Some remarks about the fusible property of noncommutative polynomial extensions

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Algunas observaciones sobre la propiedad fusible de extensiones polinomiales no conmutativas

Abstract: In this work, we focus on three objectives: first, we investigate fusible property on skew PBW extensions, recognizing conditions that guarantee the property for this family of algebras. Later, we turn our attention to the study of two very important ring families: the weak Σ-rigid rings defined in [RS18b] and (Σ, Δ)-compatible rings introduced in [HKA17] and [RS18a], respectively. We establish several results that characterize some important elements such as units, nilpotents, idempotents and zero divisors in skew PBW extensions over (Σ, Δ)-compatible rings. We extend some descriptions of these elements presented for skew polynomial rings in [HHA17]. The study of these elements leads us to find a more general notion of annihilator, for which we investigate analogous properties to those that define the Baer, quasi-Baer, p.p and p.q.-Baer rings, extending some results presented in [OB12]. Finally, having in mind a more general concept of associated prime ideal presented in [OB12], we study this generalization of the associated primes and eventually characterize these ideals in skew PBW extensions over (Σ, Δ)-compatible rings.

Resumen: En el presente trabajo, nos enfocamos en tres objetivos: primero, investigamos acerca de la propiedad fusible sobre extensiones PBW torcidas, reconociendo condiciones suficientes que garanticen dicha propiedad para esta familia de álgebras. Después, centramos nuestra atención en el estudio de dos familias de anillos muy importantes: los anillos débil Σ-rígidos definidos en [RS18b] y los anillos (Σ, Δ)-compatibles introducidos en [HKA17] y [RS18a], respectivamente. Establecemos resultados que caracterizan algunos elementos importantes como unidades, nilpotentes, idempotentes y divisores de cero en extensiones PBW torcidas sobre anillos (Σ, Δ)-compatibles. Extendemos de esta forma algunas descripciones presentadas de estos elementos en anillos de polinomios torcidos en [HHA17]. El estudio de estos elementos nos lleva a encontrarnos con una noción más general de anulador para la cual investigamos propiedades análogas a las que definen a los anillos de Baer, quasi-Baer, p.p and p.q.-Baer en extensiones PBW torcidas, extendiendo algunos resultados presentados en [OB12]. Finalmente, teniendo en mente un concepto más general de ideal primo asociado presentado en [OB12], estudiamos esta generalización de los primos asociados y eventualmente caracterizamos dichos ideales en extensiones PBW torcidas sobre anillos (Σ, Δ)-compatibles.

Keywords: Fusible property, skew PBW extension, weak Σ-rigid ring, (Σ, Δ)-compatible ring, unit, nilpotent, zero divisor, idempotent, weak annihilator, associated prime ideal.

Palabras clave: Propiedad fusible, extensión PBW torcida, anillo débil Σ-rígido, anillo (Σ, Δ)-compatible, unidad, nilpotente, divisor de cero, idempotente, anulador débil, ideal primo asociado.

Dedicatory

To my mom, Martha.

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Contents

Introduction

In 1990, Faith and Pillay [FP90] studied the commutative rings for which the FPF property is satisfied (every finitely generated faithful module is a generator for the category R-Mod of all right R-modules) and as a consequence, they showed a characterization of the commutative rings for which the set of zero divisors of a commutative ring R is an ideal. In [GM16], the following important statement is made: if the set of zero divisors is not an ideal in the ring R, it must be because some zero divisor element can be written as the sum of a zero divisor element and a nonzero divisor element. These rings that accept a type of decomposition as we mentioned previously are called *Fusible rings*. Great variety of results that relate fusible rings with structures studied in the theory of noncommutative algebra. As examples of these structures that are related, we have the *p.p.* rings and *p.q.* Baer rings.

Kaplansky [Kap65] introduced the Baer rings which satisfy that the right (left) annihilator of every nonempty subset is generated by an idempotent. Closely related to Baer rings are the *p.p.* rings. A ring R is called a right *p.p.* ring, if the right annihilator of each element of R is generated by an idempotent. Birkenmeier [BKP01] defined a ring to be called a right (left) principally quasi-Baer (or simply right (left) *p.q.*-Baer) ring, if the right annihilator of each principal right (left) ideal of R is generated by an idempotent.

These concepts have their roots in functional analysis, having close links to C^*-algebras and von Neumann algebras. Large classes of rings satisfying the Baer property examples include von Neumann algebras and the endomorphisms rings of semisimple modules. Examples of *p.p.* rings also include large classes, such as all Baer rings (see [Ber10]). The previous observation shows the importance of the study of the idempotent elements and the notion of annihilator to know the structure of a ring. Motivated by all of the above and following the same ideas, Ouyang [Ouy10] introduced the notion of nilpotent annihilator as a generalization of annihilators and the notion nilpotent *p.p.* rings. A ring R is said to be a nilpotent *p.p.* ring, if the nilpotent annihilator of each element of R does not equal R, then it is generated as a right ideal by a nilpotent.

The study of fusible property and rings Baer, *p.p.* rings, and right *p.q.* Baer is related to the identification of some important elements in a ring such as units, idempotents, nilpotents, etc. In the literature, we can find that an important work has been to characterize the elements in a polynomial ring. In the commutative case, Atiyah and Macdonald [AM69], p. 10-11, presented a complete characterization of these elements for polynomial rings. In [HHA17], the authors characterize some elements in a skew polynomial ring on a right duo ring.

In the works mentioned above, we note that the compatibility condition on the coefficient ring and some conditions on the set of nilpotent elements are a key part in the development of these investigations.

In [OB12], some results were presented which established sufficient conditions to guarantee that properties analogous to those of rings Baer, *p.p.* rings, and right *p.q.* Baer are stable on skew polynomial rings. Additionally, in this work a very important concept in algebra is studied, such as the notion of associated prime ideal. More generally, based on the notion of weak annihilator, a more general concept of associated prime ideal is defined and, as a main result, these ideals for skew polynomial rings are characterized.

The main objects of study in this work are skew PBW extensions. This family of rings was introduced by Lezama and Gallego [GL11] with the intention of studying a broader family of algebras and constructing a Gröebner bases theory for its ideals. Skew PBW extensions are an interesting generalization of the PBW extensions introduced by Bell and Goodearl [BG88] and the skew polynomial rings of injective type defined by Ore [Ore33].

We establish the structure of our work as follows: In Chapter 1, Section 1.1, we study fusible property by doing a detailed study of some notions and propositions initially stated in a commutative context. Next, we study fusible property and its different connections in noncommutative algebra. Many of these definitions and facts are introduced in [GM16] and [KM19]. In Section 1.2, we mention skew polynomial rings, investigating their most important properties and relations with fusible rings. Finally, in Section 1.3, we study PBW extensions as a motivation to study our objects of interest.

In Chapter 2, we present some of the main results. In Section 2.1, we introduce skew PBW extensions, carrying out a study of some of their properties and interesting results that relate the filtration-graduation technique and the Goldie's theory. In this section, we also present some results obtained that relate the fusible property with this family of rings. In Section 2.2, we study the Σ-rigid and weak Σ-rigid rings. These rings were introduced in [Rey15] and [RS18b], respectively. We investigate some of their properties and characterize some elements in these extensions. In Section 2.3, we study (Σ, Δ)-compatible rings. This family of rings has been studied in [HKA17] and [RS18a]. We present some properties already known in the literature and carrying out a similar treatment to characterize certain elements of the skew PBW extensions on these rings. Our original results contribute to the study developed in [HHR20], [RS19b], [LR20b] and [RS20].

Later, in Chapter 3, we present the notion of weak annihilator, which was introduced in [OB12]. We study some important properties of this concept and extending several results that relate this notion to skew PBW extensions. Finally, we investigate a more general notion of associated prime ideals and characterize these ideals in a skew PBW extension over (Σ, Δ)-compatible rings. The main result of this section extends the work done in [OB12] for skew polynomial rings over (σ, δ)-compatible rings and also partially generalizes the characterization of associated prime ideal for skew PBW extension over (Σ, Δ)-compatible modules presented in [Ram19]. Again, the results presented in this section are new for skew PBW extensions, and coincide with results presented in [OB12].

All results are new for skew PBW extensions and all they are similar to others existing in the literature. It is important to say that all new results established in this work have been submitted for publication.

Notations and conventions. Throughout this document, all rings are supposed to be associative with identity and unless stated otherwise, noncommutative. Further, $\Bbbk$ denotes a field and $M_n(R)$ denotes the set of matrices of size $n \times n$ with entries in a ring R. The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the usual numerical systems.

Further, if we consider an element r of a ring R, the subsets $l_R(r) = \{a \in R \mid ar = 0\}$ and $r_R(r) = \{a \in R \mid ra = 0\}$, denote the left annihilator and the right annihilator of r in R, respectively. If $r_R(r) \neq 0$, we say r is a left zero divisor; otherwise, it is a left nonzero divisor (similarly, the right case is defined). Let $Z_l(R)$ denote the set of left zero divisors and $Z_l^*(R)$ denote the set of left nonzero divisors of R. Analogously, the right case is defined and it is denoted by $Z_r(R)$ and $Z_r^*(R)$, respectively. In a commutative ring, it is satisfied that $Z_r(R) = Z(R) = Z_r(R)$, where $Z(R)$ denotes the set of zero divisors. We use $U(R)$, $\mathrm{Idem}(R)$, $\mathrm{nil}_*(R)$, $\mathrm{nil}(R)$ to denote the set of all unit elements of R, the set of all idempotent elements of R, the prime radical, the set of all nilpotent elements of R, respectively. A ring R is called 2-primal, if $\mathrm{nil}_*(R) = \mathrm{nil}(R)$. In addition, a ring R is NI, if $\mathrm{nil}(R)$ forms an ideal. Some of these notions are defined and studied in Appendix A.1.

CHAPTER 1

Fusible rings

In this chapter, our purpose is to study fusible rings and some families of polynomial rings that are the object of study throughout this work. In Section 1.1, we investigate fusible property in different algebraic objects. For this, we have organized and presented some of the most important results that relate the fusible property in commutative and noncommutative contexts. These facts are taken and adapted from the works [GM16] and [KM19]. In Section 1.2, we introduce skew polynomial rings. This family of rings has as examples a great variety of noncommutative rings of the polynomial type. We investigate some of the most important algebraic properties that have been studied in the literature on these objects. Finally, in Section 1.3, we present the PBW extensions which arise as a generalization of certain families of algebras which satisfy the PBW theorem (see [BG88]). These class of rings are an important motivation for us throughout this work.

1.1 Fusible rings

In this section, we present the notion of fusible ring, which was introduced by E. Ghashghaei and W. McGovern in [GM16]. In addition, we also state a generalization of fusible rings presented by M.T Koşan and J. Matczuk in [KM19]. We study important properties of these two families of rings and their important relationships with different algebraic objects presented in the literature. An interesting parallel is made between fusible property in the commutative and noncommutative fields. The different results are taken and adapted from [GM16] and [KM19].

We begin with the formal definition of the central concept of this work.

Definition 1.1.1 ([GM16], Definition 2.1)**.** Let R be a ring. We call a nonzero element $a \in R$ **left fusible**, if it can be expressed as the sum of a left zero divisor and a nonleft zero divisor in R. We call a ring R **left fusible**, if every nonzero element of R is left fusible. **Right fusible** rings are defined analogously. A ring R which is both right and left fusible is called **fusible ring**.

The following examples are adapted from [GM16], Remark 2.2.

Examples 1.1.2. We present below some examples of fusible rings.

(i) Recall that $r \in R$ is called **regular**, if r is not zero divisor. Thus, every regular element has a trivial left fusible representation given by $r = 0 + r$. Therefore, it is left fusible. Analogously, it is right fusible. A ring where every element is regular is known as a **domain**. Thus, every domain is a fusible ring.

(ii) An element $e \in R$ is said to be **idempotent**, if $e^2 = e$. Every idempotent element has a left fusible representation given by $e = (1 - e) + (2e - 1)$. Therefore, every Boolean ring is a fusible ring. Recall that R is called **Boolean ring**, if $a^2 = a$, for all $a \in R$.

(iii) Recall that a ring R is said to be **special almost clean**, if each element a can be decomposed as the sum of a regular element r and an idempotent e with $aR \cap eR = 0$ (see [AV13], p. 851). Thus, every special almost clean ring is fusible.

Remark 1.1.3. L. Gillman and M. Jerison [GJ60] studied algebraic properties of the ring of continuous functions at real value $C(X, \mathbb{R})$, and the ring of continuous functions and bounded to real value denoted as $C^*(X, \mathbb{R})$, both rings defined on a topological space X arbitrary. Following the ideas presented in [GJ60], it was shown that for any topological space X, $C(X, \mathbb{A})$ is a fusible ring, where $\mathbb{A}$ subring of $\mathbb{R}$ (see [GM16], Theorem 4.3).

Now, we introduce the notion of *unit fusible* ring. This family of rings is contained in the class of fusible rings. The following definition is adapted from [GM16], Definition 5.1.

Definition 1.1.4 ([GM16], Definition 5.1)**.** A nonzero element of a ring R is called **left unit fusible**, if it can be expressed as the sum of a left zero divisor and an unit in R. A ring R is called **left unit fusible**, if every nonzero element of R is left unit fusible. A ring which R is both right unit and left unit fusible is called **unit fusible**.

Examples 1.1.5. We present below some examples of unit fusible rings.

(i) Every unit element has a trivial unit fusible representation given by $u = 0 + u$. Therefore, it is unit fusible. A ring where every element is unit is known as a **division ring**. Hence, every division ring is a unit fusible ring.

(ii) Every idempotent element unit fusible: for each $e \in R$, we get $e = (1 - e) + (2e - 1)$ is an unit fusible representation. Hence, every Boolean ring is a unit fusible ring.

Now, we introduce the notion of *regular fusible ring*. This class of rings is introduced in [KM19], in order to continue studying the fusible property and its relationship with different algebraic objects. Additionally, a new description of semiprime left Goldie rings is presented (see Appendix A.3). Later, we study several results that are of great importance in this work.

Definition 1.1.6 ([KM19], Definition 2.1)**.** A ring R is said to be **regular left fusible**, if for any nonzero element $r \in R$, there exists a regular (i.e., left and right regular) element $s \in R$ such that the element sr is left fusible, i.e. $sr = c + w$, where c is a left zero divisor and w is left regular. **Regular right fusible** rings are defined analogously. A ring R which is both right and left regular fusible is called **regular fusible ring**.

Remark 1.1.7. Since every left regular element is left fusible, it is enough to observe that for every nonzero left zero divisor, there exists a regular element s in R such that the element sa has a left fusible representation.

We present some examples of regular left fusible rings.

Examples 1.1.8. Every left fusible ring is regular left fusible ring. If $r = a + w$ is a left fusible representation, in particular is a left regular fusible representation taking $s = 1$. From the above, it follows that the examples presented in Examples 1.1.2 and Examples 1.1.5 are also left regular fusible rings.

The following example shows that the class of left fusible rings is strictly smaller than the class of regular left fusible rings.

Example 1.1.9 ([KM19], Example 2.2)**.** Let $\Bbbk$ be a field and consider the $\Bbbk$-algebra $R = \Bbbk\langle x, y\rangle$ subject to the relations $x^2 = 0$ and $xy - yx \neq 0$. Cohn [Coh73] showed that the set of all left zero divisors of R is equal to xR and hence, the sum of any two left zero divisors is a left zero divisor. Thus R is not left fusible. Furthermore, y is a regular element of R and $ry = 0 + ry$ is a left fusible decomposition of ry, for any nonzero element $r \in R$. Thus R is a regular left fusible ring.

1.1.1 Commutative algebra

Our purpose is to study and present some interesting facts in commutative algebra that are related to fusible rings, unit fusible rings and regular fusible rings. The different definitions and results that we present below are adapted from [GM16] and [KM19]. We start with the notion of prime ideal, which is a very important concept in algebra and geometry.

Definition 1.1.10 ([GW04], p. 47)**.** A **prime ideal** in a ring R is any proper ideal P of R such that, whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A **prime ring** is a ring in which 0 is a prime ideal.

Definition 1.1.11 ([GW04], p. 49)**.** A **minimal prime ideal** in a ring R is any prime ideal of R that does not properly contain any other prime ideals. The set of all minimal prime ideals of R is denoted by $\operatorname{Min}(R)$

First, let us recall some interesting and familiar facts from the study of prime ideals in a commutative context. It is known that $Z(R) = \bigcup \operatorname{Min}(R)$. Also, for R to be reduced it is equivalent to saying that $\bigcap \operatorname{Min}(R) = 0$ (see [AM69], Proposition 4.7). The following result is very interesting since it provides conditions on $\operatorname{Min}(R)$ that guarantee the fusible property in the class of reduced rings.

Proposition 1.1.12 ([GM16], Theorem 3.7)**.** *A commutative ring R with only finitely many minimal prime ideals is reduced if and only if R is fusible.*

Proof. Suppose that R is reduced. First, let us show that $Z(R) = \bigcup_{i=1}^{n} P_i$, where P_i are all minimal prime ideals of R. Let $x \in R$ be a zero divisor. The idea is to show that x belongs to a minimal prime ideal P_i. Let $y \in R$ such that $xy = 0$ with $y \neq 0$.

If x does not belong to any minimal prime, then from $xy = 0 \in P_i$, for P_i a minimal prime, we get $y \in P_i$, so $y \in \bigcap_{i=1}^{n} P_i = \text{nil}(R)$. Hence, y is nilpotent element and thus $y = 0$, since R is reduced. This is a contradiction. Therefore, $x \in P_i \subseteq \bigcup_{i=1}^{n} P_i$, for some P_i with $1 \leq i \leq m$. The other containment follows from the well known fact that every minimal prime consists of zero divisors (see [AM69], Proposition 4.7).

Let $x \in Z(R)$. We can assume that $\{P_i\}_i$ have been indexed in such a way that there exists an integer m where $1 \leq m \leq n$ such that $x \in \bigcap_{i=1}^{m} P_i$ and $x \notin \bigcup_{i=m+1}^{n} P_i$. Since $\{P_i\}_i$ are minimal prime ideals then $\bigcap_{i=1}^{m} P_i \neq 0$ and by Prime avoidance lemma, which states that if an ideal I is contained in a union of finitely many prime ideals $\{P_i\}$, then it is contained in P_i, for some i. We have $\bigcap_{i=m+1}^{n} P_i \nsubseteq \bigcup_{i=1}^{m} P_i$. Thus there exists $y \in \cap_{i=m+1}^{n} P_i \setminus \bigcup_{i=1}^{m} P_i$. Hence, $x + y$ is regular and $y \in Z(A)$, which implies that $x = (-y) + (x + y)$ is a fusible representation. This means that R is fusible.

Now, suppose that R is fusible. Let us see that R is reduced reasoning by the method of reduction to the absurd. Consider an element $a \in R$ such that $a^2 = 0$ with $a \neq 0$. Since R is fusible and $a \neq 0$, there exists z a zero divisor of R and r a nonzero divisor of R, such that $a = z+r$. This means that $0 = a^2 = (r+z)^2 = r^2+2rz+z^2$. Now, as z is a zero divisor, there exists $z' \in R$, such that $zz' = z'z = 0$, whence $0 = (r^2 + 2rz + z^2)z' = r^2z'$, which implies that r is zero divisor, a contradiction. Thus, $a = 0$ and hence R is reduced. □

Next, we introduce the notion of a *potent ring* and study its relationship with fusible rings. Some properties of these rings were studied in [Jac56].

Definition 1.1.13 ([CS17], p. 136)**.** A ring R is called **potent**, if for each $r \in R$ there exists an integer $1 < n$, for which $r^n = r$.

Proposition 1.1.14 ([GM16], Proposition 3.1)**.** *Let R be a potent ring. Then R is a fusible commutative ring.*

Proof. Suppose that R is potent. Let $r \in Z(R)$ be nonzero element and $n \in \mathbb{N}$ such that $r^n = r$. Since $(r+1-r^{n-1})(r^{2n-3}+1-r^{n-1}) = 1$, then $r+1-r^{n-1}$ is an unit element of R . On the other hand, $r^{n-1}-1 \in Z(R)$. Therefore, we have that $r = (r^{n-1}-1)+(r+1-r^{n-1})$ which is a fusible representation. The proof that every potent ring is commutative can be found in [Jac56], p. 217. Thus, we conclude that R is a fusible commutative ring. □

Now, we recall that a ring R is called **periodic**, if for every element $a \in R$ there exist $n, m \in \mathbb{N}$ with $n < m$ such that $a^n = a^m$ (see [CS17], p. 134). A periodic ring is potent if and only if it is reduced (see [Bel76], Lemma 1 (b)). From the previous observation, we can note that periodic rings are not necessarily commutative. The following result characterizes the periodic rings that satisfy the fusible property.

Proposition 1.1.15 ([GM16], Corollary 3.2)**.** *The periodic ring R is fusible if and only if R is potent.*

Proof. Every potent ring is a fusible ring by Proposition 1.1.14. Let R be a periodic ring and suppose that R is fusible. Let us show that R is potent. Since R is fusible, by Proposition 1.1.12, we get that R is a reduced ring. Finally, we have that R is reduced and periodic by hypothesis. Therefore, the previous observation guarantees that R is a potent ring. This concludes the proof. □

Next, we study some very important facts that relate unit fusible rings to algebraic objects in a commutative context. Before, we remember some important notions. If R is a commutative ring, then $R[x]$ denotes the **polynomials ring** in an indeterminate x, where the variable commutes with the coefficients (see [Hun80], p. 149).

Analogously, the ring $R[[x]]$ denotes the **ring of formal power series** in an indeterminate x, where the variable commutes with coefficients. Its elements are called **power series** (see [Hun80], p. 154). On the other hand, given R a commutative ring, we define the **Laurent power series** ring $R[[x, x^{-1}]]$ in which each element has a unique representation as $\sum_{i\in\mathbb{Z}} a_i x^i$ with $a_{-n} = 0$, for all but finitely many $n \in \mathbb{N}$, and where $ax = xa$ (see [MR01], p. 22).

The following result is taken and adapted from [GM16], Proposition 5.10.

Proposition 1.1.16 ([GM16], Proposition 5.10)**.** *Let R be a commutative ring. Then the following assertions hold:*

(i) *The rings $R[x]$ and $R[[x]]$ are never unit fusible.*

(ii) *The ring $R[[x, x^{-1}]]$ is unit fusible if and only if R is unit fusible.*

Proof. We follow the ideas presented in [GM16], Proposition 5.10.

(i) Let us denote $\mathrm{nil}(R)$ the set of nilpotent elements of the ring R. Now, we recall the characterization of the units in the following rings (see [AM69], p. 11).

$$U(R[x]) = \{r_0 + r_1x + \cdots + r_nx^n \mid r_0 \in U(R),\ r_i \in \mathrm{nil}(R), \quad \text{for} \quad 1 \leq i \leq n\},$$

$$U(R[[x]]) = \left\{r_0 + r_1x + r_2x^2 \cdots \mid r_0 \in U(R)\right\}.$$

Now, if $x = f(x) + g(x)$ is a fusible representation in $R[[x]]$ with $f(x) \in U(R[[x]])$, then $f(x) = r_0 + r_1x + \cdots$ with $r_0 \in U(R)$. Hence, $g(x) = -r_0 - (r_1 - 1)x - r_2x^2 - \cdots$ where $-r_0 \in U(R)$. Therefore, $g(x) \in U(R[[x]])$ which is a contradiction. Additionally, it is known that every zero divisor of $R[x]$ is annihilated by an element of R. Hence, using an argument similar to the previous, if $x = f(x) + g(x)$ is a unit fusible representation in $R[x]$ with $f(x) \in U(R[x])$ and $f(x) = r_0 + r_1x + \cdots + r_nx^n$ with $r_0 \in U(R)$, then $g(x) = -r_0 - (r_1 - 1)x - r_2x^2 - \cdots - r_nx^n$ with $g(x) \in Z(R[X])$ which is a contradiction, since there exists $b \in R$ such that $g(x)b = 0$. In particular, $-r_0b = 0$, which is not possible. Therefore, we have that x is not unit fusible in $R[x]$. Thus, both $R[x]$ and $R[[x]]$ are not unit fusible rings.

(ii) Suppose R is unit fusible. Let $f(x) = f_{-m}x^{-m} + \cdots + f_0 + \cdots + f_mx^m \in R[[x, x^{-1}]]$ and let $n \in \mathbb{Z}$ the least integer such that $f_n \neq 0$ with $-m \leq n \leq m$. Now, since R is unit fusible ring, then $f_n = u_n + z_n$ where $u_n \in R$ and $z_n \in Z(R)$. Let us consider $g(x)$ replacing f_n with u_n, i.e, $g(x) = u_nx^n + \cdots + f_mx^m \in U(R[[x, x^{-1}]])$. Since $f(x) - g(x) = z_nx^n \in Z(R[[x, x^{-1}]])$, we have $f(x) = g(x) + (f(x) - g(x))$ is unit fusible representation in $R[[x, x^{-1}]]$. Hence, $R[[x, x^{-1}]]$ is unit fusible.

Let $z \in Z(R)$ and suppose that $R[[x, x^{-1}]]$ is unit fusible. Let us show that R is unit fusible. Write $z = f(x) + g(x)$ with $f(x) \in U(R[[x, x^{-1}]])$ and $g(x) \in Z(R[[x, x^{-1}]])$, where $f(x) = f_n x^n + f_{n+1}x^{n+1} + \cdots$ and $g(x) = g_m x^m + g_{m+1}x^{m+1} + \cdots$. First, if $n < 0$, then $m = n$ and $g_n = -f_n \in U(R)$. Moreover, this argument shows that $m \leq n$. Thus, it follows that $0 \leq m \leq n$. If $n = 0$, then $z = f_0 + g_0$ is a unit fusible representation of z in R.

Now, assume that $n > 0$ and $m = 0$. Observe that $g(x) = z - f_n x^n - f_{n+1}x^{n+1} - \cdots$. Since $g(x)$ is a zero divisor, there exist $t(x) = t_0 + t_1 x + t_2 x^2 + \cdots$ such that $t_0 \neq 0$ and $g(x)t(x) = 0$. In particular, $zt^n - f_n t_0 x^n = 0$ this means that $zt_n - f_n t_0 = 0$. Multiplying both sides by t_0, we get $zt_n t_0 = f_n t_0^2$ and using that $f_n \in U(R)$, we obtain $t_0^2 = f_n^{-1} z t_n t_0 = f_n^{-1} t_n z t_0 = 0$ which means that t_0 is a nilpotent element of $R[[x, x^{-1}]]$, and since R is reduced, it follows that $t_0 = 0$, a contradiction. Therefore, $n = m = 0$ and so, $z = f_0 + g_0$ is a unit fusible representation in R, that is, R is unit fusible.

□

In order to present the following result, we remind the reader some important notions about rings. The following notions can be consulted in [GW04], p. 106.

Remark 1.1.17. We recall that, if R is a ring and S is the set of regular elements of R, then $Q_l(R)$ is called the **classical ring of left quotients** of R: if d is invertible in Q, for every $d \in S$, each $q \in Q$ can be factored $q = d^{-1}a$, for some $d \in S$ and $a \in R$. **Classical ring of right quotients** are defined similarly and it is denoted $Q_r(R)$. If a ring R has a left or right classical ring of quotients, then it is unique up to isomorphism and it is denoted by $Q(R)$. Finally, a ring R is said to be **classical**, if $R = Q(R)$.

Next, we present some important results that characterize the fusible property on certain structures well known in the literature. The following result is taken and adapted from [GM16], Proposition 5.11.

Proposition 1.1.18 ([GM16], Proposition 5.11)**.** *Let R be a commutative reduced ring. Then the following assertions hold:*

(i) *If R is local, then R is unit fusible if and only if R is a field.*

(ii) *If R is a domain, then R is unit fusible if and only if R is a field.*

(iii) *If R is classical, then R is fusible if and only if R is unit fusible.*

(iv) *If R is a fusible ring, then $Q(R)$ is unit fusible.*

Proof. We follow the ideas established in [GM16], Proposition 5.11.

(i) If R is a field, then every element in R is invertible. Hence, $r = 0 + r$ is an unit fusible representation, for all $r \in R$, so R is unit fusible. Suppose R is unit fusible and let $r \in R$. Since R is unit fusible, $r = s + u$ is an unit fusible representation with s a zero divisor of R and u an unit of R. Since R is local, so the sum of any two non units in R is a non unit. Hence $r - s = u \notin U(R)$ which is a contradiction. Thus, R is a field.

(ii) Again, every field is an unit fusible ring. Now, let $r \in R$. Suppose R unit fusible and let $r = s + u$ an unit fusible representation with s zero divisor of R and u an unit of R. As s is a zero divisor, there exists $s' \in R$ with $s' \neq 0$ such that $ss' = 0$.

Hence, multiplying right by s', we have $rs' = (s+u)s' = ss' + us' = us'$ and since u is a unit, there is $u' \in A$ such that $uu' = u'u = 1$. So, multiplying left by u', we have $u'rs' = u'us' = s'$. Therefore, $(u'r - 1)s' = 0$. This implies $u'r - 1 = 0$ since R is domain and $s' \neq 0$. Thus, $u'r = 1$ and analogously it shows that $ru' = 1$. Hence r is unity, so R is a field.

(iii) Let $r \in R$. Suppose R unit fusible and let $r = s + u$ an unit fusible representation in R which is also a fusible representation in R. Since s is a zero divisor of R and u is an unit of R. In particular, u is not a zero divisor, so R is fusible.

(iv) Suppose R is a fusible ring and let $s^{-1}r \in Q(R)$. Since $r \in R$ and R is fusible, $r = a + w$ is a fusible representation in R with a zero divisor and w non zero divisor. Hence, $s^{-1}r = s^{-1}(a + w) = s^{-1}a + s^{-1}w$. Let us show that $s^{-1}r = s^{-1}(a + w) = s^{-1}a + s^{-1}w$ is an unit fusible representation in R. As a is a zero divisor of R, there exists $c \in R$ with $c \neq 0$ such that $ca = 0$. Then, we have $cs^{-1}a = s^{-1}ca = 0$, so $s^{-1}a$ is a zero divisor in $Q(R)$. Now, if $s^{-1}w$ is a zero divisor, there is $s_1^{-1}t \in Q(R)$ such that $s_1^{-1}ts^{-1}w = (s_1 s)^{-1}tw = 0$, so w is a zero divisor, which is a contradiction. Hence, $s^{-1}w$ is a regular element in $Q(R)$ and thus $s^{-1}w$ is unit. Then, $s^{-1}r = s^{-1}(a + w) = s^{-1}a + s^{-1}w$ is an unit fusible representation in R and $Q(R)$ is unit fusible.

□

Next, we study some very important facts that relate regular fusible rings to algebraic objects in a commutative context. The following results are adapted from [KM19]. The next proposition provides conditions for a commutative ring to be reduced.

Proposition 1.1.19 ([KM19], Lemma 2.11)**.** *Every commutative regular fusible ring is reduced.*

Proof. Suppose that R is a commutative regular fusible ring. Let us see that R is reduced, reasoning by contradiction. Consider an element $a \in R$ such that $a^2 = 0$ with $a \neq 0$. If R is regular fusible and $a \neq 0$, then there exists s, a regular element, such that sa is fusible, that is, there exists z a zero divisor of R and r a nonzero divisor of R such that $sr = z + r$. Since $a^2 = 0$, we get that $0 = (sa)^2 = (z + r)^2 = z^2 + 2zr + r^2$. Now, if z is a zero divisor, there exists $z' \in A$ so that $zz' = z'z = 0$. Hence, we have that $0 = (z^2 + 2zr + r^2)z' = z^2 + 2zz'r + r^2z' = r^2z'$, which implies that $0 = r^2z'$. This means that r is a zero divisor, which is a contradiction. Therefore, we conclude that $a = 0$ and thus R is reduced. □

In [KM19], the following equivalence of the notions of fusible ring and regular fusible ring is presented. Note the importance of the reducibility of R and the existence of finitely many minimal primes ideals.

Proposition 1.1.20 ([KM19], Proposition 2.12)**.** *Let R be a commutative ring with only finitely many minimal prime ideals. The following conditions are equivalent:*

(i) *R is fusible.*

(ii) *R is regular fusible.*

(iii) *R is reduced.*

Proof. We follow the ideas developed in [KM19], Proposition 2.12.

- (i) $\Rightarrow$ (ii) Every left fusible ring is a regular left fusible ring by taking $s = 1$.
- (ii) $\Rightarrow$ (iii) This implication follows from Proposition 1.1.19.
- (iii) $\Rightarrow$ (i) The proposition follows from Proposition 2.3.24.

□

1.1.2 Noncommutative algebra

In this section, our goal is to study some interesting facts in noncommutative algebra that relate to fusible property. At the end of this part, we present a very important description of semiprime left Goldie rings. The reader may refer to Appendix A.3 to recall some key notions. Again, the following results are adapted from [GM16] and [KM19].

We begin by studying some results related to left (right) fusible rings.

Proposition 1.1.21 ([GM16], Proposition 2.4)**.** *Let R be a ring with comparability relation between right annihilators of its elements. Then R is left fusible if and only if R is a domain.*

Proof. In Example 1.1.2, we observe that every domain is left fusible. Suppose that R is left fusible and let us consider a nonzero element $a \in Z_l(R)$. If A is left fusible and $a \neq 0$, then there exist $z \in Z_l(R)$ and $s \in Z_l^*(R)$ such that $a = z + s$. Now, by assumption $r_R(a) \subseteq r_R(z)$ or $r_R(z) \subseteq r_R(a)$. If we consider that $r_R(a) \subseteq r_R(z)$, then $r_R(a) \cap r_R(z) \subset r_R(s)$, since $a - z = s$. In addition, $r_R(a) \cap r_R(z) \neq \{0\}$, because $a, z \in Z_l(R)$ and $r_R(a) \subseteq r_R(z)$. Hence, we have that $r_R(s) \neq \{0\}$, that is, s is a left zero divisor, which is a contradiction. Similarly, if we assume that $r_R(z) \subseteq r_R(a)$, we get the same contradiction. Thus, we conclude $a = 0$, whence $Z_l(R) = \{0\}$. This means that A is a domain. □

Remark 1.1.22. A ring R is said to be **right (left) uniserial** if and only if its right (left) ideals are totally ordered under inclusion (see [Fai76], Definition 25.1.12). From the previous result, it follows that a right uniserial ring is left fusible if and only if it is a domain.

A right ideal I of a ring R **essential** (or **large**), if $J \cap I \neq 0$, for every nonzero right ideal J of R, and it is denoted by $I <_e R$. The right singular ideal of a ring R is $\text{Sing}(R_R) = \{x \in R \mid r_R(x) <_e R_R\}$. It is well known that $\text{Sing}(R_R)$ is a two sided ideal of R (see [GW04], p. 91). A ring R is called **right nonsingular**, if $\text{Sing}(R_R) = 0$.

Proposition 1.1.23 ([GM16], Proposition 2.11)**.** *If R is a left fusible ring, then R is a right nonsingular.*

Proof. Let R be a left fusible ring and $x \in \text{Sing}(R_R)$ with $x \neq 0$. Thus, there exists $z \in Z_l(R)$ and $r \in Z_l^*(R)$ such that $x = z + r$ is a fusible decomposition. Since $x - z = r$, it follows that $r_R(x) \cap r_R(z) \subseteq r_R(x - z) = r_R(r) = \{0\}$. Hence $r_R(x) \cap r_R(z) = \{0\}$. But $r_R(x) \leq_e R$ and $r_R(z) \neq \{0\}$ which is a contradiction. Therefore, we conclude that $x = 0$ and so $\text{Sing}(R_R) = \{0\}$. □

Now, we study some interesting properties related to left (right) unit fusible rings.

Definition 1.1.24 ([Azu54], p. 34)**.** A ring R is called **unit regular**, if for each $a \in R$ there exists u, a unit element of R, such that $aua = a$.

In [CK01], a characterization of this class of rings is presented. This important result states that a ring is unit regular if and only if every element a of R can be written as $a = e + u$ such that $aR \cap eR = 0$, where e is an idempotent and u a unit in R. The proof of this claim is extensive. The reader may check the proof of this fact in [CK01], Theorem 1. From the previous observation, an immediate result is the following:

Proposition 1.1.25 ([GM16], Proposition 5.3)**.** *Every unit regular ring is unit fusible.*

Next, we introduce the concept of the *strongly π-regular* ring. We study some previous results and present an interesting relationship with the unit fusible rings.

Definition 1.1.26 ([Nic99], p. 3584)**.** An element $a \in R$ is said to be **strongly π-regular** in R, if there exist a positive integer n and $r \in R$ such that $a^n = a^{n+1}r$. A ring R is called **strongly π-regular**, if every element is strongly π-regular

The following proposition characterizes the strongly π-regular elements.

Proposition 1.1.27 ([Nic99], Proposition 1)**.** *Let R be a ring. For an element $a \in R$, the following conditions are equivalent:*

(i) *a is strongly π-regular.*

(ii) *There exists $n \geq 1$ such that $a^n = eu = ue$, where $e^2 = e$, u is a unit of R and a, e and u commute with each other.*

The following statement describes the strongly π-regular elements in a similar way to Proposition 1.1.27 with a very interesting additional property.

Proposition 1.1.28 ([Nic99], Theorem 1)**.** *Let R be a ring. If a is strongly π-regular then $a = e + u$, where $e^2 = e$, u is a unit element of R and $eu = ue$.*

The next result characterizes the strongly π-regular non nilpotents elements in terms of the unit fusible elements. We follow the ideas presented in [GM16], Proposition 5.4

Proposition 1.1.29 ([GM16], Proposition 5.4.)**.** *Let R be a ring. If a is strongly π-regular and non nilpotent, then a is unit fusible.*

Proof. Let us consider $a \in R$ a nonzero element. Since $a \neq 0$ is strongly π-regular, there exists $n \geq 0$ such that $a^n = eu$ where $e^2 = e$, u is a unit element of R and e, u and a all commute, by Proposition 1.1.27. If a is a zero divisor, then $e \neq 1$. Now, let $z = 1 - e$ and $r = a - (1 - e)$. Hence, by Proposition 1.1.28, it follows that $z \in Z(R)$ and r is a unit of R. This means that $a = r + z$ is unit fusible. □

Before presenting the following result, we want to recall an important notion. In any ring R, the ideal defined by the intersection of all maximal ideals of R is called the **Jacobson radical** of R, denoted $\mathfrak{J}(R)$ (see [GW04], p. 61). A ring R is said to be **Jacobson semisimple** provided that $\mathfrak{J}(R) = 0$ (see [GW04], p. 62).

The following result shows that being unit fusible is a sufficient condition to be a Jacobson semisimple ring.

Proposition 1.1.30 ([GM16], Proposition 5.5)**.** *Let R be a unit left fusible ring. Then R is a Jacobson semisimple ring.*

Proof. Suppose that that $a \in \mathfrak{J}(R)$ with $a \neq 0$. If A is left unit fusible, then there exists $z \in Z_l(R)$ and u an unit element, where $a = z + u$. Since z is not a unit, it belongs to a left maximal M ideal of R, that is, $z \in M$, for some left maximal ideal M. But $\mathfrak{J}(R)$ is the intersection of all left maximal ideals and this means that $a - z \in M$. Hence $u \in M$, a contradiction. □

Now, we study some important properties and results related to regular left fusible. These facts relate commonly studied objects of algebra to fusible property. Finally, we present an interesting relationship of left regular fusible rings with Goldie's theory. The following statements are taken from [KM19].

Definition 1.1.31 ([GW04], p. 81)**.** A **multiplicative set** in a ring R is a subset $S \subseteq R$ such that $1 \in S$ and S is closed under multiplication.

The following notions can be consulted in [GW04], p. 62 and 167.

Remark 1.1.32. Let $S \subseteq R$ be a multiplicative set. Then S is a **left Ore set**, if it satisfies that for all $r_1 \in R$, $s_1 \in S$, there exist $r_2 \in R$ and $s_2 \in S$ such that $r_1 s_2 = s_1 r_2$ (the above property is commonly known as **the Ore's condition**). If S is a left Ore set, then $S^{-1}R$ is called the **left Ore localization** of R by regular element. The operations of $S^{-1}R$ are given by $as + bt := \frac{ca+db}{u}$, where $u := cs = dt \in S$, for some c, $d \in R$ (the Ore's condition applied to s and t), and $\frac{a}{s}\frac{b}{t} := \frac{cb}{us}$, where $ua = ct$, for some $u \in S$ and $c \in R$ (the Ore's condition applied to a and t).

Proposition 1.1.33 ([KM19], Proposition 2.8)**.** *Let R be a ring. Then the following assertions hold:*

(i) *If R is left fusible, then so is $S^{-1}R$.*

(ii) *If R is regular left fusible, then $Q_l(R)$ is left fusible.*

(iii) *If $S^{-1}R$ is regular left fusible, then R is regular left fusible.*

(iv) *Suppose every regular element of R is invertible. Then R is regular left fusible if and only if R is left fusible.*

Proof. We follow the ideas presented in [KM19], Proposition 2.8.

(i) Suppose R is left fusible and let $s^{-1}r \in S^{-1}R$. Then r has a left fusible decomposition, say $r = a + w$. Therefore, $s^{-1}r = s^{-1}(a + w)$. In addition, we have $s^{-1}(a + w) = s^{-1}a + s^{-1}w$, since by definition $s^{-1}a + s^{-1}w := (s's)^{-1}(s'a + r'w)$, where $s' \in S$, $r' \in R$ and also $s's = r's$. It is clear that $r' = s' = 1$ and therefore $s^{-1}(a + w) = s^{-1}r = s^{-1}a + s^{-1}w$. Let us show that $s^{-1}r = s^{-1}a + s^{-1}w$ is a left fusible representation of $s^{-1}r$. Since $a \in R$ is zero divisor, there exists $c \in R$ such that $ca = 0$ with $c \neq 0$. Thus, $(cs)s^{-1}a = c(ss^{-1})a = ca = 0$, so $s^{-1}a$ is a left zero divisor in $S^{-1}R$. On the other hand, if $s^{-1}w$ is a zero divisor, there exists $s_1^{-1}t_1 \in S^{-1}R$ such that $s_1^{-1}t_1s^{-1}w = 0$ but $s_1^{-1}t_1s^{-1}w := (s's_1)^{-1}t'w$ where $s' \in S$, $t' \in R$ and $s't_1 = r's$. Hence $(s's_1)^{-1}t'w = 0$ which implies $t'w = 0$, so w is zero divisor which is a contradiction. Therefore $s^{-1}w$ is not zero divisor. Thus, $s^{-1}r = s^{-1}a + s^{-1}w$ is a left fusible representation.

(ii) Suppose R is regular left fusible and let $s^{-1}r \in Q_l(R)$. Then, there exists a regular element $t \in R$ such that tr has a fusible representation, i.e., $tr = a + w$. Since t is a regular element, ts is a regular element and hence $(ts)^{-1}$ exists. Then, we have $(ts)^{-1}(tr) = (ts)^{-1}(a+w)$. In addition, $(ts)^{-1}(tr) = s^{-1}t^{-1}(tr) = s^{-1}r$ and therefore $s^{-1}r = (ts)^{-1}(a + w)$. On the other hand, similar to the part (i), $(ts)^{-1}(a + w) = (ts)^{-1}a + (ts)^{-1}w$. So, $s^{-1}r = (ts)^{-1}a + (ts)^{-1}w$. Analogous to the previous proof, it is possible to verify that $s^{-1}r = (ts)^{-1}a + (ts)^{-1}w$ is a left fusible representation. Thus, $Q_l(R)$ is left fusible.

(iii) Suppose $S^{-1}R$ is regular left fusible and let $r \in R$. Then, there exists a regular element $s_1^{-1}t_1 \in S^{-1}R$ such that the element $z = s_1^{-1}t_1r$ has a left fusible decomposition, i.e., $s_1^{-1}t_1r = s_2^{-1}a_2 + s_3^{-1}w_3$. Taking the left common denominator, it is possible write $z = s^{-1}tr = s^{-1}a + s^{-1}w$ for suitable $s \in S$, $t, a, w \in R$. Let us show that $tr = a+w$ is a left fusible decomposition. If t is zero divisor, there is $c \in R$ such that $ct = 0$, so $(cs)s^{-1}t = ct = 0$ which is a contradiction, since $s^{-1}t$ is a regular element. Additionally, since $s^{-1}a$ is zero divisor, there is $s_1^{-1}t_1 \in S^{-1}R$ such that $0 = s_1^{-1}t_1s^{-1}a = (s's_1)^{-1}r'a = 0$, for $s' \in S$, $r' \in R$ where $s't_1 = r's$. Therefore, $r'a = 0$, so a is a zero divisor of R. Similarly, we can show that w is not zero divisor. Hence, this implies that the element tr has a left fusible representation in R, so R is regular left fusible.

(iv) We have observed previously that every left fusible ring is regular left fusible ring. Now, suppose R is regular left fusible and let $r \in R$. Since R is regular left fusible, there exists a regular element $t \in R$ such that tr has a fusible representation, i.e., $tr = a + w$. Also, since every regular element of R is invertible, there is $t^{-1} \in R$ such that $t^{-1}(tr) = t^{-1}(a + w)$. We have $t^{-1}(tr) = r$ and similar to what has been done before, we can see that $t^{-1}(a + w) = t^{-1}a + t^{-1}w$, so $r = t^{-1}a + t^{-1}w$. Analogously to the proof presented above, it can be seen that $r = t^{-1}a + t^{-1}w$ is a left fusible representation in R, and so R is left fusible ring.

□

The following result provides an interesting equivalency of the notions fusible and regular fusible on the left quotient ring $Q_l(R)$.

Proposition 1.1.34 ([KM19], Corollary 2.9)**.** *Suppose R has the left quotient ring $Q_l(R)$. The following conditions are equivalent:*

(i) *R is regular left fusible.*

(ii) *$Q_l(R)$ is left fusible.*

(iii) *$Q_l(R)$ is regular left fusible.*

Proof. It is an immediate consequence of the previous proposition. □

Two important concepts in the study of Goldie's theory are related to this new class of regular fusible rings. The following facts are presented in [KM19].

Proposition 1.1.35 ([KM19], Lemma 2.11 and Proposition 2.13)**.** *Let R be a ring. Then the following assertions hold:*

(i) *Every unit-regular ring is regular left fusible.*

(ii) *Every regular left fusible ring R is left nonsingular.*

Proof. We follow the ideas established in [KM19], Lemma 2.11 and Proposition 2.13.

(i) Let R be a unit regular ring and $r \in R$, i.e., $r = rur$ where $u \in U(R)$. Then $e = ur$ is an idempotent since $e^2 = (ur)^2 = urur = ur = e$. Therefore, if follows that $ur = e = (1 - 2e) + (e - 1)$ is a left fusible decomposition. This fact shows that R is regular left fusible. Now, since every regular element in R is invertible this implies R is a left fusible.

(ii) Suppose $a \in \mathrm{Sing}(R_R)$ with $a \neq 0$. Since R is regular left fusible, there exists an element $s \in R$ such that sa is left fusible, i.e., $sa = r + w$ with $r \in Z_l(R)$ and $w \notin Z_l(R)$. Since $\mathrm{Sing}(R_R)$ is a two sided ideal, $sa \in \mathrm{Sing}(R_R)$. Hence, $r_R(sa) \cap r_R(r) \neq 0$ since $r_R(r) \neq 0$. Thus, there is $z \in r_R(sa) \cap r_R(r)$ such that $0 = (sa - r)z = wz$, and so w is a zero divisor, which is a contradiction. Therefore, $a = 0$, that is, R is left nonsingular.

□

As a result of the propositions presented above and the important Goldie's theorem, a characterization of semiprime left Goldie rings in terms of regular left fusible rings is presented. The reader can find the notion of semiprime ring in Definition A.3.11 and the concept of Goldie's ring in Definition A.3.14. Newt, we recall the definition of Artinian ring.

Definition 1.1.36 ([GW04], p. 73)**.** *A* module A is **Artinian** provided A satisfies the descending chain condition (or DCC) on submodules, i.e., there does not exist a properly descending infinite chain $A_1 > A_2 > \cdots$ of submodules of A. A ring R is **right (left) Artinian** provided the right module R_R (left module ${}_RR$) is Artinian. If both conditions hold, R is called an **Artinian ring**.

Proposition 1.1.37 ([KM19], Theorem 2.14)**.** *For a semiprime ring R, the following conditions are equivalent:*

(i) *R is left Goldie.*

(ii) *R is regular left fusible and has finite left Goldie dimension*

Proof. We follow the ideas developed in [KM19], Theorem 2.14.

- (i) ⇒ (ii) Suppose R is a left Goldie ring. Then, by Proposition A.3.19, it follows that $Q_l(R)$ is a semisimple Artinian ring. Since, every semisimple ring is unit regular ring, then $Q_l(R)$ is a unit regular ring. Hence, we get that $Q_l(R)$ is regular left fusible, by Proposition 1.1.35. Thus, we have that R is regular left fusible, by Proposition 1.1.34.
- (ii) ⇒ (i) Let R be a regular left fusible with finite left Goldie dimension. Then, by Proposition 1.1.35, R is left nonsingular. Therefore, by Proposition A.3.19, we have R satisfies the ascending chain condition in left annihilators. Hence, we conclude that R is left Goldie.

□

1.2 Skew polynomial rings

Introduced by Ore [Ore33], skew polynomial rings are a generalization of the classic polynomials ring. The important detail is that not necessarily the variable commutes with coefficients. For these rings, a rule of commutation between the variables and the coefficients is defined, which gives rise to the existence of an endomorphism and a derivation on the coefficient ring of this new ring of polynomials. In this section, we remember the definition of skew polynomial extension and state some interesting properties and highlight important examples in the literature.

Definition 1.2.1. ([GW04], p. 26). Let R be a ring and $\sigma : R \to R$ an endomorphism. An additive map $\delta : R \to R$ is called a **σ-derivation** of R, if the following identity is satisfied: $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$, for all $r, s \in R$.

Note that $\delta(1) = \delta(1 \cdot 1) = \delta(1)1 + 1\delta(1) = 2\delta(1)$, whence $\delta(1) = 0$.

Definition 1.2.2. ([GW04], p. 34). Let R be a ring, σ a ring endomorphism of R, and δ a σ-derivation of R. We write $R[x; \sigma, \delta]$ provided that,

(i) $R[x; \sigma, \delta]$ is a ring, containing R as a subring,

(ii) x is an element of $R[x; \sigma, \delta]$,

(iii) $R[x; \sigma, \delta]$ is a free left R-module with basis $\{1, x, x^2, \dots\}$,

(iv) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

A ring $R[x; \sigma, \delta]$ is called a **skew polynomial ring** over R, or an **Ore extension** of R

Remark 1.2.3 ([LLM77], Lemma 2.1)**.** Let $\sigma : R \to R$ be an endomorphism on R and δ be an σ-derivation of R. For integers i, j with $0 \leq i \leq j$, $f_j^i \in \mathrm{End}(R, +)$ will denote the map which is the sum of all possible words in σ, δ built with i letters σ and $j - i$ letters δ. For instance, $f_0^0 = 1$, $f_j^j = \sigma^j$, $f_j^0 = \delta^j$ and $f_j^{j-1} = \sigma^{j-1}\delta + \sigma^{j-2}\delta\sigma + \cdots + \delta\sigma^{j-1}$ In the general case, if $r \in R$ and $n \in \mathbb{N}$, the following formula holds:

$$x^n r = \sum_{i=0}^{n} f_i^n(r)x^i.$$

Moreover, if $r, s \in R$ and $n, m \in \mathbb{N}$, then:

$$(rx^n)(sx^m) = r\sum_{i=0}^{n} f_i^n(s)x^{i+m} \tag{1.2.1}$$

Now, we review some examples. These evidence that skew polynomial rings are, indeed, a generalization of the classic polynomials ring. They were adapted from [GW04], [Lez15] and [MR01].

Example 1.2.4. If $\sigma = i_R$, where i_R is the identity homomorphism and $\delta = 0$, then Definition 1.2.2 (iv) reduces to $xr = rx$, for all $r \in R$. This is simply the classical univariate polynomials ring over R, and we write $R[x; i_R, 0] = R[x]$. Moreover the formula (1.2.1) corresponds to usual multiplication of monomials.

Example 1.2.5. If $\delta = 0$, then Definition 1.2.2 (iv) becomes $xr = \sigma(r)x$, for all $r \in R$. In this case we write $R[x; \sigma, \delta] = R[x; \sigma]$. The expression (1.2.1) reduces to $(rx^n)(sx^m) = r\sigma^n(s)x^{n+m}$, for all $r, s \in R$ and $n, m \in \mathbb{N}$. An interesting case studied extensively in the literature is when σ is an automorphism of R. $R[x, \sigma]$ is known as **the polynomials ring of endomorphism type**.

Example 1.2.6. If $\sigma = i_R$, then Definition 1.2.2 (iv) becomes $xr = rx + \delta(r)$, for all $r \in R$. In this case we write $R[x; i_R, \delta] = R[x; \delta]$. Moreover, the expression (1.2.1) simplifies to

$$(rx^n)(sx^m) = r\sum_{k=0}^{n}\binom{n}{k}\delta^k(s)x^{n+m-k}, \text{ for all } r, s \in R \text{ and } n, m \in \mathbb{N}.$$

$R[x, \delta]$ is known as **the polynomials ring of derivation type**.

Example 1.2.7. The polynomial ring of endomorphism type shown in Example 1.2.5 can be considered in general for n variables. If $\delta_i = 0$, for all $1 \leq i \leq n$, we write $R[x_1; \sigma_1] \cdots [x_n, \sigma_n]$ to denote **the polynomials ring iterated of endomorphism type**. Where σ_i is defined as follows:

$$\sigma_i : R[x_1; \sigma_1] \cdots [x_{i-1}; \sigma_{i-1}] \to R[x_1; \sigma_1] \cdots [x_{i-1}; \sigma_{i-1}] \quad \text{for each} \quad 1 \leq i \leq n.$$

Example 1.2.8. Similarly, the polynomials ring of derivation type shown in Example 1.2.6 can be considered in general for n variables. If $\sigma_i = i_S$, for all $1 \leq i \leq n$, with $S = R[x_1; \delta_1] \cdots [x_{i-1}; \delta_{i-1}]$ and i_S the identity homomorphism, we write $R[x_1; \delta_1] \cdots [x_n, \delta_n]$ to denote **the polynomials ring iterated of derivation type**, where δ_i is defined as follows:

$$\delta_i : R[x_1; \delta_1] \cdots [x_{i-1}; \delta_{i-1}] \to R[x_1; \delta_1] \cdots [x_{i-1}; \delta_{i-1}], \quad \text{for each} \quad 1 \leq i \leq n.$$

We present some concepts of great importance in the study of skew polynomial rings.

Definition 1.2.9. ([GW04], p. 36) Let $R[x; \sigma, \delta]$ be a skew polynomial ring over R. Any nonzero element p in $R[x; \alpha, \delta]$ can be uniquely expressed in the form $p = r_n x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0$, for some non negative integer n and some elements $r_i \in R$ with $r_n \neq 0$. We define $\mathrm{lm}(p) := x^n$, the **leading monomial** of p; $\mathrm{lc}(p) := r_n$, the **leading coefficient** of p; $\mathrm{lt}(p) := r_n x^n$, the **leading term** of p; $\deg(p) := n$, the **degree** of p. If $p = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$ and $\deg(0) := -\infty$.

The next result states the universal property for skew polynomial rings.

Proposition 1.2.10 ([GW04], Proposition 2.4)**.** *Let $R[x; \sigma, \delta]$ be a skew polynomial ring. Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi(\sigma(r))y + \phi(\delta(r))$, for all $r \in R$. Then there is a unique ring homomorphism $\psi : R[x; \sigma, \delta] \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$.*

Proof. Since $R[x; \sigma, \delta]$ is a left free R-module with basis $\{x_i\}_{i\geq 0}$ and T is also a left R-module with product given by $a \cdot b := f(a)b$, $a \in R$, $b \in T$, then we can define a morphism of left R-modules $\psi : R[x; \sigma, \delta] \to T$ defined by $x^i \mapsto y^i$, through which $\sum_{i=0}^{n} a_i x^i \mapsto \sum_{i=0}^{n} f(a_i)y^i$. In [GW04], it was shown that the application ψ is a homomorphism ring, which is unique and satisfies that $\psi|_R = \phi$ and $\psi(x) = y$. □

We recall that R is a **principal right ideal domain**, if R is a domain in which all right ideals are principal. **Principal left ideal domains** are defined analogously (see [GW04], p. 39).

Some basic properties are established in the next result.

Proposition 1.2.11 ([GW04], Theorem 2.9)**.** *Let R be a ring and $R[x;\sigma,\delta]$ be a skew polynomial ring. Then the following assertions hold:*

(i) *If σ is injective and R is a domain, then $R[x;\sigma,\delta]$ is an integral domain.*

(ii) *If σ is an automorphism and R is a prime ring, then $R[x;\sigma,\delta]$ is a prime ring.*

(iii) *If σ is injective and R is a division ring, then $R[x;\sigma,\delta]$ is a principal right ideal domain.*

Proof. We follow the ideas presented in [GW04], Theorem 2.9.

(i) Let $p(x) = p_0 + p_1x + \cdots + p_mx^m \neq 0$ and $q(x) = q_0 + q_1x + \cdots + q_nx^n \neq 0$ be two elements of $R[x;\sigma,\delta]$ such that $p_m, q_n \neq 0$. Then $\mathrm{lt}(pq) = p_m\sigma^m(q_n)x^{m+n} \neq 0$, by the injectivity of σ. Hence, we have $pq \neq 0$.

(ii) Let $p(x) = a_nx^n + \cdots + a_1x + a_0$ and $q(x) = b_mx^m + \cdots + b_1x + b_0$ be elements in $R[x;\sigma,\delta]$ with a_n, b_m nonzero. Since σ in injective, it follows that $\sigma^n(b_m) \neq 0$. On the other hand, R is prime and so $a_nR\sigma^n(b_m) \neq 0$. In addition, σ is surjective, which implies that each element of this set is the leading coefficient of some element of $p(x)Rq(x)$. Hence $p(x)Rq(x) \neq 0$ and so $p(x)R[x;\sigma,\delta]q(x) \neq 0$.

(iii) Let I be a left ideal of $R[x;\sigma,\delta]$. If $I = 0$ the result is immediate. Suppose that $I \neq 0$ and let $q \in I$ with $q \neq 0$. Let $q_0 \in I$ be such that $q_0 \neq 0$ and $\deg(q_0)$ be minimal; then, given $p \in I$, $p = cq_0 + r$ for certain $c, r \in R[x;\sigma,\delta]$ with $r = 0$ or $\deg(r) < \deg(q_0)$. Note that if this last inequality is true, it would contradict the minimality of q_0, therefore $r = 0$, $p = cq_0$ and $I = \{q_0\rangle$.

□

Recall that a ring R is said to be **left Noetherian**, if any ascending chain of left ideals stabilizes ([GW04], Proposition 1.1). We mention a theorem of great importance which generalizes the known Hilbert's Basis Theorem for the commutative case. Its proof can be found in [GW04], Theorem 2.6 and [Lez15], Theorem 1.2.6.

Proposition 1.2.12 ([GW04], Theorem 2.6)**.** *If σ is an automorphism and R is right Noetherian, then $R[x;\sigma,\delta]$ is right Noetherian. Analogously for the left case.*

Proposition 1.2.13 ([GW04], Corollary 2.7)**.** *Let $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\alpha_n,\delta_n]$ be an iterated skew polynomial ring, where each σ_i is an automorphism. If R is right Noetherian, then $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\alpha_n,\delta_n]$ is right Noetherian. Analogously for the left case.*

The following result is adapted from [LM05], Theorem 3.6, and it is very important in the study of skew polynomial rings since it gives sufficient conditions for $R[x;\sigma,\delta]$ to be semiprime (resp. prime) ring (see Definition A.3.11).

Proposition 1.2.14 ([LM05], Proposition 3.6)**.** *Suppose that the ring R satisfies the ACC on left annihilators. Let σ be an injective endomorphism and δ a σ-derivation of R. Then the skew polynomial ring $R[x;\sigma,\delta]$ is prime (semiprime), provided R is prime (semiprime).*

Under the conditions of the previous proposition, it can be shown that the uniform dimension of the ring R coincides with the dimension of the skew polynomial ring $R[x;\sigma,\delta]$ (see [LM05], Theorem 3.4). With the help of the Proposition 1.2.14 and the previous observation, the following result is obtained, which gives sufficient conditions so that the property of being a Goldie's ring is transferred to $R[x;\sigma,\delta]$. The reader can find Goldie's ring in Definition A.3.14 and uniform dimension in Definition A.3.6.

Proposition 1.2.15 ([LM05], Theorem 3.8)**.** *Let R be a semiprime left Goldie ring, σ, δ an injective endomorphism and a σ-derivation of R, respectively. Then $R[x;\sigma,\delta]$ is also a semiprime left Goldie ring.*

Next, we present a very interesting fact that relates the fusible property with the skew polynomial rings. In addition, analogously this relation with the fusible property is also presented for the skew power series ring. Before, we recall that, if R is a ring and σ an endomorphism of R, then $R[[x;\sigma]]$ is called the **skew power series ring** in an indeterminate x and multiplication is defined with respect to the relation $x^i a = \sigma^i(a)x^i$, for all $i \in \mathbb{N}$ (see [MR01], p. 22).

Proposition 1.2.16 ([GM16], Proposition 2.9)**.** *If R is a left fusible ring and σ is a ring automorphism of R, then the following assertions hold:*

(i) *$R[x;\sigma]$ is left fusible ring.*

(ii) *$R[[x;\sigma]]$ is left fusible ring.*

Proof. We follow the ideas in [GM16], Proposition 2.9

(i) Let R be a left fusible ring and $f(x) = a_0 + a_1 x \cdots + a_n x^n \in Z_l(R[x;\sigma])$. Let $k \in \mathbb{N}$ be the least natural number such that $a_k \neq 0$ with $0 \leq k \leq n$. Since R is left fusible, then there exist $z_k \in Z_l(R)$ and $n_k \in Z_l^*(R)$ such that $a_k = z_k + n_k$, and therefore, $f(x) = (z_k x^k) + (n_k x^k + a_{k+1}x^{k+1} + \cdots + a_n x^n)$. Now, we define $g(x) = z_k x^k$ and $h(x) = n_k x^k + a_{k+1}x^{k+1} + \cdots + a_n x^n$. Let us show that $g(x) \in Z_l(R[x;\sigma])$ and $h(x) \in Z_l^*(R[x.\sigma])$. Since $z_k \in Z_l(R)$ there exists $d \in R$ with $d \neq 0$ such that $z_k d = 0$. Further, σ is an surjective homomorphism, so that there exists $e \in R$ with $e \neq 0$ such that $\sigma^k(e) = d$. Therefore, $g(x)e = z_x^k e = z_k \sigma^k(e)x^k = z_k d x^k = 0$ and this implies $g(x) \in Z_l(R[x;\sigma])$. Now, let us show that $h(x) \in Z_l^*(R[x;\sigma])$. If $h(x) \in Z_l(R[x;\sigma])$ then there exists $b \in R$ such that $n_k x^k b = 0$. Thus, $n_k \sigma^k(b)x^k = 0$. On the other hand, σ is an injective homomorphism and therefore $\sigma^k(b) \neq 0$, and hence n_k is a left zero divisor, which is a contradiction. Thus, $f(x) = g(x) + h(x)$ such that $g(x) \in Z_l(R[x;\sigma])$ and $h(x) \in Z_l^*(R[x;\sigma])$, that is, $R[x;\sigma]$ is a left fusible ring.

(ii) This proof is completely analogous to that presented in the case of skew polynomial rings $R[x;\sigma]$.

□

The following example shows that Proposition 1.2.16 need not to be true for any skew polynomial and shows that there exist right fusible elements which are not left fusible.

Example 1.2.17 ([GM16], Example 2.10)**.** Let R be a domain and σ be a ring endomorphism of R which is not injective. Let $r \in R$ satisfies $\sigma(r) = 0$, with $r \neq 0$. Then $xr = \sigma(r)x = 0$. In [GM16], Example 2.10 one can see that $\langle x \rangle = Z_l(R[x, \sigma])$. Since $Z_l(R[x, \sigma])$ is a left ideal, then $R[x, \sigma]$ is not a left fusible ring. By definition, it follows that x is a non right zero divisor element and hence x is a right fusible element while it is not a left fusible element.

Finally, we review some very important and well studied examples in the literature of skew polynomial rings. These are adapted from [GW04], [Lez15] and [MR01].

Example 1.2.18. A skew polynomials ring iterated $A = R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ that satisfies the following conditions:

$$\begin{aligned} \sigma_i(x_j) &= x_j, \quad j < i, \\ \delta_i(x_j) &= 0, \quad j < i, \\ \sigma_i\sigma_1 &= \sigma_1\sigma_i, \ 1 \leq i \leq n, \\ \delta_i\delta_1 &= \delta_1\delta_i, \ 1 \leq i \leq n, \end{aligned}$$

where the two last relations are restricted to R. A is know as **extension of Ore** of R. In [Lez15], Proposition 1.3.2, an equivalence between the relationships described above and the following relationships is presented:

$$\begin{aligned} x_ix_j &= x_jx_i, \ 1 \leq i, j \leq n, \\ \sigma_i(R), \delta_i(R) &\subseteq R, \ 1 \leq i \leq n. \end{aligned}$$

Therefore, under these conditions, the maps σ_i, δ_i can be seen as $\sigma_i, \delta_i : R \to R$. In addition, if $\Bbbk[t_1, \ldots, t_n]$ be a classical multivariate polynomial ring over $\Bbbk$, σ_i, δ_i are $\Bbbk$-linear and $A = \Bbbk[t_1, \ldots, t_n][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is an iterated Ore extension, then A is called an **Ore algebra**. We present concrete cases of the above.

Example 1.2.19. Let $\Bbbk[t]$ be the classical univariate polynomial ring over a field $\Bbbk$. If $\sigma_h : \Bbbk[t] \to \Bbbk[t]$ is the endomorphism defined by $\sigma_h(p(t)) = p(t-h)$, for all $p(t) \in \Bbbk[t]$, then the skew polynomial ring $S_h = \Bbbk[t][x_h; \sigma_h]$ over $\Bbbk[t]$ is known as **the algebra of shift operators**. If $p(t), q(t) \in \Bbbk[t]$, the expression (1.2.1) becomes

$$p(t)x^n q(t)x^m = p(t)q(t - nh)x^{n+m}, \quad \text{for all} \quad n, m \in \mathbb{N}.$$

Notice that S_h is an Ore algebra.

Example 1.2.20. Let $\Bbbk[t]$ be the classical univariate polynomial ring over a field $\Bbbk$ and denote by $\frac{\partial}{\partial t}$ the derivate operator with respect to t. The skew polynomial ring $A_1(\Bbbk) = \Bbbk[t][x; \frac{\partial}{\partial t}]$ over $\Bbbk[t]$ is known as **the first Weyl algebra**. If $p(t), q(t) \in k[t]$ the formula 1.2.1 becomes

$$p(t)x^n q(t)x^m = p(t) \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial t}(q(t))x^{n+m-k}, \quad \text{for all} \quad m, n \in \mathbb{N}.$$

More generally, the Ore algebra $A_n(\Bbbk) := \Bbbk[t_1, \ldots, t_n][x_1; \frac{\partial}{\partial t_1}] \cdots [x_n; \frac{\partial}{\partial t_n}]$ is know as **the n-th Weyl algebra**.

Example 1.2.21. Let $\Bbbk[t]$ be the classical univariate polynomial ring over a field $\Bbbk$. For every $h \in \Bbbk$, we define $D_h := \Bbbk[t][x; \frac{\partial}{\partial t}][x_h; \sigma_h]$, where $\sigma_h : \Bbbk[t] \to \Bbbk[t]$ is the endomorphism defined by $\sigma_h(p(t)) = p(t-h)$. Notice that $D_h = A_1(\Bbbk)[x_h; \sigma_h]$ and hence it is an Ore algebra which is known as **the mixed algebra** (also known as the algebra of delay differential operator).

Example 1.2.22. The Ore algebra defined as $D := \Bbbk[t_1 \dots t_n][x_1; \sigma_1] \cdots [x_n; \sigma_n]$, where

$$\sigma_i(p(t_1, \dots, t_n)) = p(t_1, \dots, t_{i-1}, t_i + 1, t_{i+1}, \dots, t_n), \quad \text{for} \quad 1 \leq i \leq n,$$

it is known as **the algebra for multidimensional discrete linear systems**.

1.3 PBW extensions

Next, we recall the PBW (Poincaré-Birkhoff-Witt) extensions defined by A. D. Bell and K. R. Goodearl [BG88] in [BG88]. As we mention in the previous section, skew polynomial rings describe a large number of algebras of polynomial type. However, some important examples, such as the enveloping algebra of a finite dimensional Lie algebra, are left out of this description. In [BG88], a new ring family was defined with the aim of generalizing rings of polynomial type that satisfy the PBW property and extend universal enveloping algebras.

Definition 1.3.1 ([BG88], p. 27)**.** Let R and A be rings. We say that A is a **Poincaré-Birkhoff-Witt (PBW) extension of** R, if the following conditions hold:

1. A contains R as a proper subring,
2. There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a free left R-module with basis $\text{Mon}(A) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$.
3. For each $r \in R$ and every $1 \leq i \leq n$, $x_i r - r x_i \in R$.
4. For every $1 \leq i, j \leq n$, $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$.

We write $A = R\langle x_1, \dots, x_n \rangle$, and R is called **the ring of coefficients** of the extension A.

Next, we present some examples of PBW extensions. Some of these noncommutative rings cannot be expressed as iterated Ore extensions. These examples are adapted from [BG88], [Pas87], [McC74] and [Rin63].

Example 1.3.2. Let $\Bbbk$ be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \dots, x_n\}$. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is a PBW extension over $\Bbbk$, since, for all $k \in \Bbbk$ and $x_i, x_j \in X$, $x_i k - k x_i = 0 \in \Bbbk$ and $x_i x_j - x_j x_i = [x_i, x_j] \in \mathfrak{g}$ where $[x_i, x_j] = \Bbbk x_i + \cdots + \Bbbk x_n \subseteq \Bbbk + \Bbbk x_i + \cdots + \Bbbk x_n$, for all $1 \leq i, j \leq n$. In [Dix77], Theorem 2.1.11, it was shown that $U(\mathfrak{g})$ satisfies the property PBW. Hence, $U(\mathfrak{g})$ is a PBW extension.

Examples 1.3.3. Let $\Bbbk$ be a field, $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \ldots, x_n\}$ and $U(\mathfrak{g})$ be as in the previous example. let R be a $\Bbbk$-algebra containing $\Bbbk$. The tensor product $A := R \otimes_{\Bbbk} U(\mathfrak{g})$ is a PBW extension of R and it is a particular case of **the crossed product** $R * U(\mathfrak{g})$ of R by $U(G)$, which is a PBW extension of R.

Examples 1.3.4. The **differential operator rings** $V(R, L)$, where L is a Lie algebra, which is also a free R-module equipped with a suitable Lie algebra map to derivations on R. In [Rin63], Theorem 3.1, it was shown that $V(R, L)$ satisfies the property PBW. Hence, $V(R, L)$ is a PBW extension.

Example 1.3.5. The **twisted or smash product differential operator ring** $R\#_\sigma U(\mathfrak{g})$, where $\mathfrak{g}$ is a finite dimensional Lie algebra acting on R by derivations, and σ is Lie 2-cocycle with values in R. In [McC74], Theorem 2.8, it was shown that $R\#_\sigma U(\mathfrak{g})$ satisfies the property PBW. Therefore, $R\#_\sigma U(\mathfrak{g})$ is a PBW extension.

Examples 1.3.6. The **universal enveloping ring** $\mathcal{U}(V, R, \Bbbk)$, where R is a $\Bbbk$-algebra, and V is a $\Bbbk$-vector space which is also a Lie ring containing R and $\Bbbk$ as Lie ideals with suitable relations. The enveloping ring $\mathcal{U}(V, R, \Bbbk)$ is a finite skew PBW extension over R, if $dim_{\Bbbk}(V/R)$ is finite. In [Pas87], Theorem 1.3, it was proved that $\mathcal{U}(V, R, \Bbbk)$ satisfies the property PBW. Hence, $\mathcal{U}(V, R, \Bbbk)$ is a PBW extension.

CHAPTER 2

Fusible property over skew PBW extensions

In this chapter, we present interesting results relating fusible rings to skew PBW extensions characterizing some important elements in a ring. In Section 2.1, we study some properties of PBW extensions and present interesting results that provide sufficient conditions to guarantee fusible property in this family of noncommutative rings. The fundamental tool in these results is Goldie's theory and the filtration-graduation technique. In Section 2.2, we study Σ-rigid rings and weak Σ-rigid rings, highlighting some of their properties and characterizing some important elements in a skew PBW extension over weak Σ-rigid rings. In Section 2.3, we study (Σ, Δ)-compatible rings, looking at some interesting properties of these rings. Also, we present an interesting result that relates these rings to the fusible property and characterize important elements of a skew PBW extension over (Σ, Δ)-compatible rings. All these new results have been submitted for publication as a contribution to the theory of fusible rings of noncommutative rings of polynomial type.

2.1 Skew PBW extensions

Skew PBW extensions, also known as σ-PBW extensions, are a wide class of noncommutative rings which include many rings and algebras arising in quantum mechanics and many areas in mathematics, such as the PBW extensions, Weyl algebras, enveloping algebras of finite dimensional Lie algebras, iterated Ore extensions of injective type and many others. Skew PBW extensions were introduced in [GL11], and since then several ring theoretical and homological properties of these objects have been studied by different authors. Some examples and results can be consulted in [GL11],[LR14],[RS17],[Rey15],[RS18a],[RJ18] and [LAC+13]. In this section, we study some very important and well known facts in the literature such as the relationship of skew PBW extensions with semiprime Goldie's rings and the filtration-graduation techniques used on them. The reader may see Appendix A.3 and Appendix A.2 to recall some basic concepts from these two areas. These results are adapted from [LAC+13] and [LR14]. We also present new results that provide sufficient conditions to guarantee fusible property in a skew PBW extension. Finally, we present a large number of noncommutative type algebras that satisfy these new results satisfying the fusible property.

Definition 2.1.1. ([GL11], Definition 1). Let R and A be rings. We say that A is a **skew PBW extension** (also known as σ**-PBW extension** or **SPBW**) over R, which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ and $x^0 := 1 \in \text{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any $1 \leq i, j \leq n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$ (i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)}$ of R with $x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{l=1}^n r_l^{(i,j)} x_l$.

Proposition 2.1.2 ([GL11], Proposition 3)**.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. From now on, we will write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$, and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Definition 2.1.3 ([GL11], Definition 4 and [LAR15], Definition 2.3)**.** Let A be a skew PBW extension over R.

(i) A is called **quasi-commutative**, if the conditions (iii) and (iv) in Definition 2.1.1 are replaced by the following: (iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; (iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(ii) A is called **bijective**, if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i, j \leq n$.

(iii) A is called of **endomorphism type**, if $\delta_i = 0$, for every i. In addition, if every σ_i is bijective, A is a skew PBW extension of **automorphism type**.

Remark 2.1.4 ([GL11], Section 3)**.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) Consider the families Σ and Δ in Proposition 2.1.2. Throughout the work, for any element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes composition, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) Let $\succeq$ be a total order defined on $\text{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. If f is a nonzero element of A, then f can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$ (eventually, we use expressions as $f = a_0 + a_1 Y_1 + \cdots + a_m Y_m$, with $a_i \in R$, and $Y_m \succ \cdots \succ Y_1$). With this notation, we define $\text{lm}(f) := X_m$, the **leading monomial** of f; $\text{lc}(f) := a_m$, the **leading coefficient** of f; $\text{lt}(f) := a_m X_m$, the **leading term** of f; $\exp(f) := \exp(X_m)$, the **order** of f. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^m$. Finally, if $f = 0$, then $\text{lm}(0) := 0$, $\text{lc}(0) := 0$, $\text{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \text{Mon}(A)$. Thus, we extend $\succeq$ to $\text{Mon}(A) \cup \{0\}$.

Definition 2.1.5 ([GL11], Definition 11)**.** If $\succeq$ is a total order on $\text{Mon}(A)$, we say that $\succeq$ is a **monomial order** on $\text{Mon}(A)$, if the following conditions hold:

(i) For every $x^{\beta}, x^{\alpha}, x^{\gamma}, x^{\lambda} \in \text{Mon}(A)$, $x^{\beta} \succeq x^{\alpha} \Rightarrow \text{lm}(x^{\gamma}x^{\beta}x^{\lambda}) \succeq \text{lm}(x^{\gamma}x^{\alpha}x^{\lambda})$ (the total order is compatible with multiplication).

(ii) $x^{\alpha} \succeq 1$, for every $x^{\alpha} \in \text{Mon}(A)$.

(iii) $\succeq$ is degree compatible, i.e., $|\beta| \succeq |\alpha| \Rightarrow x^{\beta} \succeq x^{\alpha}$.

The following are examples of total orders which are compatible. These were adapted from [CLO15], Definition 5 and 6.

Examples 2.1.6. Lexicographic Order. Let $\alpha, \beta \in \mathbb{Z}_{n\geq 0}$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. We say that $x^{\alpha} \geq_{grlex} x^{\beta}$, if $x^{\alpha} = x^{\beta}$ or $x^{\alpha} \neq x^{\beta}$ but $|\alpha| = |\beta|$ or $x^{\alpha} \neq x^{\beta}$, $|\alpha| = |\beta|$, and there is $0 \leq i \leq n$ such that $a_j = b_j$, for all $0 \leq j < i$ and $a_i > b_i$.

Examples 2.1.7. Graded Reverse Lex Order. Let $\alpha, \beta \in \mathbb{Z}_{n\geq 0}$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. We say that $x^{\alpha} \geq_{grevlex} x^{\beta}$, if $x^{\alpha} = x^{\beta}$ or $x^{\alpha} \neq x^{\beta}$ but $|\alpha| = |\beta|$ or $x^{\alpha} \neq x^{\beta}$, $|\alpha| = |\beta|$, and there is $0 \leq i \leq n$ such that $a_j = b_j$, for all $i < j \leq n$ and $a_i < b_i$.

Proposition 2.1.8 ([GL11], Theorem 7)**.** *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \ldots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(i) *For each $x^{\alpha} \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_{\alpha} := \sigma^{\alpha}(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^{\alpha}r = r_{\alpha}x^{\alpha} + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_{α}.*

(ii) *For each $x^{\alpha}, x^{\beta} \in \text{Mon}(A)$, there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^{\alpha}x^{\beta} = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.*

Remark 2.1.9. ([Rey15], Proposition 2.9 and Remark 2.10 (iv)). Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension. Then the following assertions hold:

(i) If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and r is an element of R, then

$$
\begin{aligned}
x^{\alpha}r = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r &= x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}\left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}\right) \\
&+ x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}\left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j}\delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r)))x_{n-1}^{j-1}\right)x_n^{\alpha_n} \\
&+ x_1^{\alpha_1}\cdots x_{n-3}^{\alpha_{n-3}}\left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j}\delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r))))x_{n-2}^{j-1}\right)x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} \\
&+ \cdots + x_1^{\alpha_1}\left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2-j}\delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))))x_2^{j-1}\right)x_3^{\alpha_3}x_4^{\alpha_4}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n} \\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r))))x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \text{id}_R \ \text{ for } \ 1 \leq j \leq n.
\end{aligned}
$$

(ii) If $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, and a_i, b_j are elements of R, when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i. This assertion follows from the expression:

$$\begin{aligned} a_i X_i b_j Y_j = {} & a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\ & + a_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\ & + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\ & + \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}(b_j)} x_n^{\alpha_{in}} x^{\beta_j} \\ & + a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b_j} x^{\beta_j}. \end{aligned}$$

The next result states the universal property for skew PBW extensions; its proof is presented in [AL15], Theorem 3.1.

Proposition 2.1.10 ([AL15], Theorem 3.1)**.** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension with parameters $\sigma_i, \delta_i, c_{ij}, d_{ij}, a_{ij}^{(k)}, 1 \leq i, j \leq n$. Let B be a ring with homomorphism $\phi : R \to B$ and elements $y_1, \ldots, y_n \in B$ such that*

(i) *$y_i \phi(r) = \phi(\sigma_i(r)) y_i + \phi(\delta_i(r))$, for every $r \in R$.*

(ii) *$y_j y_i = \phi(c_{ij}) y_i y_j + \phi(a_{ij}^{(1)}) y_1 + \cdots + \phi(a_{ij}^{(n)}) y_n + d_{ij}$.*

Then, there exists an unique ring homomorphism $\tilde{\varphi} : A \to B$ such that $\tilde{\varphi}\iota = \varphi$ and $\tilde{\varphi}(x_i) = y_i$, where ι is the inclusion of R in A.

We present the Hilbert's Basis Theorem in its version for skew PBW extensions. The proof of this important fact can be consulted in [LR14], Corollary 2.4.

Proposition 2.1.11 ([LR14], Corollary 2.4)**.** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension of R. If R is a left Noetherian ring then A is also a left Noetherian ring. Analogously the right case.*

We mention some examples of noncommutative rings of interest in theoretical physics. All these algebras are examples of skew PBW extensions (see [Rey13b], [RJ18] and [RS17] for a proof of this assertion). A detailed list of skew PBW extensions can be found in [GL11], [LR14] and [RS17]. The next example shows that under certain conditions, an iterated skew polynomial ring can be seen as a skew PBW extension.

Example 2.1.12. Any **iterated skew polynomial ring** $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is a skew PBW extension, if it satisfies the following conditions:

1. σ_i is injective, for all $1 \leq i \leq n$;
2. $\sigma_i(r), \delta_i(r) \in R$, for every $r \in R$ and $1 \leq i \leq n$;
3. $\sigma_j(x_i) = cx_i + d$, for $i < j$ with $c, d \in R$ and c has a left inverse;
4. $\delta_j(x_i) \in R + Rx_1 + \cdots + R_i$, for $i < j$.

Under these conditions, we have $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] = \sigma(R)\langle x_1,\ldots,x_n\rangle$. Note that skew PBW extensions are more general that skew polynomial rings of injective type (see [Rey13b] and [Sua17]).

Next, we present some examples of skew PBW extensions known as quantum algebras, which are of interest in theoretical physics. In [RJ18], some interesting algebraic properties are studied for this class of algebras. Furthermore, classical results on Ore extensions are extended to these families of quantum algebras. A detailed list of examples can be found in [LR14] and [Rey13a].

Example 2.1.13. Weyl algebra and extended Weyl algebra: Let $\Bbbk$ be a field. The Weyl algebra $A_n(\Bbbk) = \Bbbk[t_1\cdots t_n][x_1;\frac{\partial}{\partial t_1}]\cdots[x_n;\frac{\partial}{\partial t_n}]$. The extended Weyl algebra $B_n(\Bbbk) = \Bbbk(t_1,\cdots,t_n)[x_1;\frac{\partial}{\partial t_1}]\cdots[x_n;\frac{\partial}{\partial t_n}]$, where $\Bbbk(t_1,\cdots,t_n)$ is the field of fractions of $\Bbbk[t_1,\cdots,t_n]$, is also a PBW extension.

Example 2.1.14. Additive analogue of the Weyl algebra: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $A_n(q_1,\ldots,q_n)$ is generated by $x_1,\ldots,x_n,t_1,\ldots,t_n$ and subject to the relations:

$$\begin{aligned} x_jx_i &= x_ix_j, \quad t_jt_i = t_it_j, \quad 1\leq i,j\leq n.\\ x_jt_i &= t_ix_j, \quad i\neq j.\\ x_it_i &= q_it_ix_i+1, \quad 1\leq i\leq n.\end{aligned}$$

where $q_i\in\Bbbk\setminus\{0\}$. Thus, $A_n(q_1,\ldots,q_n)\cong\sigma(\Bbbk[t_1,\ldots,t_n])\langle x_1,\ldots,x_n\rangle$

Example 2.1.15. Multiplicative analogue of the Weyl algebra: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by $x_1,\ldots,x_n$ subject to the relations:

$$x_jx_i = \lambda_{ji}x_ix_j, \quad 1\leq i<j\leq n,$$

where $\lambda_{ji}\in\Bbbk-\{0\}$. If $n=2$ then $\mathcal{O}_2(\lambda_{ji})$ is called the **quantum plane**. We note that $\mathcal{O}_n(\lambda_{ji})\cong\sigma(\Bbbk)\langle x_1,\ldots,x_n\rangle\cong\sigma(\Bbbk[x_1])\langle x_2,\ldots,x_n\rangle$.

Example 2.1.16. Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$ is generated by I_1,I_2,I_3 subject to the following relations:

$$I_2I_1-qI_1I_2 = -q^{1/2}I_3, \quad I_3I_1-q^{-1}I_1I_3 = q^{-1/2}I_2, \quad I_3I_2-qI_2I_3 = -q^{1/2}I_1,$$

where $q\in\Bbbk-\{0\}$. In this way, $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))\cong\sigma(\Bbbk)\langle I_1,I_2,I_3\rangle$.

Example 2.1.17. Diffusion algebras: Let $\Bbbk$ be a field. The diffusion algebra A is generated by $\{D_i,x_i\mid 1\leq i\leq n\}$ over $\Bbbk$ with relations

$$\begin{gathered} x_ix_j = x_jx_i, \quad x_iD_j = D_jx_i, \quad 1\leq i,j\leq n.\\ c_{ij}D_iD_j - c_{ji}D_jD_i = x_jD_i - x_iD_j, \quad i<j, \quad c_{ij},c_{ji}\in\Bbbk^*.\end{gathered}$$

Thus, $A\cong\sigma(\Bbbk[x_1,\ldots,x_n])\langle D_1,\ldots,D_n\rangle$. This is an example of a skew PBW extension which we cannot see as a skew polynomial ring over $\Bbbk[x_1,\ldots,x_n]$.

Example 2.1.18. q-Heisenberg algebra: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathrm{H}_n(q)$ is generated by the set of variables $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$, subject to the relations:

- $x_j x_i = x_i x_j, \quad z_j z_i = z_i z_j, \quad$ for all $\ 1 \leq i, j \leq n.$
- $z_j y_i = y_i z_j, \quad z_j x_i = x_i z_j, \quad y_j x_i = x_i y_j, \quad i \neq j.$
- $z_i y_i = q y_i z_i, \quad z_i x_i = q^{-1} x_i z_i + y_i, \quad y_i x_i = q x_i y_i, \quad$ for all $\ 1 \leq i \leq n, \ q \in \Bbbk - \{0\}.$

Thus, $H_n(q) \cong \sigma(\Bbbk)\langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n\rangle$.

The family of 3-dimensional skew polynomial algebras is an interesting example of skew PBW extensions. This fact can be appreciated from its definition (see also [RS17] and [Rey13b] for more details).

Definition 2.1.19 ([Ros95], Definition C.4.3)**.** Let $\Bbbk$ be a field. A **3-dimensional skew polynomial algebra** A is a $\Bbbk$-algebra generated by x, y, z subject to the relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, $xy - \gamma yx = \nu$, such that $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk - \{0\}$ and the standard monomials $\left\{x^i y^j z^l \mid 0 \leq i, j, l\right\}$ are a $\Bbbk$-basis of the algebra.

Next proposition establishes a classification of 3-dimensional skew polynomial algebras, according to the values of their parameters.

Proposition 2.1.20 ([Ros95], Theorem C.4.3.1)**.** *If A is a 3-dimensional skew polynomial algebra, then A is one of the following algebras:*

(i) If $\alpha \neq \beta \neq \gamma$ and $\alpha \neq \gamma$, then A is defined by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, and $xy - \gamma yx = 0$.

(ii) If $\beta \neq \alpha = \gamma = 1$, then A is one of the following algebras:

- $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$
- $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$
- $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$
- $yz - zy = 0$, $zx - \beta xz = b$, $xy - yx = 0$
- $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$
- $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$,

where $a, b \in \Bbbk$ are arbitrary and all nonzero values of b give isomorphic algebras.

(iii) If $\beta \neq \alpha = \gamma \neq 1$, then A is one of the following algebras:

- $yz - \alpha zy = 0$, $zx - \beta xz = y + b$, $xy - \alpha yx = 0$
- $yz - \alpha zy = 0$, $zx - \beta xz = b$, $xy - \alpha yx = 0$,

where $b \in \Bbbk$ is arbitrary and all nonzero values of b give isomorphic algebras as well.

(iv) If $\alpha = \beta = \gamma \neq 1$, then A is defined by the relations $yz - \alpha zy = a_1x + b_1$, $zx - \beta xz = a_2y + b_2$, $xy - \alpha yx = a_3z + b_3$. If $a_i = 0$, for $i = 1, 2, 3$, then all nonzero values of b_i give isomorphic algebras.

(v) If $\alpha = \beta = \gamma = 1$, then A is one of the following algebras:

- $yz - zy = x$, $zx - xz = y$, $xy - yx = z$
- $yz - zy = 0$, $zx - xz = 0$, $xy - yx = z$
- $yz - zy = 0$, $zx - xz = 0$, $xy - yx = b$
- $yz - zy = -y$, $zx - xz = x + y$, $xy - yx = 0$
- $yz - zy = az$, $zx - xz = z$, $xy - yx = 0$,

where $a, b \in \Bbbk$ are arbitrary and all nonzero values of b give isomorphic algebras.

Example 2.1.21. The next algebras are particular cases of 3-dimensional skew polynomial algebras, and therefore, these are skew PBW extensions:

(i) **Dispin algebra**. Let $\Bbbk$ be a field. Let $U(\mathfrak{osp}(1,2))$ be the algebra generated by x, y, z together with the relations: $yz - zy = z$, $zx + xz = y$, $xy - yx = x$. Then $U(\mathfrak{osp}(1,2)) \cong \sigma(\Bbbk)\langle x_1, x_2, x_3\rangle$. Due to relations, this algebra cannot be seen as a skew polynomial ring.

(ii) **Woronowicz algebra** $\mathcal{W}_\nu(\mathfrak{sl}(2,\Bbbk))$: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,\Bbbk))$ is generated by x, y, z subject to the relations $xz - \nu^4 zx = (1 + \nu^2)x$, $xy - \nu^2 yx = \nu z$, $zy - \nu^4 yz = (1 + \nu^2)y$, where $\nu \in \Bbbk - \{0\}$ is not a root of unity.

Next, we study some results which seek to characterize quasi-commutative skew PBW extensions and also find interesting relationships between a skew PBW extension and its corresponding graduated ring. The graduation-filtration technique has been a very important tool to test some algebraic properties. Some of the following results are evidence of this fact. The reader may see Appendix A.2 to become familiar with this terminology.

The following proposition states that one can construct a quasi-commutative skew PBW extension from a given skew PBW extension of a ring R.

Proposition 2.1.22 ([LR14], Proposition 2.1)**.** *Let A be a skew PBW extension of R. Then, there exists a quasi-commutative skew PBW extension A^σ of R in n variables $z_1, \ldots, z_n$ defined by $z_i r = c_i, r z_i$ and $z_j z_i = c_{i,j} z_i z_j$, for all $1 \leq i, j \leq n$, where $c_{i,r}$, $c_{i,j}$ are the same constants that define A. Moreover, if A is bijective then A^σ is also bijective*

Proof. Let $z_1, \ldots, z_n$ be n indeterminates and we consider the set of standard monomials $M := \{z_1^{\alpha_1} \ldots z_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}^n, 1 \leq i \leq n\}$. Let A^σ be the free R-module with basis M. The product in A^σ it defined by the distributive law and the rules $rz^\alpha sz^\beta := r\sigma^\alpha(s)c_{\alpha,\beta}z^{\alpha+\beta}$, where the σ's and the constants c's are as in Proposition 2.1.8. The reader may see [LR14], Proposition 2.1 to check that this product is associative, and also that $R \subseteq A^\sigma$. Hence, A^σ is a quasi-commutative skew PBW extension of R □

Proposition 2.1.23 ([LR14], Theorem 2.2)**.** *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by*

$$F_m(A) := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A \mid \deg(f) \leq m\}, & \text{if } m \geq 1, \end{cases}$$

and the corresponding graded ring $Gr(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $Gr(A)$ is a quasi-commutative bijective skew PBW extension of R.

Proof. Let $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$, with σ_i as in Proposition 2.1.2, and let c_{ij} be the system of constants as in Definition 2.1.3 (i). Let A^σ be the associated quasi-commutative skew PBW extension defined in Proposition 2.1.22. Associated to the filtration $\{F_m(A)\}_{m\in\mathbb{N}}$, we define the graded ring $Gr(A)$ by $Gr(A) := \bigoplus_{m\in\mathbb{N}} F_m/F_{m-1}$ (see Proposition A.2.6). The reader may see [LR14], Theorem 2.2 to check that $Gr(A) \cong A^\sigma$. Therefore, by Proposition 2.1.22, we have that $Gr(A)$ is a quasi-commutative skew PBW extension. □

The next theorem characterizes the quasi-commutative skew PBW extensions.

Proposition 2.1.24 ([LR14], Theorem 2.3)**.** *Let A be a quasi-commutative skew PBW extension of a ring R. Then, we have the following assertions:*

(i) *A is isomorphic to an iterated skew polynomial ring of endomorphism type.*

(ii) *If A is bijective, then each endomorphism is bijective.*

Proof. We follow the ideas presented in [LR14], Theorem 2.3.

(i) Let $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$, with σ_i as in Proposition 2.1.2, and let c_{ij} be the system of constants as in Definition 2.1.3 (i). The idea of the proof is to build a skew polynomial ring B and by universal property (see Proposition 1.2.10), to show that $A \cong B$. The skew polynomial ring we need is $B := R[z_1;\theta_1]\cdots[z_n;\theta_n]$ where,

$$\begin{cases} \theta_1 = \sigma_1 \\ \theta_j : R[z_1;\theta_1]\cdots R[z_{j-1};\theta_{j-1}] \to R[z_1;\theta_1]\cdots R[z_{j-1};\theta_{j-1}] \\ \theta_j(z_i) := c_{i,j}z_i, \ 1 \leq i < j \leq n, \ \theta_j(r) := \sigma_j(r), \ \text{for } r \in R, \end{cases}$$

The argument of the proof consists of making induction on the number of variables z_i. This proof is extensive and therefore we omit it. The reader can see [LR14], Theorem 2.3 to check that, indeed, $A \cong B$.

(ii) First, we show that each θ_j is surjective: if $r \in R$, then $r = \theta_j(\sigma_j^{-1}(r))$, since $\theta_j(\sigma_j^{-1}(r)) := \sigma_j(\sigma_j^{-1}(r))$. Now, let $c := \sigma_j^{-1}(c_{i,j}^{-1})$, for $1 \leq i \leq j-1$. Then, we get that $\theta_j(\sigma_j^{-1}(c_{i,j}^{-1})z_i)) = \sigma_j(\sigma_j^{-1}(c_{i,j}^{-1}))c_{i,j}z_i = c_{i,j}^{-1}c_{i,j}^{-1}z_i = z_i$. Therefore, by linearity we can affirm that θ_j is surjective, for every $1 \leq i \leq n$.

Now, let us show that θ_j is injective, for each $1 \leq i \leq n$. Consider an element $p(z_1, \ldots, z_{j-1}) \in R[z_n; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$ such that $\theta_j(p(z_1, \ldots, z_{j-1})) = 0$. Since $p(z_1, \ldots, z_{j-1})$ is a finite sum of terms as $rz_1^{\alpha_1} \cdots z_{j-1}^{j-1}$, the idea is to prove that each coefficient r es zero, and so $p(z_1, \ldots, z_{j-1}) = 0$. For this, we show first that $\theta_j(rz_1^{\alpha_1} \cdots z_{j-1}^{j-1}) = \sigma_j(r)uz_1^{\alpha_1} \cdots z_{j-1}^{j-1}$, where $u \in R^*$. Note that, $\theta_j(z_i) = c_{i,j}z_i$ and $u = c_{i,j} \in R^*$; for $\theta_j(z_iz_i)$, we get that $\theta_j(z_iz_i) = c_{i,j}z_ic_{i,j}z_i = c_{i,j}\sigma_j(c_{i,j})z_i^2$, where $c_{i,j}\sigma_j(c_{i,j}) \in R^*$. Thus, we have $\theta_j(z_i^{\alpha_i+1}) = \theta_j(z_i^{\alpha_i})\theta_j(z_i) = u'z_1^{\alpha_i}c_{i,j}z_i$, where u' is invertible and constructed by induction for the case $\theta_j(z_i^{\alpha_i})$. Hence $\theta_j(z_i^{\alpha_i+1}) := u'\sigma_i^{\alpha_i}(c_{i,j})z_i^{\alpha_1+1} = uz_i^{a_i+1}$, where $u := u'\sigma_i^{\alpha_i}(c_{i,j}) \in R^*$. Finally, by induction on the number of factors there exists $u' \in R^*$ such that

$$\begin{aligned}\theta_j(rz_1^{\alpha_1} \cdots z_{j-1}^{j-1}) &= \sigma_j(r)\theta_j(z_1^{\alpha_1} \cdots z_{j-2}^{\alpha_{j-2}}\theta_j(z_{j-1}^{j-1}) \\ &= \sigma_j(r)u'z_1^{\alpha_1} \cdots z_{j-2}^{\alpha_{j-2}}u''z_{j-1}^{\alpha_{j-1}} \\ &= \sigma_j(r)u'\sigma_1^{\alpha_1} \cdots \sigma_{j-2}^{j-2}(u'')z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}} \qquad \text{with} \quad u'' \in R^*.\end{aligned}$$

Then, we conclude that $u := u'\sigma_1^{\alpha_1} \cdots z_{j-2}^{\alpha_{j-2}}(u'') \in R^*$. By the previous observation, we get $\theta_j(rz_1^{\alpha_1} \cdots z_{j-1}^{j-1}) = \sigma_j(r)uz_1^{\alpha_1} \cdots z_{j-1}^{j-1}$, with $u \in R^*$. If $\theta_j(p(z_1, \ldots, z_{j-1})) = 0$, then $\theta_j(rz_1^{\alpha_1} \cdots z_{j-1}^{j-1}) = \sigma_j(r)uz_1^{\alpha_1} \cdots z_{j-1}^{j-1} = 0$, which implies that $\sigma_j(r)u = 0$ and since $u \in R^*$, it follows that $\sigma_j(r) = 0$. Therefore, $r = 0$, by injectivity of σ_j. This guarantees that $p(z_1, \ldots, z_{j-1}) = 0$ whence θ_j is injective, for each $1 \leq j \leq n$.

□

We present the following result which has been product of this work, where we establish sufficient conditions to guarantee that a skew PBW extension is a left regular fusible ring.

Theorem 2.1.25. *Let A be a skew PBW extension of R and suppose that σ_i is bijective, for every $1 \leq i \leq n$. If R is semiprime Noetherian ring then A is a left regular fusible ring.*

Proof. Let A be a skew PBW extension. By Proposition 2.1.22, we have that the graded ring of A is isomorphic to A^σ, that is, $Gr(A) \cong A^\sigma$. Now, we saw that A^σ is a quasi-commutative skew PBW extension using Proposition 2.1.22. Thus, by characterizing of the quasi-commutative skew PBW extensions in Proposition 2.1.24, we obtain that $A^\sigma \cong R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$, whence $Gr(A) \cong R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$. Since R is Noetherian and σ_i is bijective, for every $1 \leq i \leq n$, by Proposition 1.2.13, we have that $R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$ is a Noetherian ring, and hence $Gr(A)$ is Noetherian. By Proposition A.2.7 (iii), we get that A is a left Noetherian ring. More generally, A is a left Goldie ring. On the other hand, as R is semiprime and Noetherian, this implies that $R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$ is a semiprime ring, using Proposition 1.2.15. That is, $Gr(A)$ is semiprime, which implies that A is semiprime, by Proposition A.2.7 (iv). Finally, since A is a semiprime left Goldie ring, by Proposition 1.1.37, we conclude that A is left regular fusible. □

Example 2.1.26. Let $\Bbbk$ be a field. We consider $R = \Bbbk[x, y]/\langle x^2 \rangle$ which is a Noetherian ring since $\Bbbk[x, y]$ is also. Furthermore, R is a semiprime ring, since for any two nonzero elements $r, s \in R$, we have $rys \neq 0$. Thus, by Theorem 2.1.25, we conclude that any skew PBW extension $A = R\langle z \rangle$ with σ_i bijective is a left regular fusible ring.

The following result which establish sufficient conditions to guarantee that a quasi-commutative skew PBW extension over semiprime left Goldie rings is a left regular fusible ring. The notions of semiprime ring and Goldie's ring can be consulted in Appendix A.3 (see Definitions A.3.11 and A.3.14).

Theorem 2.1.27. *Let R be a semiprime left Goldie ring and A a quasi-commutative skew PBW extension of R. Then A is a left regular fusible ring.*

Proof. Let R be a ring and A a quasi-commutative skew PBW extension over R. By Proposition 2.1.24 (i), which characterizes this particular type of skew PBW extensions, it follows that $A \cong R[x_1;\sigma_1]\cdots[x_n;\sigma_n]$ where σ_i is an injective endomorphism, for each $1 \leq i \leq n$ (see Proposition 2.1.2). Now, since R is a semiprime left Goldie ring, this implies that $R[x_1;\sigma_1]\cdots[x_n;\sigma_n]$ is semiprime left Goldie as a consequence of Proposition 1.2.15. Therefore, we have that A is a semiprime left Goldie. In particular, A has finite uniform dimension which implies that A is regular left fusible, by Proposition 1.1.37. □

In [LAC+13], two important facts presented in the literature are studied, such as Ore and Goldie's theorems, both well known and studied for skew polynomial rings. These results are extended to an even more general family of noncommutative rings of polynomial type such as skew PBW extensions. An interesting application of the article is the quantum version of the Gelfand-Kirillov conjecture for the skew quantum polynomials (see [LAC+13], Corollary 5.1). Next, we remember the following result which describes the Ore localization by regular elements of the ring R on a skew PBW extension. The results that we present below, allow us to study the fusible property in bijective skew PBW extensions.

Proposition 2.1.28 ([LAC+13], Lemma 2.6)**.** *Let R be a ring and A a skew PBW extension of R. Let S be the set of regular elements of R such that $\sigma_i(S) = S$, for every $1 \leq i \leq n$, where σ_i is defined by Proposition 2.1.2.*

(i) *If $S^{-1}R$ exists, then $S^{-1}A$ exists and it is a bijective skew PBW extension of $S^{-1}R$ with*

$$S^{-1}A = \sigma(S^{-1}R)\langle x'_1,\ldots,x'_n\rangle,$$

with $x'_i := \frac{x_i}{1}$ and the system of constants of $S^{-1}R$ is given by $c'_{i,j} := \frac{c_{i,j}}{1}$, $c'_{i,\frac{r}{s}} := \frac{\sigma_i(r)}{\sigma_i(s)}$, for all $1 \leq i,j \leq n$.

(ii) *If RS^{-1} exists, then AS^{-1} exists and it is a bijective skew PBW extension of RS^{-1} with*

$$AS^{-1} = \sigma(RS^{-1})\langle x'_1,\ldots,x'_n\rangle,$$

with $x''_i := \frac{x_i}{1}$ and the system of constants of $S^{-1}R$ is given by $c''_{i,j} := \frac{c_{i,j}}{1}$, $c''_{i,\frac{r}{s}} := \frac{\sigma_i(r)}{\sigma_i(s)}$, for all $1 \leq i,j \leq n$.

Proposition 2.1.29 ([LAC+13], Proposition 4.6)**.** *Let R be a ring such that any left regular element is regular. Let S be the set of regular elements of R such that $S^{-1}R$ exists. Then,*

(i) *$Q_l(R)$ exists if and only if $Q_l(S^{-1}R)$ exists. In such case, we have that $Q_l(R) \cong Q_l(S^{-1}R)$.*

(ii) *R is semiprime left Goldie if and only if $S^{-1}R$ is semiprime left Goldie.*

The right side version of the proposition holds.

Proposition 2.1.30 ([LAC+13], Proposition 4.7)**.** *Let R be a positive filtered ring. If $Gr(R)$ is semiprime, then R is semiprime.*

Next, we present a result of great interest that relate skew PBW extensions with the property of being left regular fusible. This result is a consequence of the statements previously presented and of the characterization of semiprime left Goldie rings in terms of regular fusible rings presented in Proposition 1.1.37.

Theorem 2.1.31. *Let R be a semiprime left Goldie ring and A a bijective skew PBW extension of R. Then A is a left regular fusible ring.*

Proof. Let S be the set of regular elements of R. By Goldie's classical theorem, we have that $Q_l(R) = S^{-1}R$ exists and it is left Artinian. Making use of Proposition 2.1.28, it follows that $S^{-1}A$ exists and it is a bijective extension of $Q_l(R)$, which means that, $S^{-1}A = \sigma(Q_l(R))\langle x_1, \dots, x_n\rangle$. Since $Q_l(R)$ is left Artinian, we have that $Q_l(R)$ is left Noetherian and so by Proposition 2.1.11, we get that $S^{-1}A$ is left Noetherian and hence left Goldie. On the other hand, we get that $Gr(\sigma(Q_l(R))\langle x_1, \dots, x_n\rangle)$ is a quasi-commutative extension of the semiprime left Goldie ring $Q_l(R)$. This implies that $Gr(S^{-1}A)$ is a semiprime ring and by Proposition 2.1.30, it follows that $S^{-1}A$ is semiprime. Thus $S^{-1}A$ is semiprime left Goldie and hence, we conclude that A is a semiprime left Goldie ring, by Proposition 2.1.29 (ii). Finally, since A is a semiprime left Goldie ring, we obtain that A is a left regular fusible ring as a consequence of Proposition 1.1.37. □

In the following tables, we mention some noncommutative algebras that are particular cases of skew PBW extensions. It is important to note that these algebras are skew PBW extensions on reduced rings, which are semiprime rings. The following examples are adapted from [LR14], [RJ18], [Rey13a] and [Sua17]. We use the symbol ✓ to mean the result holds; in other case, we use the symbol $\star$. We note for RF-QC Theorem 2.1.27 and RF-B Theorem 2.1.31.

Noncommutative ring	RF-B	RF-QC
Polynomial ring $\Bbbk[x_1, \ldots, x_n]$	✓	✓
Skew polynomial ring of derivation type	✓	⋆
Universal enveloping algebra of Lie algebras	✓	⋆
Universal enveloping rings $\mathcal{U}(V, R, \Bbbk)$	✓	⋆
Differential operator rings $V(R, L)$	✓	⋆
Tensor product $R \otimes_{\Bbbk} U(\mathfrak{g})$	✓	⋆
Crossed product $R * U(\mathfrak{g})$	✓	⋆
Twisted or smash product differential operator ring $R\#_{\sigma}U(\mathfrak{g})$	✓	⋆

Table 2.1: PBW extensions.

Noncommutative ring	RF-B	RF-QC
Weyl algebra $A_n(\Bbbk)$	✓	⋆
Quantum plane	✓	✓
Algebra of q-differential operators	✓	⋆
Algebra of shift operators S_h	✓	✓
Mixed algebra D_h	✓	⋆
Algebra for multidimensional discrete linear systems	✓	✓

Table 2.2: Ore extensions of derivation type.

Noncommutative ring	RF-B	RF-QC
Algebra of linear partial differential operators	✓	⋆
Algebra of linear partial shift operators	✓	✓
Algebra of linear partial difference operators	✓	⋆
Algebra of linear partial q-dilation operators	✓	✓
Algebra of linear partial q-differential operator	✓	⋆

Table 2.3: Operator algebras.

Noncommutative ring	RF-B	RF-QC
Additive analogue of the Weyl algebra	✓	⋆
Multiplicative analogue of the Weyl algebra	✓	✓
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk)$	✓	⋆
Dispin algebra $\mathcal{U}(osp(1, 2)$	✓	⋆
Woronowicz algebra $W_{\nu}(\mathfrak{sl}(2, \Bbbk)$	✓	⋆
Algebra $\mathbf{U}$	✓	⋆
The Complex algebra $V_q(\mathfrak{sl}(3, \mathbb{C}))$	✓	⋆
Manin algebra $\mathcal{O}_q M_2(\Bbbk)$	✓	⋆
q-Heisenberg algebra $\mathbf{H}_n(q)$	✓	⋆
Quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2, \Bbbk))$	✓	⋆
Hayashi's algebra $W_q(J)$	✓	⋆
Differential operators on a quantum space $D_{\mathbf{q}}(S_{\mathbf{q}})$	✓	⋆

Table 2.4: Quantum algebras.

2.2 Σ-Rigid and weak Σ-Rigid rings

Krempa [Kre96] defined σ-rigid rings. Recall that R is called *σ-rigid ring*, if there exists an endomorphism σ of R such that $a\sigma(a) = 0$ implies $a = 0$, for $a \in R$. It is known that σ-rigid rings are reduced (see Hong et al [HKK00]). Several properties of σ-rigid rings have been established in the literature (see [Kre96] and [HKK00]). With this definition in mind, Ouyang [Ouy08] defined weak σ-rigid rings which are a generalization of σ-rigid rings. More precisely, R is said to be *weak σ-rigid*, if $a\sigma(a) \in \text{nil}(R) \Leftrightarrow a \in \text{nil}(R)$, where $\text{nil}(R)$ is the set of nilpotent elements of R and σ is an endomorphism of a ring R. In [Ouy08], it was shown that R is σ-rigid if and only if R is weak σ-rigid and reduced. In this way, weak σ-rigid rings are a generalization of σ-rigid rings deleting the condition to be reduced. In this section, we study the notion of a weak Σ-rigid ring as a generalization of Σ-rigid rings, which have been extensively studied in the literature (see [Rey15], [RS18a], and [RS18b]). This generalization is particularly interesting since it allows to characterize types of elements in skew PBW extensions.

Since Ore extensions of injective type are particular examples of skew PBW extensions, in [Rey15] it was introduced the following definition with the purpose of studying the notion of rigidness in this more general setting.

Definition 2.2.1 ([Rey15], Definition 3.2)**.** Let R be a ring and Σ a family of endomorphisms of R. Σ is called a **rigid endomorphisms family**, if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$, where $r \in R$ and $\alpha \in \mathbb{N}^n$. A ring R is called **Σ-rigid**, if there exists a rigid endomorphism family Σ of R.

Remark 2.2.2. Note that if R is Σ-rigid, then σ^{α} is a injective homomorphism, for all $\alpha \in \mathbb{N}^n$. Let $r \in R$ such that $\sigma^{\alpha}(r) = 0$. Then, we get that $r\sigma^{\alpha}(r) = 0$. Now, since R is Σ-rigid, we have $r\sigma^{\alpha}(r) = 0$ which implies that $r = 0$. Thus, σ^{α} is a injective homomorphism. Furthermore, an Σ-rigid ring R is reduced: Let $r \in R$ such that $r^2 = 0$. Then, we have $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$ whence $r\sigma^{\alpha}(r) = 0$, and thus $r = 0$ since R is Σ-rigid.

With this in mind, we consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations in a skew PBW extension A of a ring R (see Proposition 2.1.2). The following proposition generalizes properties for σ-rigid rings and it is also of great importance throughout this work.

Proposition 2.2.3 ([Rey15], Lemma 3.3)**.** *Let R be a Σ-rigid ring and $a, b \in R$. Then:*

(i) *If $ab = 0$ then $a\sigma^{\alpha}(b) = \sigma^{\alpha}(a)b = 0$, for $\alpha \in \mathbb{N}^n$.*

(ii) *If $ab = 0$ then $a\delta^{\beta}(b) = \delta^{\beta}(a)b = 0$, for $\beta \in \mathbb{N}^n$.*

(iii) *If $ab = 0$ then $a\sigma^{\alpha}(\delta^{\beta}(b)) = a\delta^{\beta}(\sigma^{\alpha}(b)) = 0$, for every $\alpha, \beta \in \mathbb{N}^n$.*

(iv) *If $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$ for some $\theta \in \mathbb{N}^n$, then $ab = 0$.*

We present two interesting facts which are an important tool to study certain elements in a skew PBW extension. Theorems 2.2.4 and 2.2.5 generalize some properties of the Σ-rigid rings and both are results obtained during the realization of this work.

The following result generalizes [HHA17], Lemma 2.2.

Theorem 2.2.4. *Let A be a skew PBW extension of a Σ-rigid ring R. If $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$, $c, r \in R$. Then $fr = c$ if and only $a_0r = c$ and $a_ir = 0$, for every $1 \leq i \leq n$.*

Proof. Let $r \in R$ such that $fr = c$. The idea is show that $a_0r = c$ and $a_ir = 0$, for every $1 \leq i \leq n$, by using induction on n. For $n = 0$, $f = a_0$ whence $c = fr = a_0r$. Now, we suppose that the result is true for all the smaller values than $n \geq 1$. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$ such that $fr = c$. If, $c = fr = a_0r + a_1X_1r + \cdots + a_nX_nr$, this means that $\text{lc}(fr) = a\sigma^{\alpha_n}(r) = 0$. Thus, $a_nr = 0$, by Proposition 2.2.3 (iv). Again, by using Proposition 2.2.3 (iii) and Remark 2.1.9, we have $a_nX_nr = 0$. Hence, we have that $c = fr = (a_0r + a_1X_1r + \cdots + a_nX_nr) = (a_0 + a_1X_1 + \cdots + a_{n-1}X_{n-1})r$ whence $a_0r = c$ and $a_ir = 0$, for all $1 \leq i \leq n-1$ since we assume the result is true for all the smaller values than $n \geq 1$. On the other hand, if $a_0r = c$ and $a_ir = 0$, for every $1 \leq i \leq n$, we have $fr = c$ by Proposition 2.2.3 and Remark 2.1.9.

□

The following result generalizes [HHA17], Lemma 2.3.

Theorem 2.2.5. *Let A be a skew PBW extension of a Σ-rigid ring R and also $f = a_0 + a_1X_1 + \cdots + a_nX_n$ and $g = b_0 + b_1Y_1 + \cdots + b_mY_m$ be nonzero elements of A such that $fg = c \in R$. Then $a_0b_0 = c$ and $a_ib_j = 0$, for each i, j with $i + j \geq 1$.*

Proof. Let $f, g \in A$ with $f = a_0 + a_1X_1 + \cdots + a_nX_n$ and $g = b_0 + b_1Y_1 + \cdots + b_mY_m$ such that $fg = c$. We have $fg = (a_0 + a_1X_1 + \cdots + a_nX_n)(b_0 + b_1Y_1 + \cdots + b_mY_m) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ib_jY_j\right)$. We proceed by induction. If $m = 0$ or $n = 0$, then the result follows from Theorem 2.2.4. Now, let $m, n \geq 1$ and suppose that the result is true for all the smaller values than $m + n \geq 1$. If $fg = c$, we get $\text{lc}(fg) = a_n\sigma^{\alpha_n}(b_m) = 0$, and hence $a_nb_m = 0$ by Proposition 2.2.3. In addition, $a_nb_m = b_ma_n = 0$, due to the reducibility of R. Thus, we have $b_mc = b_mfg = (b_ma_0 + b_ma_1X + \cdots + b_ma_{n-1}X_{n-1})g = f_1g$. Since $f_1g = b_mc$ and $\text{lt}(f_1g) = b_ma_{n-1}X_{n-1}b_mY_m$, by induction hypothesis, we get the statement is true for f_1g, that is, $b_ma_0b_m = \cdots = b_ma_{n-1}b_m = 0$ which means that $b_ma_ib_m = 0$, for every $1 \leq i \leq n-1$. Hence, $a_i(b_ma_ib_m) = (a_ib_m)^2 = 0$ and so $a_ib_m = 0$, since R is reduced. Then, we have that $a_iX_ib_mY_m = 0$, for every $0 \leq i \leq n$, by Proposition 2.2.3 and Remark 2.1.9. Thus, $c = fg = f(b_0 + b_1Y + \cdots + b_{m-1}Y_{m-1}) = fg_1$. Again, $fg_1 = c$ and $\text{lt}(fg_1) = a_nX_nb_{m-1}Y_{m-1}$, by induction hypothesis, we get the statement is true for fg_1, that is, $a_0b_0 = c$ and $a_ib_j = 0$, for every $1 \leq i \leq n$ and $1 \leq j \leq m-1$. Therefore, $a_0b_0 = c$ and $a_ib_j = 0$, for each $1 \leq i + j \leq m + n$, and the proof is complete. □

In [Ouy08], the notion of weak σ-rigid rings was introduced for Ore extensions, which generalizes also the concept of σ-rigid ring. Following these ideas, in [RS18b] the notion of weak Σ-rigid rings for skew PBW extensions was considered, notion that also extends to Σ-rigid rings. These families of rings are a very important tool for studying skew PBW extensions. We focus our attention on these rings and some of their algebraic properties.

Definition 2.2.6. ([RS18b], Definition 3.2). Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a family of endomorphisms and Σ-derivations of R, respectively. R is called a **weak Σ-rigid** ring, if $a\sigma^{\theta}(a) \in \text{nil}(R) \Leftrightarrow a \in \text{nil}(R)$, where $a \in R$ and $\theta \in \mathbb{N}^n$.

It can be proved that every Σ-rigid ring is a weak Σ-rigid ring. However, the converse is false (see [Ouy08], Example 2.1). The next Proposition gives an equivalence between the notions of Σ-rigid rings and weak Σ-rigid rings.

Proposition 2.2.7 ([RS18b], Theorem 3.4)**.** *Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta = \{\delta_1, \ldots, \delta_n\}$ be a family of endomorphisms and Σ-derivations of R, respectively. R is Σ-rigid if and only if R is weak Σ-rigid and reduced.*

Proof. Suppose that R is Σ-rigid ring. Then R is reduced by the previous observations. Thus, let us show that R is weak σ-rigid weak. If $a \in \text{nil}(R)$, then $a = 0$, since R is reduced. Therefore $a\sigma^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$; in particular, we obtain that $a\sigma^{\alpha}(a) \in \text{nil}(R)$. Now, if $a\sigma^{\alpha}(a) \in \text{nil}(R)$, then $a\sigma^{\alpha}(a) = 0$, since R is reduced, but this implies that $a = 0$, since we are assuming that R is Σ-rigid.

On the other hand, suppose R is weak Σ-rigid and reduced. Let us show that R is Σ-rigid. Let $a \in R$ such that $a\sigma^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$, so $a\sigma^{\alpha}(a) \in \text{nil}(R)$ and therefore $a \in \text{nil}(R)$, since R is weak Σ-rigid ring, but this implies that $a = 0$, because we are assuming that R is reduced. Therefore R is a Σ-rigid ring □

We consider the finite family of injective endomorphisms Σ in a skew PBW extension A of a ring R. The following fact generalizes properties for weak σ-rigid rings (see [Ouy08], Proposition 2.3), and it also generalizes the properties of the Σ-rigid rings established in Proposition 2.2.3.

Proposition 2.2.8 ([RS18b], Proposition 3.5)**.** *If R is a NI ring which is weak Σ-rigid, then we have the following assertions:*

(i) *If $ab \in \text{nil}(R)$, then $a\sigma^{\alpha}(b), \sigma^{\beta}(a)b \in \text{nil}(R)$, for every elements $\alpha, \beta \in \mathbb{N}^n$.*

(ii) *If $\sigma^{\alpha}(a)b \in \text{nil}(R)$, for some element $\alpha \in \mathbb{N}^n$, then $ab, ba \in \text{nil}(R)$.*

(iii) *If $a\sigma^{\alpha}(b) \in \text{nil}(R)$, for some element $\alpha \in \mathbb{N}^n$, then $ab, ba \in \text{nil}(R)$.*

Next, we present remarkable examples of skew PBW extensions over weak Σ-rigid rings. The examples described are adapted from [RS18a],[LR14] and [Rey15].

Examples 2.2.9. The algebra of q-differential operators $D_{q,h}[x, y]$: Let $q, h \in \Bbbk$, $q \neq 0$. We consider $\Bbbk[y][x; \sigma, \delta]$ with $\sigma(y) := qy$ and $\delta(y) := h$. By definition of skew polynomial ring we have $xy = \sigma(y)x + \delta(y) = qyx + h$, and hence $xy - qyx = h$. We can prove that $D_{q,h}[x, y] \cong \sigma(\Bbbk[y])\langle x \rangle$. Thus, $D_{q,h}[x, y]$ is a skew PBW extension over $\Bbbk[y]$ which is Σ-rigid where $\Sigma = \{\sigma\}$. Furthermore, $\Bbbk[y]$ is clearly a weak Σ-rigid ring.

Examples 2.2.10. The algebra of shift operators S_h: Let $\Bbbk$ be a field. Consider $h \in \Bbbk$. This algebra is defined by $S_h := \Bbbk[t][x_h; \sigma_h, \delta_h]$, where $\sigma_h(p(t)) := p(t - h)$ and $\delta_h = 0$. We can show that, $S_h \cong \sigma(\Bbbk[t])\langle x_h \rangle$. In particular, S_h is a skew PBW extension over $\Bbbk[t]$ which is Σ-rigid with $\Sigma = \{\sigma_h\}$. Furthermore, $\Bbbk[t]$ is trivially a weak Σ-rigid ring.

Example 2.2.11. q-Heisenberg algebra: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathrm{H}_n(q)$ is generated by the set of variables $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$, subject to the relations:

- $x_j x_i = x_i x_j, \quad z_j z_i = z_i z_j, \quad$ for all $\ 1 \leq i, j \leq n.$
- $z_j y_i = y_i z_j, \quad z_j x_i = x_i z_j, \quad y_j x_i = x_i y_j, \quad i \neq j.$
- $z_i y_i = q y_i z_i, \quad z_i x_i = q^{-1} x_i z_i + y_i, \quad y_i x_i = q x_i y_i, \quad$ for all $\ 1 \leq i \leq n, \ q \in \Bbbk - \{0\}.$

We can prove that $H_n(q) \cong \sigma(\Bbbk)\langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n\rangle$. Additionally, $H_n(q)$ is a skew PBW extension over $\Bbbk$ which is Σ-rigid where $\Sigma = \{\sigma_i\}$ with $\sigma_i = i_{\Bbbk}$, for every $1 \leq i \leq n$. Moreover, $\Bbbk$ is trivially a weak Σ-rigid ring.

We recall that the following algebras are examples of skew PBW extensions which cannot be expressed as a skew polynomial rings.

Example 2.2.12. Let $\Bbbk$ be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \ldots, x_n\}$. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, is a PBW extension over $\Bbbk$, which is Σ-rigid, where $\Sigma = \{\sigma\}$ with $\sigma = i_{\Bbbk}$.

Example 2.2.13. Diffusion algebras: Let $\Bbbk$ be a field. The $\Bbbk$-algebra A generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is a Σ-rigid ring, with $\Sigma = \{\sigma_j\}$, where $\sigma_j(x_i) = x_i$, for every $1 \leq i, j \leq n$.

We build the following example of a skew polynomial ring on a weak σ-rigid ring.

Example 2.2.14. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. We consider $\sigma((a,b)) = (a^2, b^2)$. We can show that $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is σ-rigid ring. Now, let us define ring B as follows:

$$B = \left\{ \begin{pmatrix} (a,b) & (c,d) \\ (0,0) & (a,b) \end{pmatrix} \mid (a,b), (c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}.$$

The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B \to B$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. We note that the skew polynomial ring $B[x; \overline{\sigma}]$ is a skew PBW extension over B which is weak $\overline{\sigma}$-rigid.

The following definition is adapted from [AHK19], Definition 3.1.

Definition 2.2.15 ([AHK19], Definition 3.1)**.** Let R be a Σ-rigid ring and A a skew PBW extension over R, where $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ is a finite family of endomorphisms of R and $\Delta := \{\delta_1, \ldots, \delta_n\}$ is a finite family of Σ-derivations of R.

(i) An ideal I of R is a **Σ-ideal**, if $\sigma^{\alpha}(I) \subseteq I$; I is **Δ-ideal** if $\delta^{\alpha}(I) \subseteq I$, $\alpha \in \mathbb{N}^n$.

(ii) For a Σ-ideal I, it is said that I is **Σ-rigid**, if $a\sigma^{\alpha}(a) \in I$ implies $a \in I$, for each $a \in R$ and $\alpha \in \mathbb{N}^n$.

Motivated by the above facts, for a finite set of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ of a ring R, we first define weak Σ-rigid ideals in R which are a generalization of Σ-rigid ideals. This notion gives a characterization of the weak Σ-rigid rings in terms of left principal ideals generated by idempotent elements (see [RS18b], Proposition 3.7).

Definition 2.2.16 ([RS18b], p. 8)**.** Let R be a weak Σ-rigid ring and I a Σ-ideal. It is said that I is **weak Σ-rigid ideal**, if it follows that $a\sigma^{\alpha}(a) \in \text{nil}(R) \Leftrightarrow a \in \text{nil}(R)$ for each $a \in I$.

A very important example of Σ-rigid ideal is the following:

Example 2.2.17. If R is a weak Σ-rigid ring and $\text{nil}(R)$ is the set of nilpotent elements of R, then $\text{nil}(R)$ is a weak Σ-rigid ideal. This fact is a consequence of the properties stated in Proposition 2.2.8.

The following proposition extends some properties of Proposition 2.2.3.

Proposition 2.2.18 ([AHK19], Proposition 3.3)**.** *Let I be a weak Σ-rigid ideal of R and $a, b \in R$. Then we have the following:*

(i) *If $ab \in I$, then $a\sigma^{\alpha}(b) \in I$ and $\sigma^{\alpha}(a)b \in I$, for each $\alpha \in \mathbb{N}^n$.*

(ii) *If $ab \in I$, then $\sigma^{\alpha}(a)\delta^{\beta}(b), \delta^{\beta}(a)\sigma^{\alpha}(b) \in I$, for each $\alpha, \beta \in \mathbb{N}^n$.*

(iii) *If $a\sigma^{\theta}(b) \in I$ or $\sigma^{\theta}(a)b \in I$, for some $\theta \in \mathbb{N}^n$, then $ab \in I$.*

2.2.1 Elements in SPBW extensions over Σ-rigid and weak Σ-rigid ring

The description of some elements in skew PBW extensions has been a recent topic of great interest given the relationship of these elements with some algebraic notions investigated in the literature (see [Rey15],[RS19b] and [LR20b]). In this section, we characterize some types of elements of great importance in skew PBW extensions over Σ-rigid and weak Σ-rigid NI rings. The results that we present contribute precisely to the investigation of some notions and algebraic properties of skew PBW extensions. First, we study the nilpotent elements of a skew PBW extensions, and as a result, we present a characterization of the nilpotent elements of a PBW extension in terms of the set of nilpotent elements of its ring of coefficients R. We denote $\text{nil}(R)A := \{f \in A \mid f = \sum_{i=0}^{m} a_i X_i,\ a_i \in \text{nil}(R)\}$.

Theorems 2.2.19 and 2.2.20 characterize the nilpotent elements in a skew PBW extension over Σ-rigid and weak Σ-rigid rings. Furthermore, both are facts that were obtained during the realization of this work.

Theorem 2.2.19. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a weak Σ-rigid NI ring. Then $f = a_0 + a_1X_1 + \cdots + a_nX_n \in \text{nil}(A)$ if and only if $a_i \in \text{nil}(R)$, for all $0 \leq i \leq n$.*

Proof. First, Let us show that $\text{nil}(A) \subseteq \text{nil}(R)A$. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$, where $a_i \in R$, for every $1 \leq i \leq m$, $a_m \neq 0$, with $X_m \succ X_{m-1} \succ \cdots \succ X_1$ such that $f \in \text{nil}(A)$. This means that there exists $r \neq 0$ such that $f^r = 0$.

It can be shown that $\mathrm{lc}(f^r) = a_m\sigma^{\alpha_m}(a_m)\sigma^{2\alpha_m}(a_m)\cdots\sigma^{(r-1)\alpha_m}(a_m) = 0 \in \mathrm{nil}(R)$. Since R is weak Σ-rigid ring, we have $a_m^r \in \mathrm{nil}(R)$ and hence $a_m \in \mathrm{nil}(R)$. Now, let us write $f = Q + a_mX_m$ with $Q = \sum_{i=0}^{m-1} a_iX_i \in A$ and $\deg(Q) < |\alpha_m|$. Then $0 = f^t = Q^t + h$, for some $h \in A$. We observed that when we compute every summand of h we obtain products of the coefficient a_m in $\sigma's$ and $\delta's$ depending of the coordinates of α_m. Since $a_m \in \mathrm{nil}(R)$ and $\mathrm{nil}(R)$ is Σ-ideal, we have $h \in \mathrm{nil}(R)A$. Hence, $Q^r \in \mathrm{nil}(R)A$ whence $a_{m-1}\sigma^{\alpha_{m-1}}(a_{m-1})\cdots\sigma^{(t-1)\alpha_{m-1}}(a_{m-1}) \in \mathrm{nil}(R)$ and by using Proposition 2.2.18, we get $a_{m-1} \in \mathrm{nil}(R)$. Following this process, we obtain $a_i \in \mathrm{nil}(R)$, for each $1 \leq i \leq m$ and therefore $f \in \mathrm{nil}(R)A$.

Now, let us prove that $\mathrm{nil}(R)A \subseteq \mathrm{nil}(A)$. Let $f = \sum_{i=0}^{m} a_iX_i \in A$, where $a_i \in R$, $a_m \neq 0$, with $X_m \succ X_{m-1} \succ \cdots \succ X_1$ such that $f \in \mathrm{nil}(R)A$. Then, we have $a_i \in \mathrm{nil}(R)$, for every $1 \leq i \leq n$. Now, Let k_i be the minimum integer positive such that $a_i^{k_i} = 0$, for every $1 \leq i \leq n$ and $k := \max\{k_i \mid 1 \leq i \leq n\}$. Let us show that $f^{(m+1)k+1} = 0$ whence $f \in \mathrm{nil}(A)$. Since the expression for f have $m+1$ terms, when we realize the product $f^{(m+1)k+1}$, each term of the sum that forms the polynomial is a product of $(n+1)k+1$ factors of the form $\sum_{j=0}^{(n+1)k+1} a_{i,j}X_{i,j}$. Hence, when we compute $f^{(m+1)k+1}$ every product involves at least k elements a_i, for some i. Thus, we obtain that every one of these products is equal to zero and therefore each sum is equal to zero by using Proposition 2.3.9 and Remarks 2.1.9. Thus, every term of $f^{(m+1)k+1}$ is equal to zero and hence $f \in \mathrm{nil}(A)$, whence $\mathrm{nil}(R)A \subseteq \mathrm{nil}(A)$. □

Theorem 2.2.20. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a Σ-rigid NI ring. Then* $\mathrm{nil}(A) = \{0\}$.

Proof. This fact follows from Theorem 2.2.19 and Proposition 2.2.7. □

Remark 2.2.21. The following examples illustrate the previous results.

(i) **The algebra of q-differential operators $D_{q,h}[x,y]$:** The $\Bbbk$-algebra $D_{q,h}[x,y]$ is a skew PBW extension over $\Bbbk[y]$ which is Σ-rigid (see Example 2.2.9). Hence, by Theorem 2.2.20, we get that $\mathrm{nil}(D_{q,h}[x,y]) = \mathrm{nil}(\Bbbk[y])[x;\sigma,\delta]$. Since $\Bbbk[y]$ is a reduced ring, it follows that $\mathrm{nil}(\Bbbk[y]) = \{0\}$, and hence $\mathrm{nil}(D_{q,h}[x,y]) = \{0\}$.

(ii) **The algebra of shift operators S_h**: The $\Bbbk$-algebra S_h is a skew PBW extension over $\Bbbk[t]$ which is Σ-rigid (see Example 2.2.10). Hence, by Theorem 2.2.20, we get that $\mathrm{nil}(S_h) = \mathrm{nil}(\Bbbk[t])[x_h;\sigma_h,\delta_h]$. Since $\Bbbk[t]$ is a reduced ring, it follows that $\mathrm{nil}(\Bbbk[t]) = \{0\}$, and hence $\mathrm{nil}(S_h) = \{0\}$.

(iii) **q-Heisenberg algebra**: The $\Bbbk$-algebra $H_n(q)$ is a skew PBW extension over $\Bbbk$ which is Σ-rigid (see Example 2.2.11). Therefore, by Theorem 2.2.20, we get that $\mathrm{nil}(H_n(q)) = \mathrm{nil}(\Bbbk)H_n(q)$. It is clear that $\mathrm{nil}(\Bbbk) = \{0\}$, so $\mathrm{nil}(H_n(q)) = \{0\}$.

(iv) Let $\Bbbk$ be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \ldots, x_n\}$. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, is a PBW extension over $\Bbbk$, which is Σ-rigid (see Example 2.2.12). Thus, it follows that $\mathrm{nil}(\mathcal{U}(\mathfrak{g})) = \mathrm{nil}(\Bbbk)\mathcal{U}(\mathfrak{g})$, by Theorem 2.2.20. Hence, as $\mathrm{nil}(\Bbbk) = \{0\}$, this implies that $\mathrm{nil}(\mathcal{U}(\mathfrak{g})) = \{0\}$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is a Σ-rigid ring (see Example 2.2.13). Thus, by Theorem 2.2.19, we have that $\operatorname{nil}(A) = \operatorname{nil}(\Bbbk[x_1, \ldots, x_n])A$. Having in mind that $\operatorname{nil}(\Bbbk[x_1, \ldots, x_n]) = \{0\}$ since $\Bbbk[x_1, \ldots, x_n]$ is reduced, we obtain $\operatorname{nil}(A) = \{0\}$.

(vi) In the Example 2.2.14, we have that $B[x; \overline{\sigma}]$ can be seen as a skew PBW extension over B which is a weak Σ-rigid ring. Therefore, by Theorem 2.2.19, we obtain that $\operatorname{nil}(B[x; \overline{\sigma}]) = \operatorname{nil}(B)\langle x \rangle$. Furthermore, the set of nilpotent elements of B is represented as follows

$$\operatorname{nil}(B) = \left\{ \begin{pmatrix} (0,0) & (c,d) \\ (0,0) & (0,0) \end{pmatrix} \mid (c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}.$$

Thus, the set of nilpotent elements of $B[x; \overline{\sigma}]$ is as follows

$$\operatorname{nil}(B[x; \overline{\sigma}]) = \left\{ \sum_{i=0}^{n} \begin{pmatrix} (0,0) & (c,d) \\ (0,0) & (0,0) \end{pmatrix} x^i \mid (c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}.$$

Now, we study the idempotent elements of skew PBW extensions over weak Σ-rigid NI rings. First, we recall that an ideal I of a ring R is called **nil ideal**, if every element of I is nilpotent (see [Lam01], Definition 4.9). Next, we enunciate a result that describes the idempotent elements of a quotient ring by a nil ideal.

Proposition 2.2.22 ([Lam01], Theorem 21.28)**.** *Let I be a nil ideal in R and $a \in R$ be such that $\overline{a} \in \overline{R} = R/I$ is an idempotent. Then there exists an idempotent $e \in aR$ such that $\overline{e} = \overline{a} \in \overline{R}$.*

Proof. We follow the ideas presented in [Lam01], Theorem 21.28. Let $a, b \in R$ such that $b = 1 - a$. Thus, we have $ab = ba = a - a^2 \in I$, so $(ab)^n = 0$ for some integer $n \geq 1$ since I is a nil ideal. Now, as $ab = ba$ for $b = a - 1$, we have:

$$1 = (a+b)^{2n} = \sum_{i=0}^{2n} r_k a^{k-i} b^k \qquad \text{with} \quad r_k = \binom{2n}{k} \in \mathbb{Z}^+$$

Now, let us write $e = r_0 a^{2n} + \cdots + r_n a^n b^n$ and $f = r_{n+1} a^{n-1} b^{n+1} + \cdots + r_{2n} b^{2n}$. Since $a^m b^m = b^m a^m = 0$ because $ab = ba$ and $(ab)^n = 0$, it follows that $ef = 0$ and so $e = e(e+f) = e^2$. Therefore, $ab \in I$ implies that $\overline{e} = \overline{a^{2n}} = \overline{a}$. □

The following theorem gives a partial description of the idempotent elements in a skew PBW extension over weak Σ-rigid rings. Theorem 2.2.23 has not been found in the literature and hence, we consider it as a result of this work.

Theorem 2.2.23. *Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension, where R is a weak Σ-rigid NI ring. Let $f = a_0 + a_1 X_1 + \cdots + a_n X_n$ be an idempotent element of A. Then $a_i \in \operatorname{nil}(R)$, for each $1 \leq i \leq n$, and there exists an idempotent element $e \in R$ such that $\overline{a_0} = \overline{e} \in \overline{R} = R/\operatorname{nil}(R)$.*

Proof. Since R is a weak Σ-rigid NI ring, we have $\operatorname{nil}(R)$ is a weak Σ-rigid ideal of R. Now, as $\overline{R} = R/\operatorname{nil}(R)$ is reduced and weak $\overline{\Sigma}$-rigid it follows that $\overline{R}$ is a $\overline{\Sigma}$-rigid ring, by using Proposition 2.2.7. On the other hand, if $\overline{f^2} = \overline{f}$ in $\sigma(\overline{R})\langle x_1, \ldots, x_n\rangle$, we have $(\overline{1} - \overline{f})\overline{f} = 0$ which implies that $\overline{f} = \overline{a_0}$ in $\sigma(\overline{R})\langle x_1, \ldots, x_n\rangle$ and $\overline{a_i} = \overline{0}$, for each $1 \leq i \leq n$, by Proposition 2.2.5. Hence, by Proposition 2.2.22, we conclude $\overline{a_0} = \overline{e}$ in $R/\operatorname{nil}(R)$, for some idempotent $e \in R$. □

The following theorems describe the idempotent elements in skew PBW extension over Σ-rigid and weak Σ-rigid rings, respectively. Theorems 2.2.24 and 2.2.25 are presented as results obtained during the realization of this work.

Theorem 2.2.24. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a weak Σ-rigid NI ring. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$. If $f^2 = f$, then $f = e + h$ with $e \in \operatorname{Idem}(R)$ and $h \in \operatorname{nil}(A)$.*

Proof. Let $f = \sum_{i=0}^{n} a_iX_i \in A$ such that $f^2 = f$. By Theorem 2.2.23, we have $a_i \in \operatorname{nil}(R)$, for each $1 \leq i \leq n$ and also, there exist an idempotent element $e \in R$ and a nilpotent element $b \in R$ such that $a_0 = e + b$. Let us denote $h = b + a_1X_1 + \cdots + a_nX_n \in A$. Then, we have that $f = e + h$. Since R is a NI ring which is weak Σ-rigid, it follows that $\operatorname{nil}(R)\langle x_1, \ldots, x_n\rangle = \operatorname{nil}(A)$, by Theorem 2.2.19. Hence, we conclude $h \in \operatorname{nil}(A)$. □

Theorem 2.2.25. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a Σ-rigid NI ring. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$. If $f^2 = f$ if and only if $f = a_0$ with $a_0 \in \operatorname{Idem}(R)$.*

Proof. This fact follows from Theorem 2.2.24 and Proposition 2.2.7. □

Remark 2.2.26. We observe at some interesting examples that illustrate the previous result.

(i) **The algebra of q-differential operators $D_{q,h}[x, y]$:** The $\Bbbk$-algebra $D_{q,h}[x, y]$ is a skew PBW extension over $\Bbbk[y]$ which is Σ-rigid (see Example 2.2.9). Hence, by Theorem 2.2.25, we get that $\operatorname{Idem}(D_{q,h}[x, y]) = \operatorname{Idem}(\Bbbk[y])$. This implies that $\operatorname{Idem}(D_{q,h}[x, y]) = \operatorname{Idem}(\Bbbk[y]) = \{0, 1\}$.

(ii) **The algebra of shift operators S_h**: The $\Bbbk$-algebra S_h is a skew PBW extension over $\Bbbk[t]$ which is Σ-rigid (see Example 2.2.10). Hence, by Theorem 2.2.25 it follows that $\operatorname{Idem}(S_h) = \operatorname{Idem}(\Bbbk[t])$ and so, we obtain that $\operatorname{Idem}(S_h) = \operatorname{Idem}(\Bbbk[t]) = \{0, 1\}$.

(iii) **q-Heisenberg algebra**: The $\Bbbk$-algebra $H_n(q)$ is a skew PBW extension over $\Bbbk$ which is Σ-rigid (see Example 2.2.11). Thus, by Theorem 2.2.25, we get that $\operatorname{Idem}(H_n(q)) = \operatorname{Idem}(\Bbbk)$, whence $\operatorname{Idem}(H_n(q)) = \operatorname{Idem}(\Bbbk) = \{0, 1\}$.

(iv) Let $\Bbbk$ be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \ldots, x_n\}$. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, is a PBW extension over $\Bbbk$, which is Σ-rigid (see Example 2.2.12). Thus, by Theorem 2.2.25, it follows that $\operatorname{Idem}(\mathcal{U}(\mathfrak{g})) = \operatorname{Idem}(\Bbbk)$. Hence, we conclude that $\operatorname{Idem}(\mathcal{U}(\mathfrak{g})) = \{0, 1\}$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is a Σ-rigid ring (see Example 2.2.13). Thus, by Theorem 2.2.25, we have that $\operatorname{Idem}(A) = \operatorname{Idem}(\Bbbk[x_1, \ldots, x_n])$ which implies that $\operatorname{Idem}(A) = \{0, 1\}$.

(vi) In the Example 2.2.14, we have $B[x; \overline{\sigma}]$ can be seen as a skew PBW extension over B which is a weak Σ-rigid ring. Therefore, by Theorem 2.2.24, we obtain that $\operatorname{nil}(B)[x; \overline{\sigma}]X + \operatorname{Idem}(B) \subseteq \operatorname{Idem}(B[x; \overline{\sigma}])$. Furthermore, the set of idempotent elements of B is represented as follows

$$\operatorname{Idem}(B) = \left\{ \begin{pmatrix} (a,b) & (0,0) \\ (0,0) & (a,b) \end{pmatrix} \mid (a,b) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}.$$

Thus, we have the following inclusion

$$\operatorname{Idem}(B[x; \overline{\sigma}]) \supseteq \left\{ \sum_{i=1}^{n} \begin{pmatrix} (0,0) & (c,d) \\ (0,0) & (0,0) \end{pmatrix} x^i + \begin{pmatrix} (a,b) & (0,0) \\ (0,0) & (a,b) \end{pmatrix} \right\},$$

where $(a,b), (c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Next, we present an interesting property that allows us to describe the unit elements in a skew PBW extension. Theorem 2.2.27 is a result obtained from this work.

Theorem 2.2.27. *Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension, where R is a weak Σ-rigid NI ring. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n$ and $g = b_0 + b_1Y_1 + \cdots + b_mY_m$ be nonzero polynomials of A such that $fg = c \in R$. If b_0 is a unit in R, then $a_1, a_2, \ldots, a_n$ are all nilpotent.*

Proof. Assume that b_0 is a unit in R. Let us show that $a_1, a_2, \ldots, a_n$ are all nilpotent. Since R is NI ring, so $\operatorname{nil}(R)$ is an ideal of R. Hence $\overline{R} = R/\operatorname{nil}(R)$ is a reduced ring. Since R is weak Σ-rigid, $\operatorname{nil}(R)$ is weak Σ-rigid ideal. Then R is $\overline{\Sigma}$-rigid, by Proposition 2.2.7. Now, fs $fg = c \in R$, we have $\overline{f}\overline{g} = \overline{c}$ in $\sigma(\overline{R})\langle x_1, \ldots, x_n \rangle$. Thus $\overline{a_0}\overline{b_0} = \overline{c}$ and $\overline{a_i}\overline{b_j} = \overline{0}$, for each $i + j \geq 1$, by Theorem 2.2.5. Hence $\overline{a_i} = 0$ for each $i \geq 1$, since b_0 is unit. Therefore, we conclude a_i is nilpotent for each $i \geq 1$. □

As a consequence of the previous theorem, we can describe the idempotent elements in a skew PBW extension on Σ-rigid and weak Σ-rigid rings. Theorem 2.2.28 and Theorem 2.2.29 are presented as results obtained during the realization of this thesis.

Theorem 2.2.28. *Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension, where R is a weak Σ-rigid NI ring. If a polynomial $f \in A$ is unit, then its constant term is a unit and the other coefficients are nilpotent.*

Proof. Suppose R is a NI ring and weak Σ-rigid. Then, we get that $\operatorname{nil}(R)$ is a weak Σ-rigid ideal of R whence $\overline{R} = R/\operatorname{nil}(\mathrm{R})$ is a reduced ring and weak $\overline{\Sigma}$-rigid. Hence, $\overline{R}$ is $\overline{\Sigma}$-rigid, by using Proposition 2.2.7. Now, let $f = \sum_{i=1}^{n} a_iX_i \in A$ such that f is an unit. Then, there exists $g = \sum_{j=0}^{m} b_jY_j \in A$ such that $fg = 1 = gf$. Hence, we have $\overline{f}\overline{g} = \overline{1}$ in $\sigma(\overline{R})\langle x_1, \ldots, x_n \rangle$ and so $\overline{a_0}\overline{b_0} = \overline{1}$ and $\overline{a_i}\overline{b_j} = 0$, for every $i + j \geq 1$, by Theorem 2.2.5. Then, we get that $\overline{a_0}$ is a unit element of $\overline{R}$ and $a_1, \ldots, a_n$ are nilpotent, by using Theorem 2.2.27. Since $\operatorname{nil}(R) \subseteq J(A)$ and $\overline{a_0}$ is a unit element of $\overline{R}$, we get a_0 is a unit of R. □

In [AM69], p. 11, units for polynomial rings over commutative rings with identity are characterized. Given a commutative ring R with unity and $R[x]$ the ring of polynomials with coefficients in R, a polynomial $f(x) = a_0+a_1x+\cdots+a_nx^n \in R[x]$ is a unit if and only if a_0 is unit and a_i are nilpotent elements. In particular, if R is a reduced commutative ring, it follows that $U(R[x]) = U(R)$. The following result generalizes this particular case since the rigidness condition forces the ring of coefficients to be reduced (see Remark 2.2.2). This shows the importance of the reducibility condition.

Theorem 2.2.29. *Let R be a NI ring which is Σ-rigid and A a skew PBW extension over R. Then a polynomial $f \in A$ is unit if and only if its constant term is a unit and the other coefficients are zero.*

Proof. This fact follows from Theorem 2.2.28 and Proposition 2.2.7. □

Remark 2.2.30. We illustrate the previous results with some examples.

(i) **The algebra of q-differential operators $D_{q,h}[x, y]$:** The $\Bbbk$-algebra $D_{q,h}[x, y]$ is a skew PBW extension over $\Bbbk[y]$ which is Σ-rigid (see Example 2.2.9). Hence, by Theorem 2.2.29, we get that $U(D_{q,h}[x, y]) = U(\Bbbk[y])$. Since $\Bbbk[y]$ is a reduced ring. it follows that $U(\Bbbk[y]) = \Bbbk^*$ and therefore $U(D_{q,h}[x, y]) = U(\Bbbk[y]) = \Bbbk^*$.

(ii) **The algebra of shift operators S_h**: The $\Bbbk$-algebra S_h is a skew PBW extension over $\Bbbk[t]$ which is Σ-rigid (see Example 2.2.10). Hence, by Theorem 2.2.29, we get that $U(S_h) = U(\Bbbk[t])$. Having in mind that $\Bbbk[t]$ is a reduced ring, we obtain $U(\Bbbk[t]) = \Bbbk^*$ and thus, we conclude $U(S_h) = U(\Bbbk[t]) = \Bbbk^*$.

(iii) **q-Heisenberg algebra**: The $\Bbbk$-algebra $H_n(q)$ is a skew PBW extension over $\Bbbk$ which is Σ-rigid (see Example 2.2.11). Therefore, by Theorem 2.2.29, we obtain that $U(H_n(q)) = U(\Bbbk)$. Since $\Bbbk$ is a field, this means that $U(\Bbbk) = \Bbbk^*$ and hence, we have that $U(H_n(q)) = U(\Bbbk) = \Bbbk^*$.

(iv) Let $\Bbbk$ be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \ldots, x_n\}$. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, is a PBW extension over $\Bbbk$, which is Σ-rigid (see Example 2.2.12). Thus, by Theorem 2.2.29, it follows that $U(\mathcal{U}(\mathfrak{g})) = U(\Bbbk)$. Hence, we conclude that $U(\mathcal{U}(\mathfrak{g})) = \Bbbk^*$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is a Σ-rigid ring (see Example 2.2.13). Thus, by Theorem 2.2.29, we have that $U(A) = U(\Bbbk[x_1, \ldots, x_n])$ which implies that $U(A) = \Bbbk^*$.

(vi) In the Example 2.2.14, we have $B[x; \overline{\sigma}]$ can be seen as a skew PBW extension over B which is a weak Σ-rigid ring. Therefore, by Theorem 2.2.28, we obtain that $\mathrm{nil}(B)[x, \overline{\sigma}]X+U(B) \subseteq U(B[x; \overline{\sigma}])$. Furthermore, the set of units of B is represented as follows

$$U(B) = \left\{ \begin{pmatrix} (1,1) & (c,d) \\ (0,0) & (1,1) \end{pmatrix} \mid (c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}.$$

Thus, the set of units of $B[x;\overline{\sigma}]$ is as follows

$$U(B[x;\overline{\sigma}]) \supseteq \left\{ \sum_{i=1}^{n} \begin{pmatrix} (0,0) & (r,s) \\ (0,0) & (0,0) \end{pmatrix} x^i + \begin{pmatrix} (1,1) & (c,d) \\ (0,0) & (1,1) \end{pmatrix} \right\},$$

where $(r,s),(c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Theorem 2.2.31. *Let R be a NI ring which is weak Σ-rigid and A a skew PBW extension over R. If $fg = 0$, then there exists $s \in R \setminus \{0\}$ such that $a_i s \in \operatorname{nil}(R)$, for all $0 \leq i \leq n$.*

Proof. Since R is NI, then $\operatorname{nil}(R)$ is an ideal of R. Furthermore, we have that $\operatorname{nil}(R)$ is a weak Σ-rigid ideal, since R is a weak Σ-rigid ring. Thus, $\overline{R} = R/\operatorname{nil}(R)$ is $\overline{\Sigma}$-rigid, by using Proposition 2.2.7. Let $f = \sum_{i=0}^{n} a_i X_i$, $g = \sum_{j=0}^{m} b_j Y_j \neq 0$ such that $fg = 0$. If we denote by $\overline{f}, \overline{g}$ their corresponding polynomials in $\sigma(\overline{R})\langle x_1, \ldots, x_n \rangle$, then we have $\overline{f}\,\overline{g} = \overline{0}$. Since $\overline{R} = R/\operatorname{nil}(R)$ is $\overline{\Sigma}$-rigid, there exists $\overline{s} \in \overline{R} \setminus \{0\}$ such that $\overline{a_i}\,\overline{s} = \overline{0}$, for all $1 \leq i \leq n$. Therefore, we conclude $a_i s \in \operatorname{nil}(R)$, for all $1 \leq i \leq n$. □

Theorem 2.2.32. *Let R be a NI ring which is Σ-rigid and A a skew PBW extension over R. If $fg = 0$, then there exists $s \in R \setminus \{0\}$ such that $a_i s = 0$, for all $0 \leq i \leq n$.*

Proof. This fact follows from Theorem 2.2.31 and Proposition 2.2.7. □

Remark 2.2.33. We observe some interesting examples that illustrate the previous result.

(i) **The algebra of q-differential operators $D_{q,h}[x,y]$:** The $\Bbbk$-algebra $D_{q,h}[x,y]$ is a skew PBW extension over $\Bbbk[y]$ which is Σ-rigid (see Example 2.2.9). Hence, by Theorem 2.2.32, we get that $Z(D_{q,h}[x,y]) = Z(\Bbbk[y])$, and since $\Bbbk[y]$ is a domain, it follows that $Z(\Bbbk[y]) = \{0\}$ and thus $Z(D_{q,h}[x,y]) = Z(\Bbbk[y]) = \{0\}$.

(ii) **The algebra of shift operators S_h**: The $\Bbbk$-algebra S_h is a skew PBW extension over $\Bbbk[t]$ which is Σ-rigid (see Example 2.2.10). Hence, by Theorem 2.2.32, it follows that $Z(S_h) = Z(\Bbbk[t])$. Since $\Bbbk[t]$ is a domain, we have that $Z(\Bbbk[t]) = \{0\}$ which implies that $Z(S_h) = Z(\Bbbk[t]) = \{0\}$.

(iii) **q-Heisenberg algebra**: The $\Bbbk$-algebra $H_n(q)$ is a skew PBW extension over $\Bbbk$ which is Σ-rigid (see Example 2.2.11). Hence, by Theorem 2.2.32, we get $Z(H_n(q)) = Z(\Bbbk)$. Since $\Bbbk$ is a field, we have that $Z(\Bbbk) = \{0\}$, which implies that $Z(H_n(q)) = \{0\}$.

(iv) Let $\Bbbk$ be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1, \ldots, x_n\}$. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, is a PBW extension over $\Bbbk$, which is Σ-rigid (see Example 2.2.12). Thus, by Theorem 2.2.32, it follows that $Z(\mathcal{U}(\mathfrak{g})) = Z(\Bbbk)$. Hence, we conclude that $Z(\mathcal{U}(\mathfrak{g})) = \{0\}$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$, which is a Σ-rigid ring (see Example 2.2.13). Thus, by Theorem 2.2.32, we have that $Z(A) = Z(\Bbbk[x_1, \ldots, x_n])$, which implies that $Z(A) = \{0\}$.

(vi) In the Example 2.2.14, we have $B[x;\overline{\sigma}]$ can be seen as a skew PBW extension over B which is a weak Σ-rigid ring. Therefore, by Theorem 2.2.28, we obtain that $f(x) = \sum_{i=0}^{n} a_i x^i \in Z(B[x;\overline{\sigma}])$, if there exists $s \in B$ such that $a_i s \in \text{nil}(B)$, with $s \in B$ and $a_i \in B$, for all $0 \leq i \leq n$. Now, since $\text{nil}(B)$ is an ideal of B, it follows that

$$a_i s = \begin{pmatrix} (a_i, b_i) & (c_i, d_i) \\ (0,0) & (a_i, b_i) \end{pmatrix} \begin{pmatrix} (0,0) & (r,s) \\ (0,0) & (0,0) \end{pmatrix} \in \text{nil}(B),$$

for all $a_i \in B$ with $0 \leq i \leq n$. Thus, we conclude that $Z(B[x;\overline{\sigma}]) = \text{nil}(B)[x;\overline{\sigma}]$.

2.3 (Σ, Δ)-Compatible rings

With the aim of generalizing the σ-rigid rings in the context of Ore extensions, Annin [Ann04] introduced the notion of compatibility: a ring R is called **σ-compatible**, if for every $a, b \in R$, we have $ab = 0$ if and only if $a\sigma(b) = 0$; R is called **δ-compatible**, if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, R is called **(σ, δ)-compatible**. As an interesting result, (σ, δ)-compatible rings generalize the σ-rigid rings for the case of Ore extensions. Reyes and Suárez [RS18a] and Hashemi, Khalilnezhad and Alhevaz [HKA17] introduced independently the (Σ, Δ)-compatible rings which are a natural generalization of (σ, δ)-compatible rings, since skew PBW extensions generalize Ore extensions of injective type. Some examples of this type of rings and important results can be found in [HKA17], [RS18a] and [Ram19].

Next, we present some interesting results that relate this interesting class of rings and we also present a detailed description of some types of elements in skew PBW extensions over (Σ, Δ)-compatible rings.

Definition 2.3.1 ([RS18a], Definition 3.2)**.** Consider a ring R with a finite family of endomorphisms Σ and a finite family of Σ-derivations Δ. Then,

(1) R is said to be **Σ-compatible**, if for each $a, b \in R$, $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, where $\alpha \in \mathbb{N}^n$ and $\sigma^{\alpha}(a) = \sigma^{\alpha_1}(\sigma^{\alpha_2}(\sigma^{\alpha_3}(\cdots(\sigma^{\alpha_n}(a)))$, for all $a \in R$;

(2) R is said to be **Δ-compatible**, if for each $a, b \in R$, $ab = 0$ implies $a\delta^{\beta}(b) = 0$, where $\beta \in \mathbb{N}^n$ and $\delta^{\beta}(a) = \delta^{\beta_1}(\delta^{\beta_2}(\delta^{\beta_3}(\cdots(\delta^{\beta_n}(b)))$, for all $b \in R$.

If R is both Σ-compatible and Δ-compatible, R is called **(Σ, Δ)-compatible**.

Next, we present remarkable examples of skew PBW extensions over (Σ, Δ)-compatible rings. The examples described are adapted from [RS18a],[LR14] and [HKA17].

Examples 2.3.2. Multiplicative analogue of the Weyl algebra: The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by $x_1, \ldots, x_n$ subject to the relations: $x_j x_i = \lambda_{ji} x_i x_j$, $1 \leq i < j \leq n$, where $\lambda_{ji} \in \Bbbk \setminus \{0\}$. We note that $\mathcal{O}_n(\lambda_{ji})$ is isomorphic to the iterated skew polynomial ring $\Bbbk[x_1][x_2, \sigma_2] \cdots [x_n, \sigma_n]$ where $\sigma_j(x_i) = \lambda_{ji} x_i \quad 1 \leq i < j \leq n$. Thus, we conclude that $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle \cong \sigma(\Bbbk[x_1])\langle x_2, \ldots, x_n\rangle$. In addition, $\mathcal{O}_n(\lambda_{ji})$ is a skew PBW extension over $\Bbbk[x_1]$ which is (Σ, Δ)-compatible where $\Sigma = \{\sigma_j\}$ with $\sigma_j(x_1) = \lambda_{j1} x_1$, for every $1 < j \leq n$ and $\Delta = \{\delta_i\}$, where $\delta_i = 0$, for every $1 \leq i \leq n$.

Example 2.3.3. Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$ is generated by I_1, I_2, I_3 subject to the relations

$$I_2I_1 - qI_1I_2 = -q^{1/2}I_3,\ I_3I_1 - q^{-1}I_1I_3 = q^{-1/2}I_2,\ I_3I_2 - qI_2I_3 = -q^{1/2}I_1,$$

where $q \in \Bbbk - \{0\}$. We can prove that $\mathcal{U}'(\mathfrak{so}(3,\Bbbk)) \cong \sigma(\Bbbk)\langle I_1, I_2, I_3\rangle$. Moreover, $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$ is a skew PBW extension over $\Bbbk$ which is (Σ,Δ)-compatible where $\Sigma = \{\sigma_i\}$ with $\sigma_i = i_{\Bbbk}$, for all $1 \leq i \leq 3$, and $\Delta = \{\delta_i\}$ with $\delta_i = 0$, for every $1 \leq i \leq 3$.

Example 2.3.4. 3-dimensional skew polynomial algebra A: Let $\Bbbk$ be a field. The $\Bbbk$-algebra A is generated by x, y, z subject to the relations: $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$ and $xy - \gamma yx = \nu$, such that $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk - \{0\}$. We can show that $A \cong \sigma(\Bbbk)\langle x, y, z\rangle$. Furthermore, A is a skew PBW extension over $\Bbbk$ which is (Σ,Δ)-compatible with $\Sigma = \{\sigma_i\}$ where $\sigma_i = i_{\Bbbk}$, for every $1 \leq i \leq 3$ and $\Delta = \{\delta_i\}$ where $\delta_i = 0$, for every $1 \leq i \leq 3$.

Example 2.3.5. The algebra of differential operators $D_q(S_q)$ on a quantum space S_q. Let $q = [q_{ij}]$ be a matrix with entries in $\Bbbk^*$ such that $q_{ii} = 1 = q_{ij}q_{ji}$, for all $1 \leq i, j \leq n$. The $\Bbbk$-algebra S_q is generated by x_i, $1 \leq i \leq n$, subject to the relations $x_ix_j = q_{ij}x_jx_i$. The algebra $D_q(S_q)$ of q-differential operators on S_q is defined by $\partial_i x_j - q_{ij}x_j\partial_i = \delta_{ij}$, for all i, j. The relations between ∂_i, are given by $\partial_i\partial_j = q_{ij}\partial_j\partial_i$, for all i, j. Thus, we can show that $D_q(S_q) \cong \sigma(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle)\langle\partial_1, \ldots, \partial_n\rangle$. Note that $D_q(S_q)$ is a skew PBW extension over $\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle$ which is a (Σ,Δ)-compatible ring, where $\Sigma = \{\sigma_i\}$ with $\sigma_i(x_j) = q_{ij}x_j$, for all $1 \leq i, j \leq n$ and $\Delta = \{\delta_i\}$ with $\delta_i(x_j) = \delta_{ij}$, for all $1 \leq i, j \leq n$ and $\delta_{i,j}$ is the Kronecker's delta.

We recall that the following algebra is an example of a skew PBW extension which cannot be expressed as a skew polynomial ring.

Example 2.3.6. Diffusion algebras: Let $\Bbbk$ be a field. The diffusion algebra A is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ with relations

$$\begin{gathered} x_ix_j = x_jx_i,\quad x_iD_j = D_jx_i,\quad 1 \leq i, j \leq n. \\ c_{ij}D_iD_j - c_{ji}D_jD_i = x_jD_i - x_iD_j,\quad i < j,\quad c_{ij}, c_{ji} \in \Bbbk^*. \end{gathered}$$

Thus, we can prove that $A \cong \sigma(\Bbbk[x_1, \ldots, x_n])\langle D_1, \ldots, D_n\rangle$. In addition, A is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ,Δ)-compatible with $\Sigma = \{\sigma_i\}$ where $\sigma_j(x_i) = x_i$, for every $1 \leq i, j \leq n$ and $\Delta = \{\delta_i\}$ where $\delta_j(x_i) = 0$, for every $1 \leq i, j \leq n$.

We build a skew polynomial ring over a (Σ,Δ)-compatible ring which is non reduced.

Example 2.3.7. Let $\Bbbk$ be a field and $\Bbbk[t]$ the polynomial ring over $\Bbbk$. We consider $\sigma(p(t)) = p(t)$, the identity homomorphism and $\delta(t) = 1$. Since $\Bbbk[t]$ is reduced, it can be shown that $\Bbbk[t]$ is (σ,δ)-compatible ring. Now, let us define ring B as follows:

$$B = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in \Bbbk[t] \right\}.$$

The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B \to B$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of B is also extended to $\overline{\delta} : B \to B$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We note that the skew polynomial ring $B[x; \overline{\sigma}, \overline{\delta}]$ is a skew PBW extension over B which is $(\overline{\sigma}, \overline{\delta})$-compatible.

Proposition 2.3.8 ([RS18a], Proposition 3.4)**.** *Let Σ be a family of endomorphisms of a ring R and let Δ be a family of Σ-derivations of R. If R is Σ-rigid, then R is (Σ, Δ)-compatible.*

We follow the ideas in [RS18a], Proposition 3.4.

Proof. Suppose R is a Σ-rigid ring. Since every Σ-rigid ring is reduced then R is reduced. Moreover, R is a reversible ring (see Definition A.1.1). Let $a, b \in R$ such that $ab = 0$. Since σ_i is a injective endomorphism, for every i, then $\sigma^\alpha(ab) = 0$, for all $\alpha \in \mathbb{N}^n$. Therefore, we have $a\sigma^\alpha(b)\sigma^\alpha(a\sigma^\alpha(b)) = a\sigma^\alpha(ba)\sigma^\alpha(\sigma^\alpha(b)) = 0$ and as R is Σ-rigid, so $a\sigma^\alpha(b) = 0$. Similarly, if $ba = 0$, then $\sigma^\alpha(a)b = 0$. Conversely, let $a, b \in R$ such that $a\sigma^\alpha(b) = 0$. We have $ba\sigma^\alpha(b)\sigma^\alpha(a) = ba\sigma^\alpha(ba) = 0$, whence $ba = 0$. Now, as R is reversible then $ab = 0$. Finally, let $a, b \in R$ such that $ba = 0$. Then, as $\delta_i(ab) = 0$ this implies that $0 = \delta_i(ba) = \delta_i(b)a + \sigma_i(b)\delta_i(a)$, for all $1 \leq i \leq n$. Hence, multiplying right by $\sigma_i(b)\delta_i(a)$, we have $(\sigma_i(b)\delta_i(a))^2 = -\delta_i(b)a\sigma_i(b)\delta_i(a) = 0$, since $a\sigma_i(b) = 0$. Now, as R is reduced, so $\sigma_i(b)\delta_i(a) = 0$, i.e., $\delta_i(b)a = 0$ whence $a\delta_i(b) = 0$ by the reversibility condition of R. □

Proposition 2.3.9 ([RS18a], Proposition 3.8)**.** *Let R be a (Σ, Δ)-compatible ring. For every $a, b \in R$, we have:*

(i) *if $ab = 0$, then $a\sigma^\theta(b) = \sigma^\theta(a)b = 0$, for each $\theta \in \mathbb{N}^n$.*

(ii) *If $\sigma^\beta(a)b = 0$ for some $\beta \in \mathbb{N}^n$, then $ab = 0$.*

(iii) *If $ab = 0$, then $\sigma^\theta(a)\delta^\beta(b) = \delta^\beta(a)\sigma^\theta(b) = 0$, for every $\theta, \beta \in \mathbb{N}^n$.*

(iv) *If $a\sigma^\theta(b) = \sigma^\theta(a)b = 0$, for some $\theta \in \mathbb{N}^n$, then $ab = 0$.*

We know that if Σ is a family of endomorphisms of a ring R and Δ is a family of Σ-derivations of a Σ-rigid ring, then R is (Σ, Δ)-compatible ring. However, the reciprocal is false (see [RS18a], Example 3.6). However, we present the following result that establishes an equivalence between these notions under the condition of reducibility.

Proposition 2.3.10 ([RS18a], Theorem 3.9)**.** *If A is a skew PBW extension of a ring R, then the following statements are equivalent:*

(i) *R is reduced and (Σ, Δ)-compatible;*

(ii) *R is Σ-rigid;*

(iii) *A is reduced.*

Proof. We follow the ideas in [RS18a], Proposition 3.9.

- (i) $\Rightarrow$ (ii) Suppose that R is a reduced and (Σ, Δ)-compatible ring. Let $r \in R$ such that $r\sigma^{\alpha}(r) = 0$, for all $\alpha \in \mathbb{N}^n$. Let us show that $r = 0$. From Proposition 2.3.9 (i) it follows that $(\sigma^{\alpha}(r))^2 = \sigma^{\alpha}(r)\sigma^{\alpha}(r) = 0$, for every $\alpha \in \mathbb{N}^n$. Now, since R is reduced, we have $\sigma^{\alpha}(r) = 0$, and using the injectivity of σ^{α}, we have $r = 0$.

- (ii) $\Rightarrow$ (i) It follows from Proposition 2.3.8 and Definition 3.1 in [RS18a].

- (ii) $\Rightarrow$ (iii) Let R be Σ-rigid ring and suppose that A is not a reduced ring. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$ with $a_i \in R$, $a_m \neq 0$ and $X_1 \prec \cdots \prec X_{m-1} \prec X_m$ such that $f^2 = 0$. Then, $f^2 = (a_mX_m + \cdots + a_1X_1 + a_0)(a_mX_m + \cdots + a_1X_1 + a_0) = a_m\sigma^{\alpha_m}(a_m)[c_{\alpha_m,\alpha_m}x^{2\alpha_m} + p_{\alpha_m,\alpha_m}] + a_mp_{\alpha_m,a_m}X_m +$ other terms of order less than X_mX_m , where $p_{\alpha_m,a_m} = 0$ or $\deg(p_{\alpha_m,a_m}) < |\alpha_m|$ if $p_{\alpha_m,a_m} \neq 0$, and $p_{\alpha_m,\alpha_m} = 0$ or $\deg(p_{\alpha_m,\alpha_m}) < |\alpha_m + \alpha_m|$ if $p_{\alpha_m,\alpha_m} \neq 0$. Since $\mathrm{lc}(f^2) = a_m\sigma^{\alpha_m}(a_m)c_{\alpha_m,\alpha_m} = 0$ and $c_{i,j}$ is invertible, we have $a_m\sigma^{\alpha_m}(a_m) = 0$. Now, as $a_m\sigma^{\alpha_m}(a_m) = 0$, then $\sigma^{\alpha_m}(a_m)\sigma^{\alpha_m}(a_m) = 0$ from Proposition 2.3.9 (i). Thus, $\sigma^{\alpha_m}(a_m^2) = 0$ and since σ^{α} is injective, for all $\alpha \in \mathbb{N}^n$, we obtain $a_m^2 = 0$. Finally, since R is reduced, so $a_m = 0$ which is a contradiction. Hence, A is reduced, as desired.

- (iii) $\Rightarrow$ (ii) If A is a reduced ring, then R is also reduced as a subring. Let us show that that R is a Σ-rigid ring. Let $a \in R$ such that $a\sigma^{\alpha}(a) = 0$, for all $\alpha \in \mathbb{N}^n$. Multiplying to right by $x^{\alpha}a$ and multiplying to left by $\sigma^{\alpha}(a)x^{\alpha}$, we obtain $(\sigma^{\alpha}(a)x^{\alpha}a)^2 = \sigma^{\alpha}(a)x^{\alpha}a\sigma^{\alpha}(a)x^{\alpha}a = 0$ and so $\sigma^{\alpha}(a)x^{\alpha}a = 0$ since A is reduced. Thus, $0 = \sigma^{\alpha}(a)x^{\alpha}a = \sigma^{\alpha}(a)[\sigma^{\alpha}(a)x^{\alpha} + p_{\alpha,a}] = (\sigma^{\alpha}(a))^2x^{\alpha} + \sigma^{\alpha}(a)p_{\alpha,a}$, with $p_{\alpha,a} = 0$, or $\deg(p_{\alpha,a}) < |\alpha|$ if $p_{\alpha,a} \neq 0$. Therefore $(\sigma^{\alpha}(a))^2 = 0$ whence $\sigma^{\alpha}(a) = 0$. Now, since σ^{α} is injective, for all $\alpha \in \mathbb{N}^n$, we have $a = 0$. Thus, R is a Σ-rigid ring.

□

Motivated by the above facts, for a set of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ of a ring R, we recall the (Σ, Δ)-compatible ideals in R which are a generalization of Σ-rigid ideals.

Definition 2.3.11 ([AHK19], Definition 3.1)**.** Let R be a (Σ, Δ)-compatible ring and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ a skew PBW extension over R, where $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ is a finite family of endomorphisms of R and $\Delta := \{\delta_1, \ldots, \delta_n\}$ is a finite family of Σ-derivations of R.

(i) It is said that an ideal I of R is **Σ-ideal**, if $\sigma^{\alpha}(I) \subseteq I$, for each $\alpha \in \mathbb{N}^n$; I is **Δ-ideal**, if $\delta^{\alpha}(I) \subseteq I$, for each $\alpha \in \mathbb{N}^n$; I is **Σ-invariant** if $\sigma^{-\alpha}(I) = I$, for each $\alpha \in \mathbb{N}^n$. If I is both Σ and Δ-ideal, we say that I is a (Σ, Δ)-ideal.

(ii) For an ideal I, we say that I is **Σ-compatible**, if for each $a, b \in R$ and $\alpha \in \mathbb{N}^n$, $ab \in I \Leftrightarrow a\sigma^{\alpha}(b) \in I$. Moreover, I is said to be **Δ-compatible** ideal, if for each $a, b \in R$ and $\alpha \in \mathbb{N}^n$, $ab \in I \Rightarrow a\delta^{\alpha}(b) \in I$. If I is both Σ-compatible and Δ-compatible, then we say that I is **(Σ, Δ)-compatible**.

A very important example of (Σ, Δ)-compatible ideal is the following:

Example 2.3.12. If R be a (Σ, Δ)-compatible ring and $\mathrm{nil}(R)$ the set of nilpotent elements of R. Then, $\mathrm{nil}(R)$ is a (Σ, Δ)-compatible ideal. This fact is a consequence of the properties stated in Proposition 2.3.9

.

It is not difficult to observe that R is a Σ-rigid ring if and only if $\{0\}$ is Σ-rigid ideal of R. Also, R is a Σ-compatible (resp., Δ-compatible) ring if and only if $\{0\}$ is Σ-compatible (resp., Δ-compatible) ideal of R.

Remark 2.3.13. Some important observations about Σ-ideals are the following:

(i) Let $I \subseteq R$. We denote the set of all elements of A with coefficients in I by $I\langle x_1, \ldots, x_n\rangle$. If A is a skew PBW extension of a ring R and I is a (Σ, Δ)-ideal of R, then one can show that $I\langle x_1, \ldots, x_n\rangle$ is an ideal of A.

(ii) If I is a (Σ, Δ)-ideal of R, then for every $1 \leq i \leq n$, $\sigma_i : R/I \to R/I$ defined by $\overline{\sigma_i}(a + I) = \sigma_i(a) + I$, is an endomorphism and $\delta_i : R/I \to R/I$ defined by $\overline{\delta_i}(a + I) = \delta_i(a) + I$, is a $\overline{\sigma_i}$-derivation.

Proposition 2.3.14 ([AHK19], Lemma 3.2)**.** *Let I be a Σ-compatible ideal of a ring R. Then I is Σ-invariant.*

Proof. Let us show that $\sigma_i^{-1}(I) = I$, for every $1 \leq i \leq n$. Let $a \in \sigma_i^{-1}(I)$, then $\sigma_i(a) \in I$. Since I is Σ-compatible ideal, we have $1\sigma_i(a) \in I$, whence $a \in I$. Hence, I is a Σ-invariant ideal of R, as desired. □

Proposition 2.3.15 ([AHK19], Proposition 3.3)**.** *Let I be a (Σ, Δ)-compatible ideal of R and $a, b \in R$. Then we have the following:*

(i) *If $ab \in I$ then $a\sigma^\alpha(b) \in I$ and $\sigma^\alpha(a)b \in I$, for each $\alpha \in \mathbb{N}^n$.*

(ii) *If $ab \in I$ then $\sigma^\alpha(a)\delta^\beta(b), \delta^\beta(a)\sigma^\alpha(b) \in I$, for each $\alpha, \beta \in \mathbb{N}^n$.*

(iii) *If $a\sigma^\theta(b) \in I$ or $\sigma^\theta(a)b \in I$, for some $\theta \in \mathbb{N}^n$, then $ab \in I$.*

Proof. We follow the ideas in [AHK19], Proposition 3.3.

(i) Let $a, b \in R$ such that $ab \in I$. Let us show that $a\sigma_i(b) \in I$ and $\sigma_i(a)b \in I$, for every $1 \leq i \leq n$. If $ab \in I$, then $\sigma_i(ab) = \sigma_i(a)\sigma_i(b) \in I$, since I is a Σ-ideal. Therefore, $\sigma_i(a)b \in I$, for every $1 \leq i \leq n$ by Σ-compatibility of I. On the other hand, since I is (Σ, Δ)-compatible ideal, if $ab \in I$, then $a\sigma_i(b) \in I$.

(ii) Let $a, b \in R$ such that $ab \in I$. Let us show that $\sigma_i(a)\delta_i(b) \in I$ and $\delta_i(a)\sigma_i(b) \in I$. If $ab \in I$, then by (i) and Δ-compatibility of I, we have $\sigma_i(a)\delta_i(b) \in I$, for every $1 \leq i \leq n$. Additionally, $\delta_i(a)b = \delta_i(ab) - \sigma_i(a)\delta_i(b) \in I$, for each $1 \leq i \leq n$, since $\delta_i(ab), \sigma_i(a)\delta_i(b) \in I$. Hence, $\delta_i(a)b \in I$, for every $1 \leq i \leq n$. Since I is an Σ-compatible ideal, so $\delta_i(a)\sigma_i(b) \in I$, for each $1 \leq i \leq n$.

(iii) Let $a, b \in R$ such that $a\sigma_i(b) \in I$ or $\sigma_i(a)b \in I$, for every $1 \leq i \leq n$. Let us show that $ab \in I$. Suppose $\sigma_i(a)b \in I$, for each $1 \leq i \leq n$. Then, $\sigma_i(a)\sigma_i(b) \in I$ by Σ-compatibility of I and so $\sigma_i(ab) \in I$, for every $1 \leq i \leq n$. Hence, $ab \in \sigma_i^{-1}(I) = I$, since I is Σ-invariant, by Lemma 2.3.14. Thus, $ab \in I$, as desired.

□

Next, we present two interesting results with the aim of providing sufficient conditions to guarantee fusible property for skew PBW extensions over (Σ, Δ)-compatible rings. Before, we present some definitions and results that are used for our purpose.

Definition 2.3.16 ([RS16], Definition 3.1)**.** Let A be a skew PBW extension of a ring R. R is called a **Σ-skew Armendariz** ring, if for elements $f = \sum_{i=1}^{m} a_i X_i$ and $g = \sum_{i=1}^{t} b_i Y_i$ in A, the equality $fg = 0$ implies $a_i \sigma^{\alpha_i}(b_j) = 0$, for all $0 \leq i \leq m$ and $0 \leq j \leq t$, where $\alpha_i = \exp(X_i)$.

With the aim of establishing the next proposition, consider the following notation. For a skew PBW extension over a ring R, we define $\mathrm{Ann}_r(2^R) = \{r_R(U) \mid U \subseteq R\}$, for any subset U of R and $\mathrm{Ann}_r(2^A) = \{r_A(V) \mid V \subseteq A\}$, for any subset V of A. Analogously, $\mathrm{Ann}_l(2^R)$ and $\mathrm{Ann}_l(2^A)$ are defined for the left case.

Proposition 2.3.17 ([RS19a], Theorem 3.21)**.** *If A is a skew PBW extension over a (Σ, Δ)-compatible and Σ-skew Armendariz ring R, then:*

(i) $\varphi : \mathrm{Ann}_r(2^R) \to \mathrm{Ann}_r(2^A)$; $C \mapsto CA$ *is bijective.*

(ii) $\phi : \mathrm{Ann}_l(2^R) \to \mathrm{Ann}_l(2^A)$; $D \mapsto AD$ *is bijective.*

Proof. We follow the ideas presented in [RS19a], Theorem 3.21].

(i) Let $\varphi : \mathrm{Ann}_r(2^R) \to \mathrm{Ann}_r(2^A)$ defined by $C \mapsto CA$, for every $C \in \mathrm{Ann}_r(2^R)$. The sketch of the proof is as follows: We consider $\varphi' : \mathrm{Ann}_r(2^A) \to \mathrm{Ann}_r(2^R)$ defined by $B \mapsto B \cap R$, for all $B \in \mathrm{Ann}_r(2^A)$. We can verify that φ is well defined and assuming (Σ, Δ)-compatibility of R. In addition, it can be proved that φ' is well defined with which φ and φ' are well defined functions that satisfy that $\varphi'\varphi = i_R$, and so φ is injective. Now, if $B \in \mathrm{Ann}_r(2^A)$, then $B = r_A(J)$, for some $J \subseteq A$. Let us show that $\varphi(B_1) = B$, where B_1 is the set of coefficients of elements of B. Let $f = \sum_{i=0}^{m} a_i X_i \in J$ and $g = \sum_{j=0}^{t} b_j Y_j \in B$. Then $fg = 0$, and using that R is Σ-skew Armendariz and (Σ, Δ)-compatible, it follows that $a_i b_j = 0$, for every a_i and b_j. This means that $J_1 B_1 = 0$, and so $B_1 \subseteq r_R(J_1)$, where J_1 is the set of coefficients of elements of J. On the other hand, The (Σ, Δ)-compatibility of R implies that $r_R \subseteq B_1 R$ whence $r_R(J_1) = B_1 R$, that is, $r_A(J) = B_1 A$ which shows that φ is surjective. Therefore φ is bijective.

(ii) The proof is similar to the established in (i).

□

Proposition 2.3.18 ([RS19a], Corollary 3.22)**.** *If A is a skew PBW extension over a (Σ, Δ)-compatible and Σ-skew Armendariz ring R, then R satisfies the ascending chain condition on right (left) annihilators if and only if so does A.*

Proof. It follows from Proposition 2.3.17. □

We present the following theorem which establishes conditions for a skew PBW extension over a prime left Goldie ring to have the fusible property. This result has been obtained as a consequence of this work.

Theorem 2.3.19. *Let R be a prime left Goldie ring which is (Σ, Δ)-compatible Σ-skew Armendariz ring and A a skew PBW extension over R with σ_i injective endomorphism, for each $1 \leq i \leq n$. If A is semiprime, then A is a left regular fusible ring.*

Proof. Suppose that R is a prime ring and let A be a skew PBW extension over R. Consider $Gr(A)$ the associated graded ring of A. First, we get that $Gr(A) \cong R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$, by Proposition 2.1.23, and Proposition 2.1.24. Now, since R is a semiprime left Goldie ring, by Proposition 1.2.14, we have $R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$ is semiprime left Goldie and $\operatorname{udim}(R) = \operatorname{udim}(R[x_1; \sigma_1] \cdots [x_n; \sigma_n]) = \operatorname{udim}(Gr(A))$. In addition, by Proposition A.3.6, it is satisfied that $\operatorname{udim}(A) \leq \operatorname{udim}(R)$ and since $\operatorname{udim}(R) < \infty$, then A has finite uniform dimension. On the other hand, since R is (Σ, Δ)-compatible and Σ-skew Armendariz ring, by Proposition 2.3.18, there is a bijective correspondence between the annihilators of R and the annihilators of A. Therefore, since R is a left Goldie ring, in particular satisfies the ascending chain condition on left annihilator, then A also satisfies ACC. Thus, since A has finite uniform dimension and also satisfies ACC, then it follows that A is a regular fusible ring by using Proposition 1.1.37. □

2.3.1 Elements in skew PBW extensions over (Σ, Δ)-compatible rings

The search to characterize some elements in a noncommutative ring of polynomial type is a research topic of great interest given the connections of these descriptions with different concepts. In the case of skew polynomial rings, in [HHA17] the authors characterized different elements in this class of rings. For skew PBW extensions, different works have been developed in relation to this topic (see [HHR20], [LR20a] and [RS20]). In this section, we describe some types of elements of great importance in skew PBW extension over (Σ, Δ)-compatible NI rings. First, we study the nilpotent elements of a skew PBW extensions, and as a result, we present a characterization of the nilpotent elements of a PBW extension in terms of the set of nilpotent elements of its ring of coefficients R.

In [OB12], Lemma 2.6, the nilpotent elements for skew polynomial rings over (σ, δ)-compatible rings were characterized. We present a generalization of this result.

Theorem 2.3.20. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. Then $f = a_0 + a_1X_1 + \cdots + a_nX_n \in \operatorname{nil}(A)$ if and only if $a_i \in \operatorname{nil}(R)$, for all $0 \leq i \leq n$.*

Proof. Let us show that $\operatorname{nil}(A) \subseteq \operatorname{nil}(R)\langle x_1, \ldots, x_n\rangle$. Let $f = \sum_{i=0}^{m} a_iX_i \in A$, where $a_i \in R$, for every $1 \leq i \leq m$, $a_m \neq 0$, with $X_m \succ X_{m-1} \succ \cdots \succ X_1$ such that $f \in \operatorname{nil}(A)$. Then, there exists $r \neq 0$ such that $f^r = 0$. It can be shown that $\operatorname{lc}(f^r) = a_m\sigma^{\alpha_m}(a_m)\sigma^{2\alpha_m}(a_m)\cdots\sigma^{(r-1)\alpha_m}(a_m) = 0$. Since R is Σ-compatible ring, we have $a_m^t = 0$ and hence $a_m \in \operatorname{nil}(R)$. Now, let us write $f = Q + a_mX_m$ with $Q = \sum_{i=0}^{m-1} a_iX_i \in A$ and $\deg(Q) < |\alpha_m|$. Then $0 = f^t = Q^t + h$, for some $h \in A$. We observe that, when we compute every summand of h we obtain products of the coefficient a_m in $\sigma's$ and $\delta's$ depending of the coordinates of α_m. Since $a_m \in \operatorname{nil}(R)$ and $\operatorname{nil}(R)$ is (Σ, Δ)-ideal, we have $h \in \operatorname{nil}(R)\langle x_1, \ldots, x_n\rangle$. Hence, we have that $Q^r \in \operatorname{nil}(R)\langle x_1, \ldots, x_n\rangle$ whence it follows that $a_{m-1}\sigma^{\alpha_{m-1}}(a_{m-1})\cdots\sigma^{(t-1)\alpha_{m-1}}(a_{m-1}) \in \operatorname{nil}(R)$ and by using Proposition 2.3.9, we get $a_{m-1} \in \operatorname{nil}(R)$. Following this process, we obtain $a_i \in \operatorname{nil}(R)$, for each $1 \leq i \leq m$, and therefore, $f \in \operatorname{nil}(R)\langle x_1, \ldots, x_n\rangle$.

Now, let us prove that $\operatorname{nil}(R)\langle x_1,\dots x_n\rangle \subseteq \operatorname{nil}(A)$. Let $f=\sum_{i=0}^{m} a_iX_i \in A$, where $a_i \in R$, $a_m \neq 0$, with $X_m \succ X_{m-1} \succ \cdots \succ X_1$ such that $f \in \operatorname{nil}(R)\langle x_1,\cdots x_n\rangle$. Then, we have $a_i \in \operatorname{nil}(R)$, for every $1 \leq i \leq n$. Now, Let k_i be the minimum integer positive such that $a_i^{k_i}=0$, for every $1 \leq i \leq n$ and $k := \max\{k_i \mid 1 \leq i \leq n\}$. Let us show that $f^{(m+1)k+1}=0$, whence $f \in \operatorname{nil}(A)$. Since the expression for f have $m+1$ terms, when we realize the product $f^{(m+1)k+1}$, each term of the sum that forms the polynomial is a product of $(n+1)k+1$ factors of the form $\sum_{j=0}^{(n+1)k+1} a_{i,j}X_{i,j}$. Hence, when we compute $f^{(m+1)k+1}$ every product involves at least k elements a_i, for some i. Thus, we obtain that every one of these products is equal to zero and therefore each sum is equal to zero by using Proposition 2.3.9 and Remarks 2.1.9. Thus, every term of $f^{(m+1)k+1}$ is equal to zero and hence $f \in \operatorname{nil}(A)$, whence $\operatorname{nil}(R)\langle x_1,\dots x_n\rangle \subseteq \operatorname{nil}(A)$. □

Next, we generalize [OB12], Corollary 2.2.

Theorem 2.3.21. *Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension, where R is a (Σ,Δ)-compatible NI ring. Then:*

(i) $\operatorname{nil}(A)$ *is an ideal of* A.

(ii) $\operatorname{nil}(A)=\operatorname{nil}(R)\langle x_1,\dots,x_n\rangle$.

Proof. This fact follows from Theorem 2.3.20. □

The following result is obtained during the realization of this work. In addition, the following theorem generalizes [OB12], Corollary 2.3. We need to assume that the elements $c_{i,j}$ of Proposition 2.1.8 (ii) are central and invertible in R.

Theorem 2.3.22. *Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension, where R is a (Σ,Δ)-compatible NI ring, and $f=\sum_{i=1}^{m} a_iX_i$, $g=\sum_{j=i}^{n} b_jY_j \in A$. Then $fg \in \operatorname{nil}(A)$ if and only if $a_ib_j \in \operatorname{nil}(R)$, for all i,j with $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof. Let $f,g \in A$ given by $f=\sum_{i=0}^{n} a_iX_i$ and $g=\sum_{j=0}^{m} b_jY_j$ such that $fg \in \operatorname{nil}(A)$. We have that, $fg=\sum_{k=0}^{n+m}\left(\sum_{i+j=k} a_iX_ib_jY_j\right) \in \operatorname{nil}(A)=\operatorname{nil}(R)A$. It can be shown that $\operatorname{lc}(fg)=a_n\sigma^{\alpha_n}(b_m)c_{\alpha_n,\beta_m} \in \operatorname{nil}(R)$, and since the elements $c_{i,j}$ are in the center of R, then c_{α_n,β_m} are also in the center of R, whence $a_n\sigma^{\alpha_n}(b_m) \in \operatorname{nil}(R)$, and by Proposition 2.3.9 (iv), it follows that $a_nb_m \in \operatorname{nil}(R)$. The idea is to prove that $a_pb_q \in \operatorname{nil}(R)$, for $p+q \geq 0$. The proof of this fact is by induction. Suppose that $a_pb_q \in \operatorname{nil}(R)$, for $p+q=n+m, n+m-1, n+m-2, \cdots, k+1$, for some $k>0$. By Proposition 2.3.9 (iii), we obtain $a_pX_pb_qY_q \in \operatorname{nil}(R)A$, for these values of $p+q$. In this way, we consider the sum of the products $a_uX_ub_vY_v$, where $u+v=k, k-1, k-2, \dots, 0$. Using a similar reasoning as [Rey15], Proposition 3.6, we can see that the element $a_u\sigma^{\alpha_u}(b_v)c_{\alpha_u,\beta_v}$ belongs to $\operatorname{nil}(R)$. Again, since the elements $c_{i,j}$ are central and left invertible, $a_u\sigma^{\alpha_u}(b_v) \in \operatorname{nil}(R)$, and by Proposition 2.3.9 (iii), we have, $a_ib_j \in \operatorname{nil}(R)$, for $0 \leq i \leq n$ and $0 \leq j \leq m$.

Conversely, for the elements f, g previously defined, suppose that $a_ib_j \in \operatorname{nil}(R)$. When we compute every summand of $a_iX_ib_jY_j$, we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i by Remark 2.1.9, and since $a\sigma^{\alpha}(\delta^{\beta}(b))$ and $a\delta^{\beta}(\sigma^{\alpha}(b))$ are elements of $\operatorname{nil}(R)$ by Proposition 2.3.9 (iii), then every coefficient of each term of the expansion fg given by $fg = \sum_{k=0}^{m+t}\left(\sum_{i+j=k} a_iX_ib_jY_j\right)$, is an element of $\operatorname{nil}(R)$. Therefore, by Theorem 2.3.20, we conclude that fg is an element of $\operatorname{nil}(A)$. □

Remark 2.3.23. We study some examples that illustrate the previous theorem.

(i) **Multiplicative analogue of the Weyl algebra**: The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is a skew PBW extension over $\Bbbk[x_1]$ which is (Σ, Δ)-compatible (see Example 2.3.2). Hence, by Theorem 2.3.20, we get that $\operatorname{nil}(\mathcal{O}_n(\lambda_{ji})) = \operatorname{nil}(\Bbbk[x_1])\mathcal{O}_n(\lambda_{ji})$. Since $\Bbbk[x_1]$ is a reduced ring, it follows that $\operatorname{nil}(\Bbbk[x_1]) = \{0\}$, and hence $\operatorname{nil}(\mathcal{O}_n(\lambda_{ji})) = \{0\}$.

(ii) **Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$**: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$ is a skew PBW extension over $\Bbbk$ which is (Σ, Δ)-compatible (see Example 2.3.3). By Theorem 2.3.20, it follows that $\operatorname{nil}(\mathcal{U}'(\mathfrak{so}(3,\Bbbk))) = \operatorname{nil}(\Bbbk)\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$. Having in mind that $\Bbbk$ is a reduced ring, we obtain that $\operatorname{nil}(\Bbbk) = \{0\}$. Hence, we conclude that $\operatorname{nil}(\mathcal{U}'(\mathfrak{so}(3,\Bbbk))) = \{0\}$.

(iii) **3-dimensional skew polynomial algebra** A: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk$ which is (Σ, Δ)-compatible (see Example 2.3.4). Hence, by Theorem 2.3.20, it follows that $\operatorname{nil}(A) = \operatorname{nil}(\Bbbk)A$, and as $\Bbbk$ is a reduced ring, this implies that $\operatorname{nil}(\Bbbk) = \{0\}$. Therefore, we conclude that $\operatorname{nil}(A) = \{0\}$.

(iv) **The algebra of differential operators $D_q(S_q)$ on a quantum space S_q**. The algebra $D_q(S_q)$ of q-differential operators on S_q is a skew PBW extension over $\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle$ which is a (Σ, Δ)-compatible ring (see Example 2.3.5). Hence, by Theorem 2.3.20, it follows that $\operatorname{nil}(D_q(S_q)) = \operatorname{nil}(\sigma(\Bbbk)\langle x_1, \ldots x_n\rangle)D_q(S_q)$. Again, considering Theorem 2.3.20, we get that $\operatorname{nil}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle) = \operatorname{nil}(\Bbbk)\langle x_1, \ldots x_n\rangle$. Since $\operatorname{nil}(\Bbbk) = \{0\}$, this implies that $\operatorname{nil}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle) = 0$ which guarantees that $\operatorname{nil}(D_q(S_q)) = \operatorname{nil}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle)D_q(S_q) = \{0\}$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ, Δ)-compatible (see Example 2.3.6). By Theorem 2.3.20, we get that $\operatorname{nil}(A) = \operatorname{nil}(\Bbbk[x_1, \ldots, x_n])A$. Considering that $\Bbbk[x_1, \ldots, x_n]$ is reduced, this guarantees that $\operatorname{nil}(\Bbbk[x_1, \ldots, x_n]) = \{0\}$. Hence, we obtain that $\operatorname{nil}(A) = \{0\}$.

(vi) In the Example 2.3.7, $B[x; \overline{\sigma}, \overline{\delta}]$ can be seen as a skew PBW extension over B which is (Σ, Δ)-compatible. It follows from Theorem 2.3.20 that $\operatorname{nil}(B[x; \overline{\sigma}, \overline{\delta}]) = \operatorname{nil}(B)\langle x\rangle$. Furthermore, the set of nilpotent elements of B is represented as follows

$$\operatorname{nil}(B) = \left\{ \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} \mid p(t) \in \Bbbk[t] \right\}.$$

Thus, the set of nilpotent elements of $B[x; \overline{\sigma}, \overline{\delta}]$ is as follows

$$\operatorname{nil}(B[x; \overline{\sigma}, \overline{\delta}]) = \left\{ \sum_{i=0}^{n} \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} x^i \mid p(t) \in \Bbbk[t] \right\}.$$

We investigate the idempotent elements in skew PBW extensions over (Σ, Δ)-compatible rings. Before, we presented the following characterization of the idempotent elements. Our next theorem generalizes [HHA17], Proposition 3.2.

Theorem 2.3.24. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. Let $f = \sum_{i=0}^{n} a_i X_i$ be an idempotent element of A. Then $a_i \in \operatorname{nil}(R)$, for each $1 \leq i \leq n$ and there exists an idempotent element $e \in R$ such that $\overline{a_0} = \overline{e} \in R/\operatorname{nil}(R)$.*

Proof. Since R is a NI ring and (Σ, Δ)-compatible, we have $\operatorname{nil}(R)$ is a (Σ, Δ)-compatible ideal of R. Now, as $\overline{R} = R/\operatorname{nil}(R)$ is reduced and $(\overline{\Sigma}, \overline{\Delta})$-compatible whence $\overline{R}$ is a $\overline{\Sigma}$-rigid ring, by using Proposition 2.3.10. On the other hand, if $\overline{f^2} = \overline{f}$ in $\sigma(\overline{R})\langle x_1, \dots, x_n\rangle$, we have $(\overline{1} - \overline{f})\overline{f} = 0$ which implies that $\overline{f} = \overline{a_0}$ in $\sigma(\overline{R})\langle x_1, \dots, x_n\rangle$ and $\overline{a_i} = \overline{0}$ for each $1 \leq i \leq n$, by Proposition 2.2.5. Hence, by Theorem 2.2.22, we conclude $\overline{a_0} = \overline{e}$ in $R/\operatorname{nil}(R)$, for some idempotent $e \in R$. □

The following theorem describes the idempotent elements in skew PBW extensions over (Σ, Δ)-compatible rings. This characterization has been obtained in carrying out this work and generalizes [HHA17], Theorem 3.3.

Theorem 2.3.25. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. Let $f = \sum_{i=0}^{n} a_i X_i \in A$. If $f^2 = f$, then $f = a_0$ with $a_0 \in \operatorname{Idem}(R)$.*

Proof. Let $f = \sum_{i=0}^{n} a_i X_i \in A$ such that $f^2 = f$. By Theorem 2.3.24, we have $a_i \in \operatorname{nil}(R)$, for each $1 \leq i \leq n$ and also, there exist an idempotent element $e \in R$ and a nilpotent element $b \in R$ such that $a_0 = e + b$. Let us denote $h = b + a_1 X_1 + \cdots + a_n X_n \in A$. Then, we have that $f = e + h$. Since R is a NI ring which is (Σ, Δ)-compatible, this guarantees that $\operatorname{nil}(R)\langle x_1, \dots, x_n\rangle = \operatorname{nil}(A)$, by Proposition 2.3.20. Hence, we conclude $h \in \operatorname{nil}(A)$. Let us show that $h = 0$. If $h \neq 0$, then there exists an integer $n \geq 2$ such that $h^n = 0 \neq h^{n-1}$. Now, since e is an idempotent element of R and R is (Σ, Δ)-compatible ring, we have $\delta^{\beta}(e) = 0$, for all $\beta \in \mathbb{N}^n$ and $\sigma^{\alpha}(e) = e$, for all $\alpha \in \mathbb{N}^n$, which implies that $he = eh$. Thus, if $f^2 = f$, we have that $0 = (e + h)(1 - e - h) = -eh + h - he - h^2 = h - 2eh - h^2$. Hence, $h^2 = (1 - 2e)g$. By multiplying right by h^{n-2} to $h^2 = (1 - 2e)h$, we get $0 = h^n = (1 - 2e)h^{n-1}$ and since $1 - 2e$ is invertible, then $h^{n-1} = 0$, which is a contradiction. Therefore $h = 0$ and so $f = a_0$. □

Remark 2.3.26. We consider some examples that illustrate the previous result.

(i) **Multiplicative analogue of the Weyl algebra**: The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is a skew PBW extension over $\Bbbk[x_1]$ which is (Σ, Δ)-compatible (see Example 2.3.2). Hence, by Theorem 2.3.25, we get that $\operatorname{Idem}(\mathcal{O}_n(\lambda_{ji})) = \operatorname{Idem}(\Bbbk[x_1]))$. In addition, we have that $\operatorname{Idem}(\Bbbk[x_1]) = \{0, 1\}$ which implies that $\operatorname{Idem}(\mathcal{O}_n(\lambda_{ji})) = \{0, 1\}$.

(ii) **Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$**: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$ is a skew PBW extension over $\Bbbk$ which is (Σ, Δ)-compatible (see Example 2.3.3). Thus, by Theorem 2.3.25, it follows that $\operatorname{Idem}(\mathcal{U}'(\mathfrak{so}(3, \Bbbk))) = \operatorname{Idem}(\Bbbk)$. On the other hand, $\operatorname{Idem}(\Bbbk) = \{0, 1\}$ which guarantees that $\operatorname{Idem}(\mathcal{U}'(\mathfrak{so}(3, \Bbbk))) = \{0, 1\}$.

(iii) **3-dimensional skew polynomial algebra** A: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk$ which is (Σ, Δ)-compatible (see Example 2.3.4). Hence, by Theorem 2.3.25, it follows that $\operatorname{Idem}(A) = \operatorname{Idem}(\Bbbk)$. Clearly $\operatorname{Idem}(\Bbbk) = \{0, 1\}$, which implies that $\operatorname{Idem}(A) = \{0, 1\}$.

(iv) **The algebra of differential operators $D_q(S_q)$ on a quantum space S_q**. The algebra $D_q(S_q)$ of q-differential operators on S_q is a skew PBW extension over $\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle$ which is a (Σ, Δ)-compatible ring (see Example 2.3.5). Hence, by Theorem 2.3.25, it follows that $\operatorname{Idem}(D_q(S_q)) = \operatorname{Idem}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle)$. Again, considering Theorem 2.3.25, we get that $\operatorname{Idem}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle) = \operatorname{Idem}(\Bbbk)$. Since $\operatorname{Idem}(\Bbbk) = \{0, 1\}$, it follows that $\operatorname{Idem}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle) = \{0, 1\}$, which guarantees that $\operatorname{Idem}(D_q(S_q)) = \operatorname{Idem}(\sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle) = \{0, 1\}$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ, Δ)-compatible (see Example 2.3.6). It follows from Theorem 2.3.25 that $\operatorname{Idem}(A) = \operatorname{Idem}(\Bbbk[x_1, \ldots, x_n])$. Futher, we know that $\operatorname{Idem}(\Bbbk[x_1, \ldots, x_n]) = \{0, 1\}$ which implies that $\operatorname{Idem}(A) = \{0, 1\}$.

(vi) In the Example 2.3.7, $B[x; \overline{\sigma}, \overline{\delta}]$ can be seen as a skew PBW extension over B which is (Σ, Δ)-compatible. It follows from Theorem 2.3.25 that $\operatorname{Idem}(B[x; \overline{\sigma}, \overline{\delta}]) = \operatorname{Idem}(B)$. Furthermore, the set of idempotent elements of B is represented as follows

$$\operatorname{Idem}(B) = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t) \in \{0, 1\} \text{ and } p(t)q(t)p(t) = 0 \right\}.$$

Thus, the set of idempotent elements of $B[x; \overline{\sigma}, \overline{\delta}]$ is as follows

$$\operatorname{Idem}(B[x; \overline{\sigma}, \overline{\delta}]) = \left\{ \begin{pmatrix} a & q(t) \\ 0 & a \end{pmatrix} \mid a \in \{0, 1\} \text{ and } a^2 q(t) = 0 \right\}.$$

Next, we study the unit elements in a skew PBW extension over a (Σ, Δ)-compatible ring. Before, we present two preliminary results which we have obtained in carrying out this work. The following fact generalizes a result presented in [HHA17]

Theorem 2.3.27. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. Let $f = \sum_{i=0}^{n} a_i X_i$ and $g = \sum_{j=0}^{m} b_j Y_j$ be nonzero polynomials of A such that $fg = c \in R$. If b_0 is a unit in R, then $a_1, a_2, \ldots, a_n$ are all nilpotent.*

Proof. Assume that b_0 is a unit in R. Let us show that $a_1, a_2, \ldots, a_n$ are all nilpotent. Since R is NI ring and (Σ, Δ)-compatible, we have $\operatorname{nil}(R)$ is an (Σ, Δ)-compatible ideal of R. Hence $\overline{R} = R/\operatorname{nil}(R)$ is a reduced ring and also (Σ, Δ)-compatible. With this in mind, by using Proposition 2.3.10, we obtain that $\overline{R}$ is a $\overline{\Sigma}$-rigid ring. Since $fg = c \in R$, we have $\overline{f}\,\overline{g} = \overline{c}$ in $\sigma(\overline{R})\langle x_1, \ldots, x_n\rangle$ and hence $\overline{a_0}\,\overline{b_0} = \overline{c}$ and $\overline{a_i}\,\overline{b_j} = \overline{0}$, for each $i + j \geq 1$, by using Theorem 2.2.5. Therefore, we get $\overline{a_i} = 0$ for each $i \geq 1$, since b_0 is unit whence a_i is nilpotent for each $i \geq 1$. □

The definition of Levitzki radical and weakly 2-primal ring can be found in Definition A.1.33 and Definition A.1.34, respectively. We recall that Levitzki radical is denoted as L-rad(R).

Theorem 2.3.28. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible 2-primal ring. Then we have* $\mathrm{nil}(R)A = L\text{-}rad(A) = \mathrm{nil}(A)$.

Proof. Since R is 2-primal, then A is a 2-primal ring, by [AHK19], Theorem 11. On the other hand, every 2-primal ring is a weakly 2-primal ring, thus $\mathrm{nil}(A) = \mathrm{L\text{-}rad}(A)$ and $\mathrm{nil}(R)A = \mathrm{nil}(A)$, since R is a 2-primal ring which is (Σ, Δ)-compatible. Therefore $\mathrm{nil}(R)A = \mathrm{L\text{-}rad}(A)$. □

We establish the following characterization of the units in a skew PBW extension on a (Σ, Δ)-compatible ring. This result has been obtained in carrying out this work and generalizes [HHA17], Theorem 2.8.

Theorem 2.3.29. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible 2-primal ring. Then $f \in A$ is unit if and only if its constant term is a unit and the other coefficients are nilpotent.*

Proof. Suppose R is a 2-primal and (Σ, Δ)-compatible. Then, $\mathrm{nil}(R)$ is a (Σ, Δ)-compatible ideal of R whence $\overline{R} = R/\mathrm{nil(R)}$ is a reduced ring and $(\overline{\Sigma}, \overline{\Delta})$-compatible. Hence, $\overline{R}$ is $\overline{\Sigma}$-rigid, by using Proposition 2.3.10. Now, let $f = \sum_{i=1}^{n} a_i X_i \in A$ such that f is an unit. Then, there exists $g = \sum_{j=0}^{m} b_j Y_j \in A$ such that $fg = 1 = gf$, this implies that $\overline{f}\,\overline{g} = \overline{1}$ in $\sigma(\overline{R})\langle x_1, \ldots, x_n\rangle$ and so $\overline{a_0}\overline{b_0} = \overline{1}$ and $\overline{a_i}\overline{b_j} = 0$, for every $i + j \geq 1$, by using Theorem 2.2.5. Then, we get that $\overline{a_0}$ is a unit element of $\overline{R}$ and $a_1, \ldots, a_n$ are nilpotent, by Theorem 2.3.27. Since $\mathrm{nil}(R) \subseteq J(A)$, by Theorem 2.3.28 and $\overline{a_0}$ is a unit element of $\overline{R}$, we have that $a_0 \in U(R)$. Reciprocally, let a_0 be a unit element and $a_1, \ldots, a_n$ be nilpotent elements of R. Then $f = a_1X_1 + \cdots + a_nX_n \in \mathrm{nil}(R)\langle x_1, \ldots, x_n\rangle = L - \mathrm{rad}(A)$ by Theorem 2.3.28. Also, we get that $\mathrm{L} - \mathrm{rad}(A) \subseteq J(A)$ by [Lam01], Lemma 10.31, so $a_nX_n + \cdots + a_1X_1 \in J(A)$. Therefore, we conclude $f = a_nX_n + \cdots + a_1X_1 + a_0$ is a unit element of A. □

Remark 2.3.30. We observe at some interesting examples that illustrate the previous result.

(i) **Multiplicative analogue of the Weyl algebra**: The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is a skew PBW extension over $\Bbbk[x_1]$ which is (Σ, Δ)-compatible (see Example 2.3.2). Hence, by Theorem 2.3.29, we get that $U(\mathcal{O}_n(\lambda_{ji})) = \mathrm{nil}(\Bbbk[x_1])\mathcal{O}_n(\lambda_{ji}) + U(\Bbbk[x_1])$. In this way, since $\Bbbk[x_1]$ is a reduced ring, it follows that $\mathrm{nil}(\Bbbk[x_1]) = \{0\}$ and therefore $U(\mathcal{O}_n(\lambda_{ji})) = U(\Bbbk[x_1]) = \Bbbk^*$.

(ii) **Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$**: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$ is a skew PBW extension over $\Bbbk$ which is (Σ, Δ)-compatible (see Example 2.3.3). It follows from Theorem 2.3.29 that $U(\mathcal{U}'(\mathfrak{so}(3, \Bbbk))) = \mathrm{nil}(\Bbbk)\mathcal{U}'(\mathfrak{so}(3, \Bbbk)) + U(\Bbbk)$. Now, considering that $\Bbbk$ is a reduced ring, this implies that $\mathrm{nil}(\Bbbk) = \{0\}$. Thus, we conclude that $U(\mathcal{U}'(\mathfrak{so}(3, \Bbbk))) = U(\Bbbk) = \Bbbk^*$.

(iii) **3-dimensional skew polynomial algebra** A: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk$ which is (Σ, Δ)-compatible (see Example 2.3.4). Hence, by Theorem 2.3.29, it follows that $U(A) = \mathrm{nil}(\Bbbk)A + U(\Bbbk)$. As $\Bbbk$ is reduced, this implies that $\mathrm{nil}(\Bbbk) = \{0\}$. Therefore, we obtain that $U(A) = U(\Bbbk) = \Bbbk^*$.

(iv) **The algebra of differential operators $D_q(S_q)$ on a quantum space S_q.** The algebra $D_q(S_q)$ of q-differential operators on S_q is a skew PBW extension over $\sigma(\Bbbk)\langle x_1,\dots,x_n\rangle$ which is a (Σ,Δ)-compatible ring (see Example 2.3.5). Hence, by Theorem 2.3.29, it follows that $U(D_q(S_q)) = \operatorname{nil}(D_q(S_q)) + U(\sigma(\Bbbk)\langle x_1,\dots x_n\rangle)$. Now, considering again Theorem 2.3.29 and Remark 2.3.26 (iv), we obtain that $U(D_q(S_q)) = \operatorname{nil}(\Bbbk)\langle x_1,\dots x_n\rangle + U(\Bbbk)$. Having in mind that $\Bbbk$ is a reduced ring, we conclude that $U(D_q(S_q)) = U(\Bbbk) = \Bbbk^*$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk[x_1,\dots,x_n]$ which is (Σ,Δ)-compatible (see Example 2.3.6). It follows from Theorem 2.3.29 that $U(A) = \operatorname{nil}(\Bbbk[x_1,\dots,x_n])A + U(\Bbbk[x_1,\dots,x_n])$. Again. $\Bbbk[x_1,\dots,x_n]$ is a reduced ring, and so $\operatorname{nil}(\Bbbk[x_1,\dots,x_n]) = \{0\}$. Hence, we obtain $U(A) = U(\Bbbk[x_1,\dots,x_n]) = \Bbbk^*$.

(vi) In the Example 2.3.7, $B[x;\overline{\sigma},\overline{\delta}]$ can be seen as a skew PBW extension over B which is (Σ,Δ)-compatible. As a consequence of Theorem 2.3.29, we have that $U(B[x;\overline{\sigma},\overline{\delta}]) = \operatorname{nil}(B)[x;\overline{\sigma},\overline{\delta}] + U(B)$. Furthermore, the set of unit elements of B is represented as follows

$$U(B) = \left\{ \begin{pmatrix} a & q(t) \\ 0 & a \end{pmatrix} \mid p(t) \in \Bbbk \text{ and } q(t) = -a^2 r(t), \text{ for some } r(t) \in \Bbbk[t], \right\}$$

Thus, the set of unit elements of $B[x;\overline{\sigma},\overline{\delta}]$ is as follows

$$U(B[x;\overline{\sigma},\overline{\delta}]) = \left\{ \sum_{i=1}^{n} \begin{pmatrix} 0 & s(t) \\ 0 & 0 \end{pmatrix} x^i + \begin{pmatrix} a & q(t) \\ 0 & a \end{pmatrix} \mid s(t) \in \Bbbk[t], \begin{pmatrix} a & q(t) \\ 0 & a \end{pmatrix} \in U(B) \right\}.$$

The following theorem characterizes the zero divisors in a skew PBW extension on a (Σ,Δ)-compatible rings. This result has been obtained in carrying out this work.

Theorem 2.3.31. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension, where R is a (Σ,Δ)-compatible NI ring. If $fg = 0$, then there exists $s \in R\setminus\{0\}$ such that $a_i s \in \operatorname{nil}(R)$ for all $0 \le i \le n$.*

Proof. Since R is a NI ring, then $\operatorname{nil}(R)$ is an ideal of R. In addition, we have that $\operatorname{nil}(R)$ is an (Σ,Δ)-compatible ideal, since R is a (Σ,Δ)-compatible ring. There, $\overline{R} = R/\operatorname{nil}(R)$ is $\overline{\Sigma}$-rigid, by using Proposition 2.3.10. Let $f = \sum_{i=0}^{n} a_i X_i$, $g = \sum_{j=0} b_j Y_j \neq 0$ such that $fg = 0$. If we denote by $\overline{f},\overline{g}$ their corresponding polynomials in $\sigma(\overline{R})\langle x_1,\dots,x_n\rangle$, then we have $\overline{f}\,\overline{g} = \overline{0}$. Since $\overline{R} = R/\operatorname{nil}(R)$ is $\overline{\Sigma}$-rigid, there exists $\overline{s} \in \overline{R}\setminus\{0\}$ such that $\overline{a_i s} = \overline{0}$ for all $1 \le i \le n$. Hence $a_i s \in \operatorname{nil}(R)$ for all a_i, $1 \le i \le n$. □

Remark 2.3.32. The following examples illustrate the above theorem.

(i) **Multiplicative analogue of the Weyl algebra**: The $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ is a skew PBW extension over $\Bbbk[x_1]$ which is (Σ,Δ)-compatible (see Example 2.3.2). Hence, by Theorem 2.3.31, we get that $Z(\mathcal{O}_n(\lambda_{ji})) = \mathrm{Z}(\Bbbk[x_1])$. In this way, since $\Bbbk[x_1]$ is a domain, it follows that $Z(\Bbbk[x_1]) = \{0\}$ and therefore $Z(\mathcal{O}_n(\lambda_{ji})) = \mathrm{Z}(\Bbbk[x_1]) = \{0\}$.

(ii) **Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$**: Let $\Bbbk$ be a field. The $\Bbbk$-algebra $\mathcal{U}'(\mathfrak{so}(3,\Bbbk))$ is a skew PBW extension over $\Bbbk$ which is (Σ,Δ)-compatible (see Example 2.3.3). Hence, by Theorem 2.3.31, it follows that $Z(\mathcal{U}'(\mathfrak{so}(3,\Bbbk))) = Z(\Bbbk)$. Considering that $\Bbbk$ is a domain, this implies that $Z(\Bbbk) = \{0\}$. Thus, we obtain $Z(\mathcal{U}'(\mathfrak{so}(3,\Bbbk))) = \{0\}$.

(iii) **3-dimensional skew polynomial algebra** A: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk$ which is (Σ,Δ)-compatible (see Example 2.3.4). It follows from Theorem 2.3.31 that $Z(A) = Z(\Bbbk)$. Again, $\Bbbk$ is a domain, this guarantees that $Z(\Bbbk) = \{0\}$. Therefore, we conclude that $Z(A) = Z(\Bbbk) = \{0\}$.

(iv) **The algebra of differential operators $D_q(S_q)$ on a quantum space S_q**. The algebra $D_q(S_q)$ of q-differential operators on S_q is a skew PBW extension over $\sigma(\Bbbk)\langle x_1,\dots,x_n\rangle$ which is a (Σ,Δ)-compatible ring (see Example 2.3.5). Hence, by Theorem 2.3.31, it follows that $Z(D_q(S_q)) = Z(\sigma(\Bbbk)\langle x_1,\dots x_n\rangle)$. Again, considering Theorem 2.3.31, we get that $Z(\sigma(\Bbbk)\langle x_1,\dots x_n\rangle) = Z(\Bbbk)$. Since $Z(\Bbbk) = \{0\}$, this means that $Z(\sigma(\Bbbk)\langle x_1,\dots x_n\rangle) = \{0\}$, which guarantees that $Z(D_q(S_q)) = \{0\}$.

(v) **Diffusion algebras**: The $\Bbbk$-algebra A is a skew PBW extension over $\Bbbk[x_1,\dots,x_n]$ which is (Σ,Δ)-compatible (see Example 2.3.6). Hence, by Theorem 2.3.31, we get that $Z(A) = Z(\Bbbk[x_1,\dots,x_n])$. Since $\Bbbk[x_1,\dots,x_n]$ is a domain, this means that $Z(\Bbbk[x_1,\dots,x_n]) = \{0\}$ and hence, it follows that $Z(A) = Z(\Bbbk[x_1,\dots,x_n]) = \{0\}$.

(vi) In the Example 2.3.7, $B[x;\overline{\sigma},\overline{\delta}]$ can be seen as a skew PBW extension over B which is (Σ,Δ)-compatible. Therefore, by Theorem 2.3.31, we have that $f(x) = \sum_{i=0}^{n} a_i x^i \in Z(B[x;\overline{\sigma},\delta])$, if there exist $s \in B$ such that $a_i s \in \operatorname{nil}(B)$, with $s \in B$ and $a_i \in B$, for all $0 \leq i \leq n$. Now, since $\operatorname{nil}(B)$ is an ideal of B, this means that

$$a_i s = \begin{pmatrix} p_i(t) & q_i(t) \\ 0 & p_i(t) \end{pmatrix} \begin{pmatrix} 0 & r(t) \\ 0 & 0 \end{pmatrix} \in \operatorname{nil}(B),$$

for all $a_i \in B$ with $0 \leq i \leq n$. Thus, we conclude that $Z(B[x;\overline{\sigma},\overline{\delta}]) = \operatorname{nil}(B)[x;\overline{\sigma},\overline{\delta}]$.

2.3.2 Examples

An important observation is that many algebras polynomial type of great importance in the literature can be seen as skew PBW extensions over rings that are usually reduced. We have seen previously that under reducibility conditions over R, the notions weak Σ-rigid and (Σ,Δ)-compatible coincide. For this reason, we want to present the following examples that particularly cover the results obtained for both rings. We clarify again that this is possible thanks to the reducibility condition on the ring coefficients.

Noncommutative ring	nil(A)	$U(A)$	Idem(A)	$Z(A)$
Skew polynomial ring of derivation type	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Universal enveloping algebra of lie algebras	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Universal enveloping rings $\mathcal{U}(V,R,\mathbb{k})$	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Differential operator rings $V(R,L)$	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Tensor product $R\otimes_{\mathbb{k}} U(\mathfrak{g})$	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Crossed product $R * U(\mathfrak{g})$	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Twisted or smash product differential operator ring $R\#_\sigma U(\mathfrak{g})$	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$

Table 2.5: PBW extensions.

Noncommutative ring	nil(A)	$U(A)$	Idem(A)	$Z(A)$
Weyl algebra $A_n(\mathbb{k})$	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Quantum plane	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra of q-differential operators	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra of shift operators S_h	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Mixed algebra D_h	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra for multidimensional discrete linear systems	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$

Table 2.6: Types of elements in Ore extensions of derivation type.

Noncommutative ring	nil(A)	$U(A)$	Idem(A)	$Z(A)$
Algebra of linear partial differential operators	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra of linear partial shift operators	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra of linear partial difference operators	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra of linear partial q-dilation operators	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$
Algebra of linear partial q-differential operator	$\{0\}$	$\mathbb{k}^*$	$\{0,1\}$	$\{0\}$

Table 2.7: Types of elements in operator algebras.

CHAPTER 3

Weak annihilator over skew PBW extensions

In this chapter, our purpose is to study a weaker notion of annihilator about an important family of rings such as skew PBW extensions. In Section 3.1, we study some interesting properties that relate this concept of weak annihilator to skew PBW extensions. In Section 3.2, taking advantage of this weak notion of annihilator, we study a more general concept of associated prime ideals. Finally, we present a result of great importance in which we characterize these new associated prime ideals in a skew PBW extension under certain conditions on the ring of coefficients. Some important definitions in this section are taken from [OB12]. The results of this chapter are all product of this work. Given the great importance of the notion of weak annihilator and the characterization of certain associated prime ideals, these results have been submitted for publication.

3.1 Weak annihilator

Ouyang and Birkenmeier [OB12] introduced the notion of weak annihilator which is a generalization of annihilator. They described sufficient conditions to ensure that the weak annihilator over Ore extensions are generated as left ideals by nilpotent elements of the ring R. Motivated by the results in [OB12], in this section, we first study the notion of weak annihilator, which are a generalization of annihilator. We present a generalization of these results in the case of a skew PBW extension on a ring R.

We begin with the central concept of this chapter.

Definition 3.1.1. ([OB12], Definition 2.1). Let R be a ring. For $X \subseteq R$, It is defined $N_R(X) = \{a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X\}$, which is called the **weak annihilator** of X in R. If X is singleton, say $X = \{r\}$, we use $N_R(r)$ to denote $N_R(\{r\})$.

Remark 3.1.2. Let R be a ring. For any subset X of a ring R, it can be shown that $N_R(X) = \{a \in R \mid xa \in \text{nil}(R)\} = \{b \in R \mid bx \in \text{nil}(R)\}$, for all $x \in X$, and also it follows that $l_R(X) \subseteq N_R(X)$ and $r_R(X) \subseteq N_R(X)$. If R is a reduced ring, then $\text{nil}(R) = 0$ and so $r_R(X) = N_R(X) = l_R(X)$, for any subset X of R. In addition, for any subset $X \subseteq R$, we have that $N_R(X)$ is an ideal of R in case $\text{nil}(R)$ is an ideal.

Now, we present some examples which show that the notion of a weak annihilator is not a trivial generalization of annihilator.

Example 3.1.3 ([OB12], Example 2.1)**.** Let $\mathbb{Z}$ be the ring of integers and $T_2(\mathbb{Z})$ the 2×2 upper triangular matrix ring over $\mathbb{Z}$. Let $X = \left\{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\right\} \subseteq T_2(\mathbb{Z})$. Then, we have that $r_{T_2(\mathbb{Z})}(X) = 0$ and $N_{T_2(\mathbb{Z})}(X) = \left\{\left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\right\}$. Therefore, $r_{T_2(\mathbb{Z})}(X) \neq N_{T_2(\mathbb{Z})}(X)$.

Now, we present some properties of the weak annihilator.

Proposition 3.1.4 ([OB12], Proposition 2.1)**.** *Let X, Y be subsets of R. Then we have the following:*

(i) *$X \subseteq Y$ implies $N_R(Y) \subseteq N_R(X)$.*

(ii) *$X \subseteq N_R(N_R(X)))$.*

(iii) *$N_R(X) = N_R(N_R(N_R(X)))$.*

For a skew PBW extension over a ring R, we define $\mathrm{NAnn}_R(R) = \{N_R(U) \mid U \subseteq R\}$ and $\mathrm{NAnn}_A(A) = \{N_A(V) \mid V \subseteq A\}$. The following theorem establishes a bijective correspondence between weak annihilators of a skew PBW extension and weak annihilators of its ring of coefficients. Our theorem generalizes Proposition 2.3.17.

Theorem 3.1.5. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. The correspondence $\varphi : \mathrm{NAnn}_R(R) \to \mathrm{NAnn}_A(A)$, given by $\varphi(N_R(U)) = N_R(U)A$, for every $N_R(U) \in \mathrm{NAnn}_R(R)$, is bijective.*

Proof. First, let us show that φ is well defined. The idea is show that $\varphi(N_R(U)) = N_A(U)$, i.e. $N_A(U) = N_R(U)A$, for every subset U of R. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in N_R(U)A$ and $r \in U$. Since $a_i \in N_R(U)$, for all $0 \leq i \leq n$, this implies that $ra_i \in \mathrm{nil}(R)$. Thus, by Theorem 2.3.20, we get that $rf = ra_0 + ra_1X_1 + \cdots + ra_nX_n \in \mathrm{nil}(R)A = \mathrm{nil}(A)$. Hence, we conclude $N_R(U)A \subseteq N_A(U)$. Now, let us prove that $N_A(U) \subseteq N_R(U)A$. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in N_A(U)$, then $rf = ra_0 + ra_1X_1 + \cdots + ra_mX_m \in \mathrm{nil}(A) = \mathrm{nil}(R)A$, for every element r of U. This means that $ra_i \in \mathrm{nil}(R)$, for all $0 \leq i \leq n$, and so we have $a_i \in N_R(U)$, for all $0 \leq i \leq n$, whence $f \in N_R(U)A$. Therefore, $N_A(U) = N_R(U)A$. Now, let $U, V \subseteq R$ such that $\varphi(N_R(U)) = \varphi(N_R(V))$. This implies that $N_A(U) = N_A(V)$. In particular, we obtain $N_R(U) = N_R(V)$. Thus, φ is injective. Finally, let us prove that φ is surjective. Let $N_A(V) \in \mathrm{NAnn}_A(A)$, and $g = b_0 + b_1Y_1 + \cdots + b_mY_m \in N_A(V)$, for a subset V of A. Then $fg \in \mathrm{nil}(A)$, for all $f = a_0 + a_1X_1 + \cdots + a_nX_n \in V$, and by Theorem 2.3.22, $a_ib_j \in \mathrm{nil}(R)$, for each i, j. Thus, $b_j \in N_R(C_V)$, for all $0 \leq j \leq t$, whence $g \in N_R(C_V)A$, and so $N_A(V) \subseteq N_R(C_V)A$. Now, since that $N_R(C_V)A \subseteq N_A(V)$, we have that $N_A(V) = N_R(C_V)A = \varphi(N_R(C_V))$, which means that φ is surjective, and hence φ is bijective. □

Given A a skew PBW extension over a ring R, the following fact provides sufficient conditions to guarantee that $N_A(U)$ is generated by a nilpotent element, for all $U \subseteq A$. Our theorem generalizes [OB12], Theorem 2.1.

Theorem 3.1.6. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. If for each subset $X \nsubseteq \operatorname{nil}(R)$, $N_R(X)$ is generated as an ideal by a nilpotent element, then for each subset $U \nsubseteq \operatorname{nil}(A)$, $N_A(U)$ is generated as an ideal by a nilpotent element.*

Proof. Let U be a subset of A with $U \nsubseteq \operatorname{nil}(A)$. Then by Corollary 2.3.21, we have $C_U \nsubseteq \operatorname{nil}(R)$. Therefore, there exists $c \in \operatorname{nil}(R)$ such that $N_R(C_U) = cR$. Let us show that $N_A(U) = cA$. Let $f = a_0 + \cdots + a_nX_n \in U$ and $g = b_0 + \cdots + b_mY_m \in A$. We see that $fcg \in \operatorname{nil}(A)$. First, observe that $cg = cb_0 + \cdots + cb_mY_m$ and, on the other hand, we have that

$$fcg = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_icb_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(cb_j)X_iY_j + a_ip_{\alpha_i,cb_j}Y_j\right)$$

Since $a_icb_j \in \operatorname{nil}(R)$, for every i, j, Proposition 2.3.15 implies that $a_i\sigma^{\alpha_i}(cb_j) \in \operatorname{nil}(R)$. Furthermore, $a_i\sigma^{\alpha}(\delta^{\beta}(cb_j))$ and $a_i\delta^{\beta}(\sigma^{\alpha}(cb_j))$ are elements of $\operatorname{nil}(R)$, for every $\alpha, \beta \in \mathbb{N}^n$. In addition, by Remark 2.1.9, the polynomial p_{α_i,cb_j} involves elements obtained evaluating σ's and δ's (depending on the coordinates of α_i) in the element cb_j. Thus, by the previous observation $a_ip_{\alpha_i,cb_j} \in \operatorname{nil}(R)$, for every i, j and therefore, $fcg \in \operatorname{nil}(R)A = \operatorname{nil}(A)$. Hence, $cg \in N_A(U)$. Now, let us consider $f = a_0 + a_1X_1 + \cdots + a_nX_n \in N_A(U)$. By definition, for all $g = b_0 + b_1Y_1 + \cdots + b_mY_m$, it is satisfied that $fg \in \operatorname{nil}(A)$. By Corollary 2.3.21, $fg \in \operatorname{nil}(R)A$, whence

$$fg = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ib_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)X_iY_j + p_{\alpha_i,b_j}Y_j\right) \in \operatorname{nil}(R)A$$

Given that $a_i\sigma^{\alpha_i}(b_j) \in \operatorname{nil}(R)$, for every i, j. Proposition 2.3.15, guarantees $a_ib_j \in \operatorname{nil}(R)$. Therefore, $a_i \in N_R(C_U) = cR$ for each $0 \leq i \leq n$. This means that there exists $r_i \in R$ such that $a_i = cr_i$. Hence, $f = a_0 + a_1X_1 + \cdots + a_nX_n = c(r_0 + r_1X_1 + \cdots + r_nX_n) \in cA$. Therefore, we obtain that $f \in cA$. □

Remark 3.1.7. We present two examples that illustrate the above theorem.

Example 3.1.8. Diffusion algebras: Let $\Bbbk$ be a field. The diffusion algebra A is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ, Δ)-compatible. Since $\Bbbk[x_1, \ldots, x_n]$ is reduced, it follows that $\operatorname{nil}(\Bbbk[x_1, \ldots, x_n]) = \{0\}$ and $N_{\Bbbk[x_1,\ldots,x_n]}(X) = r_{\Bbbk[x_1,\ldots,x_n]}(X)$, for any subset $X \subseteq \Bbbk[x_1, \ldots, x_n]$. This means that for any subset $\{0\} \neq U \subseteq \Bbbk[x_1, \ldots, x_n]$, we get that $N_{\Bbbk[x_1,\ldots,x_n]}(U) = r_{\Bbbk[x_1,\ldots,x_n]}(U) = \{0\}$. Thus, by Theorem 3.1.6, we have that $N_A(V)$ is generated by a nilpotent element for any subset $V \subseteq A$ with $V \nsubseteq \operatorname{nil}(A)$.

Example 3.1.9. Let $\Bbbk$ be a field and $\Bbbk[t]$ the polynomial ring over $\Bbbk$. We consider $\sigma(p(t)) = p(t)$, the identity homomorphism and $\delta(t) = 1$. Having in mind that $\Bbbk[t]$ is reduced, we can show that $\Bbbk[t]$ is a (σ, δ)-compatible ring. Now, let us define ring B as follows:

$$B = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in \Bbbk[t] \right\}$$

The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B \to B$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of B is also extended to $\overline{\delta} : B \to B$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We note that the skew polynomial ring $B[x; \overline{\sigma}, \overline{\delta}]$ is a skew PBW extension over B which is $(\overline{\sigma}, \overline{\delta})$-compatible (see Example 2.3.7). We can prove that for any subset $X \subseteq B$ with $X \nsubseteq \text{nil}(B)$ it is satisfied that $N_R(X) = rB$, where $r \in \text{nil}(B)$. Thus, it follows from Theorem 3.1.6 that $N_{B[x;\overline{\sigma},\overline{\delta}]}U = wB[x; \overline{\sigma}, \overline{\delta}]$, for any subset $U \subseteq B[x; \overline{\sigma}, \overline{\delta}]$, where $U \nsubseteq \text{nil}(B[x; \overline{\sigma}, \overline{\delta}])$ and $w \in \text{nil}(B[x; \overline{\sigma}, \overline{\delta}])$.

The following theorem is a consequence of Theorem 3.1.6, considering skew PBW extensions of the endomorphism type. Our theorem characterizes skew PBW extensions for which every weak annihilator of a subset U of A is generated by a nilpotent element. This result generalizes [OB12], Theorem 2.2.

Theorem 3.1.10. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of endomorphism type, where R is a Σ-compatible NI ring. Then the following statements are equivalent:*

(i) *For each subset $X \nsubseteq \text{nil}(R)$, $N_R(X)$ is generated as an ideal by a nilpotent element.*

(ii) *For each subset $U \nsubseteq \text{nil}(A)$, $N_A(U)$ is generated as an ideal by a nilpotent element.*

Proof. By Theorem 3.1.6, it suffices to show (ii) $\Rightarrow$ (i). Let X be a subset of R with $X \nsubseteq \text{nil}(R)$. Then $X \nsubseteq \text{nil}(A)$. So there exists $f(x) = a_0 + a_1X_1 + \cdots + a_mX_m \in \text{nil}(A)$ such that $N_A(X) = f(x)A$. Note that $f(x) = a_0 + a_1X_1 + \cdots + a_mX_m \in \text{nil}(A)$, whence $a_i \in \text{nil}(R)$, for all $0 \leq i \leq m$ by Corollary 2.3.21. We may assume that $a_0 \neq 0$. Now, let us show that $N_R(X) = a_0R$. Since $a_0 \in \text{nil}(R)$ and $\text{nil}(R)$ is an ideal of R, we obtain $pa_0R \subseteq \text{nil}(R)$, for each $p \in X$ whence $a_0R \subseteq N_R(X)$. If $m \in N_R(X)$, then $m \in N_A(X)$. Thus, there exists $g(x) = b_0 + b_1Y_1 + \cdots + b_nY_n \in A$ such that

$$m = fg = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ib_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)X_iY_j\right)$$

Therefore, we conclude that $m = a_0b_0 \in a_0R$, and thus $N_R(X) \subseteq a_0R$. This means that $N_R(X) = a_0R$, where $a_0 \in \text{nil}(R)$.

□

Next, we present a theorem which follows the ideas of the previous results, but this time interested in studying weak annihilators of principal right ideals. Our following result generalizes [OB12], Theorem 2.3.

Theorem 3.1.11. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. If for each principal right ideal $pR \nsubseteq \text{nil}(R)$, $N_R(pR)$ is generated as an ideal by a nilpotent element, then for each principal right ideal $fA \nsubseteq \text{nil}(A)$, $N_A(fA)$ is generated as an ideal by a nilpotent element.*

Proof. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$ with $fA \nsubseteq \text{nil}(A)$. We show that $N_A(fA)$ is generated as an ideal by a nilpotent element. If $a_iR \subseteq \text{nil}(R)$, for all $0 \leq i \leq n$, by Corollary 2.3.21, we have that $fA \subseteq \text{nil}(A)$, which is a contradiction.

So, there exists $0 \leq i \leq n$ such that $a_iR \nsubseteq \operatorname{nil}(R)$. Thus, there exists $c \in \operatorname{nil}(R)$ such that $N_R(a_iR) = cR$. Let us show that $N_A(fA) = cA$. Let $g = b_0 + b_1Y_1 + \cdots + b_mY_m$ and $h = c_0 + c_1Z_1 + \cdots + c_tZ_t$ elements of A. By Theorem 3.1.6, $gch \in \operatorname{nil}(R)A = \operatorname{nil}(A)$ and by Corollary 2.3.21, for every $f \in A$, $fgch \in \operatorname{nil}(A) = \operatorname{nil}(R)A$, since that $\operatorname{nil}(A)$ is an ideal. Therefore, we obtain that $ch \in N_A(fA)$, that is, $cA \subseteq N_A(fA)$. Let $p = p_0 + p_1Y_1 + \cdots + p_mY_m \in N_A(fA)$. Then, we have $fAp \subseteq \operatorname{nil}(A)$, for every $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$. In particular, we have $fRp \subseteq \operatorname{nil}(A)$. Let $r \in R$. Then, we get $rp = rp_0 + rp_1Y_1 \cdots + rp_mY_m$, thus we obtain that

$$frp = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_irp_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(rp_j)X_iY_j + a_ip_{\alpha_i,rp_j}Y_j\right) \in \operatorname{nil}(R)A$$

Therefore, $a_i\sigma^{\alpha_i}(rp_j) \in \operatorname{nil}(R)$ and so $a_irp_j \in \operatorname{nil}(R)$, for every $0 \leq i \leq n$ and $0 \leq j \leq m$. In particular, we obtain that $p_j \in N_R(a_iR) = cR$ and thus there exists $r_j \in R$ such that $p_j = cr_j$ Hence, $p = p_0 + p_1Y_1 + \cdots + p_mY_m = c(r_0 + r_1Y_1 + \cdots + r_mY_m) \in cA$. Therefore, we conclude that $p \in cA$ and so $N_A(fA) \subseteq cA$. □

Remark 3.1.12. We study two examples that illustrate the previous theorem.

Example 3.1.13. Diffusion algebras: Let $\Bbbk$ be a field. The diffusion algebra A generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ, Δ)-compatible. Having in mind that $\Bbbk[x_1, \ldots, x_n]$ is a reduced ring, it follows that $N_{\Bbbk[x_1,\ldots,x_n]}(p(x)\Bbbk[x_1, \ldots, x_n]) = r_{\Bbbk[x_1,\ldots,x_n]}(p(x)\Bbbk[x_1, \ldots, x_n])$, for any principal right ideal, where $p(x) \in \Bbbk[x_1, \ldots, x_n]$. Since $\Bbbk[x_1, \ldots, x_n]$ is a domain, this guarantees that $N_{\Bbbk[x_1,\ldots,x_n]}(p(x)\Bbbk[x_1, \ldots, x_n]) = r_{\Bbbk[x_1,\ldots,x_n]}(p(x)\Bbbk[x_1, \ldots, x_n]) = \{0\}$. Thus, by Theorem 3.1.6, we have that $N_A(fA)$ is generated by a nilpotent element for any principal right ideal $fA \nsubseteq \operatorname{nil}(A)$ with $f \in A$.

Example 3.1.14. Let $\Bbbk$ be a field and $\Bbbk[t]$ the polynomial ring over $\Bbbk$. We consider $\sigma(p(t)) = p(t)$, the identity homomorphism and $\delta(t) = 1$. Since $\Bbbk[t]$ is reduced, we can prove that $\Bbbk[t]$ is a (σ, δ)-compatible ring. Now, let us define ring B as follows:

$$B = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in \Bbbk[t] \right\}$$

The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B \to B$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of B is also extended to $\overline{\delta} : B \to B$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We note that the skew polynomial ring $B[x; \overline{\sigma}, \overline{\delta}]$ is a skew PBW extension over B which is $(\overline{\sigma}, \overline{\delta})$-compatible (see Example 2.3.7). Let $p \in B$, such that $pB \nsubseteq \operatorname{nil}(B)$, where $p = \left(\begin{smallmatrix} p(t) & q(t) \\ 0 & p(t) \end{smallmatrix}\right)$, for some $p(t), q(t) \in \Bbbk[t]$ and $p(t) \neq 0$. By a routine calculations, we obtain

$$N_B(pB) = \left\{ \begin{pmatrix} 0 & q(t) \\ 0 & 0 \end{pmatrix} \mid q(t) \in \Bbbk[t] \right\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B,$$

where $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \operatorname{nil}(B)$. Therefore, it follows from Theorem 3.1.6 that $N_{B[x;\overline{\sigma},\overline{\delta}]}(fB[x; \overline{\sigma}, \overline{\delta}])$ is generated by a nilpotent element for any principal right ideal $fB[x; \overline{\sigma}, \overline{\delta}] \nsubseteq \operatorname{nil}(B[x; \overline{\sigma}, \overline{\delta}])$ with $f \in B[x; \overline{\sigma}, \overline{\delta}]$.

For skew PBW extensions of endomorphism type, we characterize such extensions for which every weak annihilator of a principal right ideal is generated by a nilpotent element. The following result generalizes [OB12], Theorem 2.4.

Theorem 3.1.15. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of endomorphism type, where R is a Σ-compatible NI ring. Then the following statements are equivalent:*

(i) *For each principal right ideal $pR \nsubseteq \operatorname{nil}(R)$, $N_R(pR)$ is generated as an ideal by a nilpotent element.*

(ii) *For each principal right ideal $fA \nsubseteq \operatorname{nil}(A), N_A(fA)$ is generated as an ideal by a nilpotent element.*

Proof. By Theorem 3.1.11, it suffices to show (ii) $\Rightarrow$ (i). Let pR be a principal right ideal of R with $pR \nsubseteq \operatorname{nil}(R)$. In particular, we have that $pR \nsubseteq \operatorname{nil}(A)$. Therefore, there exists $f = a_0 + a_1X_1 + \cdots + a_mX_m \in \operatorname{nil}(A)$ such that $N_A(pR) = fA$. With this in mind, let us note that $f = a_0 + a_1X_1 + \cdots + a_mX_m \in \operatorname{nil}(A)$, whence $a_i \in \operatorname{nil}(R)$, for all $0 \leq i \leq m$ by Corollary 2.3.21. We may assume that $a_0 \neq 0$. Let us show that $N_R(pR) = a_0R$. Since $a_0 \in \operatorname{nil}(R)$ and $\operatorname{nil}(R)$ is an ideal of R, we obtain $pra_0R \subseteq \operatorname{nil}(R)$, for each $r \in R$. Thus $a_0R \subseteq N_R(pR)$. If $m \in N_R(pR)$, then $m \in N_A(pR)$. Thus, there exists $g = b_0 + b_1Y_1 + \cdots + b_nY_n \in A$ such that

$$m = fg = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ib_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)X_iY_j\right)$$

Hence, we have $m = a_0b_0 \in a_0R$, whence $N_R(pR) \subseteq a_0R$. Therefore, we conclude that $N_R(pR) = a_0R$ where $a_0 \in \operatorname{nil}(R)$. □

Finally, we study the weak annihilator of an element. The following theorem establishes conditions for which the weak annihilator of an element in a skew PBW extension is generated by a nilpotent element. Our result generalizes [OB12], Theorem 2.5.

Theorem 3.1.16. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. If for each $p \notin \operatorname{nil}(R)$, $N_R(p)$ is generated as an ideal by a nilpotent element, then for each $f \notin \operatorname{nil}(A)$, $N_A(f)$ is generated as an ideal by a nilpotent element.*

Proof. Let $f = a_0 + a_1X_1 + \cdots + a_nX_n \in A$ with $f \notin \operatorname{nil}(A)$. We show that $N_A(f)$ is generated as an ideal by a nilpotent element. If $a_iR \subseteq \operatorname{nil}(R)$ for all $0 \leq i \leq n$, the by Corollary 2.3.21, we see that $f \in \operatorname{nil}(A)$, which is a contradiction. Hence, there exist $0 \leq i \leq n$ such that $a_i \notin \operatorname{nil}(R)$. Thus, there exists $c \in \operatorname{nil}(R)$ such that $N_R(a_i) = cR$. The idea is to show that $N_A(f) = cA$. If $h = c_0 + c_1Z_1 + \cdots + c_tZ_t \in A$, by Proposition 3.1.6, we have $fch \in \operatorname{nil}(R)A = \operatorname{nil}(A)$. Therefore, we obtain that $ch \in N_A(f)$.

Let $p = p_0 + p_1Y_1 + \cdots + p_mY_m \in N_A(f)$. Then, $fp \in \text{nil}(A)$ for $f = a_0 + \cdots + a_nX_n \in A$. Thus, we obtain that

$$fp = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ip_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(p_j)X_iY_j + a_ip_{\alpha_i,p_j}Y_j\right) \in \text{nil}(R)A$$

Therefore, $a_i\sigma^{\alpha_i}(p_j) \in \text{nil}(R)$ and so $a_ip_j \in \text{nil}(R)$ for every $0 \leq i \leq n$ and $0 \leq j \leq m$. In particular, we obtain that $p_j \in N_R(a_i) = cR$ and thus there exist $r_j \in R$ such that $p_j = cr_j$. Hence, $p = p_0 + p_1Y_1 + \cdots + p_mY_m = c(r_0 + r_1Y_1 + \cdots + r_mY_m) \in c \cdot A$. Therefore, it follows that $p \in cA$. □

Remark 3.1.17. We establish the following examples that illustrate the previous results.

Example 3.1.18. Diffusion algebras: Let $\Bbbk$ be a field. The diffusion algebra A is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ, Δ)-compatible. Having in mind that $\Bbbk[x_1, \ldots, x_n]$ is reduced, we have that $N_{\Bbbk[x_1,\ldots,x_n]}(p(x)) = r_{\Bbbk[x_1,\ldots,x_n]}(p(x))$, for any $p(x) \in \Bbbk[x_1, \ldots, x_n]$. Since $\Bbbk[x_1, \ldots, x_n]$ is a domain, this guarantees that $N_{\Bbbk[x_1,\ldots,x_n]}(p(x)) = r_{\Bbbk[x_1,\ldots,x_n]}(p(x)) = \{0\}$. It follows from Theorem 3.1.6 that $N_A(f)$ is generated by a nilpotent element for any $f \in A$, where $f \notin \text{nil}(A)$.

Example 3.1.19. Let $\Bbbk$ be a field and $\Bbbk[t]$ the polynomial ring over $\Bbbk$. We consider $\sigma(p(t)) = p(t)$, the identity homomorphism and $\delta(t) = 1$. Since $\Bbbk[t]$ is reduced, we can show that $\Bbbk[t]$ is (σ, δ)-compatible ring. Now, let us define ring B as follows:

$$B = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in \Bbbk[t] \right\}$$

The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B \to B$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of B is also extended to $\overline{\delta} : B \to B$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We note that the skew polynomial ring $B[x; \overline{\sigma}, \overline{\delta}]$ is a skew PBW extension over B which is $(\overline{\sigma}, \overline{\delta})$-compatible (see Example 2.3.7). Let $p \in B$, such that $p \notin \text{nil}(B)$, where $p = \left(\begin{smallmatrix} p(t) & q(t) \\ 0 & p(t) \end{smallmatrix}\right)$, for some $p(t), q(t) \in \Bbbk[t]$ and $p(t) \neq 0$. Again, we can prove that

$$N_B(p) = \left\{ \begin{pmatrix} 0 & q(t) \\ 0 & 0 \end{pmatrix} \mid q(t) \in \Bbbk[t] \right\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B,$$

where $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \text{nil}(B)$. Hence, it follows from Theorem 3.1.6 that $N_{B[x;\overline{\sigma},\overline{\delta}]}(f)$ is generated by a nilpotent element for any element $f \in B[x; \overline{\sigma}, \overline{\delta}]$, where $f \notin \text{nil}(B[x; \overline{\sigma}, \overline{\delta}])$.

Given a skew PBW extension of endomorphism type over a ring R, the following theorem characterizes extensions for which every weak annihilator of an element is generated by a nilpotent element. Our result generalizes [OB12], Theorem 2.6.

Theorem 3.1.20. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of endomorphism type, where R is a Σ-compatible NI ring. Then the following statements are equivalent:*

(i) *For each $p \notin \operatorname{nil}(R)$, $N_R(p)$ is generated as an ideal by a nilpotent element.*

(ii) *For each $f \notin \operatorname{nil}(A)$, $N_A(f)$ is generated as an ideal by a nilpotent element.*

Proof. By Theorem 3.1.16, it suffices to show (ii) $\Rightarrow$ (i). Let $p \in R$ with $p \notin \operatorname{nil}(R)$, whence $p \notin \operatorname{nil}(A)$. Thus, there exists $f = a_0 + a_1X_1 + \cdots + a_mX_m \in \operatorname{nil}(A)$ such that $N_A(p) = fA$. We note that $f = a_0 + a_1X_1 + \cdots + a_mX_m \in \operatorname{nil}(A)$ and so $a_i \in \operatorname{nil}(R)$, for all $0 \leq i \leq m$ by Corollary 2.3.21. We may assume that $a_0 \neq 0$. Now, we show that $N_R(p) = a_0R$. Since $a_0 \in \operatorname{nil}(R)$ and $\operatorname{nil}(R)$ is an ideal of R, we obtain $pa_0R \subseteq \operatorname{nil}(R)$, that is, $a_0 \cdot R \subseteq N_R(p)$. If $m \in N_R(p)$, then, we get that $m \in N_A(p)$. Thus, there exists $g = b_0 + b_1Y_1 + \cdots + b_nY_n \in A$ such that

$$m = fg = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ib_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)X_iY_j\right)$$

Hence, we have $m = a_0b_0 \in a_0R$, and thus $N_R(p) \subseteq a_0R$. Therefore, we conclude that $N_R(p) = a_0R$ where $a_0 \in \operatorname{nil}(R)$. □

3.2 Nilpotent associated primes

Over the years, the study of associated prime ideals has been an important tool in the study of results and properties in areas such as algebra and geometry. Given a right R-module N_R, we recall that the **right annihilator** of N_R is denoted by $r_R(N_R) = \{a \in R \mid Na = 0\}$. We say that N_R is **prime**, if $N_R \neq 0$ and $r_R(N_R) = r_R(N_R')$, for every nonzero submodule $N_R' \subseteq N_R$ (see [Ann04], Definition 1.1). Let M_R be a right R-module. An ideal $\mathscr{P}$ of R is called an **associated prime** of M_R, if there exists a prime submodule $N_R \subseteq M_R$ such that $\mathscr{P} = r_R(N_R)$. The set of associated primes of M_R is denoted by $\operatorname{Ass}(M_R)$ (see [Ann04], Definition 1.2). Associated primes are well known in commutative algebra for their important role in the primary decomposition. Brewer and Heinzer [BH74] proved that, over a commutative ring R, the associated primes ideals of the polynomial ring $R[x]$ are all extended: that is, every $\mathscr{P} \in \operatorname{Ass}(R[x])$ may be expressed as $\mathscr{P} = \mathscr{P}_0[x]$, where $\mathscr{P}_0 = \mathscr{P} \cap R \in \operatorname{Ass}(R)$. Faith [Fai00] presented proof of this same fact using different tools.

Annin [Ann04] extended the result presented by them, making a description of the associated prime ideals for skew polynomial modules. Ouyang and Birkenmeier [OB12] introduced the notion of nilpotent associated primes, which are a generalization of associated primes. They describe all nilpotent associated primes of the Ore extension ring $R[x;\sigma,\delta]$ in terms of the nilpotent associated primes of the ring R. Finally, Ramirez [Ram19] and [NRR20] investigated these ideals in a more general structure, establishing a characterization of the associated prime ideals in a skew PBW extension over an R-module, generalizing several important results presented by Annin [Ann04].

We note that the characterization of the associated primes ideals presented by [Ram19] is performed for polynomial type structures on two classes of rings: weak Σ-rigid and (Σ, Δ)-compatible rings.

As we saw earlier, the study of associated prime ideals in noncommutative polynomial type structures has been of great interest. Motivated by the results in [BH74],[Fai00], [Ann04] [OB12] and [Ram19], we first study the notion of nilpotent associated primes, which are a generalization of associated primes. We next describe all nilpotent associated primes of a skew PBW extension in terms of the nilpotent associated primes of its ring of coefficients. This part of the work has the purpose of giving a description of the nilpotent associated prime ideals in skew PBW extensions, generalizing the results presented by [OB12] and, in a certain sense, also the results presented in [Ram19].

Now, we present two central concepts of this part of our work.

Definition 3.2.1. ([OB12], Definition 3.1) Let I be a right ideal of a nonzero ring R. We say that I is a **right quasi-prime ideal**, if $I \nsubseteq \operatorname{nil}(R)$ and $N_R(I) = N_R(I')$, for every right ideal $I' \subseteq I$ and $I' \nsubseteq \operatorname{nil}(R)$.

Definition 3.2.2. ([OB12], Definition 3.2) Let $\operatorname{nil}(R)$ be an ideal of a ring R. An ideal $\mathscr{P}$ of R is called a **nilpotent associated prime** of R, if there exist a right quasi-prime ideal I such that $\mathscr{P} = N_R(I)$. The set nilpotent associated primes of R is denoted by $\operatorname{NAss}(R)$.

Let $m = m_0 + m_1X_1 + \cdots + m_kX_k + \cdots + m_nX_n \notin \operatorname{nil}(R)A$. Let be x^{α_k} leading monomial of X_k, with $X_n \succ X_{n-1} \succ \cdots \succ X_1$ and leading coefficient $m_k \neq 0$. If $m_k \notin \operatorname{nil}(R)$ and $m_i \in \operatorname{nil}(R)$, for all $i > k$, then we say that the **nilpotent degree** of m is k. To simplify notations, we write $\operatorname{Ndeg}(m)$ for the nilpotent degree of m. If $m \in \operatorname{nil}(R)A$, then we define $\operatorname{Ndeg}(m) = -1$.

Now, we present a new definition following the ideas of [OB12], Definition 3.3, where it is established for skew polynomial ring over (σ, δ)-compatible rings. This for the purpose of studying this notion in skew PBW extension.

Definition 3.2.3. Let R be a (Σ, Δ)-compatible ring and A a skew PBW extension of R. Let $m = m_0 + m_1X_1 + \cdots + m_kX_k + \cdots + m_nX_n \notin \operatorname{nil}(R)A$ and the nilpotent degree of m be k. If $N_R(m_k) \subseteq N_R(m_i)$, for all $i \leq k$, then we say m is a **nilpotent good polynomial**.

The following result extends [OB12], Lemma 3.1.

Theorem 3.2.4. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, where R is a (Σ, Δ)-compatible NI ring. For any $m = m_0 + m_1X_1 + \cdots + m_kX_k + \cdots + m_nX_n \notin \operatorname{nil}(R)A$, there exists $r \in R$ such that mr is a nilpotent good polynomial.*

Proof. Let us assume the result is false and let $m = m_0 + \cdots + m_kX_k + \cdots + m_nX_n \notin \operatorname{nil}(A)$ be a counterexample of minimal nilpotent degree $\operatorname{Ndeg}(m) = k$, that is, mr is not a nilpotent good polynomial. In particular, if $r = 1$, we have that m is not a nilpotent good polynomial. Hence, there exists $i < k$ such that $N_R(m_k) \nsubseteq N_r(m_i)$. Thus, we can find $b \in R$ such that $b \in N_R(m_k)$ and $b \notin N_R(m_i)$, whence $m_ib \notin \operatorname{nil}(R)$ and $m_kb \in \operatorname{nil}(R)$.

Note that the degree k coefficient of mb is $m_k\sigma^{\alpha_k}(b) + \sum_{i=k+1}^{n} m_i p_{\alpha_i,b}$. Now, $m_k\sigma^{\alpha_k}(b) \in \text{nil}(R)$ due to the (Σ,Δ)-compatibility of R. On the other hand, we have $\text{Ndeg}(m) = k$, thus $m_i \in \text{nil}(R)$, for all $i > k$. This means that $m_i p_{\alpha_i,b} \in \text{nil}(R)$, for all $i > k$ and so $\sum_{i=k+1}^{n} m_i p_{\alpha_i,b} \in \text{nil}(R)$, since $\text{nil}(R)$ is an ideal of R. Therefore, mb has nilpotent degree at most $k-1$. Since $m_i b \notin \text{nil}(R)$, we have $mb \notin \text{nil}(R)A$. By the minimality of k, we have that there exist $c \in R$ with mbc nilpotent good polynomial. But this contradicts the fact that m is a counterexample, since $bc \in R$ and $m(bc)$ is nilpotent good polynomial which is a contradiction. □

The following theorem characterizes the nilpotent associated primes ideals in a skew PBW extension over a (Σ,Δ)-compatible ring. This result was obtained in the realization of this work and generalizes [OB12], Theorem 3.1.

Theorem 3.2.5. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension, where R is a (Σ,Δ)-compatible NI ring. Then*

$$\text{NAss}(A) = \{\mathscr{P}A \mid \mathscr{P} \in \text{NAss}(R)\}.$$

Proof. With the aim of establishing the desired equality, we proof the two implications.

- Let $\mathscr{P} \in \text{NAss}(R)$. By Definition 3.2.2, there exists a right ideal $I \nsubseteq \text{nil}(R)$ with I a right quasi-prime ideal of R and $\mathscr{P} = N_R(I)$. Let us show that $\mathscr{P}A = N_A(IA)$ and also that IA is a quasi-prime ideal. We show first that $\mathscr{P}A = N_A(IA)$. Let $i = a_0 + a_1X_1 + \cdots + a_mX_m \in IA$ and let $f = b_0 + b_1Y_1 + \cdots + b_nY_n \in \mathscr{P}A$. Then,

$$if = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_iX_ib_jY_j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\sigma^{\alpha_i}(b_j)X_iY_j + a_ip_{\alpha_i,b_j}Y_j\right)$$

  Since $a_ib_j \in \text{nil}(R)$ for every i,j, Proposition 2.3.15 implies that $a_i\sigma^{\alpha_i}(b_j) \in \text{nil}(R)$, Furthermore, the polynomial p_{α_i,b_j} involves elements obtained evaluating σ's and δ's (depending on the coordinates of α_i) in the element b_j by Remark 2.1.9. This means that $a_ip_{\alpha_i,b_j} \in \text{nil}(R)$ for every i,j and therefore, we obtain $if \in \text{nil}(R)A = \text{nil}(A)$. Hence, $\mathscr{P}A \subseteq N_A(IA)$. Conversely, let $f = b_0 + b_1Y_1 + \cdots + b_mY_m \in N_A(IA)$, then $if \in \text{nil}(A) = \text{nil}(R)A$, for all $i = a_0 + a_1X_1 + \cdots + a_nX_n \in IA$, whence $a_i\sigma^{\alpha_i}(b_j) \in \text{nil}(R)$, which implies that $a_ib_j \in \text{nil}(R)$, by Proposition 2.3.15. Since $a_i \in I$, we have $b_j \in N_R(I) = \mathscr{P}$. Therefore, $f \in \mathscr{P}A$ and thus $N_A(IA) \subseteq \mathscr{P}A$. Hence, we conclude that $\mathscr{P}A = N_A(IA)$.

  Since the ideal I is a right quasi-prime ideal, we have $I \nsubseteq \text{nil}(R)$. Thus, $IA \nsubseteq \text{nil}(A)$. Let us show that for any right ideal U of R it is satisfied that, if $U \nsubseteq \text{nil}(A)$ and $U \subseteq I \cdot A$, then $N_A(U) = N_A(IA)$. Let us see first that $N_A(IA) \subseteq N_A(U)$. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in N_A(IA)$, this means that $if \in \text{nil}(A)$, for all $i \in IA$. In particular, since $U \subseteq IA$, then $if \in \text{nil}(A)$, for all $i \in U$, whence $f \in N_A(U)$. Hence, we have $N_A(IA) \subseteq N_A(U)$.

Conversely, let $C_U \subseteq R$ consisting of all coefficients of elements of U. Let us first consider $\mathscr{P}'$ the right ideal of R generated by C_U. Since $U \nsubseteq \operatorname{nil}(A) = \operatorname{nil}(R)A$, this means that $C_U \nsubseteq \operatorname{nil}(R)$, and hence $\mathscr{P}' \subseteq I$ and $\mathscr{P}' \nsubseteq \operatorname{nil}(R)$. Thus, as I is a right quasi-prime ideal this implies that $N_R(\mathscr{P}') = N_R(I) = \mathscr{P}$. Now, if $f = a_0 + a_1X_1 + \cdots + a_mX_m \in N_A(U)$ and $u = u_0 + u_1Y_1 + \cdots + u_nY_n \in U$, then $uf \in \operatorname{nil}(A)$, whence $u_i\sigma^{\beta_j}(a_j) \in \operatorname{nil}(R)$ and therefore $u_ia_j \in \operatorname{nil}(R)$, for all $0 \leq i \leq n$, $0 \leq j \leq m$. Since $\operatorname{nil}(R)$ of NI ring is an ideal, $u_ia_j \in \operatorname{nil}(R)$ implies $a_ju_i \in \operatorname{nil}(R)$ and thus $u_iRa_j \in \operatorname{nil}(R)$ gives that $(u_iRa_j)^2 \in \operatorname{nil}(R)$. Therefore, we obtain

$$a_j \in N_R(\mathscr{P}') = N_R(I) = \mathscr{P} \text{ for all } 0 \leq j \leq m.$$

Let $i = b_0 + b_1Z_1 + \cdots + b_rZ_r \in IA$. Then, we have that $b_ma_j \in \operatorname{nil}(R)$ and thus $b_m\sigma^{\alpha}(\delta^{\beta}(a_j))$ and $a_m\delta^{\beta}(\sigma^{\alpha}(a_j))$ are elements of $\operatorname{nil}(R)$, for every $\alpha, \beta \in \mathbb{N}^n$. Therefore, $if \in \operatorname{nil}(R)A = \operatorname{nil}(A)$, this means that $f \in N_A(IA)$. Hence, we conclude $N_A(U) \subseteq N_A(IA)$. Thus, we have proved that $\mathscr{P}A = N_A(IA)$ and also that IA is a quasi-prime ideal.

- Let $I \in \operatorname{NAss}(A)$. By Definition 3.2.2, there exists a right ideal $J \nsubseteq \operatorname{nil}(A)$ with J a right quasi-prime ideal of A and $I = N_A(J)$. Let $m = m_0 + m_1X_1 + \cdots + m_kX_k + \cdots + m_nX_n \notin \operatorname{nil}(A) = \operatorname{nil}(R)A$ and $m \in J$. Since $J \nsubseteq \operatorname{nil}(A)$, we may assume that m is nilpotent good and $\operatorname{Ndeg}(m) = k$, by Theorem 3.2.4. We consider $J_0 = mA$ the principal right ideal of A generated by m. Since $m \notin \operatorname{nil}(A) = \operatorname{nil}(R)A$ this implies that $J_0 = mA \nsubseteq \operatorname{nil}(R)A = \operatorname{nil}(A)$, whence $N_A(J) = N_A(J_0) = N_A(mA) = I$ because J is quasi-prime ideal. Now, we consider the right ideal m_kR, and let us denote $U = N_R(m_kR)$.

  Let us prove first that $I = UA$. Let $g = b_0 + b_1Y_1 + \cdots + b_lY_l \in UA$. Since $b_j \in U$, then $m_kRb_j \in \operatorname{nil}(R)$ for all $0 \leq j \leq l$. Furthermore, m is nilpotent good polynomial and $\operatorname{Ndeg}(m) = k$, hence $m_iRb_j \in \operatorname{nil}(R)$, for all $0 \leq i \leq k$, and $0 \leq j \leq l$. On the other hand, for all $i > k$, $m_i \in \operatorname{nil}(R)$. Thus, we get $m_iRb_j \in \operatorname{nil}(R)$, for all $0 \leq i \leq n$ and $0 \leq j \leq l$. Now, for any element $h \in A$ where $h = h_0 + h_1Z_1 + \cdots + h_pZ_p$, it is satisfied that $m_ih_db_j \in \operatorname{nil}(R)$, for all $0 \leq i \leq n$, $0 \leq d \leq p$ and $0 \leq j \leq l$. Therefore, by (Σ, Δ)-compatibility of R, we obtain $mhg \in \operatorname{nil}(A)$. This implies that $g \in N_A(mA) = I$, and so $UA \subseteq I$. Conversely, let $g = b_0 + b_1Y_1 + \cdots + b_lY_l \in I$. Since $mRg \in \operatorname{nil}(A)$, it follows that $m_iRb_j \in \operatorname{nil}(R)$, for all $0 \leq i \leq n$, and $0 \leq j \leq l$. Thus, we have that $b_j \in N_R(m_kR)$, for all $0 \leq j \leq l$, and so $g \in UA$. Hence, we conclude $I \subseteq UA$ which implies that $I = UA$.

  Now, let us show that m_kR is a quasi-prime ideal. Let m_kR the principal right ideal of A generated by m_k. Since $m_k \notin \operatorname{nil}(R)$, we have $m_kR \nsubseteq \operatorname{nil}(R)$. The idea is to show that for any right ideal $Q \subseteq m_kR$ with $Q \nsubseteq \operatorname{nil}(R)$, it follows that $N_R(Q) = N_R(m_kR)$. Assume that a right ideal $Q \subseteq m_kR$, and $Q \nsubseteq \operatorname{nil}(R)$. Then $N_R(m_kR) \subseteq N_R(Q)$ by Proposition 3.1.4. Now, we show that $N_R(Q) \subseteq N_R(m_kR)$. Let $\mathscr{W}$ be the following set $\mathscr{W} = \{mr \mid r \in Q\}$, and let $\mathscr{W}A$ the right ideal of A generated by $\mathscr{W}$.

First, we note that $\mathscr{W}A \subseteq mA$. Since $Q \nsubseteq \text{nil}(R)$, there exists $a \in R$ such that $m_k a \in Q$ and $m_k a \notin \text{nil}(R)$. If $m_k(m_k a) \in \text{nil}(R)$, then we have $m_k a \in \text{nil}(R)$ which contradicts to the fact that $m_k a \notin \text{nil}(R)$. Thus $m_k(m_k a) \notin \text{nil}(R)$, and hence $m(m_k a) \notin \text{nil}(A)$, by Corollary 2.3.22. This implies that $\mathscr{W}A \nsubseteq \text{nil}(A)$. Since J is quasi-prime ideal, we obtain $N_A(\mathscr{W}A) = N_A(mA) = I$.

Suppose $q \in N_R(Q)$. Then $rq \in \text{nil}(R)$, for each $r \in Q$. Now, for any $mrf \in \mathscr{W}A$ where $f = a_0 + a_1Y_1 + \cdots + a_lY_l \in A$, the term of mrf is $m_iX_ira_jY_j$. The idea is to show that $m_iX_ira_jY_j \in \text{nil}(R)$. Since $rq \in \text{nil}(R)$ and $\text{nil}(R)$ is an ideal, it follows that

$$rq \in \text{nil}(R) \Rightarrow qr \in \text{nil}(R) \Rightarrow ra_j(qr)a_jq \in \text{nil}(R) \Rightarrow ra_jq \in \text{nil}(R).$$

If $ra_jq \in \text{nil}(R)$, then $m_ira_jq \in \text{nil}(R)$. Thus, due to the (Σ, Δ)-compatibility of R, we have that $m_iX_ira_jY_jq \in \text{nil}(R)A$ which implies that $mrfq \in \text{nil}(R)A = \text{nil}(A)$. Hence, for any $\sum mr_if_i \in \mathscr{W}A$ it follows that $\sum(mr_if_i)q \in \text{nil}(A)$. Therefore, $q \in N_A(\mathscr{W}A) = I = UA$, and so $q \in U = N_R(m_kR)$. So $N_R(Q) \subseteq N_R(m_kR)$, and this implies that $N_R(Q) = N_R(m_kR)$. Hence, we conclude that m_kR is quasi-prime ideal.

□

Remark 3.2.6. We present two examples that illustrate the above theorem.

Example 3.2.7. Diffusion algebras: Let $\Bbbk$ be a field. The diffusion algebra A is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over $\Bbbk$ is a skew PBW extension over $\Bbbk[x_1, \ldots, x_n]$ which is (Σ, Δ)-compatible with $\Sigma = \{\sigma_i\}$ where $\sigma_j(x_i) = x_i$, for every $1 \leq i, j \leq n$ and $\Delta = \{\delta_i\}$ where $\delta_j(x_i) = 0$, for every $1 \leq i, j \leq n$. From Theorem 3.2, it follows that $\text{NAss}(A) = \text{NAss}(\Bbbk[x_1, \ldots, x_n])A$. Now, we must study the nilpotent associated primes of $\Bbbk[x_1, \ldots, x_n]$. Note that $\Bbbk[x_1, \ldots, x_n]$ is a skew PBW extension over $\Bbbk$ which is trivially (Σ, Δ)-compatible. Thus, $\text{NAss}(\Bbbk[x_1, \ldots, x_n]) = \text{NAss}(\Bbbk)[x_1, \ldots, x_n]$. Since $\Bbbk$ is a field, we have $\text{nil}(\Bbbk) = \{0\}$ and so the only quasi-prime ideal of $\Bbbk$ is the same. Hence, if $\mathscr{P} \in \text{NAss}(\Bbbk)$, then the following equalities are satisfied: $\mathscr{P} = N_{\Bbbk}(\Bbbk) = r_{\Bbbk}(\Bbbk) = \{0\}$. This implies that $\text{NAss}(\Bbbk[x_1, \ldots, x_n]) = \text{NAss}(\Bbbk)[x_1, \ldots, x_n] = \{0\}$ whence it follows that $\text{NAss}(A) = \text{NAss}(\Bbbk[x_1, \ldots, x_n])A = \{0\}$.

Example 3.2.8. Let $\Bbbk$ be a field and $\Bbbk[t]$ the polynomial ring over $\Bbbk$. We consider $\sigma(p(t)) = p(t)$, the identity homomorphism and $\delta(t) = 1$. Since $\Bbbk[t]$ is reduced, it can be shown that $\Bbbk[t]$ is (σ, δ)-compatible ring. Now, let us define ring B as follows:

$$B = \left\{ \begin{pmatrix} p(t) & q(t) \\ 0 & p(t) \end{pmatrix} \mid p(t), q(t) \in \Bbbk[t] \right\}$$

The endomorphism σ of B is extended to the endomorphism $\overline{\sigma} : B \to B$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of B is also extended to $\overline{\delta} : B \to B$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We note that the skew polynomial ring $B[x; \overline{\sigma}, \overline{\delta}]$ is a skew PBW extension over B which is $(\overline{\sigma}, \overline{\delta})$-compatible (see Example 2.3.7).

It follows from Theorem 3.2 that $\mathrm{NAss}(B[x;\overline{\sigma},\overline{\delta}]) = \mathrm{NAss}(B)[x;\overline{\sigma},\overline{\delta}]$. We can prove that the different right ideals of B are:

$$I_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} B, \qquad I_2 = \begin{pmatrix} 0 & 0 \\ 0 & q(t) \end{pmatrix} B, \qquad I_3 = \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} B,$$

$$I_4 = \begin{pmatrix} 0 & p(t) \\ 0 & q(t) \end{pmatrix} B, \qquad I_5 = \begin{pmatrix} q(t) & p(t) \\ 0 & 0 \end{pmatrix} B, \qquad I_6 = \begin{pmatrix} q(t) & p(t) \\ 0 & s(t) \end{pmatrix} B,$$

for some $p(t), q(t)$ in $\Bbbk[t]$. Additionally, we can see by doing routine calculations that the only quasi-prime ideals of B are: I_2, I_4, I_5 and I_6. Furthermore, it is not difficult to see that the following equalities are satisfied $N_R(I_2) = N_R(I_4) = N_R(I_5) = N_R(I_6) = I_3$, where $I_3 = \mathrm{nil}(B)$. Thus, it follows that $\mathrm{NAss}(B) = \{\mathrm{nil}(B)\} = \{I_3\}$. Therefore, we conclude that $\mathrm{NAss}(B[x;\overline{\sigma},\overline{\delta}]) = \mathrm{nil}(B)[x;\overline{\sigma},\overline{\delta}] = I_3[x;\overline{\sigma},\overline{\delta}]$.

APPENDIX A

Appendix

A.1 Classes of rings

In this section, we study some classes of rings that have taken great importance recently in the study of noncommutative rings of polynomial type. We also study Armendariz rings introduced in [CR97] which generalize fields and integral domains. These notions are of great importance throughout our work. In this section R is considered as an associative ring with unity. The following definitions are adapted from [Mar03].

Definition A.1.1 ([Sza19], Definition 2.1)**.** Let R be a ring. We say that R is:

(i) **Reduced**, if for all $a \in R$, $a^2 = 0$ implies $a = 0$.

(ii) **Symmetric**, if for all $a, b, c \in R$, $abc = 0$ implies $bac = 0$.

(iii) **Reversible**, if for all $a, b \in R$, $ab = 0$ implies $ba = 0$.

(iv) **Semicommutative**, if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$.

(v) **Abelian**, if each idempotent of R is central (an element $a \in R$ is **central**, if $ab = ba$, for all $b \in R$).

Next, we present some results that relate these different classes of rings. The details of these results and interesting examples can be found in [Mar03].

Proposition A.1.2 ([AC99], Theorem I.3)**.** *Let R be a ring. If R is a reduced ring, then R is symmetric.*

Proof. Suppose that R is reduced and let $a, b, c \in R$ such that $abc = 0$. Let us show that $bac = 0$. Since $abc = 0$, we have $c(abc)ab = 0$; then $(cab)^2 = 0$, and since R is reduced, $cab = 0$. Again, since $cba = 0$, we have $aba(cab)ac = (abac)^2 = 0$ and so $abac = 0$, using the reducibility if R. Now, since $abac = 0$, then $bacb(abac)ba = (bacba)^2 = 0$ and so $(bacba) = 0$. And finally, since $bacba = 0$, we have $(bacba)c = (bac)^2 = 0$ and as R is reduced, we have $bac = 0$. □

The class of symmetric rings contains properly the class of reduced rings. The following example can be found in [HKKL05], Theorem 2.3.

Example A.1.3. Let R be a ring and n any positive integer. According to [HKKL05], Theorem 2.3, if R is reduced, then $\frac{R[x]}{\langle x^n \rangle}$ is a symmetric ring, where $\langle x^n \rangle$ is the ideal generated by x^n. In particular, if $\Bbbk$ is a field, $\frac{\Bbbk[x]}{\langle x^2 \rangle}$ is symmetric but is not reduced, since $\overline{x}$ is a nilpotent element with $\overline{x} \neq \overline{0}$.

Proposition A.1.4 ([Kwa07], Corollary 2.6)**.** *Let R be a ring. If R is a symmetric ring, then R is a reversible ring.*

Proof. Suppose R is a symmetric ring and consider $ab = 0$ with $a, b \in R$. Let us show that $ba = 0$. Since $ab = 0$, then $(ab)r = abr = 0$, for all $r \in R$. As R is symmetric, this implies $bar = 0$. In particular, for $r = 1$, we have $ba = 0$. Therefore, R is a reversible ring. □

Reversible rings contains properly the class of symmetric rings. The following example can be found in [Mar03] and [Mar02].

Example A.1.5 ([Mar02], Example 5)**.** Let $\Bbbk$ be a field. We consider the $\Bbbk$-algebra free $F = \Bbbk \langle x, y, z \rangle$, and let

$$I = (FxF)^2 + (FyF)^2 + (FzF)^2 + FxyzF + FyzxF + FzxyF \subset F.$$

In [Mar02], it is shown that $R = F/I$ is a reversible ring which not symmetric ring.

Proposition A.1.6 ([KL03], Lemma 1.4)**.** *Let R be a ring. If R is a reversible ring, then R is a semicommutative ring.*

Proof. Suppose R is a reversible ring and let $ab = 0$. Let us show that $aRb = 0$, i.e, $arb = 0$, for all $r \in R$. Since $ab = 0$ and R is reversible, we have $ba = 0$. Now, $r(ba) = 0$, for all $r \in R$, and hence $(rb)a = r(ba) = 0$. Thus, we have $a(rb) = arb = 0$, since R is reversible. Therefore, $aRb = 0$, i.e, R is semicommutative ring. □

The class of semicommutative rings contains properly the class of reversible rings. The following example can be found in [Mar03].

Example A.1.7 ([Mar03], Example 3.11)**.** Let S be any reduced ring, and fix any $n \in \mathbb{N}$. Let

$$R = S \langle x, y \rangle / \langle x^{n+1}, y^{n+1}, yx \rangle.$$

In [Mar03], it is shown that R is a semicommutative ring which is not reversible ring.

Proposition A.1.8 ([BK10], Corollary 3.6)**.** *Let R be a ring. If R is a semicommutative ring, then R is Abelian.*

Proof. Suppose that R is a semicommutative ring and let e be an idempotent element of R. Since $e(e-1) = e^2 - e = 0$, we have $er(e-1) = 0$, for all $r \in R$, by the semicommutativity of R, i.e, $ere = er$. Also, since $1 - e$ is an idempotent element, then $(1-e)^2 - 1 + e = 0$. Hence, we have $(e-1)e = 0$, and therefore $(e-1)re = 0$, for all $r \in R$, i.e, $ere = re$. Thus, $er = ere = re$, and so e is central. □

The class of Abelian rings contains properly the class of semmicommutative rings. The following example can be found in [HLS02].

Example A.1.9 ([HLS02], Example 14)**.** Let $\Bbbk$ be a field and $R = \Bbbk\langle x, y, z\rangle$ be the free algebra of polynomials with zero constant terms in indeterminates x, y, z over $\Bbbk$ subject to relations $xy - yx \neq 0$, $xz - zx \neq 0$ and $yz - zy \neq 0$. Since there are no constant terms, R is a ring without identity and so we consider an ideal of $\Bbbk$ de R, say $I = \langle zz, xz, zrz\rangle$, for all $r \in R$. Let $S = (\Bbbk + R)/I$. We claim that S is an Abelian ring which not semicommutative ring.

Given the propositions studied above, we obtain the following implications that represents the relations between these rings.

$$\text{Reduced} \Rightarrow \text{Symmetric} \Rightarrow \text{Reversible} \Rightarrow \text{Semicommutative} \Rightarrow \text{Abelian}.$$

Now, we show an interesting connection between these different classes of rings and the Armendariz rings introduced in [CR97].

Definition A.1.10 ([CR97], Definition 1.1)**.** A ring R is an **Armendariz ring**, if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$, for every i and j.

Now, we present the relation between reduced and Armendariz rings.

Proposition A.1.11 ([Arm74], Lemma 1)**.** *Let R be a ring. If R is a reduced ring, then R is Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Suppose that $f(x)g(x) = 0$ and R a reduced ring. Moreover, we can assume that $n = m$. Making the usual product of $f(x)$ with $g(x)$, we obtain the system of equations:

$$\begin{aligned} a_0 b_0 &= 0 \\ a_1 b_0 + a_0 b_1 &= 0 \\ \vdots \qquad &\quad \vdots \\ a_n b_0 + \cdots + a_0 b_n &= 0. \end{aligned}$$

Since R is a reduced ring, we have $a_0b_0 = 0$, and so $b_0a_0 = 0$. Now, left multiplying the second of these equations by b_0, we have $b_0a_1b_0 = 0$ and so $(a_1b_0)^2 = 0$, and this implies $a_1b_0 = 0$. Analogously, we have $a_ib_0 = 0$, for all $1 \leq i \leq n$. Therefore, the previous system of equations is reduced to the following:

$$\begin{aligned} a_0b_1 &= 0 \\ a_1b_1 + a_0b_2 &= 0 \\ \vdots \qquad &\quad \vdots \\ a_{n-1}b_1 + \cdots + a_0b_n &= 0. \end{aligned}$$

Again, since $a_0b_1 = 0$, then $b_1a_0 = 0$ by using reducibility of R. We conclude from the second equation that $a_1b_1 = 0$. Repeating this process, we have $a_ib_1 = 0$, for all $1 \leq i \leq n$. Finally, we can conclude that $a_ib_j = 0$, for all $1 \leq i, j \leq n$. □

The class of Armendariz rings, contains properly the class of reduced rings. The details of the following example can be consulted in [CR97].

Example A.1.12 ([CR97], Proposition 2.1)**.** For each integer n, $\mathbb{Z}/n\mathbb{Z}$ is an Armendariz ring, which is not reduced whenever n is a natural number which is not square free.

Now, we present the relation between Armendariz rings and Abelian rings.

Proposition A.1.13. *Let R be a ring. If R is an Armendariz ring, then R is an Abelian ring.*

Proof. Let $a \in R$ and e an element idempotent of R. Consider the polynomials $f_1(x) = e - ea(1-e)x$, $g_1(x) = (1-e) + ea(1-e)x$, $f_2(x) = (1-e) - (1-e)aex$ and $g_2(x) = e + (1-e)aex \in R[x]$. Then $f_1(x)g_1(x) = e(1-e) + ea(1-e)x - ea(1-e)^2x - ea(1-e)ea(1-e)x^2 = 0$. Similarly, $f_2(x)g_2(x) = 0$. Now, since R is an Armendariz ring, we have $e(ea(1-e)) = ea(1-e) = ea - eae = 0$ and so $ea = eae$. On the other hand, since $f_2(x)g_2(x) = 0$, then $(1-e)(1-e)ae = (1-e)ae = ae - eae = 0$ and so, $ae = eae$. Therefore, R is Abelian ring. □

The previous results allow us to determine relationships between reduced, Armendariz and Abelian rings:

$$\text{Reduced} \Rightarrow \text{Armendariz} \Rightarrow \text{Abelian}.$$

In this part, we consider a skew version of some classes of rings, with respect to a ring endomorphism α. When α is the identity endomorphism, this coincides with the notion of ring. These rings arise as a natural generalization of the rings studied previously. Many of its properties have been described in the following articles: [Kwa07], [BK10], [BHK09], [KL03] and [CR97]. In addition, they have been a great tool for the study of noncommutative rings of polynomial type.

Definition A.1.14 ([Kre96],p. 7)**.** An endomorphism α of a ring R is called **rigid**, if for $r \in R$ the condition $r\alpha(r) = 0$ implies $r = 0$. A ring R is called **α-rigid**, if there exists a rigid endomorphism α of R.

Remark A.1.15. Note that if R is α-rigid, then α is a injective homomorphism: Let $r \in R$ such that $\alpha(r) = 0$. Then, we get that $r\alpha(r) = 0$. Now, as R is α-rigid, we have $r\alpha(r) = 0$ which implies that $r = 0$. Thus, α is a injective homomorphism. Furthermore, an α-rigid ring R is reduced: Let $r \in R$ such that $r^2 = 0$. Then, we have $0 = r\alpha(r^2)\alpha(\alpha(r)) = r\alpha(r)\alpha(r)\alpha(\alpha(r)) = r\alpha(r)\alpha(r\alpha(r))$ whence $r\alpha(r) = 0$, and thus $r = 0$ since R is α-rigid.

In [Kwa07], the notion of α-symmetric ring is introduced as a generalization of symmetric rings and α-rigid rings. These rings are of great importance in the study of algebraic properties of polynomial type rings.

Definition A.1.16 ([Kwa07], Definition 2.1)**.** An endomorphism α of a ring R is called **right** (resp., **left**) **symmetric**, if $abc = 0$ implies $ac\alpha(b) = 0$ (resp., $\alpha(b)ac = 0$), for $a, b, c \in R$. A ring R is called **right** (resp., **left**) **α-symmetric**, if there exists a **right** (resp., **left**) **symmetric** endomorphism α of ring R. A ring R is **α-symmetric**, if it is right and left α-symmetric.

Proposition A.1.17. *An α-rigid ring is α-symmetric.*

Proof. Assume that $abc = 0$ for $a, b, c \in R$. Then, we obtain $bac = 0$, since R is reduced and hence symmetric. Thus, we get that $ac\alpha(b)\alpha(ac\alpha(b)) = ac\alpha(bac)\alpha^2(b) = 0$. Since R is α-rigid, we have that $ac\alpha(b) = 0$ and thus R is right α-symmetric. Now, suppose that $abc = 0$, for $a, b, c \in R$. Since R is right α-symmetric, we get that $ac\alpha(b) = 0$ and so $1ac\alpha(b) = 0$ which implies that $\alpha(b)\alpha(ac) = 0$ where $\alpha(b)\alpha(ac) = \alpha(bac) = 0$. Now, as α is a injective homomorphism, then $bac = 0$ whence $0 = ac(bac)b = (acb)^2$. Hence, if R is α-rigid (in particular, R reduced), then $acb = 0$. Thus, R es symmetric. More generally, R is α-symmetric. □

In [BHK09], the notion of α-reversible ring is introduced as a generalization of α-rigid rings as well as an extension of reversible rings. This class of rings is of great importance in literature. Much research has been done on type of rings.

Definition A.1.18 ([BHK09], Definition 2.1)**.** A ring R is called **right** (respectively, **left**) **α-reversible**, if whenever $ab = 0$, for $a, b \in R$, then $b\alpha(a) = 0$ (respectively, $\alpha(b)a = 0$).

The ring R is called **α-reversible**, if it is both right and left α-reversible.

Proposition A.1.19 ([Kwa07], Theorem 2.5)**.** *An α-symmetric ring is α-reversible.*

Proof. Suppose that $ab = 0$, for $a, b \in R$. Hence $rab = 0$, for all $r \in R$. In particular, if $r = 1$, we have $1ab = 0$. Since R is right α-symmetric, then $1b\alpha(a) = b\alpha(a) = 0$. Therefore, R is right α-reversible. Similarly, if $ab = 0$, then $ab1 = 0$. Since R is left α-symmetric, then $\alpha(b)a1 = \alpha(b)a = 0$. Therefore, R is left α-reversible. Hence R is α-reversible. □

In [BK10], the notion of α-semicommutative ring with the endomorphism α is defined as a generalization of α-rigid ring and an extension of semicommutative ring.

Definition A.1.20 ([BK10], Definition 2.1)**.** An endomorphism α of a ring R is called **semicommutative**, if $ab = 0$ implies $aR\alpha(b) = 0$, for $a, b \in R$. A ring R is called **α-semicommutative**, if there exists a semicommutative endomorphism α of R.

Next, we present the relation between α-reversible rings and α-semicommutative rings. It is appropriate to note that the condition on R of being a reduced ring plays an important role in the following result.

Proposition A.1.21 ([BK10], Proposition 3.1)**.** *Let R be a ring. If R is a reduced α-reversible ring, then R is α-semicommutative.*

Proof. Suppose that $ab = 0$, for $a, b \in R$, and let R be a reduced α-reversible ring. Let $r \in R$. Then $\alpha(b)a = 0$ since R is α-reversible, and so $\alpha(b)ar = 0$. Thus, $(ar\alpha(b))^2 = ar(\alpha(b)ar)\alpha(b) = 0$, and hence $ac\alpha(b) = 0$ by reducibility of R. Therefore R is α-semicommutative. □

Finally, we present the relation between α-semicommutative rings and Abelian rings.

Proposition A.1.22 ([BK10], Theorem 3.5)**.** *Let R be a ring. If R is an α-semicommutative ring R with $\alpha(1) = 1$, then R is Abelian.*

Proof. Let e be an idempotent element of R and suppose R an α-semicommutative ring. Then, we have $e^2 = e$ since e is an idempotent element. This implies $e - e^2 = e(1 - e) = 0$ and so $eR\alpha(1 - e) = 0$, by using α-semicommutativity of R. Now, $(1 - e)e = 0$ implies $(1 - e)R\alpha(e) = 0$, since R is α-semicommutative. Therefore, $er(1 - e) = (1 - e)re = 0$, for all $r \in R$. This implies that $er = re$, for all $r \in R$, and hence R is Abelian.

□

By the previous results, we obtain the following implications that represents the relations between these rings in their skew version.

$$\alpha\text{-rigid} \Rightarrow \alpha\text{-symmetric} \Rightarrow \alpha\text{-reversible} \overset{Reduced}{\Rightarrow} \alpha\text{-semicommutative} \overset{\alpha(1)=1}{\Rightarrow} \text{Abelian}.$$

In [HKR06] the notion α-Armendariz ring is defined analogously to the Armendariz ring concept, this time for the purpose of studying the ring $R[x; \alpha]$. A number of properties of this generalization are established, and connections of properties of an α-Armendariz ring R with those of the ring $R[x; \alpha]$ are investigated. We recall that a ring R with a ring endomorphism $\alpha : R \to R$, a **skew polynomial ring** $R[x; \alpha]$ of R is the ring obtained by giving the polynomial ring over R, the new multiplication $xr = \alpha(r)x$ for all $r \in R$. The α-Armendariz rings are a generalization of the α-rigid rings and the Armendariz rings. Several authors have defined more general notions of property of Armendariz. Later, we mention a property of an Armendariz on an ring R whose associated polynomial ring is endowed with a family of endomorphisms.

Definition A.1.23 ([HKR06], Definition 1.1)**.** Let α be an endomorphism of a ring R. R is called **α-Armendariz**, if for $f(x) = \sum_{i=0}^{n} a_i x^i i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; \alpha]$, $pq = 0$ implies $a_i b_j = 0$, for all $0 \leq i \leq m$ and $0 \leq j \leq n$.

Proposition A.1.24 ([HKR06], Proposition 1.3)**.** *Let R be an α-Armendariz ring. For $a, b \in R$, we have the following:*

(i) *If $ab = 0$, then $\alpha^n(a)b = 0$, for any positive integer n.*

(ii) *If $a\alpha^m(b) = 0$ for some positive integer m, then $ab = 0$.*

Proof. We follow the ideas presented in [HKR06], Proposition 1.3.

(i) Suppose $ab = 0$. Let us show that $\alpha(a)b = 0$. Let $p(x) = \alpha(a)x \in R[x;\alpha]$ and $q(x) = bx \in R[x;\alpha]$. Then $p(x)q(x) = \alpha(a)\alpha(b)x^2 = \alpha(ab)x^2 = 0$. Since R is α-Armendariz, it follows that $\alpha(a)b = 0$.

(ii) Suppose $a\alpha^m(b) = 0$, for some positive integer m. Let $p(x) = ax^m \in R[x;\alpha]$ and $q(x) = bx \in R[x;\alpha]$. Then $p(x)q(x) = a\alpha^m(b)x^{m+1} = 0$, which implies that $ab = 0$, since R is an α-Armendariz ring.

□

Next, we present the relation between α-rigid and α-Armendariz rings.

Proposition A.1.25. *Let R be a ring. An α-rigid ring R is an α-Armendariz ring.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x,\alpha]$. Suppose that $f(x)g(x) = 0$ and that R is an α-rigid ring. Moreover, we can assume that $n = m$. Making the usual product of $f(x)$ with $g(x)$, we obtain the equalities,

$$\begin{aligned} a_0b_0 &= 0 \\ a_1\alpha(b_0) + a_0b_1 &= 0 \\ a_2\alpha^2(b_0) + a_1\alpha(b_1) + a_0b_2 &= 0 \\ \vdots \qquad & \quad \vdots \\ a_n\alpha^n(b_0) + \cdots + a_0b_n &= 0. \end{aligned}$$

Since R is a reduced ring, if $a_0b_0 = 0$, then $b_0a_0 = 0$. Now, left multiplying the second of these equations by b_0 and right multiplying by $\alpha(a_1)$, we have $b_0a_1\alpha(b_0)\alpha(a_1) = b_0a_1\alpha(b_0a_1) = 0$, and this implies $b_0a_1 = 0$, since R is an α-rigid ring. By reducibility of R and Proposition A.1.24, we have $a_1\alpha(b_0) = 0$. Analogously, we have $a_ib_0 = 0$ and $a_i\alpha^i(b_0) = 0$, for all $1 \leq i \leq n$. Therefore, the previous system of equations is reduced to the following:

$$\begin{aligned} a_0b_1 &= 0 \\ a_1\alpha(b_1) + a_0b_2 &= 0 \\ \vdots \qquad & \quad \vdots \\ a_{n-1}\alpha^{n-1}(b_1) + \cdots + a_0b_n &= 0. \end{aligned}$$

Again, $a_0b_1 = 0$ implies $b_1a_0 = 0$, by using reducibility of R. We conclude from the second equation that $a_1b_1 = 0$. By using reducibility of R and Proposition A.1.24, we have $a_1\alpha(b_1) = 0$, and repeating this process, we have $a_ib_1 = 0$, for all $1 \leq i \leq n$. Finally, we obtain that $a_ib_j = 0$, for all $1 \leq i, j \leq n$. □

We present the relation between the α-Armendariz rings and Abelian rings.

Proposition A.1.26 ([HKR06], Corollary 1.4)**.** *Let R be an α-Armendariz ring. Then $\alpha(1) = 1$, where 1 is the identity of R. In this case, $\alpha(e) = e$, for any $e = e^2 \in R$.*

Proof. Let R be an α-Armendariz rings. Since $(1 - \alpha(1))\alpha(1) = 0$, let us show that $\alpha(1-\alpha(1))\alpha(1) = 0$. Let $p(x) = \alpha(1-\alpha(1))x \in R[x, \alpha]$ and $q(x) = \alpha(1)x \in R[x, \alpha]$. Then, $p(x)q(x) = \alpha(1 - \alpha(1))x\alpha(1)x = \alpha(1 - \alpha(1))\alpha^2(1)x^2 = \alpha((1 - \alpha(1))\alpha(1))x^2 = 0$. Since R is an α-Armendariz ring, so $\alpha(1 - \alpha(1))\alpha(1) = 0$. Now, we have $0 = \alpha(1 - \alpha(1))\alpha(1) = \alpha((1 - \alpha(1))1) = \alpha(1 - \alpha(1))$, and since α is injective, we have $\alpha(1) - 1 = 0$. Therefore, we obtain that $\alpha(1) = 1$.

Similarly, let $e \in R$ such that $e^2 = e$. This implies $e(1 - e) = 0$ and $(1 - e)e = 0$. Let us show that $\alpha(e)(1 - e) = 0$ and $\alpha(1 - e)e = 0$. Let $p(x) = \alpha(e)x \in R[x, \alpha]$ and $q(x) = (1 - e)x \in R[x, \alpha]$. Then $p(x)q(x) = \alpha(e)x(1 - e)x = \alpha(e)\alpha(1 - e)x^2 = \alpha(e(1 - e))x^2 = 0$ since $e(1 - e) = 0$. Therefore, $\alpha(e)(1 - e) = 0$. Likewise, if we have $p(x) = \alpha(1 - e)x, q(x) = ex \in R[x, \alpha]$, then $p(x)q(x) = 0$ and thus $\alpha(1 - e)e = 0$. Now, if $\alpha(e)(1 - e) = 0$ then $\alpha(e) = \alpha(e)e$. On the other hand, if $\alpha(1 - e)e = 0$, then $e = \alpha(e)e$ since $\alpha(1) = 1$. Hence, $\alpha(e) = \alpha(e)e = e$, i.e, $\alpha(e) = e$. □

Proposition A.1.27 ([HKR06], Lemma 3.1)**.** *Let R be a ring. If R is an α-Armendariz ring, then R is an Abelian ring.*

Proof. Let $a \in R$ and e an idempotent element of R. Let $f_1(x) = e - ea(1 - e)x, f_2(x) = (1-e)-(1-e)aex, g_1(x) = 1-e+ea(1-e)x, g_2(x) = e+(1-e)aex \in R[x, \alpha]$, where e is an idempotent in R. Then $f_1(x)g_1(x) = 0$ and $f_2(x)g_2(x) = 0$. Since R is α-Armendariz, we have $ea(1 - e)\alpha(1 - e) = 0$. By Proposition A.1.26, $\alpha(1 - e) = 1 - e$ and so $ea(1 - e) = 0$. In the same way, $f_2(x)g_2(x) = 0$ implies that $(1 - e)ae = 0$. Then $ae = eae = ea$, so R is Abelian. □

Next, we study the concept of 2-primal ring. This concept and other nearby concepts are of great importance in our work. The study of 2-primal rings was initiated Shin [Shi73]. He proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime. The study of 2-primal condition was continued by Birkenmeier, Heatherly and Lee [BHL92]. They provided various examples relating to this equivalent condition. Shin also showed that every proper ideal of a ring R is 2-primal if and only if every prime ideal of R is completely prime. Some of the fundamental properties of 2-primal rings are presented in [Sun91], [Shi73], [Mar03] and [BHL92]. Given a ring R, we use the symbol $\mathrm{nil}(R)$ to denote the set of all nilpotent elements in R, the symbol $\mathrm{nil}_*(R)$ denotes the prime radical of a ring R, $\mathrm{nil}^*(R)$ its upper nil-radical, and L-rad(R) its Levitzki radical, respectively.

Definition A.1.28 ([BHL92], Definition 1.1)**.** An ideal I of R is a **2-primal ideal**, if $\text{nil}_*(R/I) = \text{nil}(R/I)$. If the zero ideal of R is a 2-primal, the R is a **2-primal ring**.

Proposition A.1.29. *A ring R is 2-primal if and only if the factor ring R/nil_*R is reduced.*

Proof. Suppose R is 2-primal ring and $\overline{a}^2 = 0$ where $\overline{a} \in R/\text{nil}_*(R)$. Since $\overline{a}^2 = 0$, then $a^{2n} \in \text{nil}_*(R)$, for some $1 \leq n$. Also, $\text{nil}_*(R)$ is a nil ideal, hence we have $(a^{2n})^m = a^{2nm} = 0$, for some $1 \leq m$. Thus, a is nilpotent and so $a \in \text{nil}_*(R)$. Therefore $\overline{a} = 0$ in $R/\text{nil}_*(R)$, so $R/\text{nil}_*(R)$ is reduced. Conversely, if $R/\text{nil}_*(R)$ is reduced, then for any nilpotent element $a \in R$, a is nilpotent in R/nil_*R. Thus, we have $a \in \text{nil}_*R$, so R is 2-primal ring. □

We present some interesting examples that are of great importance throughout this work. Recall that a ring R is **right duo** (respectively **left duo**), if any right (resp. left) ideal of R is a two sided ideal. If R is left and right duo, then we say that R is a **duo ring**.

Example A.1.30. Every right duo ring is a 2-primal ring.

Proof. By Proposition A.1.29, it is sufficient to show that $\overline{a}^2 = 0 \Rightarrow \overline{a} = 0$, where $\overline{a} \in \overline{R} = R/\text{nil}_*(R)$. Since aR is an ideal in R, we have $Ra \subseteq aR$. Therefore, $(\overline{a}\overline{R})^2 = \overline{a}\overline{R}\overline{a}\overline{R} \subseteq \overline{a}(\overline{a}\overline{R})\overline{R} = 0$. Since $\overline{R}$ is reduced, then $\overline{R}$ is a semiprime ring. It follows that $\overline{a}\overline{R} = 0$, and so $a = 0 \in R$, as desired. □

Example A.1.31. Every reversible ring is 2-primal ring.

Proof. Let R be a reversible ring, and let $a \in R$ be a nilpotent element, say with $a^n = 0$. By reversibility, $a^r x_1 a^s = 0$ whenever $r + s = n$ where $x_1 \in R$. Another application of reversibility, we have $a^r x_1 a^s x_2 a^t = 0$ whenever $r + s + t = n$ where $x_2 \in R$. Following this process repeatedly, it follows that $x_1 a x_2 \cdots x_{n-1} a x_n = 0$ with $x_1, \ldots, x_n \in R$. This implies that $(RaR)^n = 0$, so $RaR \subseteq \text{nil}_*(R)$. In particular, $a \in \text{nil}_*(R)$. □

Example A.1.32. From the examples above, it follows that commutative, reduced, symmetric and semicommutative rings are example of 2-primal rings.

Now, we study a more general notion of 2-primal rings which we called *weakly 2-primal rings*. In [CC11], this notion is introduced which is of great importance in the study of a more general concept of semiconmutativity in skew polynomial rings.

Definition A.1.33 ([Lam01], p. 177)**.** The **Levitzki radical** is defined as the largest locally nilpotent ideal, and is denoted as L-rad(R).

Definition A.1.34 ([CC11], Definition 3.1)**.** A ring R is called **weakly 2-primal**, if the set of nilpotent elements in R coincides with its Levitzki radical, i.e, $\text{nil}(R) = \text{L-rad}(R)$.

Example A.1.35 ([CC11], Example 3.1)**.** Every 2-primal ring is a weakly 2-primal ring.

A.2 Graded and filtered rings

Since the filtration-graduation technique was used in this work, in this Appendix we review the basic notions regarding filtered and graded rings and algebras. Definitions and results are adapted from [Lez15] and [MR01].

Definition A.2.1 ([Lez15], Definition 2.1.1)**.** A ring R is said to be $\mathbb{Z}$**-graded**, if it has a family of $\{R_p\}_{p\in\mathbb{Z}}$ subgroups of its additive group R^+ satisfying the following conditions:

(i) $R_pR_q \subseteq R_{p+q}$, for any $p, q \in \mathbb{Z}$.

(ii) $R = \bigoplus_{p\in\mathbb{Z}} R_p$.

For $p \in \mathbb{Z}$, R_p is called **the homogeneous component of degree** p and the elements R_p are said to be **homogeneous of degree** p. The $\{R_p\}_{p\in\mathbb{Z}}$ family is denotes a graduation of R. If $R_p = 0$ for $p < 0$, that is, $R = \bigoplus_{p\in\mathbb{N}} R_p$, R is said to be a **positively graded ring**. Let $\Bbbk$ be a field, if R is a $\Bbbk$-algebra, R is said to be $\mathbb{Z}$**-graded**, if in addition to the above conditions we have that R_p is a $\Bbbk$-subspace of R. Let R and T be $\mathbb{Z}$-graded rings with graduations $\{R_p\}_{p\in\mathbb{Z}}$ and $\{T_p\}_{p\in\mathbb{Z}}$, respectively. A ring homomorphism $f : R \to T$, is called **graded**, if $f(R_p) \subseteq T_p$, for every $p \in \mathbb{Z}$.

Proposition A.2.2 ([Lez15], Proposition 2.1.4)**.** *Let $R = \bigoplus_{p\in\mathbb{Z}} R_p$ be a $\mathbb{Z}$-graded ring and I an ideal of R. Then, the following assertions are equivalent:*

(i) $I = \bigoplus_{p\in\mathbb{Z}}(R_p \cap I)$.

(ii) *If $a \in I$, then every homogeneous component of a is also an element of I.*

(iii) *I is generated (as ideal) by homogeneous elements.*

(iv) *The quotient R/I is $\mathbb{Z}$-graded, with $(R/I)_p = (R_p + I))/I$, for every $p \in \mathbb{Z}$.*

We now define filtered rings.

Definition A.2.3 ([Lez15], Definition 2.2.1)**.** A ring R is said to be $\mathbb{Z}$**-filtered**, if there exists a family of subgroups $\{F_p(R)\}_{p\in\mathbb{Z}}$ of its additive subgroup R^+ which satisfies:

(i) $F_p(R)F_q(R) \subseteq F_{p+q}(R)$, for every $p, q \in R$

(ii) $\bigcup_{p\in\mathbb{Z}} F_p(R) = R$

(iii) For $p \leq q$, $F_p(R) \subseteq F_q(R)$

(iv) $1 \in F_0(R)$

The family $\{F_p(R)\}_{p\in\mathbb{Z}}$ is called a **filtration** of R. The filtration is called **separated**, if $\bigcap_{p\in\mathbb{Z}} F_p(R) = 0$. If $F_{-1}(R) = 0$, R is a **positively filtered ring**. If $\Bbbk$ is a field and R is a $\Bbbk$-algebra, then R is $\mathbb{Z}$**-filtered**, if it further satisfies $F_p(R)$ being a $\Bbbk$-subspace of R. Let R and T be two filtered rings (resp. $\Bbbk$-algebras) with filtrations $\{F_p(R)\}_{p\in\mathbb{Z}}$ and $\{F_p(T)\}_{p\in\mathbb{Z}}$, respectively. A ring (resp. $\Bbbk$-algebras) homomorphism $f : R \to T$ is called **filtered**, if $f(F_p(R)) \subseteq F_p(T)$, for every $p \in \mathbb{Z}$.

Remark A.2.4. From the previous definition, it follows that $F_0(R)$ is a subring of R. First, the sum is closed since $F_0(R)$ is an Abelian group. By Definition A.2.3 (i), we have that $F_0(R)F_0(R) \subseteq F_0(R)$, and thus the product is closed. Finally, $1 \in F_0(R)$ by the definition. Note that all positive filtering is separate. Also, composition of filtered homomorphisms is again a filtered homomorphism.

Proposition A.2.5 ([Lez15], Proposition 2.2.2)**.** *Every graded ring R is filtered.*

Proof. Let R be a graded ring with graduation $\{R_p\}_{p\in\mathbb{Z}}$. Then R is filtered, with filtration given by $\{F_p(R)\}_{p\in\mathbb{Z}}$ where $F_p(R) := \bigoplus_{p\leq n} R_n$. □

Proposition A.2.6 ([Lez15], Proposition 2.2.3)**.** *If R is a filtered ring, then there exists a graded ring $Gr(R)$ associated to R.*

Proof. Let R be a filtered ring with filtration $\{F_p(R)\}_{p\in\mathbb{Z}}$. We define the collection Abelian groups $Gr(R)_p := F_p(R)/F_{p-1}(R)$, with $p \in \mathbb{Z}$. Now, we consider $Gr(R) := \bigoplus_{p\in\mathbb{Z}} Gr(R)_p$, which is also an Abelian group. The product in $Gr(R)$ is defined by distributivity and multiplication of homogeneous elements as follows:

$$\begin{aligned} F_p(R)/F_{p-1}(R) \times F_q(R)/F_{q-1}(R) &\to F_{p+q}(R)/F_{p+q-1}(R) \\ (a + F_{p-1}(R), b + F_{q-1}(R)) &\mapsto ab + F_{p+q-1}(R) \end{aligned}$$

In [Lez15], Proposition 2.2.3, it is verified that this product is well defined. Hence, given the construction, $Gr(R)$ is a graded ring. In addition, if R is a $\Bbbk$-algebra, every $Gr(R)_p$ is a quotient of $\Bbbk$-subspaces.

□

The following proposition shows the importance of the filtration-graduation technique. A large number of facts have been presented in the literature, which show how graded and filtered rings are a great tool in studying some properties of rings, modules, and algebras. It is interesting to note that if a graded ring has a certain property, see that the associated ring also has that property.

Proposition A.2.7 ([Lez15], Theorem 2.4.4; [LAC+13], Proposition 4.7)**.** *Let R be a filtered ring. Then, the following statements are satisfied:*

(i) *If $Gr(R)$ is a domain, then R is a domain.*

(ii) *If $Gr(R)$ is a prime ring, then R is a prime ring.*

(iii) *If $Gr(R)$ is a semiprime ring, then R is a semiprime ring.*

(iv) *If $Gr(R)$ is right Noetherian, then R is right Noetherian.*

A.3 Goldie's theory

Goldie's theory has played a fundamental role being a very important tool in the study of noncommutative algebraic objects which are mainly Noetherian. In this section, we recall some concepts and properties of Goldie's theory, as well as present some interesting facts that relate Goldie's theory to skew PBW extensions. We follow the ideas presented in [GW04], [Rey14] and [Lez15].

Definition A.3.1. ([GW04], p. 89). Let R be a ring. If N is a submodule of a right R-module M such that, for all nonzero submodules X of M, one has $N \cap X \neq 0$, then N is an **essential submodule** of M, and M is an **essential extension** of N. We write $N \leq_e M$.

Example A.3.2 ([GW04], Proposition 5.6)**.** The classic example of an essential submodule is $\mathbb{Z} \subseteq \mathbb{Q}$ (considered as $\mathbb{Z}$-modules), or more generally a domain R considered as a submodule of its field of fractions (again as R-modules).

Next, we recall some interesting properties related to essential submodules.

Proposition A.3.3. *Let R be a ring. The following statements are satisfied:*

(i) *Let M be a nonzero R-module and $0 \neq N \leq_e M$. If $N \leq N' \leq M$, then $N' \leq_e M$.*

(ii) *Let M be a nonzero R-module. If $N \leq M$, then there exists $N' \leq M$ such that $N \cap N' = 0$ and $N \bigoplus N' \leq_e M$.*

(iii) *If $N_i \leq_e M_i$, for every $1 \leq i \leq t$, then $N_1 \bigoplus \cdots \bigoplus N_t \leq_e M_1 \bigoplus \cdots \bigoplus M_t$.*

Definition A.3.4 ([GW04], Proposition 4.2)**.** An R-module M is called **semisimple**, if every submodule of M is a direct summand of M. A ring R is called **semisimple**, If R is semisimple as an R-module.

We present a characterization of the semisimple modules in terms of their essential submodules.

Proposition A.3.5 ([GW04], Corollary 5.9)**.** *An R-module $M \neq 0$ is semisimple if and only if M possesses no proper essential submodules.*

Proof. Suppose that M is semisimple. If N is any proper submodule of M, according to Proposition 4.2, [GW04], which states that M is semisimple if and only if every submodule of M is a direct summand of M, we have that $M = N \bigoplus N'$ for some nonzero submodule N'. Since $N' \cap N = 0$, we obtain that $N \leq_e M$. Now, if M has no proper essential submodules, then according to Corollary 5.8, [GW04], which states that any submodule of a module M is a direct summand of an essential submodule of M, in particular we have that every submodule of M is a direct summand. Therefore, using Proposition 4.2, [GW04] again, we conclude M is semisimple. □

Definition A.3.6 ([GW04], p. 96)**.** A module U is **uniform**, if $U \neq 0$ and each nonzero submodule of U is an essential submodule. This is equivalent to U not containing a direct sum of nonzero submodules. We recall that a module M is said to have **finite uniform dimension**, if it contains no infinite direct sum of nonzero submodules.

Proposition A.3.7. *Let M be an nonzero R-module of finite uniform dimension. Then M has a uniform submodule.*

The following theorem allows defining the uniform dimension of a module of finite uniform dimension.

Proposition A.3.8 ([Lez15], Theorem 3.2.5)**.** *Let M be a module of finite uniform dimension. So, there is an integer $n \geq 1$ such that:*

(i) *M contains a finite direct sum $U_1 \bigoplus \cdots \bigoplus U_n$ of uniform submodules of M which is essential in M.*

(ii) *Any direct sum of nonzero submodules of M has at most n summands.*

(iii) *A direct sum of uniform submodules of M is essential in M if and only if it has precisely n summands.*

Remark A.3.9. The integer $n \geq 1$ of the previous proposition is called the **uniform dimension** or **Goldie dimension** of M, noted $\text{rudim}(M) = n$.

Next, we present some important statements that characterize or relate the uniform dimension of a skew PBW extension and its corresponding ring of coefficients. These propositions are part of a research on uniform dimension in skew PBW extensions in [Rey14]. The following result characterizes the uniform dimension of a bijective skew PBW extension over a right Noetherian domain ring. We recall that, if R is a right Noetherian domain, then $\text{rudim}(R) = 1$ (see [Lez15], Corollary 3.2.11).

Proposition A.3.10 ([Rey14], Proposition 3.2)**.** *If A is a bijective skew PBW extension of a right Noetherian domain R, then the uniform dimension of A is 1, that is,* $\text{rudim}A = 1$.

Proof. Since R is Noetherian, so is A by using Proposition 2.1.11. In addition, we obtain that A is also a domain, by Proposition 4.1, [LR14], which states that if R is a domain, so is A. Therefore, we conclude $\text{rudim}(A) = 1$. □

Next, we recall the notion of semiprime ideal and semiprime ring which arise as generalizations of prime ideal and prime ring. Semiprime rings are of great importance in the study of noncommutative algebra.

Definition A.3.11. ([GW04], p. 51). A **semiprime ideal** in a ring R is any ideal of R which is an intersection of prime ideals. A **semiprime ring** is any ring in which 0 is a semiprime ideal. Note that an ideal P in a ring R is semiprime if and only if R/P is a semiprime ring.

Example A.3.12. Subrings, products, and localizations of reduced rings are semiprime rings. The ring of integers $\mathbb{Z}$ is a semiprime ring. Every field and every polynomial ring over a field (in arbitrarily many variables) is a semiprime ring.

We present an important characterization of semiprime ideals.

Proposition A.3.13 ([Lez15], Proposition 3.3.2)**.** *Let I be a bilateral ideal of a ring R. The following statements are equivalent:*

(i) *I is semiprime.*

(ii) *If $x \in A$ is such that $xRx \subseteq I$, then $x \in I$.*

(iii) *If J is a bilateral ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*

(iv) *If J is a right ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*

(v) *If J is a left ideal of R such that $J^2 \subseteq I$, then $J \subseteq I$.*

(vi) *If J is a bilateral ideal of R such that $J^2 \subseteq I$ and $I \subseteq J$, then $I = J$.*

We present the central notion of this appendix.

Definition A.3.14. ([GW04], p. 115) A **right Goldie ring** is any ring R such that R_R has finite uniform dimension and R has the ascending chain condition on left annihilators.

Example A.3.15 ([GW04], Proposition 6.9)**.** If R is a right Noetherian ring, then R is right Goldie ring. Analogously for the left case.

The following proposition provides an upper bound for the uniform dimension of a skew PBW extension over prime right Goldie rings.

Proposition A.3.16 ([Rey14], Theorem 3.5)**.** *Let R be a prime right Goldie ring. If A is a bijective skew PBW extension of R, then uniform dimension of A is less or equal than uniform dimension of R.*

In [GW04] is presented the following proposition, which is a characterization of the essential ideals in a semiprime right Goldie ring.

Proposition A.3.17 ([GW04], Proposition 6.13)**. (Goldie's Regular Element Lemma)** *Let R be a semiprime right Goldie ring. Then, a nonzero right ideal I of R is essential if and only if it contains a regular element.*

Proposition A.3.18 ([Lez15], Proposition 3.3.13 and Proposition 3.3.14)**.** *Let R be a ring and S be a set of elements which are not zero divisors such that S satisfies the right Ore condition. then:*

(i) *If RS^{-1} is right Goldie, then R is right Goldie.*

(ii) *If $I \leq_e R_R$, then $IS^{-1} \leq_e (RS^{-1})_{RS^{-1}}$.*

(iii) *If $J \leq_e (RS^{-1})_{RS^{-1}}$, then $I \leq_e R_R$ with $I = \left\{r \in R \mid \frac{r}{1} \in J\right\}$.*

In [GW04], Proposition 6.9 says that any ring which has a semisimple classical right quotient ring must be a semiprime right Goldie ring. The converse statement is the main content of famous Goldie's Theorem.

Proposition A.3.19 ([Lez15], Theorem 3.3.15)**. (Goldie's Theorem)** *Let R be a ring. Then, the following statements are satisfied:*

(i) *$Q_r(R)$ exists and it is semisimple if and only if R is semiprime right Goldie.*

(ii) *Let R be semiprime right Goldie. $Q_r(R)$ is simple if and only if R is prime.*

(iii) *$Q_r(R)$ exists and it is simple right Artinian if and only if R is prime right Goldie.*

Conclusions and future work

In this work, we have presented some results which provide sufficient conditions for a skew PBW extension to have fusible property. A future work is to investigate the geometric meaning of this decomposition that characterizes the fusible property, and how that geometric notion is represented in families that satisfy this property. More exactly, the concept of zero divisor has been studied in the literature and a good part of this research has focused on the study of graphs associated with the set of zero divisors. In [NA00], for example, the set of zero divisors was studied, but a more general notion of annihilator was suggested. The definition of annihilator that is specifically considered in the study of the graph associated with the set of zero divisors is the notion of weak annihilator defined in [Ouy10]. A work in mind is to define a more general notion of fusible ring based on the notion of weak annihilator.

The study of the elements in a ring showed the importance of the study of certain rings that satisfy conditions of compatibility or weak rigidness. An interesting research topic is to extend and to study these notions for modules (e.g., [RS20]). This has already been done in works such as [Ram19], [NRR20] and [Rey19], where the compatibility condition on modules was studied, and recently, in works as [LR20a] where the notion of σ-rigid on modules was introduced.

Drawing a parallel between the research carried out on the associated prime ideals and descriptions presented in [Ram19] and [NRR20] about these ideals, we see that our work is more general regarding the notion of associated prime but not in the structure, given that the above works presented a characterization of associated prime ideals in skew PBW extensions over modules. Other possible work future is to study under which conditions it is possible to describe nilpotent associated primes ideals in skew PBW extensions over modules. Likewise, an interesting question we can ask is how are characterized these nilpotent associated prime ideals in skew PBW extensions over (Σ, Δ)-compatible modules.

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