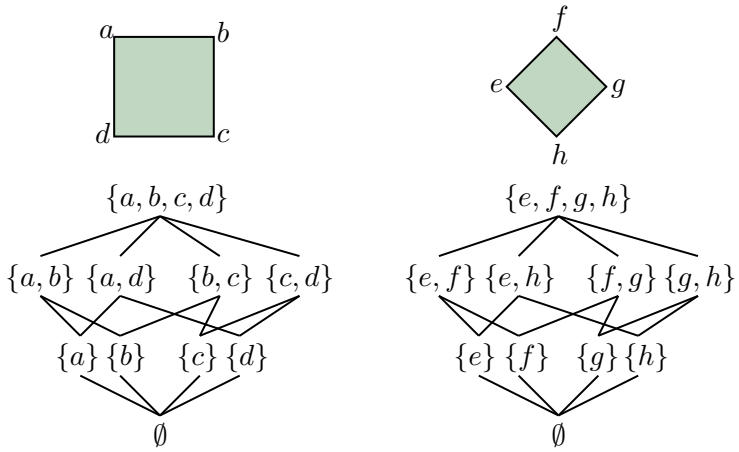


Figure 5.5: Face lattice of the square and rhombus.

For lattice polytopes, we can consider a stronger form of equivalence that preserves both the face structure and the volume.

Definition 68. Two polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ are said to be *integrally equivalent* if there exists an affine transformation $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that the restriction to P is a bijection between P and Q that preserves the lattice.

5.2 Generalized Fibonacci Polytopes

We utilize the vertex representation of polytopes to construct the family of p-generalized Fibonacci polytopes, which arise from p-generalized Fibonacci words. To the best of our knowledge, these polytopes have not been previously studied in the literature.
Let p and n be natural numbers and let $\mathcal{B}_{p,n}$ be the collection of p-generalized Fibonacci words of size n. For each word $\omega = \omega_1\omega_2 \dots \omega_n \in \mathcal{B}_{p,n}$, we associate the vector $v_\omega = (\omega_2, \dots, \omega_n) \in \mathbb{R}^{n-1}$. The set of such vectors is given by $V_{p,n} \coloneqq \{v_\omega : \omega \in \mathcal{B}_{p,n}\}$.

Definition 69. For $p, n \in \mathbb{N}$, the polytope $P_{p,n} \subset \mathbb{R}^n$ is defined as:

$$P_{p,n} \coloneqq \mathrm{Conv}(V_{p,n+1}),$$

where $V_{p,n+1}$ is the set of vectors associated with p-generalized Fibonacci words of size $n+1$. These polytopes are referred to as *p-generalized Fibonacci polytopes*.

Example 70. The polytopes $P_{p,1}$ are segments formed by the points p and $p-1$. Figure 5.6 illustrates the generalized Fibonacci polytopes $P_{p,2}$ for $p = 2, 3, 4$.

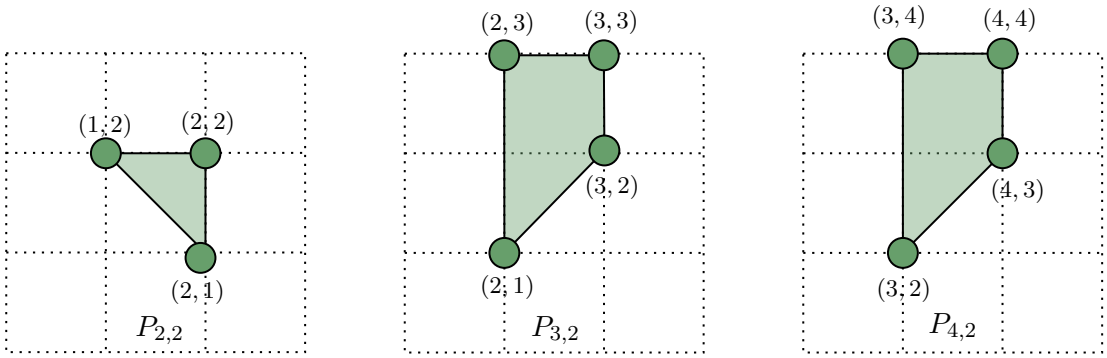


Figure 5.6: Generalized Fibonacci polytopes in the plane.

Figure 5.7 depicts the polytopes $P_{p,3}$ for $p = 2, 3, 4$

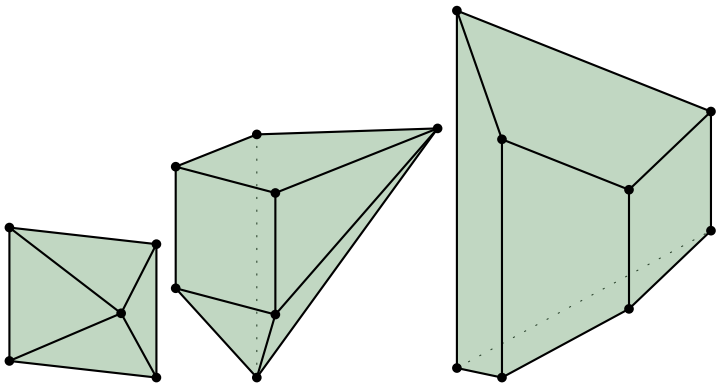


Figure 5.7: Generalized Fibonacci polytopes in the space.

See the file *Fibonaccipolytopes.ipynb* in the repository for the systematic generation of these objects.

Theorem 71. *Let $p \geq 2$ be a fixed number. Then, the polytopes $P_{i,p-1}$, for $i \geq p$, are integrally equivalent.*

Proof. Take distinct numbers $j, k \geq p$ with $j < k$. Consider the affine transformation

$$T_{j,k} : (x_1, x_2, \ldots, x_p) \mapsto \mathcal{I}_{p\times p}(x_1, \ldots, x_p) + (\underbrace{k-j, \ldots, k-j}_{p \text{ times}}).$$

The polytopes $P_{j,p-1}$ and $P_{k,p-1}$ are constructed from the generalized Fibonacci words $\mathcal{B}^j$ and $\mathcal{B}^k$, both of length p, respectively. Since $j, k \geq p$, no word can contain the subwords $1j$ or $1k$. Consequently, each of the sets $V_{j,p}$ and $V_{k,p}$ contains 2^{p-1} vectors. Moreover, if $x \in V_{j,p}$, then the vector

$$x + (\underbrace{k-j, \ldots, k-j}_{p \text{ times}}) \in V_{k,p}.$$

Therefore, when restricted to $P_{j,p-1}$, the transformation $T_{j,k}$ is a bijection. Additionally, since the entries of $T_{j,k}$ are integers, it maps lattice points to lattice points.

□

5.2.1 Volume

A powerful tool for computing the volume of lattice polytopes is Ehrhart theory, developed by the French mathematician Eugène Ehrhart in the 1960s. For further reference, see [7, 31].

Definition 72. Let P be a polytope and $t \in \mathbb{N}$. The dilation polytope tP is the polytope obtained by scaling P by a factor of t in each direction.

Example 73. Figure 5.8 ilustrates the dilations of a polytope in the plane.

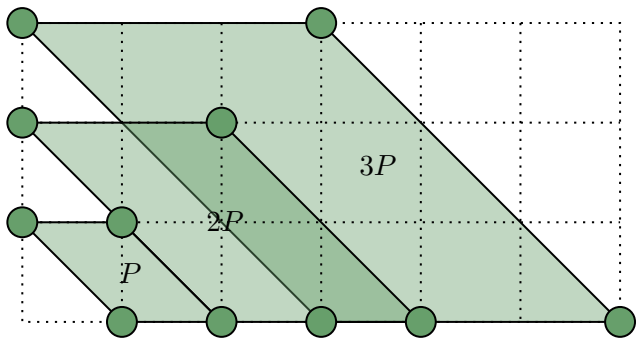


Figure 5.8: dilation polytopes.

Ehrhart theory studies properties of the polytope such as the volume, by counting the lattice points in its dilations. Let P be a d-dimensional polytope in $\mathbb{R}^n$ and let $L(P,t) \coloneqq |tP \cap \mathbb{Z}^n|$ be the number of lattice points in the dilated polytope tP, including those on the boundary.

Theorem 74 (Ehrhart, 1962)**.** *If P is a d-dimensional polytope with integer vertices, then there exist rational numbers $L_0, L_1, \ldots, L_d$ such that*

$$L(P,t) = L_d t^d + L_{d-1} t^{d-1} + \cdots + L_1 t + L_0, \tag{5.1}$$

This polynomial is called the Ehrhart polynomial associated to the polytope P. Moreover, the coefficient L_d corresponds to the volume of the d-dimensional polytope P, the coefficient L_{d-1} is half the sum of the $(d-1)-$ dimensional volume of the facets of P, and $L_0 = 1$.

Another interesting object is the *Ehrhart series*, defined as the generating function:

$$Ehr_P(z) \coloneqq 1 + \sum_{t \geq 1} L(P,t) z^t.$$

It allows to compactly manipulate the number of lattice points appearing in the dilations of the polytope.

Example 75. Consider the dilation tP of the polytope shown in Figure 5.8. This polytope has $t+1$ lattice points on each of the levels $y = 0, \ldots, p$. Therefore, the Ehrhart polynomial is

$$L(P,t) = (t+1)^2 = t^2 + 2t + 1.$$

Notice that P has a volume of 1 and 4 lattice points on its boundary. The Ehrhart series is given by:

$$Ehr_p(z) = 1 + \sum_{t \geq 1} (t+1)^2 z^t = \frac{1+z}{(1-z)^3}.$$

The characterization of the Ehrhart polynomial and the Ehrhart series for generalized Fibonacci polytopes is an open problem. In the following lines, we provide some partial results and conjectures according to our exploration in Sagemath, see for reference the file FibonaccipolytopesEhrhart.ipynb in the repository.

Theorem 76. *The Ehrhart polynomial for the generalized Fibonacci polytopes $P_{p,2}$ is*

$$L(P_{p,2},t) = \begin{cases} \frac{1}{2}t^2 + \frac{3}{2}t + 1, & \text{if} \quad p = 2 \\ \frac{3}{2}t^2 + \frac{5}{2}t + 1, & \text{if} \quad p > 2. \end{cases}$$

Proof. Observe that the dilation polytope $tP_{2,2}$ is a triangle with vertices at $(2t, 2t)$, $(2t, t)$, and $(t, 2t)$. This triangle contains $t + 1 - y$ lattice points at each level $y = t, t+1, \ldots, 2t$. Therefore, the number of lattice points in $tP_{2,2}$ is given by

$$L(P_{2,2},t) = \sum_{i=1}^{t+1} i = \frac{1}{2}(t+1)(t+2) = \frac{1}{2}t^2 + \frac{3}{2}t + 1.$$

For $p \geq 3$, the dilation polytope $tP_{p,2}$ is defined by the vertices (tp, tp), $(tp, t(p-1))$, $(t(p-1), tp)$, and $(t(p-1), t(p-2))$. This polytope consists of two parts: a square with vertices (tp, tp), $(tp, t(p-1))$, $(t(p-1), tp)$, and $(t(p-1), t(p-1))$, which has $t+1$ lattice points at each of the $t+1$ levels from $y = t(p-1)$ to $y = tp$; and a triangle with vertices $(t(p-1), t(p-2))$, $(t(p-1), t(p-1))$, and $(tp, t(p-1))$. Thus, the number of lattice points in $tP_{p,2}$ is

$$L(P_{p,2},t) = (t+1)^2 + \frac{t(t+1)}{2} = \frac{3}{2}t^2 + \frac{5}{2}t + 1.$$

□

Corollary 77. *The normalized volume of $P_{2,2}$ is 1 and the normalized volume of $P_{p,2}$, for $p \geq 3$ is 3.*

Corollary 78. *The Ehrhart series for $P_{p,2}$ is*

$$Ehr_{P_{p,2}}(z) = \begin{cases} \frac{1}{(1-z)^3}, & \text{if} \quad p = 2 \\ \frac{1+2z}{(1-z)^3}, & \text{if} \quad p > 2. \end{cases}$$

Proof. We start finding the Ehrhart series for $P_{p,2}$. Substituting $L_{P_{p,2}}(t)$:

$$Ehr_{P_{2,2}}(z) = \sum_{t=0}^{\infty} \left(\frac{1}{2}t^2 + \frac{3}{2}t + 1 \right) z^t.$$

Breaking into separate series:

$$Ehr_{P_{2,2}}(z) = \frac{1}{2} \sum_{t=0}^{\infty} t^2 z^t + \frac{3}{2} \sum_{t=0}^{\infty} t z^t + \sum_{t=0}^{\infty} z^t.$$

We use the geometric series in each part:

$$\sum_{t=0}^{\infty} z^t = \frac{1}{1-z}.$$

To find the series involving t, we use the derivative

$$\frac{1}{(1-z)^2} = \frac{d}{dz} \left(\frac{1}{1-z} \right) = \frac{d}{dt} \left(\sum_{t=0}^{\infty} z^t \right) = \sum_{t=0}^{\infty} t z^{t-1} = \frac{1}{z} \sum_{t=0}^{\infty} t z^t.$$

Therefore:

$$\sum_{t=0}^{\infty} t z^t = \frac{z}{(1-z)^2}.$$

For the series involving t^2, we use that:

$$\sum_{t=0}^{\infty} t^2 z^t = z \frac{d}{dz} \left(\frac{z}{(1-z)^2} \right) = \frac{z(z+1)}{(1-z)^3}$$

Therefore,

$$Ehr_{P_{2,2}}(z) = \frac{1}{2} \cdot \frac{z(z+1)}{(1-z)^3} + \frac{3}{2} \cdot \frac{z}{(1-z)^2} + \frac{1}{1-z}.$$

Simplifying, it follows that the Ehrhart series $Ehr_{P_{p,2}}(z)$ is:

$$Ehr_{P_{2,2}}(z) = \frac{1}{(1-z)^3}$$

A similar procedure is used to obtain that for $p > 2$:

$$Ehr_{P_{p,2}}(z) = \frac{1+2z}{(1-z)^3}$$

□

Finding a general characterization for the Ehrhart polynomial of p-generalized Fibonacci polytopes remains open. We conjecture that the volume of $P_{2,n}$ is given by the coefficients in the series expansion of $\sec(x) + \tan(x)$, which is related to the number of alternating permutations. See for reference the entry A000111 at OEIS. More specifically:

Conjecture 79. The volume of $P_{2,n}$ is given by

$$Vol(P_{2,n}) = [x^n] \left(\sec(x) + \tan(x)\right).$$

Sagemath provides robust tools for working with polytopes, which we leveraged to compute the Ehrhart polynomials of generalized Fibonacci polytopes for various values of p and n. Detailed computations can be found in the file *ehrhartpolynomials.ipynb*, available in the repository. Below, we present some of the polynomials obtained for different combinations of p and n:

n	Ehrhart Polynomial
1	$t+1$
2	$\frac{1}{2}(t+1)(t+2)$
3	$\frac{1}{3}(t+1)(t+\frac{3}{2})(t+2)$
4	$\frac{5}{24}(t+1)(t+2)(t^2+3t+\frac{12}{5})$
5	$\frac{2}{15}(t+1)(t+\frac{3}{2})(t+2)(t^2+3t+\frac{5}{2})$

Table 5.1: Ehrhart Polynomials for $p = 2$.

n	Ehrhart Polynomial
1	$t+1$
2	$\frac{3}{2}(t+\frac{2}{3})(t+1)$
3	$\frac{5}{3}(t+1)(t^2+\frac{11}{10}t+\frac{3}{5})$
4	$\frac{25}{12}(t+1)(t^3+\frac{41}{25}t^2+\frac{6}{5}t+\frac{12}{25})$
5	$\frac{53}{20}(t+1)(t^4+\frac{697}{318}t^3+\frac{683}{318}t^2+\frac{67}{53}t+\frac{20}{53})$

Table 5.2: Ehrhart Polynomials for $p = 3$.

n	Ehrhart Polynomial
1	$t+1$
2	$\frac{3}{2}(t+\frac{2}{3})(t+1)$
3	$\frac{8}{3}(t+\frac{1}{2})(t+\frac{3}{4})(t+1)$
4	$\frac{101}{24}(t+1)(t^3+\frac{153}{101}t^2+\frac{82}{101}t+\frac{24}{101})$
5	$\frac{104}{15}(t+1)(t^4+\frac{387}{208}t^3+\frac{18}{13}t^2+\frac{137}{208}t+\frac{15}{104})$

Table 5.3: Ehrhart Polynomials for $p = 4$.

n	Ehrhart Polynomial
1	$t+1$
2	$\frac{3}{2}(t+\frac{2}{3})(t+1)$
3	$\frac{8}{3}(t+\frac{1}{2})(t+\frac{3}{4})(t+1)$
4	$\frac{125}{24}(t+\frac{2}{5})(t+\frac{3}{5})(t+\frac{4}{5})(t+1)$
5	$\frac{49}{5}(t+1)(t^4+\frac{44}{21}t^3+\frac{31}{21}t^2+\frac{47}{98}t+\frac{5}{49})$

Table 5.4: Ehrhart Polynomials for $p = 5$.

n	Ehrhart Polynomial
1	$t+1$
2	$\frac{3}{2}(t+\frac{2}{3})(t+1)$
3	$\frac{8}{3}(t+\frac{1}{2})(t+\frac{3}{4})(t+1)$
4	$\frac{155}{24}(t+\frac{1}{6})(t+\frac{2}{6})(t+\frac{3}{6})(t+\frac{4}{6})(t+1)$
5	$\frac{47}{10}(t+1)(t^5+\frac{38}{19}t^4+\frac{24}{19}t^3+\frac{93}{76}t^2+\frac{17}{19}t+\frac{5}{47})$

Table 5.5: Ehrhart Polynomials for $p = 6$.

n	Ehrhart Polynomial
1	$t+1$
2	$\frac{3}{2}(t+\frac{2}{3})(t+1)$
3	$\frac{8}{3}(t+\frac{1}{2})(t+\frac{3}{4})(t+1)$
4	$\frac{265}{24}(t+\frac{1}{7})(t+\frac{2}{7})(t+\frac{3}{7})(t+\frac{4}{7})(t+\frac{5}{7})(t+1)$
5	$\frac{97}{10}(t+1)(t^5+\frac{56}{29}t^4+\frac{41}{29}t^3+\frac{31}{58}t^2+\frac{23}{29}t+\frac{10}{97})$

Table 5.6: Ehrhart Polynomials for $p = 7$.

We also use Sagemath to compute the volume of $P_{p,n}$ for different values of p and n. See for reference the file *Volumepolytopes.ipynb* in the repository.

n/p	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1
2	1/2	3/2	3/2	3/2	3/2	3/2	3/2
3	1/3	5/3	8/3	8/3	8/3	8/3	8/3
4	5/24	25/12	101/24	125/24	125/24	125/24	125/24
5	2/15	53/20	104/15	49/5	54/5	54/5	54/5
6	61/720	119/36	927/80	13517/720	16087/720	16807/720	16807/720
7	17/315	2089/504	6116/315	182989/5040	5893/126	16069/315	16384/315

Table 5.7: Volumes of $P_{p,n}$ for different values of p and n.

Conjecture 80. For any integer $n < p$ and $p \geq 3$, the volume of the polytope $P_{p,n}$ is given by the formula:

$$\mathrm{Vol}(P_{p,n}) = \frac{n^{n-1}}{n!}.$$

Conjecture 81. The volume of $P_{p,n}$ satisfies:

$$[x^n](\sec x + \tan x) \leq \mathit{Vol}(P_{p,n}) \leq \frac{n^{n-1}}{n!}.$$

5.3 Faces

Using SageMath, we generate and analyze the f-vector of $P_{p,n}$ across various values of p and n, see for reference the file *fvector.ipynb* in the repository. The results of these computations are

presented below, offering valuable insights into the structure and properties of these polytopes. The data obtained through these calculations not only supports our conjectures but also provides tools for the systematic study of the faces of the generalized Fibonacci polytopes.

n	f-vector
1	$(1, 2, 1)$
2	$(1, 3, 3, 1)$
3	$(1, 5, 8, 5, 1)$
4	$(1, 8, 18, 17, 7, 1)$
5	$(1, 13, 38, 48, 30, 9, 1)$

Table 5.8: f-vector for $p = 2$

n	f-vector
1	$(1, 2, 1)$
2	$(1, 4, 4, 1)$
3	$(1, 7, 12, 7, 1)$
4	$(1, 13, 32, 30, 11, 1)$
5	$(1, 24, 79, 104, 63, 16, 1)$

Table 5.9: f-vector for $p = 3$

n	f-vector
1	$(1, 2, 1)$
2	$(1, 4, 4, 1)$
3	$(1, 8, 12, 6, 1)$
4	$(1, 15, 34, 28, 9, 1)$
5	$(1, 29, 88, 105, 58, 14, 1)$

Table 5.10: f-vector for $p = 4$

n	f-vector
1	$(1, 2, 1)$
2	$(1, 4, 4, 1)$
3	$(1, 8, 12, 6, 1)$
4	$(1, 16, 32, 24, 8, 1)$
5	$(1, 31, 85, 90, 45, 11, 1)$

Table 5.11: f-vector for $p = 5$

n	f-vector
1	$(1, 2, 1)$
2	$(1, 4, 4, 1)$
3	$(1, 8, 12, 6, 1)$
4	$(1, 16, 32, 24, 8, 1)$
5	$(1, 32, 80, 80, 40, 10, 1)$

Table 5.12: f-vector for $p = 6$

n	f-vector
1	$(1, 2, 1)$
2	$(1, 4, 4, 1)$
3	$(1, 8, 12, 6, 1)$
4	$(1, 16, 32, 24, 8, 1)$
5	$(1, 32, 80, 80, 40, 10, 1)$

Table 5.13: f-vector for $p = 7$

Based on the information obtained about the f-vectors, we propose the following conjectures:

Conjecture 82. The number of vertices of the polytope $P_{p,n}$ is given by $F_{p,n+1}$. That is, the points corresponding to the words in $\mathcal{B}_{p,n+1}$ form a linearly independent set.

Conjecture 83. The number of i-dimensional faces of the polytope $P_{p,n}$ for $p \geq 3$ is given by:

$$f_i = \begin{cases} 1, & \text{if } i = -1 \text{ (corresponding to the empty face)}, \\ 2^{n-i}\binom{n}{i}, & \text{if } 0 \leq i \leq n. \end{cases}$$

The table obtained by varying p and n appears in the OEIS sequence A038207. This sequence also counts the number of faces of each dimension of the n-dimensional hypercube, suggesting a possible structural relationship between $P_{p,n}$ and hypercubes.

These conjectures highlight interesting combinatorial properties of $P_{p,n}$ and open the door for further exploration of its structure and geometric interpretation.

Chapter 6

Conclusion

6.1 Summary of Findings

In this thesis, we have explored the generalized Fibonacci numbers under the scope of the ECO theory and the combinatorial properties of objects enumerated by generalized Fibonacci numbers, focusing on words, bargraphs and polytopes. Our investigation led to several key findings:

- **Generating Functions for Bargraphs:** We successfully derived explicit formulas for generating functions related to the statistics of area, semiperimeter, number of internal points and the grade of the points in generalized Fibonacci bargraphs. These formulas offer new insights into the distribution and behavior of generalized Fibonacci bargraphs, providing a more nuanced understanding of their combinatorial properties.
- **Polytopes and Conjectures:** While our study of polytopes did not yield definitive theorems, we formulated several conjectures that suggest intriguing patterns and properties. These conjectures highlight the potential for deeper exploration and understanding of the geometric aspects of polytopes derived from generalized Fibonacci words.
- **Implementation in SageMath:** We developed and implemented codings in SageMath to generate and study the combinatorial objects related to our research. This implementation facilitated the exploration of various aspects of generalized Fibonacci words, bargraphs and polytopes, including their generation and enumeration.

6.2 Implications and Contributions

The findings of this thesis make several important contributions to the field of combinatorics:

- **Theoretical Contributions:** The derived generating functions for bargraphs enhance our understanding of their combinatorial statistics and provide a foundation for future

research. The conjectures about polytopes, while not yet proven, offer a starting point for exploring the complex relationships between combinatorial and geometric structures.

- **Connection to ECO Theory:** The integration of production systems and ECO theory with Fibonacci numbers contributes to the broader understanding of combinatorial structures and their generation processes. This approach provides a valuable perspective on how combinatorial objects are produced and enumerated.

6.3 Limitations

Several limitations were encountered during the research:

- **Challenges with Polytopes:** The conjectures regarding polytopes remain unproven, and further research is needed to address these open questions. The complexity of polytopes in higher dimensions presents significant challenges for deriving definitive results.
- **Scope of Generating Functions:** Although we derived formulas for various statistics of bargraphs, there are other statistics, see for reference [27], which require further exploration and different approaches for their enumeration.

6.4 Future Research Directions

Based on our findings, several avenues for future research are suggested:

- **Polytopes and Conjectures:** Future research should focus on proving or refining the conjectures about polytopes. New methodologies or theoretical approaches may be required to address these challenges and uncover deeper insights.
- **Expansion of Generating Functions:** Further investigation into additional statistics or types of bargraphs could reveal new properties and relationships. Exploring generating functions for other combinatorial objects may also yield valuable results.

6.5 Final Thoughts

This research journey has provided valuable insights into the combinatorial properties of Fibonacci numbers and their related structures. The process has been intellectually stimulating and has broadened my understanding of combinatorial research. I am grateful for the support and guidance received throughout this work.

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