$$\Omega s_{TT}^{\leftarrow} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge \mathbf{ff}_{2\_T} \wedge (\mathbf{f}_1 < 0) \wedge (\mathbf{f}_2 < 0)\} \quad (4.46)$$

The points of the 440, 444, 446 and 447 types belong to that classification.

For the following points, the sliding direction is not specified; instead, the individual direction of the vectors is specified. Some of these vectors have equal length and direction of components opposite to that of the DB, thereby resulting in a static point, where the evolution over the DB stops.

Point  ($\Omega s_{TT\rightarrow}^{\leftarrow}$)

$$\Omega s_{TT\rightarrow}^{\leftarrow} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge \mathbf{ff}_{2\_T} \wedge (\mathbf{f}_1 > 0) \wedge (\mathbf{f}_2 < 0)\} \quad (4.47)$$

The points of the 040, 044, 046 and 047 types belong to that classification.

Point  ($\Omega s_{TT\leftarrow}^{\rightarrow}$)

$$\Omega s_{TT\leftarrow}^{\rightarrow} \Rightarrow \{\mathbf{ff}_{1\_T} \wedge \mathbf{ff}_{2\_T} \wedge (\mathbf{f}_1 < 0) \wedge (\mathbf{f}_2 > 0)\} \quad (4.48)$$

The points of the 400, 404, 406, and 407 types belong to that classification.

Null-null point

If both vectors become null or vanish.

Point  (Ωs_{VV})

$$\Omega s_{VV} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge \mathbf{ff}_{2\_X}\} \quad (4.49)$$

The points of the 001, 041, 401, and 441 types belong to this classification.

Anti-collinear points

These are static points composed of vectors having opposite directions but equal length, which means that they are anti-collinear. These points are common in the swaps of sliding segments.

They are sub-divided into inclined and perpendicular vectors in relation with the tangent vector to the DB. Further, they are attractive and repulsive.

Some of this class of points are attractive and meets the condition required to be a sliding point; however, these points are not sliding, and hence, they have a different expression.

Point  ($\Omega_{ACS}/$)

$$\Omega_{ACS}/\Rightarrow \{\mathbf{AC} \wedge (\mathbf{f}_{1\_T} > 0)\} \tag{4.50}$$

The points of the 158 type belong to this classification.

Point  ($\Omega_{ACS}\backslash$)

$$\Omega_{ACS}\backslash\Rightarrow \{\mathbf{AC} \wedge (\mathbf{f}_{1\_T} < 0)\} \tag{4.51}$$

The points of the 378 type belong to this classification.

This class of points also meets the condition required to be a sliding point; however, these points are not sliding points. Further, these points are repulsive.

Point  ($\Omega_{ACU}/$)

$$\Omega_{ACU}/\Rightarrow \{\mathbf{AC} \wedge (\mathbf{f}_{1\_T} < 0)\} \tag{4.52}$$

The points of the 518 type belong to this classification.

Point  ($\Omega_{ACU}\backslash$)

$$\Omega_{ACU}\backslash\Rightarrow \{\mathbf{AC} \wedge (\mathbf{f}_{2\_T} < 0)\} \tag{4.53}$$

The points of the 738 type belong to this classification.

The anti-collinear perpendicular points are common in dynamics where a sliding segment changes its direction.

Point  ($\Omega_{ACS}|$)

$$\Omega_{ACS}|\Rightarrow \{\mathbf{AC} \wedge (\mathbf{f}_{1\_T} = 0) \wedge (\mathbf{f}_{1\_N} > 0)\} \tag{4.54}$$

The points of the 260 and 265 type belong to this classification.

Point  ($\Omega_{ACU}|$)

$$\Omega_{ACU} \mid\Rightarrow \{\mathbf{AC} \wedge (\mathbf{f}_{1\_T} = 0) \wedge (\mathbf{f}_{1\_N} < 0)\} \quad (4.55)$$

The points of the 620 and 625 type belong to this classification.

Other important points.

These points are important because they are present in the union of two different dynamics forming splitting curves.

Point  (Ω_{SVP})

$$\Omega_{SVP} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge \mathbf{ff}_{2\_N} \wedge (\mathbf{f}_2 < 0)\} \quad (4.56)$$

Point  (Ω_{UVP})

$$\Omega_{UVP} \Rightarrow \{\mathbf{ff}_{1\_X} \wedge \mathbf{ff}_{2\_N} \wedge (\mathbf{f}_2 > 0)\} \quad (4.57)$$

4.4 Identification of the different dynamics over the DB

Once all the individual points are classified, the process of finding the relationship among points located in the DB and the most significant dynamics of a system begins. We have the following hypothesis: if each type of point has been coded and the border between two regions is constituted by points that in turn have a relation with the dynamics of both vector fields, then the DB curve has unique chains related to each class of dynamics. These unique chains maintain the information of all the local dynamics in the sequence of its elements.

Classification of sequences of ternas of elements in the DB.

The most important dynamics within the DB are initially confirmed by the changes in the types of segments that constitute the DB. In planar systems, the DB is a curve; in three-dimensional systems, it is a surface; in *n*-dimensional systems, the DB is a hyper-surface of $(n - 1)$ dimensions.

When the DB boundary is tracked from one side to the other is found various types of dynamics. Each dynamics has their respective classified points. Next we will recognize six sets of dynamics. Each set constituted by two different dynamics.

The changes along the DB in the types of dynamics are (see also (82) (83):

1. Change in the crossing direction from Z_1 to Z_2 or from Z_2 to Z_1.
2. Start or end of an attractive sliding segment.
3. Start or end of a repulsive segment.
4. Change in the direction of an attractive sliding segment.
5. Change in the direction of a repulsive segment.
6. Change in the type of segment from attractive to repulsive or vice versa.

These dynamics show a common characteristic: they are constituted by two types of segments and between them, there is a point. The combination of segment-point-segment is unique for each set, and hence, it can be coded. In order to use the codification, the DB is tracked, and thus, the class of dynamics presented in the neighbourhood of the DB is detected.

4.4.1 Crossing flow

In a nonsmooth dynamical system, it is possible to find over the DB a crossing dynamics from the region Z_1 to the Z_2 and at the same time, next to the previous, other crossing dynamic from the region Z_2 to the Z_1. See figure 18. In the analysis of the DB, when a crossing segment is found next to another crossing segment of contrary direction, we say, we have a change of direction of a crossing dynamics. An analysis of the types of points found on the DB showed that in the position where the direction of the flow is changed, a special point is found. By listing the classes of points of the segments and the special point, we can obtain a sequence of the transition.

Note: since here, we will begin to use a graphics that takes three pictures over different positions of a point on the DB. These points have associated a draw of the vector $\mathbf{f}_1$ and $\mathbf{f}_2$. Also the draw of the the trajectory of the evolution of each vector is included. The difference of two vectors is achived using different colors and shapes; red- round and blue-square.

A possible process of the change in the direction of the crossing dynamics from Z_1 to Z_2 is as follows:

1. Left side of figure 18 with x = -1. Over the DB, on the left side of the point of change, both vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ point toward the region Z_1.
2. As we advance over the DB, to the right (without getting the point of change), both vector have a relatively small angle with the DB.
3. Center of figure, with x=0. At the point of the change, both vectors are parallel to the DB. At this point, the point exhibits double tangency. In this case, $\mathbf{f}_{1\_T} < 0$ and $\mathbf{f}_{2\_T} > 0$ but other cases are possible.

4. Right side of the figure, with x = 1. On the right of the point of change, the vectors have a rotated relative position by an angle superior to the tangency angle, and now, both vectors point to the region Z_2.

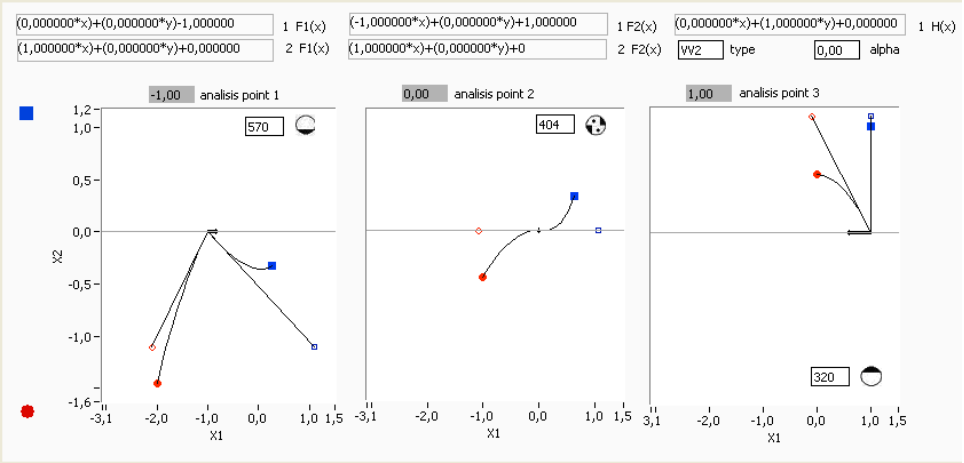


Figure 18. Different shots in the DB of a change in direction in a system with crossing dynamics.

The detection of this phenomenon in the DB is confirmed when the next sequence of points is found: 570/570////570/**404**/320////320. The duplication of the symbol /// means that are presenting many points of the same type. The diagram of the flow corresponding with this phenomenon will be coded in the next paragraphs as ${}^{S}\Phi_{CDC(2),}$.

The sequence of the changes in the crossing dynamics will be coded because there are other possibilities and the code is necessary to develop numerical tools to identify these phenomena.

Sequence ${}^{S}\Phi_{CDC(1)}$

The segments or the orbits are symbolized with the letter Φ with –S" indicating sequence. In this case, the symbol is accompanied with two numerical index indicating the region in which begins the action and the region that the orbit crosses. For example, Φc_{12} or Φc_{21}. See figure 19. In order to indicate that there has been a change in direction, the letters DC are added. The sequence is symbolized as

${}^{S}\Phi_{CDC(1)}$={ (Φc_{12}) ($\Omega s_{TT\rightarrow}{}^{\leftarrow}$) ($\Phi c_{21}$) } or {  } (4.58)

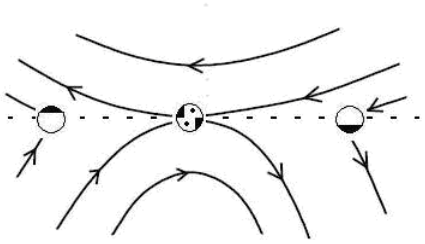


Figure 19. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(1)}$.

Note: the underline accompaning the symbol indicates that instead a point it is a segment.

Other dynamics related to the changes in the direction of crossing segments are as follows:

Sequence ${}^{S}\Phi_{CDC(2)}$

The figure 20 shows an example of the sequence type ${}^{S}\Phi_{SIF(2)}$. In this, in the left side of the DB, we have a crossing dynamics Φc_{21} from the region Z_2 to Z_1 . The point where the dynamics change is a point of double tangency with a contrary direction $\Omega s_{TT\leftarrow}^{\rightarrow}$. This point have the following vectors associated: a vector $\mathbf{f}_1$ tangential pointing to the left and a vector $\mathbf{f}_2$ tangential pointing to the right. Toward the right of the point of change, we have a crossing dynamics $\Phi c_{1,2}$ from Z_1 to Z_2 . The equilibrium points (focus) of each vector field are in their own region.

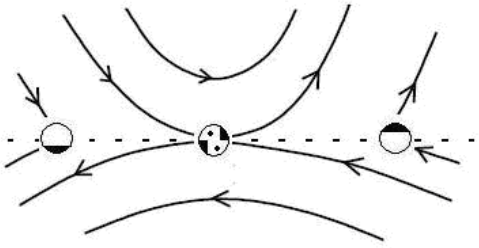


Figure 20. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(2)}$.

The sequence Φ is symbolized as

${}^{S}\Phi_{CDC(2)} = \{(\Phi c_{2,1})\ (\Omega s_{TT\leftarrow}^{\rightarrow})\ (\Phi c_{1,2})\}$ or $\{\underline{\ } \ \ \underline{\ }\}$ (4.59)

Sequence ${}^{S}\Phi_{CDC(3)}$

The figure 21 shows an example of the sequence type ${}^{S}\Phi_{SIF(3)}$. In this, in the left side of the DB, we have a crossing dynamics $\Phi c_{1,2}$ from the region Z_1 to Z_2. The point where the dynamics change is a point of a double tangency of a contrary signal $\Omega s_{TT\leftarrow}^{\rightarrow}$. This point have the following vectors associated: a vector $\mathbf{f}_1$ tangential pointing to the left and a vector $\mathbf{f}_2$ tangential pointing to the right. Toward the right of the point of change, we have a crossing dynamics $\Phi c_{2,1}$ from Z_2 to Z_1 . The equilibrium points (focus) of each vector field are in the contrary region.

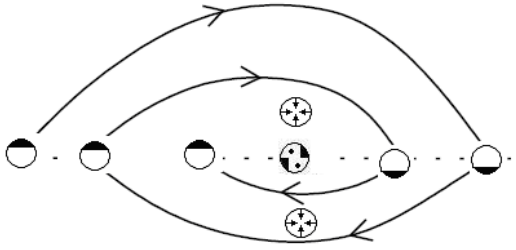


Figure 21. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(3)}$.

The sequence Φ is symbolized as

$${}^{S}\Phi_{CDC(3)} = \{(\Phi c_{1,2})\ (\Omega s_{TT\leftarrow}^{\rightarrow})\ (\Phi c_{2,1})\} \text{ or } \{\underline{\quad}\ \underline{\quad}\} \qquad (4.60)$$

Sequence ${}^{S}\Phi_{CDC(4)}$

The figure 22 shows an example of the sequence type ${}^{S}\Phi_{SIF(4)}$. In this, in the left side of the DB, we have a crossing dynamics $\Phi c_{2,1}$ from the region Z_2 to Z_1. The point where the dynamics change is a point of a double tangency of a contrary signal $\Omega s_{TT\rightarrow}^{\leftarrow}$. This point have the following vectors associated: a vector $\mathbf{f}_1$ tangential pointing to the right and a vector $\mathbf{f}_2$ tangential pointing to the left. Toward the right of the point of change, we have a crossing dynamics $\Phi c_{1,2}$ from Z_1 to Z_2 . The equilibrium points (focus) of each vector field are in the contrary region.

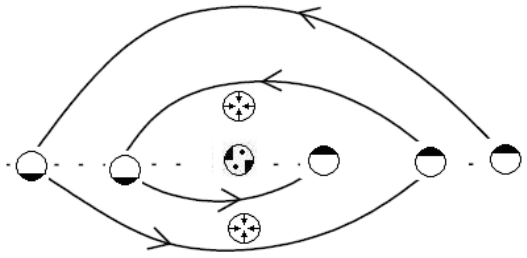


Figure 22. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(4)}$.

The sequence Φ is symbolized as

${}^{S}\Phi_{CDC(4)} = \{(\Phi c_{2,1})\ (\Omega s_{TT\rightarrow}^{\leftarrow})\ (\Phi c_{1,2})\}$ or $\{$  $\}$ (4.61)

Sequence ${}^{S}\Phi_{CDC(5)}$

The figure 23 shows an example of the sequence type ${}^{S}\Phi_{SIF(5)}$. In this, in the left side of the DB, we have a crossing dynamics $\Phi c_{1,2}$ from the region Z_1 to Z_2. The point where the dynamics change is a point of a double tangency pointing to the right $\Omega s_{TT\rightarrow}$. This point have the following vectors associated: a vector $\mathbf{f}_1$ tangential pointing to the right and a vector $\mathbf{f}_2$ tangential pointing to the rigth. Toward the right of the point of change, we have a crossing dynamics $\Phi c_{2,1}$ from Z_2 to Z_1. The equilibrium points (focus) of the vector field $\mathbf{f}_1$ is the region Z_1 and the focus of the vector field $\mathbf{f}_2$ is in the region Z_1.

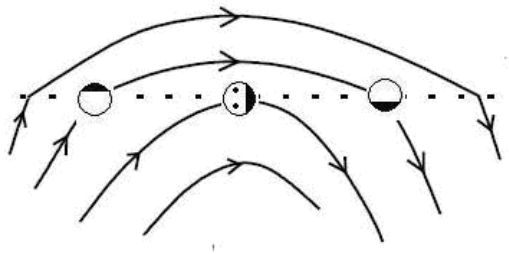


Figure 23. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(5)}$.

The sequence Φ is symbolized as

${}^{S}\Phi_{CDC(5)} = \{(\Phi c_{1,2})\ (\Omega s_{TT\rightarrow})\ (\Phi c_{2,1})\}$ or $\{$  $\}$ (4.62)

Sequence ${}^{S}\Phi_{CDC(6)}$

The figure 24 shows an example of the sequence type ${}^{S}\Phi_{SIF(6)}$. In this, in the left side of the DB, we have a crossing dynamics $\Phi c_{1,2}$ from the region Z_1 to Z_2. The point where the

dynamics change is a point of a double tangency pointing to the left $\Omega s_{TT\rightarrow}^{\rightarrow}$. This point have the following vectors associated: a vector $\mathbf{f}_1$ tangential and a vector $\mathbf{f}_2$ tangential. Both vector point to the left. Toward the right of the point of change, we have a crossing dynamics $\Phi c_{2,1}$ from Z_2 to Z_1. The equilibrium points (focus) of the vector field $\mathbf{f}_1$ is the region Z_2 and the focus of the vector field $\mathbf{f}_2$ is in the region Z_2.

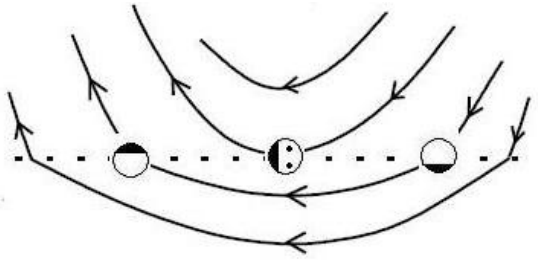


Figure 24. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(6)}$.

The sequence Φ is symbolized as

$${}^{S}\Phi_{CDC(6)}=\{(\Phi c_{1,2})\ (\Omega s_{TT\leftarrow})\ (\Phi c_{21})\}\ \text{or}\ \{\underline{\ } \ \ \underline{\ }\} \qquad (4.63)$$

Sequence ${}^{S}\Phi_{CDC(7)}$

The figure 25 shows an example of the sequence type ${}^{S}\Phi_{SIF(7)}$. In this, crossing dynamics from Z_2 to Z_1 is observed on the left side of the point of change. At the point of change, a double tangency pointing to the right is found. On the right of the point of change, crossing dynamics from Z_1 to Z_2 is observed. The equilibrium points (focus) of the vector fields are in the region Z_2.

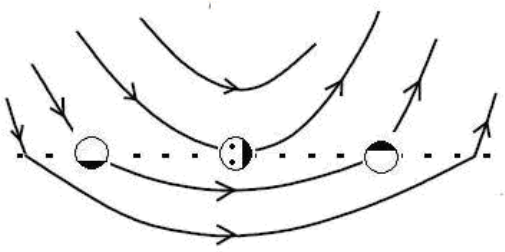


Figure 25. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(7)}$.

The sequence Φ is symbolized as

$^{S}\Phi_{CDC(7)}=\{(\Phi c_{2,1})\ (\Omega s_{TT\rightarrow})\ (\Phi c_{1,2})\}$ or $\{$  $\}$ (4.64)

Sequence $^{S}\Phi_{CDC(8)}$

The figure 26 shows an example of the sequence type $^{S}\Phi_{SIF(8)}$. In this, crossing dynamics from Z_2 to Z_1 is observed on the side of the point of change. At the point of change, a double tangency pointing to the left is found. On the right side of the point of change, crossing dynamics from Z_1 to Z_2 is observed. The equilibrium points (focus) of the vector fields are in the region Z_1.

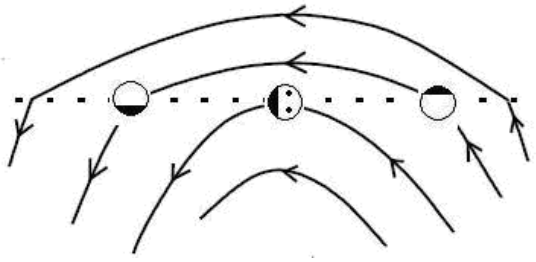
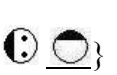


Figure 26. Flow of a system with change in the direction of a crossing-dynamics-type $^{S}\Phi_{CDC(8)}$.

The sequence Φ is symbolized as

$^{S}\Phi_{CDC(8)}=\{(\Phi c_{2,1})\ (\Omega s_{TT\leftarrow})\ (\Phi c_{1,2})\}$ o $\{$  $\}$ (4.65)

Sequence $^{S}\Phi_{CDC(9)}$

The figure 27 shows an example of the sequence type $^{S}\Phi_{SIF(9)}$. In this, crossing dynamics from Z_1 to Z_2 is observed on the left of the point of change. At the point of change, a double vanished vector is found. On the right of the point of change, crossing dynamics from Z_2 to Z_1 is observed. The equilibrium points (focus) of both vector fields are found on the DB.

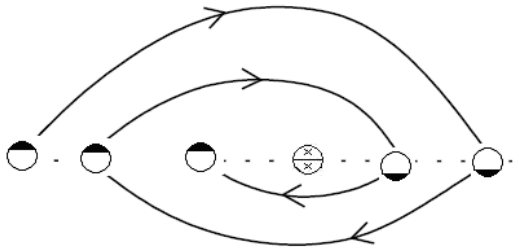


Figure 27. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(9)}$.

The sequence Φ is symbolized as

${}^{S}\Phi_{CDC(9)}=\{(\Phi c_{1,2})\ (\Omega s_{VV})\ (\Phi c_{2,1})\}$ o $\{$  $\}$ (4.66)

Sequence ${}^{S}\Phi_{CDC(10)}$

The figure 28 shows an example of the sequence type ${}^{S}\Phi_{SIF(10)}$. In this, crossing dynamics from Z_1 to Z_2  is observed on the left of the point of change. At the point of change, a double vanished vector  is found. On the right of the point of change, crossing dynamics from Z_2 to Z_1  is observed. The equilibrium points (focus) of both vector fields are found on the DB.

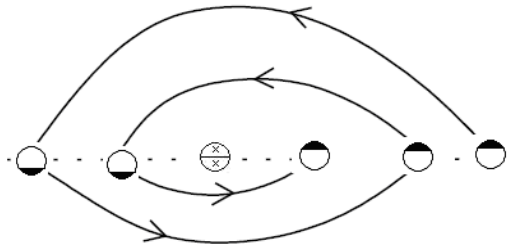
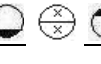


Figure 28. Flow of a system with change in the direction of a crossing-dynamics-type ${}^{S}\Phi_{CDC(10)}$.

The sequence Φ is symbolized as

${}^{S}\Phi_{CDC(10)} = \{(\Phi c_{2,1})\ (\Omega s_{VV})\ (\Phi c_{1,2})\}$ or $\{$  $\}$ (4.67)

4.4.2 Start and end of attractive sliding

In a nonsmooth dynamical system, it is possible to find over the DB a crossing dynamics and at the same time, next to the previous, an attractive sliding dynamics. In the analysis of the DB, when a crossing segment is found next to an attractive sliding segment, we say, we have the start or the end of a sliding segment. The direction adopted over the DB determines whether it is the start or the end of the segment.

A possible process for identifying the start or the end of an attractive sliding segment presents the following shots:

1. Left side of figure 29 with x = -1.5. Over the DB, on the left side of the point of change, both vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ point toward the region Z_1.
2. As it advances over the DB, to the right, near the point of change, the vector $\mathbf{f}_1$ has a rotated relative position in the CCW direction.
3. Center of figure, with x=-1,0. At the point of the initialization of the sliding segment, the vector $\mathbf{f}_1$ is tangent to the DB, with direction to the left.
4. Right side of the figure, with x = -0,50. On the right side of the point of initialization of the sliding segment, the vector $\mathbf{f}_1$, now points to the opposite region, the Z_2.

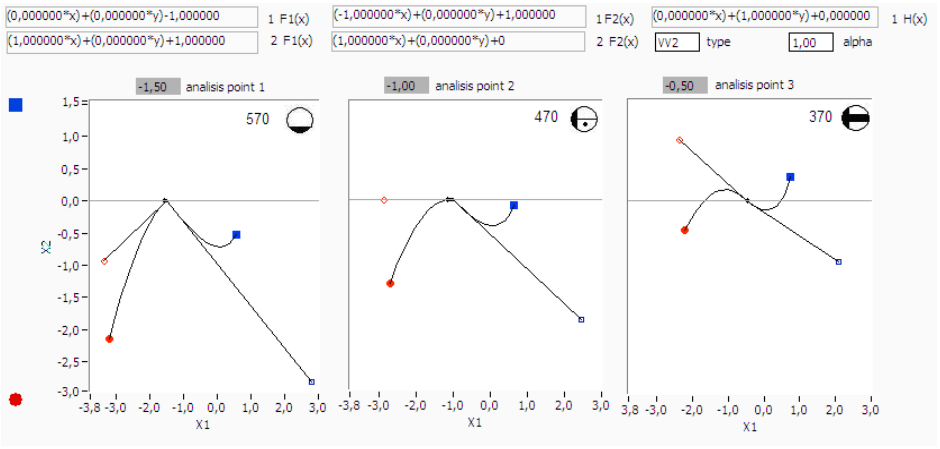


Figure 29. Different shots in the DB of a system with a starting sliding segment.

The detection of this phenomenon on the DB is confirmed when the next sequence of points is found: 570/570////570/**470**/370////370. The flow diagram corresponding to this phenomenon will be coded in the next paragraphs as ${}^{S}\Phi_{SIF(5)}$.

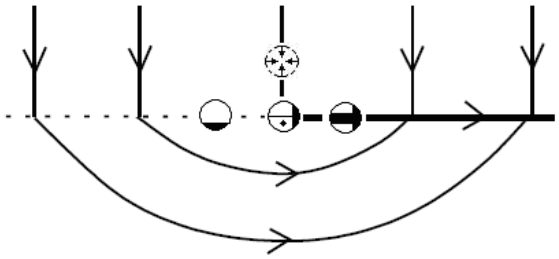


Figure 30. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(1)}$.

Sequence ${}^{S}\Phi_{SIF(1)}$

The segments or the sliding orbits are symbolized with the letters Φs. The letter ‗S' accompanying the letter ‗Φ' means sliding. The second ‗S' means attractive or stable. The other sub-index indicates the regions that are separated by the DB. For example, $\Phi s_{1,2}$. In order to indicate whether the class of dynamics is a start or an end, the letters IF are added. Thus, the sequence is symbolized as

$${}^{S}\Phi_{SIF(1)} = \{(\Phi c_{2,1})\ (\Omega ss\rightarrow_{T})\ (\Phi ss_{\rightarrow 1,2})\} \text{ or } \{\underline{\ominus}\ \ominus\ \underline{\ominus}\} \qquad (4.68)$$

The other dynamics including starts or ends of sliding segments are as follows:

Sequence ${}^{S}\Phi_{SIF(2)}$

The figure 31 shows an example of the sequence type ${}^{S}\Phi_{SIF(2)}$. In this, in the left side of the DB, we have an attractive or stable sliding segment $\Phi ss_{\leftarrow 1,2}$ that moves toward the left. The starting point of the sliding dynamics is the point $\Omega ss\leftarrow_{T}$. This point have the following vectors associated: a vector $\mathbf{f}_1$ tangential and a vector $\mathbf{f}_2$ pointing toward the DB. Toward the right of the starting point, we have a crossing dynamics from Z_2 to Z_1, $\Phi c_{2,1}$, The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in region Z_2.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(2)} = \{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss\leftarrow_{T})(\Phi c_{2,1})\} \text{ or } \{\underline{\ominus}\ \ominus\ \underline{\ominus}\} \qquad (4.69)$$

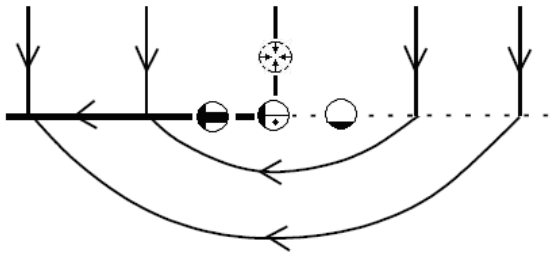


Figure 31. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(2)}$.

Sequence ${}^{S}\Phi_{SIF(3)}$

The figure 32 shows an example of the sequence type ${}^{S}\Phi_{SIF(3)}$. In this, crossing dynamics $\Phi c_{1,2}$ from Z_1 to Z_2, is observed on the left of the point of change. At the point of change $\Omega ss\rightarrow^{T}$, we have a vector $\mathbf{f}_2$ tangent to the DB joined to a vector $\mathbf{f}_1$ pointing to the DB. On the right of the point of change, we have an attractive sliding dynamics $\Phi ss_{\rightarrow 1,2}$ moving to the left on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in region Z_1.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(3)} = \{(\Phi c_{1,2})\ (\Omega ss\rightarrow^{T})(\Phi ss_{\rightarrow 1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.70)$$

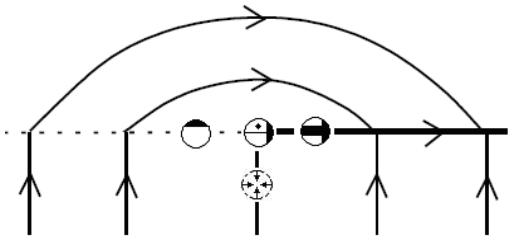


Figure 32. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(3)}$.

Sequence ${}^{S}\Phi_{SIF(4)}$

The figure 33 shows an example of the sequence type ${}^{S}\Phi_{SIF(4)}$. In this, an attractive or stable sliding dynamics $\Phi ss_{\leftarrow 1,2}$ moving to the left on the DB is observed on the left side of the point of change. At the point of change $\Omega ss_{\leftarrow}^{T}$, a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing to the DB is found. On the right side of the point of change we have a crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in region Z_1.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(4)} = \{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss\leftarrow^{T})(\Phi c_{1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.71)$$

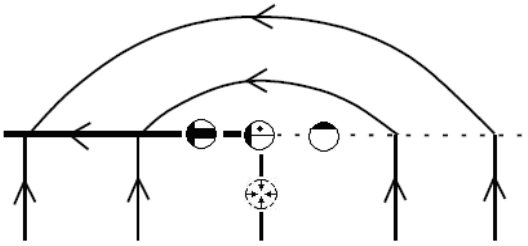


Figure 33. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(4)}$.

Sequence ${}^{S}\Phi_{SIF(5)}$

The figure 34 shows an example of the sequence type ${}^{S}\Phi_{SIF(5)}$. In this, crossing dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1 is observed on the left side of the point of change. At the point of change $\Omega ss\leftarrow_T$, a tangential vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing to the DB is found. On the right of the point of change we have an attractive sliding dynamics $\Phi ss_{\leftarrow 1,2}$ moving to the left on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the region Z_1.

The sequence Φ is symbolized as

${}^{S}\Phi_{SIF(5)} = \{(\Phi c_{2,1})\ (\Omega ss\leftarrow_T)(\Phi ss_{\leftarrow 1,2})\}$ or $\{$  $\}$ (4.72)

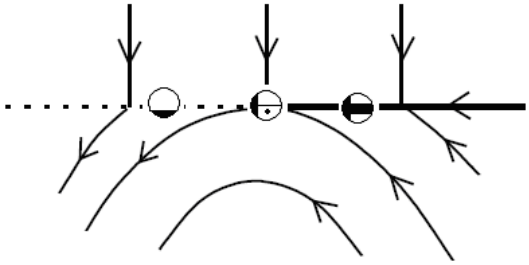


Figure 34. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(5)}$.

Sequence ${}^{S}\Phi_{SIF(6)}$

The figure 35 shows an example of the sequence type ${}^{S}\Phi_{SIF(6)}$. In this, an attractive or stable sliding dynamics $\Phi ss_{\rightarrow 1,2}$ moving to the right on the DB is observed on the left of the point of change. At the point of change $\Omega ss\rightarrow_T$, a tangential vector in the $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing to the DB is found. On the right of the point of change we have a crossing

dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the region Z_1.
The sequence Φ is symbolized as

$$^{S}\Phi_{SIF(6)} = \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss \rightarrow_{T})(\Phi c_{2,1})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \quad (4.73)$$

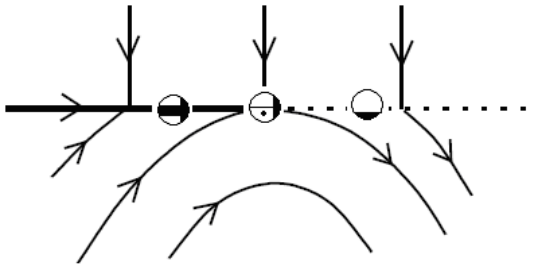


Figure 35. Flow of a system with a starting or an end of a sliding dynamics of the type $^{S}\Phi_{SIF(6)}$.

Sequence $^{S}\Phi_{SIF(7)}$

The figure 36 shows an example of the sequence type $^{S}\Phi_{SIF(7)}$. In this, crossing dynamics $\Phi c_{2,1}$ from region Z_1 to Z_2 is observed on the left of the point of change. At the point of change $\Omega ss \leftarrow^{T}$, a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing to the DB is found. On the right of the point of change, we have an attractive sliding dynamics $\Phi ss_{\leftarrow 1,2}$ moving to the left on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in the region Z_2.

The sequence Φ is symbolized as

$$^{S}\Phi_{SIF(7)} = \{(\Phi c_{1,2})\ (\Omega ss \leftarrow^{T})(\Phi ss_{\leftarrow 1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \quad (4.74)$$

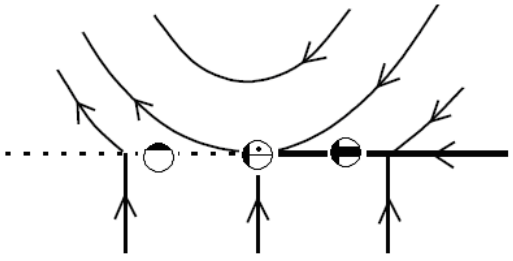


Figure 36. Flow of a system with a starting or an end of a sliding dynamics of the type $^{S}\Phi_{SIF(7)}$.

Sequence ${}^{S}\Phi_{SIF(8)}$

The figure 37 shows an example of the sequence type ${}^{S}\Phi_{SIF(8)}$. In this, an attractive or stable sliding dynamics $\Phi ss_{\rightarrow 1,2}$ moving to the right on the DB is observed on the left of the point of change. At the point of change $\Omega ss \rightarrow^{T}$, a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing to the DB is observed. On the right of the point of change we have a crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in the region Z_2.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(8)} = \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss \rightarrow^{T})(\Phi c_{1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.75)$$

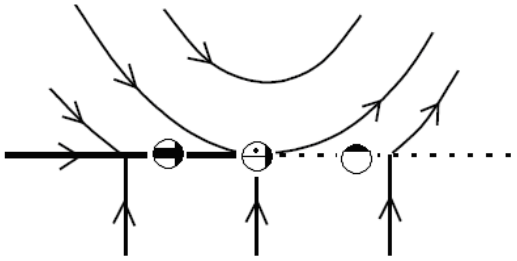


Figure 37. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(8)}$.

Sequence ${}^{S}\Phi_{SIF(9)}$

The figure 38 shows an example of the sequence type ${}^{S}\Phi_{SIF(9)}$. In this, crossing dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1 is observed on the left of the point of change. At the point of change $\Omega ss_{V} \rightarrow_{T}$, a vanished vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing to the DB is obseved. On the right of the point of change we have an attractive sliding segment $\Phi ss_{\rightarrow 1,2}$ moving to the right on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the DB.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(9)} = \{(\Phi c_{2,1})\ (\Omega ss_{V} \rightarrow_{T)}(\Phi ss_{\rightarrow 1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.76)$$

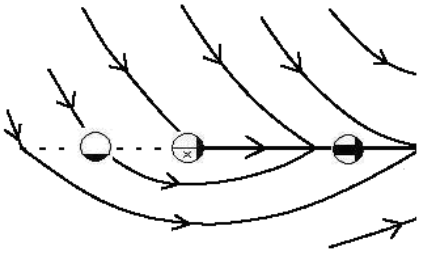


Figure 38. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(9)}$.

Sequence ${}^{S}\Phi_{SIF(10)}$

The figure 39 shows an example of the sequence type ${}^{S}\Phi_{SIF(10)}$. In this, an attractive or stable sliding dynamics $\Phi ss_{\leftarrow 1,2}$ moving to the left on the DB is observed on the left of the point of change. At the point of change $\Omega ss_{V\leftarrow T}$ a vanished vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing to the DB is found. On the right of the point of change we have a crossing dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is on the DB.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(10)} = \{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss_{V\leftarrow T})(\Phi c_{2,1})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.77)$$

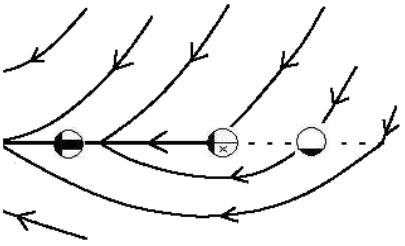


Figure 39. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(10)}$.

Sequence ${}^{S}\Phi_{SIF(11)}$

The figure 40 shows an example of the sequence type ${}^{S}\Phi_{SIF(11)}$. In this, crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2 is observed on the left of the point of change. At the of change $\Omega ss_{V\rightarrow}{}^{T}$, is found a vanished vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing to the DB. On the right of the point of change we have an attractive sliding dynamics $\Phi ss_{\rightarrow 1,2}$ moving to the right on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is on the DB.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SIF(11)} = \{(\Phi c_{1,2)}\ (\Omega ss_{V\rightarrow}{}^{T})(\Phi ss_{\rightarrow 1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.78)$$

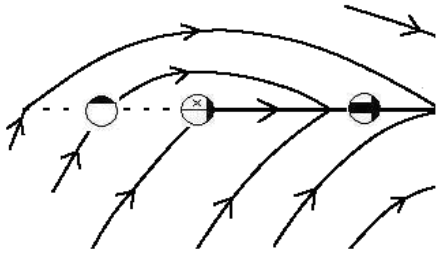
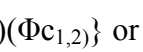


Figure 40. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(11)}$.

Sequence ${}^{S}\Phi_{SIF(12)}$

The figure 41 shows an example of the sequence type ${}^{S}\Phi_{SIF(12)}$. In this, an attractive or stable sliding dynamics $\Phi ss_{\leftarrow 1,2}$ moving to the left on the DB is observed on the left of the point of change. At the point of change $\Omega ss_{V}\leftarrow^{T}$ is found a vanished vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing to the DB. On the right of the point of change we have a crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SIF(12)} = \{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss_{V}\leftarrow^{T})(\Phi c_{1,2})\}$ or $\{$  $\}$ (4.79)

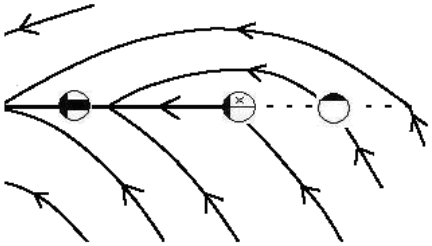


Figure 41. Flow of a system with a starting or an end of a sliding dynamics of the type ${}^{S}\Phi_{SIF(12)}$.

There are systems that instead of have a focus on the DB, they have a stable node and show similar sequence within the DB as those described before. The node appears on the DB as a vanished point. Figure 42 shows a group with this class of dynamics. In Section 4.7 will be observed that sequences of the type ${}^{S}\Phi_{SIF(9\text{-}12)}$ generates a different bifurcation pattern compared with bifurcations of the systems as presented in Figure 4.30.

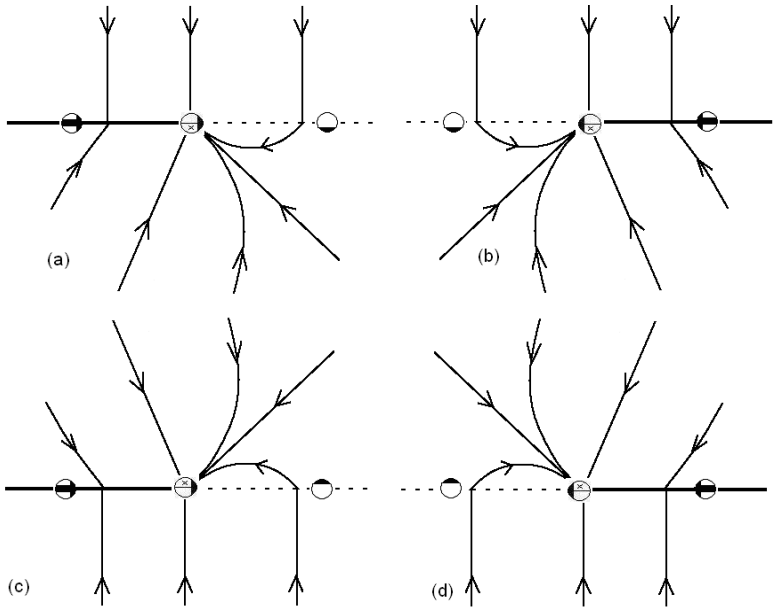


Figure 42. Systems with start or end of sliding segment and nodes over the DB.

4.4.3 Start and end of a repulsive segment

In a nonsmooth dynamical system, is possible to find simultaneously, over the DB a crossing dynamics, and and the same time, next to the previous, a repulsive or unstable dynamic. During the analysis of the DB, when a crossing segment is found next to a repulsive segment, it is said, we have the start or the end of a repulsive segment. The direction adopted over the DB determines whether it is the start or the end.

A possible process for identifying the start or end of a repulsive segment presents the following different shots:

1. Left side of figure 43 with x = -0.5. Over the DB, on the left side of the point of initialization of the repulsive segment, both vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ point toward the region Z_1.
2. As we advances over the DB, to the right, to the point of initialization of the repulsive segment, the vector $\mathbf{f}_2$ have a rotated relative position in the CCW direction.
3. Center of figure, with x=0. At the point of initialization of the repulsive segment, the vector $\mathbf{f}_2$ is tangent to the DB, with direction to the right.
4. Right side of the figure, with x = 0,5. On the right side of the point of initialization of the repulsive segment, the vector $\mathbf{f}_2$ points to its own region, the Z_2.

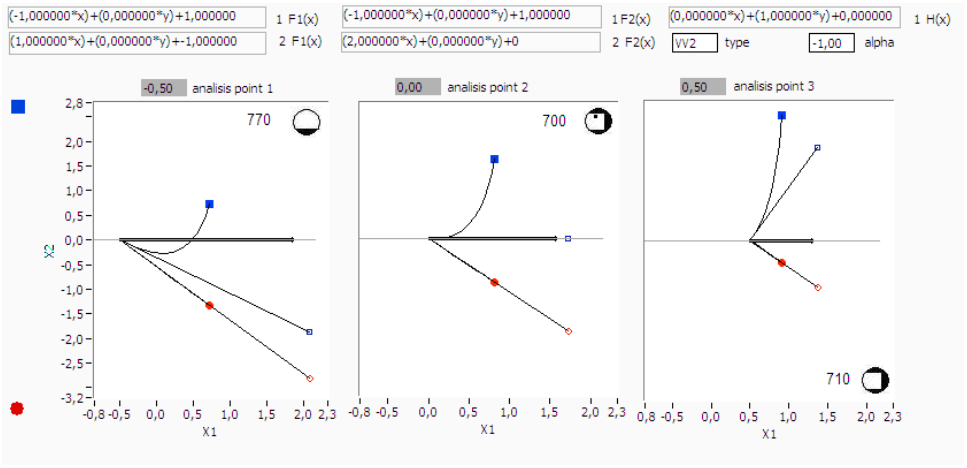


Figure 43. Different shots in the DB of a system with a starting repulsive segment.

The detection of this phenomenon in the DB is confirmed when the next sequence of points is found: 770/770////770/**700**/710////710. The diagram of the flow corresponding to this phenomenon will be coded in the next paragraphs as ${}^{S}\Phi_{SIF(5)}$.

Sequence ${}^{S}\Phi_{SUIF(1)}$

The segments or the repulsive orbits are symbolized with the letters Φs_U. The letter ‗S' accompanying the letter ‗Φ' means sliding. The second letter, ‗U', means repulsive or unstable. The other sub-index indicates the regions that are splitted by the DB (for example, $\Phi s_{U1,2}$). Letters ‗I' and ‗F' are added in order to indicate whether the class of dynamics presented is a start or an end, respectively. Thus, the sequence is symbolized as

$${}^{S}\Phi_{SUIF(1)} = \{(\Phi c_{2,1})\ (\Omega s_U \rightarrow^{T})\ (\Phi s_{U\rightarrow 1,2})\} \text{ or } \{\underline{\text{◒}}\ \text{◑}\ \underline{\text{◑}}\} \qquad (4.80)$$

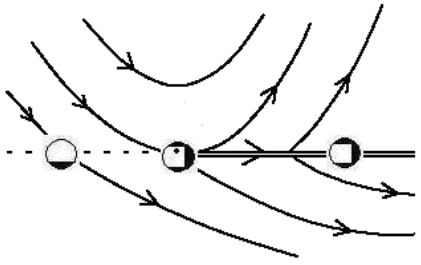


Figure 44. Flow of a system with a start or an end of a repulsive dynamics type ${}^{S}\Phi_{SIF(1)}$.

The other dynamics, including starts or ends of repulsive segments, are as follows:

Sequence ${}^{S}\Phi_{SUIF(2)}$

The figure 45 shows an example of the sequence type ${}^{S}\Phi_{SUIF(2)}$. In this, a repulsive segment $\Phi s_{U\leftarrow 1,2}$ observed on the left of the point of change. At point of change $\Omega s_{U}\leftarrow^{T}$, is found a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing away from the DB, toward the region Z_1. On the right of the point of change we have a crossing dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in Z_2.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF(2)} = \{(\Phi s_{U\leftarrow 1,2})\ (\Omega s_{U}\leftarrow^{T})(\Phi c_{2,1})\}$ or $\{$ $\}$ (4.81)

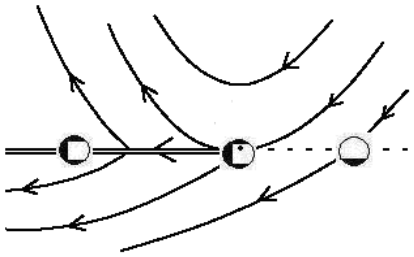


Figure 45. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(2)}$.

Sequence ${}^{S}\Phi_{SUIF(3)}$

The figure 46 shows an example of the sequence type ${}^{S}\Phi_{SUIF(3)}$. In this, crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2 is observed on the left of the point of change. At the point of change $\Omega s_{U}\rightarrow_{T}$ is found a tangential vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing away from the DB, toward the region Z_2. On the right side of the point of change we have a repulsive dynamics $\Phi s_{U\rightarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in Z_1.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF(3)} = \{(\Phi c_{1,2})\ (\Omega s_{U}\rightarrow_{T})(\Phi s_{U\rightarrow 1,2})\}$ or $\{$  $\}$ (4.82)

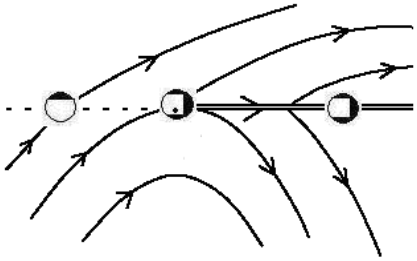


Figure 46. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(3)}$.

Sequence ${}^{S}\Phi_{SUIF(4)}$.

The figure 47 shows an example of the sequence type ${}^{S}\Phi_{SUIF(4)}$. In this, a repulsive dynamics $\Phi s_{U\leftarrow 1,2}$ is observed on the left of the point of change. At the point of change $\Omega s_{U\leftarrow T}$ is observed a tangential vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing away from the DB, toward the region Z_2. On the right of the point of change we have a crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in region Z_1.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SUIF\,(4)} = \{(\Phi s_{U\leftarrow 1,2})\,(\Omega s_{U\leftarrow T})(\Phi c_{1,2})\} \text{ or } \{\underline{\quad}\;\underline{\quad}\} \tag{4.83}$$

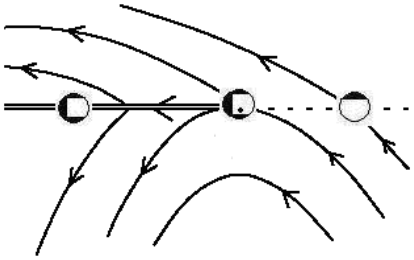


Figure 47. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(4)}$.

Sequence ${}^{S}\Phi_{SUIF(5)}$.

The figure 48 shows an example of the sequence type ${}^{S}\Phi_{SUIF(5)}$. In this, a repulsive dynamics $\Phi_{SU\rightarrow 1,2}$ is observed on the left of the point of change. At the point of change t $\Omega_{SU\rightarrow T}$ is found a tangential vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing away from the DB, toward the region Z_2. On the right side of the point of change we have a crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the region Z_2.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SUIF\,(5)} = \{(\Phi_{SU\rightarrow 1,2})\,(\Omega_{SU\rightarrow T})(\Phi c_{1,2})\} \text{ or } \{\underline{\quad}\;\underline{\quad}\} \tag{4.84}$$

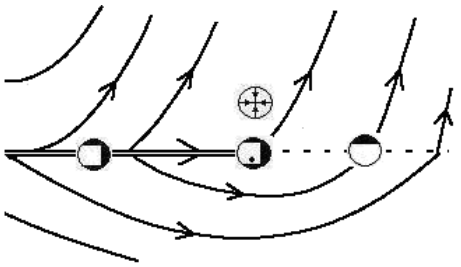


Figure 48. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(5)}$.

Sequence ${}^{S}\Phi_{SUIF(6)}$.

The figure 49 shows an example of the sequence type ${}^{S}\Phi_{SUIF(6)}$. In this, crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2 is observed on the left side of the point of change. At the point of change $\Omega_{SU\leftarrow T}$ is found a tangential vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing away from the DB, toward the region Z_2. On the right side of the point of change we have a repulsive dynamics $\Phi s_{U\rightarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the region Z_2.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF(6)} = \{(\Phi c_{1,2})\ (\Omega_{SU\leftarrow T})(\Phi_{SU\leftarrow 1,2})\}$ or $\{$  $\}$ (4.85)

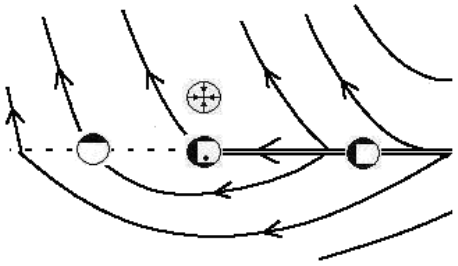


Figure 49. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(6)}$.

Sequence ${}^{S}\Phi_{SUIF(7)}$.

The figure 50 shows an example of the sequence type ${}^{S}\Phi_{SUIF(7)}$. In this, a repulsive dynamics $\Phi s_{U\leftarrow 1,2}$ is observed on the left side of the point of change. At the point of change $\Omega_{SU\rightarrow}{}^{T}$ is found a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing away from the DB, toward the region Z_1. On the right side of the point of change, we have a crossing dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in the region Z_1.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF(7)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SU\rightarrow}{}^{T})(\Phi c_{2,1})\}$ or $\{$  $\}$ (4.86)

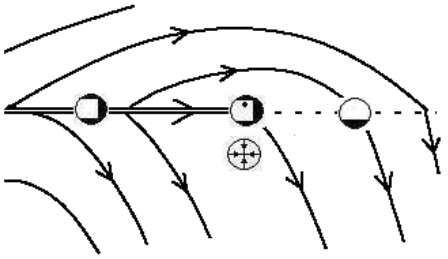


Figure 50. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(7)}$.

Sequence ${}^{S}\Phi_{SUIF(8)}$

The figure 51 shows an example of the sequence type ${}^{S}\Phi_{SUIF(8)}$. In this, crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2 is observed on the left of the point of change. At the point of change $\Omega_{SU\leftarrow}{}^{T}$ is found a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing away from the DB, toward the region Z_1. On the right side of the point of change, we have a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is in the region Z_1.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF\,(8)} = \{(\Phi c_{2,1})\ (\Omega_{SU\leftarrow}{}^{T})(\Phi_{SU\leftarrow 1,2})\}$ or $\{$  $\}$ (4.87)

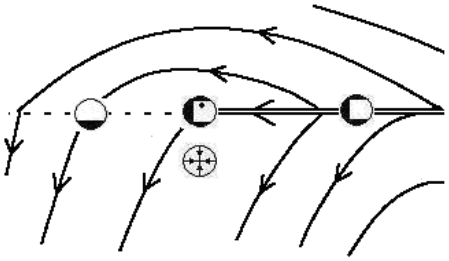


Figure 51. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(8)}$.

Sequence ${}^{S}\Phi_{SUIF(9)}$

The figure 52 shows an example of the sequence type ${}^{S}\Phi_{SUIF(9)}$. In this, a repulsive dynamics $\Phi s_{U\rightarrow 1,2}$ is observed on the left side of the point of change. At the point of change $\Omega_{SUV\rightarrow X}$ is found a vanished vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing away from the DB . On the right side of the point of change we have a crossing dynamics $\Phi c_{1,2}$ from region Z_1 to $Z_{2,}$. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF\,(9)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SUV\rightarrow X})(\Phi c_{2,1})\}$ or $\{$  $\}$ (4.88)

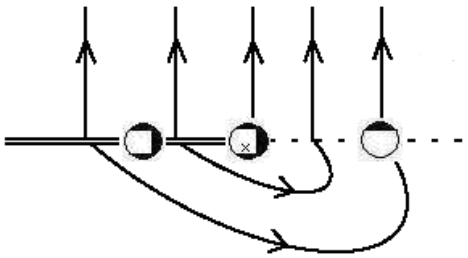


Figure 52. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(9)}$.

Sequence ${}^{S}\Phi_{SUIF(10)}$.

The figure 53 shows an example of the sequence type ${}^{S}\Phi_{SUIF(10)}$. In this, crossing dynamics $\Phi c_{1,2}$ from region Z_1 to Z_2 is observed on the left of the point of change. At point of change $\Omega_{SUV}\leftarrow_X$ is found a vanished vector $\mathbf{f}_1$ joined to a vector $\mathbf{f}_2$ pointing away from the DB. On the right of the point of change, we have a repulsive dynamics $\Phi s_{U\leftarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUIF\,(10)} = \{(\Phi c_{2,1})\ (\Omega_{SUV}\leftarrow_X)\ (\Phi_{SU\leftarrow 1,2})\}$ or $\{$  $\}$ (4.89)

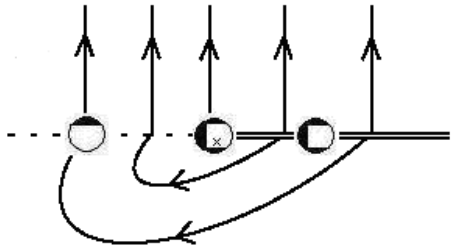


Figure 53. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(10)}$.

Sequence ${}^{S}\Phi_{SUIF(11)}$

The figure 54 shows an example of the sequence type ${}^{S}\Phi_{SUIF(11)}$. In this, a repulsive dynamics $\Phi s_{U\rightarrow 1,2}$ is observed on the left side of the point of change. At the point of change $\Omega_{SUV}\rightarrow^X$ is found a vanished vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing away from the DB. On the right of the point of change, we have a crossing dynamics $\Phi c_{2,1}$ from region Z_2 to $Z_{1,}$. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is on the DB.

${}^{S}\Phi_{SUIF\,(11)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SUV}\rightarrow^X)(\Phi c_{2,1})\}$ or $\{$  $\}$ (4.90)

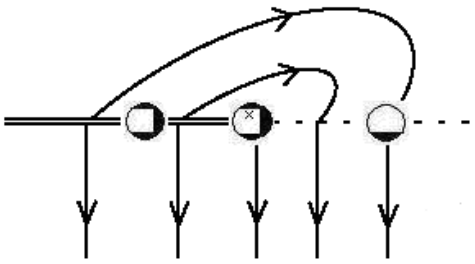


Figure 54. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(11)}$.

Sequence ${}^{S}\Phi_{SUIF(12)}$

The figure 55 shows an example of the sequence type ${}^{S}\Phi_{SUIF(12)}$. In this, crossing dynamics $\Phi c_{2,1}$ from region Z_2 to Z_1 is observed on the left side of the point of change. At the point of change $\Omega_{SUV}\leftarrow^{X}$ is found a tangential vector $\mathbf{f}_2$ joined to a vector $\mathbf{f}_1$ pointing away from the DB, to the region Z_1. On the right of the point of change, we have a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_2$ is on the DB.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SUIF\,(12)} = \{(\Phi c_{2,1})\,(\Omega_{SUV}\leftarrow^{X})(\Phi_{SU\leftarrow 1,2})\} \text{ or } \{\underline{\quad}\;\;\underline{\quad}\} \qquad (4.91)$$

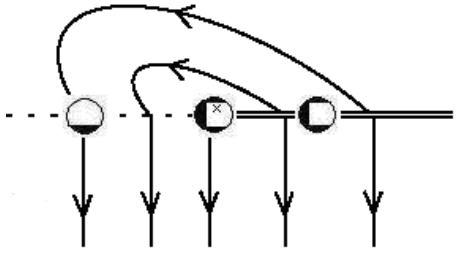


Figure 55. Flow of a system with a start or an end of a repulsive dynamics of the type ${}^{S}\Phi_{SUIF(12)}$.

There are systems that instead of have a focus on the DB, they have a stable node and show similar dynamics within the DB as those described before. The node appears on the DB as a vanished point. Figure 56 shows a group with this class of dynamics. In Section 4.7 will be observed that sequences of the type ${}^{S}\Phi_{SUIF(9\text{-}12)}$ generates a different bifurcation pattern compared compared with bifurcations of the systems as presented in the Figure

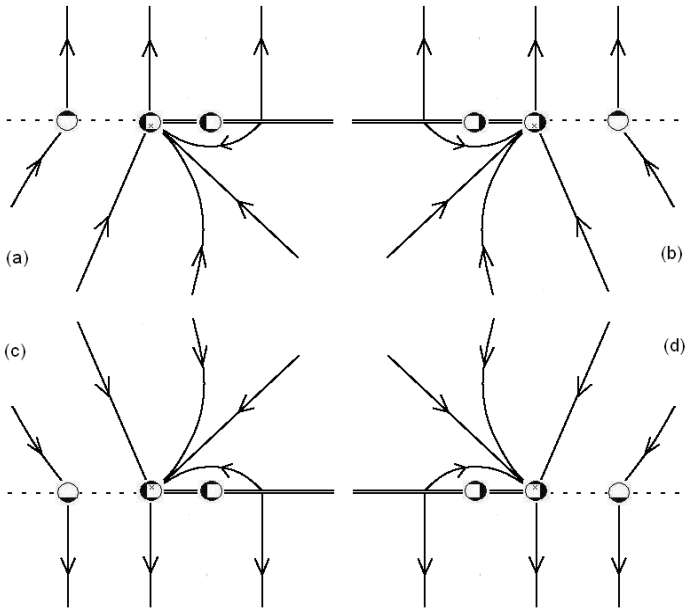


Figure 56. Systems with start or end of repulsive segments and nodes over the DB.

4.4.4 Change from an attractive sliding segment to a repulsive segment

In nonsmooth dynamical systems is possible to find simultaneously, over the DB an attractive sliding dynamics, and at the same time, next to the previous, a repulsive segment or unstable segment. During the analysis of the DB, when an attractive sliding segment is found next to a repulsive segment, we say, we have a change from an attractive dynamic to a repulsive dynamic.

A possible process responsible for this change of an attractive sliding segment to a repulsive segment has the following different shots :

1. Left side of figure 57 with x = -0,5. Over the DB, on the left of the point of change, the vector $\mathbf{f}_1$ points to the opposite region Z_2, and the vector $\mathbf{f}_2$ points to the opposite region Z_1. The calculation of the Filippov with $\mathbf{f}_1$ and $\mathbf{f}_2$ gives a vector G(x) with an orientation toward the right.
2. As we advance over the DB toward the right, to the point of change, both vectors have a rotated relative position in CCW direction.
3. Center of figure, with x=0. At the point of change, both vectors are tangent to the DB. The $\mathbf{f}_1$ vector has a negative direction, and $\mathbf{f}_2$ vector has a positive direction. The calculation of the Filippov gives a vector G(x)=0 because both the vectors have different directions. Some cases both vectors have the same direction. This characteristic is called a singular point in the terminology of the Filippov systems.

4. Right side of the figure, with x = 0,5. On the right side of the point of change, the vector $\mathbf{f}_1$ points to its own region, the Z_1, and the vector $\mathbf{f}_2$ points to its own region, the Z_2. The calculation of the Filippov solution gives a vector G(x) with an orientation toward the left.

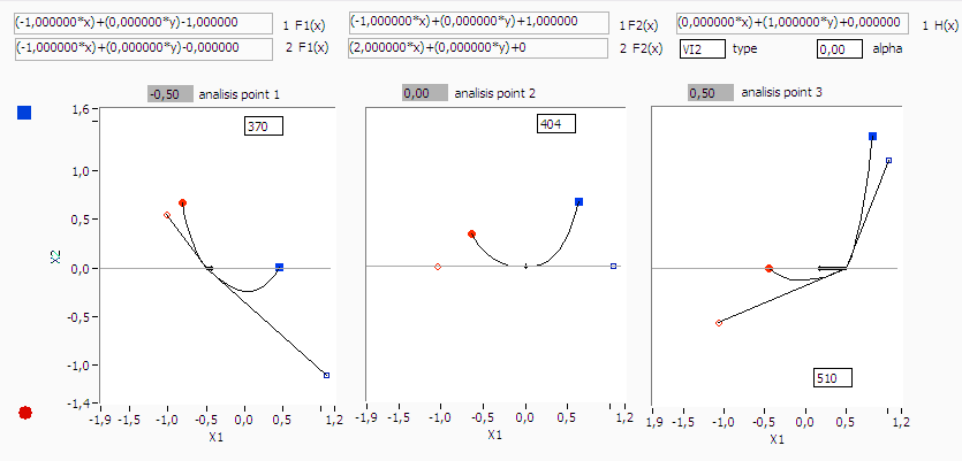


Figure 57. Different shots in the DB of a system with a change of attractive to repulsive segment.

The detection of this phenomenon in the DB is confirmed when the next sequence of points is found: 370/370////370/**404**/510////510/510. The diagram of the flow corresponding to this phenomenon has not been coded yet.

Sequence ${}^{S}\Phi_{SI\,(1)}$

The segments or the sliding orbits are symbolized with the letters _Φs'. In order to indicate that there is the change from an attractive sliding segment to a repulsive segment, the letter _I' is added. Thus, the first sequence is symbolized as

$${}^{S}\Phi_{SI\,(1)} = \{(\Phi_{SS\rightarrow 1,2})\ (\Omega_{STT}\rightarrow)\ (\Phi_{SU\rightarrow 1,2})\} \text{ or } \{\underline{\ominus}\ \ominus\ \underline{\ominus}\} \qquad (4.92)$$

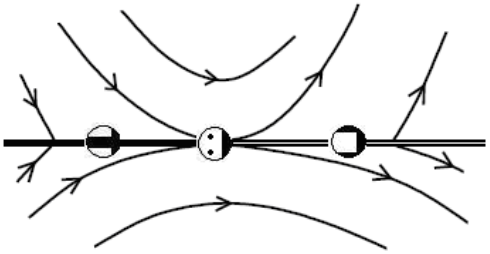


Figure 58. Flow of a system with a change of dynamics attractive to repulsive of the type ${}^{S}\Phi_{SI\,(1)}$.

The other dynamics that include changes from attractive to repulsive segments are as follows:

Sequence ${}^{S}\Phi_{SI\,(2)}$

The figure 59 shows an example of the sequence type ${}^{S}\Phi_{SI\,(2)}$ In this, a repulsive dynamics $\Phi s_{U\leftarrow 1,2}$ is observed on the left side of the point of change. At the point of change $\Omega_{STT}\leftarrow$ is found two tangential vectors with a negative direction. On the right side of the point of change, we have an attractive sliding dynamics $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB. The equilibrium point (focus) of the vector field $\mathbf{f_1}$ is in the region Z_1, and the equilibrium point of the vector field $\mathbf{f}_2$ is in Z_2.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SI\,(2)} = \{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{STT}\leftarrow)(\Phi_{SS\leftarrow 1,2})\} \text{ or } \{\underline{\ } \ : \ \underline{\ }\} \qquad (4.93)$$

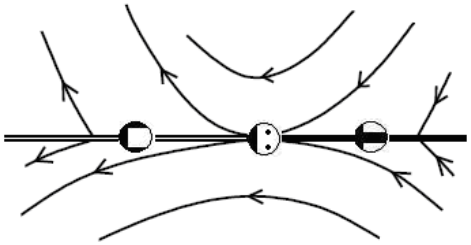


Figure 59. Flow of a system with a change of dynamics attractive to repulsive of the type ${}^{S}\Phi_{SI\,(2)}$.

Sequence ${}^{S}\Phi_{SI\,(3)}$

The figure 60 shows an example of the sequence type ${}^{S}\Phi_{SI\,(3)}$. In this, an attractive sliding segment $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB is observed on the left side of the point of change. At the point of change $\Omega_{SVT}{}^{\rightarrow}{}_{X}$ is found a tangential vector $\mathbf{f}_2$ joined to a vanished vector $\mathbf{f}_1$. On the right of the point of change, we have a repulsive segment $\Phi_{SU\leftarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is on the DB, and the equilibrium point of the vector field $\mathbf{f}_2$ is in the region Z_2.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SI\,(3)} = \{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{SVT}{}^{\rightarrow}{}_{X})(\Phi_{SU\leftarrow 1,2})\} \text{ or } \{\underline{\ } \ \otimes \ \underline{\ }\} \qquad (4.94)$$

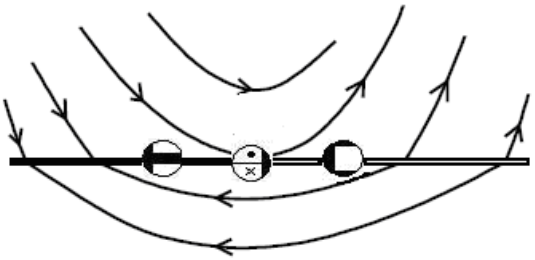


Figure 60. Flow of a system with a change of dynamics attractive to repulsive of the type ${}^{S}\Phi_{SI\,(3)}$.

Sequence ${}^{S}\Phi_{SI\,(4)}$.

The figure 61 shows an example of the sequence type ${}^{S}\Phi_{SI\,(4)}$. In this, a repulsive dynamics $\Phi_{SU\rightarrow 1,2}$ is observed on the left of the point of change. At the point of change $\Omega_{SVT}{}^{\leftarrow}{}_{X}$ is found a tangential vector $\mathbf{f}_2$ joined to a vanished vector $\mathbf{f}_1$. On the right side of the point of change, we have an attractive sliding dynamics $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is on the DB, and the equilibrium point of the vector field $\mathbf{f}_2$ is in the region Z_2.

The sequence Φ is symbolized as

${}^{S}\Phi_{SI\,(4)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SVT}{}^{\leftarrow}{}_{X})(\Phi_{SS\rightarrow 1,2})\}$ or $\{\underline{\ }\ \underline{\ }\}$ (4.95)

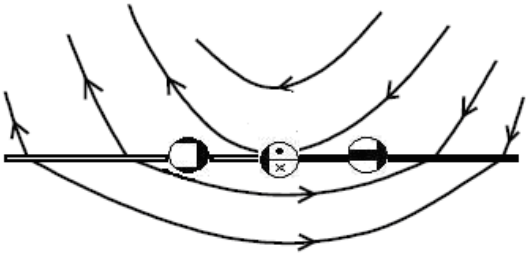


Figure 61. Flow of a system with a change of dynamics attractive to repulsive of the type ${}^{S}\Phi_{SI\,(4)}$.

Sequence ${}^{S}\Phi_{SI\,(5)}$

The figure 62 shows an example of the sequence type ${}^{S}\Phi_{SI\,(5)}$. In this, an attractive sliding dynamics $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB is observed on the left side of the point of change. At the point of change $\Omega_{STV}{}^{\rightarrow X}$ is found a tangential vector $\mathbf{f}_1$ joined to a vanished vector $\mathbf{f}_2$. On the right side of the point of change, we have a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the region Z_1, and the equilibrium point of the vector field $\mathbf{f}_2$ is on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SI\,(5)} = \{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{STV}{}^{\rightarrow X})(\Phi_{SU\leftarrow 1,2})\}$ or $\{\underline{\ }\ \underline{\ }\}$ (4.96)

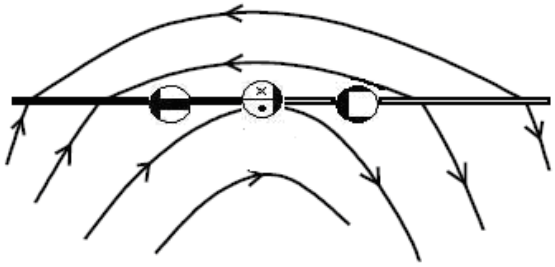


Figure 62. Flow of a system with a change of dynamics attractive to repulsive of the type ${}^{S}\Phi_{SI\,(5).}$

Sequence ${}^{S}\Phi_{SI\,(6).}$

The figure 63 shows an example of the sequence type ${}^{S}\Phi_{SI\,(6).}$ In this, a repulsive segment $\Phi_{SU\rightarrow 1,2}$ is observed on the left side of the point of change. At the point of change $\Omega_{STV}{}^{\leftarrow X}$ is found a tangential vector $\mathbf{f}_1$ joined to a vanished vector $\mathbf{f}_2$. On the right side of the point of change, we have an attractive sliding dynamics $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB. The equilibrium point (focus) of the vector field $\mathbf{f}_1$ is in the region Z_1, and the equilibrium point of the vector field $\mathbf{f}_2$ is on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SI\,(6)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{STV}{}^{\leftarrow X})(\Phi_{SS\rightarrow 1,2})\}$ or $\{$  $\}$ (4.97)

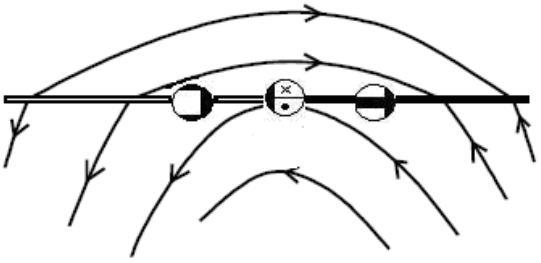


Figure 63. Flow of a system with a change of dynamics attractive to repulsive of the type ${}^{S}\Phi_{SI\,(6).}$

4.4.5 Change in direction of attractive sliding segment

In nonsmooth dynamical system is possible to find simultaneously, over the DB, two concurrent attractive sliding dynamics; one of them moving toward the right, and the other moving to the left. During the analysis of the DB, when two attractive sliding segments with different directions are found and one follows the other, we say, we have a change in the direction of the attractive dynamics.

A possible process that induces the change in the direction of an attractive sliding segment has the following different shots:

1. Left side of figure 64 with x = -0,6. Over the DB, on the left side of the point of change, both vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ point to the opposite regions. The calculation of the Filippov solution using $\mathbf{f}_1$ and $\mathbf{f}_2$ gives a vector G(x) with an orientation toward the right.
2. As we advance over the DB, toward the right, to the point of the change, both vectors have a rotated relative position: $\mathbf{f}_1$ in the CCW direction and $\mathbf{f}_2$ in the CW direction.
3. Center of figure, with x=-0,38. At the point of the change, both vectors become anti-collinear, and the calculation of the Filippov solution gives a null vector G(x)=0, which means that this is a static point where there is no sliding in spite of that fact that both vectors point to opposite regions. This characteristic is called a pseudo-equilibria point in the terminology of the Filippov systems.
4. Right side of figure with x = -0,2. Over the DB, on the right of the point of change, both vectors point to the contrary region. The calculation of the Filippov solution gives a vector G(x) with an orientation toward the left.

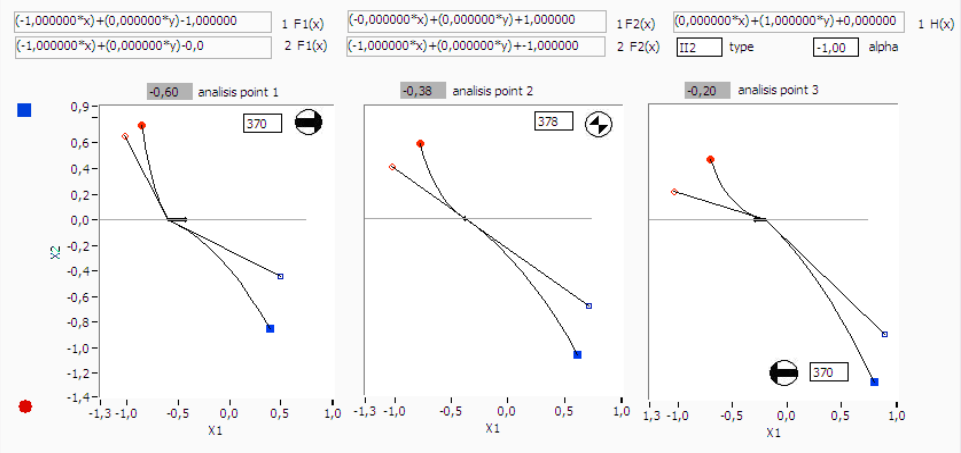


Figure 64. Different shots in the DB of a system with a change of direction of a sliding segment.

The detection of this phenomenon in the DB is confirmed when the next sequence of points is found: 370/370////370/**378**/370////370. The diagram of the flow corresponding to this phenomenon will be coded in the next paragraphs as ${}^{S}\Phi_{SSE\,(2)}$.

Sequence ${}^{S}\Phi_{SSE\,(1)}$.

The segments or the sliding orbits are symbolized with the letters _Φs'. The letter _S' accompanying the letter _Φ' means sliding. The second _S' means attractive or stable. In

order to indicate that the class of dynamics is a change in direction, the letter ‗E' is added. Thus, the sequence is symbolized as

$^{S}\Phi_{SSE\,(1)} = \{(\Phi_{SS\rightarrow 1,2})\,(\Omega_{ACS}/)\,(\Phi_{SS\leftarrow 1,2})\}$ or $\{$  $\}$ (4.98)

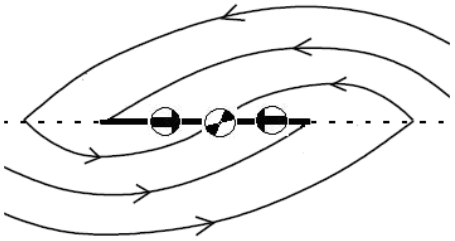


Figure 65. Flow of a system with a change of direction of an attractive dynamics of the type $^{S}\Phi_{SSE\,(1)}$.

The other dynamics, including starts or ends of sliding segments, are as follows:

Sequence $^{S}\Phi_{SSE\,(2)}$.

The figure 66 shows an example of the sequence type $^{S}\Phi_{SSE\,(2)}$. In this, an attractive sliding dynamics $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB is observed on the left of the point of change. At the point of change Ω_{ACS}\ two anti-collinear vectors pointing to each other, with an inclination in the range Θ_{2L} is found. On the right of the point of change, we have an attractive sliding segment $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB.

The sequence Φ is symbolized as

$^{S}\Phi_{SSE\,(2)} = \{(\Phi_{SS\rightarrow 1,2})\,(\Omega_{ACS}\backslash)(\Phi_{SS\leftarrow 1,2})\}$ or $\{$  $\}$ (4.99)

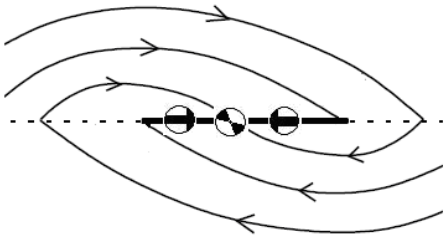


Figure 66. Flow of a system with a change of direction of an attractive dynamics of the type $^{S}\Phi_{SSE\,(2)}$.

Sequence $^{S}\Phi_{SSE\,(3)}$.

The figure 67 shows an example of the sequence type ${}^{S}\Phi_{SSE\ (3)}$. In this, an attractive sliding dynamics $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB in observed in the left side of the point of change. At the point of change $\Omega_{ACS}\backslash$ two anti-collinear vectors pointing to each other, with an inclination in the range Θ_{2L} is found. On the right side of the point of change, we have an attractive sliding dynamics $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SSE\ (3)} = \{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{ACS}\backslash)(\Phi_{SS\rightarrow 1,2})\} \text{ or } \{\underline{\ominus}\ \oplus\ \underline{\ominus}\} \qquad (4.100)$$

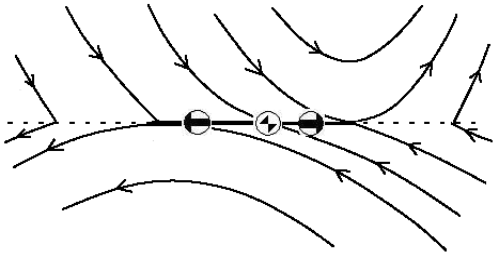


Figure 67. Flow of a system with a change of direction of an attractive dynamics of the type ${}^{S}\Phi_{SSE\ (3)}$.

Sequence ${}^{S}\Phi_{SSE\ (4)}$

The figure 68 shows an example of the sequence type ${}^{S}\Phi_{SSE\ (4)}$. In this, an attractive sliding dynamics $\Phi_{SS\leftarrow 1,2}$ moving to the left of the DB is observed on the left side of the point of change. At the point of change $\Omega_{ACS}/$ two anti-collinear vectors pointing to each other, with an inclination in the range Θ_{2R} is found. On the right side of the point of change, we have an attractive sliding dynamics $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB.

The sequence Φ is symbolized as

$${}^{S}\Phi_{SSE\ (4)} = \{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{ACS}/)(\Phi_{SS\rightarrow 1,2})\} \text{ or } \{\underline{\ominus}\ \oplus\ \underline{\ominus}\} \qquad (4.101)$$

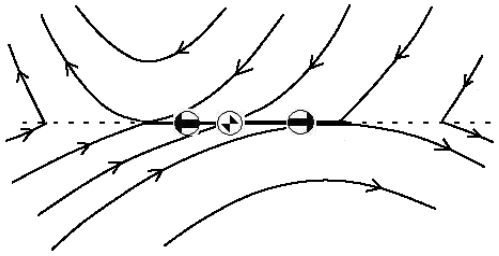


Figure 68. Flow of a system with a change of direction of an attractive dynamics of the type ${}^{S}\Phi_{SSE\ (4)}$.

Sequence ${}^{S}\Phi_{SSE\ (5)}$

The figure 69 shows an example of the sequence type ${}^{S}\Phi_{SSE\,(5)}$. In this, an attractive sliding dynamics $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB is observed on the left side of the point of change. At the point of change $\Omega_{ACS}|$ two anti-collinear vectors pointing to each other, with a perpendicular inclination in the range Θ_{90} is found . On the right side of the point of change we have an attractive sliding dynamics $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SSE\,(5)} = \{(\Phi_{SS\rightarrow 1,2})\,(\Omega_{ACS}|)(\Phi_{SS\leftarrow 1,2})\}$ or $\{\underline{\ominus}\ \otimes\ \underline{\ominus}\}$ (4.102)

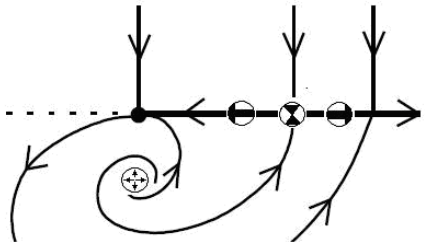


Figure 69. Flow of a system with a change of direction of an attractive dynamics of the type ${}^{S}\Phi_{SSE\,(5)}$.

Sequence ${}^{S}\Phi_{SSE\,(6)}$

The figure 70 shows an example of the sequence type ${}^{S}\Phi_{SSE\,(6)}$. In this, an attractive sliding segment $\Phi_{SS\rightarrow 1,2}$ moving to the right on the DB is observed on the left side of the point in which the dynamics changes. At the point of change $\Omega_{ACS}|$ two anti-collinear vectors pointing to each other, with a perpendicular inclination in the range Θ_{90} is found. On the right side of the point of change we have an attractive sliding dynamics $\Phi_{SS\leftarrow 1,2}$ moving to the left on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SSE\,(6)} = \{(\Phi_{SS\rightarrow 1,2})\,(\Omega_{ACS}|)(\Phi_{SS\leftarrow 1,2})\}$ or $\{\underline{\ominus}\ \otimes\ \underline{\ominus}\}$ (4.103)

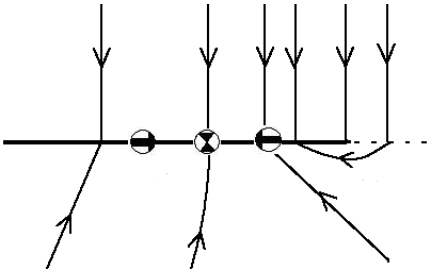


Figure 70. Flow of a system with a change of direction of an attractive dynamics of the type ${}^{S}\Phi_{SSE\,(6)}$.

4.4.6 Change in direction in a repulsive segment

In nonsmooth dynamical system is possible to find simultaneously, over the DB, two repulsive segments dynamics; one of them pointing toward the right, and the other pointing to the left. During the analysis of the DB, when two repulsive segments with different directions are found and one follows the other, we say, we have a change in the direction of the repulsive dynamics.

A possible process that induces the change in the direction of a repulsive segment has the following different shots:

1. Left side of figure 71 with x = 0,1. Over the DB, on the left side of the point of change, both vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ point to the opposite regions. The calculation of the Filippov solution using $\mathbf{f}_1$ and $\mathbf{f}_2$, gives a vector G(x) with an orientation toward the right.
2. As we advance over the DB, toward the right, to the point of the change, both vectors have a rotated relative position: $\mathbf{f}_1$ in CW and $\mathbf{f}_2$ in the CCW direction.
3. Center of figure, with x=0,37. At the point of the change, both vectors become anti-collinear and the calculation of the Filippov solution gives in a null vector G(x).
4. Right side of figure with x = 0,6. On the right of the point of change, both vectors continue point to the opposite regions. The Filippov solution gives a vector G(x) with an orientation toward the left.

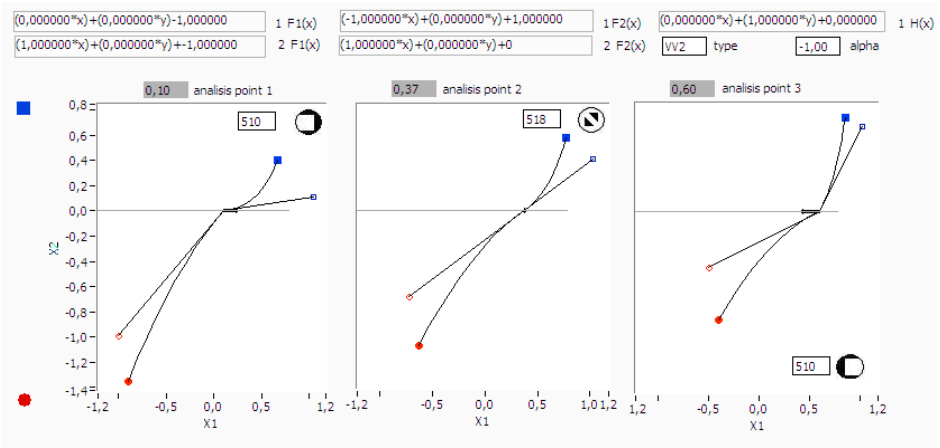


Figure 71. Different shots in the DB of a system with a change of direction of a repulsive segment.

The detection of this phenomenon in the DB is confirmed when the next sequence of points is found: 510/510////510/**518**/510////510. The diagram of the flow corresponding to this phenomenon will be coded in the next paragraphs as $^{S}\Phi_{SUD\,(1)}$.

Sequence $^{S}\Phi_{SUD\,(1)}$

The segments or the sliding orbits are symbolized with the letters ‗Φs'. The letter ‗S' accompanying the letter ‗Φ' means sliding. The second letter ‗U' means repulsive or unstable. The other sub-index indicates the regions of both sides of the DB, (for example, $\Phi s_{1,2)}$. In order to indicate that the class of dynamics is a change in the direction, the letter ‗D' is added. Thus, the sequence is symbolized as

$$^{S}\Phi_{SUD\,(1)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{ACU}/)\ (\Phi_{SU\leftarrow 1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.104)$$

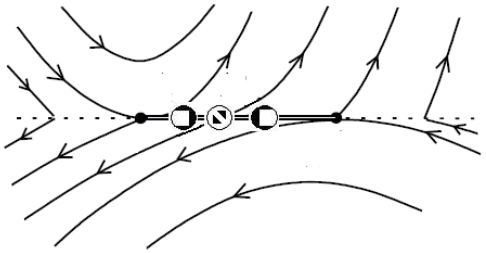


Figure 72. Flow of a system with a change of direction of a repulsive dynamics of the type $^{S}\Phi_{SUD\,(1)}$.

The other dynamics, including starts or ends of sliding segments, are as follows:

Sequence $^{S}\Phi_{SUD\,(2)}$

The figure 73 shows an example of the sequence type $^{S}\Phi_{SUD\ (2)}$. In this, a repulsive dynamics $\Phi_{SU\rightarrow 1,2}$ moving to the right on the DB and is observed on the left side of the point of change. At the point of change $\Omega_{ACU}\backslash$ two anti-collinear vectors pointing in opposite directions, with an inclination in the range Θ_{1R} is found . On the right of the point of change we have a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$ moving to the left on the DB.

The sequence Φ is symbolized as

$$^{S}\Phi_{SUD\,(2)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{ACU}\backslash)(\Phi_{SU\leftarrow 1,2})\} \text{ or } \{\underline{\ } \ \underline{\ }\} \qquad (4.105)$$

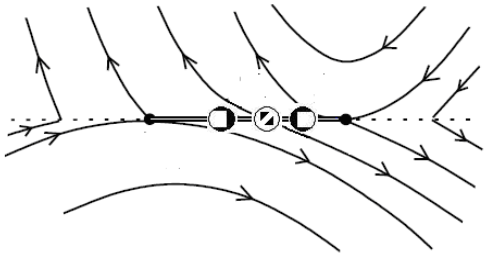


Figure 73. Flow of a system with a change of direction of a repulsive dynamics of the type ${}^{S}\Phi_{SUD\,(2)}$.

Sequence ${}^{S}\Phi_{SUD\,(3)}$

The figure 74 shows an example of the sequence type ${}^{S}\Phi_{SUD\,(3)}$. In this, a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$ moving to left on the DB is observed on the left side of the point of change. At the point of change Ω_{ACU} two anti-collinear vectors pointing in opposite directions, with an inclination in the range Θ_{1R} \ is found . On the right side of the point of change we have a repulsive dynamics $\Phi_{SU\rightarrow 1,2}$ moving to the right on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUD\,(3)} = \{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{ACU}\backslash)(\Phi_{SU\rightarrow 1,2})\}$ or $\{$  $\}$ (4.106)

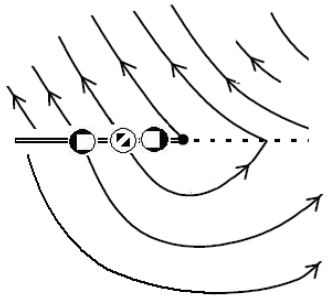


Figure 74. Flow of a system with a change of direction of a repulsive dynamics of the type ${}^{S}\Phi_{SUD\,(3)}$.

Sequence ${}^{S}\Phi_{SUD\,(4)}$

The figure 75 shows an example of the sequence type ${}^{S}\Phi_{SUD\,(4)}$. In this, a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$ moving to the left on the DB and is observed on the left of the point of change. At the point of change $\Omega_{ACU}/$ two anti-collinear vectors pointing in opposite directions, with an inclination in the range Θ_{1L} is found. On the right side of the point of change we have a repulsive dynamics $\Phi_{SU\rightarrow 1,2}$ moving to the right on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUD\,(4)} = \{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{ACU}/)(\Phi_{SU\rightarrow 1,2})\}$ or $\{$  $\}$ (4.107)

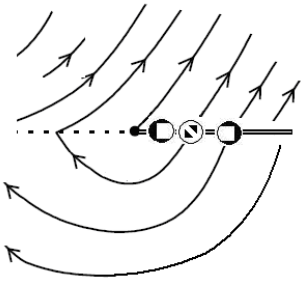


Figure 75. Flow of a system with a change of direction of a repulsive dynamics of the type ${}^{S}\Phi_{SUD\,(4)}$.

Sequence ${}^{S}\Phi_{SUD\,(5)}$

The figure 76 shows an example of the sequence type ${}^{S}\Phi_{SUD\,(5)}$. In this, a repulsive dynamics $\Phi_{SU\rightarrow 1,2}$ moving to the right on the DB is observed on the left of the point of change. At the point of change $\Omega_{ACU}|$ two anti-collinear vectors pointing in opposite directions, perpendicular to the DB is found. On the right of the point of change, we have a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$ moving to the left on the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUD\,(5)} = \{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{ACU}|)(\Phi_{SU\leftarrow 1,2})\}$ or { } (4.108)

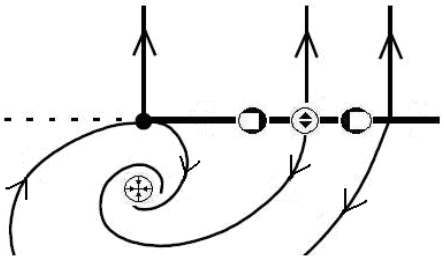


Figure 76. Flow a system with a change of direction of a repulsive dynamics of the type ${}^{S}\Phi_{SUD\,(5)}$.

Sequence ${}^{S}\Phi_{SUD\,(6)}$

The figure 77 shows an example of the sequence type ${}^{S}\Phi_{SUD\,(6)}$. In this, a repulsive dynamics $\Phi_{SU\leftarrow 1,2}$ moving to the left on the DB is observed on the left side of the point of change. At the point of change $\Omega_{ACS}|$ two anti-collinear vectors pointing in opposite directions, perpendicular to the DB is found. On the right side of the point of change is a repulsive segment $\Phi_{SU\rightarrow 1,2}$ that moves toward the right of the DB.

The sequence Φ is symbolized as

${}^{S}\Phi_{SUD\,(6)} = \{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{ACS}|)(\Phi_{SU\rightarrow 1,2})\}$ or { } (4.109)

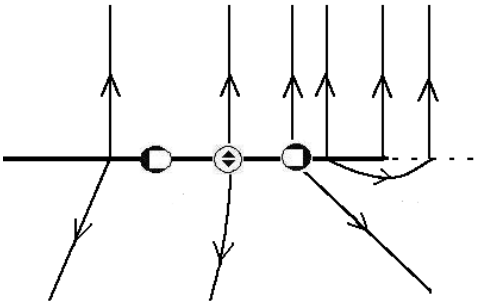
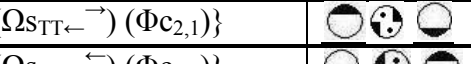
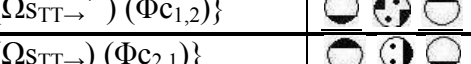
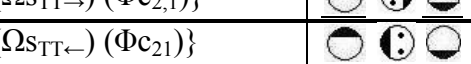
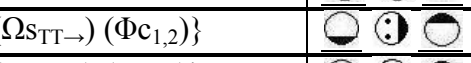
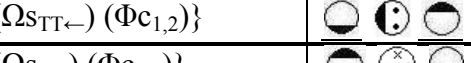
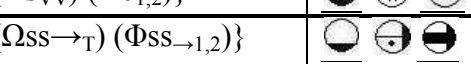
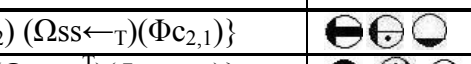
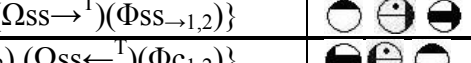
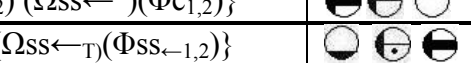
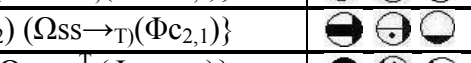
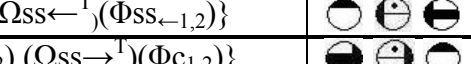
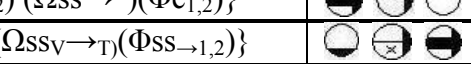
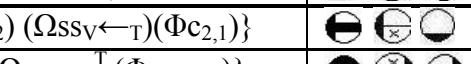
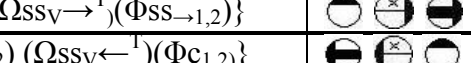
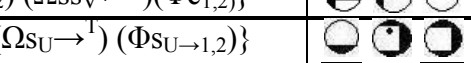
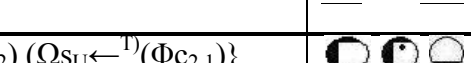
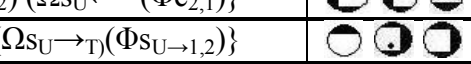
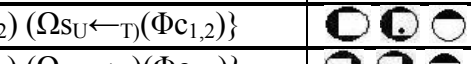
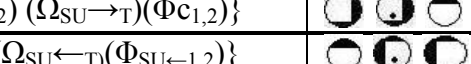
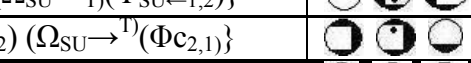
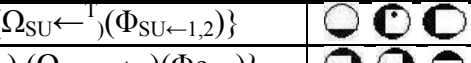
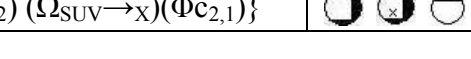


Figure 77. Flow of a system with a change of direction of a repulsive dynamics of the type ${}^{S}\Phi_{SUD\,(6)}$.

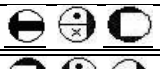
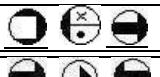
Summarizing the last sections, with the synthesis method, we could define the first 52 forms of sequences over the DB of planar systems, but as the process continued other dynammics containing other types of singular and special points, more DBs, sequences with more elements and higher dimensions can be added. Thus, the database of sequences (see table 4.1) will be bigger and more reliable. Meanwhile, we will use these sequences in the next section to analyse standard forms of local bifurcations.

Table 4.1. Database of sequences over the DB

Dynamics	**Sequence**	**Text symbol**	**Graphic symbol**
Change of crossing dynamics	${}^{S}\Phi_{CDC(1)}$	$(\Phi c_{12})\ (\Omega s_{TT\rightarrow}{}^{\leftarrow})\ (\Phi c_{21})$	
	${}^{S}\Phi_{CDC(2)}$	$\{(\Phi c_{2,1})\ (\Omega s_{TT\leftarrow}{}^{\rightarrow})\ (\Phi c_{1,2})\}$	
	${}^{S}\Phi_{CDC(3)}$	$\{(\Phi c_{1,2})\ (\Omega s_{TT\leftarrow}{}^{\rightarrow})\ (\Phi c_{2,1})\}$	
	${}^{S}\Phi_{CDC(4)}$	$\{(\Phi c_{2,1})\ (\Omega s_{TT\rightarrow}{}^{\leftarrow})\ (\Phi c_{1,2})\}$	
	${}^{S}\Phi_{CDC(5)}$	$\{(\Phi c_{1,2})\ (\Omega s_{TT\rightarrow})\ (\Phi c_{2,1})\}$	
	${}^{S}\Phi_{CDC(6)}$	$\{(\Phi c_{1,2})\ (\Omega s_{TT\leftarrow})\ (\Phi c_{21})\}$	
	${}^{S}\Phi_{CDC(7)}$	$\{(\Phi c_{2,1})\ (\Omega s_{TT\rightarrow})\ (\Phi c_{1,2})\}$	
	${}^{S}\Phi_{CDC(8)}$	$\{(\Phi c_{2,1})\ (\Omega s_{TT\leftarrow})\ (\Phi c_{1,2})\}$	
	${}^{S}\Phi_{CDC(9)}$	$\{(\Phi c_{1,2})\ (\Omega s_{VV})\ (\Phi c_{2,1})\}$	
	${}^{S}\Phi_{CDC(10)}$	$\{(\Phi c_{2,1})\ (\Omega s_{VV})\ (\Phi c_{1,2})\}$	
Start and end of attractive sliding segment	${}^{S}\Phi_{SIF(1)}$	$\{(\Phi c_{2,1})\ (\Omega ss\rightarrow_{T})\ (\Phi ss_{\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SIF(2)}$	$\{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss\leftarrow_{T})(\Phi c_{2,1})\}$	
	${}^{S}\Phi_{SIF(3)}$	$\{(\Phi c_{1,2})\ (\Omega ss\rightarrow^{T})(\Phi ss_{\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SIF(4)}$	$\{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss\leftarrow^{T})(\Phi c_{1,2})\}$	
	${}^{S}\Phi_{SIF(5)}$	$\{(\Phi c_{2,1})\ (\Omega ss\leftarrow_{T})(\Phi ss_{\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SIF(6)}$	$\{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss\rightarrow_{T})(\Phi c_{2,1})\}$	
	${}^{S}\Phi_{SIF(7)}$	$\{(\Phi c_{1,2})\ (\Omega ss\leftarrow^{T})(\Phi ss_{\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SIF(8)}$	$\{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss\rightarrow^{T})(\Phi c_{1,2})\}$	
	${}^{S}\Phi_{SIF(9)}$	$\{(\Phi c_{2,1})\ (\Omega ss_{V}\rightarrow_{T})(\Phi ss_{\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SIF(10)}$	$\{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss_{V}\leftarrow_{T})(\Phi c_{2,1})\}$	
	${}^{S}\Phi_{SIF(11)}$	$\{(\Phi c_{1,2})\ (\Omega ss_{V}\rightarrow^{T})(\Phi ss_{\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SIF(12)}$	$\{(\Phi ss_{\leftarrow 1,2})\ (\Omega ss_{V}\leftarrow^{T})(\Phi c_{1,2})\}$	
Start and end of a repulsive segment	${}^{S}\Phi_{SUIF(1)}$	$\{(\Phi c_{2,1})\ (\Omega s_{U}\rightarrow^{T})\ (\Phi s_{U\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SUIF(2)}$	$\{(\Phi s_{U\leftarrow 1,2})\ (\Omega s_{U}\leftarrow^{T})(\Phi c_{2,1})\}$	
	${}^{S}\Phi_{SUIF(3)}$	$\{(\Phi c_{1,2})\ (\Omega s_{U}\rightarrow_{T})(\Phi s_{U\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SUIF(4)}$	$\{(\Phi s_{U\leftarrow 1,2})\ (\Omega s_{U}\leftarrow_{T})(\Phi c_{1,2})\}$	
	${}^{S}\Phi_{SUIF(5)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SU}\rightarrow_{T})(\Phi c_{1,2})\}$	
	${}^{S}\Phi_{SUIF(6)}$	$\{(\Phi c_{1,2})\ (\Omega_{SU}\leftarrow_{T})(\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SUIF(7)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SU}\rightarrow^{T})(\Phi c_{2,1})\}$	
	${}^{S}\Phi_{SUIF(8)}$	$\{(\Phi c_{2,1})\ (\Omega_{SU}\leftarrow^{T})(\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SUIF(9)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SUV}\rightarrow_{X})(\Phi c_{2,1})\}$	

	${}^{S}\Phi_{SUIF(10)}$	$\{(\Phi c_{2,1})\ (\Omega_{SUV}\leftarrow_{X})\ (\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SUIF(11)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SUV}\rightarrow^{X})(\Phi c_{2,1})\}$	
	${}^{S}\Phi_{SUIF(12)}$	$\{(\Phi c_{2,1)}\ (\Omega_{SUV}\leftarrow^{X)})(\Phi_{SU\leftarrow 1,2})\}$	

Continuation of table

Dynamic	**Sequence**	**Text symbol**	**Graphic symbol**
Change from an attractive sliding segment to a repulsive segment	${}^{S}\Phi_{SI\ (1)}$	$\{(\Phi_{SS\rightarrow 1,2})\ (\Omega_{STT}\rightarrow)\ (\Phi_{SU\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SI\ (2)}$	$\{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{STT}\leftarrow)(\Phi_{SS\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SI\ (3)}$	$\{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{SVT}\overrightarrow{\ }_{X})(\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SI\ (4)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{SVT}\overleftarrow{\ }_{X})(\Phi_{SS\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SI\ (5)}$	$\{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{STV}\overrightarrow{\ }^{X})(\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SI\ (6)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{STV}\overleftarrow{\ }^{X})(\Phi_{SS\rightarrow 1,2)}\}$	
Change in direction of attractive sliding segment	${}^{S}\Phi_{SSE\ (1)}$	$\{(\Phi_{SS\rightarrow 1,2})\ (\Omega_{ACS}/)\ (\Phi_{SS\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SSE\ (2)}$	$\{(\Phi_{SS\rightarrow 1,2})\ (\Omega_{ACS}\backslash)(\Phi_{SS\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SSE\ (3)}$	$\{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{ACS}\backslash)(\Phi_{SS\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SSE\ (4)}$	$\{(\Phi_{SS\leftarrow 1,2})\ (\Omega_{ACS}/)(\Phi_{SS\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SSE\ (5)}$	$\{(\Phi_{SS\rightarrow 1,2})\ (\Omega_{ACS}\ |)(\Phi_{SS\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SSE\ (6)}$	$\{(\Phi_{SS\rightarrow 1,2})\ (\Omega_{ACS}\ |)(\Phi_{SS\leftarrow 1,2})\}$	
Change in direction in a repulsive segment	${}^{S}\Phi_{SUD\ (1)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{ACU}\ /)\ (\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SUD\ (2)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{ACU}\backslash)(\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SUD\ (3)}$	$\{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{ACU}\backslash)(\Phi_{SU\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SUD\ (4)}$	$\{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{ACU}/)(\Phi_{SU\rightarrow 1,2})\}$	
	${}^{S}\Phi_{SUD\ (5)}$	$\{(\Phi_{SU\rightarrow 1,2})\ (\Omega_{ACU}\ |)(\Phi_{SU\leftarrow 1,2})\}$	
	${}^{S}\Phi_{SUD\ (6)}$	$\{(\Phi_{SU\leftarrow 1,2})\ (\Omega_{ACS}\ |)(\Phi_{SU\rightarrow 1,2})\}$	

4.5 Analysis of the DB when there is a change of parameters

With the classification and codification done in the previous sections, it is possible to develop a method to determine the dynamics which are present in the neighbourhood of the DB, without the use of the integration process.

Now, this quick determination is used for detecting changes in the dynamics. When a parameter is changed in the system equations, a change in the coding is possible and a comparison with previous results too. Therefore, it is possible to detect bifurcations.

The procedure is as follows:

1. It is defined the length over the DB that will be analized.
2. It is defined the range of analysis of the parameter.
3. The size of the increment in the change of the parameter value is defined.
4. The procedure is initialized from an initial point through the DB by using each of the parameter values.

For each value of the parameter the result is a sequence of coded data that revels the quantity, the class and the order of the points in the DB.

It is defined as equivalence between two systems if in from a change in the parameter are no changes in:

1. The quantity of components on the DB; the points and the segments are considered elements.
2. The class of each element in the sequence.
3. The order in the sequence of the components in the DB.

Contrary, are declared non-equivalence systems and a local bifurcation the event in which the quantity or the class or the order of the sequence varies.

Before we continue with the explanation of the abovementioned method, we will check the qualitative definition of local sliding bifurcation presented in (3) with the previously presented concept of equivalence in terms of the procedure followed.

In (3), is presented a concept of topological equivalence of the vector fields for planar systems. It is based in the presence of homeomorphism that maps the state portrait of one system onto the state portrait of the other, preserving orientation of the orbits. Also are included the mapping of the sliding segment and the discontinuity boundary from one system to the other. After, in a qualitative way is defined local sliding bifurcation as result

of various facts that conduce to nonequivalent topological between two systems as result in a little change in the parameter. These are:

1. The appearance of disappearance of a sliding segment.
2. The collision of pseudo-equilibria and tangent points in a sliding segment.
3. The approaches and hit of standard hyperbolic equilibrium from one of the regions to the discontinuity boundary.

Doing a comparison between the previous items of the definition of local sliding bifurcation and the rules of the procedure of the method, the second ones are wide and enough to detect all the cases presented due to each case breakes at least one of the three rules of sequences.

Continuing, the results of the evaluation of the method can be presented in a graph or diagram to visualize the dynamic of the bifurcations.

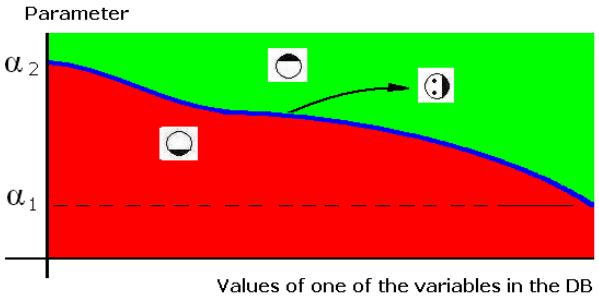


Figure 78. Bifurcation diagram associated to the change in the direction of a crossing segment.

Figure 78 is a bifurcation diagram that shows the types of points in different areas of the diagram. These areas are built by plotting one of the state variables on one axis and the parameter values on the other.

It should be noted that planar systems, the DB is a curve that may or may not be a straight line and may or may not be parallel to one of the axes of the plane.

The result of a gathering of points are presented using a diagram due the method is being tested using an software application, in which it is possible to determine the representation of the change in the dynamics as a function of the position on the DB and the parameter values.

The different areas of the bifurcation diagram are representing the presences of points of the same classification. Colours are associated with the dynamics occupying the biggest areas of the diagrams with the goal to determine easily the neighbourhoods.

Figure 79. Colours associated with the dynamics.

For example, figure 78 shows a diagram of a DB where initially, for $\alpha < \alpha_1$, we have crossing dynamics $\Phi c_{2,1}$ from region Z_2 to region Z_1. When the parameter reaches α_1 for some values of the variable, a double tangency point $\Omega s_{TT}\rightarrow$ appears. This point generates a separatrix in the bifurcation diagram. Then, for values of parameter $\alpha_1 < \alpha < \alpha_2$ the point $\Omega s_{TT}\rightarrow$ can be continued. Additionally, next to the right of this point we have crossing dynamics $\Phi c_{1,2}$. If we further increase the parameter higher than α_2, point $\Omega s_{TT}\rightarrow$ disappears and we only have crossing dynamics of $\Phi c_{1,2}$ type.

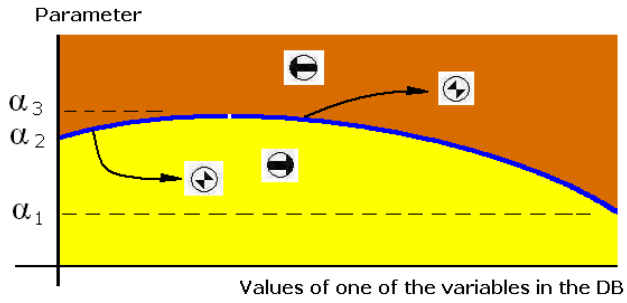


Figure 80. Bifurcation diagram associated to the change in the direction of an attractive sliding segment.

Another example is shows in figure 80 in which is illustrated diagram of a DB where initially, for $\alpha < \alpha_1$, we have sliding dynamics moving toward the right. When the parameter reaches α_1, appears an anti-collinear point $\Omega_{ACS}\backslash$. This point generates a separatrix in the bifurcation diagram. Then, for values of parameter $\alpha_1 < \alpha < \alpha_2$ the point $\Omega_{ACS}\backslash$ can be continued. Additionally, next to the rigth of this point we have a sliding dynamics moving to the left. If we increase the parameter over α_2, point $\Omega_{ACS}/$ appear but from the left side of diagram. If we further increase the parameter over α_3, point $\Omega_{ACS}\backslash$ and $\Omega_{ACS}/$ disappears and we only have a slidinig dynamic moving to left.

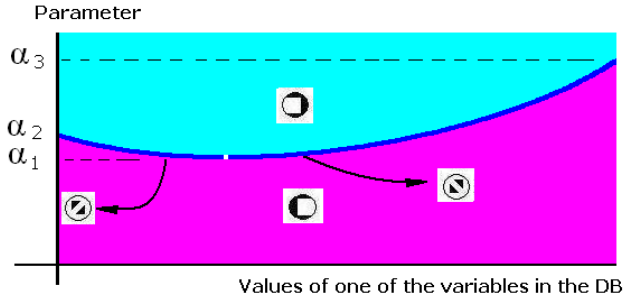


Figure 81. Bifurcation diagram associated to the change in the direction of a repulsive segment.

Another example is shows in figure 81 in which is illustrated diagram of a DB where initially (for $\alpha < \alpha_1$) we have a repulsive pointing to the left. When the parameter reaches α_1, appears two anti-collinear points $\Omega_{ACU}\backslash$ and $\Omega_{ACU}/$. This point generates a separatrix in the bifurcation diagram. Then, for values of parameter $\alpha_1 < \alpha < \alpha_2$ the point separatrix can be continued. Additionally, in the between the points we have a repulsive dynamics pointing to the right. If we increase the parameter over α_2, point $\Omega_{ACU}\backslash$ disappear. If we further increase the parameter over α_3, point $\Omega_{ACU}\backslash$ disappears and we only have a repulsive dynamic pointing to rigth.

The diagram can have wider ranges in the DB and it can have simultaneously more than two types of dynamics. Figure 82 shows an example in which for high values of $\alpha > \alpha_3$ or low values of $\alpha < \alpha_1$, only one type of dynamics is observed, but for values between α_1 and α_3, three types of dynamics are observed along with a strong change in when $\alpha = \alpha_2$.

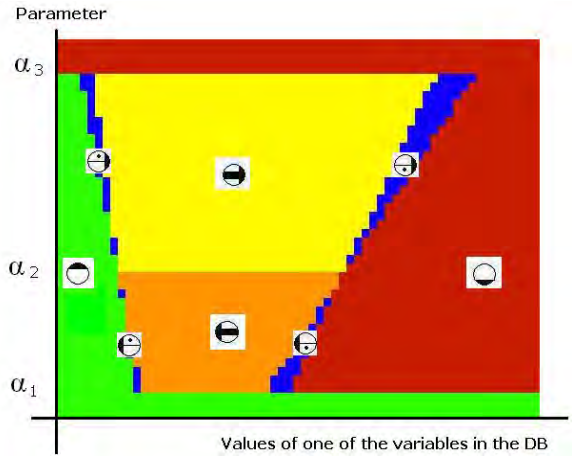


Figure 82. Bifurcation diagram with multiple types of segments in the DB.

The construction of diagrams for detecting bifurcations is not limited to one variable and one parameter. It is possible to construct three-dimensional diagram with two variables and

one parameter or one variable and two parameters, as is shown in Figure 83. In these cases, the different regions are three-dimensional and contain specific dynamics identified by a symbol.

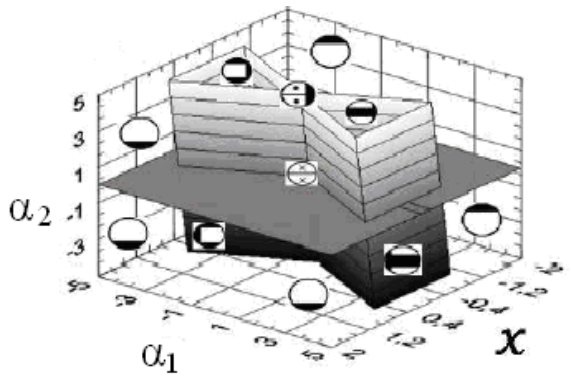


Figure 83. Three-Dimensional Bifurcation diagram of one variable and two parameters.

4.6 Tracking the separatrix curve

The separatrix curve is also called border in this text, in the sections of bifurcation of 2-D zones inside discontinuity boundaries. The diagrams presented in the previous paragraph has been built using the method of brute force, for each value of the parameter, all the range of the variable is scanned. Now, in the cases in which the analysis is complex or the system is extremely large, another method that is more efficient in terms of the processing time should be used. This method is a continuation method that tracks for a specific separatrix curve and for that uses the same theory of ternas. In figure 84 is shown how from an initial point an algorithm changes the values of the variable and the parameter and with information of the database evaluates where the next point of the separatrix can be. In next sections, this issue will be explained a little more. See chapter 5, –eontinuation of cycles" (see also (84), (85)).

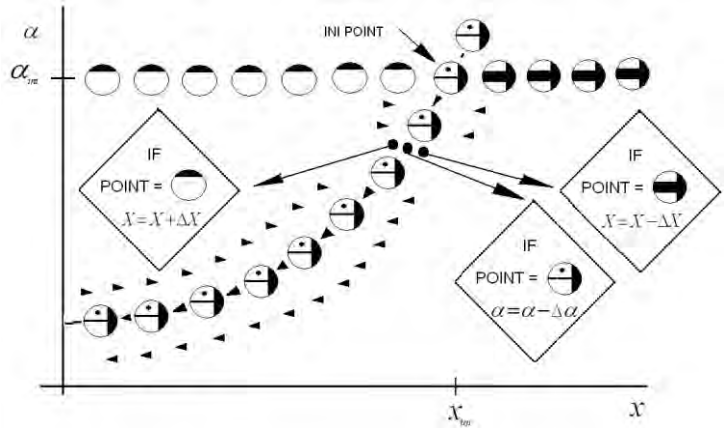


Figure 84. Tracking process of a separatrix curve.

4.7 Validation of the method using the catalogue of local bifurcations

In order to test the results obtained in the previous section, it is necessary to use a considerably large set of results obtained by other methods or tools and use it as problems to be solved with the proposed method. A set of well-classified local bifurcations, including normal forms, was presented by Kuznetsov *et al.* (3); this is one of the best options for the abovementioned task (see also (83)).

4.7.1 Boundary-focus bifurcations

The classified BF bifurcations are related to the collision of a focus with the DB and to the effect of change in the local dynamics.

BF_1

The normal form of the BF_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=x_1-2x_2\\ \dot{x}_2=4x_1\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.110)$$

$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with this dynamics is presented in Figure 85. In this diagram and all the following diagrams in this section, the variable *x* will be analyzed in the range of (-1.0, 1.0) and the parameter will be analyzed in the range of (-1.0, 1.0).

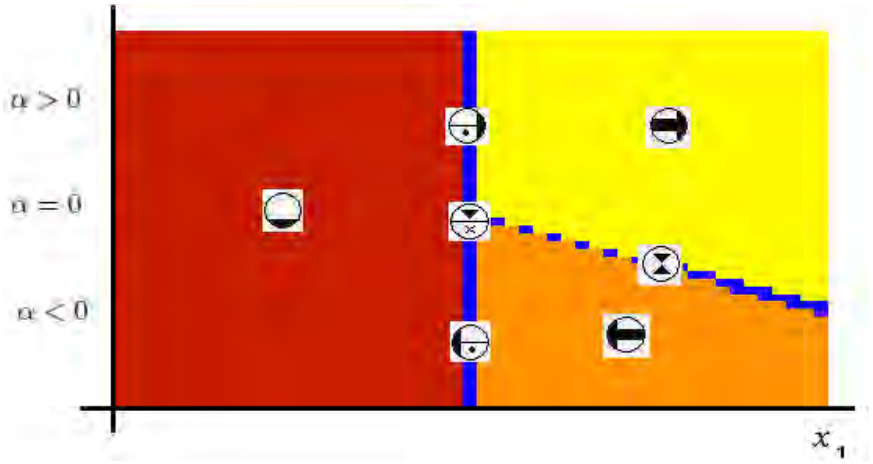


Figure 85. Bifurcation diagram of the form of the bifurcations type BF_1.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi ss_{\leftarrow 1,2})\ (\Omega_{ACS}\ |)\ (\Phi ss_{\rightarrow 1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss_{\rightarrow 1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.111)

From a total of 19 local bifurcations, only two bifurcations posed difficulties in the process of detection. This method is unable to distinguish between bifurcations BF_1 and BF_2 because the differences are not within the DB; they are outside the DB. In order to discriminate between these cases, it is necessary to use an additional process to the method for local bifurcation. This additional process is discussed in the next chapter. This process is also useful for analysing global bifurcations.

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BF_3

The normal form of the BF_3 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=-x_1-2x_2\\ \dot{x}_2=4x_1+2x_2\end{array}\right\}$$
$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.112)$$
$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 86.

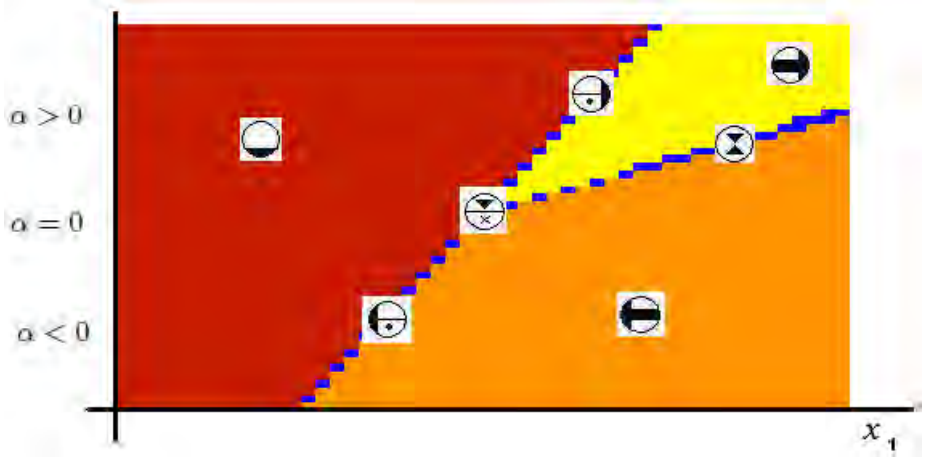


Figure 86. Bifurcation diagram of the form of the bifurcations type BF_3.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss\leftarrow_{T})\ (\Phi ss\leftarrow_{1,2})\}$ o
$^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss\leftarrow_{1,2})\}$ o
$^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\,|)\ (\Phi ss\leftarrow_{1,2})\}$
or
$^{S}\alpha<0 \Rightarrow \{$ $\}$
$^{S}\alpha=0 \Rightarrow \{$ $\}$
$^{S}\alpha>0 \Rightarrow \{$ $\}$ (4.113)

BF_4

The normal form of the BF_4 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=x_1-2x_2\\ \dot{x}_2=3x_1\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=0\\ \dot{x}_2=1\end{array}\right\} \qquad (4.114)$$

$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 87.

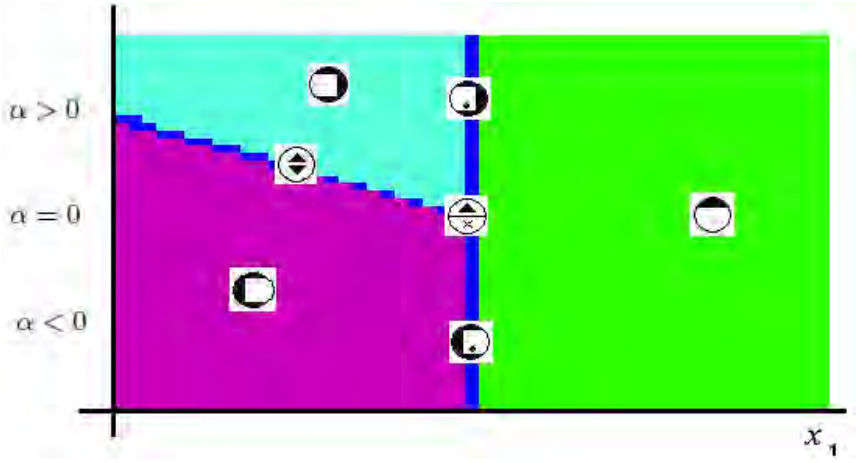


Figure 87. Bifurcation diagram of the form of the bifurcations type BF_4.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi s_{U\leftarrow 1,2})\ (\Omega su^{\leftarrow}{}_{T})\ (\Phi c_{1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi s_{U\leftarrow 1,2})\ (\Omega_{SVP})\ (\Phi c_{1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi s_{U\leftarrow 1,2})\ (\Omega_{ACU}|)\ (\Phi s_{U\rightarrow 1,2})\ (\Omega su^{\rightarrow}{}_{T}|)\ (\Phi c_{1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.115)

BF_5

The normal form of the BF_5 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{matrix}\dot{x}_1=-x_1-2x_2\\ \dot{x}_2=4x_1+2x_2\end{matrix}\right\}$$
$$f_2(x,\alpha)=\left\{\begin{matrix}\dot{x}_1=0\\ \dot{x}_2=1\end{matrix}\right\} \qquad (4.116)$$
$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 88.

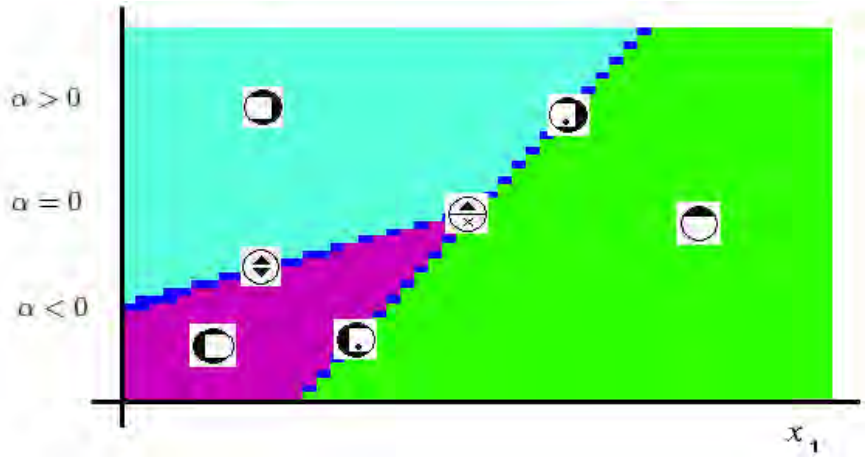


Figure 88. Bifurcation diagram of the form of the bifurcations type BF_5.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi s_{U\rightarrow 1,2})\ (\Omega_{ACU}|)\ (\Phi s_{U\leftarrow 1,2})\ (\Omega su^{\leftarrow}{}_{T})\ (\Phi c_{1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi s_{U\rightarrow 1,2})\ (\Omega_{UVP})\ (\Phi c_{1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi s_{U\rightarrow 1,2})\ (\Omega su^{\rightarrow}{}_{T})\ (\Phi c_{1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {} 4.117)

4.7.2 Boundary-node bifurcations

The classified BN bifurcations are related to the collision of a node with the DB and to the effect of or change in the local dynamics.

BN_1

The normal form of the BN_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=-3x_1-x_2\\ \dot{x}_2=-x_1-3x_2\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.118)$$

$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 89.

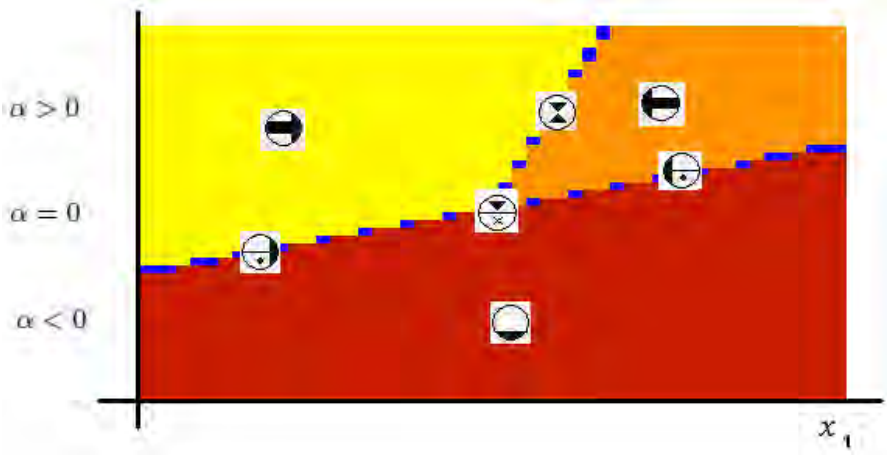


Figure 89. Bifurcation diagram of the normal form of the bifurcations type BN_1.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi c_{2,1})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{SVP})\ (\Phi c_{2,1})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}|)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi c_{2,1})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }

$^{S}\alpha>0 \Rightarrow \{$  $\}$ (4.119)

BN_2

The normal form of the BN_2 bifurcations is

$$f_1(x,\alpha)=\begin{Bmatrix}\dot{x}_1=-3x_1-x_2\\ \dot{x}_2=-x_1-3x_2\end{Bmatrix}$$

$$f_2(x,\alpha)=\begin{Bmatrix}\dot{x}_1=0\\ \dot{x}_2=1\end{Bmatrix} \qquad (4.120)$$

$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 90.

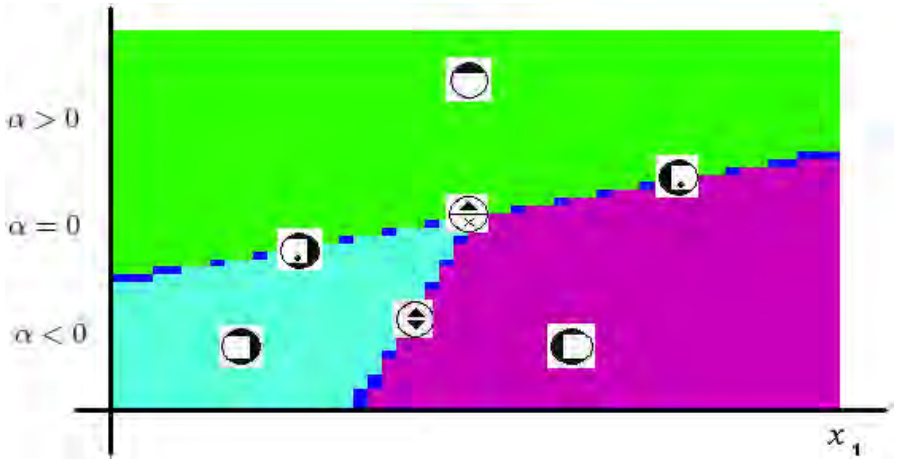
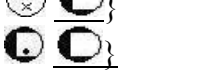


Figure 90. Bifurcation diagram of the form of the bifurcations type BN_2.

By analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi c_{1,2})\ (\Omega su^{\rightarrow}{}_{T})\ (\Phi s_{U\rightarrow 1,2})\ (\Omega_{ACU}|)\ (\Phi s_{U\leftarrow 1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi c_{1,2})\ (\Omega_{UVP})\ (\Phi s_{U\leftarrow 1,2})\}$
$^{S}\alpha>0 \Rightarrow (\Phi c_{1,2})\ (\Omega su^{\leftarrow}{}_{T})\ (\Phi s_{U\leftarrow 1,2})\}$
or
$^{S}\alpha<0 \Rightarrow \{$  $\}$
$^{S}\alpha=0 \Rightarrow \{$  $\}$
$^{S}\alpha>0 \Rightarrow \{$ $\}$ (4.121)

4.7.3 Boundary-saddle bifurcations

The classified BS bifurcations are related to the collision of a saddle node with the DB and to the effect of change in the local dynamics.

BS_1

The normal form of the BS_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=-x_1+3x_2\\ \dot{x}_2=3x_1-2x_2\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.122)$$

$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 91.

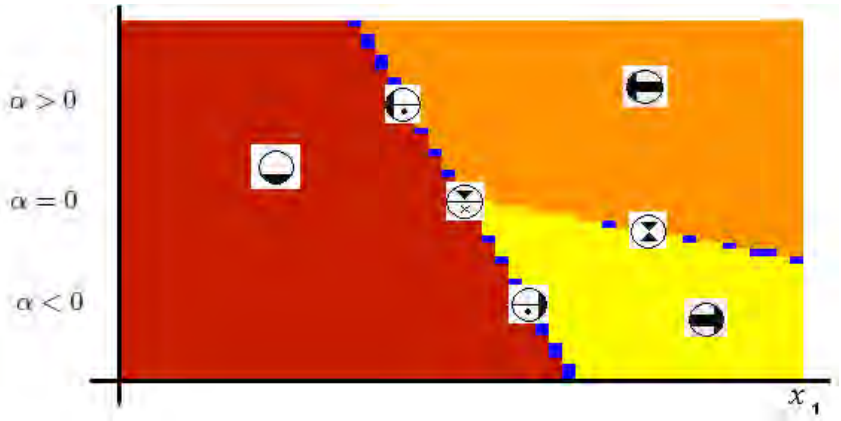


Figure 91. Bifurcation diagram of the form of the bifurcations type BS_1.

By analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}|)\ (\Phi ss_{\leftarrow 1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss_{\leftarrow 1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }

$^{S}\alpha>0 \Rightarrow$ {  } (4.123)

BS_2

The normal form of the BS_2 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=-2x_1-x_2\\ \dot{x}_2=x_1+x_2\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.124)$$

$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 92.

Figure 92. Bifurcation diagram of the form of the bifurcations type BS_2.

Analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow$ {$(\Phi c_{2,1}(\Omega ss^{\rightarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}|)\ (\Phi ss_{\leftarrow 1,2})$}
$^{S}\alpha=0 \Rightarrow$ {$(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss_{\leftarrow 1,2})$}
$^{S}\alpha>0 \Rightarrow$ {$(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi ss_{\leftarrow 1,2})$}
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.125)

BS_3

The normal form of the BS_3 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=x_1-3x_2\\ \dot{x}_2=-3x_1+x_2\end{array}\right\}$$
$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.126)$$
$$H(x,\alpha)=x_2+\alpha$$

The diagram associated with these dynamics is presented in figure 93.

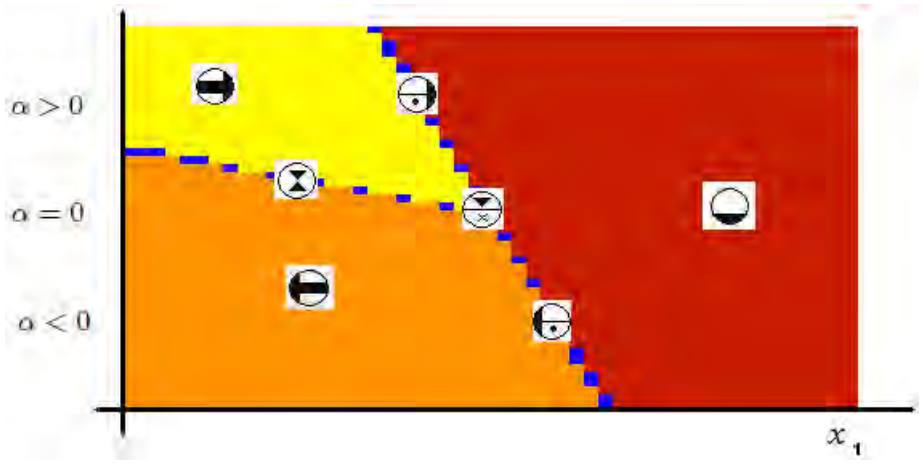


Figure 93. Bifurcation diagram of the form of the bifurcations type BS_3.

By analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\leftarrow 1,2})\,(\Omega ss^{\leftarrow}{}_{T})\,(\Phi c_{2,1})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\leftarrow 1,2})\,(\Omega_{SVP})(\Phi c_{2,1})\}$
$^{S}\alpha>0 \Rightarrow \{((\Phi ss_{\leftarrow 1,2})\,(\Omega_{ACS}|^{)}\,(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi c_{2,1})\}$
or
$^{S}\alpha<0 \Rightarrow$
$^{S}\alpha=0 \Rightarrow$
$^{S}\alpha>0 \Rightarrow$ (4.127)

4.7.4 Double-tangency bifurcations

The classified DT bifurcations are related to the presence of a point in the DB with two associated vectors of type attractive and a tangent.

DT_1

The normal form of the DT_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=1\\ \dot{x}_2=\alpha+4x_1^2\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \qquad (4.128)$$

$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 94.

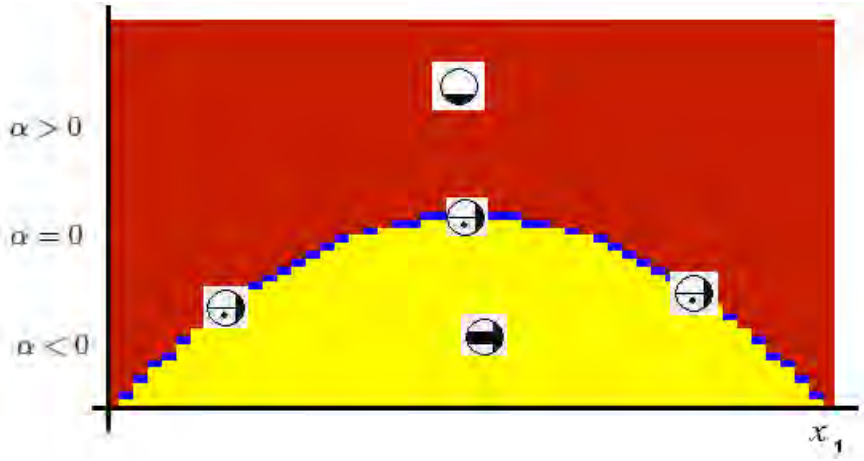


Figure 94. Bifurcation diagram of the form of the bifurcations type DT_1.

Analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi c_{2,1})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi c_{2,1})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\}$
or
$^{S}\alpha<0 \Rightarrow$ 
$^{S}\alpha=0 \Rightarrow$ 

$^{S}\alpha>0 \Rightarrow$ {} (4.129)

DT2

The normal form of the DT_2 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=1\\ \dot{x}_2=-\alpha+4x_1^2\end{array}\right\}$$
$$f_2(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=0\\ \dot{x}_2=-1\end{array}\right\} \quad (4.130)$$
$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 95.

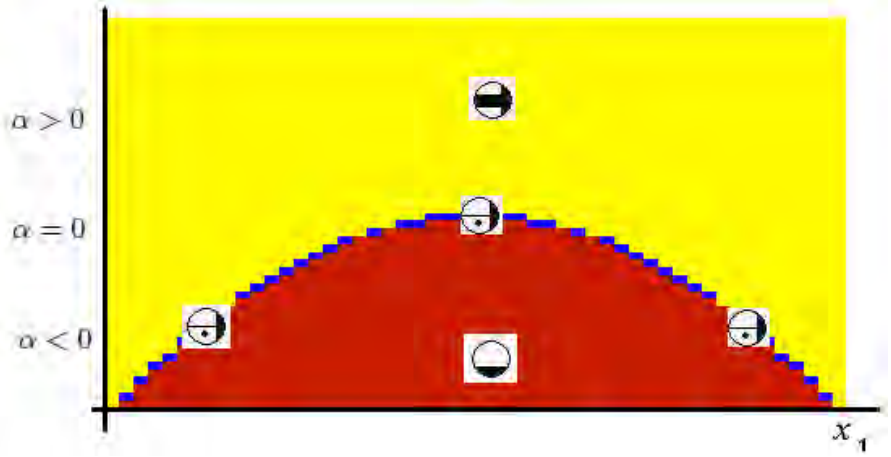
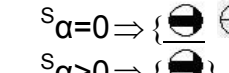


Figure 95. Bifurcation diagram of the form of the bifurcations type DT_2.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow$ {($\Phi ss_{\rightarrow 1,2}$) ($\Omega ss^{\rightarrow}{}_{T}$) ($\Phi c_{2,1}$) ($\Omega ss^{\rightarrow}{}_{T}$) ($\Phi ss_{\rightarrow 1,2}$)}
$^{S}\alpha=0 \Rightarrow$ {($\Phi ss_{\rightarrow 1,2}$) ($\Omega ss^{\rightarrow}{}_{T}$) ($\Phi ss_{\rightarrow 1,2}$)}
$^{S}\alpha>0 \Rightarrow$ {($\Phi ss_{\rightarrow 1,2}$)}
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {} (4.131)

4.7.5 Visible-visible bifurcations

The classified VV bifurcations are related to the presence in the DB of a point with two associated vectors of the type tangential .

VV_1

The normal form of the VV_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=1\\ \dot{x}_2=-(\alpha+x_1)\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=1-x_1\\ \dot{x}_2=x_1\end{array}\right\} \qquad (4.132)$$

$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 96.

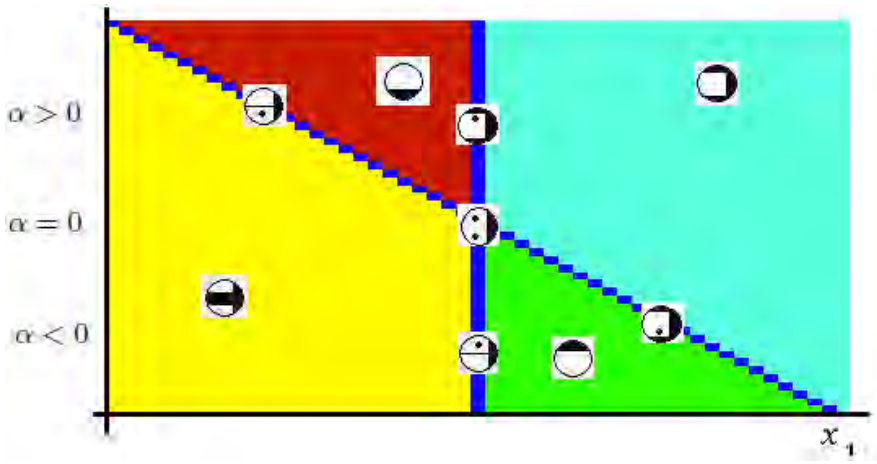
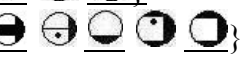


Figure 96. Bifurcation diagram of the normal form of the bifurcations type VV_1.

Analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\ (\Omega su^{\rightarrow}{}_{T})\ (\Phi s_{U\rightarrow 1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{SVP})\ (\Phi s_{U\rightarrow 1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }

$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.133)

VV_2

The normal form of the VV_2 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=-1\\ \dot{x}_2=(\alpha+x_1)\end{array}\right\}$$
$$f_2(x,\alpha)=\left\{\begin{array}{c}\dot{x}_1=1-x_1\\ \dot{x}_2=x_1\end{array}\right\} \quad (4.134)$$
$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 97.

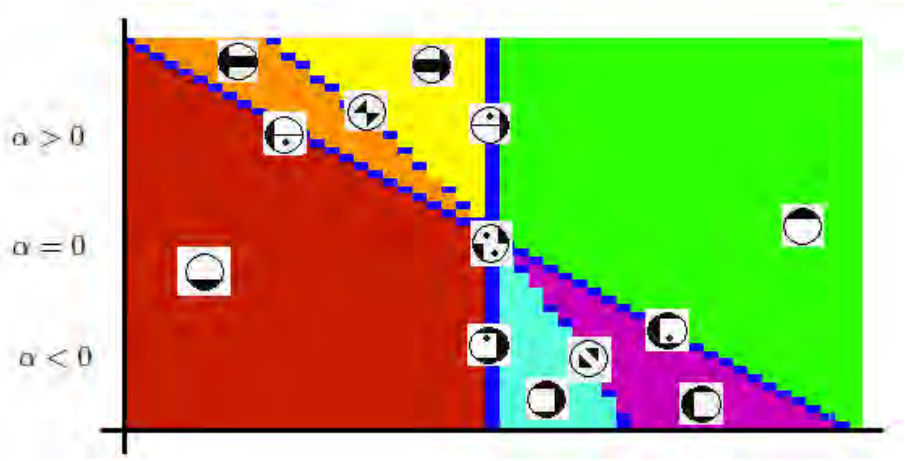


Figure 97. Bifurcation diagram of the form of the bifurcations type VV_2.

By analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow$ {($\Phi c_{2,1}$) ($\Omega su^{\rightarrow T}$) ($\Phi s_{U\rightarrow 1,2}$) ($\Omega_{ACU}$ /) ($\Phi s_{U\leftarrow 1,2}$) ($\Omega su^{\leftarrow}{}_{T}$) ($\Phi c_{1,2}$)}
$^{S}\alpha=0 \Rightarrow$ {($\Phi c_{2,1}$) ($\Omega s_{TT\leftarrow}{}^{\rightarrow}$) ($\Phi c_{1,2}$)}
$^{S}\alpha>0 \Rightarrow$ {($\Phi c_{2,1}$) ($\Omega ss^{\leftarrow}{}_{T}$) ($\Phi ss_{\leftarrow 1,2}$) ($\Omega_{ACS}$ \) ($\Phi ss_{\rightarrow 1,2}$) ($\Omega ss^{\rightarrow T}$) ($\Phi c_{1,2}$)}
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }

$^{S}\alpha>0 \Rightarrow$ {  } (4.135)

4.7.6 Visible-invisible bifurcations

The classified VI bifurcations are related to the presence of a point in the DB with two associated vectors of the type vanished and tangent.

VI_1

The normal form of the VI_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1-x_1\\ \dot{x}_2=(\alpha+x_1)\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1-x_1\\ \dot{x}_2=2x_1\end{array}\right\} \qquad (4.136)$$

$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 98.

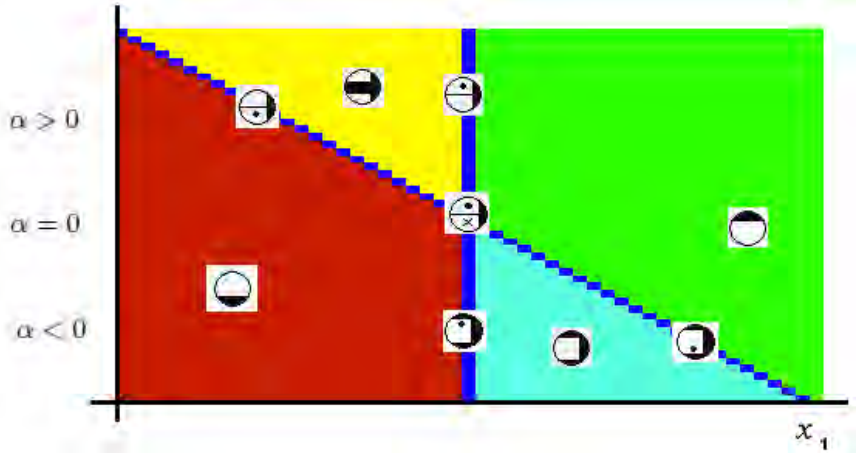


Figure 98. Bifurcation diagram of the form of the bifurcations type VI_1.

Analyzing this diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\ (\Omega su^{\rightarrow}{}_{T})\ (\Phi c_{1,2})\}$

$^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega_{SVT}{}^{\rightarrow}{}_{X})\ (\Phi c_{1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.137)

VI_2

The normal form of the VI_2 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=-1-x_1\\ \dot{x}_2=-(\alpha+x_1)\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1-x_1\\ \dot{x}_2=2x_1\end{array}\right\} \qquad (4.138)$$

$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 99.

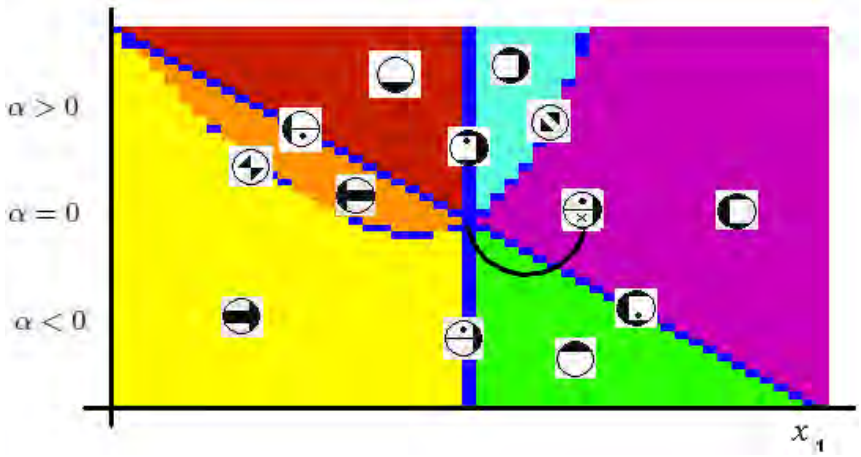


Figure 99. Bifurcation diagram of the form of the bifurcations type VI_2.

Analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\ (\Omega su^{\leftarrow}{}_{T})\ (\Phi s_{U\leftarrow 1,2})\}$

$^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\backslash)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega_{SVT}{}^{\rightarrow}{}_{X})\ (\Phi s_{U\leftarrow 1,2})\}$

$^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\backslash)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\ (\Omega_{ACU}/)\ (\Phi s_{U\leftarrow 1,2})$

or

$^{S}\alpha<0 \Rightarrow \{$  $\}$

$^{S}\alpha=0 \Rightarrow \{$  $\}$

$^{S}\alpha>0 \Rightarrow \{$  $\}$ (4.139)

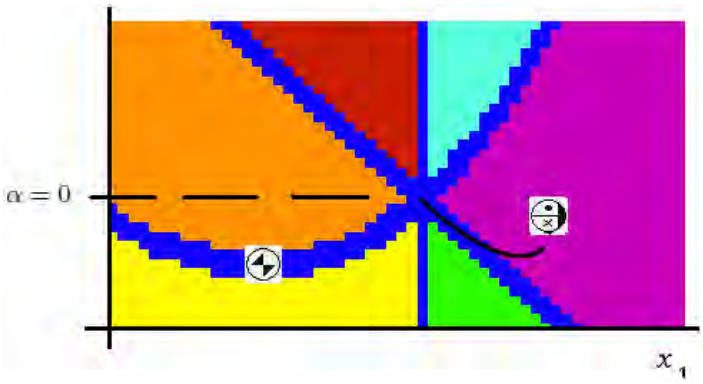


Figure 100. Zoom of the diagram in figure 99.

In this normal form, there are two analysis regions. The first one is used for negative values for α that are far away from zero, and the second one is used for negative values for α close to zero. The first case was already presented, but the second case shows a difference in the first data element of the sequence.

$^{S}\alpha<0 \Rightarrow \{$  $\}$

As observed, the first five elements of this sequence are by themselves a bifurcation, that is, there is double bifurcation (see the enlarged image in Figure 100).

Space left intentionally blank.

VI_3

The normal form of the VI_3 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1+x_1\\ \dot{x}_2=-\alpha-2x_1\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1-x_1\\ \dot{x}_2=x_1\end{array}\right\} \qquad (4.140)$$

$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 101.

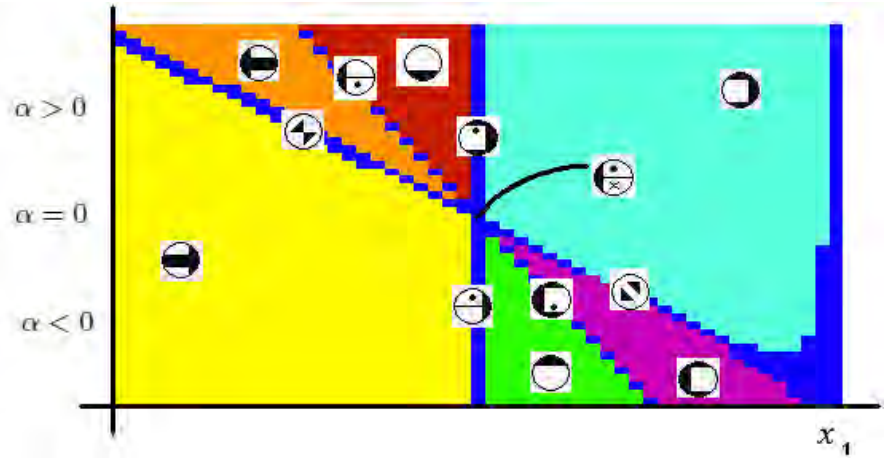


Figure 101. Bifurcation diagram of the form of the bifurcations type VI_3.

Analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\ (\Omega su^{\leftarrow}{}_{T})\ (\Phi s_{U\leftarrow 1,2})\ (\Omega_{ACU}/)\ (\Phi s_{U\rightarrow 1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{SVT}{}^{\leftarrow}{}_{X})\ (\Phi s_{U\rightarrow 1,2})\}$
$^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\backslash)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\}$
or
$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {} (4.141)

4.7.7 Invisible-invisible bifurcations

The classified II bifurcations are related to the presence in the DB of a point with two associtated vectors of the type vanished.

II_1

The normal form of the II_1 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=-1-x_1\\ \dot{x}_2=-x_1\end{array}\right\}$$

$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1\\ \dot{x}_2=\alpha-x_1\end{array}\right\} \qquad (4.142)$$

$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 102.

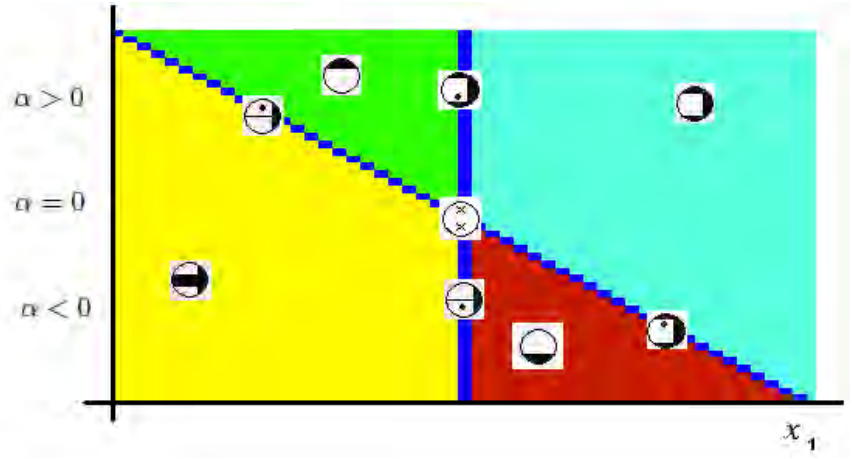
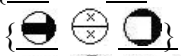


Figure 102. Bifurcation diagram of the form of the bifurcations type II_1.

Analyzing the diagram, we find that the sequence in the DB is as follows:

$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\}$
$^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega s_{VV})\ (\Phi s_{U\rightarrow 1,2})\ \}$
$^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\}$
or

$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.143)

II_2

The normal form of the II_2 bifurcations is

$$f_1(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1+x_1\\ \dot{x}_2=x_1\end{array}\right\}$$
$$f_2(x,\alpha)=\left\{\begin{array}{l}\dot{x}_1=1\\ \dot{x}_2=\alpha-x_1\end{array}\right\} \quad (4.144)$$
$$H(x,\alpha)=x_2$$

The diagram associated with these dynamics is presented in figure 103.

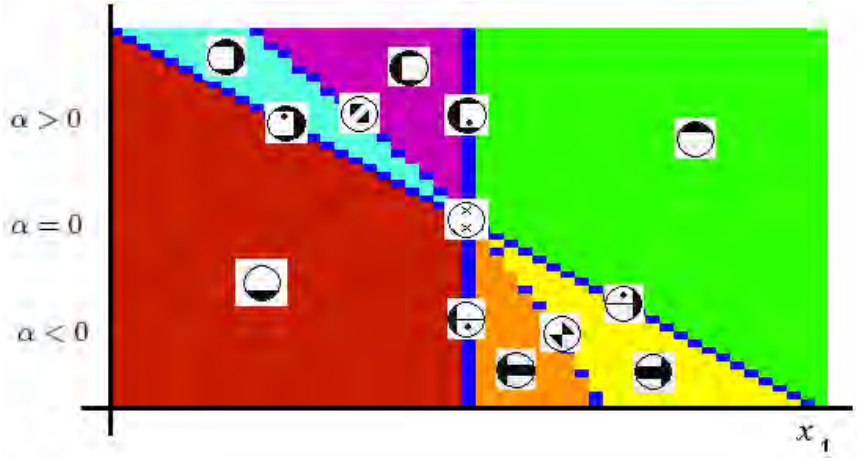


Figure 103. Bifurcation diagram of the form of the bifurcations type II_2.

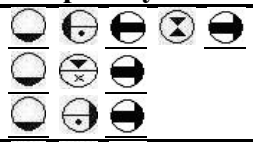
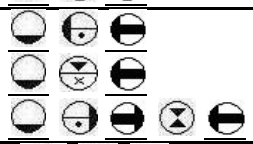
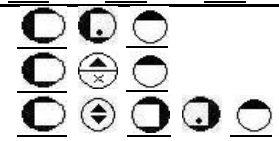
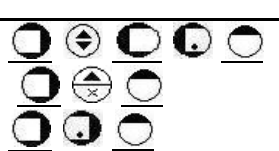
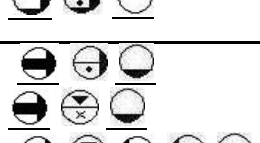
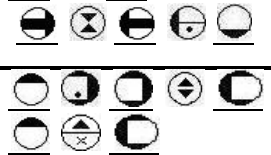
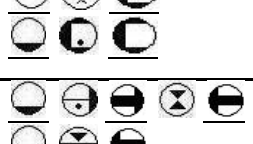
Analyzing the diagram, we find that the sequence in the DB is as follows:

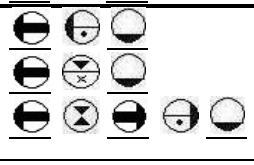
$^{S}\alpha<0 \Rightarrow$ {($\Phi c_{2,1}$) ($\Omega ss^{\leftarrow}{}_{T}$) ($\Phi ss_{\leftarrow 1,2}$) ($\Omega_{ACS}$ /) ($\Phi ss_{\rightarrow 1,2}$) ($\Omega ss^{\rightarrow T}$) ($\Phi c_{1,2}$)}
$^{S}\alpha=0 \Rightarrow$ {($\Phi c_{2,1}$) (Ωs_{VV}) ($\Phi c_{1,2}$)}
$^{S}\alpha>0 \Rightarrow$ {($\Phi c_{2,1}$) ($\Omega su^{\rightarrow T}$) ($\Phi s_{U\rightarrow 1,2}$) ($\Omega_{ACU}$ \) ($\Phi s_{U\leftarrow 1,2}$) ($\Omega su^{\leftarrow}{}_{T}$) ($\Phi c_{1,2}$)}
or

$^{S}\alpha<0 \Rightarrow$ {  }
$^{S}\alpha=0 \Rightarrow$ {  }
$^{S}\alpha>0 \Rightarrow$ {  } (4.145)

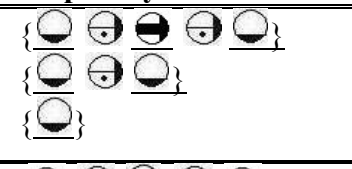
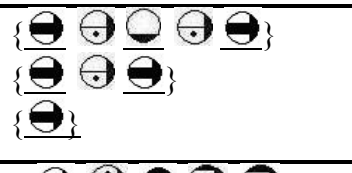
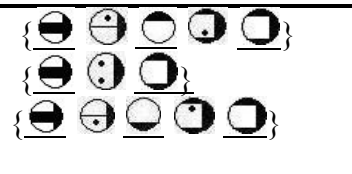
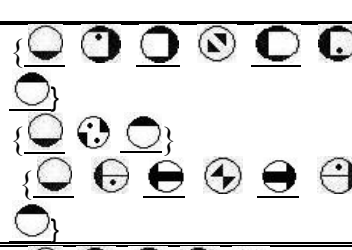
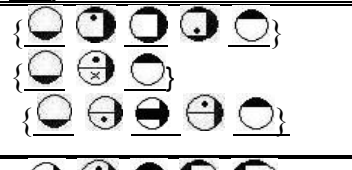
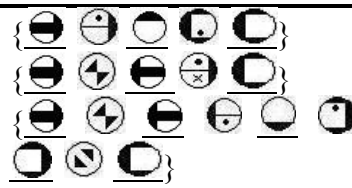
As a Database of this section we have the table 4.2, which is used as source of information by the method SPT to detect what class of bifurcation is presenting at the moment that in the process of gathering, a terna of sequences over the DB is registered.

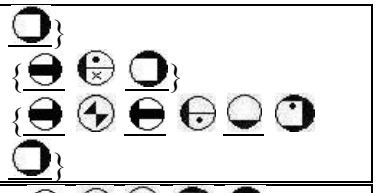
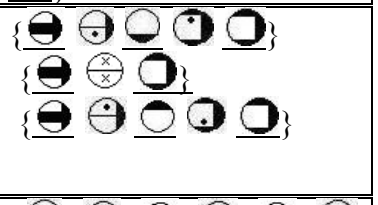
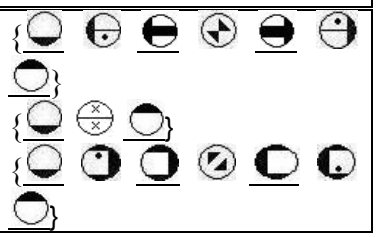
Table 4.2. Database of the local sliding bifurcation.

Type	Text symbol	Graphic symbol
BF_1	$\{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_T)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega_{ACS}\,|)\ (\Phi ss_{\rightarrow 1,2})\}$ $\{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss_{\rightarrow 1,2})\}$ $\{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_T)\ (\Phi ss_{\rightarrow 1,2})\}$	
BF3	$\{(\Phi c_{2,1})\ (\Omega ss\leftarrow_T)\ (\Phi ss\leftarrow_{1,2})\}$ $\{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss\leftarrow_{1,2})\}$ $\{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_T)\ (\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\,|)\ (\Phi ss\leftarrow_{1,2})\}$	
BF4	$^{S}\alpha<0 \Rightarrow \{(\Phi s_{U\leftarrow 1,2})\ (\Omega su^{\leftarrow}{}_T)\ (\Phi c_{1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi s_{U\leftarrow 1,2})\ (\Omega_{SVP})\ (\Phi c_{1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi s_{U\leftarrow 1,2})\ (\Omega_{ACU}\,|)\ (\Phi s_{U\rightarrow 1,2})\ (\Omega su^{\rightarrow}{}_T\,|)\ (\Phi c_{1,2})\}$	
BF5	$^{S}\alpha<0 \Rightarrow \{(\Phi s_{U\rightarrow 1,2})\ (\Omega_{ACU}\,|)\ (\Phi s_{U\leftarrow 1,2})\ (\Omega su^{\leftarrow}{}_T)\ (\Phi c_{1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi s_{U\rightarrow 1,2})\ (\Omega_{UVP})\ (\Phi c_{1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi s_{U\rightarrow 1,2})\ (\Omega su^{\rightarrow}{}_T)\ (\Phi c_{1,2})\}$	
BN1	$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow}{}_T)\ (\Phi c_{2,1})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{SVP})\ (\Phi c_{2,1})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\,|)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega ss^{\leftarrow}{}_T)\ (\Phi c_{2,1})\}$	
BN_2	$^{S}\alpha<0 \Rightarrow \{(\Phi c_{1,2})\ (\Omega su^{\rightarrow}{}_T)\ (\Phi s_{U\rightarrow 1,2})\ (\Omega_{ACU}\,|)\ (\Phi s_{U\leftarrow 1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi c_{1,2})\ (\Omega_{UVP})\ (\Phi s_{U\leftarrow 1,2})\}$ $^{S}\alpha>0 \Rightarrow (\Phi c_{1,2})\ (\Omega su^{\leftarrow}{}_T)\ (\Phi s_{U\leftarrow 1,2})\}$	
BS_1	$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\rightarrow}{}_T)\ (\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\,|)\ (\Phi ss_{\leftarrow 1,2})_{\}}$ $^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss_{\leftarrow 1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_T)\ (\Phi ss_{\rightarrow 1,2})\}$	
BS_2	$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1}(\Omega ss^{\rightarrow}{}_T)\ (\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\,|)\ (\Phi ss_{\leftarrow 1,2})_{\}}$ $^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega_{SVP})\ (\Phi ss_{\leftarrow 1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_T)\ (\Phi ss_{\leftarrow 1,2})\}$	

		
BS_3	$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\leftarrow 1,2})\,(\Omega ss^{\leftarrow}{}_{T})\,(\Phi c_{2,1})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\leftarrow 1,2})\,(\Omega_{SVP})(\Phi c_{2,1})\}$ $^{S}\alpha>0 \Rightarrow \{((\Phi ss_{\leftarrow 1,2})\,(\Omega_{ACS}\vert^{)}\,(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi c_{2,1})\}$	

Continuación of table 4.2

Type	**Text symbol**	**Graphic symbol**
DT1	$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi c_{2,1})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi c_{2,1})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\}$	
DT2	$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi c_{2,1})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi ss_{\rightarrow 1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi ss_{\rightarrow 1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\}$	
VV_1	$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow T})\,(\Phi c_{1,2})\,(\Omega su^{\rightarrow}{}_{T})\,(\Phi s_{U\rightarrow 1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega_{SVP})\,(\Phi s_{U\rightarrow 1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi c_{2,1})\,(\Omega su^{\rightarrow T})$ $(\Phi s_{U\rightarrow 1,2})\}$	
VV_2	$\{(\Phi c_{2,1})\,(\Omega su^{\rightarrow T})\,(\Phi s_{U\rightarrow 1,2})\,(\Omega_{ACU}/)\,(\Phi s_{U\leftarrow 1,2})\,(\Omega su^{\leftarrow}{}_{T})$ $(\Phi c_{1,2})\}$ $\{(\Phi c_{2,1})\,(\Omega s_{TT\leftarrow}{}^{\rightarrow})\,(\Phi c_{1,2})\}$ $\{(\Phi c_{2,1})\,(\Omega ss^{\leftarrow}{}_{T})\,(\Phi ss_{\leftarrow 1,2})\,(\Omega_{ACS}\backslash)\,(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow T})$ $(\Phi c_{1,2})\}$	
VI_1	$^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\,(\Omega su^{\rightarrow T})\,(\Phi s_{U\rightarrow 1,2})\,(\Omega su^{\rightarrow}{}_{T})\,(\Phi c_{1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\,(\Omega_{SVT}{}^{\rightarrow}{}_{X}{}^{)}\,(\Phi c_{1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\,(\Omega ss^{\rightarrow}{}_{T})\,(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow T})\,(\Phi c_{1,2})\}$	
VI_2	$^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow T})\,(\Phi c_{1,2})\,(\Omega su^{\leftarrow}{}_{T})$ $(\Phi s_{U\leftarrow 1,2})\}$ $^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega_{ACS}\backslash)\,(\Phi ss_{\leftarrow 1,2})\,(\Omega_{SVT}{}^{\rightarrow}{}_{X}{}^{)}$ $(\Phi s_{U\leftarrow 1,2})\}$ $^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\,(\Omega_{ACS}\backslash)\,(\Phi ss_{\leftarrow 1,2})\,(\Omega ss^{\leftarrow}{}_{T})\,(\Phi c_{2,1})$ $(\Omega su^{\rightarrow T})\,(\Phi s_{U\rightarrow 1,2})\,(\Omega_{ACU}/)$	
VI_3	$\{(\Phi ss_{\rightarrow 1,2})\,(\Omega ss^{\rightarrow T})\,(\Phi c_{1,2})\,(\Omega su^{\leftarrow}{}_{T})\,(\Phi s_{U\leftarrow 1,2})\,(\Omega_{ACU}/)$	

	$(\Phi s_{U\rightarrow 1,2})\}$ $\{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{SVT}{}^{\leftarrow}{}_{X})\ (\Phi s_{U\rightarrow 1,2})\}$ $\{(\Phi ss_{\rightarrow 1,2})\ (\Omega_{ACS}\ \backslash)\ (\Phi ss_{\leftarrow 1,2})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\}$	
II_1	${}^{S}\alpha<0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow}{}_{T})\ (\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\}$ ${}^{S}\alpha=0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega s_{VV})\ (\Phi s_{U\rightarrow 1,2})\ \}$ ${}^{S}\alpha>0 \Rightarrow \{(\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\ (\Omega su^{\rightarrow}{}_{T})\ (\Phi s_{U\rightarrow 1,2})\}$	
II_2	${}^{S}\alpha<0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega ss^{\leftarrow}{}_{T})\ (\Phi ss_{\leftarrow 1,2})\ (\Omega_{ACS}\ /)\ (\Phi ss_{\rightarrow 1,2})\ (\Omega ss^{\rightarrow T})\ (\Phi c_{1,2})\}$ ${}^{S}\alpha=0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega s_{VV})\ (\Phi c_{1,2})\}$ ${}^{S}\alpha>0 \Rightarrow \{(\Phi c_{2,1})\ (\Omega su^{\rightarrow T})\ (\Phi s_{U\rightarrow 1,2})\ (\Omega_{ACU}\ \backslash)\ (\Phi s_{U\leftarrow 1,2})\ (\Omega su^{\leftarrow}{}_{T})\ (\Phi c_{1,2})\}$	

4.8 Sliding dynamics in the DB of three-dimensional systems

Sliding (attracting and repulsive), and crossing segments in planar systems were presented in the previous sections. Further, in three-dimensional systems, the segments become sectors or areas. In n-dimensional systems, the segments become hyper-surfaces of (n – 1) dimensions.

The results obtained in the previous sections are also useful for studying local sliding bifurcations in higher dimensions. But higher-dimension systems have richer dynamics in the DB than those of planar systems. Higher-dimension dynamic systems can in turn show all types of bifurcations—smooth, nonsmooth, and sliding nonsmooth bifurcations. Therefore, this is a big research area. In this section, we will introduce the effect of the extension of the results previously obtained when they are applied to higher-dimensional systems (see also (84) (85)).

Sliding hyper-surfaces capture the dynamics of an *n*-dimensional system in a hyper-surface of (*n – 1)* dimensions. These sub-manifolds determine the dynamics of the full system, if we are close enough to the sub-manifold. Moreover, for example in 3-dimensional systems, the hyper-surface can contain a stable limit cycle in an attractive sliding region, which avoids the orbits to return to the full-dimension state space (as a pseudo-equilibrium in planar systems). This means that the hyper-surface is locally stable, which can be very useful in the field of control systems.

System representation

In this section, we perform a generalization of the mathematical framework given in the previous sections.

Let $\dot{\mathbf{x}} = f_j(\mathbf{x}, \boldsymbol{\alpha})$ (4.146)

n: order of the system with $i = 1, 2, \ldots, n$
$m \in R$: number of vector fields with $j = 1, 2, 3, \ldots, m$
Z_j: sub-regions where the vector fields are defined.
$\boldsymbol{\alpha}$: the vector of parameters.

Let $\Sigma_{j,j}$ be the symbol of a set of hyper-surfaces represented by the algebraic equation $H_{j,j}(\mathbf{x}) = 0$ that corresponds to the DB between the regions. We will assume that the set of hyper-surfaces have no other intersections than a point.

The sub-indices j,j are used to introduce the possibility that a sub-region can have more than one neighbourhood DB. Each region is defined in $\mathbf{R}^3$, and its border is limited by a group of DBs (see figure 104 for a 2D example).

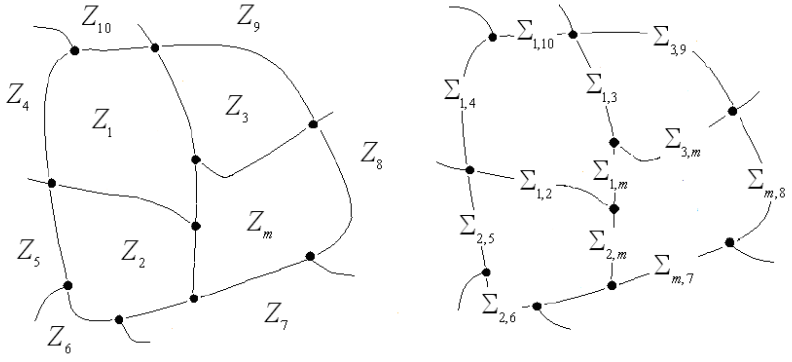


Figure 104. System with multiple sub-regions: (a) symbols for sub-region (b) symbols for the DBs.

In the simplest case, the shape of the DB can be a flat surface, perpendicular to one of the axes of the state space. In the general case, the DB is a surface without flatness and with an orientation that changes according to its nature.

$\dot{\mathbf{x}} = f_j(\mathbf{x}, \boldsymbol{\alpha})$: such that $x \in Z_j$ (4.147)
and
$Z_j = \{x \in \mathbf{R}^n \text{ such that } (H_{j,j}(x) < 0)\}$ (4.148)

The equations to defines the DB in systems with multiple region can have various inequalities and some of them can use condition different that ~~–m~~inor than" and can be assigned to other variable different that x_1.

Dynamics of Zones

Assume a system with $n = 3$ and $j = 2$, described as earlier and with the DB $\Sigma_{1,2}$ as a surface composed of two subsets or zones:

Ψ_c is a zone of points such that $\boldsymbol{x} \in H_{1,2}(\boldsymbol{x})$; and we have a dynamic crossing from Z_1 to Z_2. For these points we have:

$$\{\langle (H\_T),(\mathbf{f}_1(\mathbf{x},\boldsymbol{\alpha}))\rangle * \langle (H\_T),(\mathbf{f}_2(\mathbf{x},\boldsymbol{\alpha}))\rangle\} > 0 \qquad (4.149)$$

Ψ_s is a zone of points such that $\boldsymbol{x} \in H_{1,2}(\boldsymbol{x})$ and we have attracting sliding dynamics. For these points we have:

$$\{\langle (H\_T),(\mathbf{f}_1(\mathbf{x},\boldsymbol{\alpha}))\rangle * \langle (H\_T),(\mathbf{f}_2(\mathbf{x},\boldsymbol{\alpha}))\rangle\} < 0 \qquad (4.150)$$

H_T is a vector tangential to the DB at the analysis point, and $\langle\ ,\ \rangle$ denotes the scalar product.

In the study of the DB $\Sigma_{1,2}$ between two regions Z_1 and Z_2, it is necessary to analyse

1. Which are the changes generated for an orbit when it touches the DB as the value of the parameters changes from $\boldsymbol{\alpha}_1$ to $\boldsymbol{\alpha}_2$?
2. What is the change in the size and the type of the dynamics of the zones Ψ_c and Ψ_s when the value of the parameters changes from $\boldsymbol{\alpha}_1$ to $\boldsymbol{\alpha}_2$?

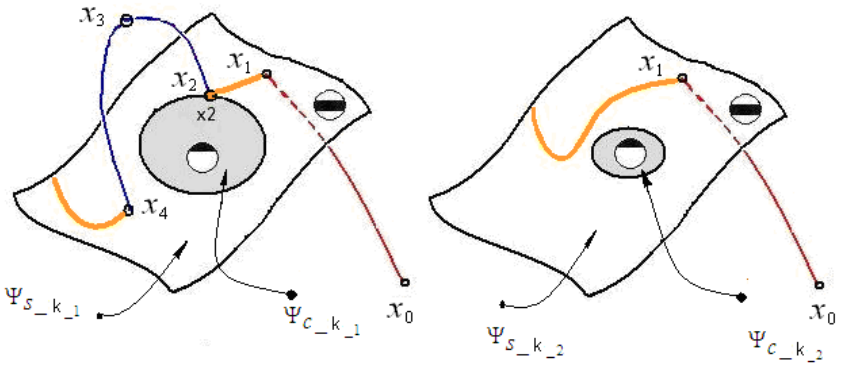


Figure 105. Bifurcation of an orbit crossing the DB without bifurcation of areas.

The study of an orbit crossing a DB in which there is a sliding area has been recently studied by Dibernardo *et al.* (20), (5), (6) but the relation between the size of the zones and the presence of zones with sliding generated by a change in the parameter (which is the topic of this section) has not received so much attention.

The difference between both classes of dynamics can be observed in figure 105. It illustrates a DB $\Sigma_{1,2}$ of a system with two vector fields Z_1 and Z_2. The DB has two areas with different types of dynamics—attractive sliding Ψ_{SS} in the external part and crossing $\Psi c_{1,2}$ in the internal part (shown in grey). The difference between the left and right sides of the figure is due to a change in the parameter value from $\boldsymbol{\alpha}_1$ to $\boldsymbol{\alpha}_2$. This change results also in a change in the size of the areas in the DB.

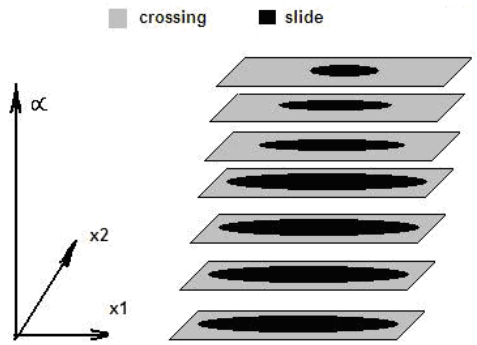


Figure 106. Changes in zones of the DB without bifurcation of zones.

Let $\Sigma_{j,j}$ be the DB described by $H_{j,j}(\mathbf{x}) = 0$ and assume that it has different sets of points (where the Filippov vector field has the same dynamics) such that can be represented by the union of disjoint subregions

$$\Psi = \Psi c_{1,2} \cup \Psi c_{2,1} \cup \Psi_{SS} \cup \Psi s_U \cup \Psi o \qquad (4.151)$$

where Ψ_{SS} are zones of stable (or attractive) sliding points, Ψs_U are zones of unstable (or repulsive) sliding points and Ψo contains only the singular and special points as they were defined in sections 4.1 to 4.3.

Then, a bifurcation of the areas in the DB $\Sigma_{a,b}$ of a three-dimensional nonsmooth system is defined as a change in the type, distribution, or quantity of sub-zones of points composing the DB as a consequence of a change in the value of a parameter.

Remark: Three types of sliding bifurcations need to be considered when we deal with systems of dimension three or higher:

1. Bifurcation of an orbit that has a first contact with a sliding region in the DB and after a variation in a parameter has sliding segments in the DB (an example was illustrated in figure 105).

2. Bifurcations of zones in the DB as it was just defined (see figure 107).

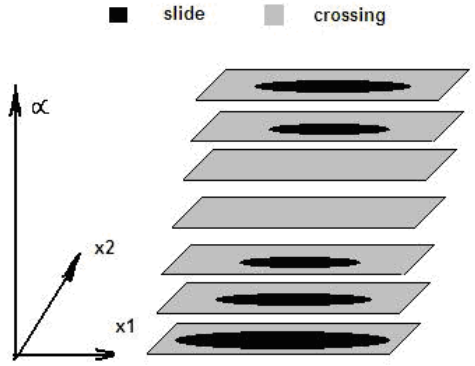


Figure 107. Changes in zones of the DB with bifurcation of zones.

3. Bifurcation of an orbit that is entirely in the DB. Here, the bifurcation can be smooth or nonsmooth. From figure 108, we observe two states of a system that illustrate the change in the quantity and the stability of the equilibrium in the DB.

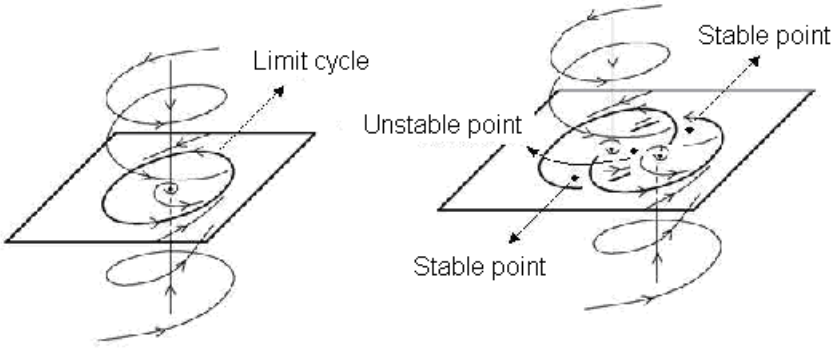


Figure 108. Smooth bifurcation of an orbit in the DB.

DB with one type dynamics

Research of complex dynamics can be started with a list of the basic types of dynamics in the DB, and then follow with the dynamics derived from an interaction among several regions with different dynamics in the DB.

The basic one-type dynamics are described in the following.

The vector fields in the discontinuity boundary can

1. Point to the same direction towards the DB (crossing). See figure 109.

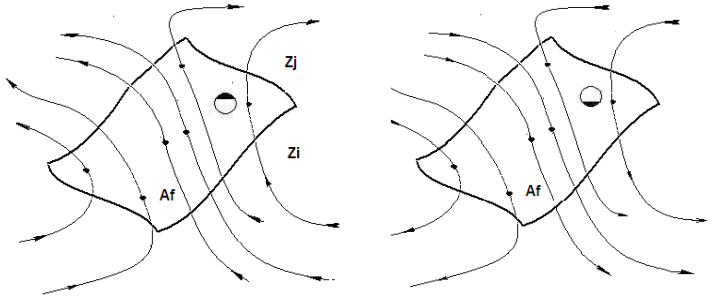


Figure 109. Dynamics moving toward the DB from one side.

2. Point in the reverse direction, moving away from the DB (repulsive, escaping). See Figure 110.

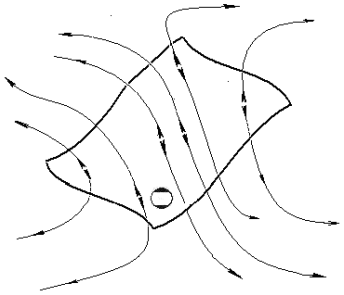


Figure 110. Dynamics moving away from the DB.

3. Point in the reverse direction, (attractively) sliding in the DB. See figure 111.

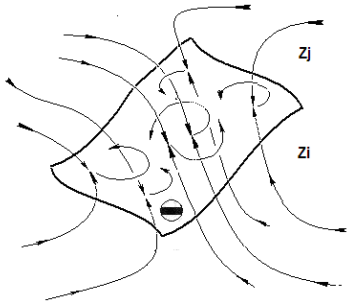


Figure 111. Dynamics moving over the DB.

Thus we have the four possibilities shown in figures 108-111 with their respective symbols derived on the basis of the theory developed in the previous sections.

Note: In planar systems, the DB is a curve that has a logical defined orientation toward one side or toward the other side from a specific point on the DB. In three-dimensional systems, the DB is a surface, and the direction of a trajectory over the DB can be defined by using a four-direction system, as mentioned in section 4.1. However, as we deal with a higher dimensional system, the orientation becomes a problem.

The table 4.3 shows the sysmbol adopted for the basics types of points in three dimensional systems and it counterpart in 2D systems

Table 4.3 Symbols for points in the DB of 3D systems and their counterpart in 2D systems.

Type of system or type of points.	Symbol in 3D	Possibilities in 2D
Crossing from Z_1 to Z_2		
Crossing from Z_2 to Z_1		
Attractive sliding		
Repulsive		
Double tangency		
Anti-collinear attractive		
Anti-collinear repulsive		
Attractive tangent		
Attractive tangent		
Attractive null		
Attractive null		
Tangent null		
Null tangent		
Null null		

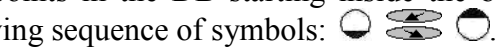
DB with more than two types of dynamics

The study of dynamical systems is useful for the comprehension of a phenomenon in the range of values where a real system can operate. With this aim, it is possible to capture all the states and behaviours of a system in order to approximately predict the performance of the real system. If the DB has a high number of different symbols, it is possible to find several types of dynamics in the DB which leads to complex behaviour.

Let Ψ_f and Ψ_g be two arbitrary zones in the DB with different dynamics. Since four basic dynamics are available, there are eight combinations, which are the following:

1. Crossing from Z_1 to Z_2 in the area Ψ_f and crossing from Z_2 to Z_1 in the area Ψ_g.
2. Crossing from Z_2 to Z_1 in the area Ψ_f and crossing from Z_1 to Z_2 in the area Ψg.
3. Repulsive in the area Ψ_f and attractive sliding in the area Ψ_g.
4. Attractive in the area Ψ_f and repulsive in the area Ψ_g.
5. Crossing from Z_1 to Z_2 in the area Ψ_f and attractive sliding in the area Ψ_g.
6. Crossing from Z_2 to Z_1 in the area Ψ_f and attractive sliding in the area Ψ_g.
7. Crossing from Z_1 to Z_2 in the area Ψ_f and repulsive in the area Ψ_g.
8. Crossing from Z_2 to Z_1 in the area Ψ_f and repulsive in the area Ψ_g.

Figure 112 shows an example of a DB with a crossing zone next to another crossing zone having opposite directions. Their common border (closed curve in fig.) has all points of double-tangency type  (or of one of the other groups of points).

In this case, the symbol has no directional information because the orientation varies continuously. If we consider a curve of points in the DB starting inside the border and ending outside the border we get the following sequence of symbols:  .

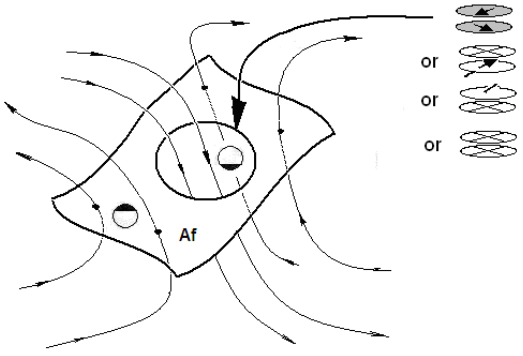


Figure 112. DB with dynamics of type and .