UNIVERSIDAD NACIONAL DE COLOMBIA

ATIYAH-SINGER INDEX THEOREM AND MINIMAL MORSE FUNCTIONS

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Chapter 1

Introduction

The thesis is divided into two main sections. The first one is a monograph on the Atiyah-Singer theorem. The second part is a research that aims to find a relationship between Morse theory and the heat equation.

1 Atiyah-Singer index theorem

The Atiyah-Singer index theorem is one of the deepest and most beautiful results in modern geometry. This theorem was announced by Michael Atiyah and Isadore Singer in 1963, and states that for an elliptic differential operator on a compact smooth manifold, the analytical index is equal to the topological index, giving a fundamental relationship between differential geometry, partial differential equations, differential topology, operator algebras, and has links to many others fields, including mathematical physics, number theory, etc.

The Atiyah-Singer index theorem has three main components, one analytical (the study of differential operators on manifolds), a topological one (K-theory and the construction of the characteristic class associated to a given differential operator) and a geometric one (by choosing a representative for the characteristic class). The following diagram illustrates the interaction among these components.

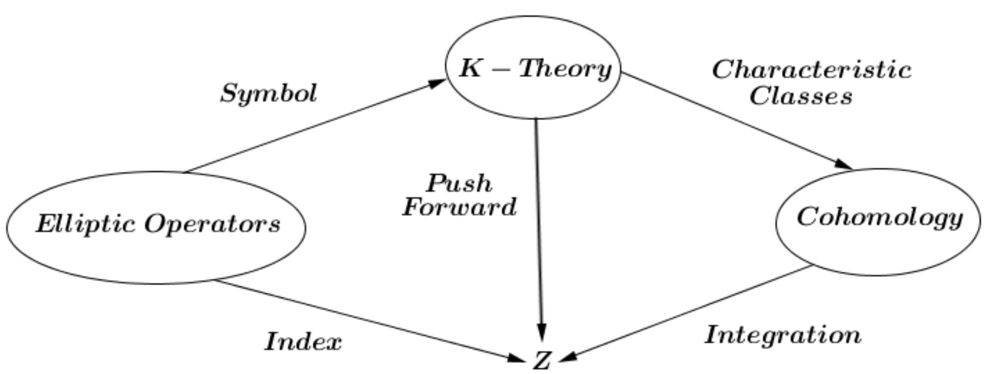


Figure 1.1: Diagram

In order to explain each component of the diagram 1.1, the thesis is organized as follows:

In chapter 2 we will give a detailed introduction to real and complex vector bundles, starting from the basic definitions and key examples, ending with the theorem of classification of them. This theorem allows us to define characteristic classes in chapter 4. We end the chapter discussing principal G-bundles and connections on them, with the aim of establishing the basic material for discussing the Chern-Weil theory in Chapter 4.

In Chapter 3 we discuss the first constituent of the diagram 1.1, namely we define K-theory in degree zero for a topological space X. In order to extend K-theory in degrees different from zero, we will discuss the Bott periodicity theorem in complex K-theory and we will see that K-theory defines a generalized cohomology theory. We discuss the Thom isomorphism theorem in K-theory which together with the Whitney embedding theorem and the tubular neighborhood theorem allow the definition of push-forward in K-Theory, giving the arrow from K-theory to the integer numbers $\mathbb{Z}$. Moreover, we will discuss the Atiyah-Bott-Shapiro construction, which will allow us to establish the relationship between elliptic differential operators discussed in Chapter 6 and the K-theory.

In Chapter 4 we discuss the Chern and Pontryagin characteristic classes defined respectively on real and complex vector bundles, which intuitively measure the non-triviality of such bundles. In addition, we define the Chern character Ch, which establishes a morphism of rings between the K-theory and cohomology of a topological space X, establishing one of the arrows in diagram 1.1. Finally we interpret geometrically the Chern class and the Pontryagin class by the Chern-Weil homomorphism, which is discussed in detail using material developed in chapter 2 on connections on principal G-bundles. More precisely, given a complex (real) principal G-bundle on a smooth manifold X, we rewrite the Chern (Pontryagin) class of such bundle in terms of the curvature of a connection on the principal G-bundle.

To define the arrow between cohomology and the integer numbers $\mathbb{Z}$ shown in the diagram 1.1, in Chapter 5 we define multiplicative sequences and genera in order to establish the Todd class and the $\hat{A}$ class, which will appear as a "correction" term for the commutativity of right side of the diagram 1.1. Finally we will discuss Pontryagin numbers which allow us to discuss one of the main instance of the Atiyah-Singer index theorem, namely the Hirzebruch signature theorem.

In Chapter 6 we will discuss elliptic differential operators on sections of vector bundles on smooth manifolds, giving their basic properties and defining the principal symbol of those operators. Also we define the analytical index of elliptic operators and we sketch the proof on the well definition of the analytical index (see [Ban-Crainic] for more details). By the Atiyah-Bott-Shapiro construction developed in Chapter 3, given the principal symbol of an elliptical differential operator on a smooth manifold X, we can define a class in K-theory with compact support on the cotangent bundle of X, and then sending that class by push-forward, we define an element in the integer numbers $\mathbb{Z}$; thus defining the topological index of an elliptic differential operator. Finally we announced the Atiyah-Singer index theorem, which relates the analytical index with the topological index of an elliptic differential operator on a smooth manifold X. This theorem establishes the commutativity of the left side of the diagram 1.1.

The commutativity of the right side of the diagram 1.1 will be discussed in Chapter 7, giving the cohomological formulation of the Atiyah-Singer index theorem. Finally we discuss some instances of the Atiyah-Singer index theorem, namely, the Gauss-Bonnet's theorem, the Chern-Gauss-Bonnet's theorem, the Hirzebruch's signature theorem and the Hirzebruch-Riemann-Roch's theorem.

2 Minimal Morse functions in homogeneous Riemannian spaces via the heat equation

Let (M, g) be an oriented, compact, connected smooth manifold, without boundary. In each tangent space $T_p(M)$, g determines a bilinear map which we denote by $\langle -,-\rangle_g$. Given a smooth map f in M, remember that its gradient is defined as the vector field in $T(M)$, the tangent space of M, that satisfies $\langle \mathrm{grad}(f), \zeta\rangle_g = \zeta(f)$, for all $\zeta \in T(M)$. We denoted by ∇_ζ the connection of *Levi-Civita* determined by the metric g. For all smooth vector fields X in M, the divergence of X is defined as $\mathrm{div}(X)(p) = \mathrm{trace}(\zeta \longmapsto \nabla_\zeta X)$, for $\zeta \in T_p(M)$. The Laplacian of a smooth map $f : M \to \mathbb{R}$ is defined as $\Delta_g f = \mathrm{div}(\mathrm{grad}(f))$. The heat equation in (M, g) is the partial differential equation $\partial f/\partial t = \Delta_g f$, where $f : M \times [0, \infty) \to \mathbb{R}$ denotes a smooth map in $M \times (0, \infty)$, with *initial condition* $f_0 : M \times \{0\} \to \mathbb{R}$ of type L^2 in $M \times \{0\}$ [ch].

It is well known that solutions of this equation can be obtained as follows. First, it can be seen that the eigenvalues of the operator Δ_g are real numbers, non-negative and discrete [ch], $\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_i < \cdots$. Moreover, for each λ_i, the associated eigenspace is of finite dimension. Let $B = \cup B_i$ be an orthonormal basis for $L^2(M)$, which is obtained as the union of basis $B_i = \{\phi_{i,j} : j = 1, ..., m_i\}$ for the eigenspace of λ_i.

The general solution is written in the form

$$f = \sum\nolimits_{i\geq 0} e^{-\lambda_i t} \sum\nolimits_{j=1}^{m_i} \langle f_0, \phi_{i,j}\rangle_{L^2} \phi_{i,j} \qquad (1)$$

where f_0 denotes the initial condition, and $\langle -,-\rangle_{L^2}$ denotes the inner product $L^2(M)$. As M is compact, it can be shown that B_0 is a space of dimension 1, formed by the constant functions.

Notice that when $t \to \infty$, from the general form of the solution it follows that $e^{\lambda_1 t} f$ tends in the metric C^∞ to the orthogonal projection of f in the subspace generated by $B_0 \cup B_1$; i.e., to a map of the form $h = c_0 + \sum_{j=1}^{m_1} c_j \phi_{1,j}$ (2), where $c_j = \langle f_0, \phi_{1,j}\rangle_{L^2}$, called the *first part* of f_0. An initial condition f_0 is called generic, if their first part is a function of Morse.

We show that in the case $M = S^n$, the n-dimensional sphere, with the metric that inherits from $\mathbb{R}^{n+1}$, thus as in the n-torus T^n with the plane metric ($T^n = \mathbb{R}^n/\mathbb{Z}^n$), for almost all m_1-tuples $(c_1, \ldots, c_{m_1})$, $h = c_0 + \sum_{j=1}^{m_1} c_j \phi_{1,j}$ is a Morse functon, which satisfies the following key results:

Theorem 1.1. With the above notation, the map h has a number of critical points equal to the category of the manifold M, i.e., equal to the minor number of possible critical points for a Morse function in M.

For the well-known Morse inequalities [Mi], this number must be greater than or equal to $b(M) = \sum_{k=1}^{\dim(M)} \dim_{\mathbb{R}} H^i(M, \mathbb{R})$, the sum of the Betti numbers. [Ve-Ca].

Chapter 2

Fiber Bundles

In this chapter we will give a detailed introduction to real and complex vector bundles, starting from the basic definitions and key examples, ending with the theorem of classification of them. This theorem allows us to define characteristic classes in chapter 4. We end the chapter discussing principal G-bundles and connections on them, with the aim of establishing the basic material for discussing the Chern-Weil theory in Chapter 4.

1 Basic definitions

In this chapter we present a notion that will be central to this thesis: the concept of a fiber bundle. This concept captures the intuition of topological spaces that are locally the product of two topological spaces, but that generally do not present a global behaviour as the product of two topological spaces. We will closely follow Hatcher's book [Hat09] and [NashSen87].

Definition 2.1. By a **fiber bundle** or a **locally trivial bundle** we understand a triple of topological spaces (E, X, F), where E, X and F are called respectively the **total space**, the **base space** and the **fiber**, and a surjective continuous map $p : E \to X$ which satisfies the following: around each point $x \in X$ there exist a neighborhood U of x and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ called a **local trivialization** such that the following diagram commutes (π_1 is projection onto U)

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"'] & U \times F \arrow[dl, "\pi_1"] \\ U. & \end{tikzcd}$$

The set of trivializations $\{(U_i, \varphi_i)\}$ is called a **trivialization cover** of the bundle.

Remark 2.1.

- We note that the local behaviour of the topological space E as the product of two topological spaces is captured by the homeomorphism $\varphi : p^{-1}(U) \to U \times F$.
- It follows from this definition that $E_x := p^{-1}(x)$ is homeomorphic to the fiber F for all $x \in X$.
- For the category of smooth manifolds, a **smooth fiber bundle** is a fiber bundle where the triplet (E, X, F) are smooth manifolds, the map $p : E \to X$ is a surjective submersion and the local trivializations φ_i are diffeomorphisms.

Notation 1. We will use $\mathbb{K}$ to denote the real numbers field or the complex numbers field.

For our purposes we require a particular case of fiber bundles, namely, vector bundles.

Definition 2.2. A fiber bundle (E, X, F) with continuous map $p : E \to X$ is a real (complex) **vector bundle** of rank k on X, if its fiber $E_x := p^{-1}(x) \subset E$ is a real (complex) k-dimensional vector space, and the local trivializations $\{(U_i, \varphi_i)\}$ are such that the map φ restricted to E_x is a vector space isomorphism $E_x \to \{x\} \times \mathbb{K}^n$.

By abuse of notation, from now on we will give just the function $p : E \to X$ of a vector bundle to refer to it.

Below we will give some examples of vector bundles:

Example 2.1. The **product bundle** or **trivial bundle** $E = X \times \mathbb{R}^n$ with $p : E \to X$ the projection onto the first factor and fiber $\mathbb{R}^n$. The trivialization cover consist of the one unique element, namely, the total space $U = X$ with the identity map $\mathrm{id} : p^{-1}(X) = E \to X \times \mathbb{R}^n$. This bundle is also denoted by $\underline{n}$.

Example 2.2. The **tangent bundle** of the unit sphere

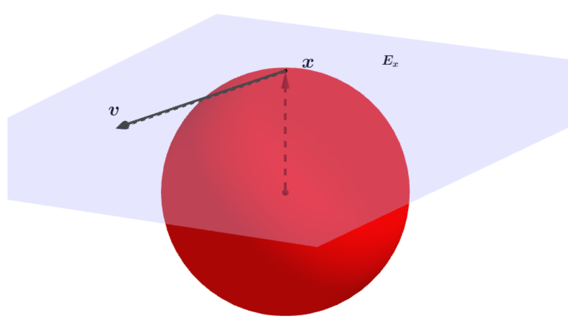
$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | \ |x| = 1\}$$

is the vector bundle $p : E \to \mathbb{S}^n$ with $E = \{(x, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} | \ x \perp v\}$, sends $(x, v) \mapsto x$, and we think of v as a tangent vector to $\mathbb{S}^n$ by translating it so that its tail is at the head of x, on $\mathbb{S}^n$.

To construct local trivializations, choose any point $x \in \mathbb{S}^n$ and let $U_x \subset \mathbb{S}^n$ be the open hemisphere containing x and bounded by the hyperplane through the origin orthogonal to x. Define

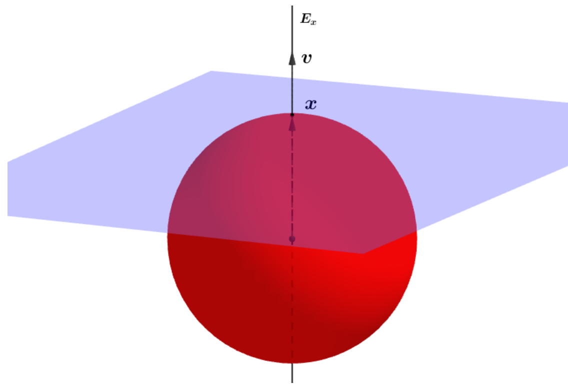
$$\begin{array}{cccc} \varphi_x : & p^{-1}(U_x) & \to & U_x \times p^{-1}(x) \cong U_x \times \mathbb{R}^n \\ & (y, v) & \mapsto & (y, \pi_x(v)) \end{array}$$

where π_x is orthogonal projection onto the hyperplane $p^{-1}(x)$. Then φ_x is a local trivialization since π_x restricts to an isomorphism of $p^{-1}(y)$ onto $p^{-1}(x)$ for each $y \in U_x$.



Example 2.3. The **normal bundle** to $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, is the bundle $p : E \to \mathbb{S}^n$ with E consisting of pairs $(x, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1}$ such that v is perpendicular to the tangent plane to $\mathbb{S}^n$ at x, or in other words, $v = tx$ for some $t \in \mathbb{R}$. Therefore the fiber F of this bundle is the vector space $\mathbb{R}$, thus the normal bundle is a special case of **line bundles**, i.e., vector bundles whose fiber is the vector space $\mathbb{R}$.

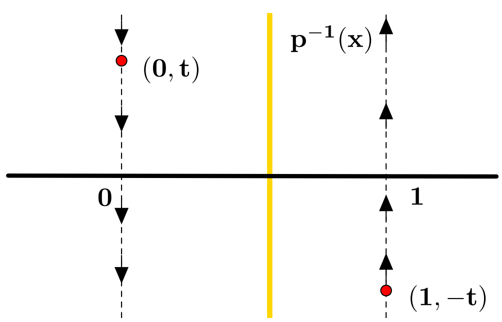
The map $p : E \to \mathbb{S}^n$ is again given by $p(x, v) = x$. As in the previous example, local trivializations $\varphi_x : p^{-1}(U_x) \to U_x \times \mathbb{R}$ can be obtained by orthogonal projection of the fibers $p^{-1}(y)$ onto $p^{-1}(x)$ for $y \in U_x$.



Example 2.4. Set $I = [0, 1]$ and let $E = (I \times \mathbb{R})/\sim$ be the infinite Möbius strip, where $(0, t) \sim (1, -t)$ for all $t \in \mathbb{R}$, and $X = I/\sim$ with $0 \sim 1$ ($X \cong \mathbb{S}^1$). Notice that the projection in the first coordinate $I \times \mathbb{R} \to I$ induces a morphism $p : E \to X$ such that the following diagram commutes

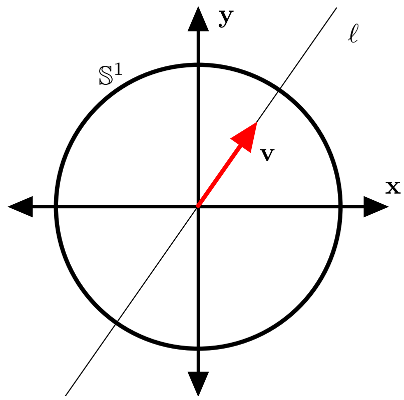
$$\begin{tikzcd} I \times \mathbb{R} \arrow[r, "\pi"] \arrow[d] & (I \times \mathbb{R})/\sim \arrow[d, dashed, "p"] \\ I \arrow[r, "\pi"] & I/\sim . \end{tikzcd}$$

The map $p : E \to \mathbb{S}^1$ given by $[x, t] \mapsto [x]$ is a continuous map and is a 1-dimensional vector bundle or line bundle called the **Möbius bundle**, where the trivialization cover is given by two overlapping open arcs U_1, U_2 in $\mathbb{S}^1$ with local trivializations $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}$ given by $[t, s] \mapsto ([t], s)$, for $i = 1, 2$.



Example 2.5. The **real projective n-space** $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$ through the origin. Since each such line intersects the unit sphere $\mathbb{S}^n$ in a pair of antipodal points, we can also regard this space as the quotient space of $\mathbb{S}^n$ in which antipodal pairs of points are identified, i.e., $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$, with $x \sim -x$ for all $x \in \mathbb{S}^n$. The **canonical line bundle** $p : E \to \mathbb{RP}^n$ has as its total space $E := \{(\ell, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} |\ v \in \ell\}$ and send $(\ell, v) \mapsto \ell$. Again local trivializations can be defined by orthogonal projection.

Notice that we can change in this example the real numbers $\mathbb{R}$ by the complex numbers $\mathbb{C}$ obtaining the **complex projective n-space** $\mathbb{CP}^n$.



The following example will be the most important for us, bearing in mind that one of our goals is proving Bott's periodicity Theorem.

Example 2.6. Taking in the previous example $n = 1$, we have the space $\mathbb{CP}^1$ define as follows

$$\mathbb{CP}^1 := (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$$

where $(z_0, z_1) \sim (\lambda z_0, \lambda z_1)$, $\lambda \in \mathbb{C} \setminus \{0\}$. We denote the class of (z_0, z_1) by $[z_0, z_1]$.

The **canonical line bundle** on $\mathbb{CP}^1$ is define as follow

$$\begin{array}{ccc} H & \to & \mathbb{CP}^1 \\ (\ell, v) & \mapsto & \ell \end{array}$$

where $H := \{(\ell, v) \in \mathbb{CP}^1 \times \mathbb{C}^2 | \ v \in \ell\}$. Notice that we can describe the points $[z_0, z_1] \in \mathbb{CP}^1$ as $z = z_0/z_1 \in \mathbb{C} \cup \{\infty\}$, and by stereographic projection we have that $\mathbb{C} \cup \{\infty\} = \mathbb{S}^2$, where $\mathbb{S}^2$ is the Riemann sphere. Thus the bundle H is a bundle on $\mathbb{S}^2$.

Now we will give the exact notion that will allow to compare the vector bundles.

Definition 2.3. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles on the same space X. Then a **bundle morphism** (or bundle map) from E_1 to E_2 over X is a continuous map $h : E_1 \to E_2$ such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "h"] \arrow[d, "p_1"'] & E_2 \arrow[dl, "p_2"] \\ X & \end{tikzcd}$$

that is $p_2 \circ h = p_1$, or equivalently for any point $x \in X$, h maps the fiber $(E_1)_x$ to the fiber $(E_2)_x$; and furthermore we require the map $h_x : (E_1)_x \to (E_2)_x$ to be a linear map for all $x \in X$. We denote by $\mathrm{Hom}_X(E_1, E_2)$ the vector space of bundle morphisms between the vector bundles E_1 and E_2 over X and when $E_1 = E_2$ we set $\mathrm{End}(E_1) := \mathrm{Hom}_X(E_1, E_1)$.

Definition 2.4. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles on the same space X. A bundle morphism $h : E_1 \to E_2$ is a **bundle isomorphism** between these vector bundles if there is an inverse $h' : E_2 \to E_1$ of h which is also a bundle homomorphism. If there is a bundle isomorphism $h : E_1 \to E_2$ then E_1 and E_2 are said to be **isomorphic** vector bundles, and we use the notation $E_1 \cong E_2$ to indicate that E_1 and E_2 are isomorphic. An isomorphism from a vector bundle E to itself is called a **bundle automorphism** or **gauge transformation** of E, and we denoted the set of automorphisms of E by $\mathrm{Aut}(E)$.

Remark 2.2. Notice that if the bundle morphism $h : E_1 \to E_2$ has an inverse bundle morphism, then the map h is a homeomorphism and $h_x : (E_1)_x \to (E_2)_x$ is a linear isomorphism for all $x \in X$.

Lemma 2.1. A bundle morphism $h : E_1 \to E_2$ between vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ over the same base space X is a bundle isomorphism if it takes each fiber $(E_1)_x$ to the corresponding fiber $(E_2)_x$ by a linear isomorphism.

Proof. All we have to prove is that h^{-1} is a continuous map. This is clear since if we restrict to an open set $U \subset X$ over which E_1 and E_2 are trivial, then composing with local trivializations $\varphi_1|_U$ and $\varphi_2|_U$ of p_1 and p_2 respectively

$$\begin{tikzcd}
U \times \mathbb{K}^n \arrow[drr, "\pi_1"'] & E_1|_U \arrow[l, "\varphi_1|_U"'] \arrow[r, "h"] \arrow[dr, "p_1|_U"] & E_2|_U \arrow[r, "\varphi_2|_U"] \arrow[d, "p_2|_U"] & U \times \mathbb{K}^n \arrow[dl, "\pi_1"] \\
& & U &
\end{tikzcd}$$

reduces to the case that h is a continuous map $U \times \mathbb{K}^n \to U \times \mathbb{K}^n$ of the form $h(x, v) = (x, g_x(v))$, with $g_x \in \mathrm{GL}_n(\mathbb{K})$, and g_x depend continuously on x. Therefore the inverse matrix g_x^{-1} also depends continuously on x since g_x^{-1} is $1/\det(g_x)$ times the adjoint matrix of g_x. Hence $h^{-1}(x, v) = (x, g_x^{-1}(v))$ is a continuous map. $\square$

Some examples of isomorphic vector bundles are given next:

Example 2.7. The normal bundle of $\mathbb{S}^n$ in $\mathbb{R}^n$ is isomorphic to the product bundle $\mathbb{S}^n \times \mathbb{R}$ by the map $(x, tx) \mapsto (x, t)$.

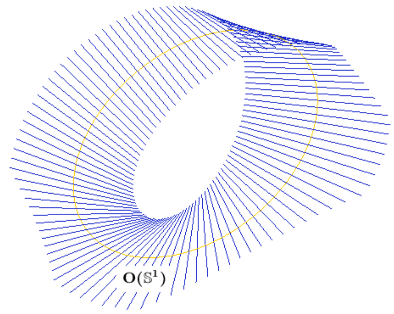
Example 2.8. The tangent bundle to $\mathbb{S}^1$ is also isomorphic to the trivial bundle $\mathbb{S}^1 \times \mathbb{R}$, via $(e^{i\theta}, ite^{i\theta}) \mapsto (e^{i\theta}, t)$, for $e^{i\theta} \in \mathbb{S}^1$ and $t \in \mathbb{R}$.

In what follows we discuss an approach that will allow us to know when two vector bundles are isomorphic. For this purpose we have the following definition.

Definition 2.5. Let $p : E \to X$ be a vector bundle. A **section** of p is a continuous map $s : X \to E$ such that to each $x \in X$ a vector $s(x) \in E_x$ is assigned, i.e $p \circ s = id_X$. We denote by $\Gamma(E)$ the set of sections of the vector bundle E.

Every vector bundle $p : E \to X$ has a canonical section, namely, the **zero section** $\mathbf{O}_E : X \to E$ whose value is the zero vector in each fiber. The zero section is identified with its image $\mathbf{O}_E(X)$, a subspace of E, therefore $p|_{\mathbf{O}_E(X)} : \mathbf{O}_E(X) \to X$ sending the zero vector of E_x to x is a homeomorphism. Let $h : E_1 \to E_2$ be an isomorphism between two vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$, as $h_x : (E_1)_x \to (E_2)_x$ is a linear isomorphism for all $x \in X$, then h sends the zero section $\mathbf{O}_{E_1}(X)$ of p_1 to the zero section $\mathbf{O}_{E_2}(X)$ of p_2, and as $h : E_1 \to E_2$ is a homeomorphism we have the homeomorphism $E_1 \setminus \mathbf{O}_{E_1}(X) \cong E_2 \setminus \mathbf{O}_{E_2}(X)$. Thus we can distinguish non-isomorphic bundles comparing the complement of the zero section in both vector bundles.

Example 2.9. Let $p : E \to \mathbb{S}^1$ be the Möbius bundle, then the zero section of p is the set $\mathbf{O}_p(\mathbb{S}^1) := \{[x, 0] \in E_x |\ x \in [0, 1]\}$.



$E \setminus \mathbf{O}_p(\mathbb{S}^1)$ is homeomorphic to a cylinder, which is a connected space. In the other hand, the trivial bundle $t : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ has zero section $\mathbf{O}_t(\mathbb{S}^1) = \mathbb{S}^1 \times \{0\}$, thus $(\mathbb{S}^1 \times \mathbb{R}) \setminus \mathbf{O}_t(\mathbb{S}^1)$ is the disconnected space $(\mathbb{S}^1 \times \mathbb{R}_+) \cup (\mathbb{S}^1 \times \mathbb{R}_-)$, where $\mathbb{R}_+$ and $\mathbb{R}_-$ are the positive and negative real numbers respectively. Therefore, the trivial bundle on $\mathbb{S}^1$ and the Möbius bundle are not isomorphic.

Remark 2.3. For an isomorphism $h : E_1 \to E_2$, as $h_x : (E_1)_x \to (E_2)_x$ is an linear isomorphism for all $x \in X$, then h sends non-vanishing sections to non-vanishing sections.

Example 2.10. For a tangent bundle $p : E \to \mathbb{S}^n$, a section is a tangent vector field on $\mathbb{S}^n$. By the hairy ball theorem if n is an even number with $n \geq 0$, then each continuous vector field s on the real sphere $\mathbb{S}^n$ vanishes at some point $v \in \mathbb{S}^n$, i.e., $s(v) = 0$. As the trivial bundle has a non-vanishing section and by the above theorem the tangent bundle $p : E \to \mathbb{S}^n$ does not have a non-vanishing section for n a positive even number, then these vector bundles are not isomorphic.

Lemma 2.2. Let $p : E \to X$ be a vector bundle of rank n. The vector bundle p is isomorphic to the trivial bundle if and only if it has n sections $s_1, ..., s_n$ such that the vectors $s_1(x), ..., s_n(x)$ are linearly independent in each fiber E_x.

Proof. In one direction this is evident since the trivial bundle certainly has such sections and an isomorphism of vector bundles takes linearly independent sections to linearly independent sections. Conversely, if one has n linearly independent sections s_i, the map $h : X \times \mathbb{R}^n \to E$ given by $h(x, t_1, ..., t_n) = \sum_i t_i s_i(x)$ is a linear isomorphism in each fiber, and is continuous since its composition with a local trivialization $p^{-1}(U) \to U \times \mathbb{R}^n$ is continuous. So by the latter lemma h is an isomorphism. $\square$

Remark 2.4. Notice that the previous lemma also makes sense for complex vector bundles.

Example 2.11. The tangent bundle on $\mathbb{S}^1$ is isomorphic to the trivial bundle, since the tangent bundle on $\mathbb{S}^1$ has the section

$$\begin{array}{cccc} s: & \mathbb{S}^1 & \longrightarrow & E \\ & (x_1, x_2) & \mapsto & (-x_2, x_1). \end{array}$$

Example 2.12. The tangent bundle on $\mathbb{S}^3$ is isomorphic to the trivial bundle, since the tangent bundle on $\mathbb{S}^3$ has the sections

$$\begin{array}{cccc} s_1: & \mathbb{S}^3 & \to & E \\ & (x_1, x_2, x_3, x_4) & \mapsto & (-x_2, x_1, -x_4, x_3), \end{array}$$

$$\begin{array}{cccc} s_2: & \mathbb{S}^3 & \to & E \\ & (x_1, x_2, x_3, x_4) & \mapsto & (-x_3, x_4, x_1, -x_2), \end{array}$$

$$\begin{array}{rccc} s_3: & \mathbb{S}^3 & \to & E \\ & (x_1,x_2,x_3,x_4) & \mapsto & (-x_4,-x_3,x_2,x_1), \end{array}$$

and s_1, s_2, s_3 are linear independent everywhere.

Now we move to the category of smooth manifolds M.

Definition 2.6. A **local frame** of a vector bundles $p: E \to M$ over an open set $U \subset X$ is an ordered n-tuple $s_1, ..., s_n$ of smooth section of E over U so that for each $x \in U$, $s_1(x), ..., s_n(x)$ form a basis of E_x.

Remark 2.5. Notice that by the Lemma 2.2 we have a local frame on each local trivialization of a vector bundle.

Using local frames, we can give the following criterion for the smoothness of sections.

Lemma 2.3. Let $p: E \to M$ be a vector bundle on a smooth manifold M. A section $s \in \Gamma(E)$ is smooth if and only if for any $x \in M$, there is a neighborhood U of x and a local frame $s_1, ..., s_n$ of E over U so that $s = f_1 s_1 + \cdots + f_n s_n$ for some smooth functions $f_1, ..., f_n$ defined on U.

Proof. Let $\varphi|_U : p^{-1}(U) \to U \times \mathbb{K}^n$ be the local trivialization associated to the frame $s_1, ..., s_n$. Then the smoothness of $f_1, ..., f_n$ is equivalent to the smoothness of $\varphi|_U \circ s|_U$. Since $\varphi|_U$ is a diffeomorphism, $s|_U$ is smooth. On the other hand, if s is smooth and $\varphi|_U$ is a local trivialization, we let $s_1, ..., s_k$ be the frame corresponding to this local trivialization constructed as above in Lemma 2.2. Then $f_1, .., f_n$ are coefficients of $\varphi|_U \circ s|_U$ in this basis. Since s and $\varphi|_U$ are both smooth, f_i's are smooth. □

Finally we give the key observation, which is a simple consequence of Tietze's extension theorem:

Lemma 2.4. Suppose that X is a compact hausdorff space, Y a compact subspace of X and E, F vector bundles over X. If $\varphi : E|_Y \to F|_Y$ is an isomorphism the φ can by extended as an isomorphism to an open $U \supset Y$.

Remark 2.6. Suppose that $X = X_1 \cup X_2$ where all spaces are compact Hausdorff and set $A = X_1 \cup X_2$. Assume that E_i is a vector bundle over X_i and $\varphi : E_1|_A \to E_2|_A$ is an isomorphism. Then we define a vector bundle $E_1 \cup_\varphi E_2$ over X as follows. As a topological space $E_1 \cup_\varphi E_2$ is the quotient of the disjoint union of E_1 and E_2 modulo the equivalence relation $v \sim \varphi(v)$. There is a natural projection $\pi : E_1 \cup_\varphi E_2 \to X$ and the fibers are naturally vector spaces. One can prove that this family of vector spaces is in fact locally trivial and moreover, it has the following properties:

1. If E is a vector bundle over X and $E_i = E|_{X_i}$, then the identity map $\mathrm{id} : E_1|_A \to E_2|_A$ is an isomorphism and
$$E \cong E_1 \cup_{\mathrm{id}} E_2.$$
2. The construction is natural in the sense that if there are isomorphisms $\beta_i : E_i \to E_i'$ such that $\beta_i \varphi = \varphi' \beta_i$ then:
$$E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'.$$
3. If (E_i, φ) and (E_i', φ') are two such data then:
$$(E_1 \cup_\varphi E_2) \oplus (E_1 \cup_\varphi E_2) \cong (E_1 \oplus E_1') \cup_{\varphi \oplus \varphi'} (E_2 \oplus E_2'),$$
$$(E_1 \cup_\varphi E_2) \otimes (E_1 \cup_\varphi E_2) \cong (E_1 \otimes E_1') \cup_{\varphi \times \varphi'} (E_2 \oplus E_2'),$$
$$(E_1 \cup_\varphi E_2)^* \cong (E_1^*) \cup_{(\varphi^*)^{-1}} E_2^*.$$

4. The isomorphism class of the vector bundle $E_1 \cup_\varphi E_2$ depends only on the homotopy class of the isomorphism φ.

2 Cocycles

Let $p : E \to X$ be a vector bundle and consider two local trivializations $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ and $\varphi_\beta : p^{-1}(U_\beta) \to U_\beta \times \mathbb{K}^n$ such that $U_\alpha \cap U_\beta \neq \emptyset$.

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times \mathbb{K}^n \arrow[dr, "\pi_1"'] & E_{U_\alpha \cap U_\beta} \arrow[l, "\varphi_\beta"'] \arrow[r, "\varphi_\alpha"] \arrow[d, "p|_{U_\alpha \cap U_\beta}"'] & (U_\alpha \cap U_\beta) \times \mathbb{K}^n \arrow[dl, "\pi_1"] \\
& U_\alpha \cap U_\beta. &
\end{tikzcd}$$

Then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a continuous invertible map of the form

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^n \to (U_\alpha \cap U_\beta) \times \mathbb{K}^n.$$

Now let $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{K}^n$, then fix x and only allow v to vary, the map $\varphi_\alpha \circ \varphi_\beta^{-1}$ for fixed x is now just a invertible map from $\mathbb{K}^n$ to $\mathbb{K}^n$. Therefore we have define a map called the **cocycle map**

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{K}) \tag{2.1}$$

where $\mathrm{GL}_n(\mathbb{K})$ is the general linear group of $\mathbb{K}$, such that

$$\begin{array}{cccc} \varphi_\alpha \circ \varphi_\beta^{-1} : & (U_\alpha \cap U_\beta) \times \mathbb{K}^n & \to & (U_\alpha \cap U_\beta) \times \mathbb{K}^n \\ & (x, v) & \mapsto & (x, g_{\alpha\beta}(x)v). \end{array}$$

This map is called a **transition function**. It is not hard to verify that the cocycle maps for a vector bundle satisfy the following conditions:

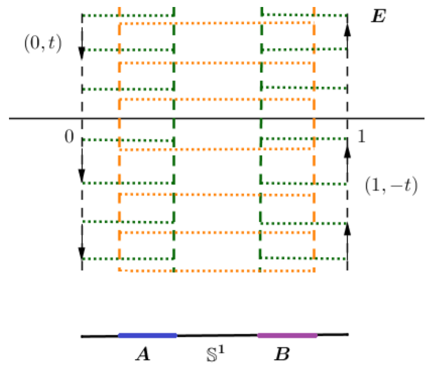
- $g_{\alpha\alpha}(x) = x$ for all $x \in U_\alpha$.
- $g_{\alpha\beta}(x)^{-1} = g_{\beta\alpha}(x)$ for all $x \in U_\alpha \cap U_\beta$.
- $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

The last one is know as the **cocycle condition**.

Example 2.13. Let $p : E \to \mathbb{S}^1$ be the Möbius bundle, where the trivialization cover is given by two overlapping open arcs U_1 and U_2 in $\mathbb{S}^1$ with local trivializations $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}$ given by $[t, s] \mapsto ([t], s)$, for $i = 1, 2$. Then the intersection of two open sets U_1 and U_2 are two disjoint open arcs A and B. By moving a fibre F once round the Möbius strip, if we do this we find that F has been reflected in its midpoint, this reflection is a manifestation of the group element $-1 \in \mathbb{R}$; clearly $(-1)^2 = 1$ the identity element, so that we end up with the group $\{1, -1\}$ of just two elements. The point is that if one moves once round the Möbius strip once, the one must have passed through A and B once. Then in changing from coordinates in A to coordinates in B, or vice versa, on must use the transition function g_{12} or g_{21} given by

$$g_{12}(x) = \begin{cases} 1 & \text{, if } x \in A \\ -1 & \text{, if } x \in B \end{cases}$$

$g_{21}(x) = g_{21}^{-1}(x)$ and $g_{11} = g_{22} = 1$.



The reason cocycles are important is that if one is given the base space X, the fiber $\mathbb{K}^n$ and the cocycles $g_{\alpha\beta}$, then we can reconstruct the bundle E' to which these objects belong. For this we need to find the trivializations φ_α, the total spaces E' and the continuous surjective map p. Set

$$E' := \Big(\bigsqcup_{\alpha\in I}(U_\alpha \times \mathbb{K}^n)\Big)\Big/\sim,$$

where $\sim$ is given by: $(x, v) \sim (y, v')$ iff $x = y$ and $v' = g_{\alpha\beta}(x)v$, for all $(x, v) \in U_\alpha \times \mathbb{K}^n$ and $(y, v') \in U_\beta \times \mathbb{K}^n$, and E' carries the quotient topology. An element of E' is written by $[x, v]$, so we can now define the continuous projection map

$$\begin{array}{cccc} p: & E' & \to & X \\ & [x,v] & \mapsto & x. \end{array}$$

Finally we define

$$\begin{array}{cccc} \varphi_\alpha^{-1}: & U_\alpha \times \mathbb{K}^n & \to & p^{-1}(U_\alpha) \\ & (x,v) & \mapsto & [(x,v)]. \end{array}$$

Clearly $p\varphi_\alpha^{-1}(x, v) = x$. We have hence constructed the bundle whose cocycles are $g_{\alpha\beta}$, thus from now on we can specify a bundle by giving only the base space X, the fiber $\mathbb{K}^n$ and the cocycles $g_{\alpha\beta}$. Moreover, if $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{K})\}$ are the cocycles arising from a vector bundle $p : E \to X$ with trivialization cover $\{U_i, \varphi_i\}$, then the vector bundle E' constructed in the previous paragraph is isomorphic to E. To prove this, we define

$$\begin{array}{cccc} \widetilde{f}: & E & \to & E' \\ & v & \mapsto & [\varphi_\alpha(v)] \end{array}$$

if $p(v) \in U_\alpha$. If $p(v) \in U_\alpha \cap U_\beta$, then

$$[\varphi_\beta(v)] = [\varphi_{\alpha\beta}(\varphi_\alpha(v))] = [\varphi_\alpha(v)] \in E'$$

thus the map $\widetilde{f}$ is well-defined. Clearly $p' \circ \widetilde{f} = p$. Let $q : \bigsqcup_{\alpha\in I}(U_\alpha \times \mathbb{K}^n) \to E'$ be the quotient map and $q_\alpha := q|_{U_\alpha\times\mathbb{K}^n}$, we note that

$$q_\alpha^{-1} \circ \widetilde{f} \circ \varphi^{-1} : U_\alpha \times \mathbb{K}^n \to U_\alpha \times \mathbb{K}^n$$

is the identity map, thus $\widetilde{f}$ is a continuous map. Since the restriction of q_α and φ_α to every fiber are vector space isomorphisms, it follows that so is $\widetilde{f}$. Finally by the Lemma 2.1, the map $\widetilde{f}$ is a vector bundle isomorphism between E and E'.

Notation 2. We will refer to the vector bundle obtained by the base space X, the fiber $\mathbb{K}^n$ and the cocycles $g_{\alpha\beta}$ as the **gluing cocycle** and we will denote it by $(X, \mathcal{U}, g_\bullet, \mathbb{K}^n)$, where $\mathcal{U}$ is the family on which the cocycles are defined.

The second important aspect of the cocycles of a vector bundle is that more than one set of cocycles $g_{\alpha\beta}$ can induce the same bundle. To see this let $p : E \to X$ and $p' : E' \to X$ be two isomorphic vector bundles with fiber $\mathbb{K}^n$ and trivialization cover $\{U_\alpha, \varphi_\alpha\}$ and $\{U_\alpha, \psi_\alpha\}$ respectively. Then the map $\varphi_\alpha \circ \psi_\alpha^{-1} : U_\alpha \times \mathbb{K}^n \to U_\alpha \times \mathbb{K}^n$ restricted to a fixed $x \in U_\alpha$ gives rise to an invertible map from $\mathbb{K}^n$ to $\mathbb{K}^n$, therefore we have define the maps $\lambda_\alpha : U_\alpha \to \mathrm{GL}_n(\mathbb{K})$ given by $x \mapsto (\varphi_\alpha \circ \psi_\alpha^{-1})|_x$. Thus by definition for $x \in U_\alpha \cap U_\beta$ we have

$$\begin{aligned}
\lambda_\alpha^{-1}(x) \circ g_{\alpha\beta}(x) \circ \lambda_\beta(x) &= ((\varphi_\alpha \circ \psi_\alpha^{-1})(x))^{-1} \circ ((\varphi_\alpha \circ \psi_\beta^{-1})(x)) \circ (\varphi_\beta \circ \psi_\beta^{-1})(x) \\
&= (\psi_\alpha \circ \psi_\beta)(x) \\
&= g'_{\alpha\beta}(x).
\end{aligned}$$

This discussion motivates the following definition.

Definition 2.7. We say that the cocycle sets $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ for the vector bundles $p : E \to X$ and $p' : E \to X$ respectively are **equivalent**, denote by $\{g_{\alpha\beta}\} \sim \{g'_{\alpha\beta}\}$, iff there are continuous maps $\{\lambda_\alpha : U_\alpha \to \mathrm{GL}_n(\mathbb{K})\}$ such that $g'_{\alpha\beta}(x) = \lambda_\alpha^{-1}(x) \circ g_{\alpha\beta}(x) \circ \lambda_\beta(x)$ for all $x \in U_\alpha \cap U_\beta$ and each $\alpha,\ \beta$.

Thus we show that:

Proposition 2.1. The vector bundles $p : E \to X$ and $p' : E' \to X$ are isomorphic iff there exist a cover $\{U_\alpha\}$ of X such that $\{g_{\alpha\beta}\} \sim \{g'_{\alpha\beta}\}$.

3 Operations on vector bundles

We first discuss the sum of vector bundles with the future goal of defining a semigroup on the set of isomorphism classes of n-dimensional real (complex) vector bundles on the base space X, denoted by $\mathrm{Vect}^n_{\mathbb{R}}(X)$ in the real case and $\mathrm{Vect}^n_{\mathbb{C}}(X)$ in the complex case. Here we give some important observations allowing us to define the sum operation between vector bundles:

- Given a vector bundle $p : E \to X$ and a subspace $Y \subset X$, then $p : p^{-1}(Y) \to Y$ is clearly a vector bundle. We call this the restriction of E over Y.
- Given vector bundles $p_1 : E_1 \to X_1$ and $p_2 : E_2 \to X_2$, then the product $p_1 \times p_2 : E_1 \times E_2 \to X_1 \times X_2$ is also a vector bundle with fiber the product $p_1^{-1}(x_1) \times p_2^{-1}(x_2)$, for each $x_1 \in X_1,\ x_2 \in X_2$. Note that $\varphi_\alpha \times \varphi_\beta$ is a local trivialization for $E_1 \times E_2$ where $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ and $\varphi_\beta : p^{-1}(U_\beta) \to U_\beta \times \mathbb{K}^m$ are local trivializations for E_1 and E_2 respectively.

If E_1 and E_2 both have the same base space X, the restriction of the product $E_1 \times E_2$ over the diagonal $X = \{(x, x) \in X \times X\}$ is exactly

$$E_1 \oplus E_2 := \{(v_1, v_2) \in E_1 \times E_2 |\ p_1(v_1) = p_2(v_2)\}.$$

Therefore by previous observations we have the following definition:

Definition 2.8. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be vector bundles over the same base space. We define the vector bundle

$$\begin{array}{ccc} E_1 \oplus E_2 & \to & X \\ (v_1, v_2) & \mapsto & p_1(v_1) = p_2(v_2). \end{array}$$

The fibers of this bundle are the direct sums of the fibers of E_1 and E_2. This vector bundle is called the **Whitney sum** of the vector bundles p_1 and p_2.

If $\{g^i_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_{n_i}(\mathbb{K})\}$ are the cocycles for each E_i and $i = 1, 2$, then the cocycle associated with the Whitney sum is the cocycle of the block diagonal matrix functions

$$\begin{array}{cccc} g^1_{\beta\alpha} \oplus g^2_{\beta\alpha} : & U_\alpha \cap U_\beta & \to & GL_{n_1+n_2}(\mathbb{K}) \\ & x & \mapsto & g^1_{\beta\alpha}(x) \oplus g^2_{\beta\alpha}(x). \end{array}$$

Example 2.14. The direct sum of trivial vector bundles is again a trivial bundle.

But the direct sum of two nontrivial vector bundles may be trivial, as the following example illustrates:

Example 2.15. Consider the tangent vector bundle and the normal vector bundle over $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The direct sum between these vector bundles is the trivial bundle $\mathbb{S}^n \times \mathbb{R}^{n+1}$, since elements of the direct sum are triples $(x, v, tx) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with $x \perp v$ and the map $(x, v, tx) \mapsto (x, v + tx)$ gives an isomorphism of the direct sum bundle with $\mathbb{S}^n \times \mathbb{R}^{n+1}$.

In addition to direct sum, we can define tensor product over vector bundles.

Definition 2.9. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be two vector bundles with cocycles $\{g^i_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_{n_i}(\mathbb{K})\}$ for each E_i and $i = 1, 2$. Then the cocycles for the **tensor bundle** denoted by $E_1 \otimes E_2$ are the tensor product function

$$\begin{array}{cccc} g^1_{\beta\alpha} \otimes g^2_{\beta\alpha} : & U_\alpha \cap U_\beta & \to & GL_{n_1 \cdot n_2}(\mathbb{K}) \\ & x & \mapsto & g^1_{\beta\alpha}(x) * g^2_{\beta\alpha}(x) \end{array}$$

where $*$ denote the Kronecker product of the two matrix $g^1_{\beta\alpha}(x)$ and $g^2_{\beta\alpha}(x)$.

It is routine to verify that the tensor product operation for vector bundles over a fixed base space is commutative, associative, and has an identity element up to isomorphism, the trivial line bundle. It is also distributive with respect to direct sum.

Another construction that will be used in this work is the **conjugate bundle** $\overline{E} \to X$ of a complex vector bundle $E \to X$. As topological space, E and $\overline{E}$ are the same, but the scalar multiplication on $\overline{E}$ is given by $\lambda(v) := \overline{\lambda} v$, where $\overline{\lambda} v$ is the scalar multiplication in E. So the local trivializations for $\overline{E}$ are obtained from local trivializations for E by composing with the map $\mathbb{C}^n \to \mathbb{C}^n$ given by $z \mapsto \overline{z}$ in each fiber. The cocycles $\overline{g}_{\beta\alpha}$ of $\overline{E}$ are the complex conjugates of the cocycles $g_{\beta\alpha}$ of E. For line bundles, as $z\overline{z} = 1$, with $|z| = 1$, then $E \otimes \overline{E}$ is isomorphic to the trivial line bundle.

Now we give another construction of vector bundle over two vector bundles $p : E \to X$ and $p' : E' \to X$ on X. We denote the space of continuous linear maps from the fiber E_x to the fiber E'_x by $\mathrm{Hom}(E_x, E'_x)$, for all $x \in X$. We define the set $\mathrm{Hom}(E, E') := \coprod_{x \in X} \mathrm{Hom}(E_x, E'_x)$ and define the projection $q : \mathrm{Hom}(E, E') \to X$ given by $\mathrm{Hom}(E_x, E'_x) \mapsto x$ for all $x \in X$. Consider open subset $U \subset X$ over which $E|_U$ and $E'|_U$ are trivial, with local trivializations given by $\varphi : E|_U \to U \times \mathbb{K}^n$ and $\psi : E'|_U \to U \times \mathbb{K}^m$. Then we define the bijections $\phi_U : q^{-1}(U) \to U \times \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m)$ given on x fibers by the linear isomorphism $\mathrm{Hom}(E_x, E'_x) \to \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m)$ given by $f \mapsto \psi \circ f \circ \varphi^{-1}$. We force ϕ_U to be a homeomorphism. Thus we have the vector bundle $q : \mathrm{Hom}(E, E') \to X$, called the **hom-bundle**, with fibers $\mathrm{Hom}(E_x, E'_x)$.

Remark 2.7. Is not difficult to see that the sections of the hom-bundle $\mathrm{Hom}(E, E')$ is a bundle morphism $E \to E'$, i.e.,

$$\Gamma(\mathrm{Hom}(E, E')) = \mathrm{Hom}_X(E, E').$$

Finally, we discuss the inner products on vector bundles.

Definition 2.10. If E is a vector space over $\mathbb{R}$, a **positive-definite inner product** on E is a symmetric bilinear map

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & E \times E & \to & \mathbb{R} \\ & (v, w) & \mapsto & \langle v, w \rangle \end{array}$$

such that $\langle v, v \rangle > 0$ for all $v \in E \setminus \{0\}$.

If E is a vector space over $\mathbb{C}$, a **non-degenerate Hermitian inner product** on E is a map

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & E \times E & \to & \mathbb{C} \\ & (v, w) & \mapsto & \langle v, w \rangle \end{array}$$

wich is $\mathbb{C}$-antilinear in the first input, $\mathbb{C}$-linear in the second input and such that $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v \in E \setminus \{0\}$.

From now on we will refer to inner product to discuss the real (positive-definite inner product) and complex (non-degenerate Hermitian inner product) case.

Definition 2.11. An **inner product** on a vector bundle $p : E \to X$ is a map $\langle \cdot, \cdot \rangle : E \otimes E \to \mathbb{K}$ such that for all $x \in X$ restricts to an inner product $\langle \cdot, \cdot \rangle_x : E_x \otimes E_x \to \mathbb{K}$.

Proposition 2.2. If the base space X is paracompact space, then an inner product exists for a vector bundle $p : E \to X$.

Proof. Let $\langle \cdot, \cdot \rangle_{\mathbb{R}} : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ be the standard inner product on $\mathbb{R}^n$, that is $\langle e_i, e_j \rangle = \delta_{ij}$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$ and let $\{U_\alpha, \varphi_\alpha\}$ be a trivialization cover of X. Given $x \in X$, for all neighborhood U_α of x we define

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle_\alpha : & E_x \times E_x & \to & \mathbb{R} \\ & (v, w) & \mapsto & \langle v, w \rangle_\alpha = \langle \varphi_\alpha|_x(v), \varphi_\alpha|_x(w) \rangle_{\mathbb{R}}. \end{array}$$

Finally, let $\{\lambda_\beta\}$ be a partition of unity subordinate to $\{U_\alpha\}$, then we define

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & E \times E & \to & X \\ & (v, w) & \mapsto & \langle v, w \rangle = \sum_{\beta,\ x \in U_\beta} \lambda_\beta \ p(v) \ \langle v, w \rangle_\beta. \end{array}$$

$\square$

Definition 2.12. Let $p : E \to X$ be a vector bundle. The map $p' : E' \to X$ is a **subbundle** of p if $E' \subset E$ is a vector subspace of E which is itself a bundle with respect to the restriction of $p|_{E'}$.

Definition 2.13. If E is a subbundle of F over X, then the **quotient bundle** F/E is a vector bundle with fiber F_x/E_x.

Proposition 2.3. Let $p : E \to X$ be a vector bundle on paracompact base space X and $E_0 \subset E$ is a vector subbundle, then there is a vector subbundle $E_0^\perp \subset E$ such that $E_0 \oplus E_0^\perp \cong E$.

Proof. With respect to a chosen inner product $\langle\cdot,\cdot\rangle$ on E, let $E_0^{\perp} := \{v \in E|\ \langle v,w\rangle = 0 \text{ for all } w \in E_0\}$ be the orthogonal complement of E_0 in E. We claim that the natural projection $E_0^{\perp} \to X$ is a vector bundle. First we assume E is the trivial bundle $X \times \mathbb{R}^n$ and that E_0 is a vector bundle of dimension m. Then for each point $x_0 \in X$ there is a neighborhood U_{x_0} such that the vector bundle $E_0 \to X$ has m independent local sections $x \mapsto (x, s_i(x))$ with extended to set of n independent local sections of E by choosing the sections $s_{m+1}, ..., s_n$ first in the fiber E_{x_0}, then taking the same vectors for all nearby fibers, since $s_1, ..., s_m, s_{m+1}, ..., s_n$ are independent at x_0 they will remain independent for nearby x by continuity of the determinant function. Using the given inner product, we apply the Gram-Schmidt orthogonalization process to $s_1, ..., s_m, s_{m+1}, ..., s_n$ in each fiber to obtain new sections s'_i, where the first m of them are a basis for E_0 in each fiber. The sections s'_i allow us to define a local trivialization $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$ with $\varphi(x, s'_i(x)) = (x, e_i)$, where e_i is the i^{th} standard basis vector of $\mathbb{R}^n$. Thus φ carries E_0 to $U \times \mathbb{R}^m$ and $E_0^{\perp}$ to $U \times \mathbb{R}^{n-m}$, so $\varphi|_{E_0^{\perp}}$ is a local trivialization of $E_0^{\perp}$. Therefore $E_0^{\perp} \to X$ is a vector bundle and $E_0 \oplus E_0^{\perp}$ is isomorphic to E via the map $(v,w) \mapsto v + w$ by the Lemma 2.1. ☐

Proposition 2.4. For each vector bundle $p : E \to X$ with X paracompact space there exist a vector bundle $E' \to X$ such that $E \oplus E'$ is the trivial bundle.

Proof. Let $\{U_\alpha, \varphi_\alpha\}$ be a trivialization cover of the bundle $p : E \to X$. Let U_x be the neighborhood of x over that E is trivial. By Urysohn's lemma there is a map $g_x : X \to [0,1]$ that is 0 outside U_x and nonzero at x. Letting x vary, the sets $g_x^{-1}(0,1]$ form an open cover of X. By compactness this has a finite subcover $\{U_i, g_i\}_{i=1,...,k}$ say. We define $f_i : E \to \mathbb{R}^n$ by $f_i(v) := g_i(p(v))(\pi_i\varphi_i(v))$, for all $i = 1, ..., k$, where π_i is the projection of the local trivialization $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ to $\mathbb{R}^n$. Then f_i is a linear injection on each fiber over $g_i^{-1}(0,1]$, so we define a map $f : E \to \mathbb{R}^{n^k}$ given by $v \mapsto (f_1(v), ..., f_k(v))$, then f is a linear injection on each fiber. Finally we define the map $F : E \to X \times \mathbb{R}^N$ given by $v \mapsto (p(v), f(v))$. Since projection of $\mathbb{R}^N$ onto the i-th $\mathbb{R}^N$ factor gives the second coordinate of local trivialization over $g_i^{-1}(0,1]$, the image of F is a subbundle of the product $X \times \mathbb{R}^N$. Thus we have E isomorphic to a subbundle of $X \times \mathbb{R}^N$, and by previous lemma there is a complementary subbundle E' with $E \oplus E'$ isomorphic to $X \times \mathbb{R}^N$. ☐

Remark 2.8. This construction can be made in the same way on $\mathbb{C}$.

4 Pullback bundles

In this section we discuss the notion of a pullback bundle. The idea is that if we have a fiber bundle on a topological space Y and a continuous map $f : X \to Y$ we get a fiber bundle whose fiber over $x \in X$ is the fiber over $f(x)$. The formalization of this is the following:

Definition 2.14. Let $p : E \to Y$ be a fiber bundle with fiber F and $f : X \to Y$ a continuous map. Defining the space $f^*(E)$ by

$$f^*(E) := \{(x,v) \in X \times E|\ f(x) = p(v)\}$$

and a map $\widetilde{p} : f^*(E) \to X$ by $(x,v) \mapsto x$ we get a fiber bundle with fiber F, called the **pullback bundle**. Indeed, if (U_i, φ_i) is a trivialization cover of E then $(f^{-1}U_i, \psi_i)$ where

$$\psi_i : \widetilde{p}^{-1}(f^{-1}U_i) \to (f^{-1}U_i) \times F$$

is given by $\psi_i(x,v) = (x, \pi_2(\varphi_i(v)))$, is a trivialization cover of $f^*(E)$. So we have the following commutative diagram

$$\begin{tikzcd} f^*(E) \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{p}"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\widetilde{f} : f^*(E) \to E$ is given by $\widetilde{f}(x,v) = v$.

Now we study the homotopy properties of the pullback.

Lemma 2.5. A vector bundle $p : E \to X \times [a,b]$ is trivial if its restrictions over $E_1 := E|_{X\times[a,c]}$ and $E_2 := E|_{X\times[c,b]}$ are both trivial for some $c \in (a,b)$.

Proof. Let $\varphi_1 : E_1 \to X \times [a,c] \times \mathbb{R}^n$ and $\varphi_2 : E_2 \to X \times [c,b] \times \mathbb{R}^n$ be isomorphisms. For define an isomorphism in all of E we need that the previous isomorphisms agree in $p^{-1}(X \times \{c\})$, so we replace φ_2 by its composition with the isomorphism $X \times [c,b] \times \mathbb{R}^n \to X \times [c,b] \times \mathbb{R}^n$ which on each slice $X \times \{x\} \times \mathbb{R}^n$ is given by $\varphi_1\varphi_2^{-1} : X \times \{c\} \times \mathbb{R}^n \to X \times \{c\} \times \mathbb{R}^n$. Once φ_1 and φ_2 agree on $E_1 \cap E_2$, thus they define a trivialization of E. $\square$

Lemma 2.6. For a vector bundle $p : E \to X \times I$, there exists an open cover $\{U_\alpha\}$ of X so that each restriction $p^{-1}(U_\alpha \times I) \to U_\alpha \times I$ is trivial.

Proof. By compactness of $[0,1]$, for each $x \in X$ we can find open neighborhoods $U_{x,1}, ..., U_{x,k}$ in X and a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that the bundle is trivial on $U_{x,i} \times [t_{i-1}, t_i]$. Then by previous lemma the bundle is trivial on $U_\alpha \times I$, where $U_\alpha = U_{x,1} \cap \cdots \cap U_{x,k}$. $\square$

Proposition 2.5. The restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact.

Proof. By the previous lemma we can choose an open cover U_α of X so that E is trivial on each $U_\alpha \times I$. As X is a paracompact space there is a countable cover $\{V_i\}_{i\geq 1}$ of X and a partition of unity $\{\lambda_i\}$ with λ_i support in V_i, such that each V_i is a disjoint union of open sets each contained in some U_α, thus E is trivial on each $V_i \times I$. For $i \geq 0$, let $\varphi_i = \lambda_1 + \cdots + \lambda_i$, we set X_i the graph of φ_i the subspace of $X \times I$ consisting of points of the form $(x, \varphi_i(x))$, and let $p_i : E_i \to X_i$ be the restriction $E|_{X_i}$. Since E is trivial on $V_i \times I$, the natural projection homeomorphism $X_i \to X_{i-1}$ lifts to a homeomorphism $h_i : E_i \to E_{i-1}$ which is the identity outside $p^{-1}(U_i)$ and which takes each fiber of E_i isomorphically onto the corresponding fiber of E_{i-1}. Since near each point $x \in X$ only finitely many λ_i's are nonzero, so there is a neighborhood of x in which all but finitely many h_i's are the identity. Therefore the infinite composition $h = h_1h_2\cdots$ is then well-defined isomorphism from $E|_{X\times\{1\}}$ to $E|_{X\times\{0\}}$. $\square$

Theorem 2.1. Given a vector bundle $p : E \to X$ and homotopic maps $f_0, f_1 : Y \to X$, then the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if Y is paracompact space.

Proof. Let $F : Y \times I \to X$ be a homotopy from f_0 to f_1. The restrictions of $F^*(E)$ over $Y \times \{0\}$ and $Y \times \{1\}$ are then $f_0^*(E)$ and $f_1^*(E)$ respectively, thus by previous proposition these bundles are isomorphic. $\square$

Using the characteristic property of a pullback we have the elementary properties of pullbacks:

- $(fg)^*(E) \cong g^*(f^*(E))$.

- $\mathrm{id}^*(E) \cong E$.
- $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$.
- $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$.

Corollary 2.1. Let $f, g : X \to Y$ be two homotopic maps. Then $f^* = g^*$.

Proof. If $H : X \times I \to Y$ is a homotopy from f to g and E' is a vector bundle on Y, then consider the bundle $E = H^*(E')$. By above theorem $E|_{X\times\{0\}}$ and $E|_{X\times\{1\}}$ are isomorphic. But they are respectively equal to $f^*(E')$ and $g^*(E')$. □

Corollary 2.2. A homotopy equivalence $f : Y \to X$ of paracompact spaces induces a bijection $f^* : \mathrm{Vect}^n(X) \to \mathrm{Vect}^n(Y)$. In particular, every vector bundle on a contractible paracompact base space is trivial.

Proof. If g is a homotopy inverse of f then we have $f^*g^* = \mathrm{id}^* = \mathrm{id}$ and $g^*f^* = \mathrm{id}^* = \mathrm{id}$. □

5 Clutching functions

The idea of this part is to construct vector bundles $E \to \mathbb{S}^k$ over the sphere of dimension k in $\mathbb{R}^{k+1}$. We can write $\mathbb{S}^k$ as the union of its upper and lower hemispheres D^k_+ and D^k_-, with $D^k_+ \cap D^k_- = \mathbb{S}^{k-1}$. Given a map $f : \mathbb{S}^{k-1} \to \mathrm{GL}_n(\mathbb{R})$, we define E_f as the quotient of the disjoint union

$$D^k_+ \times \mathbb{R}^n \amalg_f D^k_- \times \mathbb{R}^n / \sim$$

obtained by identifying $(x, v) \in \partial D^k_- \times \mathbb{R}^n$ with $(x, f(x)v) \in \partial D^k_+ \times \mathbb{R}^n$. So the elements in E_f are classes $[(x, v)]$, with $x \in D^k_\pm,\ v \in \mathbb{R}^n$. We have the natural projection

$$\begin{array}{ccc} E_f & \to & \mathbb{S}^k \\ [(x,v)] & \mapsto & x \end{array}$$

which is a vector bundle. The map $f : \mathbb{S}^{k-1} \to \mathrm{GL}_n(\mathbb{R})$ is called **clutching function** for E_f.

Remark 2.9. This construction can be made in the same way on $\mathbb{C}$.

Example 2.16. Let H be the canonical line bundle over the Riemann sphere $\mathbb{CP}^1 = \mathbb{S}^2$. Points of the form $z_0/z_1 = z$ in the disk D^2_0 inside the unit circle $\mathbb{S}^1 \subset \mathbb{C}$, i.e. $|z| \le 1$ which is in correspondence with $[z_0, z_1] \in \mathbb{CP}^1$, but $(z_0, z_1) \sim (z_0/z_1, 1)$ with $\lambda = z_1$, so we have that $[z_0/z_1] = [z, 1]$, $|z| \le 1$. On the other hand, points of the form $w = z_0/z_1$ in the disk D^2_∞ outside of $\mathbb{S}^1$, then $1/w = z = z_1/z_0$ belonging within $\mathbb{S}^1$, by the above this corresponds to $[z_1/z_0, 1] = [z_1, z_0] = [1, z_0/z_1] = [1, z^{-1}] \in \mathbb{CP}^1$, and $|z^{-1}| \ge 1$ since $|w| \ge 1$. Hence we can write $\mathbb{CP}^1 = \mathbb{S}^2 = D^2_0 \cup D^2_\infty$ with

$$D^2_0 = \{[z, 1] |\ |z| \le 1\},$$

$$D^2_\infty = \{[1, z^{-1}] |\ |z^{-1}| \ge 1\}.$$

Over D^2_0 and D^2_∞ we have respectively the trivializations

$$([z, 1], (z, 1)) \mapsto [z, 1],$$

$$([1, z^{-1}], (1, z^{-1})) \mapsto [1, z^{-1}],$$

of the canonical line bundle. Over $D_0^2 \cap D_\infty^2 = \mathbb{S}^1$ spent from the trivialization of D_∞^2 to the trivialization of D_0^2 multiplying by z. Hence the canonical line bundle on $\mathbb{S}^2$ have the clutching function $f(z) = (z)$.

Lemma 2.7. If f and g are homotopic clutching functions then E_f and E_g are isomorphic.

Proof. Let $f, g : \mathbb{S}^{k-1} \to \mathrm{GL}_n(\mathbb{R})$ be clutching functions , since $f \sim g$ let $F : \mathbb{S}^{k-1} \times I \to \mathrm{GL}_n(\mathbb{R})$ be a homotopy between f to g, i.e. $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. For the same previous construction, we take $\mathbb{S}^k \times I = (D_+^k \cup D_-^k) \times I$, with $(D_+^k \cap D_-^k) \times I = \mathbb{S}^{k-1} \times I$. We define E_F as the quotient of the disjoint union

$$D_+^k \times I \times \mathbb{R}^n \amalg_F D_-^k \times I \times \mathbb{R}^n$$

obtained by identifying $(x, a, v) \in \partial D_+^k \times I \times \mathbb{R}^n$ with $(x, a, F(x, a)v) \in \partial D_-^k \times I \times \mathbb{R}^n$, where we denote the classes under $\sim$ by $[x, a, v]$. Thus we have the vector bundle

$$\begin{array}{ccc} E_F & \to & \mathbb{S}^k \times I \\ [(x, a, v)] & \mapsto & (x, a). \end{array}$$

Note that E_F on $\mathbb{S}^k \times \{0\}$ is such that $(x, 0, v) \sim (x, 0, F(x, 0)v) = (x, 0, f(x)v)$, i.e. E_F is restricted to E_f over $\mathbb{S}^k \times \{0\}$ and the same form E_F is restricted to E_g over $\mathbb{S}^k \times \{1\}$. Therefore $E_f \cong E_g$ by the Proposition 2.5. $\square$

Thus if we denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$, then the association $f \to E_f$ gives a well-defined map $\phi : [\mathbb{S}^{k-1}, \mathrm{GL}_n(\mathbb{R})] \to \mathrm{Vect}^n_{\mathbb{R}}(\mathbb{S}^k)$. The same constructions are obtained in the complex case.

Proposition 2.6. The map $\phi : [\mathbb{S}^{k-1}, \mathrm{GL}_n(\mathbb{C})] \to Vect^n_{\mathbb{C}}(\mathbb{S}^k)$ given by $f \mapsto E_f$ is a bijection.

Proof. We define a inverse $\psi : \mathrm{Vect}^n_{\mathbb{C}}(\mathbb{S}^k) \to [\mathbb{S}^{k-1}, \mathrm{GL}_n(\mathbb{C})]$. Let $p : E \to \mathbb{S}^k$ be a n dimensional complex vector bundle, then $p_1 : E_+ \to D_+^k$ and $p_2 : E_- \to D_-^k$ are trivial, since D_+^k and D_-^k are contractile. Thus we take trivializations $\varphi_\pm : E_\pm \to D_\pm^k \times \mathbb{C}^n$, then

$$\begin{array}{cccc} \varphi_+\varphi_-^{-1}: & \mathbb{S}^{k-1} \times \mathbb{C}^n & \to & \mathbb{S}^{k-1} \times \mathbb{C}^n \\ & (x, v) & \mapsto & (x, g(x)v) \end{array}$$

induces a map $g : \mathbb{S}^{k-1} \to \mathrm{GL}_n(\mathbb{C})$ such that its homotopy class is by definition an element in $[\mathbb{S}^{k-1}, \mathrm{GL}_n(\mathbb{C})]$. Therefore we define $\psi(E) = g$. $\square$

Example 2.17. Let E_f and E_g be vector bundles on $\mathbb{S}^k$ with clutching functions $f, g : \mathbb{S}^{k-1} \to \mathrm{GL}_n(\mathbb{C})$. We show that $E_{fg} \oplus \underline{n} \cong E_f \oplus E_g$ where fg is the clutching function given by $(fg)(x) = f(x)g(x)$ for all $x \in \mathbb{S}^{k-1}$. The bundle $E_f \oplus E_g$ has clutching function $f \oplus g : \mathbb{S}^{k-1} \to \mathrm{GL}_{2n}(\mathbb{C})$ where $(f \oplus g)(x)$ is the matrix

$$\begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix}$$

for all $x \in \mathbb{S}^{k-1}$. Also the bundle $E_{fg} \oplus \underline{n}$ has clutching function given by the matrix

$$\begin{pmatrix} f(x)g(x) & 0 \\ 0 & id_{n\times n} \end{pmatrix}.$$

Since $\mathrm{GL}_{2n}(\mathbb{C})$ is path-connected, we can find a path $\alpha : [0,1] \to \mathrm{GL}_{2n}(\mathbb{C})$ such that $\alpha(0) = id_{2n\times 2n}$ and $\alpha(1)$ is the matrix which interchanges the two factors of $\mathbb{C}^n \times \mathbb{C}^n$

$$\begin{pmatrix} 0 & id_{n\times n} \\ id_{n\times n} & 0 \end{pmatrix}.$$

Then the matrix product $(f \oplus id_{n\times n})\alpha(t)(id_{n\times n} \oplus g)\alpha(t)$ is such that:

- For $t = 0$, $(f \oplus id_{n\times n})\alpha(0)(id_{n\times n} \oplus g)\alpha(0)$ is the matrix $f \oplus g$.
- For $t = 1$, we have that $\alpha(1)(id_{n\times n} \oplus g)$ is the matrix

$$\begin{pmatrix} 0 & g(x) \\ id_{n\times n} & 0 \end{pmatrix}$$

therefore $\alpha(1)(id_{n\times n} \oplus g)\alpha(1)$ is the matrix $g \oplus id_{n\times n}$, thus

$$(f \oplus id_{n\times n})\alpha(1)(id_{n\times n} \oplus g)\alpha(1)$$

is the matrix $fg \oplus id_{n\times n}$.

Then $(f \oplus id_{n\times n})\alpha(t)(id_{n\times n} \oplus g)\alpha(t)$ gives a homotopy from $f \oplus g$ to $fg \oplus id_{n\times n}$, thus by the Lemma 2.7 we conclude that $E_{fg} \oplus \underline{n} \cong E_f \oplus E_g$.

As a particular case, by the Example 2.16 the canonical line bundle $H \to \mathbb{S}^2$ has the clutching function $f : \mathbb{S}^1 \to \mathbb{C}$ given by $f(z) = (z)$. Hence the bundles $(H \otimes H) \oplus \underline{1}$ and $H \oplus H$ has clutching functions $\mathbb{S}^1 \to \mathrm{GL}_2(\mathbb{C})$ given respectively by the matrix

$$\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.$$

By previous general argument we have that $(H \otimes H) \oplus \underline{1} \cong H \oplus H$.

6 The universal bundle

The aim of this section is to show that if X is a paracompact space, then all n-dimensional vector bundles $p : E \to X$ are obtainable as pullbacks of a special n-dimensional vector bundle. First we define the objects on which we will work.

Definition 2.15. The **Stiefel manifold** $V_n(\mathbb{C}^k)$ is the space of n-tuples (called n**-frames**) of orthonormal vectors in $\mathbb{C}^k$. The **Grassmann manifold** $G_n(\mathbb{C}^k)$, for $0 \leq n \leq k$, as a set is the collection of all n-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^k$.

Note that $V_n(\mathbb{C}^k)$ is a subset of the product of n copies of the unit sphere $\mathbb{S}^{k-1} \subset \mathbb{C}^k$, therefore $V_n(\mathbb{C}^k)$ has the subspace topology. An element in $V_n(\mathbb{C}^k)$ can be seen as a matrix X of dimension $k \times n$ satisfying the algebraic equation $X^T X = I_{n\times n}$, hence $V_n(\mathbb{C}^k)$ is a compact space since is a closed space in the compact space given by n copies of $\mathbb{S}^{k-1}$. Now we consider the natural surjection $V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ sending an n-frame to the subspace it spans, therefore $G_n(\mathbb{C}^k)$

is topologized by giving it the quotient topology with respect to this surjection. So $G_n(\mathbb{C}^k)$ is compact as well.

Now fix $k \geq 0$ and consider the inclusions $\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \cdots$ which induce the inclusions $V_n(\mathbb{C}^k) \subset V_n(\mathbb{C}^{k+1}) \subset \cdots$ and inclusions $G_n(\mathbb{C}^k) \subset G_n(\mathbb{C}^{k+1}) \subset \cdots$.

Definition 2.16. The **infinite Stiefel space** and the **infinite grassmannian space** is define respectively by

$$V_n(\mathbb{C}^\infty) := \mathrm{colim}_{k\to\infty} V_n(\mathbb{C}^k),$$

$$G_n(\mathbb{C}^\infty) := \mathrm{colim}_{k\to\infty} G_n(\mathbb{C}^k)$$

with the direct limit topology. So in the Grassmannian case, a set in $G_n(\mathbb{C}^\infty)$ is open iff it intersects each $G_n(\mathbb{C}^k)$ in an open set.

Remark 2.10. Notice that the same constructions works for the real numbers $\mathbb{R}$ instead of complex numbers $\mathbb{C}$.

An important observation that will be used later is as follows:

Remark 2.11. The projection maps $p : V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ sending an n-frame to the subspace it spans, are fiber bundles. To show this, if we fix an n-plane $\ell \in G_n(\mathbb{C}^k)$ and choose an orthonormal basis for ℓ , then we obtain continuously varying orthonormal bases for all n-planes ℓ' in a neighborhood U of ℓ by projecting the basis for ℓ orthogonally onto ℓ' to obtain a non-orthonormal basis for ℓ' , then applying the Gram–Schmidt process to this basis to make it orthonormal. The formulas for the Gram–Schmidt process show that it is continuous. Having orthonormal bases for all n-planes in U , we can use these to identify these n-planes with $\mathbb{C}^n$, hence n-frames in these n-planes are identified with n-frames in $\mathbb{C}^n$, and so $p^{-1}(U)$ is identified with $U \times V_n(\mathbb{C}^n)$. This argument works for $k = \infty$ as well as for finite k.

Define

$$E_n(\mathbb{C}^k) := \{(\ell, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k |\ v \in \ell\}.$$

Again the inclusions $\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \cdots$ give inclusions $E_n(\mathbb{C}^k) \subset E_n(\mathbb{C}^{k+1}) \subset \cdots$ and we set

$$E_n(\mathbb{C}^\infty) := \mathrm{colim}_{k\to\infty} E_n(\mathbb{C}^k)$$

with the direct limit topology.

Lemma 2.8. The projection $p : E_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ given by $p(\ell, v) = \ell$, is a vector bundle.

Proof. We take the orthogonal projection $\pi_\ell : \mathbb{C}^k \to \ell$ for $\ell \in G_n(\mathbb{C}^k)$, and set

$$U_\ell := \{\ell' \in G_n(\mathbb{C}^k) |\ \pi_\ell(\ell') \text{ has dimension } n\}.$$

Note that in particular $\ell \in U_\ell$. We will show that U_ℓ is open in $G_n(\mathbb{C}^k)$ and the map

$$\begin{array}{cccc} \varphi : & p^{-1}(U_\ell) & \to & U_\ell \times \ell \simeq U_\ell \times \mathbb{C}^n \\ & (\ell', v) & \mapsto & (\ell', \pi_\ell(v)) \end{array}$$

is a local trivialization of $E_n(\mathbb{C}^k)$. To show that U_ℓ is open we show that its preimage in $V_n(\mathbb{C}^k)$ is open. It preimage consists of orthonormal frames $v_1, ..., v_n$ such that $\pi_\ell(v_1), ..., \pi_\ell(v_n)$ are independent. In other words if we take a matrix A of π_ℓ given by fixing the standard basis in the domain $\mathbb{C}^k$ and any basis in its range ℓ, the previous condition on the frames is then that the $n \times n$ matrix with columns $Av_1, ..., Av_n$ has nonzero determinant. By continuity of the determinant map it follows that there exist an open set in $V_n(\mathbb{C}^k)$ where the frames $v_1, .., v_n$ have nonzero determinant. So U_ℓ is open in $G_n(\mathbb{C}^k)$.

It is clear that φ is a bijection which is a linear isomorphism on each fiber. Now we show that φ and φ^{-1} are continuous. By definition for $\ell' \in U_\ell$ there is a unique invertible linear map $L_{\ell'} : \mathbb{C}^k \to \mathbb{C}^k$ restricting to π_ℓ on ℓ' and the identity on $\ell^\perp \in \ker\pi_\ell$. We claim that $L_{\ell'}$ regard as a $k \times k$ matrix, depends continuously on ℓ'. Namely, we can write $L_{\ell'}$ as a product AB^{-1} where B sends the standard basis to $v_1, ..., v_n, v_{n+1}, ..., v_k$ with $v_1, ..., v_n$ an orthonormal basis for ℓ' and $v_{n+1}, ..., v_k$ a fixed basis for $\ell^\perp$, and A sends the standard basis to $\pi_\ell(v_1), ..., \pi_\ell(v_n), v_{n+1}, ..., v_k$. Both A and B depend continuously on $v_1, ..., v_n$. Since matrix multiplication and matrix inversion are continuous operations , it follows that the product $L_{\ell'} = AB^{-1}$ depends continuously on $v_1, ..., v_n$. But since $L_{\ell'}$ depends only on ℓ', not on the basis $v_1, ..., v_n$ for ℓ', it follows that $L_{\ell'}$ depends continuously on ℓ' since $G_n(\mathbb{C}^k)$ has the quotient topology from $V_n(\mathbb{C}^k)$. As we have $\varphi(\ell', v) = (\ell', \pi_\ell(v)) = (\ell', L_{\ell'}(v))$, we see that φ is continuous. Similarly $\varphi^{-1}(\ell', w) = (\ell', L_{\ell'}^{-1}(w))$ and $L_{\ell'}^{-1}$ depend continuously on ℓ', matrix inversion being continuous, so φ^{-1} is continuous. □

Corollary 2.3. The projection $p : E_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ given by $p(\ell, v) = \ell$, is a vector bundle.

Proof. Let U_ℓ be the union of U_ℓ's increasing k. The local trivialization φ constructed above for finite k then fit together to give a local trivialization over this U_ℓ. Continuity being automatic since we use the weak topology. □

Simplifying notation we will write G_n for $G_n(\mathbb{C}^\infty)$ and E_n for $E_n(\mathbb{C}^\infty)$. Remember that we use the notation $[X, Y]$ for the set of homotopy classes of maps $f : X \to Y$.

Definition 2.17. A vector bundle $p : E \to X$ is **universal** if:

- Given a vector bundle $q : E_1 \to Y$, then there is a continuous function $f : Y \to X$ such that $E_1 \cong f^*(E)$.
- If $f, g : Y \to X$ are such that $f^*(E) \cong g^*(E)$, then f and g are homotopic functions.

Finally we prove the main theorem of this section.

Theorem 2.2. For paracompact X, the map

$$\begin{array}{ccc} [X, G_n] & \to & \mathrm{Vect}^n(X) \\ [f] & \mapsto & f^*(E_n) \end{array}$$

is a bijection.

Proof. To show surjectivity first note that for a n-dimensional vector bundle $p : E \to X$, is equivalent to finding maps $f : X \to G_n$ such that $E \cong f^*(E_n)$ and $g : E \to \mathbb{C}^\infty$ that is a linear injection on each fiber. To show this, suppose first that we have a map $f : X \to G_n$ such that $E \cong f^*(E_n)$, then we consider the following commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[dr, "p"'] & f^*(E_n) \arrow[r, "\widetilde{f}"] \arrow[d] & E_n \arrow[r, "\pi"] \arrow[d] & \mathbb{C}^\infty \\ & X \arrow[r, "f"] & G_n & \end{tikzcd}$$

where $\pi(\ell, v) = v$ and $\widetilde{f}(x, w) = w$. We define $g : E \to \mathbb{C}^\infty$ as the composition across the top row in the diagram, so by definition of π and $\widetilde{f}$ we have that g is linear injection on each fiber. Conversely, given a map $g : E \to \mathbb{C}^\infty$ that is a linear injection on each fiber, we define

$$\begin{array}{rccc} f: & X & \to & G_n \\ & x & \mapsto & g(p^{-1}(x)) \end{array}$$

where $g(p^{-1}(x))$ is the n-plane. f is well define since g is linear injection on each fiber. Note that by definition for each $x \in X$ the fiber in $f^*(E_n)$ is the n-dimensional plane ℓ such that $g(p^{-1}(x)) = \ell$, so the map g gives us a correspondence between the fibers of E and $f^*(E_n)$. Therefore $E \cong f^*(E_n)$. With this in hand for an n-dimensional vector bundle $p : E \to X$, let $\{U_i\}$ be a countable open covering of X such that E is trivial over each U_i with trivialization $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^n$, and let $\{\lambda_i\}$ be a partition of unity with compact support in U_i. Let $g_i : p^{-1}(U_i) \to \mathbb{C}^n$ be the composition $g_i = \pi_2 \circ \lambda_i$. We define $\hat{f}_i : E \to \mathbb{C}^n$ by

$$\hat{f}_i(v) = \begin{cases} \lambda_i(p(v))g_i(v) & \text{if } p(v) \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

Each $\hat{f}_i$ is continuous and $\hat{f}_i|_{p^{-1}(x)}$ is lineal, so finally we define $f : E \to \mathbb{C}^\infty$ given by $v \mapsto (\hat{f}_1(v), \hat{f}_2(v), ...)$. This map is a linear injection on each fiber.

For injectivity, if we have isomorphisms $E \cong f_0^*(E_n)$ and $E \cong f_1^*(E_n)$ for two maps $f_0, f_1 : X \to G_n$, then these give maps $g_0, g_1 : E \to \mathbb{C}^\infty$ that are linear injections on fibers. We claim g_0 and g_1 are homotopic through maps g_t that are linear injections on fibers. Define

$$\begin{array}{rccc} L_t: & \mathbb{C}^\infty & \to & \mathbb{C}^\infty \\ & (x_1, x_2, ...) & \mapsto & (1-t)(x_1, x_2, ...) + t(x_1, 0, x_2, 0, ...). \end{array}$$

For each t this is a linear map whose kernel is easily computed to be 0, so L_t is injective. Composing the homotopy L_t with g_0 moves the image of g_0 into the odd-numbered coordinates. Similarly we can homotope g_1 into the even-numbered coordinates. Keeping the notation g_0 and g_1 for the new functions obtained, let $g_t = (1-t)g_0 + tg_1$. This is linear and injective on fibers for each t since g_0 and g_1 are linear and injective on fibers. Thus f_0 and f_1 are homotopic via $f_t(x) = g_t(p^{-1}(x))$. □

By this theorem the vector bundles over a fixed base space are classified by homotopy classes of maps into G_n. Because of this, G_n is called the classifying space for n-dimensional vector bundles and $E_n \to G_n$ is called the universal bundle.

7 Principal bundles and connections

This section contains the basic elements for discussing of the Chern-Weil construction in chapter 4. For our purposes we work in the category of smooth manifolds.

Definition 2.18. Let M be a smooth manifold and G a matrix Lie group, i.e., it is a closed subgroup of some $\mathrm{GL}_n(\mathbb{K})$. A **principal G-bundle** over M is a fiber bundle $p : P \to M$ with trivializing cover $\{U_\alpha\}$ together with a smooth right action $\rho : P \times G \to P$ of G on P, that for simplicity we will denote $\rho(q, g) = q \cdot g$, satisfying

- ρ preserves p-fibers in the sense that $p(q \cdot g) = p(q)$, and acts freely and transitively on them. Thus, the fibers are exactly the orbits of G.
- For every $x \in M$ there is a open neighborhood U_α (of the trivializing cover) of x and a diffeomorphism $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ such that

$$\varphi_\alpha(q) = (p(q), g_\alpha(q))$$

for all $q \in p^{-1}(U_\alpha)$, where $g_\alpha : p^{-1}(U_\alpha) \to G$ satisfies $g_\alpha(qh) = g_\alpha(q)h$, for all $h \in G$, i.e., g_α is a **G-equivariant map**.

Example 2.18. Let M be a smooth manifold. Then $\pi : M \times G \to M$ is a principal G-bundle called the trivial bundle.

Example 2.19. If G is a compact Lie group acting freely and smoothly on a compact smooth manifold M, then $\pi : M \to M/G$ is a principal G-bundle.

Example 2.20. Let $p : E \to M$ be a principal G-bundle. Suppose that G acts on F smoothly. Consider the **balanced product**

$$E \times_G F := (E \times F)/ \sim$$

where $(e, f) \sim (g \cdot e, g \cdot f)$ for $e \in E, f \in F$ and $g \in G$. We define

$$\begin{array}{rccc} \pi: & E \times_G F & \to & M \\ & [(e,f)] & \mapsto & p(e). \end{array}$$

This function is smooth and $\pi^{-1}(x) \cong F$, for all $x \in M$. Further $\pi : E \times_G F \to M$ is trivial locally, i.e, there is an open cover $U = \{U_\alpha\}_\alpha$ of M and diffeomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "\pi|_{\pi^{-1}(U_\alpha)}"'] & U_\alpha \times F \arrow[dl, "pr_1"] \\ U_\alpha. & \end{tikzcd}$$

So $\pi : E \times_G F \to M$ is a fiber bundle.

Definition 2.19. Let $p : P \to M$ and $q : Q \to M$ be principal G-bundles. A **morphism** between these principal G-bundles is a G-equivariant smooth function $\tau : P \to Q$ such that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\tau"] \arrow[d, "p"] & Q \arrow[dl, "q"] \\ M. & \end{tikzcd}$$

We say that $\tau : P \to Q$ is an isomorphism if τ is a diffeomorphism.

Proposition 2.7. Every morphism $\tau : P \to Q$ between principal G-bundles is an isomorphism.

Proof. We need only prove that $\tau : P \to Q$ is a diffeomorphism. Suppose that $P = M \times G$ and $Q = M \times G$, i.e. P and Q are trivial principal G bundles (note that the general case is obtained by local triviality). We note that $\tau(x, 1) = (x, f(x))$, where $f(x) \in G$. By G-invariance

$$\tau(x, g) = \tau((x, 1) \cdot g) = \tau(x, 1) \cdot g = (x, f(x)) \cdot g = (x, f(x) \cdot g)$$

Further, by the same argument

$$\tau^{-1}(x, g) = (x, f(x)^{-1} \cdot g).$$

□

Let $p : P \to M$ be a pricipal G-bundle and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ a local trivialization of M. Consider two local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and $\varphi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times G$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a diffeomorphism of the form

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G.$$

Now let $x \in U_\alpha \cap U_\beta$ and $1 \in G$ be the identity element, then the map $\varphi_\alpha \circ \varphi_\beta^{-1}$ for fixed x is now just a invertible map from G to G. Therefore we have define a cocycle map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

such that

$$\begin{array}{rccc} \varphi_\alpha \circ \varphi_\beta^{-1} : & (U_\alpha \cap U_\beta) \times G & \to & (U_\alpha \cap U_\beta) \times G \\ & (x, 1) & \mapsto & (x, g_{\beta\alpha}(x)). \end{array}$$

This map is called a **transition function**. As $\varphi_\alpha \circ \varphi_\beta^{-1}$ is G-equivariant, then

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1}(x, g) &= \varphi_\alpha \circ \varphi_\beta^{-1}((x, 1) \cdot g) \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1}(x, 1)) \cdot g \\ &= (x, g_{\beta\alpha}(x)) \cdot g \\ &= (x, g_{\beta\alpha}(x) \cdot g). \end{aligned}$$

It is not hard to verify that the transition functions for a principal G-bundle satisfy the following conditions:

- $g_{\alpha\alpha}(x) = x$ for all $x \in U_\alpha$.
- $g_{\alpha\beta}(x)^{-1} = g_{\beta\alpha}(x)$ for all $x \in U_\alpha \cap U_\beta$.
- $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

The last one is known as the **cocycle condition**.

As in the vector bundle case, we can specify a principal G-bundle up to isomorphism by giving only the base space M and the cocycles $g_{\alpha\beta} : U_{\alpha\beta} \to G$ (Equation 2.1) satisfying the cocycle condition. Let us see how this happens. First of all take

$$\widetilde{P} := \Big(\bigsqcup_{\alpha\in\Lambda} (U_\alpha \times G) \Big) \Big/ \sim$$

where $\sim$ is given by: $(x, g) \sim (y, h)$ iff $x = y$ and $h = g \cdot g_{\beta\alpha}(x)$, for all $(x, g) \in U_\alpha \times G$ and $(y, h) \in U_\beta \times G$. An element of $\widetilde{P}$ is written $[x, g]$. We can now define the projection

$$\begin{array}{rccc} \phi : & \widetilde{P} & \to & X \\ & [x, g] & \mapsto & x. \end{array}$$

Is not difficult to see that $\widetilde{P} \cong P$. We will denote the principal G-bundle obtained in this way by $(M, \mathcal{U}, g_\bullet, G)$, where $\mathcal{U}$ is the family of open sets on which the cocycles are defined.

Example 2.21. Let $p : E \to M$ be a smooth vector bundle with gluing data $(M, \mathcal{U}, g_\bullet, \mathbb{K}^n)$. We denote by $\mathrm{Fr}(\mathbb{K}^n)$ as the set of n-frames $\bar{e} = (e_1, ..., e_n)$ of $\mathbb{K}^n$. We have a free and transitive right action

$$\begin{array}{rcl} \mathrm{Fr}(\mathbb{K}^n)\times \mathrm{Gl}_n(\mathbb{K}) & \to & \mathrm{Fr}(\mathbb{K}^n) \\ ((e_1,e_2,...,e_n),g) & \mapsto & (e_1,e_2,...,e_n)\cdot g = (\sum_i g^i_1 e_i, ..., \sum_i g^i_n e_i) \end{array}$$

where $g = [g^i_j]_{1\le i,j\le n}$, with i denote the rows and j denote the columns. In particular, the set of frames $\mathrm{Fr}(\mathbb{K}^n)$ is naturally a smooth manifold diffeomorphic to $\mathrm{Gl}_n(\mathbb{K})$.

Then to the vector bundle $p : E \to M$ we associate the fiber bundle $\mathrm{Fr}(E) \to M$ with same base smooth manifold M, trivializing neighborhood $\mathcal{U}$ and cocycles $g_\bullet$, but with fiber $\mathrm{Fr}(\mathbb{K}^n)$ and total space given by the disjoint union

$$\mathrm{Fr}(E) := \Big(\bigsqcup_{\alpha\in I}\big(\mathrm{Fr}(\mathbb{K}^n)\times U_\alpha\big)\Big)\Big/\!\sim$$

via the equivalence relation $\sim$ given by $(\bar{e}_\alpha, x_\alpha) \sim (\bar{e}_\beta, x_\beta)$ iff $x_\alpha = x_\beta = x$ and $\bar{e}_\beta = g_{\beta\alpha}(x)\bar{e}_\alpha$. The right action makes this bundle a principal $\mathrm{Gl}_n(\mathbb{K})$-bundle.

Definition 2.20. A **section** of a principal G-bundle $p : P \to M$ is a map $s : M \to P$ such that $p \circ s = \mathrm{id}_M$. A **local section** is a section $s : U \to P|_U$ for an open subset $U \subset M$.

While vector bundles always have a section, i.e, the zero section, principal G-bundles only have sections on open sets U on which the bundle is isomorphic to the trivial bundle $U \times G$.

Proposition 2.8. Let $p : P \to U$ be a principal G-bundle with smooth right action ρ of G on P. Then there exist a section $s : U \to P$ if and only if $P \cong U \times G$ as G-bundles.

Proof. Given a section s, by the Proposition 2.7 we only need define a map of G-bundles $f : P \to U \times G$. We define $f' : U \times G \to P$ by setting $f'(u, g) = \rho(s(u), g)$. Notice that $f'(u, gh) = \rho(s(u), gh) = \rho(\rho(s(u), g), h) = \rho(f'(u, g), h)$. Thus f' is a map of G bundles and thus an isomorphism. We can let $f = (f')^{-1}$. On the other hand, if $P \cong U \times G$ via an isomorphism f, we set $s(u) = f^{-1}(u, 1)$, where $1 \in G$ is the identity element. □

Thus sections give us a way to locally find "horizontal" copies of U in P.

Definition 2.21. Let P be a smooth manifold of dimension n. Suppose that for each $x \in P$, we assign a k-dimensional, $k \le n$, subspace $D_x \subset T_xP$ such that there exist an open set $V_x \subset P$ around x, and there exist k linearly independent smooth sections (vector fields) $X_1, ..., X_k$ such that for any point $y \in V_x$, $X_1(y), ..., X_k(y)$ span D_y. A **smooth distribution** D of dimension k, is the collection of all the D_x for all $x \in P$.

Now we will give the most intuitive and geometric definition of a connection on a principal G-bundle $p : P \to M$.

Definition 2.22. Let $p : P \to M$ be a principal G-bundle. For $x \in P$, let $V_x \subset T_xP$ be the kernel of $p_* : T_xP \to T_{p(x)}M$. We call V_x the **vertical vectors**. A smooth distribution H on P is a **principal G-connection** if:

- The distribution is horizontal, i.e, $H_x \oplus V_x = T_xP$.
- The distribution is G-equivariant, i.e, for any $g \in G$, $H_{xg} = R_{g^*}H_x$, where R_{g^*} is the differential of the map $R_g : G \to G$ given by $h \mapsto hg$.

For a better understanding of the vertical vectors we need the following Lie theory machinery. For reference, can see the book [Hall].

Given a matrix Lie group G with Lie algebra $\mathfrak{g}$, given $X \in \mathfrak{g}$ there exist a smooth group homomorphism $\mathbb{R} \to G$ given by $t \mapsto \exp(tX)$ and such that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \exp(tX) = X.$$

Since G acts on P to the right, for any $x \in P$, $x \cdot \exp(tX)$ is a smooth curve in P. So we define the smooth **fundamental vector field** associated to X by

$$\sigma(X)(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} x \cdot \exp(tX).$$

For each $A \in G$, we define the map $\mathrm{ad}_A : G \to G$ given by $X \mapsto AXA^{-1}$, then we have the map $Ad_A := (\mathrm{ad}_{A^*})_e : \mathfrak{g} \to \mathfrak{g}$ in $\mathrm{End}(\mathfrak{g})$ given by $Ad_A(X) = AXA^{-1}$. Thus we define the smooth map $ad : G \to \mathrm{End}(\mathfrak{g})$ as $A \mapsto Ad_A$ called the **adjoint representation**.

Thus we can understand the vertical vectors intuitively as follows.

Proposition 2.9. Let $p : P \to M$ be a principal G-bundle and H a principal G-connection on P. A vector $Y_x \in V_x$ if and only if $Y_x = \sigma(X)(x)$ for some $X \in \mathfrak{g}$.

Proof. Let $X \in \mathfrak{g}$ and $x \in P$ be fixed. Let $\gamma(t) = x \cdot \exp(tX)$ be a curve. Note that $Y_x = \gamma'(0)$ by the definition of the fundamental vector field. Since $p(\gamma(t)) = p(x)$ is constant by G-invariance, we have that $p_*(\gamma'(0))$ is the vector zero in $T_{p(x)}M$, hence $Y_x \in V_x$.

In the other direction, notice that the rank of p_* is the dimension of M. By the rank-nullity theorem, the dimension of P is equal to the dimension of M plus the dimension of the kernel of p_*. By the first part of the proof we have already shown that the kernel of p_* is at least the dimension of G. Since the $\dim(P) = \dim(G) + \dim(M)$, we have that the kernel of p_* is exactly $\sigma(X)$ for some $X \in \mathfrak{g}$. $\square$

This definition of principal G-connection is geometrically intuitive, but is not necessarily the easiest kind of data to deal with. Instead, we will try to find a more compact way of describing a connection as a 1-form valued in the Lie algebra $\mathfrak{g}$. Before, let us state the following proposition.

Proposition 2.10. Let $p : P \to M$ be a principal G-bundle. Let $X \in \mathfrak{g}$ and $\sigma(X)$ be the fundamental vector field associated to X. Then for any $x \in P$ and $A \in G$ we have

$$R_{A^*}[\sigma(X)(x)] = \sigma(ad_{A^{-1}}(X))(x \cdot A).$$

Proof.

$$\begin{aligned}
\sigma(ad_{A^{-1}}(X))(x \cdot A) = \sigma(A^{-1}XA)(x \cdot A) &= \left.\frac{\partial}{\partial t}\right|_{t=0} (x \cdot A\exp(tA^{-1}XA)) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} (x \cdot AA^{-1}\exp(tX)A) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} (x \cdot \exp(tX)A) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} (R_A(x \cdot \exp(tX)) \\
&= R_{A^*}[\sigma(X)(p)].
\end{aligned}$$

$\square$

This proposition motivates the last condition in the following definition.

Definition 2.23. Let $p : P \to M$ be a principal G-bundle. A **connection 1-form** ω, is a 1-form valued in the Lie algebra $\mathfrak{g}$ of the Lie group G, such that for $x \in P$, $g \in G$, $A \in G$ and $X \in \mathfrak{g}$,

- ω is smooth,
- $\omega_x(\sigma(X)(x)) = X$, and
- $\omega_{x\cdot A}(R_{A^*}Y_x) = ad(A^{-1})\omega_x(Y_x)$.

We construct a smooth distribution H_x out of ω by setting $H_x := \ker\, \omega_x$. We will show that H_x is a G-connection on P.

Theorem 2.3. Every connection 1-form ω uniquely determines a G-connection on a principal G-bundle $p : P \to M$ and vice-versa.

Proof. Let ω be a 1-form, n the dimension of M and m the dimension of P. We will prove that $H_x := \ker\, \omega_x$ is a G-connection. Let $x \in P$ and choose a trivial coordinate neighbourhood U of x. Let $Y_1, ..., Y_n, ..., Y_m$ be smooth vector fields spanning T_zP for all $z \in U$. Define $Y_i' := Y_i - \omega^j(Y_i)\sigma(X_j)$, where the X_j is a basis of $\mathfrak{g}$ and ω^j are the coordinate functions of ω valued in $\mathfrak{g}$. Of course, $\omega(Y_i') = 0$. We note that the Y_i' span H, since write any vector in H as a linear combination of the Y_i's and note that the same linear combination of the Y_i' equals the chosen vector; hence H_p is smooth. In order to show the second condition in the definition of G-connection, we know that for all $v_x \in \ker\, p_*$, $v_x = \sigma(X)(x)$ for some $X \in \mathfrak{g}$. Hence the rank of ω is equal to the dimension of the Lie algebra $\mathfrak{g}$. Thus the nullity of ω equals the dimension of H_x. So it suffices to show that $H_x \cap \ker\, p_*$ is equal to the zero vector. However, this follows from the fact that $\omega(\sigma(X)) = X$. Lastly, R_{A^*} is one to one since R_A is invertible. Let $v_x \in H_x$. Then $\omega_{x\cdot A}(R_{A^*}v_x) = ad(A^{-1})\omega_x(v_x) = 0$; hence $R_{A^*}(H_x) \subset H_{x\cdot A}$. But the same holds true for $R_{A^{-1}_*}$, hence we have equality.

In the other direction, given a connection H_x, we need to define a 1-form ω and show that it is a connection 1-form. We define ω as follows: If $v_x \in \ker\, p_*$ and hence $v_x = \sigma(X)(x)$ for some $X \in \mathfrak{g}$, we define $\omega_x(v_x) = X$. If $v_x \in H_x$, define $\omega_x(v_x) = 0$ and extend this definition linearly. Notice that this gives us a 1-form valued in $\mathfrak{g}$. Let $\omega = \omega^j X_j$, where X_j are a basis for $\mathfrak{g}$ and recall that near x, $\sigma(X_1), ..., \sigma(X_k)$ and $Y_1, ..., Y_l$ are a smooth basis for TP, and $Y_1, ..., Y_l$ are smooth basis for H, so we have that $\omega^j(\sigma(X_k)) = \delta^j_k$ and $\omega^j(Y_k) = 0$ and hence ω is smooth. By definition $\omega(\sigma(X)) = X$. Finally, let $Y_x = \sigma(X)(x)$, then $\omega_{x\cdot A}(R_{A^*}Y_x) = \omega_{x\cdot A}(\sigma(ad(A^{-1})X)(x \cdot A))$. However, $ad(A^{-1})X = ad(A^{-1})\omega_x(Y_x)$, which is what we want. Instead, if $Y_x \in H_p$, then $R_{A^*}Y_x \in H_{x\cdot A}$ and hence $\omega_x(Y_x) = 0$ and $\omega_{x\cdot A}(R_{A^*}Y_x) = 0$, and the result follows. □

This connection 1-form ω is a global 1-form on P. Since for most purposes our data lives on M, we would prefer that the data of a connection also lives on M. As our fiber bundle is locally trivial, and a trivialization is equivalent to a section by the Proposition 2.8, locally we can find a section $s_\alpha : U_\alpha \to P|_{U_\alpha}$, then we can pull back ω under s_α to get a 1-form $A_\alpha := s_\alpha^*\omega$ valued in $\mathfrak{g}$ and defined on U_α. However, on the overlap two pullbacks do not agree, and this local 1-form does not transform under change of coordinates as a global 1-form, but this local 1-form transforms in a way that is not too difficult to understand.

Lemma 2.9. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivializing cover with cocycle maps $g_{\alpha\beta}$ and trivializations defined by sections $\{s_\alpha : U_\alpha \to P|_{U_\alpha}\}$. Then

$$s_\beta = s_\alpha \cdot g_{\alpha\beta}.$$

Proof. By the Proposition 2.8 we know that $s_\beta(u) = \varphi_\beta^{-1}(u, 1)$, where $1 \in G$ is the identity element. Hence

$$\begin{aligned} \varphi_\alpha s_\beta(u) = \varphi_\alpha \varphi_\beta^{-1}(u, 1) &= (u, g_{\alpha\beta}(u)) \\ &= (u, 1) \cdot g_{\alpha\beta}(u) \\ &= \varphi_\alpha(s_\alpha(u)) \cdot g_{\alpha\beta}(u) \\ &= \varphi_\alpha(s_\alpha(u) \cdot g_{\alpha\beta}(u)). \end{aligned}$$

Since φ_α is an isomorphism, we have the result. $\square$

On G, there is a canonical $\mathfrak{g}$-valued 1-form θ defined as $\theta_X = L_{X_*^{-1}}$. In our context of matrix Lie groups, this form is given by $\theta_X = X^{-1}\mathrm{d}X$. Since the left invariant vector field is smooth, then this 1- form is smooth and further

$$L_{A^*}\theta_X = \theta_X \circ L_{A^*} = L^{X_*^{-1}} \circ L_{A^*} = L_{X^{-1}A_*} = \theta_{A^{-1}X}.$$

Hence θ is left-invariant 1-form called the **Maurer-Cartan form**. Let $p : P \to M$ be a principal G-bundle with cocycle maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. We define $\theta_{\alpha\beta}$ as the pullback of θ under $g_{\alpha\beta}$, i.e., $\theta_{\alpha\beta} = g_{\alpha\beta}^*\theta$.

Finally we have the central theorem in this section.

Theorem 2.4. The restriction of the connection one form ω to $P|_{U_\alpha}$, denoted by ω_α agrees with

$$\omega_\alpha = ad_{g_\alpha^{-1}} \circ p^* A_\alpha + g_\alpha^* \theta.$$

where $g_\alpha : p^{-1}(U_\alpha) \to G$ (see Definition 2.18).

Proof. First we show that ω_α and ω agree on the image of s_α. Indeed, let $x \in U_\alpha$ and $y = s_\alpha(x)$. We have a direct sum decomposition

$$T_y P = \mathrm{im}(s_\alpha \circ \pi)_* \oplus V_y$$

so that every $v \in T_y P$ can be written uniquely as $v = (s_\alpha)_* \pi_*(v) + \bar{v}$, for a unique vertical vector $\bar{v}$. Since $g_\alpha(s_\alpha(x)) = 1$ and $(g_\alpha \circ s_\alpha)_* = 0$, applying ω_α on v we obtain

$$\begin{aligned} \omega_\alpha(v) = (p^* s_\alpha^* \omega)(v) + (g_\alpha^* \theta_1)(v) &= \omega((s_\alpha)_* p_* v) + \theta_1((g_\alpha)_* v) \\ &= \omega((s_\alpha)_* p_* v) + \theta_1((g_\alpha)_* \bar{v}) \\ &= \omega((s_\alpha)_* p_* v) + \omega(\bar{v}) \\ &= \omega(v). \end{aligned}$$

Next we show that they transform in the same way under the right action of G. By equivariance of g_α and since $p \circ R_g = \pi$, we have

$$\begin{aligned} R_g^*(\omega_\alpha)_{xg} &= ad_{g_\alpha(xg)^{-1}} \circ R_g^* p^* s_\alpha^* \omega + R_g^* g_\alpha^* \theta \\ &= ad_{(g_\alpha(x)g)^{-1}} \circ R_g^* p^* s_\alpha^* \omega + g_\alpha^* R_g^* \theta \\ &= ad_{g^{-1} g_\alpha(x)^{-1}} \circ p^* s_\alpha^* \omega + g_\alpha^*(ad_{g^{-1}} \circ \theta) \\ &= ad_{g^{-1}} \circ (ad_{g_\alpha(x)^{-1}} \circ p^* s_\alpha^* \omega + g_\alpha^* \theta) \\ &= ad_{g^{-1}} \circ (\omega_\alpha)_x. \end{aligned}$$

Therefore they agree everywhere on $P|_{U_\alpha}$. □

Now since ω is defined globally, we have that $\omega_\alpha = \omega_\beta$ on $P|_{U_\alpha}$. This allows us to relate A_α and A_β on $U_\alpha \cap U_\beta$ by

$$\begin{aligned} A_\alpha = s_\alpha^* \omega_\alpha &= s_\alpha^* \omega_\beta \\ &= s_\alpha^*(ad_{g_\beta^{-1}} \circ p^* A_\beta + g_\beta^* \theta) \\ &= ad_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^* \theta. \end{aligned}$$

For matrix groups this becomes the more familiar

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - (\mathrm{d} g_{\alpha\beta}) g_{\alpha\beta}^{-1} = g_{\beta\alpha}^{-1} \mathrm{d} g_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha}. \tag{2.2}$$

We are now interested in measuring the "oscillations" of sections of vector bundles, i.e., the particular case of connections on vector bundles. More precisely, let $E \to M$ be a complex vector bundle over a smooth manifold M, then we would like to have a procedure of measuring the rate of change of a section s of E along a direction described by a vector field X over M. This procedure is given by an operator

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E),$$

given by $(X, s) \mapsto \nabla_X s$. Moreover, if we think of the usual directional derivative, we expect that after "rescaling" the direction X the derivative along X should only rescale by the same factor, i.e.,

$$\nabla_{fX} s = f \nabla_X s,$$

for all $f \in C^\infty(M)$. Finally, in order to measuring the rate of change of a section s of E along a direction described by a vector field X, ∇ is to be a derivation and it has to satisfy a sort of Leibniz product rule. Since the only product rule that exist on an abstract vector bundle is the multiplication of a section with a smooth function, we require that

$$\nabla_X(fs) = (Xf)s + f \nabla_X s,$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Definition 2.24. Let $E \to X$ be a complex vector bundle. A **connection** ∇ on the vector bundle E is map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

given by $(X, s) \mapsto \nabla_X s$, such that

$$\nabla_{fX} s = f \nabla_X s,$$

$$\nabla_X(fs) = (Xf)s + f \nabla_X s,$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Example 2.22. Let $p : E \to M$ be a trivial bundle of rank k. Then by Lemma 2.2 there exist global sections $\{s_1, ..., s_k\}$ of the bundle E such that $\{s_j(x)\}$ is a basis for E_x for all points $x \in M$, i.e., $\{s_j\}$ is a frame of E. So for any $s \in \Gamma(E)$, we have $s = \sum_j f_j s_j$ for some $f_j \in C^\infty(M)$, for all $1 \leq j \leq k$. We define a bilinear map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ by

$$\nabla_X s = \nabla_X \left(\sum_j f_j s_j \right) := \sum_j X(f_j) s_j.$$

We check that ∇ is indeed a connection on E:

$$\nabla_{fX} s = \nabla_{fX} \left(\sum_j f_j s_j \right) = \sum_j f X(f_j) s_j = f \sum_j X(f_j) s_j = f \nabla_X s,$$

$$\nabla_X(fs) = \nabla_X \left(f \sum_j f_j s_j \right) = \sum_j X(f f_j) s_j = X(f) \sum_j f_j s_j + f \sum_j X(f_j) s_j = X(f) s + f \ \nabla_X s.$$

Lemma 2.10. Any convex linear combination of two connections on a vector bundle $E \to M$ is a connection. More precisely, let ∇^1, ∇^2 be two connections on E and $\rho_1, \rho_2 \in C^\infty(M)$ be two functions with $\rho_1 + \rho_2 = 1$. Then $\nabla := \rho_1 \nabla^1 + \rho_1 \nabla^2$ is a connection.

Proof.

$$\nabla(fX, s) = \rho_1 \ \nabla^1_{fX} s + \rho_2 \ \nabla^2_{fX} s = f \nabla(X, s),$$

$$\begin{aligned} \nabla(X, fs) = \rho_1 \ \nabla^1_X(fs) + \rho_2 \ \nabla^2_X(fs) &= \rho_1(X(f)s + f \ \nabla^1_X s) + \rho_2(X(f)s + f \ \nabla^2_X s) \\ &= (\rho_1 + \rho_2) X(f) s + f(\rho_1 \nabla^1_X s + \rho_2 \nabla^2_X s) \\ &= X(f) s + f \ \nabla_X s. \end{aligned}$$

□

Corollary 2.4. Any vector bundle $p : E \to M$ has a connection.

Proof. Choose a cover $\{U_\alpha\}$ on M such that $E|_{U_\alpha}$ is trivial. Let ∇^α be a connection on $E|_{U_\alpha}$, as in the Example 2.22. Let $\{\rho_\beta\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then supp $\rho_\beta \subset U_\alpha$, for some $\alpha = \alpha(\beta)$. We define the map $\nabla : \Gamma(TX) \times \Gamma(E) \to \Gamma(E)$ by

$$\nabla_X s = \sum_\beta \rho_\beta (\nabla^\alpha_{X_U} s|_{U_\alpha}).$$

This is indeed a connection, since a convex linear combination of any finite number of connections is a connection by previous lemma. □

Now we will discuss another way to define connections.

Definition 2.25. Let $E \to M$ and $F \to M$ be two vector bundles. We say that a map $T : \Gamma(E) \to \Gamma(F)$ is **tensorial** if T is $\mathbb{R}$-linear and $C^\infty(M)$-linear, i.e., for any $f \in C^\infty(M)$ we have that $T(fs) = f \ T(s)$, for all sections $s \in \Gamma(E)$.

Lemma 2.11. Let $E \to M$ and $F \to M$ be two vector bundles. If $T : \Gamma(E) \to \Gamma(F)$ is tensorial then there is a vector bundle map $\phi : E \to F$ so that

$$(T(s))(x) = \phi(s(x)),$$

for all $s \in \Gamma(E)$ and $x \in M$. And conversely, any vector bundle map $\phi : E \to F$ defines a tensorial map on sections $T_\phi : \Gamma(E) \to \Gamma(F)$ by $T_\phi(s) = \phi \circ s$.

Proof. We first show that T is local. If $s \in \Gamma(E)$ vanishes on an open set $U \subset M$ then $T(s)$ vanishes on U as well. Pick a point $x \in U$ and a smooth function $\rho \in C^\infty(M)$ with supp $\rho \subset U$ and $\rho = 1$ on a neighborhood V of x, with $V \subset U$. Then ρs is identically zero everywhere. Hence

$$0 = T(\rho s)(x) = \rho(x)T(s)(x) = T(s)(x).$$

Since $x \in U$ is arbitrary $T(s)|_U = 0$.

Since T is local and E, F are locally trivial, we may assume that E and F are, in fact, trivial. That is $E = M \times \mathbb{R}^k$ and $F = M \times \mathbb{R}^l$. Moreover the sections of E and F are simply k-tuples and l-tuples of functions. We want to define a vector bundle map $\phi : E \to F$. Then $\phi : M \times \mathbb{R}^k \to M \times \mathbb{R}^l$ has to be the form

$$\phi(x, v) = (x, A(x)v)$$

where $A : M \to \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ is smooth, with the property that

$$T(f_1, ..., f_k)(x) = A(x)(f_1(x), ..., f_k(x))^T$$

for all $x \in M$. To define $A(x)$, define the jth column of $A(x)$ to be the l-tuple of functions $T(e_j)$, where e_j is the section of E that assigns to every point the jth basis vector $(0, ..., 0, 1, 0, ..., 0)$, with 1 in the j-th position. ☐

Remark 2.12. The previous lemma generalizes furthermore. Let $E_1, ..., E_k$ and F be vector bundles over a manifold M and

$$T : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to \Gamma(F)$$

a k-linear map which is tensorial in each slot, $T(f_1 s_1, ..., f_k s_k) = f_1 \cdots f_k T(s_1, ..., s_k)$, for all $s_i \in \Gamma(E_i)$ and $f_i \in C^\infty(M)$. Then for every $x \in M$ there is a unique k-linear map $T_x : (E_1)_x \times \cdots \times (E_k)_x \to F_x$ with $T_x(s_1(x), ..., s_k(x)) = (T(s_1, ..., s_k))(x)$. Globally this means that there is a vector bundle map $\phi : E_1 \otimes \cdots \otimes E_k \to F$ so that $T(s_1, ..., s_k)(x) = \phi(s_1(x) \otimes \cdots \otimes s_k(x))$, for all $x \in M$ and all sections $s_i \in \Gamma(E_i)$.

The Remark 2.12 and the Remark 2.7 give us the following proposition.

Proposition 2.11. Let $E_1, E_2, ...E_k$ and F be vector bundles over a manifold M. Then there is a bijection between k-linear tensorial maps

$$T : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to \Gamma(F)$$

and sections of the bundle $E_1^* \otimes \cdots \otimes E_k^* \otimes F \to M$.

In our particular case, let $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ be a connection on a complex vector bundle $E \to M$. For any $s \in \Gamma(E)$, the map

$$\nabla_s : \Gamma(TM) \to \Gamma(E)$$

given by $X \mapsto \nabla_X s$, is C^∞-linear and so by the above proposition, we have that $\nabla_s \in \Gamma(T^*M \otimes E)$. Summarizing, we can formulate the following definition.

Definition 2.26. A **covariant derivative** or **linear connection** on a complex vector bundle E, is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

such that for all $f \in C^\infty(M)$ and all sections $s \in \Gamma(E)$, we have

$$\nabla(fs) = df \otimes s + f\nabla s.$$

For any vector bundle E over M we set

$$\Omega^k(E) := \Gamma(\bigwedge^k T^*M \otimes E),$$

and we will refer to these sections as differential k-forms with coefficients in the vector bundle E.

Proposition 2.12. Let E be a vector bundle. The space $\mathrm{Conn}(E)$ of linear connections on E is an affine space modeled on $\Omega^1(\mathrm{End}(E))$.

Proof. By the Corollary 2.4 $\mathrm{Conn}(E)$ is not empty. To check that $\mathrm{Conn}(E)$ is an affine space, we consider two connections ∇^0 and ∇^1. Their difference $A = \nabla^1 - \nabla^0$ is an operator

$$A : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

satisfying $A(fs) = fA(s)$, for all $s \in \Gamma(E)$ and $f \in C^\infty(M)$. Thus

$$A \in \Gamma(E^* \otimes T^*M \otimes E) \cong \Gamma(T^*M \otimes E^* \otimes E) \cong \Omega^1(E^* \otimes E) \cong \Omega^1(\mathrm{End}(E)).$$

Conversely, given $\nabla^0 \in \mathrm{Conn}(E)$ and $A \in \Omega^1(\mathrm{End}(E))$ one can verify that the operator

$$\nabla^A = \nabla^0 + A : \Gamma(E) \to \Omega^1(E)$$

is a linear connection. $\square$

Chapter 3

K-Theory

In this chapter we discuss the first constituent of the diagram 1.1, namely we define K-theory in zero degree of complex vector bundles on a topological space X. In order to extend K-theory in degrees different from zero, we will discuss the Bott periodicity theorem in complex K-theory and we will see that this K-theory defines a generalized cohomology theory. We discuss the Thom isomorphism theorem in K-theory which together with Whitney embedding theorem and the tubular neighborhood theorem allow the definition of push-forward in K-Theory, giving the arrow from K-theory to the integer numbers $\mathbb{Z}$. Moreover, we will discuss the Atiyah-Bott-Shapiro construction, which will allow us to establish the relationship between elliptic differential operators discussed in Chapter 6 and the K-theory.

1 The basic construction

The first generalized cohomology theory to be studied in detail was K-theory. It has been very influential in algebraic topology and in particular in index theory. In this section we will study some of the basic properties of K-theory, including Bott periodicity theorem and the Thom isomorphism for K-theory. We will closely follow Atiyah [Ati67].

Convention 1. In this section all spaces are assumed to be compact Hausdorff spaces.

Definition 3.1. Let S be a commutative semigroup. The **Grothendieck construction** of S, denoted by $G(S)$, is an abelian group together with a morphism of semigroups $\iota : S \to G(S)$ such that given any group H and a morphism of semigroups $\phi : S \to H$ there is a unique group homomorphism $\hat{\phi} : G(S) \to H$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\phi"'] & G(S) \arrow[d, "\hat{\phi}"] \\ & H. \end{tikzcd}$$

Notice that the above property clearly characterizes the map $\iota : S \to G(S)$ up to isomorphism. On the other hand, in order to prove that a pair with this property exists we can construct it explicitly. Let us define $G(S)$ to be the set of equivalence classes of pairs $(x, y) \in S \times S$ subject to the equivalence relation given by $(x, y) \sim (u, v)$ if and only if there exists $s \in S$ such that $s + x + v = s + u + y$. The elements of $G(S)$ are denoted by $[x, y]$ and can be thought as the formal subtraction $x - y$. The addition on $G(S)$ is given by $[x, y] + [u, v] = [x + u, y + v]$ and the canonical map is given by

$$\begin{array}{rccc} \iota: & S & \to & G(S) \\ & s & \mapsto & (s,0). \end{array}$$

The Grothendieck construction is of course functorial, i.e., a morphism of semigroups $\phi : S \to H$ induces in an obvious way a group homomorphism $G(\phi) : G(S) \to G(H)$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\phi"'] & G(S) \arrow[d, "G(\phi)"] \\ H \arrow[r, "\iota"'] & G(H). \end{tikzcd}$$

In case the semigroup S is a semiring, the abelian group $G(S)$ inherits the structure of a ring.

Example 3.1. If $S = (\mathbb{N}, +)$, then $G(S) = \mathbb{Z}$.

Example 3.2. Let S be a group, then $G(S) = S$.

Definition 3.2. Let X be a compact Hausdorff space. We denote by $\mathrm{Vect}_{\mathbb{C}}(X)$ the semigroup of isomorphism classes of complex vector bundles on X, i.e.

$$\mathrm{Vect}_{\mathbb{C}}(X) := \bigcup_{n \geq 0} \mathrm{Vect}^n_{\mathbb{C}}(X).$$

Notice that $\mathrm{Vect}_{\mathbb{C}}(X)$ have commutative semigroup structure with the Whitney sum

$$\begin{array}{rccc} \oplus: & \mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X) & \to & \mathrm{Vect}_{\mathbb{C}}(X) \\ & ([E],[F]) & \mapsto & [E \oplus F]. \end{array}$$

Thus we define the **K-group** or **K-theory** of X as:

$$K^0(X) := G(\mathrm{Vect}_{\mathbb{C}}(X)).$$

The element $[E, F] \in K^0(X)$ is called a **virtual bundle**. In this case the neutral element is represented by the class $[E, E]$, and the inverse element for $[E, F]$ is denoted by $-[E, F] = [F, E]$.

Since the abelian semigroup $\mathrm{Vect}_{\mathbb{C}}(X)$ has a multiplication given by the tensor product of vector bundles, the group $K^0(X)$ has the structure of a commutative ring. Notice that we are using the convention that vector bundles over non connected spaces may have different rank in different connected components.

Remark 3.1. The previous construction also makes sense for real vector bundles, in which case

$$\mathrm{KO}^0(X) := G(\mathrm{Vect}_{\mathbb{R}}(X)).$$

Both the real and complex cases we denote the semigroup on the set of isomorphism classes of n-dimensional real (complex) vector bundles by $\mathrm{Vect}(X)$.

Example 3.3. Let X be the space that consists of a unique point, say $X = \{*\}$, and $n \geq 0$ an integer number. Notice that each vector bundle over $\{*\}$ of rank n is isomorphic to $\{*\} \times \mathbb{C}^n$, then

$$\mathrm{Vect}(\{*\}) := \bigcup_{n \geq 0} \mathrm{Vect}^n(\{*\}) \cong \mathbb{N}$$

Thereby $K^0(\{*\}) = G(\mathbb{N}) = \mathbb{Z}$.

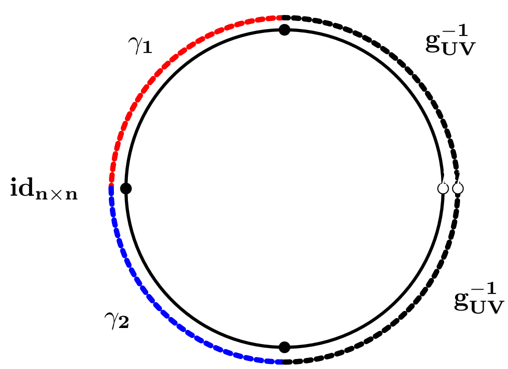
From now on we denote the n-dimensional trivial bundle by $\underline{n}$.

Example 3.4. We will calculate the K-theory $K^0(\mathbb{S}^1)$ by showing that for each $n \geq 0$, we have that $\mathrm{Vect}^n(\mathbb{S}^1) = [\underline{n}]$, in which case we have the isomorphism $\mathrm{Vect}(\mathbb{S}^1) \cong \mathbb{N}$ given by $[\underline{n}] \mapsto n$. Therefore

$$K^0(\mathbb{S}^1) = G(\mathrm{Vect}(\mathbb{S}^1)) = G(\mathbb{N}) = \mathbb{Z}.$$

Let $p : E \to \mathbb{S}^1$ be an n-dimensional complex vector bundle on $\mathbb{S}^1$. Set $U = \mathbb{S}^1 \setminus \{(1,0)\}$ and $V = \mathbb{S}^1 \setminus \{(-1,0)\}$ overlapping open sets of $\mathbb{S}^1$. As U and V are contractile, then by the Corollary 2.2 the restrictions $E|_U$ and $E|_V$ are trivial, thus we can consider the cocycles $\{g_{UV} : U \cap V \to \mathrm{GL}_n(\mathbb{C})\}$ corresponding to the trivialization open cover $\{U, V\}$. To show that $\mathrm{Vect}^n(\mathbb{S}^1) = [\underline{n}]$, by the Proposition 2.1 we want to find maps $\lambda_U : U \to \mathrm{GL}_n(\mathbb{C})$ and $\lambda_V : V \to \mathrm{GL}_n(\mathbb{C})$ such that $g_{UV} = \lambda_U^{-1} \circ \lambda_V$. By path-connectedness of $\mathrm{GL}_n(\mathbb{C})$ we can find the continuous functions $\gamma_1 : [\pi/2, \pi] \to \mathrm{GL}_n(\mathbb{C})$ and $\gamma_2 : [-\pi, -\pi/2] \to \mathrm{GL}_n(\mathbb{C})$ such that $\gamma_1(\pi/2) = g_{UV}(\mathrm{e}^{\pi \mathrm{i}/2})$, $\gamma_1(\pi) = id_{n\times n}$ and $\gamma_2(-\pi/2) = g_{UV}(\mathrm{e}^{-\pi\mathrm{i}/2}), \gamma_2(-\pi) = id_{n\times n}$. Thus we define the continuous map $\lambda_U : U \longrightarrow \mathrm{GL}_n(\mathbb{C})$ given by

$$\lambda_U(x) = \begin{cases} g_{UV}(x)^{-1} & x = \exp(t\mathrm{i}),\ -\pi/2 < t < \pi/2,\ t \neq 0, \\ \gamma_1(t) & x = \exp(t\mathrm{i}),\ \pi/2 \leq t \leq \pi, \\ \gamma_2(t) & x = \exp(t\mathrm{i}),\ -\pi \leq t \leq -\pi/2, \end{cases}$$



and $\lambda_V : V \longrightarrow \mathrm{GL}_n(\mathbb{C})$·given by

$$\lambda_V(x) = \begin{cases} 1 & x = (1,0), \\ \lambda_U(x) g_{UV}(x) & x \neq (1,0),\ x \neq (-1,0). \end{cases}$$

By definitions of λ_U and λ_V is clear that $g_{UV} = \lambda_U^{-1} \circ \lambda_V$.

Given a vector bundle E on X, we will write $[E] \in K^0(X)$, for its class in K-theory $[E, \underline{0}]$, where $\underline{n}$ denote the trivial vector bundle of rank n. When no confusion is possible, we will simply write E for its class in K-theory.

Lemma 3.1. Any element in $K^0(X)$ can be written in the form $[E] - [F]$, where E, F are bundles over X.

Proof.

$$\begin{aligned} [E, F] = [E \oplus \underline{0}, F \oplus \underline{0}] &= [E, \underline{0}] + [\underline{0}, F] \\ &= [E, \underline{0}] - [F, \underline{0}] \\ &= [E] - [F]. \end{aligned}$$

□

Given any $[E]-[F] \in K^0(X)$, by the Proposition 2.4 there exists some vector bundle G such that $F \oplus G = \underline{n}$, therefore:

$$[E]-[F] = [E \oplus G] - [F \oplus G] = [E \oplus G] - [\underline{n}].$$

Thus any element in $K^0(X)$ can be written in the form $[H]-[\underline{n}]$, for some vector bundle H on X. On the other hand, if E and F are vector bundles over X such that $[E]=[F]$ in $K^0(X)$, by construction there exists a vector bundle G over X such that $E \oplus G \cong F \oplus G$ in $\mathrm{Vect}(X)$. Let G' be a vector bundle such that $G \oplus G' \cong \underline{n}$, then

$$\begin{aligned} E \oplus G \oplus G' &\cong F \oplus G \oplus G' \\ E \oplus \underline{n} &\cong F \oplus \underline{n}. \end{aligned}$$

This discussion motivate the following definition.

Definition 3.3. We say that two vector bundles E, F over X are **stably equivalent** if and only if

$$E \oplus \underline{n} \cong F \oplus \underline{n},$$

for some natural number n.

So by the previous discussion two vector bundles E and F are stably equivalent iff $[E]=[F]$ in $K^0(X)$.

Example 3.5. The tangent bundle $T\mathbb{S}^2$ over $\mathbb{S}^2$ is stably equivalent to $\underline{2}$, since $T\mathbb{S}^2 \oplus \underline{1} \cong \underline{3} = \underline{2} + \underline{1}$.

Notice that if $f : X \to Y$ is a continuous map between compact Hausdorff spaces, then the pullback operation induces a morphism of semigroups

$$\begin{array}{cccc} f^* : & \mathrm{Vect}(Y) & \to & \mathrm{Vect}(X) \\ & [E] & \mapsto & [f^*E] \end{array}$$

and by functoriality of the Grothendieck construction this morphism induces a group homomorphism from $K^0(Y)$ to $K^0(X)$, denoted by abuse of notation $f^* : K^0(Y) \to K^0(X)$

$$\begin{tikzcd} Vect(Y) \arrow[r, "i"] \arrow[d, "f^*"'] & K^0(Y) \arrow[d, "f^*"] \\ Vect(X) \arrow[r, "j"'] & K^0(X). \end{tikzcd}$$

By the Corollary 2.1 and previous observations to this we see that this homomorphism is such that:

- $id^* = id$.
- $(g \circ f)^* = f^* \circ g^*$.
- The homomorphism $f^* : K(Y) \to K(X)$ depends only on the homotopy class of the map f.

Thus, we conclude that the assignment $X \mapsto K(X)$ defines a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

2 Bott periodicity

The central result in K-theory is Bott's periodicity theorem which is essential in extending the group $K^0(X)$ to a cohomology theory. In its simplest form it states that there is an isomorphism between $K^0(X \times \mathbb{S}^2)$ and $K^0(X) \otimes K^0(\mathbb{S}^2)$.

We know by the Example 2.17 that the canonical line bundle H over $\mathbb{CP}^1$ satisfies the relation $(H \otimes H) \oplus \underline{1} \cong H \oplus H$, which in $K^0(\mathbb{S}^2)$ can be written $H^2 + 1 = 2H$ or $(H-1)^2 = 0$. This gives us a natural ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K^0(\mathbb{S}^2)$ where $\mathbb{Z}[H]/(H-1)^2$ is the ring of polynomials in H with integer coefficients, modulo the relation $(H-1)^2 = 0$.

On the other hand we define the **external product** by

$$\begin{array}{cccc} \mu: & K^0(X) \otimes K^0(Y) & \to & K^0(X \times Y) \\ & a \otimes b & \mapsto & \pi_1^*(a)\pi_2^*(b) \end{array}$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection maps. We will also write $a * b := \mu(a \otimes b)$. μ is a ring homomorphism since

$$\begin{aligned} \mu((a \otimes b)(c \otimes d)) = \mu(ac \otimes bd) &= \pi_1^*(ac)\pi_2^*(bd) \\ &= \pi_1^*(a)\pi_1^*(c)\pi_2^*(b)\pi_2^*(d) \\ &= \pi_1^*(a)\pi_2^*(b)\pi_1^*(c)\pi_2^*(d) \\ &= \mu(a \otimes b)\mu(c \otimes d). \end{aligned}$$

In particular notice that an additive basis for $\mathbb{Z}[H]/(H-1)^2$ is $\{1, H\}$. This means we can define a homomorphism called μ by abuse of notation

$$\mu : K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K^0(X) \otimes K^0(\mathbb{S}^2) \to K^0(X \times \mathbb{S}^2) \tag{3.1}$$

Where the second map is the external product. Our purpose is to demonstrate that this homomorphism μ is in fact an isomorphism. We will divide the proof into the following steps:

- Showing that every vector bundle over $X \times \mathbb{S}^2$ is isomorphic to one of the form $[E, f]$ where E is a vector bundle over X and f is a clutching function.
- Reduce the clutching function f in $[E, f]$ to a Laurent clutching function ℓ such that $[E, f] \cong [E, \ell] = [E, z^{-m}q] \cong [E, q] \otimes \widehat{H}^{-m}$ for a polynomial clutching function q in the variable z and where $\widehat{H}^{-m}$ denotes the pullback of H^n via the projection $X \times \mathbb{S}^2 \to \mathbb{S}^2$.
- Obtaining the decomposition $[E, q] \oplus [nE, Id] \cong [(n+1)E, L^n q]$ for a linear clutching function $L^n q$. Which combined with the previous step gives

$$[E, q] \otimes \widehat{H}^{-m} \cong [(n+1)E, L^n q] \otimes \widehat{H}^{-m} - [nE, Id] \otimes \widehat{H}^{-m}.$$

- Given a linear clutching function $a(x)z + b(x)$ and a bundle $[E, a(x)z + b(x)]$, show that there exist a splitting $E \cong E_+ \oplus E_-$ such that $[E, a(x) + b(x)] \cong [E_+, Id] \oplus [E_-, z]$. Thus we have that

$$[(n+1)E, L^n q] \cong [((n+1)E)_+, Id] \oplus [((n+1)E)_-, z].$$

- Proving that μ is bijective.

In the proof of Bott periodicity theorem we follow closely Hatcher's book [Hat09].

First step: We will create vector bundles over $X \times \mathbb{S}^2$ by gluing together two vector bundles over $X \times D^2_{\pm}$ keeping in mind the same idea used in the section of clutching functions.

Given a vector bundle $p : E \to X$, consider the product vector bundle $p \times \mathrm{id}_{\mathbb{S}^1} : E \times \mathbb{S}^1 \to X \times \mathbb{S}^1$ and let $f : E \times \mathbb{S}^1 \to E \times \mathbb{S}^1$ be an automorphism of this product vector bundle.

$$\begin{tikzcd} E \times \mathbb{S}^1 \arrow[r, "f"] \arrow[d, "p\times \mathrm{id}_{\mathbb{S}^1}"'] & E \times \mathbb{S}^1 \arrow[dl, "p\times \mathrm{id}_{\mathbb{S}^1}"] \\ X \times \mathbb{S}^1. & \end{tikzcd}$$

Thus for each $x \in X$ and $z \in \mathbb{S}^1$, f specifies an isomorphism $f(x,z) : p^{-1}(x) \to p^{-1}(x)$. Taking $X \times D^2_- \times E$ and $X \times D^2_+ \times E$ where D^2_+ and D^2_- are respectively the north and south hemisphere of $\mathbb{S}^2$, then we glue these two spaces via f by

$$\left(X \times \partial D^2_+ \times E \bigcup_f X \times \partial D^2_- \times E\right) \Big/ \sim$$

where $(x,z,v) \in X \times \partial D^2_- \times E$ is identified with $(x,z,f(x,z)(v)) \in X \times \partial D^2_+ \times E$. Define E_f as the set of the classes under this relation, so the vector bundle over $X \times \mathbb{S}^2$ is

$$\begin{array}{ccc} E_f & \to & X \times \mathbb{S}^2 \\ [x,z,v] & \mapsto & (x,z). \end{array}$$

We write this bundle as $[E,f]$, and call f a **clutching function** for $[E,f]$. If $f_t : E \times \mathbb{S}^1 \to E \times \mathbb{S}^1$ is a homotopy of clutching functions f_0 and f_1, then using the same previous construction we can construct a vector bundle over $X \times \mathbb{S}^2 \times I$ such that restricted to $X \times \mathbb{S}^2 \times \{0\}$ and $X \times \mathbb{S}^2 \times \{1\}$ the vector bundles $[E,f_0]$ and $[E,f_1]$ are obtained respectively, thus by Proposition 2.5 $[E,f_0] \cong [E,f_1]$. Notice that when X is a point the previous construction is the particular case of clutching functions over $\mathbb{S}^k$. From construction it is evident that

$$[E_1,f_1] \oplus [E_2,f_2] \cong [E_1 \oplus E_2, f_1 \oplus f_2].$$

Example 3.6. By elementary calculation we have the following essential facts:

- $[E, \mathrm{id}]$ is the external product $E * \underline{1} = \mu(E \otimes \underline{1})$, or equivalently the pullback of E via the projection $X \times \mathbb{S}^2 \to X$.
- By the Example 2.17 we have that $[\underline{1}, z] \cong H$, where $\underline{1}$ is the trivial line bundle over the space X consisting of a single point, z is multiplication by z in $\mathbb{S}^1 \subset \mathbb{C}$, and H is the canonical line bundle over $\mathbb{S}^2 = \mathbb{CP}^1$. More generally we have $[\underline{1}, z^n] \cong H^n$, the n-fold tensor product of H with itself. The formula $[\underline{1}, z^n] \cong H^n$ holds also for negative n if we define $H^{-1} = [\underline{1}, z^{-1}]$, which is justified by the fact that $H \otimes H^{-1} \cong \underline{1}$.
- $[E, z^n] \cong E * H^n = \mu(E \otimes H^n)$ for $n \in \mathbb{Z}$.
- Generalizing this, $[E, z^n f] \cong [E,f] \otimes \widehat{H}^n$ where $\widehat{H}^n$ denotes the pullback of H^n via the projection $X \times \mathbb{S}^2 \to \mathbb{S}^2$.

  Notice that by the last two items we have $[E, \mathrm{id}] \otimes \widehat{H}^n \cong [E, z^n] \cong E * H^n$ and $[E, z] \otimes \widehat{H}^n \cong [E, z^{n+1}] \cong E * H^{n+1}$. This observation will be used to prove the surjectivity of μ.

Now we give the main fact in this step.

Proposition 3.1. Every vector bundle $E' \to X \times \mathbb{S}^2$ is isomorphic to the vector bundle $[E, f]$ for some vector bundle $E \to X$ and clutching function $f : E \times \mathbb{S}^1 \to E \times \mathbb{S}^1$.

Proof. Let $\mathbb{S}^2 = D_+ \cup D_-$. We define $E_\alpha := E'|_{X\times D_\alpha}$ for $\alpha = +, -$ and $E = E'|_{X\times\{1\}}$. As D_α is contractible then the projection $X \times D_\alpha \to X \times \{1\}$ is homotopic to the identity map $X \times D_\alpha \to X \times D_\alpha$, therefore by the Theorem 2.1 E_α is isomorphic to the pullback of E by the projection $X \times D_\alpha \to X \times \{1\}$, and by elementary calculation this pullback is $E \times D_\alpha$. Thus we have the isomorphism $h_\alpha : E_\alpha \to E \times D_\alpha$, which gives us the clutching function

$$f := (h_+|_{E\times\mathbb{S}^1}) \circ (h_-|_{E\times\mathbb{S}^1})^{-1} : E \times \mathbb{S}^1 \to E \times \mathbb{S}^1.$$

Is a matter of calculation to show that $E' \cong [E, f]$. □

Second step: Now we show that any clutching function $f : E \times \mathbb{S}^1 \to E \times \mathbb{S}^1$ can be approximated by a **Laurent polynomial clutching function** which have the form $\ell(x, z) = \sum_{|i|\leq n} a_i(x)z^i$ where $a_i : E \to E$ restrict to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. For this we approximate each continuous function $f : X \times \mathbb{S}^1 \to \mathbb{C}$ for X a compact space by a continuous Laurent polynomial function

$$\begin{array}{cccc} \ell : & X \times \mathbb{S}^1 & \to & \mathbb{C} \\ & (x, z) & \mapsto & \sum_{|n|<N} a_n(x)z^n, \qquad z = e^{i\theta} \end{array}$$

where $a_n : X \to \mathbb{C}$ is given by

$$x \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(x, e^{i\theta})e^{-in\theta}d\theta.$$

For $r > 0$ let

$$u(x, r, \theta) := \sum_{n\in\mathbb{Z}} a_n(x)r^{|n|}e^{in\theta}.$$

Notice that for $0 < r < 1$ by compactness of $X \times \mathbb{S}^1$ we have that $|f(x, e^{i\theta})|$ is bounded and hence also $|a_n(x)|$, thus by comparison with the geometric series $\sum_n r^n,\ 0 < r < 1$, this series converges absolutely as (x, θ) ranges over $X \times [0, 2\pi]$. The following lemma will allow us to conclude that sums of finitely many terms in the series for $u(x, r, \theta)$ with r near to 1 will give the desired approximations to f by Laurent polynomial functions.

Lemma 3.2. As $r \to 1$, $u(x, r, 0) \to f(x, e^{i\theta})$ uniformly in x and 0.

Proof. For $r < 1$ we have

$$\begin{aligned} u(x, r, \theta) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} r^{|n|}e^{in(\theta-t)}f(x, e^{it})dt \\ &= \int_0^{2\pi} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-t)}f(x, e^{it})dt \end{aligned}$$

where the order of summation and integration can be interchanged since the series in the latter formula converges uniformly, by comparison with the geometric series $\sum_n r^n$. Define the Poisson kernel

$$P(r, \varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\varphi}$$

for $0 \leq r < 1$ and $\varphi \in \mathbb{R}$. Then

$$u(x,r,\theta) = \int_0^{2\pi} P(r,\theta - t) f(x, e^{it}) dt.$$

By summing the two geometric series for positive and negative n in the formula for $P(r,\varphi)$, one compute that

$$P(r,\varphi) = \frac{1}{2\pi}\left(\frac{1-r^2}{1-2rcos\varphi + r^2}\right).$$

Three basic facts about $P(r,\varphi)$ which we shall need are:

- As a function of φ, $P(r,\varphi)$ is even, of period 2π, and monotone decreasing on $[0,\pi]$, since the same is true of $cos\varphi$ which appears in the denominator of $P(r,\varphi)$ with a minus sign. In particular we have $P(r,\varphi) \geq P(r,\pi) > 0$ for all $r < 1$.
- $\int_0^{2\pi} P(r,\varphi) d\varphi = 1$ for each $r < 1$, as one sees by integrating the series for $P(r,\varphi)$ term by term.
- For fixed $\varphi \in (0,\pi)$, $P(r,\varphi) \to 0$ as $r \to 1$ since the numerator of $P(r,\varphi)$ approaches 0 and the denominator approaches $2 - 2\cos\varphi \neq 0$.

Now to show uniform convergence of $u(x,r,\theta)$ to $f(x,e^{i\theta})$ we first observe that, using the second point above, we have

$$\begin{aligned}|u(x,r,\theta) - f(x,e^{i\theta})| &= |\int_0^{2\pi} P(r,\theta-t) f(x,e^{it})dt - \int_0^{2\pi} P(r,\theta-t) f(x,e^{i\theta})dt| \\ &\leq \int_0^{2\pi} P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|dt.\end{aligned}$$

Given $\varepsilon > 0$, there exist a $\delta > 0$ such that $|f(x,e^{it}) - f(x,e^{i\theta})| < \varepsilon$ for $|t-\theta| < \delta$ and all x, since f is uniformly continuous on the compact space $X \times \mathbb{S}^1$. Let I_δ denote the integral $\int P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|dt$ over the interval $|t-\theta| \leq \delta$ and let I'_δ denote this integral over the complement of the interval $|t-\theta| \leq \delta$ in an interval of length 2π. Then we have

$$I_\delta \leq \int_{|t-\theta|\leq\delta} P(r,\theta-t)\varepsilon dt \leq \varepsilon \int_0^{2\pi} P(r,\theta-t)dt = \varepsilon.$$

By the first numeral in the previous properties of Poisson kernel the maximum value of $P(r,\theta-t)$ on $|t-\theta| \geq \delta$ is $P(r,\delta)$. So

$$I'_\delta \leq P(r,\delta) \int_0^{2\pi} |f(x,e^{it}) - f(x,e^{i\theta})|dt.$$

The integral here has a uniform bound for all x and θ since f is bounded. Thus by the last paragraph we can make $I'_\delta \leq \varepsilon$ by taking r close enough to 1. Therefore $|u(x,r,\theta) - f(x,\theta)| \leq I_\delta + I'_\delta \leq 2\varepsilon$. $\square$

Before showing the key result in this step, we state some facts:

- The endomorphisms of $E \times \mathbb{S}^1$ form a complex vector space $End(E \times \mathbb{S}^1)$ with norm $\|\alpha\| = \sup_{|v|=1}|\alpha(v)|$ for a Hermitian inner product on E. By the triangle inequality, the balls in $\mathrm{End}(E \times \mathbb{S}^1)$ are convex.

- The subspace $\mathrm{Aut}(E\times\mathbb{S}^1)$ of automorphisms is open in the topology defined by this norm since it is the preimage of $(0,\infty)$ under the continuous map

$$\begin{array}{ccc}\mathrm{End}(E\times\mathbb{S}^1) & \to & [0,\infty)\\ \alpha & \mapsto & \inf_{(x,z)\in X\times\mathbb{S}^1}|\det(\alpha(x,z))|.\end{array}$$

Proposition 3.2. Every vector bundle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent polynomial clutching function ℓ. Laurent polynomial clutching functions ℓ_0 and ℓ_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy $\ell_t(x,z)=\sum_i a_i(x,t)z^i$.

Proof. In order to show that every vector bundle $[E,f]$ is isomorphic to $[E,\ell]$ for some Laurent polynomial clutching function ℓ, notice that by previous observations it is enough to show that Laurent polynomials are dense in $\mathrm{End}(E\times\mathbb{S}^1)$, since a sufficiently close Laurent polynomial approximation ℓ to f will then be homotopic to f via the linear homotopy $t\ell+(1-t)f$ through clutching functions. Let $\{(U_i,\varphi_i\}$ be a trivialization cover of $[E,f]$ with isomorphism $\varphi_i: p^{-1}(U_i)\to U_i\times\mathbb{C}^{n_i}$. Taking the standard basis $\{e_j\}$ of $\mathbb{C}^{n_i}$ we can apply Gram-Schmidt process to $\varphi_i^{-1}(e_j)$, in such way that φ_i takes the chosen inner product in $p^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$. Now, let $\{\lambda_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$ and let X_i be the support of λ_i, a compact set in U_i. Note that via φ_i for each $(x,z)\in X_i\times\mathbb{S}^1$ we have the linear map $f(x,z):E_x\to E_x$, which can be viewed as a matrix, where the entries of this matrix define functions $X_i\times\mathbb{S}^1\to\mathbb{C}$. By the previous lemma we can find Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ viewed as matrix for $x\in X_i$. It follows easily that ℓ_i approximates f in the norm $\|\cdot\|$. From the Laurent polynomial approximations ℓ_i over X_i we form by means partition of unity arguments the convex linear combination $\ell=\sum_i\lambda_i\ell_i$, a Laurent polynomial approximating f over all of $X\times\mathbb{S}^1$. So ℓ and f are homotopic via the linear homotopy $t\ell+(1-t)f$ where the important fact is that this homotopy is through clutching functions.

Finally, if ℓ_0 and ℓ_1 are homotopic Laurent polynomial clutching functions through clutching functions, we can approximate by similar arguments a homotopy from ℓ_0 and ℓ_1, viewed as an automorphism of $E\times\mathbb{S}^1\times I$, by a Laurent polynomial homotopy ℓ'_t, then combining this with linear homotopies from ℓ_0 to ℓ'_0 and ℓ_1 to ℓ'_1 to obtain a homotopy ℓ_t from ℓ_0 to ℓ_1. ☐

The last basic observation in this part is that each Laurent polynomial clutching function $\ell(x,z)=\sum_{|i|\le m}a_i(x)z^i$ can be written in the form $\ell(x,z)=z^{-m}q(x,z)$ for a polynomial clutching function $q(x,z)=a_m(x)z^m+\cdots+a_0(x)$. So we have $[E,\ell]\cong[E,q]\otimes\widehat{H}^{-m}$ by the Example 3.6. Thereby in short we have:

$$[E,f]\cong[E,q]\otimes\widehat{H}^{-m}$$

for $q=a_m(x)z^m+\cdots+a_0(x)$ and $\widehat{H}^{-m}$ denotes the pullback of H^m via the projection $X\times\mathbb{S}^2\to\mathbb{S}^2$.

Third step: In this step we reduce polynomial clutching functions to linear clutching functions by the following result:

Proposition 3.3. Let $q(x,z)=a_n(x)z^n+\cdots+a_0(x)$ be a polynomial clutching function of degree at most n, then

$$[E,q]\oplus[nE,\mathrm{id}]\cong[(n+1)E,L^nq]$$

for a linear clutching function L^nq.

Proof. It is clear that the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q(x,z) \end{pmatrix}$$

is a matrix representation of a clutching function for $[nE, \mathrm{id}] \oplus [E, q]$. On the other hand the matrix

$$B = \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ a_n(x) & a_{n-1}(x) & a_{n-2}(x) & \cdots & a_1(x) & a_0(x) \end{pmatrix}$$

defines an endomorphism of $(n+1)E$ by interpreting the (i,j) entry of the matrix as a linear map from the j^{th} summand of $(n+1)E$ to the i^{th} summand, with the entries 1 denoting the identity $E \to E$ and z denoting z times the identity, for $z \in \mathbb{S}^1$. We can pass from the matrix B to matrix A by a sequence of elementary row and column operations in the following way:

- In B, add z times the first column to the second column, then z times the second column to the third. And general, add z times the i^{th} column to the i^{th} column for $1 \le i \le n$. So we obtain the following matrix:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ a_n(x) & a_n(x)z + a_{n-1}(x) & \cdots & r(x,z) & q(x,z) \end{pmatrix}$$

  where $r(x,z) = a_n(x)z^{n-1} + a_{n-1}(x)z^{n-2} + \cdots + a_2(x)z + a_1(x)$.

- For each $i \le n$, subtract the multiple $a_n(x)z^{i-1} + a_{n-1}(x)z^{i-2} + \cdots + a_{n-i+2}(x)z + a_{n-i+1}(x)$ of the i^{th} row from the last row to make all the entries in the last row 0 except for the final $q(x,z)$. Thus we obtain the matrix A.

Notice that these row and column operations are not quite elementary row and column operations in the traditional sense since the entries of the matrices are not numbers but linear maps. However, by restricting to a fiber of E an choosing a basis in this fiber, each entry in the matrices A and B becomes a matrix itself, so the expanded version of A has nonzero determinant, and since each of the preceding row and column operations can be realized by a sequence of traditional row and column operations on the expanded matrices and in addition this operations preserve determinant, the expanded version of B is also invertible in each fiber. This means that B is a automorphism of $(n+1)E$ for each $z \in \mathbb{S}^1$, and therefore determines a clutching function that have the form $a(x)z+b(x)$ and we denote by $L^n q$, so it is a linear clutching function. As the elementary row and column operations can be achieved by continuous one-parameter families such operations preserving determinant, then the matrices A and B define homotopic clutching functions, producing isomorphic bundles, it is

$$[E,q]\oplus[nE,\mathrm{id}]\cong[(n+1)E,L^n q].$$

$\square$

Fourth step: The purpose of this part is to show that there is a splitting $E\cong E_+\oplus E_-$ such that $[E,a(x)z+b(x)]\cong[E_+,\mathrm{id}]\oplus[E_-,z]$.

Lemma 3.3. Let $b:E\to E$ be an endomorphism having no eigenvalues on the unit circle $\mathbb{S}^1$. Then there are unique subbundles E_+ and E_- of E such that:

a). $E=E_+\oplus E_-$.

b). $b(E_\pm)\subset E_\pm$.

c). The eigenvalues of $b|_{E_+}$ all lie outside $\mathbb{S}^1$ and the eigenvalues of $b|_{E_-}$ all lie inside $\mathbb{S}^1$.

Proof. Consider first the algebraic situation of a linear transformation $T:V\to V$ with characteristic polynomial $q(t)$. Assuming $q(t)$ has no roots on $\mathbb{S}^1$, we may factor $q(t)$ in $\mathbb{C}[t]$ as $q_+(t)q_-(t)$ where $q_+(t)$ has all its roots outside $\mathbb{S}^1$ and $q_-(t)$ has all its roots inside $\mathbb{S}^1$. Let $V_\pm$ be the kernel of $q_\pm(T):V\to V$. Since q_+ and q_- are relatively prime in $\mathbb{C}[t]$, there are polynomials r and s with $rq_++sq_-=1$. From $q_+(T)q_-(T)=q(T)=0$, we have $\mathrm{Im}q_-(T)\subset\mathrm{Ker}q_+(T)$, and the opposite inclusion follows from $r(T)q_+(T)+q_-(T)s(T)=\mathrm{Id}$. Thus $\mathrm{Ker}q_+(T)=\mathrm{Im}q_-(T)$, and similarly $\mathrm{Ker}q_-(T)=\mathrm{Im}q_+(T)$. From $q_+(T)r(T)+q_-(T)s(T)=\mathrm{Id}$ we see that $\mathrm{Im}q_+(T)+\mathrm{Im}q_-(T)=V$, and from $r(T)q_+(T)+s(T)q_-(T)=\mathrm{Id}$ we deduce that $\mathrm{Ker}q_+(T)\cap\mathrm{Ker}q_-(T)=0$. Hence $V=V_+\oplus V_-$. We have $T(V_\pm)\subset V_\pm$ since $q_\pm(T)(v)=0$ implies $q_\pm(T)(T(v))=T(q_\pm(T)(v))=0$. All eigenvalues of $T|V_\pm$ are roots of $q_\pm$ since $q_\pm(T)=0$ on $V_\pm$. Thus conditions (a)-(c) hold for V_+ and V_-.

To see the uniqueness of V_+ and V_- satisfying (a)–(c), let $q'_\pm$ be the characteristic polynomial of $T|V_\pm$, so $q=q'_+q'_-$. All the linear factors of $q'_\pm$ must be factors of $q_\pm$ by condition (c), so the factorizations $q=q'_+q'_-$ and $q=q_+q_-$ must coincide up to scalar factors. Since $q'_\pm(T)$ is identically zero on $V_\pm$, so must be $q_\pm(T)$, hence $V_\pm\subset\mathrm{Ker}q_\pm(T)$. Since $V=V_+\oplus V_-$ and $V=\mathrm{Ker}q_+(T)\oplus\mathrm{Ker}q_-(T)$, we must have $V_\pm=\mathrm{Ker}q_\pm(T)$. This establishes the uniqueness of $V_\pm$.

As T varies continuously through linear transformations without eigenvalues on $\mathbb{S}^1$, its characteristic polynomial $q(t)$ varies continuously through polynomials without roots in $\mathbb{S}^1$. In this situation we assert that the factors $q_\pm$ of q vary continuously with q, assuming that q, q_+ , and q_- are normalized to be monic polynomials. To see this we shall use the fact that for any circle C in $\mathbb{C}$ disjoint from the roots of q, the number of roots of q inside C , counted with multiplicity, equals the degree of the map $\gamma:C\to\mathbb{S}^1$, $\gamma(z)=q(z)/|q(z)|$. To prove this fact consider first the case of a small circle C about a root $z=a$ of multiplicity m, so $q(t)=p(t)(t-a)^m$ with $p(a)\neq 0$. The homotopy

$$\gamma_s(z)=\frac{p(sa+(1-s)z)(z-a)^m}{|p(sa+(1-s)z)(z-a)^m|}$$

gives a reduction to the case $(t-a)m$, where it is clear that the degree is m. As the circle C is gradually enlarged, the map γ changes only by homotopy, leaving its degree unchanged, except when C is enlarged to enclose another root of q. In this case the new loop C is homotopic to the composition of the old loop C with a small loop about the new root, so the degree of γ for the new C is the sum of the old degree plus the degree for the small loop. The result then follows by induction.

For a small circle C about a root $z = a$ of q of multiplicity m, small perturbations of q to nearby polynomials q' produce small perturbations of the map γ to nearby maps γ' which are therefore homotopic to γ, having the same degree, hence q' also has m roots $a_1, ..., a_m$ inside C. Thus the factor $(z-a)^m$ of q becomes a factor $(z-a_1)\cdots(z-a_m)$ of the nearby q'. Since the a_i's are near a, the factor $(z-a)^m$ of q is close to the factor $(z-a_1)\cdots(z-a_m)$ of q' since multiplication of polynomials is a continuous operation. Taking the products of these factors for the roots inside or outside the unit circle, we conclude that $q'_{\pm}$ is close to $q_{\pm}$, so the factorization $q = q_+q_-$ varies continuously with q, as claimed.

Next we observe that as T varies continuously through transformations without eigenvalues in $\mathbb{S}^1$, the splitting $V = V_+ \oplus V_-$ also varies continuously. To see this, recall that $V_+ = \mathrm{Im} q_-(T)$ and $V_- = \mathrm{Im} q_+(T)$. Choose a basis $v_1, ..., v_n$ for V such that $q_-(T)(v_1), ..., q_-(T)(v_k)$ is a basis for V_+ and $q_+(T)(v_{k+1}), ..., q_+(T)(v_n)$ is a basis for V_-. For nearby T these vectors vary continuously, hence remain independent. Thus the splitting $V = \mathrm{Im} q_-(T) \oplus \mathrm{Im} q_+(T)$ continues to hold for nearby T, and so the splitting $V = V_+ \oplus V_-$ varies continuously with T. It follows that the union $E_{\pm}$ of the subspaces $V_{\pm}$ in all the fibers V of E is a subbundle, and so the proof of the lemma is complete. □

Proposition 3.4. Given a bundle $[E, a(x)z + b(x)]$, there is a splitting $E \cong E_+ \oplus E_-$ with $[E, a(x)z + b(x)] \cong [E_+, \mathrm{id}] \oplus [E_-, z]$.

Proof. The first step is to reduce to the case that $a(x)$ is the identity for all x. Notice that the substitution $z \mapsto (z+t)/(1+tz)$ for $0 \le t < 1$ takes $\mathbb{S}^1$ to itself, since if $|z| = 1$ we have

$$\left|\frac{z+t}{1+tz}\right| = \left|\frac{\overline{z}(z+t)}{1+tz}\right| = \left|\frac{1+t\overline{z}}{1+tz}\right| = \left|\frac{\overline{w}}{w}\right| = 1.$$

So applying this substitution to $a(x)z + b(x)$ and then multiplying by the nonzero scalar $1 + tz$, we obtain an invertible linear transformation

$$A(t) := (1+tz)\left(a(x)\frac{z+t}{1+tz} + b(x)\right)$$

for $0 \le t < 1$. Therefore $A(t)$ defines a homotopy of clutching functions as t goes from 0 to $t_0 < 1$, where for $t = 0$ this is equals $a(x)z + b(x)$. This give us the isomorphism $[E, a(x)z + b(x)] \cong [E, A(t_0)]$ for t_0 near to 1. But by elementary calculations we have $A(x) = (a(x) + tb(x))z + ta(x) + b(x)$. Now notice that in this expression the term $a(x) + tb(x)$ is invertible for $t = 1$ since it is the restriction of $a(x)z + b(x)$ to $z = 1$, and as the continuous function $t \mapsto \inf_{x \in X} |\det(a(x) + tb(x))|$ is nonzero for $t = 1$, hence also for t near 1. Thus $a(x) + tb(x)$ is invertible for $t = t_0$ near 1. Now we use the simple general fact that $[E, f \cdot g] \cong [E, f]$ for any isomorphism $g : E \times \mathbb{S}^1 \to E \times \mathbb{S}^1$, to this specific situation taking $f(x, z) = (a(x) + tb(x))z + ta(x) + b(x)$ and $g(x, z) = (a(x) + t_0 b(x))^{-1}$, allows us to replace the clutching function $A(t_0)$ by the clutching function $z + (t_0 a(x) + b(x))(a(x) + t_0 b(x))^{-1}$, reducing to the case of clutching function of the form $z + b(x)$. Since $z + b(x)$ is invertible for all x, $b(x)$ has no eigenvalues on the unit circle $\mathbb{S}^1$, so by previous lemma we have the splitting

$$[E, z + b(x)] \cong [E_+, z + b_+(x)] \oplus [E_-, z + b_-(x)]$$

where b_+ and b_- are the restrictions of b. Since $b_+(x)$ has all its eigenvalues outside $\mathbb{S}^1$, the formula $tz + b_+b(x)$ for $0 \le t \le 1$ defines a homotopy of clutching functions from $z + b_+(x)$ to $b_+(x)$. Hence $[E_+, z + b_+(x)] \cong [E_+, b_+(x)] \cong [E_+, \mathrm{id}]$. Similarly, $z + tb_-(x)$ defines a homotopy of clutching functions from $z + b_-(x)$ to z, so $[E_-, z + b_-(x)] \cong [E_-, z]$. So we get the desired result

$$[E, a(x)z + b(x)] \cong [E_+, \mathrm{id}] \oplus [E_-, z].$$

□

Note that the first step of reducing to the case when $a(x)$ is the identity clearly respects sums, and the other hand, the uniqueness of the $\pm$-splitting in Lemma 3.3 guarantees that it preserves sums. Therefore the splitting

$$[E, a(x)z + b(x)] \cong [E_+, \mathrm{id}] \oplus [E_-, z]$$

constructed in the previous proposition preserves direct sums, in the sense that the splitting for a sum $[E_1 \oplus E_2, (a_1z + b_1) \oplus (a_2z + b_2)]$ has $(E_1 \oplus E_2)_\pm = (E_1)_\pm \oplus (E_2)_\pm$.

Fifth step: Finally we can prove the Bott periodicity theorem.

Theorem 3.1. The homomorphism $\mu : K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K^0(X \times \mathbb{S}^2)$ is an isomorphism of rings for all compact Hausdorff spaces X.

Proof. By previous propositions we have that each vector bundle on $X \times \mathbb{S}^2$ is isomorphic to a vector bundle of the form $[E, f]$ for some vector bundle E on $\mathbb{S}^2$ and a polynomial clutching function f. Further we have in $K^0(X \times \mathbb{S}^2)$

$$\begin{aligned}
[E, f] &= [E, z^{-m}q] \\
&= [E, q] \otimes \widehat{H}^{-m} \\
&= [(n+1)E, L^n q] \otimes \widehat{H}^{-m} - [nE, \mathrm{id}] \otimes \widehat{H}^{-m} \\
&= [((n+1)E)_+, \mathrm{id}] \otimes \widehat{H}^{-m} + [((n+1)E)_-, z] \otimes \widehat{H}^{-m} - [nE, \mathrm{id}] \otimes \widehat{H}^{-m} \\
&= ((n+1)E)_+ * H^{-m} + ((n+1)E)_- * H^{1-m} - nE * H^{-m}.
\end{aligned}$$

Clearly this expression is in the image of μ. Hence μ is surjective.

For the injectivity of μ we shall construct $\nu : K^0(X \times \mathbb{S}^2) \to K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2$ such that $\nu\mu = \mathrm{id}$. The following identities allow us to define $\mu([E, f])$. For deg $q \le n$ we have:

1. $$[(n+2)E, L^{n+1}q] \cong [(n+1)E, L^n q] \oplus [E, \mathrm{id}].$$

   The matrix representation of $L^{n+1}q$ is

$$\begin{pmatrix}
1 & -z & 0 & \cdots & 0 & 0 \\
0 & 1 & -z & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -z \\
0 & a_n(x) & a_{n-1}(x) & \cdots & a_1(x) & a_0(x)
\end{pmatrix}.$$

Add z times the first column to the second column we have

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ 0 & a_n(x) & a_{n-1}(x) & \cdots & a_1(x) & a_0(x) \end{pmatrix}$$

where the first row and column give the summand $[E, \mathrm{id}]$ while the rest of the matrix gives $[(n+1)E, L^n q]$.

2.
$$[(n+2)E, L^{n+1}(qz)] \cong [(n+1)E, L^n q] \oplus [E, z].$$

The matrix representation of $L^{n+1}(zq)$ is

$$\begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ a_n(x) & a_{n-1}(x) & a_{n-2}(x) & \cdots & a_0(x) & 0 \end{pmatrix}.$$

Add z^{-1} times the last column to the next-to-last column and exchanging the last row with the next-to-last row we have

$$\begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_n(x) & a_{n-1}(x) & a_{n-2}(x) & \cdots & a_0(x) & 0 \\ 0 & 0 & 0 & \cdots & 0 & -z \end{pmatrix}.$$

The first $n+1$ block in the matrix is the representation of summand $[(n+1)E, L^n q]$ and the last row and column give the summand $[E, -z]$. But $[E, -z] \cong [E, z]$, since the clutching function $-z$ being the composition of the clutching function z with the automorphism -1 of E.

3. For $[E, \mathrm{id}]$ the summand E_- is 0 and $E_+ = E$.

   Applying the procedure at the beginning of the proof of before proposition in step fourth, which replaces a clutching function $a(x)z + b(x)$ by the clutching function $z + (t_0 a(x) + b(x))(a(x) + t_0 b(x))^{-1}$ with $0 < t_0 < 1$, in the case in that $a(x)z + b(x) = \mathrm{id}$ this yields $z + t_0^{-1}\mathrm{id}$. Since $t_0^{-1}\mathrm{id}$ has only the one eigenvalue $t_0^{-1} > 1$, we have $E_- = 0$.

4. For $[E, z]$ the summand E_+ is 0 and $E_- = E$.

   The clutching function z is in the form $z + b(x)$ with $b(x) = 0$, so 0 is the only eigenvalue of $b(x)$ and hence $E_+ = 0$.

Suppose we define

$$\begin{array}{rccc} \nu: & K^0(X \times \mathbb{S}^2) & \to & K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \\ & [E, z^{-m} q] & \mapsto & ((n+1)E)_- \otimes (H-1) + E \otimes H^{-m}. \end{array}$$

For $n \geq \deg q$. Using the fact that the $\pm$-splitting preserves direct sums, by the formulas (1) and (3) we have that $((n+2)E)_- \cong ((n+1)E)_-$, so the minus summand is independent of n, thus $\nu([E, z^{-m}q])$ does not depend on n. To see that it is independent of m we must see that it is unchanged when $z^{-m}q$ is replaced by $z^{-m-1}(zq)$. By (2) and (4) we have the first of the following qualities:

$$\begin{aligned}\nu([E, z^{-m-1}(zq)]) &= ((n+1)E)_- \otimes (H-1) + E \otimes (H-1) + E \otimes H^{-m-1} \\ &= ((n+1)E)_- \otimes (H-1) + E \otimes (H^{-m} - H^{-m-1}) \\ &\quad + E \otimes H^{-m-1} \\ &= ((n+1)E)_- \otimes (H-1) + E \otimes H^{-m} \\ &= \nu([E, z^{-m}(q)]).\end{aligned}$$

The second equality uses the relation $(H-1)^2 = 0$ which implies $H(H-1) = H-1$ and hence iterating we have that $H - 1 = H^{-m} - H^{-m-1}$ for all m. The third and fourth equalities are evident operations.

Another choice which might perhaps affect the value of $\nu([E, z^{-m}(q)])$ is the constant $t_0 < 1$ in the proof in the proposition in the before step. This could be any number sufficiently close to 1, so varying t_0 gives a homotopy of the endomorphism b in Proposition 3.4 in the previous step. This has no effect on the $\pm$-splitting since we can apply the same proposition to the endomorphism of $E \times I$ given by the homotopy. Hence the choice of t_0 does not affect $\nu([E, z^{-m}(q)])$. It remains to see that $\nu([E, z^{-m}(q)])$ depends only on the bundle $[E, z^{-m}q]$, not on the clutching function $z^{-m}q$ for this bundle. We showed that every vector bundle over $X \times \mathbb{S}^2$ has the form $[E, f]$ for a normalized clutching function f which was unique up to homotopy, and we showed that Laurent polynomial approximations to homotopic f's are Laurent-polynomial-homotopic. Applying propositions in steps third and fourth over $X \times I$ with a Laurent polynomial homotopy as clutching function, we conclude that the two bundles $((n+1)E)_-$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic. So $\nu([E, z^{-m}(q)])$ is well-defined.

As the $\pm$ splitting preserves sums and $L^n(q_1 \oplus q_2) = L^n q_1 \oplus L^n q_2$, then by definitions we have that ν takes sums to sums. It follows that ν extends to a homomorphism $K^0(X \times \mathbb{S}^2) \to K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2$.

We observe that the group $\mathbb{Z}[H]/(H-1)^2$ is generated by 1 and H, so in view of the relation $H + H^{-1} = 2$, which follows from $(H-1)^2 = 0$, we see that $K^0(\mathbb{S}^2)$ is also generated by 1 and H^{-1}. Thus in order to verify that $\nu\mu = \mathrm{id}$ it is enough to check the equation on elements of the form $E \otimes H^{-m}$ for $m \geq 0$. We have

$$\begin{aligned}\nu\mu(E \otimes H^{-m}) &= \nu([E, z^{-m}) \\ &= E_- \otimes (H-1) + E \otimes H^{-m} \\ &= E \otimes H^{-m}\end{aligned}$$

since $E_- = 0$, the polynomial q being id so that (3) applies. $\square$

3 Cohomological properties

The main reason that ordinary cohomology is computable is the Mayer-Vietoris principle, which allows one to compute the cohomology of a space in terms of that of its pieces. Roughly, the Mayer-Vietoris principle is a conservation rule for the total cohomology, and it is essential to this conservation rule that all cohomology groups are considered. In what follows, we will see that Bott periodicity provides similar long exact sequences for K-theory. We will closely follow Atiyah [Ati67] and [Ati62].

Definition 3.4. A **compact pair** (X, Y) is a pair of compact topological spaces where Y is a subspace of X. A morphism $f : (X, Y) \to (X', Y')$ is a morphism $f : X \to X'$ which sends Y to Y'.

If (X, Y) is a compact pair, we will denote by X/Y the quotient space of X where all the points in Y are identified. We will denote by $\mathcal{C}$ the category of compact spaces, by $\mathcal{C}^+$ the category of compact spaces with a base point and $\mathcal{C}^2$ the category of compact pairs. There are natural functors:

$$\begin{aligned} A : \mathcal{C} \to \mathcal{C}^2 \quad &; \quad X \mapsto (X, \varnothing), \\ B : \mathcal{C}^2 \to \mathcal{C}^+ \quad &; \quad (X, Y) \mapsto (X/Y, Y/Y). \end{aligned}$$

For any X in $\mathcal{C}$ we denote by X^+ the pointed space $B(A(X))$, which is just the disjoint union of X and a point.

In what follows we denote the K-theory K^0 simply by K.

Definition 3.5. The **reduced K-group** of a pointed space X, denoted $\widetilde{K}(X)$ is the kernel of the map $\iota^* : K(X) \to K(\{x\})$, where ι is the inclusion of the base point.

The pull back by the projection map $\pi : X \to \{x\}$ gives a splitting for ι^*, since applying the K functor to

$$\{x\} \hookrightarrow X \xrightarrow{\pi} \{x\}$$

that is the identity map on $\{x\}$, we obtain

$$K(\{x\}) \xrightarrow{\pi^*} K(X) \xrightarrow{i^*} K(\{x\})$$

i.e. $(i \circ \pi)^* = \mathrm{id}^*$,so that there is a natural isomorphism:

$$K(X) \cong \widetilde{K}(X) \oplus K(\{x\}) = \widetilde{K}(X) \oplus \mathbb{Z}.$$

Where the last equality is by the Example 3.3. This decomposition is natural with respect to based maps and therefore the reduced K-group defines a functor from pointed spaces to commutative rings.

Definition 3.6. The **relative K-group** of a pair, denoted $K(X, Y)$ is:

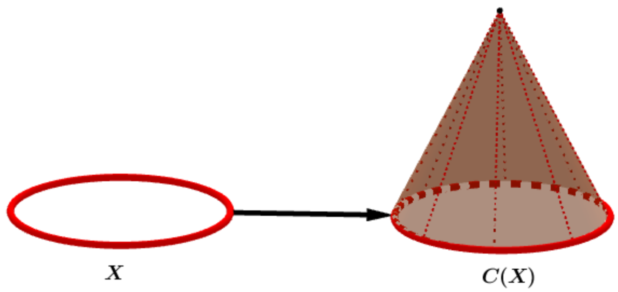
$$K(X, Y) := \widetilde{K}(X/Y).$$

Notice that for any compact space there is a natural isomorphism $K(X) \cong \widetilde{K}(X^+)$, therefore we have that in particular $K(X, \emptyset) \cong K(X)$. Furthermore, since $\widetilde{K}$ is a functor on $\mathcal{C}^+$ it follows that $K(X, Y)$ is a contravariant functor of (X, Y) in $\mathcal{C}^2$.

Before we continue we will describe some other constructions we will need.

Definition 3.7. Suppose that $X \in \mathcal{C}$, the **cone** of X, denoted $C(X)$ is the space:

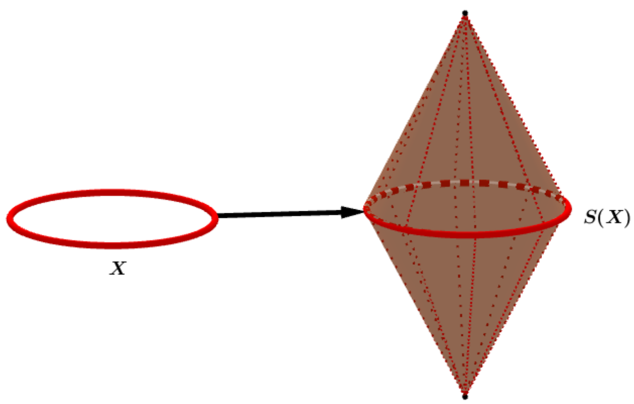
$$C(X) := X \times I / X \times \{1\}.$$



We consider the cone to be a pointed space in the obvious way. The space X can be seen as a subspace of $C(X)$ by identifying X with $X \times \{0\}$.

Definition 3.8.

- The **unreduced suspension** of X is the space $C(X)/X \times \{0\}$.



- The **smash product** of two pointed spaces (X, x) and (Y, y) is the pointed space:

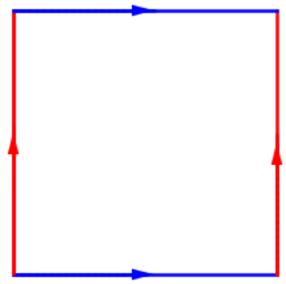
$$X \wedge Y := X \times Y/(X \times \{y\} \cup \{x\} \times Y).$$

The smash product is associative in the sense that there are natural homeomorphisms:

$$X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z,$$

which we will use freely in what follows.

Example 3.7. The smash product of two circles is a sphere. To see this we first note that $\mathbb{S}^1 \times \mathbb{S}^1$ gives the torus:



and then quotienting by $X \times \{y\} \cup \{x\} \times Y$ means identifying the edges of the square in the above diagram, and so we obtain a sphere.

We denote by I the unit interval and consider the sphere $\mathbb{S}^n$ as a pointed space via the identification $\mathbb{S}^n \cong I^n/\partial I$. There is a natural homeomorphism

$$\mathbb{S}^n \cong \mathbb{S}^1 \wedge \cdots \wedge \mathbb{S}^1.$$

Definition 3.9. The **reduced suspension** of a pointed space is:

$$S(X) := \mathbb{S}^1 \wedge X.$$

By the associativity of the smash product, iterating the reduced suspension n times has the effect of taking smash product with $\mathbb{S}^n$. We denote the iterated suspension of X by $S^n(X)$ and thus:

$$S^n(X) = S \circ \cdots \circ S(X) = \mathbb{S}^1 \wedge \cdots \wedge \mathbb{S}^1 \wedge X \cong \mathbb{S}^n \wedge X.$$

Definition 3.10. For $n \geq 0$:

$$\begin{aligned} \widetilde{K}^{-n}(X) &:= \widetilde{K}(S^n(X)) \quad \text{if} \quad X \in \mathcal{C}^+, \\ K^{-n}(X,Y) := \widetilde{K}^{-n}(X/Y) &= \widetilde{K}(S^n(X/Y)) \quad \text{if} \quad X \in \mathcal{C}^2, \\ K^{-n}(X) &:= \widetilde{K}(S^n(X^+)) \quad \text{if} \quad X \in \mathcal{C}. \end{aligned}$$

It seems that our notation for the suspension of a pointed space is ambiguous, since we denote by $S(X)$ both the reduced and the unreduced suspension of X. This abuse of notation should not be problematic from the point of view of K-theory. Suppose that $(X, x_0) \in \mathcal{C}^+$ then, by the functoriality of the cone, there is a natural inclusion map:

$$I \cong S(\{x_0\}) \to S(X),$$

such that the by collapsing $I \subset S(X)$ to a point, one obtains the reduced suspension of X. The following lemma justifies our choice of notation:

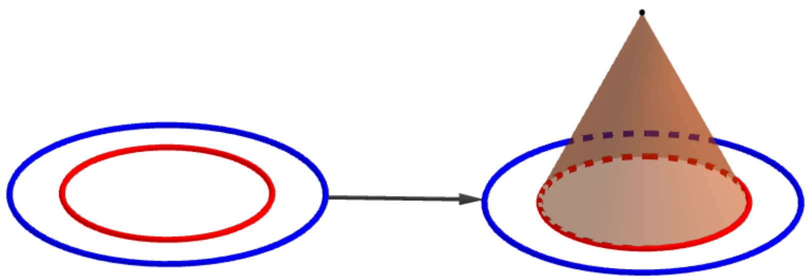
Lemma 3.4. Suppose that $(X, Y) \in \mathcal{C}^2$ and Y is a contractible space. Then the quotient map $\pi : X \to X/Y$ induces an isomorphism $\pi^* : K(X/Y) \to K(X)$.

Proof. We will construct an inverse map π_*. Suppose that E is a vector bundle over X. Then, since Y is contractible we know that $E|Y$ is a trivial bundle and we fix an isomorphism φ : $E|Y \cong \underline{n}$. We observe that, since Y is contractible, any two such isomorphisms are homotopic. We define the bundle $\pi_*(E)$ over X/Y as follows: as a space it is the quotient of E by the equivalence relation that identifies $v \in E_y$ with $v' \in E_{y'}$ if $\varphi(v) = \varphi(v')$. Since the identification is linear, the fibers of the projection $\pi_*(E) \to X/Y$ have the structure of vector spaces. We only need to check local triviality of this bundle around the point Y/Y, for this we observe that by the Lemma 2.4, φ can be extended to an open set open set $U \supset Y$. It is immediate to check that the function π_* is an inverse to π^*. □

Definition 3.11. If $(X, Y) \in \mathcal{C}^2$ we define the **relative cone** of of Y in X to be the space:

$$XUC(Y) := X \coprod C(Y)/\sim,$$

where $\sim$ is the equivalence relation that identifies a point $y \in Y \subset X$ with a point $(y, 1) \in C(Y)$.



We view the relative cone as a pointed space with base point that of $C(Y)$. There is an obvious homeomorphism:

$$X \cup C(Y)/X \cong S(Y).$$

We conclude that if $Y \in \mathcal{C}^+$ then:

$$K(X \cup C(Y), X) \cong K(C(Y)/Y) \cong \widetilde{K}(S(Y)) \cong \widetilde{K}^{-1}(Y).$$

Lemma 3.5. Suppose that $(X, Y) \in \mathcal{C}^2$ and denote by $\iota : Y \hookrightarrow X$ and $\pi : X \to X/Y$ the obvious inclusion and projection maps. Then the sequence:

$$K(X, Y) \xrightarrow{\pi^*} K(X) \xrightarrow{\iota^*} K(Y),$$

is exact.

Proof. In order to prove that $(\pi \circ \iota)^* = \iota^* \circ \pi^* = 0$ we observe that $\pi \circ \iota$ factors through the space Y/Y, whose reduced K group is zero. It remains to show that $\ker(\iota^*) \subset \operatorname{Im}(\pi^*)$. The general element in the kernel of ι^* is of the form $[E] - [\underline{n}]$ and satisfies the property that

$$E|Y \oplus \underline{m} \cong \underline{n} \oplus \underline{m}.$$

Since $E \oplus \underline{m}$ is trivial over Y, by Lemma 2.4 it is trivial over some open containing Y and therefore the bundle $E \oplus \underline{n}$ descends to the quotient X/Y, whose class in $\widetilde{K}(X, Y)$ we denote by $[E \oplus \underline{m}]_{(X/Y)}$. Set

$$\eta := [E \oplus \underline{n}]_{(X/Y)} - [\underline{m} \oplus \underline{m}] \in K(X, Y),$$

then:

$$\pi^*(\eta) = [E \oplus \underline{m}] - [\underline{m} \oplus \underline{n}] = [E] - [\underline{n}].$$

□

Corollary 3.1. If $(X, Y) \in \mathcal{C}^2$ and $Y \in \mathcal{C}^+$, then the sequence:

$$K(X,Y) \xrightarrow{\pi^*} \widetilde{K}(X) \xrightarrow{\iota^*} \widetilde{K}(Y),$$

is exact.

The following technical results will be essential in what follows.

Lemma 3.6. For any $X \in \mathcal{C}^+$ we view a function $f : X \to \mathrm{GL}_n(\mathbb{C})$ as a clutching data for the trivial bundle over the north and south cones of $S(X)$ and write E_f for the associated bundle. The map:

$$[X, \mathrm{GL}_n(\mathbb{C})] \to \mathrm{Vect}_n(S(E));\ f \mapsto E_f,$$

is an isomorphism of sets.

Proof. Let E be a vector bundle of rank k over $S(X)$. Since the north and south cones of $S(X)$ are contractible, we know the restriction of E to those subspaces is trivial. We conclude that the map above is surjective. In order to prove injectivity, suppose that there is an isomorphism $\varphi : E_f \to E_g$, and denote by φ^+, φ^- the restriction of φ to the north and south cone. Since these cones are contractible, φ^+, φ^- are homotopic to the constant function equal to the identity. Then we consider the commutative diagram of vector bundles over X:

$$\begin{tikzcd}
\underline{k} \arrow[r, "f"] \arrow[d, "\varphi^+|X"'] & \underline{k} \arrow[d, "\varphi^-|X"] \\
\underline{k} \arrow[r, "g"] & \underline{k}.
\end{tikzcd}$$

Because φ^+, φ^- are homotopic to the identity, so are there restrictions to X and we conclude that f and g are homotopic. $\square$

Lemma 3.7. Using the same notation as in the previous lemma, for any $X \in \mathcal{C}$ the map:

$$\lim_{n\to\infty} [X, \mathrm{GL}_n(\mathbb{C})] \to \widetilde{K}(S(E));\ f \mapsto [E_f] - [\underline{n}],$$

is an isomorphism of groups.

Proof. Let us first prove that the map above is a bijection of sets. In order to prove surjectivity, we observe that the general element in $\widetilde{K}(S(X))$ is of the form $[E] - [\underline{n}]$, where E is some vector bundle of rank n. By the previous lemma, E is of the form E_f for some f. Similarly, injectivity is a consequence of the lemma above. Let us now check that the map is a group homomorphism. For this it suffices to prove that the two maps $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ given by the formulas:

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

and

$$(A, B) \mapsto \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix}$$

are homotopic. A specific homotopy is given by the formula:

$$\phi_t(A, B) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix},$$

where $t \in [0, \frac{\pi}{2}]$. □

Corollary 3.2. Let $T : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by

$$T(t) = 1 - t.$$

For $Y \in \mathcal{C}^+$ we denote by $T \wedge 1 : S(Y) \to S(Y)$ the induced map. Then, for any $\alpha \in \widetilde{K}(S(Y))$:

$$(T \wedge 1)^*(\alpha) = -\alpha.$$

Proof. Under the isomorphism given by the Lemma 3.6, the map $(T \wedge 1)^*$ corresponds to the map $f \mapsto f^{-1}$, which is the inverse operation in the group $\widetilde{K}(S(Y))$. □

We are now ready to prove the existence of a long exact sequence in K-theory.

Theorem 3.2. For any $(X, Y) \in \mathcal{C}^2$ there is a long exact sequence:

$$\cdots \to K^{-n-1}(Y) \xrightarrow{\delta} K^{-n}(X, Y) \xrightarrow{\pi^*} K^{-n}(X) \xrightarrow{\iota^*} K^{-n}(Y) \cdots \to K^0(Y)$$

Proof. First we observe that it suffices to prove that for $(X, Y) \in \mathcal{C}^2$ and $Y \in \mathcal{C}^+$, there is an exact sequence:

$$\widetilde{K}^{-1}(X) \xrightarrow{\iota^*} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} K^0(X, Y) \xrightarrow{\pi^*} \widetilde{K}^0(X) \xrightarrow{\iota^*} \widetilde{K}^0(Y) \tag{3.2}$$

Indeed, for $n > 0$ one obtains the sequence above by applying sequence (3.2) to the pair $(S^n(X), S^n(Y))$. For $n = 0$ one applies (3.2) to the pair (X^+, Y^+). We will now prove the existence of the sequence (3.2). Our first task is to define the connecting map $\delta : \widetilde{K}^{-1}(Y) \to K(X, Y)$. We will apply Corollary 3.1 to the pair $(X \cup CY, X)$, so we obtain the exact sequence:

$$K(X \cup CY, X) \xrightarrow{\iota^*} \widetilde{K}(X \cup C(Y)) \xrightarrow{\pi^*} \widetilde{K}(X).$$

On the other hand, there is a natural homeomorphism

$$X \cup CY/X \cong CY/Y = S(Y),$$

which induces an isomorphism:

$$\alpha : K^{-1}(S(Y)) \cong K(X \cup CY, X).$$

Moreover, since CY is contractible then the quotient map

$$X \cup CY \to X \cup CY/CY \cong X/Y$$

induces an isomorphism:

$$\beta : K(X, Y) \to \widetilde{K}(X \cup CY).$$

We define the map δ by the property that the following diagram commutes:

$$\begin{tikzcd}
K(X \cup CY, X) \arrow[r, "\iota^*"] & \widetilde{K}(X \cup C(Y)) \arrow[r, "\pi^*"] \arrow[d, "\beta^{-1}"] & \widetilde{K}(X) \arrow[d, "\mathsf{id}"] \\
K^{-1}(Y) \arrow[u, "\alpha"] \arrow[r, "\delta"] & K(X/Y) \arrow[r, "\pi^*"] & \widetilde{K}(X).
\end{tikzcd} \tag{3.3}$$

It remains to prove that the sequence (3.2) is exact. Exactness at the last place is the content of Corollary 3.1 while exactness at the second place follows from the isomorphism of sequences (3.3). It remains to prove that the sequence is exact at the first place. We will again apply Corollary 3.1, this time to the pair

$$(X \cup CY \cup CX, X \cup CY).$$

We obtain an exact sequence:

$$K(X \cup CY \cup CX, X \cup CY) \xrightarrow{\iota^*} \widetilde{K}(X \cup C(Y) \cup CX) \xrightarrow{\pi^*} \widetilde{K}(X \cup CY).$$

On the other hand, there is a natural homeomorphism

$$X \cup CY \cup CX/(X \cup CY) \cong CX/X \cong S(X),$$

which induces an isomorphism:

$$\gamma : K^{-1}(X) \cong K(X \cup CY \cup CX, X \cup CY).$$

Moreover, since CX is contractible then the quotient map

$$X \cup CY \cup CX \to X \cup CY \cup CX/CX \cong S(Y)$$

induces an isomorphism:

$$\theta : K^{-1}(Y) \to \widetilde{K}(X \cup CY \cup CX).$$

In order to prove the exactness of the sequence (3.2) at the first place, it suffices to show that the following diagram commutes up to a sign:

$$\begin{tikzcd}
K(X \cup CY \cup CX, X \cup CY) \arrow[r, "\pi^*"] & \widetilde{K}(X \cup C(Y) \cup CX) \arrow[r, "\iota^*"] \arrow[d, "\theta^{-1}"] & \widetilde{K}(X \cup CY) \arrow[d, "\beta^{-1}"] \\
K^{-1}(X) \arrow[u, "\gamma"] \arrow[r, "\pi^*"] & K^{-1}(Y) \arrow[r, "\delta"] & K(X, Y).
\end{tikzcd} \tag{3.4}$$

First let us consider the square on the right. In view of the definition of δ, it suffices to prove that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{K}(X \cup C(Y) \cup CX) \arrow[r, "\iota^*"] \arrow[d, "\alpha\circ\theta^{-1}"'] & \widetilde{K}(X \cup CY) \arrow[d, "\mathrm{id}"] \\
K(X \cup CY/X) \arrow[r, "\pi^*"] & \widetilde{K}(X \cup CY).
\end{tikzcd}$$

This is true because it comes from a commutative diagram at the level of spaces. Finally, we need to prove that the square on the left commutes up to a sign. We will denote by C_1 and C_2 the two types of cones that appear in the construction and note that there is a commutative diagram of spaces:

$$\begin{tikzcd}
X \cup C_1Y \cup CX \arrow[rr, "\tau"] \arrow[dd] & & C_1(Y)/Y \arrow[r, "\tau"] & SY \\
& C_1Y \cup C_2Y \arrow[ul, "\tau"'] \arrow[ur, "\tau"] \arrow[dl] \arrow[dr, "\tau"] & & \\
C_2(X)/X \arrow[rr] & & C_2Y/Y \arrow[r, "\tau"] & SY.
\end{tikzcd}$$

In the above diagram, the maps denoted by τ induce isomorphisms in the K-groups. In view of Corollary 3.2, we know that the following diagram commutes up to a sign:

$$\begin{tikzcd}
& K(C_1Y/Y) \arrow[dl] & \widetilde{K}(SY) \arrow[l] \arrow[dd] \\
K(C_1Y \cup C_2Y) & & \\
& K(C_2Y/Y) \arrow[ul] & \widetilde{K}(SY). \arrow[l]
\end{tikzcd}$$

This implies what we had to prove. $\square$

We will now see how the periodicity theorem allows to extend the short exact sequence to the right.

Lemma 3.8. Suppose that $(X, Y) \in \mathcal{C}^2$. A retraction $p : X \to Y$ induces an isomorphism:

$$K^{-n}(X) \cong K^{-n}(X, Y) \oplus K^{-n}(Y)$$

Proof. Since the inclusion is a split map we know that ι^* is surjective and therefore the connecting maps are zero. The map p^* gives a splitting for the resulting short exact sequences. $\square$

Lemma 3.9. Suppose that X, Y are pointed spaces. Then, for all $n \geq 0$, the projections induce isomorphisms:

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)$$

Proof. Since X has a base point then Y is a subspace of $X \times Y$ with a retraction given by the second projection, thus by Lemma 3.8 there is an isomorphism:

$$K^{-n}(X \times Y) \cong K^{-n}(X \times Y, Y) \oplus K^{-n}(Y)$$

Also, since Y is pointed, X is a retract of $(X \times Y/Y)$ and therefore:

$$K^{-n}(X \times Y/Y) \cong K^{-n}(X \wedge Y) \oplus K^{-n}(X).$$

$\square$

By the previous Lemma we see that for X, Y pointed spaces, there is a short exact sequence:

$$0 \to \widetilde{K}^0(X \wedge Y) \to \widetilde{K}^0(X \times Y) \to \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \to 0,$$

where the map on the right is the sum of the maps induced by the inclusions.

By definition, if $a \in \widetilde{K}^0(X)$ and $b \in \widetilde{K}^0(Y)$ then the external product is $a * b = \pi_1^*(a)\pi_2^*(b) \in K^0(X \times Y)$ and has $\pi_1^*(a)$ restricting to zero in $K^0(Y)$ and $p_2^*(b)$ restricting to zero in $K^0(X)$, thus $\pi_1^*(a)\pi_2^*(b)$ restricts to zero in both $K^0(X)$ and $K^0(Y)$. In particular $a*b$ lies in $\widetilde{K}^0(X \times Y)$, and by before short exact sequence $a * b$ pulls back to a unique element of $\widetilde{K}^0(X \wedge Y)$, hence we have define the **reduced external product** $\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \wedge Y)$. From Lemma 3.9 we get the following commutative diagram

$$\begin{tikzcd}
K^0(X) \otimes K^0(Y) \arrow[r, "\cong"] \arrow[d] & (\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y)) \oplus \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \oplus \mathbb{Z} \arrow[d] \\
K^0(X \times Y) \arrow[r, "\cong"] & \widetilde{K}^0(X \wedge Y) \oplus \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \oplus \mathbb{Z}.
\end{tikzcd}$$

Therefore the reduced external product is a ring homomorphism that is a restriction of the unreduced external product. Thus, we obtain a pairing:

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \wedge Y).$$

Replacing X by X/Y and Y by X'/Y' we get to

$$K^0(X, Y) \otimes K^0(X', Y') \to K^0(X \times X', Y \times X' \cup X \times Y') \tag{3.5}$$

For $(X, Y), (X', Y') \in \mathcal{C}^2$.

Since

$$S^n X \wedge S^n Y = \mathbb{S}^n \wedge \mathbb{S}^m \wedge X \wedge Y = S^{m+n}(X \wedge Y),$$

we obtain also:

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}^{-m}(Y) \to \widetilde{K}^{-m-n}(X \wedge Y),$$

which when applied to X^+ and Y^+ becomes the pairing:

$$K^{-n}(X) \otimes K^{-m}(Y) \to K^{-m-n}(X \wedge Y),$$

for any two spaces X, Y.

By the periodicity Theorem 3.1 $K^0(\mathbb{S}^2) \otimes K^0(X) \to K^0(\mathbb{S}^2 \times X)$ is an isomorphism, so by the previous commutative diagram the reduced external product $\widetilde{K}^0(\mathbb{S}^2) \otimes \widetilde{K}^0(X) \to \widetilde{K}^0(\mathbb{S}^2 \wedge X)$ is an isomorphism, and by the Lemma 3.4 the quotient map $S^n X \to \mathbb{S}^n \wedge X$ induces an isomorphism on $\widetilde{K}^0$. Hence, $\widetilde{K}^0(\mathbb{S}^2 \wedge X) \cong \widetilde{K}^0(S^2 X)$, therefore we obtain the isomorphism $\widetilde{K}^0(\mathbb{S}^2) \otimes \widetilde{K}^0(X) \to \widetilde{K}^0(S^2 X)$. So we define the homomorphism $\beta : \widetilde{K}^0(X) \to \widetilde{K}^0(S^2 X)$ as the composition

$$\widetilde{K}^0(X) \to \widetilde{K}^0(\mathbb{S}^2) \otimes \widetilde{K}^0(X) \to \widetilde{K}^0(S^2 X)$$

where the first map define as $a \mapsto (H - 1) * a$ is an isomorphism sice $\widetilde{K}^0(\mathbb{S}^2)$ is infinite cyclic generated by $H - 1$, and the second map is the previous isomorphism. Therefore we have proved that:

Theorem 3.3. The homomorphism $\beta : \widetilde{K}^0(X) \to \widetilde{K}^0(S^2 X)$ given by $a \mapsto (H - 1) * a$ is an isomorphism for all compact Hausdorff space X.

Corollary 3.3. We have the following consequences of Bott periodicity theorem:

- $\widetilde{K}(\mathbb{S}^{2n+1}) = 0$ and $\widetilde{K}(\mathbb{S}^{2n}) \cong \mathbb{Z}$, generated by the n-fold reduced external product $(H-1) * \cdots * (H-1)$.
- The external product $\widetilde{K}(\mathbb{S}^{2k}) \otimes \widetilde{K}(X) \to \widetilde{K}(\mathbb{S}^{2k} \wedge X)$ is an isomorphism since it is an iterate of the periodicity isomorphism.
- The external product $K(\mathbb{S}^{2k}) \otimes K(X) \to K(\mathbb{S}^{2k} \times X)$ is an isomorphism.

4 Atiyah-Bott-Shapiro construction

To adapt K-theory to the theory of elliptic operators, we will give an alternative definition of the group $K(X,Y)$. We will closely follow [Ati63]. We shall work in the category of finite CW-complexes and pairs of complexes.

Definition 3.12. For $Y \subset X$ a closed subspace and for each $n \geq 0$ we define the set $\mathcal{L}_n(X,Y)$ of sequences

$$E = (0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} E_{n-2} \to \cdots \xrightarrow{\sigma_1} E_0 \to 0)$$

where the E_i are vector bundles on X, the σ_i are bundle morphisms defined on Y such that

$$0 \to E_n|_Y \xrightarrow{\sigma_n} E_{n-1}|_Y \xrightarrow{\sigma_{n-1}} \cdots \to E_1|_Y \xrightarrow{\sigma_1} E_0|_Y \to 0$$

is an exact sequence for the restriction to Y. These sequences are also denoted by

$$E = (E_n, ..., E_0; \sigma_n, ..., \sigma_1)$$

Two such elements $E = (E_n, ..., E_0; \sigma_n, ..., \sigma_1)$ and $E' = (E'_n, ..., E'_0; \sigma'_n, ..., \sigma'_1)$ are said **isomorphic** if there are bundle isomorphisms $h_i : E_i \to E'_i$ on X so that the diagram

$$\begin{tikzcd} E_i \arrow[r, "\sigma_i"] \arrow[d, "h_i"'] & E_{i-1} \arrow[d, "h_{i-1}"] \\ E'_i \arrow[r, "\sigma'_i"'] & E'_{i-1} \end{tikzcd}$$

commutes on Y for each i.

An **elementary** sequence in $\mathcal{L}_n(X,Y)$ is one in which

- $E_i = E_{i-1}$, $\sigma_i = \mathrm{id}$, for some i.
- $E_j = \{0\}$,for $j \neq i$ or $i-1$.

The direct sum $E \oplus F$ of two sequences in $\mathcal{L}_n(X,Y)$ is defined in the obvious way. We consider now the following equivalence relation:

Definition 3.13. Two elements $E, E' \in \mathcal{L}_n(X,Y)$ are defined to be **equivalent**, denoted by $E \sim E'$, if there exist elementary elements $E_1, ..., E_i, F_1, ..., F_j \in \mathcal{L}_n(X,Y)$ and an isomorphism

$$E \oplus E_1 \oplus \cdots \oplus E_i \cong E' \oplus F_1 \oplus \cdots \oplus F_j.$$

The equivalence class of an element $E = (E_n, ..., E_0; \sigma_n, ..., \sigma_1)$ will be denoted by

$$[E_n, ..., E_0; \sigma_n, ..., \sigma_1].$$

The set of equivalence classes will be denoted by $L_n(X,Y)$. The operation $\oplus$ induces on $L_n(X,Y)$ an abelian semigroup structure. If $Y=\emptyset$ we write $L_n(X)=L_n(X,\emptyset)$.

Consider the natural map $\mathcal{L}_n(X,Y)\to\mathcal{L}_{n+1}(X,Y)$ given by

$$(E_n,...,E_0;\sigma_n,...,\sigma_1)\mapsto(E_n,...,E_0,0;\sigma_n,...,\sigma_1,0).$$

In this way we get inclusions

$$\mathcal{L}_1(X,Y)\to\mathcal{L}_2(X,Y)\to\cdots\mathcal{L}_n(X,Y)\to\cdots$$

and we put $\mathcal{L}(X,Y)=\mathcal{L}_\infty(X,Y)=\lim_{n\to\infty}\mathcal{L}_n(X,Y)$. These induce homomorphisms

$$L_1(X,Y)\to L_2(X,Y)\to\cdots L_n(X,Y)\to\cdots.$$

By an equivalence relation as above applied now to sequences of finite but unbounded length, we obtain from $\mathcal{L}(X,Y)$

$$L(X,Y)=L_\infty(X,Y)=\lim_{n\to\infty}L_n(X,Y).$$

Definition 3.14. A space X with basepoint x_0 is said to be n-connected if $\pi_i(X,x_0)=0$ for $i\leq n$.

Lemma 3.10. Let $p:E\to X$ and $q:F\to X$ be vector bundles on a finite CW-complex X and $f:E\to F$ a monomorphism on Y. Then if $\dim F>\dim E+\dim X$, f can be extended to a monomorphism on X and any two such extensions are homotopic.

Proof. Consider the sub-bundle $\mathrm{Mon}(E,F)$ of $\mathrm{Hom}(E,F)$ on X whose fibre at $x\in X$ is the space of all monomorphisms $E_x\to F_x$. This fiber is homoeomorphic to $\mathrm{GL}_n/\mathrm{GL}_{n-m}$ where $n=\dim F$, $m=\dim E$, and so it is $(n-m-1)$-connected. Hence cross-sections can be extended and are all homotopic if

$$\dim X\leq n-m-1=\dim F-\dim E-1.$$

But a cross-section of $\mathrm{Mon}(E,F)$ is just a global monomorphism $E\to F$ by the Remark 2.7. $\square$

Lemma 3.11. The induced map $L_n(X,Y)\to L_{n+1}(X,Y)$ is an isomorphism for $n\geq 1$.

Proof. Let $\overline{\mathcal{L}}_{n+1}(X,Y)$ denote the subset of $\mathcal{L}_{n+1}(X,Y)$ consisting of sequences E such that

$$\dim E_n>\dim E_{n+1}+\dim X \ (1)$$

If $n\geq 1$ then given any $E\in\mathcal{L}_{n+1}(X,Y)$ we can add an elementary sequence to it so that it will satisfy (1), hence $\overline{\mathcal{L}}_{n+1}(X,Y)\to L_{n+1}(X,Y)$ is surjective. Let

$$E=(0\to E_{n+1}\xrightarrow{\sigma_{n+1}}E_n\xrightarrow{\sigma_n}E_{n-1}\to\cdots\to E_0\to 0)\in\overline{\mathcal{L}}_{n+1}$$

then by the previous lemma σ_{n+1} can be extended to a monomorphism $\sigma'_{n+1}:E_{n+1}\to E_n$ on the whole of X. Set

$$E'_n:=\mathrm{coker}\ \sigma'_{n+1}=\frac{E_n}{\mathrm{im}\ \sigma'_{n+1}},$$

let P be the elementary sequence $0 \to E_{n+1} \xrightarrow{\text{id}} E_{n+1} \to 0$ and let

$$E' = (0 \to E'_n \xrightarrow{\rho'_n} E_{n-1} \xrightarrow{\sigma_{n-1}} E_{n-2} \to \cdots \to E_0 \to 0) \in \overline{\mathcal{L}}_{n+1}$$

where ρ'_n is defined by the commutative diagram on Y

$$\begin{tikzcd} E_n \arrow[r, "\pi"] \arrow[dr, "\sigma_n"'] & E'_n \arrow[d, "\rho'_n"] \\ & E_{n-1} \end{tikzcd}$$

Consider the diagram

$$P \longrightarrow E \longrightarrow E'$$

$$\begin{tikzcd} E_{n+1} \arrow[r, "\text{id}"] \arrow[d, "\text{id}"'] & E_{n+1} \arrow[r, "0"] \arrow[d, "\sigma_{n+1}"] & 0 \arrow[d, "0"] \\ E_{n+1} \arrow[r, "\sigma'_{n+1}"] \arrow[d, "0"'] & E_n \arrow[r, "\pi"] \arrow[d, "\sigma_n"] & E'_n \arrow[d, "\rho'_n"] \\ 0 \arrow[r, "0"] \arrow[d, "0"'] & E_{n-1} \arrow[r, "\text{id}"] \arrow[d, "\sigma_{n-1}"] & E_{n-1} \arrow[d, "\sigma_{n-1}"] \\ 0 \arrow[r, "0"] \arrow[d] & E_{n-2} \arrow[r, "\text{id}"] \arrow[d] & E_{n-2} \arrow[d] \\ \vdots & \vdots & \vdots \end{tikzcd}$$

Is clear that each of the squares in this diagram commutes. We claim that $E' \in \mathcal{L}_n$. For the injectivity of ρ'_n, let $x \in \ker\, \rho'_n \subset E'_n$, then there is $x' \in E_n$ such that $\pi(x') = x$, so $\sigma_n(x') = \rho'_n(\pi(x')) = 0$, i.e., $x' \in \ker\, \sigma_n = \text{im}\, \sigma_{n+1}$, thus $x' \in \text{im}\, \sigma'_{n+1}$ on Y, therefore $x = \pi(x') = 0$. For the exactness in E_{n-1}, let $x \in \text{im}\, \rho'_n \subset E_{n-1}$, then there is $y \in E'_n$ and $y' \in E_n$ such that $\rho'_n(y) = x$ and $\pi(y') = y$, thus $\sigma_{n-1}(x) = \sigma_{n-1}(\rho'_n(\pi(y'))) = \sigma_{n-1}(\sigma_n(y')) = 0$, i.e., $x \in \ker\, \sigma_{n-1}$. Finally, if $x \in \ker\, \sigma_{n-1}$, then $x \in \text{im}\, \sigma_n$, so there is $y \in E_n$ such that $x = \sigma_n(y) = \rho'_n(\pi(y))$, thus $x \in \text{im}\, \rho'_n$.

Therefore the splitting of the exact sequence on X

$$0 \to E_{n+1} \xrightarrow{\sigma'_{n+1}} E_n \xrightarrow{\pi} E'_n \to 0$$

define an isomorphism $P \oplus E' \cong E$ in $\mathcal{L}_{n+1}(X, Y)$.

If σ''_{n+1} is another extension of σ_{n+1} leading to a sequence E'', then by previous lemma $E' \cong E''$

and this isomorphism can be taken to extend the given one on Y, i.e., the diagram

$$\begin{tikzcd} E'_n \arrow[r, "\rho'_n"] \arrow[d] & E_{n-1} \arrow[d, "\mathrm{id}"] \\ E''_n \arrow[r, "\rho''_n"'] & E_{n-1} \end{tikzcd}$$

commutes on Y. Hence $E' \cong E''$ in $\mathcal{L}_n(X,Y)$ and so we have a well-defined map $E \mapsto E'$ from the isomorphism classes in $\overline{\mathcal{L}}_{n+1}(X,Y)$ to the isomorphism classes in $\mathcal{L}_n(X,Y)$. Moreover, if

$$Q = (0 \to Q_{n+1} \to Q_n \to 0),\ R = (0 \to R_i \to R_{i-1} \to 0),\quad i \le n$$

are elementary sequences, then

$$(E \oplus Q)' \cong E',\quad (E \oplus R)' \cong E' \oplus R.$$

Hence the class of E' in $L_n(X,Y)$ depends only on the class of E in $L_{n+1}(X,Y)$. Since $\overline{\mathcal{L}}_{n+1}(X,Y) \to L_{n+1}(X,Y)$ is surjective it follows that $E \mapsto E'$ induces a map $L_{n+1}(X,Y) \to L_n(X,Y)$. From this construction it is immediate that its composition in either direction with $L_n(X,Y) \to L_{n+1}(X,Y)$ is the identity. $\square$

By induction on n and then passing to the limit we have:

Corollary 3.4. The homomorphisms $L_1(X,Y) \to L_n(X,Y)$ are isomorphisms for $1 \le n \le \infty$.

Definition 3.15. An **Euler characteristic** for $\mathcal{L}_n(X,Y)$ is a natural transformation of functors

$$\chi : L_n(X,Y) \to K(X,Y)$$

which for $Y = \emptyset$ is given by

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i.$$

Notice that if $Y = \emptyset$, $E \mapsto \sum (-1)^i E_i$ gives a well-defined map $L_n(X) \to K(X)$. Our purpose is to show that there exists a unique Euler characteristic.

Lemma 3.12. Let A be a semigroup with an identity element 1, B a group with identity element e, $\phi : A \to B$ an epimorphism with $\phi^{-1}(e) = 1$. Then ϕ is an isomorphism.

Proof. If we prove that A is a group, then $\phi(a) = \phi(b)$ implies that $\phi(ab^{-1}) = e$, $a = b$, so we have the injectivity of ϕ. For any $a \in A$, by surjectivity of ϕ there exist $a' \in A$ such that $\phi(a') = \phi(a)^{-1}$, hence $e = \phi(a)\phi(a)^{-1} = \phi(a)\phi(a') = \phi(aa')$, so $aa' = 1$. $\square$

Lemma 3.13. Let χ be an Euler characteristic for $\mathcal{L}_1(X)$, then $\chi : L_1(X) \to K(X)$ is an isomorphism.

Proof. By the Lemma 3.1 χ is an epimorphism. Suppose that for $E = (E_1, E_0; \sigma_1) \in L_1(X)$, $\chi(E) = 0$.Then $[E_0] - [E_1] = 0$ in K-theory, thus by definition there exists F in $\mathrm{Vect}(X)$ such that $E_1 \oplus F \cong E_0 \oplus F$. Hence if $P : 0 \to F \to F \to 0$ is elementary sequence, then $E \oplus P = (0 \to E_1 \oplus F \to E_0 \oplus F)$ is isomorphic to the elementary sequence defined by $E_1 \oplus F = (0 \to E_1 \oplus F \to E_1 \oplus F \to 0)$. Therefore $E \sim 0$ in $\mathcal{L}_1(X,Y)$ and so $E = 0$ in $L_1(X)$. So injectivity is a consequence of the previous lemma. $\square$

Lemma 3.14. Let χ be an Euler characteristic for $\mathcal{L}_1(X,Y)$, and let Y be the set consisting of a point x_0. Then $\chi : L_1(X,Y) \to K(X,Y)$ is an isomorphism.

Proof. By the Lemma 3.5 and the two previous lemmas it will be sufficient to show the exactness of the top line in the diagram

$$\begin{tikzcd} 0 \to L_1(X,Y) \arrow[r, "\alpha"] \arrow[d, "\chi"'] & L_1(X) \arrow[r, "\beta"] \arrow[d, "\chi"] & L_1(Y) \arrow[d, "\chi"] \\ 0 \to K(X,Y) \arrow[r, "\pi*"] & K(X) \arrow[r, "i^*"] & K(Y) \cong \mathbb{Z}. \end{tikzcd}$$

where α and β are define such that the diagram commutes. Clearly $\beta\alpha = 0$, thus im $\alpha \subset$ ker β. To show that ker $\beta \subset$ im α, we note that for any vector bundle $F \in K(X)$ we have that $i^*(F) = F|_{x_0} = n \in \mathbb{Z}$, where n determine the dimension of the bundle $F|_{x_0}$. So for $E = (E_1, E_1; \sigma_1) \in L_1(X)$, $\beta(E) = 0$ implies that $0 = i^*\chi(E) = [E_0|_{x_0}] - [E_1|_{x_0}]$, therefore dim $E_1|_{x_0} =$ dim $E_0|_{x_0}$. Hence we can find an isomorphism $\sigma : E_1|_{x_0} \to E_0|_{x_0}$, showing that $E \in$ im α. Finally for show the injectivity of α, we show that $\alpha^{-1}(0) = 0$. Let $E = (0 \to E_1 \xrightarrow{\sigma_1} E_0 \to 0)$ be an element of $\mathcal{L}_1(X,Y)$ and suppose that $\alpha(E) = 0$ in $L_1(X)$. Then $\chi\alpha(E) = 0$ in $K(X)$, and if we suppose dim $E_1 >$ dim X as before for $i = 0, 1$, there is an isomorphism $\tau : E_1 \to E_0$ on the whole of X. Then $\sigma_1\tau^{-1} \in \mathrm{Aut}(E_0|Y)$. Since Y is a point, this automorphism is homotopic to the identity and thus can be extended to an element $\rho \in \mathrm{Aut}(E_0)$. Then $\rho\tau : E_1 \to E_0$ is an isomorphism extending σ_1. Therefore E represents 0 in $L_1(X,Y)$. $\square$

Lemma 3.15. Let χ be an Euler characteristic for $\mathcal{L}_1(X,Y)$, then χ is an equivalence of functors $L_1 \to K$.

Proof. For any pair (X,Y), we consider the following commutative diagram

$$\begin{tikzcd} L_1(X/Y, Y/Y) \arrow[r, "\chi"] \arrow[d, "\phi"'] & K(X/Y, Y/Y) \arrow[d, "\psi"] \\ L_1(X,Y) \arrow[r, "\chi"'] & K(X,Y). \end{tikzcd}$$

By definition ψ is an isomorphism and χ on the top line is an isomorphism by before lemma, it will be sufficient by the Lemma 3.12 to prove that ϕ is an epimorphism. Now any element $\zeta \in L_1(X,Y)$ can be represented by a sequence $E = (0 \to E_1 \xrightarrow{\sigma_1} E_0 \to 0)$ where E_0 is a trivial product bundle. But we can define a bundle $E_1' = E_1|_{\sigma_1}$ on X/Y and a collapsed sequence $E' \in \mathcal{L}_1(X/Y, Y/Y)$ defining an element $\zeta' \in L_1(X/Y, Y/Y)$. Then $\phi(\zeta') = \zeta$ and so ϕ is an epimorphism. $\square$

Lemma 3.16. Let χ, χ' be two Euler characteristics for $\mathcal{L}_1(X,Y)$, then $\chi = \chi'$.

Proof. By the previous lemma we can define the natural automorphism $T = \chi'\chi^{-1}$ of $K(X,Y)$ which is the identity when $Y = \emptyset$. Replacing X by X/Y and considering the exact sequence for $(X/Y, Y/Y)$ we deduce that $T = \mathrm{id}$, thus $\chi' = \chi$. $\square$

Lemma 3.17. There is a bijective correspondence between Euler characteristics χ_1 and χ_n,

$1 \leq n \leq \infty$, for $\mathcal{L}_1(X,Y)$ and $\mathcal{L}_n(X,Y)$ such that the diagram

$$\begin{tikzcd} L_1(X,Y) \arrow[r] \arrow[d, "\chi_1"'] & L_n(X,Y) \arrow[dl, "\chi_n"] \\ K(X,Y) & \end{tikzcd}$$

commutes.

Proof. Follows directly from Corollary 3.4 and previous lemma. □

So, we have shown that there is at most one Euler characteristic. Now we show that it exist by giving a direct construction.

Given a compact pair (X,Y), we define for $j = 0,1$ the sets $X_j = X \times \{i\}$. Let $A = X_0 \cup_Y X_1$ be the set obtained by identifying $y \times \{0\}$ and $y \times \{1\}$ for all $y \in Y$. Then by the Lemma 3.5 we get split exact sequence

$$0 \to K(A, X_j) \xrightarrow{\rho_j^*} K(A) \underset{\pi_j^*}{\overset{\iota_j^*}{\rightleftarrows}} K(X_j) \to 0,$$

where $\pi_j^* : K(X_j) \to K(A)$ is given by the retraction $\pi_j : A \to X_j$. Taking the index $j \in \mathbb{Z}_2$, the natural map $X \to X_j$ gives an inclusion $\phi_j : (X,Y) \to (A, X_{j+1})$, which induces an isomorphism

$$\phi_j^* : K(A, X_{j+1}) \to K(X,Y).$$

Now let $E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0) \in \mathcal{L}_1(X,Y)$, then by the Remark 2.6 we construct a vector bundle F on A by putting E_j on X_j and identifying on Y by σ. Notice that the isomorphism class of F depends only on the isomorphism class of E in $\mathcal{L}_1(X,Y)$. Set $F_j = \pi_j^*(E_j)$, then $F|_{X_j} \cong F_j$ and so $F - F_j \in \ker \iota_j^*$. Therefore we define the **difference element** $d(E) \in K(X,Y)$ by

$$\rho_1^*(\phi_0^*)^{-1}\ d(E) = F - F_1.$$

Notice that d is additive, i.e., $d(E \oplus E') = d(E) + d(E')$. Also if E is elementary, i.e., $F \cong F_1$, then $d(E) = 0$. Hence d induces a natural homomorphism

$$d : L_1(X,Y) \to K(X,Y).$$

Moreover if $Y = \emptyset$, $A = X_0 \sqcup X_1$, $F = E_0 \times \{0\} \sqcup E_1 \times \{1\}$, $F_i = E_i \times \{0\} \sqcup E_i \times \{1\}$, then

$$d(E) = E_0 - E_1.$$

Thus by previous discussion d is an Euler characteristic and also:

Theorem 3.4. For any integer n with $1 \leq n \leq \infty$, there exist a unique natural homomorphism

$$\chi : L_n(X,Y) \to K(X,Y)$$

which for $Y = \emptyset$ is given by

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i.$$

Moreover χ is a isomorphism. This unique χ will be referred to as the **Euler characteristic**.

By the Lemma 3.16 we may effectively identify the χ for different n.

We say that two elements $E, F \in \mathcal{L}_n(X, Y)$ are **homotopic** if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathcal{L}_n(X \times I, Y \times I)$. So by previous theorem and homotopy invariance of $K(X, Y)$ we have that:

Lemma 3.18. Homotopic elements in $\mathcal{L}_n(X, Y)$ define the same elements in $L_n(X, Y)$. Therefore we could take $L_n(X, Y)$ for $n \geq 1$ as a definition of $K(X, Y)$.

We shall now construct the inverse of $j : L_1(X, Y) \to L_n(X, Y)$. If $E \in \mathcal{L}_n(X, Y)$, then by introducing metrics we can define the adjoint sequence E^* with maps $\sigma_i^* : E_{i-1} \to E_i$. Consider the sequence

$$F = (0 \to F_1 \xrightarrow{\tau} F_0 \to 0)$$

where $F_0 = \bigoplus_i E_{2i}$, $F_1 = \bigoplus_i E_{2i+1}$ and

$$\tau(e_1, e_3, e_5, ...) = (\sigma_1 e_1, \sigma_2^* e_2 + \sigma_3 e_3, \sigma_4^* e_3 + \sigma_5 e_5, ...).$$

Since on Y we have the descomposition

$$E_{2i} = \sigma_{2i+1}(E_{2i+1}) \oplus \sigma_{2i}^*(E_{2i-1})$$

it follows that $F \in \mathcal{L}_1(X, Y)$. If $E \in \mathcal{L}_1(X, Y)$ then $E = F$. Since two choices of metric in E are homotopic it follows by the Lemma 3.18 that F will be a representative for $j^{-1}(E)$.

Hereinafter we consider complex of vector bundles, i.e., sequences

$$0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \to E_0 \to 0$$

in which $\sigma_{i-1}\sigma_i 0$ for all i.

Lemma 3.19. Let $E_0, ..., E_n$ be vector bundles on X, with X a finite CW-complex, and

$$0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \to \cdots \to E_0 \to 0$$

a complex on Y. Then the σ_i's can be extended so that this becomes a complex on X.

Proof. By induction on the cells of $X \setminus Y$ we consider when X is obtained from Y by attaching one cell, say $X = Y \cup_f e^k$, where $f : \mathbb{S}^{k-1} \to Y$ is the attaching map. Let D^k be the unit ball in $\mathbb{R}^k$, with boundary $\mathbb{S}^{k-1}$, then X is the quotient of $Y \sqcup D^k$ by an identification map π induced by f. Therefore the bundle $\pi^* E_i$ is then the disjoint sum of $E_i|_Y$ and a trivial bundle $D^k \times \mathbb{K}_i^{m_i}$. The homomorphism $\sigma_i : E_i \to E_{i-1}$ on Y lifts to give a homomorphism $\tau_i : \mathbb{S}^{k-1} \times \mathbb{K}_i^{m_i} \to \mathbb{S}^{k-1} \times \mathbb{K}_{i-1}^{m_{i-1}}$, i.e., a map $\mathbb{S}^{k-1} \to \mathrm{Hom}(V_i, V_{i-1})$. Extend each τ_i to D^k by defining

$$\tau_i(u) = \|u\| \sigma_i(u) \qquad u \in D^k.$$

This induces an extension of the σ_i to X preserving the relations $\sigma_{i-1}\sigma_i = 0$. $\square$

Definition 3.16. We denote by $\mathcal{D}_n(X, Y)$ the set of complex of length n on X exact on Y. Two such complexes are **homotopic** if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of a element in $\mathcal{D}_n(X \times I, Y \times I)$. By restricting the homomorphisms to Y on a element in $\mathcal{D}_n(X, Y)$ we get a natural map

$$\Phi : \mathcal{D}_n(X, Y) \to \mathcal{L}_n(X, Y).$$

Lemma 3.20. $\Phi : \mathcal{D}_n(X,Y) \to \mathcal{L}_n(X,Y)$ induced a bijective map of homotopy classes.

Proof. By the previous lemma Φ is surjective. Again by previous lemma applied to the pair

$$(X \times I, X \times \{0\} \cup X \times \{1\} \cup Y \times I)$$

we see that if $\Phi(E)$ is homotopic to $\Phi(F)$, then E is homotopic to F. □

If $E \in \mathcal{D}_n(X,Y)$ and $F \in \mathcal{D}_m(X',Y')$ then $E \otimes F$ is a complex on $X \times X'$ exact on $X \times Y \cup Y \times X'$ so that

$$E \otimes F \in \mathcal{D}_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

This product is additive and compatible with homotopies. Hence it induces a bilinear product on the homotopy classes. Thus by the previous Lemma 3.20 and Lemma 3.18 it follows that it induces a natural product

$$L_n(X,Y) \otimes L_m(X',Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

Proposition 3.5. The tensor product of complexes induces a natural product

$$L_n(X,Y) \otimes L_m(X',Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X')$$

and $\chi(ab) = \chi(a)\chi(b)$.

Proof. The formula is true if $Y = Y' = \emptyset$. By the Equation 3.5 and the Theorem 3.4 the formula $\chi(ab) = \chi(a)\chi(b)$ is also true in the general case. □

Theorem 3.5. Let

$$E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0) \in \mathcal{D}_1(X,Y),$$

$$E' = (0 \to E'_1 \xrightarrow{\sigma'} E_0 \to 0) \in \mathcal{D}_1(X',Y')$$

and choose metrics in all the bundles. Let

$$F = (0 \to F_1 \xrightarrow{\tau} F_0 \to 0) \in \mathcal{D}_1(X \times X', X \times Y' \cup Y \times X')$$

be defined by

$$F_1 = E_0 \otimes E'_1 \oplus E_1 \otimes E'_0,$$

$$F_0 = E_0 \otimes E'_0 \oplus E_1 \otimes E'_1,$$

$$\tau = \begin{pmatrix} \mathrm{id} \otimes \sigma' & \sigma \otimes \mathrm{id} \\ \sigma^* \otimes \mathrm{id} & -\mathrm{id} \otimes \sigma'^* \end{pmatrix},$$

where σ^*, σ'^* denote the adjoints of σ, σ'. Then

$$\chi(F) = \chi(E)\chi(E').$$

Proof. By previous proposition $\chi(E)\chi(E') = \chi(E \otimes E')$. By construction of the inverse of $j_2 : L_1(X,Y) \to L_2(X,Y)$ turns $E \otimes E'$ into F and hence $\chi(E \otimes E') = \chi(F)$. □

5 K- theory with compact support and Bott generator

We will discuss a modification of K-theory called the K-theory with compact support that is of great importance.

Definition 3.17. For a locally compact space X we define the **K-theory with compact support** as

$$K_{cpt}(X) := \widetilde{K}(X^+),$$

where $X^+ = X \cup \{*\}$, denotes the one point compactification of X in the point $*$.

Definition 3.18. Let $p : E \to X$ a vector bundle. We define the **unit disk bundle** and the **unit sphere bundle** of E respect to some metric on E respectively by

$$\mathbb{D}(E) := \{v \in E \mid ||v|| \leq 1\},$$

$$\mathbb{S}(E) := \dot{\mathbb{D}}(E) = \{v \in E \mid ||v|| = 1\}.$$

The **Thom space** $\mathrm{Th}(E)$ of the vector bundle $p : E \to X$ is the topological space defined by

$$\mathrm{Th}(E) := \mathbb{D}(E)/\mathbb{S}(E).$$

Notice that for X compact, the Thom space of E is the one-point compactification of the total space E, i.e., $\mathrm{Th}(E) = E^+$.

Now we show briefly that the Atiyah-Bott-Shapiro construction can be done in the context of K-theory with compact support. A **virtual bundle with compact support** is a tuple $E = (E_0, E_1, ..., E_n; \sigma_1, ..., \sigma_n)$ where $E_0, E_1, ..., E_n$ are vector bundles over a space X and $\sigma_1, ..., \sigma_n$ are bundle isomorphisms $\sigma_i : E_{i-1} \to E_i$ outside a compact set $Y \subset X$ which is called the **support** of E. Such a triple is called **trivial** if the isomorphisms σ_i's are defined everywhere on X. Two triples $E = (E_0, ..., E_n; \sigma_1, ..., \sigma_n)$ and $F = (F_0, ..., F_n; \tau_1, ..., \tau_n)$ are called **isomorphic** if there exist bundle isomorphisms $\varphi_k : E_k \to F_k$ with $k = 0, 1, ..., n$ defined everywhere on X such that $\tau_k \varphi_{k-1} = \varphi_k \sigma_k$ outside a compact. Two triples E and F are called **stable isomorphic**, $E \sim F$, if there are trivial triples G_1, G_2 such that $E \oplus G_1$ and $F \oplus G_2$ are isomorphic. The equivalence classes with respect to $\sim$ form a group $L_n(X)_{cpt}$ with respect to direct sum. The element of $L_n(X)_{cpt}$ will be denoted by $E = [E_0, ..., E_n; \sigma_1, ..., \sigma_n]$. As in the previous section there are natural isomorphisms

$$L_1(X)_{cpt} \xrightarrow{\cong} L_2(X)_{cpt} \xrightarrow{\cong} \cdots \xrightarrow{\cong} K_{cpt}(X).$$

In particular, any element of $K_{cpt}(X)$ can be represented by a triple $[E_0, E_1; \sigma]$, where $\sigma : E_0 \to E_1$ is a bundle isomorphism defined in a neighborhood of infinity.

With the Atiyah-Bott-Shapiro construction in hand, we will discuss the Bott generator. Let $p : E \to M$ be a complex n-dimensional vector bundle over a manifold M and choose any Hermitian metric $\langle \cdot, \cdot \rangle$ on E. We consider the pullback bundle p^*E of the bundle $p : E \to M$ on itself and consider the bundle $\bigwedge E := \bigwedge(p^*E)$ over the manifold E. The zero section $i : M \to E$ of the bundle $p : E \to M$ gives an embedding of M in E, and the bundle p^*E over $E \setminus M$ has a non-vanishing tautological section given by $(x, z) \mapsto z$, where (x, z) is a point of the total space E defined by a point $x \in M$ and a vector $z \in E_x$. We define a bundle homomorphism $\epsilon(z) : \bigwedge E \to \bigwedge E$ acting as the exterior multiplication $\epsilon(z)u = z \wedge u$. Let $\epsilon^*(z)$ be the adjoint homomorphism with respect to the metric. It acts on $u \in E$ as $\epsilon^*(z)u = (z, u)$ and extends to the whole $\bigwedge E$ as antiderivation

$$\epsilon^*(z)(v_p \wedge v_q) = (\epsilon^*(z)v_p) \wedge v_q + (-1)^p v_p \wedge (\epsilon^*(z)v_q),$$

where $v_p \in \bigwedge^p E$ and $v_q \in \bigwedge^q E$. Thus,

$$\epsilon^*(z)(\epsilon(z)u) = \epsilon^*(z)(z \wedge u) = (\epsilon^*(z)z \wedge u) - z \wedge (\epsilon^*(z)u) = |z|^2 u - z \wedge \epsilon^*(z)u,$$

so $\epsilon^*(z)\epsilon(z) + \epsilon(z)\epsilon^*(z) = |z|^2$. We consider the homomorphism $b : \bigwedge E \to \bigwedge E$ given by $b(z) = \epsilon(z) + \epsilon^*(z)$. Since $((\epsilon(z)))^2 = ((\epsilon^*(z)))^2 = 0$, we get $(b(z))^2 = |z|^2$, so that $b(z)$ is invertible on $E \setminus M$, where M is identified with the zero section.

Definition 3.19. We have defined an element in $K(\mathbb{D}(E), \mathbb{S}(E)) \cong \widetilde{K}(\mathrm{Th}(E))$, denoted by λ_E and called the **Bott element** or **Thom class**.

6 The Thom isomorphism and push forward in K-theory

We discuss first the Thom isomorphism in K-theory and next we define the push forward in K-theory $K_{\mathrm{cpt}}(TX) \to \mathbb{Z}$, which is of fundamental importance in defining the topological index of an elliptic differential operator.

Proposition 3.6. If X is a finite cell complex with n cells, then $K^*(X)$ is a finitely generated group with at most n generators. If all the cells of X have even dimension then $K^1(X) = 0$ and $K^0(X)$ is free abelian with one basis element for each cell.

Proof. By induction on the number of cells, if X is obtained from a subcomplex A by attaching a k-cell, for some k, then for the pair (X, A) we have and exact sequence $\tilde{K}^*(X/A) \to \tilde{K}^*(X) \to \tilde{K}^*(A)$. Since $X/A = \mathbb{S}^k$, we have $\tilde{K}^*(X/A) = \mathbb{Z}$, and exactness implies that $\tilde{K}^*(X)$ requires at most one more generator than $\tilde{K}^*(A)$.

The first term of the exact sequence $K^1(X/A) \to K^1(X) \to K^1(A)$ is zero if all cells X are of even dimension, so induction on the number of cells implies that $K^1(X) = 0$. Then there is a short exact sequence $0 \to \widetilde{K}^0(X/A) \to \widetilde{K}^0(X) \to \widetilde{K}^0(A) \to 0$ with $\widetilde{K}^0(X/A) = \mathbb{Z}$. By induction $\widetilde{K}(A)$ is free, so this sequence splits, hence $\widetilde{K}^0(X) \cong \mathbb{Z} \oplus K^0(A)$. $\square$

Example 3.8. If X is $\mathbb{CP}^n$, as the cell structure of this space is given by one cell in each dimension $0, 2, 4, ..., 2n$, then by the Proposition 3.6 we have that $K^1(\mathbb{CP}^n) = 0$ and $K^0(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1}$.

Now we are interested in studying the ring structure of $K(\mathbb{CP}^n)$. From the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ we have the short exact sequence

$$0 \to K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to K(\mathbb{CP}^n) \xrightarrow{\rho} K(\mathbb{CP}^{n-1}) \to 0.$$

Lemma 3.21. $(L_n - 1)^n$ generates the kernel of the restriction map ρ, where L_n is the canonical line bundle on $\mathbb{CP}^n$.

Proof. The space $\mathbb{CP}^n$ is the quotient of the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$ under multiplication by scalars in $\mathbb{S}^1 \subset \mathbb{C}$. Instead of $\mathbb{S}^{2n+1}$ we could equally well take the boundary of the product $\mathbb{D}_0^2 \times \cdots \times \mathbb{D}_n^2$ where $\mathbb{D}_i^2$ is the unit disk in the i^{th} coordinate of $\mathbb{C}^{n+1}$, and we start the count with $i = 0$ for convenience. Then we have

$$\partial(\mathbb{D}_0^2 \times \cdots \times \mathbb{D}_n^2) = \bigcup_i (\mathbb{D}_0^2 \times \cdots \times \partial\mathbb{D}_i^2 \times \cdots \times \mathbb{D}_n^2).$$

The action of $\mathbb{S}^1$ by scalar multiplication respects this decomposition. The orbit space of $\mathbb{D}_0^2 \times \cdots \times \partial\mathbb{D}_i^2 \times \cdots \times \mathbb{D}_n^2$ under the action is a subspace $C_i \subset \mathbb{CP}^n$ homeomorphic to the product

$\mathbb{D}_0^2 \times \cdots \times \mathbb{D}_n^2$ with the factor $\mathbb{D}_i^2$ deleted. Thus we have a decomposition $\mathbb{CP}^n = \bigcup_i C_i$, with each C_i homeomorphic to $\mathbb{D}^{2n}$ and with $C_i \cap C_j = \partial C_i \cap \partial C_j$ for $i \neq j$.

Consider now $C_0 = \mathbb{D}_0^2 \times \cdots \times \mathbb{D}_n^2$. Its boundary is decomposed into the pieces $\partial_i C_0 = \mathbb{D}_1^2 \times \cdots \times \partial\mathbb{D}_i^2 \times \cdots \times \mathbb{D}_n^2$. The inclusions $(\mathbb{D}_1^2, \partial\mathbb{D}_i^2) \subset (C_0, \partial_i C_0) \subset (\mathbb{CP}^n, C_i)$ give rise to a commutative diagram

$$\begin{tikzcd}
K(\mathbb{D}_1^2, \partial\mathbb{D}_1^2) \otimes \cdots \otimes K(\mathbb{D}_n^2, \partial\mathbb{D}_n^2) \arrow[dr, "\cong"] & & \\
K(C_0, \partial_1 C_0) \otimes \cdots \otimes K(C_0, \partial_n C_0) \arrow[u, "\cong"] \arrow[r, "\cong"] & K(C_0, \partial C_0) & \\
K(\mathbb{CP}^n, C_1) \otimes \cdots \otimes K(\mathbb{CP}^n, C_n) \arrow[u] \arrow[r] \arrow[d] & K(\mathbb{CP}^n, C_1 \cup \cdots \cup C_n) \arrow[u, "\cong"] \arrow[r, "\cong"] \arrow[d] & K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \arrow[dl] \\
K(\mathbb{CP}^n) \otimes \cdots \otimes K(\mathbb{CP}^n) \arrow[r] & K(\mathbb{CP}^n) &
\end{tikzcd}$$

Where the maps from the first column to the second are the n-fold products. The upper map in the middle column is an isomorphism because the inclusion $C_0 \hookrightarrow \mathbb{CP}^n$ induces a homeomorphism $C_0/\partial C_0 \cong \mathbb{CP}^n/(C_1 \cup \cdots \cup C_n)$. The $\mathbb{CP}^{n-1}$ at the right side of the diagram sits in $\mathbb{CP}^n$ in the last n coordinates of $\mathbb{CP}^{n+1}$, so is disjoint from C_0. hence the quotient map $\mathbb{CP}^n/\mathbb{CP}^{n-1} \to \mathbb{CP}^n/(C_1 \cup \cdots \cup C_n)$ is a homotopy equivalence.

The element $x_i \in K(\mathbb{CP}^n, C_i)$ mapping downward to $L_n - 1 \in K(\mathbb{CP}^n)$, maps upward to a generator of $K(C_0, \partial_i C_0) \cong K(\mathbb{D}_i^2, \partial\mathbb{D}_i^2)$. By commutativity of the diagram, the product $x_1 \cdots x_n$ then generates $K(\mathbb{CP}^n, C_1 \cup \cdots \cup C_n)$. This means that $(L_n - 1)^n$ generates the image of the map $K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to K(\mathbb{CP}^n)$, which equals the kernel of ρ. $\square$

Proposition 3.7. $K(\mathbb{CP}^n)$ is the quotient ring $\mathbb{Z}[L_n]/(L_n - 1)^{n+1}$.

Proof. By induction on n. For $n = 1$ by the Bott periodicity Theorem 3.1 we have that $K(\mathbb{CP}^1) \cong K(\mathbb{S}^2) \cong \mathbb{Z}[L_1]/(L_1 - 1)^2$. Suppose that $K(\mathbb{CP}^{n-1}) \cong \mathbb{Z}[L_{n-1}]/(L_{n-1} - 1)^n$ with $\{1, (L_{n-1} - 1), ..., (L_{n-1} - 1)^{n-1}\}$ as additive basis, where L_n is the canonical line bundle on $\mathbb{CP}^n$. From the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ we have the short exact sequence

$$0 \to K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \cong \mathbb{Z} \to K(\mathbb{CP}^n) \xrightarrow{\rho} K(\mathbb{CP}^{n-1}) \cong \mathbb{Z}^n \to 0$$

By surjectivity of ρ and as $\rho L_n \cong L_{n-1}$, then $\rho([L_n - 1]) = [L_{n-1} - 1]$. By the Lemma 3.21, $(L_n - 1)^n$ generates $\ker(\rho)$, thus $\{1, (L_n - 1), ..., (L_n - 1)^n\}$ is an additive basis of $K(\mathbb{CP}^n)$. So as ρ is a group homomorphism we have $0 = \rho([L_{n+1} - 1]^{n+1}) = (\rho([L_{n+1} - 1]))^{n+1} = (L_n - 1)^{n+1}$. Thus $K(\mathbb{CP}^n) \cong \mathbb{Z}[L_n]/(L_n - 1)^{n+1}$. $\square$

Proposition 3.8. The exact sequence

$$\begin{tikzcd}
\widetilde{K}^*(X, A) \arrow[r] & \widetilde{K}^*(X) \arrow[dl] \\
\widetilde{K}^*(A) \arrow[u] &
\end{tikzcd}$$

is an exact sequence of $\widetilde{K}^*(X)$-modules, with the maps homomorphisms of $\widetilde{K}^*(X)$-modules.

Proof. The $\widetilde{K}^*(X)$-module structure on $\widetilde{K}^*(A)$ is defined by $\xi \cdot \alpha = i^*(\xi)\alpha$ where $i : A \hookrightarrow X$ is the inclusion and the product on the right side of the equation is multiplication in the ring $\widetilde{K}^*(A)$. To define the module structure on $\widetilde{K}^*(X, A)$, observe that the diagonal map $X \to X \wedge X$ induces a well-defined quotient map $X/A \to X \wedge X/A$, and this leads to a product $\widetilde{K}^*(X) \otimes \widetilde{K}^*(X, A) \to \widetilde{K}^*(X, A)$.

To see that the maps in the exact sequence are module homomorphisms we look at the diagram

$$\begin{tikzcd}
\widetilde{K}(S^jSA) \arrow[r] \arrow[d] & \widetilde{K}(S^j(X/A)) \arrow[r] \arrow[d] & \widetilde{K}(S^jX) \arrow[r] \arrow[d] & \widetilde{K}(S^jA) \arrow[d] \\
\widetilde{K}(S^iX \wedge S^jSA) \arrow[r] \arrow[d] & \widetilde{K}(S^iX \wedge S^j(X/A)) \arrow[r] \arrow[d] & \widetilde{K}(S^iX \wedge S^jX) \arrow[r] \arrow[d] & \widetilde{K}(S^iX \wedge S^jA) \arrow[d] \\
\widetilde{K}(S^{i+j}SA) \arrow[r] & \widetilde{K}(S^{i+j}(X/A)) \arrow[r] & \widetilde{K}(S^{i+j}X) \arrow[r] & \widetilde{K}(S^{i+j}A).
\end{tikzcd}$$

where the vertical maps between the first two rows are external product with a fixed element of $\widetilde{K}(S^iX)$ and the vertical maps between the second and third rows are induced by diagonal maps. What we must show is that the diagram commutes. For the upper two rows this follows from naturality of external product since the horizontal maps are induced by maps between spaces. The lower two rows are induced from suspensions of maps between spaces,

$$\begin{tikzcd}
X \wedge SA & X \wedge X/A \arrow[l] & X \wedge X \arrow[l] & X \wedge A \arrow[l] \\
SA \arrow[u] & X/A \arrow[l] \arrow[u] & X \arrow[l] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

so it suffices to show this diagram commutes up to homotopy. This is clear for the middle and right squares. The left square can be rewritten

$$\begin{tikzcd}
X \wedge SA & X \wedge (X \cup CA) \arrow[l] \\
SA \arrow[u] & X \cup CA \arrow[l] \arrow[u]
\end{tikzcd}$$

where the horizontal maps collapse the copy of X in $X \cup CA$ to a point, the left vertical maps sends $(a, s) \in SA$ to $(a, a, s) \in X \wedge SA$, and the right vertical map sends $x \in X$ to $(x, x) \in X \cup CA$ and $(a, s) \in CA$ to $(a, a, s) \in X \wedge CA$. Commutativity is then obvious. $\square$

We shall now present the Leray-Hirsch theorem for K-theory, which we will use in the proof of the splitting principle.

Theorem 3.6 (Leray-Hirsch)**.** Let $p : E \to X$ be a fiber bundle with X compact Hausdorff and with fiber F such that $K^*(F)$ is a free $K^*(X)$-module. Suppose that there exist classes $c_1, ..., c_k \in K^*(E)$ that restrict to a basis for $K^*(F)$ in each fiber F. If F is a finite cell complex having all cells of even dimension, then $K^*(E)$, as a module over $K^*(X)$, is free with basis $\{c_1, ..., c_k\}$.

Proof. The $K^*(X)$-module structure on $K^*(E)$ is defined by $\beta \cdot \gamma = p^*(\beta)\gamma$ for $\beta \in K^*(X)$ and $\gamma \in K^*(E)$. For a subspace $X' \subset X$, let $E' = p^{-1}(X')$. Then we have the diagram $(*)$

$$\begin{tikzcd}
\to K^*(X,X')\otimes K^*(F) \arrow[r] \arrow[d, "\Phi"] & K^*(X)\otimes K^*(F) \arrow[r] \arrow[d, "\Phi"] & K^*(X')\otimes K^*(F)\to \arrow[d, "\Phi"] \\
\cdots\to K^*(E,E') \arrow[r] & K^*(E) \arrow[r] & K^*(E')\to
\end{tikzcd}$$

where the left-hand Φ is defined by the same formula $\Phi(\sum_i b_i\otimes i^*(c_i)) = \sum_i p^*(b_i)c_i$, but with $p^*(b_i)c_i$ referring now to the relative product $K^*(E,E')\times K^*(E)\to K^*(E,E')$. The right-hand Φ is defined using the restrictions of the c_i's to the subspace E'. To see that the diagram $(*)$ commutes, we can interpolate between its two rows the row

$$\to K^*(E,E')\otimes K^*(F)\to K^*(E)\otimes K^*(F)\to K^*(E')\otimes K^*(F)\to$$

by factoring Φ as the composition $\sum_i b_i\otimes i^*(c_i)\mapsto \sum_i p^*(b_i)\otimes i^*(c_i)\mapsto \sum_i p^*(b_i)c_i$. The upper squares of the enlarged diagram then commute trivially, and the lower squares commute by the Proposition 3.8. The lower row of the diagram is of course exact. The upper row is also exact since we assume $K^*(F)$ is free, and tensoring an exact sequence with a free abelian group preserves exactness, the result of the tensoring operation being simply to replace the given exact sequence by the direct sum of a number of copies of itself.

Let us first prove the result for a product bundle $E = F\times X$. In this case Φ is just the external product, so we are free to interchange the roles of F and X. Thus we may use the previous diagram with F an arbitrary compact Hausdorff space and X a finite cell complex having all cells of even dimension, obtained by attaching a cell e^n to a subcomplex X'. The upper row of this diagram is then an exact sequence since it is obtained from the split short exact sequence $0\to K^*(X,X')\to K^*(X)\to K^*(X')\to 0$ by tensoring with the fixed group $K^*(F)$. If we can show that the left-hand Φ in the previous diagram is an isomorphism, then by induction on the number of cells of X we may assume the right-hand Φ is an isomorphism, so the five-lemma will imply that the middle Φ is also an isomorphism.

To show the left-hand Φ is an isomorphism, note first that $X/X' = \mathbb{S}^n$ so we may as well take the pair (X,X') to be $(\mathbb{D}^n,\mathbb{S}^{n-1})$. Then the middle Φ in the diagram is obviously an isomorphism, so the left-hand Φ will be an isomorphism iff the right-hand Φ is an isomorphism. When the sphere $\mathbb{S}^{n-1}$ is even-dimensional we have already shown that Φ is an isomorphism by the Corollary 3.3, and the same argument applies also when the sphere is odd-dimensional, since K^1 of an odd-dimensional sphere is K^0 of an even-dimensional sphere.

Now we turn to case for nonproducts. The proof will once again be inductive, but this time we need a more subtle inductive statement since X is just a compact Hausdorff space, not a cell complex. Consider the following condition on a compact subspace $U\subset X$: For all compact $V\subset U$ the map $\Phi : K^*(V)\otimes K^*(F)\to K^*(p^{-1}(V))$ is an isomorphism. If this is satisfied, let us call U good. By the special case already proved, each point of X has a compact neighborhood U that is good. Since X is compact, a finite number of these neighborhoods cover X, so by induction it will be enough to show that if U_1 and U_2 are good, then so is $U_1\cup U_2$. A compact $V\subset U_1\cup U_2$ is the union of $V_1 = V\cap U_1$ and $V_2 = V\cap U_2$. Consider the diagram like before for the pair (V,V_2). Since $K^*(F)$ is free, the upper row of this diagram is exact. Assuming U_2 is good, the map Φ is an isomorphism for V_2 , so Φ will be an isomorphism for V if it is an isomorphism for (V,V_2). The quotient V/V_2 is homeomorphic to $V_1/(V_1\cap V_2)$ so Φ will be an isomorphism for (V,V_2) if it is an isomorphism for $(V_1,V_1\cap V_2)$. Now look at the diagram as before for $(V_1,V_1\cap V_2)$. Assuming U_1 is good, the maps Φ are isomorphisms for V_1 and $V_1\cap V_2$. Hence Φ is an isomorphism for $(V_1,V_1\cap V_2)$, and the induction step is finished. $\square$

Let $p : E \to X$ be a complex vector bundle with fibers $\mathbb{C}^n$. We define the **projectivisation** $P(E)$ as the projective space of all lines through the origin in all the fibers of E, thus a point of $P(E)$ is a line ℓ_x in the fiber E_x. Factoring out scalar multiplication in each fiber of E minus the zero section, we topologize $P(E)$ as the quotient under this factorization. We consider the natural projection $P(p) : P(E) \to X$ sending each line ℓ_x in E_x to $x \in X$, $P(p)(\ell_x) = x$. So if U_i is a trivialization of vector bundle $p : E \to X$, with $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^n$, the above factorization induce a quotient $U \times P\mathbb{C}^{n-1}$, thereby $P(p)$ is a fiber bundle over X with fiber $P\mathbb{C}^{n-1}$.

Example 3.9. Let $E \to X$ be a vector bundle with fiber $\mathbb{C}^n$ and compact base X, and let $p : P(E) \to X$ be the associated projective bundle with fibers $\mathbb{CP}^{n-1}$. We have the canonical line bundle over $P(E)$, $\tilde{L} \to P(E)$. We know that $K^*(\mathbb{CP}^{n-1}) = \mathbb{Z}^n$ and $K^*(\mathbb{CP}^{n-1}) \cong \mathbb{Z}[L]/(L-1)^n$. Under the restriction $\rho(\tilde{L}) = L$ the subset $\{1, \tilde{L}-1, ..., (\tilde{L}-1)^{n-1}\}$ of $K^*(P(E))$ becomes a basis of $K^*(\mathbb{CP}^{n-1})$. Hence by Leray-Hirsch theorem, $K^*(P(E))$ is a free $K^*(X)$-module with basis $1, L, ..., L^{n-1}$.

We can now present the splitting principle for K-theory.

Proposition 3.9 (Splitting principle)**.** Given a vector bundle $E \to X$ of rank n with X compact Hausdorff and let $p : P(E) \to X$ the projectivisation of E. Then there is a compact Hausdorff space $F(E)$ and a map $\tilde{p} : F(E) \to X$ such that the induced map $\tilde{p}^* : K^*(X) \to K^*(F(E))$ is injective and $\tilde{p}^*(E)$ splits as a sum of line bundles.

Proof. We can construct the pullback bundle $p^*(E)$

$$\begin{tikzcd} L \subset p^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ P(E) \arrow[r, "p"] & X \end{tikzcd}$$

and it contains the canonical line bundle L as a subbundle. Hence we can split $p^*(E) \cong L \oplus E'$ where E' is the subbundle of $p^*(E)$ orthogonal to L, with respect to some choice of inner product, and has thus dimension $n-1$. We can now repeat the process as follows

$$\begin{tikzcd} p'^*(E') \arrow[r] \arrow[d] & E' \arrow[d] \\ P(E') \arrow[r, "p'"] & P(E) \end{tikzcd}$$

until the pullback bundle has dimension 1. We then obtain the following diagram

$$\begin{tikzcd} L \otimes L' \otimes \cdots \otimes L^{(n-2)} \cong \widetilde{p}^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ F(E) \arrow[r, "\widetilde{p}"] & X \end{tikzcd}$$

$\widetilde{p}^*(E) \cong L \otimes L' \otimes \cdots \otimes L^{(n-2)}$ splits as a sum of line bundles and induced map $\widetilde{p}^* = p^* \circ p'^* \circ \cdots \circ p^{*(n-2)}$ is injective since it is a composition of injective maps. $\square$

Let $p : E \to X$ be a complex vector bundle of dimension n and for each k, $1 \leq k \leq n$, consider the bundle $\bigwedge^k E$. This operation on vector bundles has the properties

i) $\bigwedge^k(E \oplus E') = \bigoplus_{i+j=k}(\bigwedge^i E) \otimes (\bigwedge^j E')$,

ii) $\lambda^0(E) = [\underline{1}]$,

iii) $\lambda^1(E) = E$,

iv) $\lambda^k(E) = 0$, where in the last equality k is the greater than the maximum dimension of the fibers of E.

Now we consider the ring $K(X)[[t]]$ of formal power series with coefficients in $K(X)$. Assigning to a vector bundle E the element

$$\lambda_t(E) = \sum_{k=0} \left[\bigwedge^k E\right] t^k,$$

where this is a finite sum by property (iv), and the property (i) says that

$$\lambda_t(E \oplus F) = \lambda_t(E)\lambda_t(F).$$

When $E = L_1 \oplus \cdots \oplus L_n$, where L_i are line bundles, this implies that $\lambda_t(E) = \prod_i \lambda_t(L_i)$, which equals $\prod_i(1 + L_i t)$ by (ii), (iii) and (iv).

Proposition 3.10. For a n-dimensional vector bundle $p : E \to X$ the ring $K(P(E))$ is isomorphic to the quotient ring

$$K^*(X)[L]/\left(\sum_i (-1)^i \lambda^i(E) L^{n-i}\right).$$

Proof. The pullback of E over $P(E)$ splits as $L \oplus E'$ for some bundle E' of dimension $n-1$, and the desired relation will be $\lambda^n(E') = 0$. To compute $\lambda^n(E') = 0$ we use the formula $\lambda^t(E) = \lambda_t(L)\lambda_t(E')$ in $K^*(P(E))[t]$, where to simplify notation we let E also denote the pullback of E over $P(E)$. The equation $\lambda_t(E) = \lambda_t(L)\lambda_t(E')$ can be rewritten as $\lambda_t(E') = \lambda_t(E)\lambda_t(L)^{-1}$ where $\lambda_t(L)^{-1} = \sum_i(-1)^i L^i t^i$ since $\lambda_t(L) = 1 + Lt$. Equating coefficients of t^n in the two sides of $\lambda_t(E') = \lambda_t(E)\lambda_t(L)^{-1}$, we get $\lambda^n(E') = \sum_i (-1)^{n-i}\lambda^i(E)L^{n-i}$. The relation $\lambda_n(E') = 0$ can be written as $\sum_i(-1)^{n-i}\lambda^i(E)L^{n-i}$, with the coefficient of L^n equal to 1, as desired. □

Finally we can state the Thom isomorphism theorem.

Corollary 3.5 (The Thom isomorphism theorem)**.** If $p : E \to X$ is a vector bundle, then $\Phi : K^*(X) \to \widetilde{K}^*(\mathrm{Th}(E))$ defined by $\Phi(x) = \lambda_E x$ is an isomorphism.

Proof. Let $P = P(E \oplus 1)$ and let H be the standard line bundle over P. By definition we have a monomorphism $H \to \pi^*(E \oplus 1)$, where $\pi : P \to X$ is the projection. Hence tensoring with H we get a section of $H \otimes \pi^*(E \oplus 1)$. Projecting onto the first factor gives therefore a natural section $s \in \Gamma(H \otimes \pi^*(E)$. Consider the exterior algebra $\bigwedge^*(H \otimes \pi^*(E))$. Each component is a vector bundle over P and exterior multiplication by s gives us a complex of vector bundles acyclic outside the subspace where $s = 0$. But this is just the image of the natural cross-section $X \to P$. If we restrict to the complement of $P(E)$ in $P(E \oplus 1)$ then H becomes isomorphic to 1 and we recover the element which defines λ_E, identifying $P(E \oplus 1) - P(E)$ with E. Now, the Thom space $\mathrm{Th}(E)$ may be identified with $P(E \oplus 1)/P(E)$. This shows that the image of λ_E under the homomorphism $\widetilde{K}(\mathrm{Th}(E)) = K(P(E \oplus 1), P(E)) \to K(P(E \oplus 1))$ is the alternating sum

$$\sum (-1)^i H^i \lambda^i E.$$

Since this element generates as an ideal the kernel of $\widetilde{K}(\mathrm{Th}(E)) = K(P(E\oplus 1), P(E)) \to K(P(E\oplus 1))$ we have the desired result. $\square$

Now we will discuss the push forward in K-theory. Let X and Y be a smooth manifolds, with X compact, and let $f : X \hookrightarrow Y$ be a smooth embedding. We consider the subbundle $TX \subset TY|_X$, where $TY|_X$ is just the ambient tangent bundle restricted to points of X, i.e., the fiber of $TY|_X \to X$ is T_xY. Then the quotient bundle (see Definition 2.13) $TY|_X/TX$ is by definition the normal bundle to X, denoted by NX.

Definition 3.20. Let $X \subset Y$ smooth manifolds. A **tubular neighborhood** of X in Y is a vector bundle $p : E \to X$ together with a smooth map $\varphi : E \to Y$ such that

- $\varphi \circ \mathbf{O}_E = i$, where $\mathbf{O}_E : X \to E$ is the zero section of the bundle $p : E \to X$ and $i : X \to Y$ is the embedding.
- There exist open sets $U \subset E$ and $V \subset Y$ with $\mathbf{O}_E(X) \subset U$ and $X \subset V$ such that $\varphi|_U : U \to V$ is a diffeomorphism.

By abuse of notation we denote a tubular neighborhood of $X \subset Y$ by the tuple (U, φ).

By the tubular neighborhood theorem [Spivak], every submanifold X of Y has a tubular neighborhood (U, φ) with vector bundle the normal bundle $NX \to X$ of X, and in fact $\varphi : NX \to V$ is a diffeomorphism. So composing with the inclusion map $V \hookrightarrow Y$ we have the continuous map $NX \to Y$, therefore we have the induced map $K_{\mathrm{cpt}}(NX) \to K_{\mathrm{cpt}}(Y)$.

Assume now that X and Y are smooth manifolds and that $f : X \hookrightarrow Y$ is a smooth proper embedding, and in addition we assume that the normal bundle $N(f(X))$ is equipped with a complex structure, hence the rank of N is equal to $\dim(Y)-\dim(X)$ is even. By taking the Thom isomorphism, the Corollary 3.5 $K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(N)$ followed by the map $K_{\mathrm{cpt}}(N) \to K_{\mathrm{cpt}}(Y)$ obtained in the previous discussion, we have the **push forward** on K-theory

$$f_! : K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(Y). \tag{3.6}$$

Now, the normal bundle to the associated proper embedding $f_* : TX \to TY$ is the pullback to TX of $N \oplus N$, where N is the normal bundle to X, and such normal bundle has complex structure given by

$$S = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

Therefore, by previous discussion, for any proper embedding of manifolds $f : X \hookrightarrow Y$, there is an associated map

$$f_! : K_{\mathrm{cpt}}(TX) \to K_{\mathrm{cpt}}(TY).$$

By Whitney embedding theorem [Spivak], for a compact smooth manifold X there exist a integer N and a smooth embedding $f : X \hookrightarrow \mathbb{R}^N$, hence we have the induced map

$$f_! : K_{\mathrm{cpt}}(TX) \to K_{\mathrm{cpt}}(T\mathbb{R}^N). \tag{3.7}$$

Now, let $g : T\mathbb{R}^N \to \{*\}$ be the canonical map taking $T\mathbb{R}^N$ to a point. As $T\mathbb{R}^N = \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{C}^N$, we can consider to $q : \mathbb{C}^N \to \{*\}$ as a complex vector bundle, and again by the Thom isomorphism theorem we have the isomorphism $K(\{*\}) \to K_{\mathrm{cpt}}(\mathbb{C}^N)$. Thus we consider the inverse of the Thom isomorphism $i_!$, $g_! : K_{\mathrm{cpt}}(T\mathbb{R}^N) \to K(\{*\}) \cong \mathbb{Z}$. Finally we have the map

$$g_! f_! : K_{\mathrm{cpt}}(TX) \to \mathbb{Z}, \tag{3.8}$$

which is of fundamental importance in defining the topological index of an elliptic differential operator.

Remark 3.2. We see that the map $g_! f_! : K_{\text{cpt}}(TX) \to \mathbb{Z}$ is independent of the choice of f. We consider $\widetilde{f} = j \circ f$, where $j : \mathbb{R}^N \hookrightarrow \mathbb{R}^{N+N'}$ is a linear inclusion. The induced map $j_! : K_{\text{cpt}}(T\mathbb{R}^N) \to K_{\text{cpt}}(T\mathbb{R}^{N+N'})$ is just the Thom isomorphism for the bundle $\mathbb{C}^{N+N'} \to \mathbb{C}^N$, and one easily checks that $\widetilde{g}_! \widetilde{f}_! = g_! f_!$ where $\widetilde{q} : T\mathbb{R}^{N+N'} \to \{*\}$. If we are given two embeddings $f_0 : X \hookrightarrow \mathbb{R}^{N_0}$ and $f_1 : X \hookrightarrow \mathbb{R}^{N_1}$, then the embeddings $j_0 f_0 : X \hookrightarrow \mathbb{R}^{N_0+N_1}$ and $j_1 f_1 : X \hookrightarrow \mathbb{R}^{N_0+N_1}$, defined as above, are isotopic. That is, $F_t = t j_1 f_1 + (1-t) j_0 f_0$, $0 \leq t \leq 1$, is a smooth family of embeddings. By homotopy invariance of K_{cpt} conclude that the topological index is independent of the choice of f.

Finally, if we restrict the element λ_E, of the Corollary 3.5, to the zero section, we recover the element $\lambda_{-1}(E) \in K^*(X)$.

Proposition 3.11. If X is compact, then for all $\xi \in K(X) = K_{\text{cpt}}(X)$, one has

$$i^* i_!(\xi) = \lambda_{-1}(E) \cdot \xi,$$

where $\lambda_{-1}(E) = \left[\bigwedge^{\text{even}} E\right] - \left[\bigwedge^{\text{odd}} E\right] \in K(X)$.

Chapter 4

Characteristic classes

We discuss the Chern and Pontryagin characteristic classes defined respectively on real and complex vector bundles, which intuitively measure the non-triviality of such bundles. In addition, we define the Chern character Ch, which establishes a morphism of rings between the K-theory and cohomology of a topological space X, establishing one of the arrows in the diagram 1.1. Finally we interpret geometrically the Chern class and the Pontryagin class by the Chern-Weil homomorphism, which is discussed in detail using material developed in chapter 2 on connections on principal G-bundles. More precisely, given a complex (real) principal G-bundle on a smooth manifold X, we rewrite the Chern (Pontryagin) class of such bundle in terms of the curvature of a connection on the principal G-bundle.

1 Chern Classes

First we will give some definitions and elementary results without proof (see [Hat01]) that will be useful to prove the Leray-Hirsch theorem.

Definition 4.1. A map $f : X \to Y$ is called a **weak homotopy equivalence** if it induces isomorphisms $\pi_n(X, x_0) \simeq \pi_n(Y, f(x_0))$, for all $n \geq 0$ and all choices of basepoint x_0. If X is a CW-complex, then the weak homotopy equivalence $f : X \to Y$ is called a **CW approximation** to Y.

Proposition 4.1. Every space X has a CW approximation.

Proposition 4.2. Let $E \to X$ be a fiber bundle with fiber F. Pick $y_0 \in E$, let $x_0 = \pi(y_0) \in X$, and identify $F = \pi^{-1}(x_0)$. Then there is a long exact sequence

$$\cdots \to \pi_{k+1}(X, x_0) \to \pi_k(F, y_0) \to \pi_k(E, y_0) \to \pi_k(X, x_0) \to \pi_{k-1}(F, y_0) \to \cdots$$

Proposition 4.3. A weak homotopy equivalence $f : X \to Y$ induces isomorphisms $f^* : H^n(Y; G) \to H^n(X; G)$, for all n and all coefficient groups G.

Definition 4.2. A **covering space** of a topological space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$, called the **covering map**, satisfying the following condition: There exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U_α by p. The empty disjoint union is allowed, so p need not be surjective.

Definition 4.3. Let $p : \widetilde{X} \to X$ be a covering map of the topological space X. A **lift** of a map $f : Y \to X$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$.

Proposition 4.4 (homotopy lifting property)**.** Given a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\widetilde{f}_0 : Y \to \widetilde{X}$ lifting f_0 , then there exists a unique homotopy $\widetilde{f}_t : Y \to \widetilde{X}$ of $\widetilde{f}_0$ that lifts f_t .

Lemma 4.1. Given a fiber bundle $p : E \to X$ and a subspace $Y \subset X$ such that (X, Y) is k-connected (see Definition 3.14), then $(E, p^{-1}(Y))$ is also k-connected.

Proof. For a map $g : (D^i, \partial D^i) \to (E, p^{-1}(Y))$ with $i \leq k$, there is by hypothesis a homotopy $f_t : (D^i, \partial D^i) \to (X, Y)$ of $f_0 = p \circ g$ to a map f_1 with image in Y. The homotopy lifting property 4.4 then gives a homotopy $g_t : (D^i, \partial D^i) \to (E, p^{-1}(Y))$ of g to a map with image in $p^{-1}(Y)$. □

Theorem 4.1 (Leray-Hirsh)**.** Let $p : E \to X$ be a fiber bundle with fiber F such that for some commutative coefficient ring R:

(a) $H^n(F; R)$ is a finitely generated free R-module for each n.

(b) There exist classes $c_j \in H^{k_j}(E; R)$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F; R)$ in each fiber F, where $i : F \to E$ is the inclusion map.

Then the map

$$\begin{array}{cccc} \Phi : & H^*(X;R) \otimes_R H^*(F;R) & \to & H^*(E;R) \\ & \sum_{i,\ j} b_i \otimes i^*(c_j) & \mapsto & \sum_{i,\ j} p^*(b_i) \smile c_j \end{array}$$

is an isomorphism.

Proof. Notice first that the function Φ is well defined, since by (b), to define the assignment on an element of $H^*(X; R) \otimes_R H^*(F; R)$ is enough to defined this over the restrictions $i^*(c_j)$ that form a basis for $H^*(F; R)$. For short notation we delete R from the notation for the cohomology group.

We first prove the result for finite-dimensional CW-complex X by induction on their dimension. The case that X is 0-dimensional is trivial. Now suppose that X has dimension n and the desired result holds for all CW-complex of dimension less than n. We define $X' \subset X$ the subspace obtained by deleting a point x_α from the interior of each n-cell e^n_α of X. Let $E' := p^{-1}(X')$ and consider the exact sequences

$$0 \to C_n(X') \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, X') \to 0$$

$$0 \to C_n(E') \xrightarrow{i} C_n(E) \xrightarrow{j} C_n(E, E') \to 0$$

This sequences induce long exact sequences in the cohomology and tensoring with the free module $H^n(F)$ preserves exactness, so the rows in the following diagram, with coefficients in R understood, are exact

$$\begin{tikzcd} \to H^*(X,X') \otimes_R H^*(F) \arrow[r, "j^* \otimes id"] \arrow[d, "\Phi"] & H^*(X) \otimes_R H^*(F) \arrow[r, "i^* \otimes id"] \arrow[d, "\Phi"] & H^*(X') \otimes_R H^*(F) \to \arrow[d, "\Phi"] \\ \cdots \to H^*(E,E') \arrow[r, "j^*"] & H^*(E) \arrow[r, "i^*"] & H^*(E') \xrightarrow{\delta} \cdots . \end{tikzcd}$$

In the definition of the first Φ in the diagram is being used the relative cup product. We have that the two squares show in the previous diagram are commutative, by naturality of Φ. In the squares given by coboundary maps

$$
\begin{tikzcd}
\cdots \arrow[r] & H^*(X') \otimes_R H^*(F) \arrow[r, "\delta\otimes id"] \arrow[d, "\Phi"] & H^*(X,X') \otimes_R H^*(F) \arrow[r] \arrow[d, "\Phi"] & \cdots \\
\cdots \arrow[r] & H^*(E') \arrow[r, "\delta"] & H^*(E,E') \arrow[r] & \cdots
\end{tikzcd}
$$

we have the following: we take $x \otimes i^*(c_j) \in H^*(X') \otimes_R H^*(F)$ and map first this element horizontally get $\delta x \otimes i^*(c_j)$ which maps vertically to $p^*(\delta x) \smile c_j$, whereas if we first map vertically we get $p^*(x) \smile c_j$ which maps horizontally to $\delta(p^*(x) \smile c_j) = \delta p^*(x) \smile c_j = p^*(\delta x) \smile c_j$ since $\delta c_j = 0$.

The idea is to show that the functions of the left and right in the first diagram are isomorphisms, then using the five-lemma we conclude that the center function is an isomorphism.

First we show that the map on the right is an isomorphism. The space X' deformation retracts onto the skeleton X^{n-1}, and the previous lemma implies that the inclusion $p^{-1}(X^{n-1}) \hookrightarrow E'$ is a weak homotopy equivalence, hence induces an isomorphism on all cohomology groups by the Proposition 4.3.

For the left function, let $U_\alpha \subset e^n_\alpha$, with $x_\alpha \in U_\alpha$, an open covering of X defining local trivializations $p^{-1}(U_\alpha) \mapsto U_\alpha \times F$. We define $U := \bigcup_\alpha U_\alpha$ and $U' = U \cap X'$. By excision we have $H^*(X,X') \simeq H^*(U,U')$, and $H^*(E,E') \simeq H^*(p^{-1}(U), p^{-1}(U'))$. Therefore we have the reduction to show that $\Phi : H^*(U,U') \otimes_R H^*(F) \to H^*(U \times F, U' \times F)$ is an isomorphism. For this we can argue again by induction, since U and U' are deformation retract onto complexes of dimension 0 and $n-1$ respectively, and by lemma 4.1 we can restrict to the bundles over these complexes, so the inductive step hold for U and U', thereby applying the five-lemma to the diagram with (X,X') replaced by (U,U'), we obtain the desired result.

Now, in the case in that X is an infinite CW-complex, since (X,X^n) is n-connected, the lemma implies that the same is true of $(E, p^{-1}(X^n))$, so $p^{-1}(X^n) \hookrightarrow E$ is a weak homotopy equivalence, induces an isomorphism on all cohomology groups. Hence in the commutative diagram

$$
\begin{tikzcd}
H^*(X) \otimes_R H^*(F) \arrow[r] \arrow[d, "\Phi"'] & H^*(X^n) \otimes_R H^*(F) \arrow[d, "\Phi"] \\
H^*(E) \arrow[r] & H^*(p^{-1}(X^n))
\end{tikzcd}
$$

the horizontal maps are isomorphisms below dimension n. Further we have shown in the previous step that the right Φ is an isomorphism, implies that the left-hand Φ is an isomorphism below dimension n. Since n is arbitrary, this gives the theorem for all CW complexes X.

Finally, if $f : A \to X$ is a CW approximation to an arbitrary base space X, with A a CW-complex, then by definition f is a weak homotopy equivalence, i.e., we have isomorphisms $\pi_n(A) \simeq \pi_n(X)$ for all $n \geq 0$. Consider the fiber bundles $E \to X$ and their pullback $f^*(E) \to A$, and consider their homotopy fiber $F \to E$ and $F \to f^*E$ respectively. This bundles by the Proposition 4.2 induce the long exact sequences of homotopy groups show in the rows of the following diagram

$$\begin{tikzcd}
\cdots \to \pi_{n+1}(X) \arrow[r] \arrow[d, "\simeq"] & \pi_n(F) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_n(E) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d, "\simeq"] & \pi_{n-1}(F) \to \cdots \arrow[d, "\mathrm{id}"] \\
\cdots \to \pi_{n+1}(A) \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(f^*(E)) \arrow[r] & \pi_n(A) \arrow[r] & \pi_{n-1}(F) \to \cdots
\end{tikzcd}$$

We can show that de square commutes, so by five-lemma we have the isomorphism $\pi_n(E) \simeq \pi_n(f^*(E))$, for all $n \geq 0$. Therefore by definition $\widetilde{f} : f^*(E) \to E$ is a weak homotopy equivalence, hence by the Proposition 4.3 induces isomorphisms $\widetilde{f}_* : H^*(E) \to H^*(f^*(E))$. Thereby the class c_j pull back to classes in $H^*(f^*(E))$ which still restrict to a basis in each fiber, and so the naturality of Φ reduces the theorem for $E \to X$ to the case of $f^*(E) \to A$. $\square$

Note that $H^*(E; R)$ is a $H^*(X; R)$-module under scalar multiplication $xc = p^*(x) \smile c$, with $x \in H^*(X)$ and $c \in H^*(E)$.

Corollary 4.1. *$H^*(E; R)$ is a free $H^*(X; R)$-module with basis $\{c_j\}$.*

Proof. We shown by easy computation that $\{c_j\}$ is linearly independent. Let $x_1, ..., x_k$ be elements of $H^*(X; R)$ such that $x_1c_1 + x_2c_2 + \cdots + x_kc_k = 0$, then $p^*(x_1) \smile c_1 + p^*(x_2) \smile c_2 + \cdots + p^*(x_k) \smile c_k = 0$. By the previous isomorphism $x_1 \otimes i^*(c_1) + x_2 \otimes i^*(c_2) + \cdots + x_k \otimes i^*(c_k) = 0$, hence $x_1 = x_2 = \cdots = x_k = 0$. $\square$

As an application of the Leray-Hirsch theorem, we will calculate the cohomology of Grassmannians. First some definitions that we need in the following theorem.

Definition 4.4. Define an n-**flag** in $\mathbb{C}^k$ to be an ordered n-tuple of orthogonal 1 dimensional vector subspaces of $\mathbb{C}^k$. Equivalently, an n-flag could be defined as a chain of vector subspaces $V_1 \subset \cdots \subset V_n$ of $\mathbb{C}^k$ where V_i has dimension i. The set of all n-flags in $\mathbb{C}^k$ forms a subspace $F_n(\mathbb{C}^k)$ of the product of n copies of $P\mathbb{C}^{k-1}$. There is a natural fiber bundle

$$F_n(\mathbb{C}^n) \to F_n(\mathbb{C}^k) \xrightarrow{p} G_n(\mathbb{C}_k)$$

where p sends an n-tuple of orthogonal lines to the n-plane it spans. The local triviality property can be verified just as was done for the analogous map $V_n(\mathbb{C}^k) \to G_n(C^k)$ in the Remark 2.11. The case $k = \infty$ is covered by the same argument.

Definition 4.5. For a graded free $\mathbb{Z}$ module $A = \bigoplus_i A_i$, define its **Poincaré series** to be the formal power series $p_A(t) = \sum_i a_i t^i$ where a_i is the rank of A_i , which we assume to be finite for all i.

The basic formula we need is that $p_{A \otimes B}(t) = p_A(t)p_B(t)$, which is immediate from the definition of the graded tensor product.

Theorem 4.2. *Let $G_n(\mathbb{C}^\infty)$ be the Grassmann manifold of n-dimensional vector subspace of $\mathbb{C}^\infty$, then $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z})$ is a polynomial ring $\mathbb{Z}[c_1, ..., c_n]$ on generators c_i of dimension $2i$.*

Proof. The plan of the proof is to apply the Leray–Hirsch theorem to a fiber bundle

$$F_n(\mathbb{C}^n) \to F_n(\mathbb{C}^\infty) \xrightarrow{p} G_n(\mathbb{C}^\infty)$$

The first step in the proof is to show that $H^*(F_n(\mathbb{C}^\infty); \mathbb{Z}) \cong \mathbb{Z}[x_1, ..., x_n]$ where x_i is the pullback of a generator of $H^2(P\mathbb{C}^\infty; \mathbb{Z})$ under the map $F_n(\mathbb{C}^\infty) \to P\mathbb{C}^\infty$ projecting an n-flag onto its

i-th line. This can be seen by considering the fiber bundle

$$P\mathbb{C}^\infty \to F_n(\mathbb{C}^\infty) \xrightarrow{p} F_{n-1}(\mathbb{C}^\infty)$$

where p projects an n-flag onto the $(n-1)$-flag obtained by ignoring its last line. The local triviality property can be verified by the argument in the Remark 2.11. The Leray–Hirsch theorem applies since the powers of x_n restrict to a basis for $H^*(P\mathbb{C}^\infty;\mathbb{Z})$ in the fibers $P\mathbb{C}^\infty$, each fiber being the space of lines in a vector subspace $\mathbb{C}^\infty$ of the standard $\mathbb{C}^\infty$. The elements x_i for $i < n$ are the pullbacks via p of elements of $H^*(F_{n-1}(\mathbb{C}^\infty);\mathbb{Z})$ defined in the same way. By induction $H^*(F_{n-1}(\mathbb{C}^\infty);\mathbb{Z})$ is a polynomial ring on these elements. From the Leray–Hirsch theorem we conclude that the products of powers of the x_i's for $1 \le i \le n$ form an additive basis for $H^*(F_{n-1}(\mathbb{C}^\infty);\mathbb{Z})$, hence this ring is the polynomial ring on the x_i's.

There is a corresponding result for $F_n(\mathbb{C}^k)$, that $H^*(F_n(\mathbb{C}^k);\mathbb{Z})$ is free with basis the monomials $x_1^{i_1}\cdots x_n^{i_n}$ with $i_j \le k-j$ for each j . This is proved in exactly the same way, using induction on n and the fiber bundle $P\mathbb{C}^{k-n} \to F_n(\mathbb{C}^k) \to F_{n-1}(\mathbb{C}^k)$. Thus the cohomology groups of $F_n(\mathbb{C}^k)$ are isomorphic to those of $P\mathbb{C}^{k-1}\times\cdots\times P\mathbb{C}^{k-n}$.

Thus, using the fiber bundle

$$F_n(\mathbb{C}^n) \to F_n(\mathbb{C}^\infty) \xrightarrow{p} G_n(\mathbb{C}^\infty)$$

the preceding calculations show that the Leray–Hirsch theorem applies, so $H^*(F_n(\mathbb{C}^\infty);\mathbb{Z})$ is a free module over $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z})$ with basis the monomials $x_1^{i_1}\cdots x_n^{i_n}$ with $i_j \le n-j$ for each j . In particular, since 1 is among the basis elements, the homomorphism p^* is injective and its image is a direct summand of $H^*(F_n(\mathbb{C}^\infty);\mathbb{Z})$. It remains to show that the image of p^* is exactly the symmetric polynomials.

To show that the image of p^* is contained in the symmetric polynomials, consider a map $\pi : F_n(\mathbb{C}^\infty) \to F_n(\mathbb{C}^\infty)$ permuting the lines in each n-flag according to a given permutation of the numbers $1,...,n$. The induced map π^* on $H^*(F_n(\mathbb{C}^\infty);\mathbb{Z}) \cong \mathbb{Z}[x_1,...,x_n]$ is the corresponding permutation of the variables x_i . Since permuting the lines in an n-flag has no effect on the n-plane they span, we have $p\circ\pi = p$, hence $\pi^*\circ p^* = p^*$, which says that polynomials in the image of p^* are invariant under permutations of the variables. As the symmetric polynomials σ_1 are algebraically independent in $\mathbb{Z}[x_1,...,x_n]$ (see [Lang]), the symmetric polynomial form a polynomial ring $\mathbb{Z}[\sigma_1,...,\sigma_n]$ where σ_i has degree i. We have shown that the image of p^* is a direct summand, so to show that p^* maps onto the symmetric polynomials it will suffice to show that the graded rings $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z})$ and $\mathbb{Z}[\sigma_1,...,\sigma_n]$ have the same rank in each dimension, where the rank of a finitely generated free abelian group is the number of $\mathbb{Z}$ summands.

As all nonzero cohomology is in even dimensions, so let us simplify notation by taking A_i to be the $2i$-dimensional cohomology of the space in question. Since the Poincaré series of $\mathbb{Z}[x]$ is $\sum_i t^i = (1-t)^{-1}$, the Poincaré series of $H^*(F_n(\mathbb{C}^\infty);\mathbb{Z})$ is $(1-t)^{-n}$. For $H^*(F_n(\mathbb{C}^n);\mathbb{Z})$ the Poincaré series is

$$(1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = \prod_{i=1}^n \frac{1-t^i}{1-t} = (1-t)^{-n}\prod_{t=1}^n(1-t^i).$$

From the additive isomorphism $H^*(F_n(\mathbb{C}^\infty);\mathbb{Z}) \cong H^*(G_n(\mathbb{C}^\infty);\mathbb{Z})\otimes H^*(F_n(\mathbb{C}^\infty);\mathbb{Z})$ we see that the Poincaré series $p(t)$ of $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z})$ satisfies

$$p(t)(1-t)^{-n}\prod_{i=1}(1-t^i) = (1-t)^{-n}$$

and hence

$$p(t) = \prod_{i=1}^{n} (1 - t^i)^{-1}.$$

This is exactly the Poincaré series of $\mathbb{Z}[\sigma_1, ..., \sigma_n]$ since σ_i has degree i. As noted before, this implies that the image of p^* is all the symmetric polynomials.

$\square$

Let $p : E \to X$ be a complex vector bundle. Recall that we define the projectivisation $P(E)$ of bundle E as the projective space of all lines through the origin in all the fibers of E, thus a point of $P(E)$ is a line ℓ_x in the fiber E_x. With this in hand we show the first key theorem in this chapter.

Proposition 4.5 (Splitting principle)**.** For each vector bundle $p : E \to X$ there is a space $F(E)$ and a map $f : F(E) \to X$ such that the pullback bundle $f^*(E) \to F(E)$ splits as a direct sum of line bundles, and $f^* : H^*(X; \mathbb{Z}) \to H^*(F(E); \mathbb{Z})$ is injective.

Proof. Let $\pi_1 : P(E) \to X$ be the projectivisation of n-dimensional vector bundle $p : E \to X$, and we consider the pullback $\pi_1^*(E)$ via π_1.

$$\begin{tikzcd} \pi_1^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ P(E) \arrow[r, "\pi_1"] & X. \end{tikzcd}$$

By definition $\pi_1^*(E) = \{(\ell, v) \in P(E) \times E |\ \pi_1(\ell) = p(v)\}$, so $L_1 := \{(\ell, v) \in P(E) \times E |\ v \in \ell\}$ is a one-dimensional subbundle of the pullback $\pi_1^*(E)$, since for $(\ell, v) \in P(E) \times E$ we have that ℓ is a line in some fiber E_x, thus $v \in E_x$, hence $\pi_1(\ell) = x = p(v)$. Now, we know that an inner product on E pull back to an inner product on the pullback bundle, so we have the splitting $\pi_1^*(E) = E_1 \oplus L_1$, with E_1 the orthogonal subbundle of $\pi_1^*(E)$ having dimension $n-1$. As we have seen, by Leray-Hirsch theorem on the vector bundle $\pi_1 : P(E) \to X$ with fiber $P\mathbb{C}^{n-1}$, we have the isomorphism $H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(P\mathbb{C}^{n-1}; \mathbb{Z}) \to H^*(P(E); \mathbb{Z})$, in particular the homomorphism $H^*(X; \mathbb{Z}) \to H^*(P(E); \mathbb{Z})$ is injective. So if $n = 2$, we are done. If $n > 2$, let $\pi_2 : P(E_1) \to P(E)$ be of projectivisation of E_1.

$$\begin{tikzcd} \pi_2^*(E_1) \arrow[r] \arrow[d] & E_1 \arrow[r, hook] \arrow[d] & \pi_1^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ P(E_1) \arrow[r, "\pi_2"] & P(E) & P(E) \arrow[r, "\pi_1"] & X. \end{tikzcd}$$

Then by the same argument on $\pi_2^*(E_1) \to P(E_1)$, we have the splitting $\pi_2^*(E_1) = E_2 \oplus L_2$, for some subbundle $E_2 \to P(E_1)$ of dimension $n - 2$. Therefore

$$(\pi_2^* \circ \pi_1^*)(E) = \pi_2^*(E_1 \oplus L_1) = \pi_2^*(E_1) \oplus \pi_2^*(L_1) = E_2 \oplus L_2 \oplus \pi_2^*(L_1)$$

After taking $n - 1$ projectivisations, we obtain a fibration

$$f := \pi_1 \circ \cdots \circ \pi_{n-1} : F(E) := P(E_{n-2}) \to X$$

such that

$$\begin{aligned} f^*(E) &= (\pi_{n-1}^* \circ \pi_{n-2}^* \circ \cdots \circ \pi_2^* \circ \pi_1^*)(E) \\ &= E_{n-1} \oplus L_{n-1} \oplus \pi_{n-1}^*(L_{k-2}) \oplus (\pi_{n-1}^* \circ \pi_{n-2}^*)(L_{k-3}) \oplus \cdots \\ &\qquad \oplus (\pi_{n-1}^* \circ \cdots \circ \pi_2^*)(L_1) \end{aligned}$$

is a sum of line bundles and the homomorphism $f^* : H^*(X;\mathbb{Z}) \to H^*(F(E);\mathbb{Z})$ is injective. $\square$

We are now able to establish the central theorems in this section. We will assume that all base spaces of vector bundles are paracompact, which is not an essential restriction since one can always pass pullback over a CW approximation to a give base space, and CW complex are paracompact.

Theorem 4.3. There is a unique sequence of functions $c_1, c_2, \ldots$ assigning to each complex vector bundle $E \to X$ a classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$, depending only on the isomorphism type of E, such that:

(a) $c_i(f^*(E)) = f^*(c_i(E))$ for a pullback $f^*(E)$.

(b) $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ for $c = 1 + c_1 + \cdots \in H^*(X;\mathbb{Z})$.

(c) $c_i(E) = 0$ if $i > \dim E$.

(d) For the canonical line bundle $E \to P\mathbb{C}^\infty$, $c_1(E)$ is a generator of $H^2(P\mathbb{C}^\infty;\mathbb{Z})$ specified in advance.

Proof. We will apply the Leray-Hirsch theorem to $P(p) : P(E) \to X$ with fiber $P\mathbb{C}^{n-1}$. To do this we need classes $x_i \in H^{2i}(P(E);\mathbb{Z})$ restricting to generators of $H^{2i}(P\mathbb{C}^{n-1};\mathbb{Z})$ in each fiber $P\mathbb{C}^{n-1}$ for $0 \le i \le n-1$. Recall that given a vector bundle $p : E \to X$ we have via the bijection $[X, G_n] \to \mathrm{Vect}^n(X)$ that $E \cong f^*(E_n)$ for some map $f : X \to G_n$, which is equivalent to have a map $g : E \to \mathbb{C}^\infty$ that is a linear injection on each fiber. Projectivizing the map g we have the map $P(g) : P(E) \to P\mathbb{C}^\infty$ which in turn induces the map $P(g)^* : H^*(P\mathbb{C}^\infty;\mathbb{Z}) \to H^*(P(E);\mathbb{Z})$. Let α be a generator of $H^2(P\mathbb{C}^\infty;\mathbb{Z})$ and let $x := P(g)^*(\alpha) \in H^*(P(E);\mathbb{Z})$. We assert that the powers x^i for $0 \le i \le n-1$ are the desired classes x_i. To show this, note that a linear injection $\mathbb{C}^n \to \mathbb{C}^\infty$ induces a embedding $P\mathbb{C}^{n-1} \to P\mathbb{C}^\infty$, which in turn induce the homomorphism $H^*(P\mathbb{C}^\infty;\mathbb{Z}) \to H^*(P\mathbb{C}^{n-1};\mathbb{Z})$, so α pullback to a generator of $H^2(P\mathbb{C}^{n-1};\mathbb{Z})$, hence α^i pull back to a generator of $H^i(P\mathbb{C}^{n-1};\mathbb{Z})$. Note that any two linear injections $\mathbb{C}^n \to \mathbb{C}^\infty$ are homotopic through linear injections, so the induced embeddings $P\mathbb{C}^{n-1} \hookrightarrow P\mathbb{C}^\infty$ of different fibers of $P(E)$ are all homotopic. In the proof of the Theorem 2.2 that any two choices of g are homotopic through maps that are linear injections on fibers, so the class x^i are independent of the choice of g. Thus we can apply the Theorem 4.1 we have that $H^*(P(E);\mathbb{Z})$ is a free $H^*(X;\mathbb{Z})$-module with basis $1, x, \ldots, x^{n-1}$. Consequently, x^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(X;\mathbb{Z})$. Thus there is a unique relation of the form

$$x^n - c_1(E)x^{n-1} + \cdots + (-1)^n c_n(E) \cdot 1 = 0$$

for certain classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$. Here $c_i(E)x^i$ means $P(p)^*(c_i(E)) \smile x^i$, by the definition of the $H^*(X;\mathbb{Z})$-module structure on $H^*(P(E);\mathbb{Z})$. By completeness we define $c_i(E) = 0$ for $i > n$ and $c_0(E) = 1$.

To prove property (a), consider a pullback $f^*(E) = E'$ of the diagram

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] \\ X' \arrow[r, "f"] & X. \end{tikzcd}$$

If $g : E \to \mathbb{C}^\infty$ is a linear injection on fibers then so is $g \circ \tilde{f}$, and it follows that $P(\tilde{f})^*$ takes the canonical class $x = x(E)$ for $P(E)$ to the canonical class $x(E')$ for $P(E')$. Then

$$\begin{aligned} P(\tilde{f})^*\big(\sum_i P(p)^*(c_i(E)) \smile x(E)^{n-i}\big) &= \sum_i P(\tilde{f})^* P(p)^*(c_i(E)) \smile P(\tilde{f})^*(x(E)^{n-i}) \\ &= \sum_i P(p')^* f^*(c_i(E)) \smile x(E')^{n-i} \end{aligned}$$

so the relation $x^n - c_1(E)x^{n-1} + \cdots + (-1)^n c_n(E) \cdot 1 = 0$ defining $c_i(E)$ pulls back to the relation $x(E')^n + f^*(c_1(E))x(E')^{n-1} + \cdots + f^*(c_n(E)) \cdot 1 = 0$ defining $c_i(E')$. By the uniqueness of this relation, $c_i(E') = f^*(c_i(E))$.

For the property (b), the inclusion of E_1 and E_2 into $E_1 \oplus E_2$ give inclusions of $P(E_1)$ and $P(E_2)$ into $P(E_1 \oplus E_2)$ with $P(E_1) \cap P(E_2) = \emptyset$. Let $U_1 := P(E_1 \oplus E_2) - P(E_1)$ and $U_2 := P(E_1 \oplus E_2) - P(E_2)$. These are open sets in $P(E_1 \oplus E_2)$ that deformation retract onto $P(E_2)$ and $P(E_1)$, respectively. A map $g : E_1 \oplus E_2 \to \mathbb{C}^\infty$ which is a linear injection on fibers restricts to such a map on E_1 and E_2, so the canonical class $x \in H^2(P(E_1 \oplus E_2); \mathbb{Z})$ for $E_1 \oplus E_2$ restricts to the canonical classes for E_1 and E_2. If E_1 and E_2 have dimension m and n, consider the classes $\omega_1 = \sum_j c_j(E_1)x^{m-j}$ and $\omega_2 = \sum_j c_j(E_2)x^{n-j}$ in $H^*(P(E_1 \oplus E_2); \mathbb{Z})$, with cup product $\omega_1\omega_2 = \sum_j [\sum_{r+s=j} c_r(E_1)c_s(E_2)]x^{m+n-j}$. By the definition of the classes $c_j(E_1)$, the classes ω_1 restricts to zero in $H^m(P(E_1); \mathbb{Z})$, hence ω_1 pulls back to a class in the relative group $H^m(P(E_1 \oplus E_2), P(E_1); \mathbb{Z}) \cong H^m(P(E_1 \oplus E_2), U_2; \mathbb{Z})$, and similarly for ω_2. The following commutative diagram with $\mathbb{Z}$ coefficients, then shows that $\omega_1\omega_2 = 0$

$$\begin{tikzcd} H^m(P(E_1 \oplus E_2), U_2) \times H^n(P(E_1 \oplus E_2), U_1) \arrow[r, "\smile"] \arrow[d] & H^{m+n}(P(E_1 \oplus E_2), U_1 \cup U_2) = 0 \arrow[d] \\ H^m(P(E_1 \oplus E_2)) \times H^n(P(E_1 \oplus E_2)) \arrow[r, "\smile"] & H^{m+n}(P(E_1 \oplus E_2)). \end{tikzcd}$$

Thus $\gamma_1\gamma_2 = \sum_j [\sum_{r+s=j} c_r(E_1)c_s(E_2)]x^{m+n-j} = 0$ is the defining relation for the Chern classes of $E_1 \oplus E_2$, and so $c_j(E_1 \oplus E_2) = \sum_{r+s=j} c_r(E_1)c_s(E_2)$.

Property (c) holds by definition. For (d), recall that the canonical line bundle is $E = \{(\ell, v) \in P\mathbb{C}^\infty \times \mathbb{C}^\infty | v \in \ell\}$. The map $P(p)$ in this case is the identity. The map $g : E \to \mathbb{C}^\infty$ which is a linear injection on fibers can be taken to be $g(\ell, v) = v$. So $P(g)$ is also the identity, hence $x(E)$ is a generator of $H^2(P\mathbb{C}^\infty; \mathbb{Z})$. The defining relation $x(E) - c_1(E) \cdot 1 = 0$ the says that $c_1(E)$ is a generator of $H^2(P\mathbb{C}^\infty; \mathbb{Z})$.

Finally, property (d) determine $c_1(E)$ for the canonical line bundle $E \to P\mathbb{C}^\infty$. Property (c) then determines all the c_i's for this bundle. Since the canonical line bundle is the universal line bundle, property (a) therefore determines the classes c_i for all line bundles. Property extend this to sums of line bundles, and finally the splitting principle 4.5 implies that the c_i's are determined for all bundles. $\square$

Definition 4.6. We call $c_i(E) \in H^{2i}(X;\mathbb{Z})$ as the i-th **Chern class** of E and

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots$$

the **total Chern class**.

Remark 4.1. Let E be a complex vector bundle, then the Chern classes of its conjugate vector bundle $c_k(\overline{E})$ is equal to $(-1)^k c_k(E)$, for all $k \geq 1$. Hence the total Chern class of $\overline{E}$ is

$$c(\overline{E}) = 1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E).$$

2 The Chern character

Let E be a complex vector bundle of rank n over a manifold M. By splitting principle we can express the total Chern class of E as

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) = \prod_{k=1}^{n}(1 + x_k)$$

so that $c_k = \sigma_k(x_1, ..., x_n)$. Consider the expression

$$\mathrm{ch}(E) = e^{x_1} + \cdots + e^{x_n} = n + \sum_{i=1}^{n} x_i + \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \cdots .$$

The term of degree k in this expression is just the symmetric polynomial

$$\mathrm{ch}^k(E) = \frac{1}{k!}\sum_{j=1}^{n} x_j^k = \frac{1}{k!} s_k(x_1, ..., x_n)$$

which can be rewritten as a universal polynomial expression in the elementary symmetric functions $c_1, ..., c_n$ by induction using Newton's formulae

$$s_m - s_{m-1}c_1 + s_{m-2}c_2 + \cdots + (-1)^{m-1}s_1 c_{m-1} + (-1)^m c_m = 0,$$

and recall that $c_k = \sigma_k(x_1, ..., x_n)$, we have that

$$\mathrm{ch}^1(c_1) = c_1,$$

$$\mathrm{ch}^2(c_1, c_2) = \frac{1}{2}(c_1^2 - 2c_2),$$

$$\mathrm{ch}^3(c_1, c_2, c_3) = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3),$$

$$\mathrm{ch}^4(c_1, c_2, c_3) = \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4).$$

In particular,

$$\mathrm{ch}(E) = n + \mathrm{ch}^1(E) + \mathrm{ch}^2(E) + \cdots$$

is a well-defined element of $H^{2*}(M;\mathbb{Q})$.

Definition 4.7. The element

$$\mathrm{ch}(E) = n+\mathrm{ch}^1(E)+\mathrm{ch}^2(E)+\cdots = n+c_1+\frac{1}{2}(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\cdots \in H^{2*}(M,\mathbb{Q})$$

is called the **Chern character** of E. The Chern character of M is defined to be the one of TM.

The importance of the Chern character lies in the fact that it respects the ring structures on the set of vector bundles.

Proposition 4.6. The Chern character is a ring homomorphism $\mathrm{ch} : K(X) \to H^{2*}(X,\mathbb{Q})$.

Proof. Due to the nature of the Grothendieck construction it suffices to show that $\mathrm{ch}(E \oplus E') = \mathrm{ch}(E) \oplus \mathrm{ch}(E')$ and $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E)\mathrm{ch}(E')$ for complex vector bundles E, E'. These two properties follow from an application of the splitting principle:

$$c(E \oplus E') = c(E)c(E') = \prod_{i=1}^{n}(1+x_i)\prod_{i=1}^{m}(1+x'_i),$$

$$\begin{aligned}
c(E \otimes E') = c(\ell_1 \oplus \cdots \oplus \ell_n) \otimes (\ell'_1 \oplus \cdots \oplus \ell'_m) &= c\left(\sum_{i\ j} \ell_i \otimes \ell'_j\right) \\
&= \prod_{i\ j} c(\ell_i \ell'_j) \\
&= \prod_{i\ j}(1 + c_1(\ell_i) + c_1(\ell'_j)) \\
&= \prod_{i\ j}(1 + x_i + x'_j)
\end{aligned}$$

where remember that $c_1(\ell \otimes \ell') = c_1(\ell) + c_1(\ell')$. Thus we obtain

$$\mathrm{ch}(E \oplus E') = \sum_{i=1}^{n} e^{x_i} + \sum_{j=1}^{m} e^{x'_j} = \mathrm{ch}(E) + \mathrm{ch}(E'),$$

$$\mathrm{ch}(E \otimes E') = \sum_{i=1}^{n}\sum_{j=1}^{m} e^{x_i + x'_j} = \left(\sum_{i=1}^{n} e^{x_i}\right)\cdot\left(\sum_{j=1}^{m} e^{x'_j}\right) = \mathrm{ch}(E)\mathrm{ch}(E').$$

∎

Now let us compute the Chern characters of several bundles. Let E be a complex vector bundle of rank n and consider the bundle $\bigwedge^k E$ of k-th exterior powers, with $1 \leq k \leq n$. Now consider the ring $K(X)[[x]]$ of formal power series in $K(X)$. We define a group homomorphism $\lambda_t : K(X) \to K(X)[[x]]$ on vector bundles by

$$\lambda_t(E) = \sum_k \left[\bigwedge^k E\right] t^k.$$

This map then satisfies $\lambda_t(E \oplus F) = \lambda_t(E)\lambda_t(F)$ due to

$$\bigwedge^k(E \oplus F) = \sum_{i+j=k}\left(\bigwedge^i E\right) \otimes \left(\bigwedge^j F\right)$$

which we derive from the corresponding equality of vector spaces.

Remark 4.2. For vector spaces we have that $\lambda_t(E)$ is a polynomial whilst for general elements in $K(X)$ this is false. For example,

$$\lambda_t(-[\ell]) = (1 + t[\ell])^{-1} = \sum (-t)^m [\ell]^m$$

as $-[\ell]$ is the inverse of $[\ell]$, which is mapped to $1+t[\ell]$, and due to the formula for the geometric series.

Using the splitting principle we see from the homomorphism property that

$$\lambda_t(E) = \prod_{i=1}^{n} \lambda_t(\ell_i) = \prod_{i=1}^{n}(1 + t[\ell_i])$$

from which we deduce the formula

$$\mathrm{ch}(\lambda_t E) = \prod_{i=1}^{n}(1 + te^{x_i}).$$

In particular, this yields

$$\mathrm{ch}(\lambda_{-1} E) = \prod_{i=1}^{n}(1 - te^{x_i}) = \mathrm{ch}\Big(\bigwedge^{\text{even}} E - \bigwedge^{\text{odd}} E\Big). \tag{4.1}$$

3 Pontryagin classes

We will make a short introduction to the Pontryagin classes with the aim of studying in the next chapter the Pontryagin numbers.

For a real vector space V, its **complexification** is the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$, which is a complex vector space with complex multiplication defined by $\alpha(v \otimes \beta) := v \otimes (\alpha\beta)$, for all $v \in V$ and $\alpha, \beta \in \mathbb{C}$. Let $a + ib$ be a complex number and $v \in V$, then $v \otimes (a + ib) = a \otimes v + i(b \otimes v)$, so every element in $V \otimes_{\mathbb{R}} \mathbb{C}$ can be written uniquely as a sum $v + iw$, with $v, w \in V$, hence we obtain the identification

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus iV. \tag{4.2}$$

For a real vector bundle $E \to X$, we define its complexification as the tensor complex bundle $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{C}$ denote by abuse of notation the product bundle $X \times \mathbb{C}$. Note that $F \otimes_{\mathbb{R}} \mathbb{C}$ is a typical fiber of $E_{\mathbb{C}}$, so with the identification 4.2 we have that $F \otimes_{\mathbb{R}} \mathbb{C} \cong F \oplus iF$, thus the underlying real vector bundle $E_{\mathbb{C}}$ is canonically isomorphic to the Whitney sum $E \oplus E$ with fiber $F \oplus F$. Conversely, given the Whitney sum $E \oplus E$, we define over this sum the complex structure $(u, v) \mapsto (-v, u)$ (escalar multiplication by the complex number i), given us the complexification $E_{\mathbb{C}}$.

Lemma 4.2. The complexification $E_{\mathbb{C}}$ of a real vector bundle E is isomorphic to its own conjugate bundle $\overline{E_{\mathbb{C}}}$.

Proof. We define the map $f : E_{\mathbb{C}} \to \overline{E_{\mathbb{C}}}$ given by $x + iy \mapsto x - iy$. Remember that as topological spaces $E_{\mathbb{C}}$ and $\overline{E_{\mathbb{C}}}$ are the same, so clearly f define a homeomorphism between these spaces. Also f is $\mathbb{R}$-linear in each fiber with

$$f(i(x + iy)) = f(-y + ix) = -y - ix = -i(x - iy) = -if(x + iy).$$

□

Now consider the total Chern class

$$c(E_{\mathbb{C}}) = 1 + c_1(E_{\mathbb{C}}) + c_2(E_{\mathbb{C}}) + \cdots$$

and by the Remark 4.1 and before lemma this is equal to

$$c(\overline{E_{\mathbb{C}}}) = 1 - c_1(E_{\mathbb{C}}) + c_2(E_{\mathbb{C}}) - \cdots$$

hece

$$2c_{2i+1}(E_{\mathbb{C}}) = 0 \tag{4.3}$$

i.e. the odd Chern classes $c_{2i+1}(E_{\mathbb{C}})$ are elements of order 2. Therefore in the following definition we ignore the odd Chern classes.

Definition 4.8. For every real vector bundle $E \to X$ and for every $i \geq 1$ we define the i-th **Pontryagin class** of E as the cohomology class

$$p_i(E) := (-1)^i c_{2i}(E_{\mathbb{C}}) \in H^{4i}(X; \mathbb{Z}).$$

By property (a) in the Theorem 4.3 we deduce that for every continuous map $f : Y \to X$, and every real vector bundle $E \to X$ we have

$$p_i(f^*(E)) = f^*(p_i(E))$$

for all $i \geq 1$.

Notation 3.
- For a group G, we denote by Torsion the torsion elements of G, i.e. the elements of finite order in the group G. Let X be a topological space, we define

$$\overline{H}^*(X) := H^*(X; \mathbb{Z})/\text{Torsion}.$$

  Note that the cup product on $H^*(X; \mathbb{Z})$ induces a cup product on $\overline{H}^*(X)$.

- We define the commutative subring with respect to the cup product

$$H^{\text{even}}(X) := \bigoplus_{k \geq 0} H^{2k}(X; \mathbb{Z}) \subset H^*(X; \mathbb{Z}).$$

  Note that every Chern class c_i belong to this ring.

Definition 4.9.
- For every complex vector bundle $E \to X$ we denote by $\overline{c}_i(E) \in \overline{H}^{2i}(X)$ the image of the i-th Chern class of E via the projection $H^{2i}(X; \mathbb{Z}) \to \overline{H}^{2i}(X)$. We will refer to $\overline{c}_i$ as the i-th **reduced Chern class** and we define the **reduced Chern polynomial** of E by

$$\overline{\mathfrak{C}}_E(t) := \sum_{k \geq 0} \overline{c}_k(E) t^k \in \overline{H}^{even}(X)[t], \; c_0(E) := 1$$

- For every real vector bundle $E \to X$ we denote by $\overline{p}_i(E) \in \overline{H}^{4i}(X)$ the image of the i-th Pontryagin class of E via the projection $H^{4i}(X; \mathbb{Z}) \to \overline{H}^{4i}(X)$. We will refer to $\overline{p}_i$ as the

i-th **reduced Pontryagin class** and we define the **reduced Pontryagin polynomial** of E by

$$\overline{\mathcal{P}}_E(t) := \sum_{k\geq 0} \overline{p}_k(E)t^{2k} \in \overline{H}^{even}(X)[t],\ p_0(E) := 1$$

The next result all us to translate results from the properties of the Chern class.

Lemma 4.3. If E is a real vector bundle then

$$\overline{\mathcal{P}}_E(t) = \overline{\mathcal{C}}_{E_{\mathbb{C}}}(it) \in \mathbb{Z}[i] \otimes_{\mathbb{Z}} \overline{H}^{even}(X)[t]. \tag{4.4}$$

Proof. By definition we have

$$\begin{aligned}\overline{\mathcal{C}}_{E_{\mathbb{C}}}(it) &:= \sum_{k\geq 0} \overline{c}_k(E_{\mathbb{C}})(it)^k \\ &= \sum_{k\geq 0}(-1)^k \overline{c}_{2k}(E_{\mathbb{C}})t^{2k} + i\sum_{k\geq 0}(-1)^k \overline{c}_{2k+1}(E_{\mathbb{C}})t^{2k+1}\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{P}}_E(t) &:= \sum_{k\geq 0} \overline{p}_k(E)t^{2k} \\ &= \sum_{k\geq 0}(-1)^k \overline{c}_{2k}(E_{\mathbb{C}})t^{2k}.\end{aligned}$$

By the Equation 4.3 we have that $\overline{c}_{2k+1}(E_{\mathbb{C}}) = 0$, therefore we obtain the desired result. $\square$

Corollary 4.2. For any pair of real vector bundles $E_0, E_1 \to X$ we have

$$\overline{\mathcal{P}}_{E_0\oplus E_1}(t) = \overline{\mathcal{P}}_{E_0}(t) \cdot \overline{\mathcal{P}}_{E_1}(t).$$

Proof.

$$\overline{\mathcal{P}}_{E_0\oplus E_1}(t) = \overline{\mathcal{C}}_{(E_0)_{\mathbb{C}}\oplus(E_1)_{\mathbb{C}}}(it) = \overline{\mathcal{C}}_{(E_0)_{\mathbb{C}}}(it) \cdot \overline{\mathcal{C}}_{(E_1)_{\mathbb{C}}}(it) = \overline{\mathcal{P}}_{E_0}(t) \cdot \overline{\mathcal{P}}_{E_1}(t).$$

$\square$

Corollary 4.3. The reduced Pontryagin classes are stable, i.e., for any real vector bundle $E \to X$ and any $n > 0$ we have

$$\overline{p}_k(E \oplus \underline{n}) = \overline{p}_k(E)$$

for any $k \geq 1$, where $\underline{n}$ is the n-dimensional trivial real bundle over X.

Proof.

$$\overline{\mathcal{P}}_{E\oplus\underline{n}}(t) = \overline{\mathcal{C}}_{(E)_{\mathbb{C}}\oplus(\underline{n})_{\mathbb{C}}}(it) = \overline{\mathcal{C}}_{(E)_{\mathbb{C}}}(it) \cdot \overline{\mathcal{C}}_{(\underline{n})_{\mathbb{C}}}(it) = \overline{\mathcal{P}}_E(t).$$

$\square$

Corollary 4.4. Let $T\mathbb{S}^n$ be the tangent bundle on $\mathbb{S}^n$, then $\overline{\mathcal{P}}_{T\mathbb{S}^n}(t) = 1$.

Proof. Let $\underline{n+1}$ and $\underline{1}$ be the trivial real bundles on $\mathbb{S}^n$ of dimension $n+1$ and 1 respectively. Then we have the isomorphism $T\mathbb{S}^n \oplus \underline{1} \cong \underline{n+1}$. Thus by stability of the reduced Pontryagin classes we have the result. $\square$

Proposition 4.7. If $E \to X$ is a complex vector bundle and $E_{\mathbb{R}}$ denotes the same bundle but viewed as a real vector bundle, then

$$\begin{aligned}\overline{\mathcal{P}}_{E_{\mathbb{R}}}(t) &= \overline{\mathcal{C}}_E(it)\overline{\mathcal{C}}_E(-it)\\ &= 1-(c_1(E)^2-2c_2(E))t^2+(2c_4(E)+c_2(E)^2-2c_1(E)c_3(E))t^4+\cdots).\end{aligned}$$

Proof. By the Equation 4.2 we have

$$\overline{\mathcal{P}}_{E_{\mathbb{R}}}(t) = \overline{\mathcal{C}}_E(it)\overline{\mathcal{C}}_{\overline{E}}(it) = \overline{\mathcal{C}}_E(it)\overline{\mathcal{C}}_E(-it).$$

□

Corollary 4.5.

$$\overline{\mathcal{P}}_{T\mathbb{CP}^n}(t) = (1+[H]^2t^2)^{n+1},\ [H]^{n+1}=0.$$

Proof. We deduce that

$$\overline{\mathcal{C}}_{T\mathbb{CP}^n}(t) = (1+[H]t)^{n+1}$$

so that

$$\overline{\mathcal{P}}_{T\mathbb{CP}^n}(t) = \overline{\mathcal{C}}_{T\mathbb{CP}^n}(\mathrm{it})\cdot\overline{\mathcal{C}}_{T\mathbb{CP}^n}(-\mathrm{it}) = (1+[H]^2t^2)^{n+1}.$$

□

4 Chern-Weil homomorphism

In the section 8 of chapter 1, we have three equivalent descriptions of a connection on a principal G-bundle $p: P \to M$:

- A G-invariant horizontal distribution $H \subset TP$.
- A G-invariant one-form ω on P with values on the Lie algebra $\mathfrak{g}$ of the Lie group G such that $\omega(\sigma(X)) = X$. We denote by $\Omega^1(P;\mathfrak{g}) := \Omega^1(P)\otimes\mathfrak{g}$ the vector space of these 1-forms ω.
- A family of one-forms $A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ on open trivializing cover $\{U_\alpha\}$ of M with values on the Lie algebra $\mathfrak{g}$ of the Lie group G satisfying 2.2 given by

$$A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - (\mathrm{d}g_{\alpha\beta})g_{\alpha\beta}^{-1} = g_{\beta\alpha}^{-1}\mathrm{d}g_{\beta\alpha} + g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha}. \tag{4.5}$$

The last formulation of a connection on a principal G-bundle will be that we will use. As equivalent gluing cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ both define on a refinement of trivializing open cover $\{U_\alpha\}$ of M, and describing the principal G-bundle $p: P \to M$, define the same family of one-forms A_α, then by simplicity we will denote by $\mathcal{A}(P)$ the set of connections on P up to equivalent gluing cocycles.

Since $A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$, we have that $A_\alpha \wedge A_\alpha \in \Omega^2(U_\alpha;\mathfrak{g}\otimes\mathfrak{g})$ and we define $[A_\alpha, A_\alpha] \in \Omega^2(U_\alpha;\mathfrak{g})$ to be the image of $A_\alpha \wedge A_\alpha$ by the Lie bracket $[\cdot,\cdot]: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ sending $X \otimes Y$ to $[X, Y]$ for all $X, Y \in \mathfrak{g}$. Thus we define:

Definition 4.10. The **curvature form** of a connection is defined as the collection $F_\alpha \in \Omega^2(U_\alpha;\mathfrak{g})$ is define by the **structural equation**

$$F_\alpha := \mathrm{d}A_\alpha + \frac{1}{2}[A_\alpha, A_\alpha].$$

Definition 4.11. Let $p : P \to M$ be a principal G-bundle. We denote by $\mathrm{Ad}(P)$ the vector bundle with standard fiber $\mathfrak{g}$ associated to P via the adjoint representation $ad : G \to \mathrm{End}(\mathfrak{g})$. In other words, if $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ are the gluing cocycles of the principal G-bundle P, then $\mathrm{Ad}(P)$ is the vector bundle defined by the open cover $\{U_\alpha\}$, and gluing cocycle $\mathrm{Ad}(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathfrak{g})$ given by $x \mapsto ad(g_{\alpha\beta}(x)) : \mathfrak{g} \to \mathfrak{g}$.

The bracket operation in the fibers of $\mathrm{Ad}(P)$ induces a bilinear map

$$[\cdot,\cdot] : \Omega^k(Ad(P)) \times \Omega^l(Ad(P)) \to \Omega^{k+l}(Ad(P)),$$

define by

$$[\omega^k \otimes X, \eta^l \otimes Y] := (\omega^k \wedge \eta^l) \otimes [X, Y], \tag{4.6}$$

for all $\omega^k \in \Omega^k(M)$, $\eta^l \in \Omega^l(M)$, and $X, Y \in \Omega^0(\mathrm{Ad}(P))$.

By easy calculation we have for any $\omega, \eta, \varsigma \in \Omega^*(\mathrm{Ad}(P))$

$$[\omega, \eta] = -1(-1)^{|\omega|\cdot|\eta|}[\eta, \omega], \tag{4.7}$$

$$[[\omega, \eta], \varsigma] = [[\omega, \varsigma], \eta] + (-1)^{|\omega|\cdot|\eta|}[\omega, [\eta, \varsigma]]. \tag{4.8}$$

and for any $\omega, \eta \in \Omega^*(U_\alpha; \mathfrak{g})$

$$\mathrm{d}[\omega, \eta] = [\mathrm{d}\omega, \eta] + (-1)^{|\omega|}[\omega, \mathrm{d}\eta]. \tag{4.9}$$

Lemma 4.4. Let $\theta = g^{-1}\mathrm{d}g$ be the Maurer-Cartan 1-form. Then

$$\mathrm{d}\theta + \frac{1}{2}[\theta, \theta] = 0 \tag{4.10}$$

$$\mathrm{d}(g^{-1}A_\alpha g) + [\theta, g^{-1}A_\alpha g] = g^{-1}(\mathrm{d}A_\alpha)g \tag{4.11}$$

The first equation is called the **Maurer-Cartan structural** equation.

Proof. Let us first introduce a new operation. Let $\mathrm{gl}(n, \mathbb{K})$ denote the associative algebra of $\mathbb{K}$-valued $n \times n$ matrices. There exists a natural operation

$$\bigwedge : \Omega^k(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K}) \times \Omega^l(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K}) \to \Omega^{k+l}(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K})$$

uniquely defined by

$$(\omega^k \otimes A) \wedge (\eta^l \otimes B) := (\omega^k \wedge \eta^l) \otimes (A \cdot B)$$

where $\omega^k \in \Omega^k(U_\alpha)$, $\eta^l \in \Omega^l(U_\alpha)$, and $A, B \in \mathrm{gl}(n, \mathbb{K})$. The space $\mathrm{gl}(n, \mathbb{K})$ is naturally a Lie algebra with respect to the commutator of two matrices. This structure induces a bracket

$$[\cdot,\cdot] : \Omega^k(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K}) \times \Omega^l(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K}) \to \Omega^{k+l}(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K})$$

defined as in the Equation 4.6. A simple computation yields the following identity

$$\omega \wedge \eta = \frac{1}{2}[\omega, \eta] \tag{4.12}$$

for all $\omega, \eta \in \Omega^1(U_\alpha) \otimes \mathrm{gl}(n, \mathbb{K})$. The Lie group lies inside $\mathrm{GL}(n, \mathbb{K})$, so that its Lie algebra $\mathfrak{g}$ lies inside $\mathrm{gl}(n, \mathbb{K})$. We can think of the map $g_{\alpha\beta}$ as a matrix valued map, so that we have by the Equation 4.12

$$\begin{aligned}\mathrm{d}\theta = \mathrm{d}(g^{-1}\mathrm{d}g) = (\mathrm{d}g^{-1}) \wedge \mathrm{d}g &= -(g^{-1} \cdot \mathrm{d}g \cdot g^{-1})\mathrm{d}g \\ &= -(g^{-1}\mathrm{d}g) \wedge (g^{-1}\mathrm{d}g) \\ &= -\theta \wedge \theta \\ &= -\frac{1}{2}[\theta, \theta].\end{aligned}$$

On the other hand, by the Equations 4.12 and 4.7 we have

$$\begin{aligned}\mathrm{d}(g^{-1}A_\alpha g) &= (\mathrm{d}g^{-1}A_\alpha) \cdot g + g^{-1}(\mathrm{d}A_\alpha)g + g^{-1}A_\alpha \mathrm{d}g \\ &= -g^{-1} \cdot \mathrm{d}g \cdot g^{-1} \wedge A_\alpha \cdot g + g^{-1}(\mathrm{d}A_\alpha)g + (g^{-1}A_\alpha g) \wedge g^{-1}\mathrm{d}g \\ &= -\theta \wedge g^{-1}A_\alpha g + g^{-1}A_\alpha g \wedge \theta + g^{-1}(\mathrm{d}A_\alpha)g \\ &= -\frac{1}{2}[\theta, g^{-1}A_\alpha g] + \frac{1}{2}[g^{-1}A_\alpha g, \theta] + g^{-1}(\mathrm{d}A_\alpha)g \\ &= -[\theta, g^{-1}A_\alpha g] + g^{-1}(\mathrm{d}A_\alpha)g.\end{aligned}$$

□

Proposition 4.8. Let $p : P \to M$ be a principal G-bundle described by gluing cocycles $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$, then:

(a) The set $\mathcal{A}(P)$ of connections on P is an affine space modelled by $\Omega^1(\mathrm{Ad}(P))$.

(b) For any connection ω given by the family of local connections $\{A_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})\}$, the collection $\{F_\alpha\}$ of curvature 2-form defines a global $\mathrm{Ad}(P)$-valued 2-form. We will denote it by F_A, and we will refer to it as the curvature of the connection ω.

(c) The **Bianchi identity**

$$\mathrm{d}F_\alpha + [A_\alpha, F_\alpha] = 0 \tag{4.13}$$

for all α.

Proof. (a) If $\{A_\alpha\}, \{B_\alpha\} \in \mathcal{A}(P)$, then their difference $C_\beta = g_{\beta\alpha}C_\alpha g_{\beta\alpha}^{-1}$, so that it defines an element of $\Omega^1(\mathrm{Ad}(P))$. Conversely, if $\{A_\alpha \in \mathcal{A}(P)\}$, and $\omega \in \Omega^1(\mathrm{Ad}(P))$, then ω is described by a collection of $\mathfrak{g}$-valued 1-forms $\omega_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$, satisfying the gluing rules

$$\omega_\beta|_{U_\alpha \cap U_\beta} = g_{\beta\alpha}\omega_\alpha|_{U_\alpha \cap U_\beta} g_{\beta\alpha}^{-1}.$$

The collection $A'_\alpha := A_\alpha + \omega_\alpha$ is then a connection on P.

(b) We need to check that the forms F_α satisfy the gluing rules $F_\beta = g^{-1}F_\alpha g$, where $g = g_{\alpha\beta} = g_{\beta\alpha}^{-1}$. We have by the Equations 4.5 and 4.7

$$\begin{aligned}F_\beta &= \mathrm{d}A_\beta + \frac{1}{2}[A_\beta, A_\beta] \\ &= \mathrm{d}(g^{-1}\mathrm{d}g + g^{-1}A_\alpha g) + \frac{1}{2}[g^{-1}\mathrm{d}g + g^{-1}A_\alpha g, g^{-1}\mathrm{d}g + g^{-1}A_\alpha g] \\ &= \mathrm{d}\theta + \frac{1}{2}[\theta, \theta] + \mathrm{d}(g^{-1}A_\alpha g) + [\theta, g^{-1}A_\alpha g] + \frac{1}{2}[g^{-1}A_\alpha g, g^{-1}A_\alpha g]\end{aligned}$$

where $\theta = g^{-1}\mathrm{d}g$ is the Maurer-Cartan form. Using Maurer-Cartan Equation 4.10 and the Equation 4.11 we have

$$F_\beta = g^{-1}(\mathrm{d}A_\alpha)g + \frac{1}{2}[g^{-1}A_\alpha g, g^{-1}A_\alpha g]$$

as desired.

(c) Using the Equalities 4.9, 4.7 and 4.9 we get

$$\begin{aligned}\mathrm{d}(F_\alpha) = \frac{1}{2}\{[\mathrm{d}A_\alpha, A_\alpha] - [A_\alpha, \mathrm{d}A_\alpha]\} = [\mathrm{d}A_\alpha, A_\alpha]& \\ = [F_\alpha, A_\alpha] - \frac{1}{2}[[A_\alpha, A_\alpha], A_\alpha]& \\ = [F_\alpha, A_\alpha].&\end{aligned}$$

□

Now we review some basic on invariant polynomials. Let V be a $\mathbb{K}$-linear vector space with basis $\{e_1, ..., e_n\}$. Consider the dual vector space V^* of V with basis $\{x_1, ..., x_n\}$ with $x_i : V \to \mathbb{K}$ given by $v = c_1e_1 + \cdots + c_ne_n \mapsto x_i(v) = c_i$. For $k \geq 1$, let $S^k(V^*)$ denote the vector space of symmetric $\mathbb{K}$-linear functions $\varphi : V \times \cdots \times V \to \mathbb{K}$. For $v_j = \sum_{i_j=1}^n c_{i_j}e_{i_j}$ in V, we have

$$\begin{aligned}\varphi(v_1, ..., v_k) &= \varphi(\sum_{i_1=1}^n c_{i_1}e_{i_1}, ..., \sum_{i_k=1}^n c_{i_k}e_{i_k}) \\ &= \sum_{i_1,...,i_k=1}^n \varphi(e_{i_1}, ..., e_{i_k})c_{i_1}\cdots c_{i_k} \\ &= \sum_{i_1,...,i_k=1}^n \varphi(e_{i_1}, ..., e_{i_k})x_{i_1}(v_1)\cdots x_{i_k}(v_k).\end{aligned}$$

Thus we can identify $S^k(V^*)$ with the space of degree k homogeneous polynomials in n-variables $\mathbb{K}[x_1, ..., x_n]_k = \mathbb{K}[V]_k$. So we can define the map $P : S^k(V^*) \to \mathbb{K}[V]_k$ given by $\varphi : V\times\cdots\times V \to \mathbb{K} \mapsto P_\varphi(x_1, ..., x_n) = \varphi(v, ..., v)$ where $v = \sum_{i=1}^n c_ie_i$.

Now, assume that $\mathcal{A}$ is a $\mathbb{K}$-algebra with unity 1. Starting with $\varphi \in S^k(V^*)$ we can produce a $\mathbb{K}$-multilinear map $\varphi = \varphi_\mathcal{A} : (\mathcal{A} \otimes V) \times \cdots \times (\mathcal{A} \otimes V) \to \mathcal{A}$, uniquely determined by $\varphi(a_1 \otimes v_1, ..., a_k \otimes v_k) = \varphi(v_1, ..., v_k)a_1a_2\cdots a_k \in \mathcal{A}$. In addition, if the algebra $\mathcal{A}$ is commutative, then $\varphi_\mathcal{A}$ is uniquely determine by the polynomial $P_\varphi(x) = \varphi_\mathcal{A}(x, ..., x)$, with $x \in \mathcal{A} \otimes V$.

Consider now a matrix Lie group G. The adjoint action of G on its Lie algebra $\mathfrak{g}$ induces an action on $S^k(\mathfrak{g}^*)$ still denoted by Ad.

Definition 4.12. We denote by $I^k(G)$ the Ad-invariant elements of $S^k(\mathfrak{g}^*)$. It consists of those $\varphi \in S^k(\mathfrak{g}^*)$ such that

$$\varphi(gX_1g^{-1}, ..., gX_kg^{-1}) = \varphi(X_1, ..., X_k).$$

for all $X_1, ..., X_k \in \mathfrak{g}$. Set

$$I^\bullet(G) := \bigoplus_{k\geq 0} I^k(G)$$

The elements of $I^\bullet(G)$ are usually called **invariant polynomials**.

Proposition 4.9. Let $\varphi \in I^k(G)$. Then for any $X, X_1, ..., X_k \in \mathfrak{g}$ we have

$$\varphi([X, X_1], X_2, ..., X_k) + \cdots + \varphi(X_1, X_2, ..., [X, X_k]) = 0 \tag{4.14}$$

Proof. The proposition follows immediately from the equality

$$\frac{d}{dt}\Big|_{t=0} \varphi(\mathrm{e}^{tX} X_1 \mathrm{e}^{-tX}, ..., \mathrm{e}^{tX} X_k \mathrm{e}^{-tX}) = 0.$$

□

If $P \in I^k(\mathfrak{g})$, U is an open subset of $\mathbb{R}^n$, and $F_i = \omega_i \otimes X_i \in \Omega^{d_i}(U; \mathfrak{g})$, $A = \omega \otimes X \in \Omega^d(U; \mathfrak{g})$, then

$$P(F_1, ..., F_{i-1}, [A, F_i], F_{i+1}, ..., F_k) = (-1)^{d(d_1 + \cdots + d_{i-1})} \omega\omega_1 \cdots \omega_k \; P(X_1, ..., [X, X_i], ..., X_k).$$

In particular, if $F_1, ..., F_{k-1}$ have even degree, we deduce that for every $i = 1, ..., k$ we have

$$P(F_1, ..., F_{i-1}, [A, F_i], F_{i+1}, ..., F_k) = \omega\omega_1 \cdots \omega_k \; P(X_1, ..., [X, X_i], ..., X_k).$$

Summing over i, and using the Ad-invariance of the polynomial P, we deduce

$$\sum_{i=1}^{k} P(F_1, ..., F_{i-1}, [A, F_i], F_{i+1}, ..., F_k) = 0 \tag{4.15}$$

for all $F_1, ..., F_{k-1} \in \Omega^{\text{even}}(U; \mathfrak{g})$, $F_k, A \in \Omega^\bullet(U; \mathfrak{g})$.

Theorem 4.4 (Chern-Weil construction)**.** Let $p : P \to M$ be a principal G-bundle, then:

(a) The form $\phi(F_A)$ is closed for any connection $A \in \mathcal{A}(P)$.

(b) If $A, A' \in \mathcal{A}(P)$, then the forms $\phi(F_A)$ and $\phi(F_{A'})$ are equivalents. In other words, the closed form $\phi(F_A)$ defines a cohomology class in $H^{2k}(M)$ which is independent of the connection $A \in \mathcal{A}(P)$.

Proof. (a) By Bianchi identity 4.13 $\mathrm{d}F_\alpha = -[A_\alpha, F_\alpha]$, by the Equation4.14 and the Leibniz' rule

$$\begin{aligned} \mathrm{d}\phi(F_\alpha, ..., F_\alpha) &= \phi(\mathrm{d}F_\alpha, F_\alpha, ..., F_\alpha) + \cdots + \phi(F_\alpha, ..., F_\alpha, \mathrm{d}F_\alpha) \\ &= -\phi([A_\alpha, F_\alpha], F_\alpha, ..., F_\alpha) - \cdots - \phi(F_\alpha, ..., F_\alpha, [A_\alpha, F_\alpha]) \\ &= 0. \end{aligned}$$

(b) Let $A, A' \in \mathcal{A}(P)$ be connections, defined by the collections of local 1-forms $A_\alpha, A'_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})$. Set $C_\alpha := A'_\alpha - A_\alpha$. For $0 \le t \le 1$ we define $A^t_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})$ by $A^t_\alpha := A_\alpha + tC_\alpha$. Thus the collection $\{A^t_\alpha\}$ defines a connection $B^t \in \mathcal{A}(P)$, and $t \mapsto B^t \in \mathcal{A}(P)$ is an path connecting A to A'. Note that $C = \{C_\alpha\} = \dot{B}^t$. We denote by $F^t := \{F^t_\alpha\}$ the curvature of B^t. A simple computation yields

$$F^t_\alpha = F^0_\alpha + t(\mathrm{d}C_\alpha + [A_\alpha, C_\alpha]) + \frac{t^2}{2}[C_\alpha, C_\alpha].$$

Hence,

$$\dot{F}^t_\alpha = \mathrm{d}C_\alpha + [A_\alpha, C_\alpha] + t[C_\alpha, C_\alpha] = \mathrm{d}C_\alpha + [A^t_\alpha, C_\alpha].$$

Consequently

$$\begin{aligned}\phi(F_\alpha^1)-\phi(F_\alpha^0) &= \int_0^1 \{\phi(\dot F_\alpha^t, F_\alpha^t, ..., F_\alpha^t)+\cdots+\phi(F_\alpha^t, ..., F_\alpha^t, \dot F_\alpha^t)\}dt\\ &= \int_0^1 \{\phi(\mathrm{d}C_\alpha, F_\alpha^t, ..., F_\alpha^t)+\cdots+\phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha)\}dt\\ &+ \int_0^1 \{\phi([A_\alpha^t, C_\alpha], F_\alpha^t, ..., F_\alpha^t)+\cdots+\phi(F_\alpha^t, ..., F_\alpha^t, [A_\alpha^t, C_\alpha])\}dt.\end{aligned}$$

Because the algebra $\Omega^{\mathrm{even}}(U_\alpha)$ is commutative, we deduce

$$\phi(\omega_{\sigma(1)}, ..., \omega_{\sigma(k)}) = \phi(\omega_1, ..., \omega_k),$$

for all permutation σ of order k and any $\omega_1, ..., \omega_k \in \Omega^{\mathrm{even}}(U_\alpha; \mathfrak{g})$. Hence

$$\phi(F_\alpha^1)-\phi(F_\alpha^0) = k\int_0^1 \phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha + [A_\alpha^t, C_\alpha])dt.$$

We claim that

$$\phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha + [A_\alpha^t, C_\alpha]) = \mathrm{d}\phi(F_\alpha^t, ..., F_\alpha^t, C_\alpha).$$

Using Bianchi identity 4.13 and the Equation 4.15 we get

$$\begin{aligned}\mathrm{d}\phi(F_\alpha^t, ..., F_\alpha^t, C_\alpha) &= \phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha)+\phi(\mathrm{d}F_\alpha^t, ..., F_\alpha^t, C_\alpha)+\cdots+\phi(F_\alpha^t, ..., \mathrm{d}F_\alpha^t, C_\alpha)\\ &= \phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha)-\phi(C_\alpha, [A_\alpha^t, F_\alpha^t], F_\alpha^t, ..., F_\alpha^t)-\\ &\cdots-\phi(C_\alpha, F_\alpha^t, ..., F_\alpha^t, [A_\alpha^t, F_\alpha^t])\\ &= \phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha+[A_\alpha^t, C_\alpha])-\phi(F_\alpha^t, ..., F_\alpha^t, [A_\alpha^t, C_\alpha])\\ &-\phi([A_\alpha^t, F_\alpha^t], F_\alpha^t, ..., F_\alpha^t, C_\alpha)-\cdots-\phi(F_\alpha^t, ..., F_\alpha^t, [A_\alpha^t, F_\alpha^t], C_\alpha)\\ &= \phi(F_\alpha^t, ..., F_\alpha^t, \mathrm{d}C_\alpha+[A_\alpha^t, C_\alpha])\\ &= \phi(\mathrm{d}C_\alpha+[A_\alpha^t, C_\alpha]), F_\alpha^t, ..., F_\alpha^t).\end{aligned}$$

Hence

$$\phi(F_\alpha^1)-\phi(F_\alpha^0) = k\int_0^1 \phi(\dot A_\alpha^t, F_\alpha^t, ..., F_\alpha^t)dt.$$

We set

$$T_\phi(A'_\alpha, A_\alpha) := \int_0^1 k\phi(\dot A_\alpha^t, F_\alpha^t, ..., F_\alpha^t)dt.$$

Since $C_\beta = g_{\beta\alpha}C_\alpha g_{\beta\alpha}^{-1}$ and $F_\beta = g_{\beta\alpha}F_\alpha g_{\beta\alpha}^{-1}$ on $U_\alpha \cap U_\beta$, we conclude from the Ad-invariance of ϕ that the collection $T_\phi(A'_\alpha, A_\alpha)$ defines a global $(2k-1)$-form on M which we denote by $T_\phi(A', A)$, and we name it the **ϕ-transgression** from A to A'. We have thus established the **transgression formula**

$$\phi(F_{A'})-\phi(F_A) = \mathrm{d}T_\phi(A', A).$$

□

In conclusion, given a principal G-bundle $p : P \to M$ on a smooth manifold M defined by gluing cocycles $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$, let $A \in \mathcal{A}(P)$ be a connection defined by the collection

$A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ satisfying the Equation 4.5, with curvature form defined by the collection $F_\alpha \in \Omega^2(U_\alpha;\mathfrak{g})$, and given $\phi \in I^k(G)$, then the Chern-Weil construction give us a closed form $\phi(F(A)) \in \Omega^{2k}(M)$, whose cohomology class is independent of the connection A. We denote this cohomology class by $\phi(P)$. Thus, the principal G-bundle P defines a map, called the **Chern-Weil correspondence**

$$\mathfrak{cw}_P : I^\bullet(G) \to H^\bullet(M)$$

defined by $\phi \mapsto \phi(P)$. In fact, $\mathfrak{cw}_P$ is a morphism of $\mathbb{R}$-algebras.
Let $G = U(n)$ be the Lie unitary group with Lie algebra $\mathfrak{g} = u(n)$ consisting of skew-hermitian matrices. The Lie group $U(n)$ acts on $u(n)$ by conjugation $U(n) \times u(n) \to u(n)$ given by $(g, X) \mapsto gXg^{-1}$. We know that every skew-hermitian matrix is conjugate to a diagonal one, and the space of diagonal skew-hermitian matrices forms a commutative Lie subalgebra of $u(n)$, namely the **Cartan subalgebra** denote by

$$\mathbf{Cartan}(u(n)) := \{\mathrm{Diag}(i\lambda_1, ..., i\lambda_n) \mid (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n\}.$$

Notice that the group of permutations of n objects acts on $\mathbf{Cartan}(u(n))$ by permutation of its diagonal entries. Thus, we say that two diagonal matrices are conjugate if and only if we can obtain one from the other by a permutation of its diagonal entries. Therefore and Ad-invariant polynomial on $u(n)$ is determined by its restriction to the Cartan algebra, i.e., we can regard every Ad-invariant polynomial on $u(n)$ as a polynomial function $P = P(\lambda_1, ..., \lambda_n)$. This polynomial is also invariant under the permutation of its variables and thus can be described as a polynomial in the symmetric quantities

$$c_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

where the variables

$$x_j = \frac{i}{2\pi}(i\lambda_j) = -\frac{\lambda_j}{2\pi}$$

are called **Chern roots**. If we set $D = D(\vec{\lambda}) = \mathrm{Diag}(i\lambda_1, ..., i\lambda_n) \in u(n)$, then

$$\det\left(1 + \frac{it}{2\pi}D\right) = 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

Instead of the elementary sums we can consider

$$s_r = \sum_i x_i^r.$$

The elementary sums can be expressed in terms of the s_r's via the Newton relation

$$\sum_{j=1}^{r} (-1)^j s_{r-j} c_j = 0,$$

in particular $s_1 = c_1$, $s_2 = c_1^2 - 2c_2$, $s_3 = c_1^2 - 3c_1c_2 + 3c_3$. Using again the matrix D we have

$$\sum_{r \geq 0} \frac{s_r}{r!} t^r = \mathrm{tr}\ \exp\left(\frac{it}{2\pi}D\right).$$

There exist a general procedure of constructing symmetric elements in $\mathbb{C}[[\lambda_1, ..., \lambda_n]]$. We consider a formal power series

$$f = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{C}[[x_1, ..., x_n]],\ a_0 = 1.$$

Then if we set $\vec{x} = (x_1, ..., x_n)$ the function the function $G_f(\vec{x}) = f(x_1)\cdots f(x_n) \in \mathbb{C}[[x_1, ..., x_n]]$ is a symmetric power series in $\vec{x}$ with leading coefficient 1. Then if $D = \mathrm{Diag}(i\vec{\lambda})$ we have

$$f\left(\frac{i}{2\pi}D\right) = \mathrm{Diag}(f(x_1), ..., f(x_n)),$$

so $f(\vec{x}) = \det\ f\left(\frac{i}{2\pi}D\right)$. We thus get an element $G_f \in \mathbb{C}[[u(n)]]^{U(n)}$ defined by

$$G_f(X) := \det\ f\left(\frac{i}{2\pi}X\right).$$

It is called the f-**genus** or the genus associated to f.

Example 4.1. When $f(x) = 1 + x$ we obtain the **Chern polynomial**. For $c \in \mathbb{C}[[u(n)^*]]^{U(n)}$ and $X \in u(n)$ we have

$$c(X) = \det\left(\mathrm{Id}_{\mathbb{C}^n} + \frac{i}{2\pi}X\right).$$

Now the **Chern character** $\mathrm{ch} \in \mathbb{C}[[u(n)^*]]^{U(n)}$,

$$\mathrm{ch}(X) = \mathrm{tr}\ \exp\left(\frac{i}{2\pi}X\right).$$

Using the Newton formula

$$\mathrm{ch} = n + c_1 + \frac{1}{2!}(c_1^2 - 2c_2) + \frac{1}{3!}(c_1 - 3c_1c_2 + 3c_3) + \cdots .$$

Consider now a rank n complex vector bundle $E \to M$ on a smooth manifold M equipped with a hermitian metric h. For each connection $\nabla \in \mathrm{Conn}(E)$, we can regard the curvature $F(\nabla)$ as a $n \times n$ matrix with entries even degree forms on M. For any $f = 1 + a_1 x + \cdots \in \mathbb{C}[[x]]$, by the Chern-Weil Theorem 4.4 we have a closed form with cohomological class is independent of the metric h and the connection ∇

$$G_f(\nabla) = \det(F(\nabla)) \in \Omega^{\mathrm{even}}(M). \tag{4.16}$$

Example 4.2. When $f(x) = 1 + x$ we obtain a non-homogeneous even degree form

$$c(\nabla) := c(F(\nabla)) = \det\left(\mathrm{Id}_E + \frac{i}{2\pi}F(\nabla)\right) \in \Omega^{\mathrm{even}}(M).$$

therefore $c(\nabla)$ is a topological invariant of E, and we recover the total Chern class of E in the case in which M is a smooth manifold. It has a decomposition into homogeneous components

$$c(E) = 1 + c_1(E) + \cdots + c_n(E),\ c_k(E) \in H^{2k}(M, \mathbb{R}).$$

We will refer to $c_k(E)$ as the k-th Chern class. Also we recover the Chern character of E as the cohomology class $\mathrm{ch}(E)$ carried by the form

$$\mathrm{ch}(\nabla) = \mathrm{tr}\,\exp\left(\frac{i}{2\pi}F(\nabla)\right) = \mathrm{rank}E + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots\,. \tag{4.17}$$

Due to the naturality of the Chern-Weil construction 4.4 we deduce that for every smooth map $f : M \to N$ and every complex vector bundle $E \to N$ we have

$$c(f^*E) = f^*c(E).$$

Chapter 5

Multiplicative sequences and genera

To define the arrow between cohomology and the integer numbers $\mathbb{Z}$ shown in the diagram 1.1, in this chapter we define multiplicative sequences and genera in order to establish the Todd class and the $\hat{A}$ class, which will appear as a "correction" term for the commutativity of right side of the diagram 1.1. Finally we will discuss Pontryagin numbers which allow us to discuss one of the main instance of the Atiyah-Singer index theorem, namely the Hirzebruch signature theorem.

1 Pontryagin numbers

Now we developed the facts that we need in order to state Hirzebruch's signature theorem.

First we recall some basic definitions in differential topology. A **manifold with boundary** is a topological pair $(M, \partial M)$ satisfying the following conditions

- The set $\mathrm{int}(M) := M \backslash \partial M$ is a topological manifold. Its dimension is called the **dimension of the manifold with boundary**.
- The set ∂M is closed in M and it is called the **boundary**. It is a manifold of dimension $\dim \partial M = \dim \mathrm{int}(M) - 1$.
- There exists an open neighborhood N of ∂M in M and a homeomorphism

  $$\Psi : (-1, 0] \times \partial M \to N$$

  such that $\Psi(\{0\} \times \partial M)$. Such a neighborhood is called a **neck** of the boundary.

Let $(M, \partial M)$ be a manifold with boundary with neck N of ∂M, we can form the noncompact manifold $\widetilde{M}$, called a **neck extension** of the manifold with boundary, by attaching the cylinder $C = (-1, 1) \times \partial M$ to the neck along the portion $(-1, 0] \times \partial M$.

Definition 5.1. A **partition** is a finite weakly decreasing sequence I of positive integers

$$I := \{i_1 \geq i_2 \geq \cdots \geq i_l \geq 1\}.$$

We say that l is the **length** of the partition and $w(I) := i_1 + \cdots + i_l$ is the **weight** of the partition. We denote by Part the set of all partitions, by Part(k) the set of partitions of weight k, and we denote by $p(k)$ its cardinality.

Notice that given any finite non-necessarily monotonic sequence of positive integers $(i_1, ..., i_l)$, we can associate a canonical partition $(i_1, ..., i_l)_{\searrow}$ obtained by rearranging the terms of the sequence in decreasing order. Thus we can define the operation $* : \mathrm{Part} \times \mathrm{Part} \to \mathrm{Part}$ given by

$$(i_1, ..., i_l) * (j_1, ..., j_m) = (i_1, ..., i_l, j_1, ..., j_m)_{\searrow}.$$

Notice that $w(I * J) = w(I) + w(J)$, for any $I, J \in \mathrm{Part}$.

For every smooth manifold M we set

$$\overline{p}_k(M) := \overline{p}_k(TM), \quad \overline{\mathcal{P}}_M(t) := \overline{\mathcal{P}}_{TM}(t).$$

Definition 5.2. The **Kronecker pairing** is a bilinear pairing $\langle -, - \rangle : H^n(X) \times H_n(X) \to \mathbb{Z}$ given by

$$([\varphi], [\alpha]) \to \langle [\varphi], [\alpha] \rangle := \varphi(\alpha).$$

Definition 5.3. For any smooth, compact oriented manifold M of dimension m and any partition $I = (i_1, ..., i_l) \in \mathrm{Part}(k)$ we define the **Pontryagin number** of M as the integers

$$P_I(M) := \begin{cases} 0 & m \neq 4w(I), \\ \langle \overline{p}_{i_1}(M) \cdots \overline{p}_{i_l}(M), [M] \rangle & m = 4w(I), \end{cases}$$

where $\langle -, - \rangle : H^m(M) \times H_m(M) \to \mathbb{Z}$ denotes the Kronecker pairing and $[M] \in H_m(M)$ denotes the orientation class.

Observe that if the dimension of M is not divisible by 4 then the Pontryagin numbers are trivial.

Proposition 5.1. If the smooth compact oriented manifold M is the boundary of a smooth compact manifold with boundary $\widehat{M}$, then all the Pontryagin numbers of M are trivial.

Proof. We assume that M has dimension $4k$. Choose an orientation class $[\widehat{M}] \in H_{4k+1}(\widehat{M}, \partial\widehat{M})$ that induces the orientation $[M]$ on M, i.e., $\partial[\widehat{M}] = [M]$. Let $\widetilde{M}$ be a neck extension of $\widehat{M}$. Observe that the normal bundle $T_M\widetilde{M}$ is a trivial real line bundles so that $(T\widetilde{M})_M \cong T_M\widetilde{M} \oplus TM$. By the stability of the reduced Pontryagin classes we deduce that $\overline{p}_i(M) = j^* p_i(\widetilde{M})$ for all $i > 0$, where $j : M \to \widetilde{M}$ denotes the natural inclusion. If $I = (i_1, ..., i_l) \in \mathrm{Part}(k)$ then

$$\begin{aligned} P_I(M) = \langle \overline{p}_{i_1}(M) \cdots \overline{p}_{i_l}(M), [M] \rangle &= \langle \overline{p}_{i_1}(M) \cdots \overline{p}_{i_l}(M), \partial[\widehat{M}] \rangle \\ &= \langle j^*(\overline{p}_{i_1}(M) \cdots \overline{p}_{i_l}(M)), \partial[\widehat{M}] \rangle \\ &= \langle \partial^\dagger j^*(\overline{p}_{i_1}(M) \cdots \overline{p}_{i_l}(M)), [\widehat{M}] \rangle, \end{aligned}$$

where $\partial^\dagger : H^{4k}(M) \to H^{4k+1}(\widehat{M}, \partial\widehat{M})$ is the connecting morphism in the long exact sequence of the pair $(\widehat{M}, \partial\widehat{M})$. Now observe that $\partial^\dagger j^* = 0$ due to the exactness of the sequence

$$H^{4k}(\widehat{M}) \xrightarrow{j^*} H^{4k}(M) \xrightarrow{\partial^\dagger} H^{4k+1}(\widehat{M}, \partial\widehat{M})).$$

□

This proposition shows that we can regard the Pontryagin numbers as morphism of groups

$$P_I : \Omega^+_{4k} \to \mathbb{Z}$$

with $I \in \mathrm{Part}(k)$. Note that by definition P_I vanishes on the torsion elements of Ω^+_{4k} an thus P_I is uniquely determined by the induced morphism

$$P_I : \Omega^+_{4k} \otimes \mathbb{Q} \to \mathbb{Q}.$$

The principal objective in this section is to show that the Pontryagin numbers $(P_I)_{I \in \mathrm{Part}(k)}$ form a linearly independent subfamily of the $\mathbb{Q}$-vector space

$$\Xi_k := \mathrm{Hom}_{\mathbb{Q}}(\Omega^+_{4k} \otimes \mathbb{Q}, \mathbb{Q}).$$

For this purpose it is convenient to think of the reduced Pontryagin polynomial of a manifold M of dimension k as a product of elementary polynomials

$$\overline{\mathcal{P}}_M(t) = \sum_{i=0}^{k} \overline{p}_i(M) z^i = \prod_{i}^{k} (1 + r_i z),\ z = t^2.$$

Thus we can think of the Pontryagin classes as elementary symmetric polynomials in the variables r_i, where we denoted by σ_j the j-th elementary symmetric polynomial in the variables $r_i,\ 1 \le i \le k$. Thus any other polynomial in these variables is then a polynomial combination of the elementary ones. For any partition $I = (i_1, i_2, ..., i_l) \in \mathrm{Part}$ we define the symmetric polynomial

$$s_I(\sigma_1, ..., \sigma_k) = \sum r_1^{i_1} \cdots r_l^{i_l} \in \mathbb{Z}[\sigma_1, ..., \sigma_k],$$

where the sum is over all monomials equivalent to $r_1^{i_1} \cdots r_l^{i_l}$, where two monomials are declared equivalent if one can be obtained from the other by a permutation of the variables $r_1, ..., r_k$.

Example 5.1. For example we have the following polynomials

$$s_{2,1} = \sum_{i \neq j} r_i^2 r_j = \sigma_1 \sigma_2 - 3\sigma_3, \quad s_{1,1} = \sum_{i<j} r_i r_j = \sigma_2.$$

From this construction, we have the immediate result:

Lemma 5.1. The polynomials $(s_I)_{I \in \mathrm{Part}(k)}$ form a basis of the free abelian group space of symmetric polynomials of degree $d \le k$ in the variables $r_1, ..., r_k$ with integral coefficients.

We denote by $\mathcal{S}_k$ the ring of symmetric polynomials with integral coefficients, and by $\mathcal{S}_k^d$ the subgroup consisting of homogeneous polynomials of degree d. For any smooth manifold M of dimension $4k$ and any partition I of weight k we set

$$s_I(M) := s_I(\overline{p}_1(M), ..., \overline{p}_k(M)), \quad S_I := \langle s_I(M), \mu_M \rangle,$$

where $\mu_M \in H_{4k}(M)$ denotes the orientation class of M. Therefore for every $I \in \mathrm{Part}(k)$ we obtain morphisms of groups

$$S_I : \Omega^+_{4k} \to \mathbb{Z}$$

and thus $\mathbb{Q}$-linear maps $S_I : \Omega^+_{4k} \otimes \mathbb{Q} \to \mathbb{Q}$. Thus by the Lemma 5.1, the linear subspace of Ξ_k spanned by $(S_I)_{I \in \mathrm{Part}(k)}$ is equal to the linear subspace spanned by the linear maps $P_I : \Omega^+_{4k} \otimes \mathbb{Q} \to \mathbb{Q}$. Therefore to show that the Pontryagin numbers $(P_I)_{I \in \mathrm{Part}(k)}$ form a linearly independent subfamily of the $\mathbb{Q}$-vector space Ξ_k it is suffices to show that the functionals S_I are linearly independent. In fact we prove something stronger; for every $I = (i_1, ..., i_l) \in \mathrm{Part}(k)$ we set

$$\mathbb{CP}^{2I} := \mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_l}$$

and we denote by $\widehat{\Omega}_4 k(\mathbb{Q})$ the subspace of Ω_{4k}^+ spanned by the cobordism classes of $\mathbb{CP}^{2I}$, with $I \in \mathrm{Part}(k)$. We will show that the vector space $\widehat{\Omega}_{4k}(\mathbb{Q})$ has dimension $p(k)$ and that the functionals S_I, with $I \in \mathrm{Part}(k)$, form a basis of

$$\widehat{\Xi}_k := \mathrm{Hom}_{\mathbb{Q}}(\widehat{\Omega}_{4k}(\mathbb{Q}), \mathbb{Q}).$$

Let $\sigma_i, \sigma_i', \sigma_i''$ be the i-th elementary symmetric polynomials in the respective variables $r_1, ..., r_n$, $r_1', ..., r_n'$ and $r_1'', ..., r_n''$ with r_j'' equal to $r_j,\ j \leq n$ and equal to $r_{j-n}',\ j > n$. From the identities

$$\sum_{i=0}^{n} \sigma_i t^i = \prod_{i=0}^{n}(1 + r_i t), \quad \sum_{j=0}^{n} \sigma_j' t^j = \prod_{j=0}^{n}(1 + r_j' t)$$

we deduce

$$\sigma_l'' = \sum_{i+j=l} \sigma_i \sigma_j'. \tag{5.1}$$

As any monomial in the variables r_i'' can be written in a unique way as a product between a monomial in the variables r_i and a monomials in the variables r_j', we have the following identity

$$s_I(\sigma_I'') = \sum_{J*K=I} s_J(\sigma_i) s_K(\sigma' j), \tag{5.2}$$

for any $I \in \mathrm{Part}(2n)$.

Lemma 5.2. Suppose M_0 and M_1 are smooth, compact oriented manifolds of dimensions $4k_0$ and $4k_1$ respectively. Then for every $I \in \mathrm{Part}(k_0 + k_1)$ we have

$$S_I(M_0 \times M_1) = \sum_{(I_0, I_1) \in \mathrm{Part}(k_0) \times \mathrm{Part} k_1, I_0 * I_1 = I} S_{I_0}(M_0) \cdot S_{I_1}(M_1) \tag{5.3}$$

Proof. Let $n = k_0 + k_1$. We denote by π_i the projection $M_0 \times M_1 \to M_i$, with $i = 0, 1$, and we set $p_j'' := \overline{p}_j(M_0 \times M_1)$, $p_i := \pi_0^* \overline{p}_i(M_0)$, $p_i' := \pi_1^* \overline{p}_i(M_1)$, then $p_l'' = \sum_{i+j=l} p_i p_j'$, and since the variables σ_i, σ_j' are algebraically independent we deduce from Equation 5.2 that

$$s_I(M_0 \times M_1) = s_I(p_i'') = \sum_{J*K=I} s_J(p_j) s_K(p_k')$$

so

$$\begin{aligned} S_I(M_0 \times M_1) &= \sum_{J*K=I} \langle s_J(p_j) s_K(p_k'), \mu_{M_0} \times \mu_{M_1} \rangle \\ &= \sum_{J*K=I} \langle s_J(p_j), \mu_{M_0} \rangle \cdot \langle s_K(p_k'), \mu_{M_1} \rangle \\ &= \sum_{J*K=I} S_J(M_0) \cdot S_J(M_1). \end{aligned}$$

□

Now we prove that the $p(k) \times p(k)$ matrix $[S_I(\mathbb{CP}^J)]_{I,J \in \mathrm{Part}(k)}$ is nonsingular. We define a partial order $\succeq$ on $\mathrm{Part}(k)$ by declaring $I = (i_1 \geq \cdots \geq i_l > 0) \succeq J = (j_1 \geq \cdots \geq j_m > 0)$, with $I, J \in \mathrm{Part}(k)$, if $I = I_1 * \cdots * I_m$, with $I_\alpha \in \mathrm{Part}(j_\alpha)$ for all $1 \leq \alpha \leq m$.

From Equation 5.3 we deduce that $S_I(\mathbb{CP}^{2J}) = 0$ if $J \not\preceq I$. This shows that the matrix $[S_I(\mathbb{CP}^J)]_{I,J\in \mathrm{Part}(k)}$ is upper triangular with respect to the above partial order on Part(k). An inductive application of Equation 5.3 we show that

$$S_{(i_1,\dots,i_l)}(\mathbb{CP}^{2i_1}\times\cdots\times\mathbb{CP}^{2i_l}) = \prod_{\nu=1}^{l} S_{i_\nu}(\mathbb{CP}^{2i_\nu}).$$

Thus the matrix $[S_I(\mathbb{CP}^J)]_{I,J\in \mathrm{Part}(k)}$ is nonsingular if and only if $S_n(\mathbb{CP}^{2n}) \neq 0$, for all n.

Lemma 5.3. For all $n \geq 1$

$$S_n(\mathbb{CP}^{2n}) = 2n+1.$$

Proof. Let σ_i be the i-th elementary symmetric polynomial in the variables $r_1, ..., r_N$, with $N \geq n$. We set $f(t) = \sum_{i=0}^{N} \sigma_i t^i$, with $\sigma_0 = 1$, so that

$$f(z) = \prod_{i=1}^{N}(1 + r_i z).$$

We then have the Newton formula

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{N} \frac{r_i}{1+r_i z} = \sum_{m\geq 0}(-1)^m s_{m+1}(M)\ z^m. \tag{5.4}$$

If we let $f(z) := \sum_{i\geq 0}\overline{p}_i(\mathbb{CP}^{2n})z^i$, then by the Corollary 4.5 we have that

$$f(z) = \sum_{i\geq 0}\overline{p}_i(\mathbb{CP}^{2n})z^i = (1+[H]^2 z)^{2n+1},\ [H]^{2n+1} = 0.$$

Therefore

$$\frac{f'(z)}{f(z)} = (2n+1)[H]^2(1+[H]^2 z)^{-1} = \sum_{m\geq 0}(-1)^m s_{m+1}(M) z^m.$$

We deduce that $s_n(M) = (2n+1)[H]^{2n}$. □

Therefore we have the final result.

Corollary 5.1. The cobordism classes of the manifold $\mathbb{CP}^{2I}$, with $I \in$ Part(k) are linearly independent in the $\mathbb{Q}$-vector space $\Omega^+_{4k}\otimes\mathbb{Q}$.

2 Multiplicative sequences

Let $A^\bullet = \bigoplus_{n\geq 0} A^n$ be a commutative graded $\mathbb{Q}$-algebra with grading $A^n \cdot A^m \subset A^{n+m}$. Then we associate to $A^\bullet$ the ring of formal power series $A^\bullet[[z]]$ with elements of the form $\sum_{n\geq 0} p_n z^n$, where $p_n \in A^n$ has degree n. The ring $A^\bullet[[z]]$ contains a multiplicative semigroup with 1,

$$A^\bullet[[z]]^\dagger := \Big\{ \sum_{n\geq 0} p_n z^n \in A^\bullet[[z]] \mid p_0 = 1\Big\}.$$

We consider the commutative, graded $\mathbb{Q}$-algebra $\mathfrak{B} := \mathbb{Q}[p_1, p_2, ...]$, where the variables p_n have degree n. We can think of $\mathfrak{B}$ as the ring of symmetric polynomials in an indefinite number of variables. Thus an element of $\mathfrak{B}[[z]]$ is of the form

$$K = K((p_n)_{n\geq 1}; z)$$

Since the variables p_n have degree n, we can write the element K as

$$K = \sum_{n\geq 0} K_n(p_1, p_2, ..., p_n)z^n,$$

where $K_n(p_1, ..., p_n)$ is homogeneous of degree n.

Definition 5.4. A **multiplicative sequence** is an element

$$K = 1 + \sum_{n\geq 1} K_n(p_1, p_2, ..., p_n)z^n \in \mathfrak{B}[[z]]^\dagger$$

such that, for any graded $\mathbb{Q}$-algebra $A^\bullet$, the map $\widehat{K} : A^\bullet[[z]]^\dagger \to A^\bullet[[z]]^\dagger$ given by

$$a(z) = 1 + \sum_{n\geq 1} a_n z^n \mapsto \widehat{K}(a(z)) := 1 + \sum_{n\geq 1} K_n(a_1, ..., a_n)z^n,$$

is a morphism of semigroups with 1.

As an immediate consequence of the definition, if K is an multiplicative sequence, then for any commutative $\mathbb{Q}$-algebra R with 1, and any sequences $(p_n)_{n\geq 0}$, $(p'_i)_{i\geq 0}$ and $(p''_j)_{j\geq 0}$ in R such that $p_0 = p'_0 = p''_0 = 1$ and $p_n = \sum_{i+j=n} p'_i p''_j$ we have for all n

$$K_n(p_1, ..., p_n) = \sum_{i+j=n} K_i(p'_1, ..., p'_i)K_j(p''_1, ..., p''_j)$$

Now let $K = 1 + \sum_{n\geq 1} K_n(p_1, ..., p_n)z^n$ be a multiplicative sequence, then for every $n \geq 1$ the polynomial $K_n(\xi, 0, ..., 0) \in \mathbb{Q}[\xi]$ is homogeneous of degree n and thus it has the form

$$K_n(\xi, 0, ..., 0) = k_n\xi^n, \;\; k_n \in \mathbb{Q}.$$

Definition 5.5. We define the power series

$$a_K(\xi) := 1 + \sum_{n\geq 1} k_n\xi^n \in \mathbb{Q}[\xi]$$

called the **symbol** of the multiplicative sequence K.

Remark 5.1. By easy computation we have the identity

$$a_K(p_1 z) = \widehat{K}(1 + p_1 z).$$

Proposition 5.2. A multiplicative sequence K is uniquely determined by its symbol $a_K(\xi)$.

Proof. We consider the graded ring $A^\bullet = \mathbb{Q}[t_1, ..., t_n]$ and we denote by σ_k the k-th elementary symmetric function in the variables $t_1, ..., t_n$, thus we have the identity in $A^\bullet$

$$1 + \sum_{k=1}^{n} \sigma_k z^k = \prod_{k=1}^{n} (1 + t_1 z)$$

therefore

$$1+\sum_{j=1}^{n} K_j(\sigma_1,...,\sigma_j)z^j = \widehat{K}(1+\sum_{j=1}^{n}\sigma_j z^j) = \prod_{j=1}^{n}\widehat{K}(1+t_j z) = \prod_{j=1}^{n} a_K(t_j z).$$

This uniquely determines the polynomial K_j since the polynomials σ_k are algebraically independent. □

Proposition 5.3. For any power series

$$a(\xi) = 1+\sum_{n\geq 1} a_n\xi^n \in \mathbb{Q}[\xi]$$

there exists a multiplicative sequence $K = K^a$ such that $a_K(\xi) = a(\xi)$.

Proof. We consider the product

$$\prod_{j=1}^{n} a(t_j z) = 1+\sum_{j=1}^{n} A_{j,n}(t_1,...,t_n)z^j + \sum_{N>n} A_{N,n}(t_1,...,t_n)z^N.$$

Observe that the polynomials $A_{j,n}$ are homogeneous of degree j and symmetric in the variables $t_1,...,t_n$, so we can express them as polynomials in the elementary symmetric functions,

$$A_{j,n}(t_1,...,t_n) = K_{j,n}(\sigma_1,...,\sigma_j),\ j\leq n. \tag{5.5}$$

Moreover, for $j\leq n<N$ we have

$$K_{j,n}(\sigma_1,...,\sigma_j) = K_{j,N}(\sigma_1,...,\sigma_j).$$

For this reason we will denote by $K_j(\sigma_1,...,\sigma_j)$ any of the polynomials $K_{j,n}$, with $j\leq n$. If instead of elementary symmetric polynomials σ_i we use the polynomials s_I, with $I\in \mathrm{Part}(j)$ introduced in the previous section, then we have

$$K_j = \sum_{I\in \mathrm{Part}(j)} a_I s_I(\sigma_1,...,\sigma_j),\ a_I = a_{i_1,...,i_l} = a_{i_1}\cdots a_{i_l}.$$

Now we define

$$K((p_i)_{i\geq 1}, z) := 1+\sum_{j\geq 1} K_j(p_1,...,p_j)z^j,$$

where K_j are the polynomials defined by the Equation 5.5, or equivalently,

$$K_j(p_1,...,p_j) := \sum_{I\in \mathrm{Part}(n)} a_I s_I(p_1,...,_n).$$

If in some ring $A^\bullet[[z]]$ we have an equality of the form

$$1+\sum_{n\geq 1} p''_n z^n = (1+\sum_{i\geq 1} p_i z^i)\cdot(1+\sum_{j\geq 1} p'_j z^j),$$

then $p''_n = \sum_{i+j=n} p_i p'_j$, and from the Equation 5.2 we deduce

$$s_I(p'') = \sum_{J*K=I} s_J(p)\cdot s_K(p').$$

Coupled with the identity $a_{I*J} = a_I a_J$ for all $I, J \in \mathrm{Part}$, this implies

$$K_n(p''_\cdot) = \sum_{i+j=n} K_i(p_\cdot)K_j(p'_\cdot)$$

so that

$$\widehat{K}(1 + \sum_{n\geq 1} p''_n z^n) = \widehat{K}(1 + \sum_{i\geq 1} p_i z^i) \cdot \widehat{K}(1 + \sum_{j\geq 1} p'_j z^j).$$

□

Now some examples.

Example 5.2. (a) The set Part(1) consist of single partition $I = 1$ and we have

$$s_1(t_1, ..., t_n) = t_1 + \cdots + t_n = \sigma_1. \tag{5.6}$$

(b) The set Part(2) consist of two elements, namely, 2 and $(1, 1)$. Then

$$s_2 = \sum t_i^2 = \sigma_1^2 - 2\sigma_2, \quad s_{1,1} = \sum_{i<j} t_i t_j = \sigma_2.$$

(c) The set Part(3) consist of three partitions, namely, $3, (2, 1)$, and $(1, 1, 1)$ and we have

$$s_{1,1,1} = \sigma_3, \quad s_3 = \sum_i t_i^3$$

and

$$s_{2,1} = \sum_{i\neq j} t_i^2 t_j = \sum_j t_j \sum_{i\neq j} t_i^2 = \sum_j t_j(s_2 - t_j^3) = s_2\sigma_1 - s_3 = \sigma_1^3 - 2\sigma_1\sigma_2 - s_3.$$

To express s_3 in terms of the elementary symmetric functions we use Newton's formula 5.4

$$\sum_j^n j\sigma_j z^{j-1} = \Big(1 + \sum_{j=1}^n \sigma_j z^j\Big) \cdot \Big(\sum_{m\geq 0} (-1)^m s_{m+1} z^m\Big).$$

we deduce $\sigma_1 = s_1, 2\sigma_2 = -s_2 + \sigma_1 s_1, \sigma_3 = s_3 - s_1 s_2 + \sigma_2 s_1$, so that

$$s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \tag{5.7}$$

and

$$s_{2,1} = \sigma_1\sigma_2 - 3\sigma_3. \tag{5.8}$$

Example 5.3. (a) Let $a(z) = 1 + z$, then the multiplicative sequence with symbol a is $K = 1 + p_1 z + p_2 z^2 + \cdots$. Indeed,

$$1 + \sum_{j=1}^n K_j(\sigma_1, ..., \sigma_j)z^j = \prod_{j=1}^n (1 + t_j z) = 1 + \sum_{j=1}^n \sigma_j z^j.$$

(b) Consider the series

$$\ell(\xi) := \frac{\sqrt{\xi}}{\tanh\sqrt{\xi}} = 1 + \sum_{k=1} \frac{2^{2k}}{(2k)!} b_{2k}\xi^k,$$

where b_n denote the Bernoulli numbers, that are defined by the recurrence relation

$$\sum_{j=1}^{n} \binom{n}{j} b_{n-j} = 0$$

where $b_0 = 1, b_1 = -\dfrac{1}{2}$. We have $b_{2j+1} = 0$ for all $j \geq 1$. Equivalently

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n\geq 0} b_n \frac{t^n}{n!}.$$

Therefore

$$\ell(z) = 1 + \frac{1}{3}z - \frac{1}{45}z^2 + \frac{2}{945}z^3 - \frac{1}{4725}z^4 + \frac{2}{93555}z^5 + O(z^6).$$

we denote by ℓ_n the coefficient of z^n in the above expansion. If we denote by $\mathbf{L}$ the multiplicative sequence with symbol $\ell(\xi)$, then we have

$$\mathbf{L}_n = \sum_{I\in \mathrm{Part}(n)} \ell_I s_I(p_1, ..., p_n).$$

Here is the explicit description of $\mathbf{L}_n$ for $n \leq 3$. We have by the Equation 5.6

$$\mathbf{L}_1 = \ell_1 s_1(p_1) = \frac{1}{3}p_1,$$

$$\mathbf{L}_2 = \ell_2 s_2 + \ell_1^2 s_{1,1} = -\frac{1}{45}(p_1^2 - 2p_2) + \frac{1}{9}p_2 = \frac{7}{45}p_2 - \frac{1}{45}p_1^2.$$

and finally by Equations 5.7, 5.8, we have

$$\begin{aligned}\mathbf{L}_3 = \ell_1^3 s_{1,1,1} + \ell_2\ell_1 s_{2,1} + \ell_3 s_3 &= \frac{1}{27}p_2 - \frac{1}{135}(p_1p_2 - 3p_3) + \frac{2}{945}(p_1^3 - 3p_1p_2 + 3p_3) \\ &= \frac{62}{945}p_3 - \frac{13}{945}p_2p_1 + \frac{2}{945}p_1^3.\end{aligned}$$

(c) Let $a(\xi) = 1 + \sum_{n\geq 1} a_n\xi^n \in \mathbb{Q}[\xi]$ be a formal power series and let $\{K_j^a\}$ be the multiplicative sequence associated to this formal power series. To each complex vector bundle E over a space X, we associate the **total K-class**

$$K_{\mathbb{C}}(E) := K(c(E)) \in H^{2*}(X; \mathbb{Q}),$$

where $c(E)$ is the total Chern class of the vector bundle E. Since $c(E \oplus E') = c(E)c(E')$, this class has the property that for any two complex vector bundles E and E' over X,

$$K_{\mathbb{C}}(E \oplus E') = K_{\mathbb{C}}(E)K_{\mathbb{C}}(E').$$

If we decompose by the splitting principle the bundle E as $E = \ell_1 \oplus \ell_2 \oplus \cdots \oplus \ell_n$, then

$$c(E) = \prod_{j=1}^{n}(1 + x_j)$$

where $x_j = c_1(\ell_j)$ for $j = 1, ..., n$. In particular, we find that

$$c_j(E) = \sigma_j(x_1, ..., x_n),\ j = 1, ..., n$$

where σ_j denotes the jth elementary symmetric function of $x_1, ..., x_n$. Thus we have that

$$K_{\mathbb{C}}(E) = a(x_1) \cdots a(x_n),$$

where $x_j = c_1(\ell_j)$ for each j.

In particular, associated to the formal power series

$$\mathrm{td}(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots$$

is the multiplicative sequence $\{\mathrm{Td}_m\}$ called the **Todd sequence**. The total Todd class is denoted by $\mathbf{Td}_{\mathbb{C}}$. Its first few terms are:

$$\mathbf{Td}_1(c_1) = \frac{1}{2}c_1$$

$$\mathbf{Td}_2(c_1, c_2) = \frac{1}{12}(c_2 + c_1^2)$$

$$\mathbf{Td}_3(c_1, c_2, c_3) = \frac{1}{24}c_2c_1$$

$$\mathbf{Td}_4(c_1, c_2, c_3, c_4) = \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4)$$

$$\mathbf{Td}_5(c_1, c_2, c_3, c_4, c_5) = \frac{1}{1440}(-c_4c_1 + c_3c_1^2 + 3c_2^2c_1 - c_2c_1^3).$$

By the Equation 4.16 the Todd sequence has the form

$$\mathbf{Td}(\nabla) := \det\left(\frac{\frac{i}{2\pi}F(\nabla)}{\exp\left(\frac{i}{2\pi}F(\nabla)\right) - \mathrm{Id}_E}\right). \tag{5.9}$$

(d) Let $a(\xi) = 1 + \sum_{n \geq 1} a_n\xi^n \in \mathbb{Q}[\xi]$ be a formal power series and let $\{K_j^a\}$ be the multiplicative sequence associated to this formal power series. To each real vector bundle E over a space X, we associate the **total K-class**

$$K(E) := K(p(E)) \in H^{4*}(X; \mathbb{Q}),$$

where $p(E)$ is the total Pontrjagin class of the vector bundle E. Since $p(E \oplus E') = p(E)p(E')$, this class has the property that for any two real vector bundles E and E' over X,

$$K(E \oplus E') = K(E)K(E').$$

Assume E is oriented and of dimension $2n$. If we decompose by the splitting principle the bundle $E \otimes \mathbb{C}$ as $\ell_1 \oplus \ell_2 \oplus \cdots \oplus \ell_n$, then

$$c(E \otimes \mathbb{C}) = \prod_{j=1}^{n} c(\ell_j)c(\bar{\ell}_j) = \prod_{j=1}^{n}(1 - x_j^2),$$

and we find that

$$p(E) = \prod_{j=1}^{n}(1 + x_j^2),$$

where $x_j = c_1(\ell_j)$. In particular, we have

$$p_j(E) = \sigma_j(x_1^2, ..., x_n^2) \tag{5.10}$$

for each j. Thus

$$K(E) = a(x_1^2) \cdots a(x_n^2), \tag{5.11}$$

where $x_j = c_1(\ell_j)$ for each j.

In particular, associated to the formal power series

$$\hat{a}(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{1}{24}x + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^2 + \cdots$$

is a multiplicative sequence $\{\hat{A}_m\}$ called the **$\hat{A}$-sequence**. The first few terms of the sequence are

$$\hat{A}_1(p_1) = -\frac{1}{24}p_1$$

$$\hat{A}_2(p_1, p_2) = \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2)$$

$$\hat{A}_3(p_1, p_2, p_3) = -\frac{1}{2^{10} \cdot 3^3 \cdot 5 \cdot 7}(16p_3 - 44p_2p_1 + 31p_1^3).$$

Given a real bundle E, the total $\hat{A}$-class of E is the sum

$$\hat{\mathbf{A}}(E) = 1 + \hat{A}_1(p_1(E)) + \hat{A}_2(p_1(E), p_2(E)) + \cdots .$$

If we write $E \otimes \mathbb{C} = \ell_1 \oplus \ell_2 \oplus \cdots \oplus \ell_n$ as above, then by the Equation 5.11

$$\hat{\mathbf{A}}(E) = \prod_{j=1}^{n} \frac{x_j/2}{\sinh(x_j/2)},$$

where of course $p_j(E) = \sigma_j(x_1^2, ..., x_n^2)$.

The Todd class and the $\hat{A}$-class are intimately related.

Proposition 5.4. For any oriented real vector bundle E it is true that

$$\mathbf{Td}_{\mathbb{C}}(E \otimes \mathbb{C}) = \hat{\mathbf{A}}(E)^2.$$

Proof. We may assume dim$E$ is even and we consider a formal splitting $E \otimes \mathbb{C} = \ell_1 \oplus \overline{\ell}_1 \oplus \cdots \oplus \ell_n \oplus \overline{\ell}_n$, since in the case in that the dimension of E is odd, we can add a trivial line bundle onto the previous decomposition. Then by definition

$$\mathbf{Td}_{\mathbb{C}}(E \otimes \mathbb{C}) = \prod_{j=1}^{n} \frac{x_j}{1 - e^{-x_j}} \frac{(-x_j)}{1 - e^{x_j}}$$

where $x_j = c_1(\ell_j)$. Multiplying by $e^{x_j/2}e^{-x_j/2}$ in the denominator gives

$$\begin{aligned} \mathbf{Td}_{\mathbb{C}}(E \otimes \mathbb{C}) &= \prod_{j=1}^{n} \left[\frac{x_j}{e^{x_j/2} - e^{-x_j/2}} \right]^2 \\ &= \prod_{j=1}^{n} \left[\frac{x_j/2}{\mathrm{sinh(x_j/2)}} \right]^2 \\ &= [\hat{\mathbf{A}}(E)]^2. \end{aligned}$$

□

3 Genera

Definition 5.6. A **genus** is a ring morphism $\gamma : \Omega^+_\bullet \to \mathbb{Q}$, where $\Omega^+_\bullet$ denotes the oriented cobordism ring.

Definition 5.7. For any formal power series $u = u_0 + u_1 + \cdots + u_n z^n + \cdots$, we denote by $\mathbf{j}^n_z u := \sum_{k \leq n} u_k z^k$ its n-th **jet** and by $[z^n]u$ the coefficient of z^n in the expansion of u, $[z]^n u := u_n$.

Proposition 5.5. Let M be a compact, oriented manifold of dimension m. Any multiplicative sequence $K = (K_n)_{n \geq 1}$ defines a genus $\gamma = \gamma_K$ such that

$$\gamma_K(M) := \begin{cases} 0 & m \neq 4s, \\ \langle K(\overline{p}_{i_1}(M) \cdots \overline{p}_{i_l}(M)), [M] \rangle & m = 4s, \end{cases}$$

for some s and $[M]$ denotes the orientation class of M. Moreover, if $a(\xi)$ is the symbol of K, then $\gamma(\mathbb{CP}^{2n})$ are the coefficients of ξ^{2n} in the formal power series $a(\xi^2)^{2n+1}$.

Proof. Suppose M_0, M_1 are two compact, oriented smooth manifolds. We set $M = M_0 \times M_1$ and we denote by π_i the natural projection $M \to M_i$, with $i = 0, 1$. We set

$$\overline{\mathcal{P}}_i(z) = 1 + \sum_{k \geq 1} \overline{p}_k(M_i) z^k, \ \overline{\mathcal{P}}(z) = 1 + \sum_{k \geq 1} \overline{p}_k(M) z^k, \ i = 0, 1.$$

Then $TM \cong \pi_0^*(TM_0) \oplus \pi_1^*(TM_1)$. From the product formula 4.2 we deduce

$$\overline{\mathcal{P}}(z) = \pi_0^* \overline{\mathcal{P}}_0(z) \cdot \pi_1^* \overline{\mathcal{P}}_1(z).$$

Hence

$$\begin{aligned} \langle K(\overline{\mathcal{P}}(z)), [M] \rangle = \langle K(\overline{\mathcal{P}}(z)), [M_0] \times [M_1] \rangle &= \langle \pi_0^* K(\overline{\mathcal{P}}_0(z)) \cdot \pi_1^* K(\overline{\mathcal{P}}_1(z)), [M_0] \times [M_1] \rangle \\ &= \langle K(\overline{\mathcal{P}}_0(z)), [M_0] \rangle \cdot \langle K(\overline{\mathcal{P}}_1(z)), [M_1] \rangle. \end{aligned}$$

If we set $z = 1$ we deduce $\gamma_k(M_0 \times M_1) = \gamma_K(M_0) \cdot \gamma_K(M_1)$. The linearity of γ is obvious. Notice that

$$\gamma_K(\mathbb{CP}^{2n}) = \langle K_n(p_1(\mathbb{CP}^{2n}), ..., p_n(\mathbb{CP}^{2n})), [\mathbb{CP}^{2n})]\rangle.$$

If we set

$$\overline{\mathcal{P}}_n(z) = 1 + \sum_{k\geq 1} \overline{p}_k(\mathbb{CP}^{2n})z^k$$

then

$$K_n = [z]^n \widehat{K}(\overline{\mathcal{P}}_n(z)) = [z]^n \widehat{K}(\mathbf{j}_z^n \overline{\mathcal{P}}_n(z)).$$

On the other hand we deduce from the Corollary 4.5 that

$$\overline{\mathcal{P}}_n(z) = \mathbf{j}_z^n (1 + [H]^2 z)^{2n+1}$$

so that

$$\begin{aligned} K_n = [z]^n \widehat{K}(\mathbf{j}_z^n (1 + [H]^2 z)^{2n+1}) = [z]^n \widehat{K}((1 + [H]^2 z)^{2n+1}) &= [z]^n (\widehat{K}(1 + [H]^2 z))^{2n+1} \\ &= ([z]^n a(z)^{2n+1})[H]^{2n} \\ &= [\xi^{2n}](a(\xi^2))^{2n+1}[H]^{2n}. \end{aligned}$$

Then

$$\gamma_K(\mathbb{CP}^{2n}) = [\xi^{2n}](a(\xi^2))^{2n+1}\langle [H]^{2n}, [\mathbb{CP}^{2n}]\rangle.$$

□

A genus γ extends to a ring morphism $\gamma : \Omega_\bullet^+ \otimes \mathbb{Q} \to \mathbb{Q}$. Remember that the subring $\widehat{\Omega}_\bullet^+$ of $\Omega_\bullet^+$ is generated by the cobordism classes $\mathbb{CP}^{2n}$, for all $n \geq 1$.

Definition 5.8. To any ring morphism $\gamma : \widehat{\Omega}_\bullet^+ \otimes \mathbb{Q} \to \mathbb{Q}$ we form the generating series

$$r^\gamma(t) := 1 + \sum_{n\geq 1} r_{2n}^\gamma t^{2n}, \quad r_{2n}^\gamma := \gamma(\mathbb{CP}^{2n}).$$

We also set

$$\mathbf{R}^\gamma := \int_0^t r^\gamma = \sum_{n\geq 0} \frac{r_{2n}^\gamma}{2n+1} t^{2n+1}, \quad \gamma_0 = 1.$$

Proposition 5.6. There exist a bijection between formal power series

$$r(t) := 1 + \sum_{n\geq 1} r_{2n} t^{2n} \in \mathbb{Q}[[z]]^\dagger$$

and ring morphisms $\widehat{\Omega}_\bullet^+ \otimes \mathbb{Q} \to \mathbb{Q}$.

Proof. By the Corollary 5.1 the collection $\{\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_l} \mid (i_1, ..., i_l) \in \mathrm{Part}(n)\}$ forms a rational basis of $\widehat{\Omega}_{4n}^+ \otimes \mathbb{Q}$, therefore the sequence r_{2n}^γ completely determines the restriction to $\widehat{\Omega}_\bullet^+ \otimes \mathbb{Q}$ of the genus γ. Conversely, any sequence of rational numbers $(r_{2n})_{n\geq 1}$ determines a unique morphism $\gamma : \widehat{\Omega}_\bullet^+ \otimes \mathbb{Q} \to \mathbb{Q}$ such that

$$\gamma(\mathbb{CP}^{2n}) = r_{2n}$$

for any $n \geq 1$. □

By the Proposition 5.3 to every power series $a \in \mathbb{Q}[[\xi]]^\dagger$ we can associate a multiplicative sequence $K = K^a$ whose symbol is a. We denote by γ^a the genus determined by K^a, and we set

$$r^a(t) := 1 + \sum_{n\geq 0} \gamma^a(\mathbb{CP}^{2n})t^{2n} \in \mathbb{Q}[[t^2]]^\dagger.$$

Thus we obtain a correspondence $\mathbb{Q}[[\xi]]^\dagger \to \mathbb{Q}[[t^2]]^\dagger$ given by $a(\xi) \mapsto r^a(t)$. We would like to give a description of the inverse of this correspondence.

Definition 5.9. For any formal power series $u = \sum_{n\geq 0} u_n t^n \in \mathbb{R}[[t]]$ we set

$$\int_0^t u := \sum_{n\geq 0} \frac{u_n}{n+1} t^{n+1}.$$

We define the composition of two formal power series $u \circ v$, with $[t^0]v = 0$,

$$(u \circ v)(t) = u_0 + u_1\left(\sum_{n\geq 1} v_n t^n\right) + u_2\left(\sum_{n\geq 1} v_n t^n\right)^2 + \cdots .$$

A formal power series u is called **formally invertible** if it has an expansion of the form

$$u(t) = u_1 t + u_2 t^2 + \cdots , \quad u_1 \neq 0.$$

In this case, a version of the implicit function theorem for formal series implies that we can find a formal series $v = v_1 t + v_2 t^2 + \cdots$, called **formal inverse**, such that $u \circ v = t = v \circ u$. We denote it by $u^{[-1]}$. The coefficients of v are found via **Lagrange inversion formula**. We write u as a ratio $u = t/q(t)$ and then we have

$$[t^n]v = \frac{1}{n}[t^{n-1}]q(t)^n.$$

Proposition 5.7. For any morphism $\gamma : \widehat{\Omega}^+_\bullet \otimes \mathbb{Q} \to \mathbb{Q}$ there exists a unique multiplicative sequence $K = K^\gamma$ such that $\gamma = \gamma_K$ on $\Omega^+_\bullet \otimes \mathbb{Q}$. More precisely, K^γ is the unique multiplicative sequence whose symbol a_γ satisfies the equation

$$\mathbf{R}^\gamma = \left(\frac{t}{a_\gamma(t^2)}\right)^{[-1]}.$$

Equivalently,

$$a_\gamma(t^2) = \frac{t}{(\mathbf{R}^\gamma)^{[-1]}}.$$

Proof. Observe first that $\mathbf{R}^\gamma$ is formally invertible because $[t^0]\mathbf{R}^\gamma = 0$ and $[t]\mathbf{R}^\gamma = 1$. From the Proposition 5.5 we deduce $r^\gamma_{2n} = [t^{2n}]a(t^2)^{2n+1}$, then

$$[t^{2n+1}]\mathbf{R}^\gamma = \frac{1}{2n+1}[t^{2n}]a(t^2)^{2n+1}.$$

Observing that

$$[t^{2n}]\mathbf{R}^\gamma = 0 = \frac{1}{2n}[t^{2n-1}]a(t^2)^{2n}$$

we deduce that

$$[t^n]\mathbf{R}^\gamma = \frac{1}{n}[t^{n-1}]a(t^2)^n$$

for all $n \geq 1$. The Lagrange inversion formula implies that $\mathbf{R}^\gamma$ must be the formal inverse $t/a(t^2)$. $\square$

We have thus produced several bijections

- $\mathbb{Q}[[\xi]]^\dagger \to$ Multiplicative sequences, given by $a \mapsto K_a$; inverse the symbol map $K \mapsto a_K$.
- $\mathrm{Hom}(\widehat{\Omega}^+_\bullet \otimes \mathbb{Q}, \mathbb{Q}) \to \mathbb{Q}[[t^2]]^\dagger$, given by $\gamma \mapsto r^\gamma = 1 + \sum_{n\geq 1} \gamma(\mathbb{CP}^{2n}) t^{2n}$.
- Multiplicative sequences $\to \mathrm{Hom}(\widehat{\Omega}^+_\bullet \otimes \mathbb{Q}, \mathbb{Q})$, given by $K \mapsto \gamma_K$; inverse $K^\gamma \mapsto \gamma$.

At this point we want to invoke R. Thom's results on the structure of the oriented cobordism ring.

Theorem 5.1 (Thom cobordism theorem)**.** If $4 \nmid n$ then Ω^+_n is a finite group. Moreover the group Ω^+_{4n}/torsion is a finitely generated free Abelian group of rank $p(n) = |\mathrm{Part(n)}|$. In particular, we have

$$\widehat{\Omega}^+_\bullet \otimes \mathbb{Q} = \Omega^+_\bullet \otimes \mathbb{Q}.$$

Putting together all the facts established so far we obtain the following result.

Corollary 5.2. For any genus $\gamma : \Omega^+_\bullet \to \mathbb{Q}$ there exists a unique multiplicative sequence K such that for any smooth, compact oriented manifold M of dimension $4n$ we have

$$\gamma(M) = \langle K_n(\overline{p}_1(M), ..., \overline{p}_n(M)), [M] \rangle.$$

The symbol $a(\xi)$ of K is related to the generating series

$$r^\gamma(t) = 1 + \sum_{n\geq 1} \gamma(\mathbb{CP}^{2n}) t^{2n}$$

via the equality

$$a(t^2) = \frac{t}{\mathbf{R}^{[-1]}}$$

where $\mathbf{R} = \int_0^t r^\gamma$.

Chapter 6

Elliptic operators

We will discuss elliptic differential operators on sections of vector bundles on smooth manifolds, giving their basic properties and defining the principal symbol of those operators. Also we define the analytical index of elliptic operators and we sketch the proof on the well definition of the analytical index (see [Ban-Crainic] for more details). Also we will define the topological index of an elliptic differential operator. Finally we announced the Atiyah-Singer index theorem, which relates the analytical index with the topological index of an elliptic differential operator on a smooth manifold X. This theorem establishes the commutativity of the left side of the diagram 1.1.

1 Differential operators on spaces of functions on a manifold

We first discuss differential operators acting on spaces of functions on an open subset $U \subset \mathbb{R}^n$, then we extend on space of functions on a manifold and finally we will move to those acting on spaces of sections of vector bundles. We also discuss the symbol of a differential operator.

For multi-indices $\alpha, \beta \in \mathbb{N}^n$ we use the following notation:

- $$|\alpha| := \sum_{j=1}^{n} \alpha_j \ ; \quad \alpha! := \prod_{j=1}^{n} \alpha_j!.$$
- We write $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ for all $1 \leq j \leq n$. If $\alpha \leq \beta$ we put
$$\binom{\beta}{\alpha} := \prod_{j=1}^{n} \binom{\beta_j}{\alpha_j}.$$
- We put $\partial_j := \partial / \partial x_j$ and
$$x^\alpha := \prod_{j=1}^{n} x_j^{\alpha_j} \ ; \quad \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

Lemma 6.1. Let $f, g \in C^\infty(U)$ and $\alpha \in \mathbb{N}^n$. then we have the Leibniz' rule

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Proof. We will proceed by induction over $k = |\alpha|$. For $k = 1$, the statement is just the ordinary product rule. If $|\alpha| = k+1$, there exist $1 \leq j \leq n$ such that α can be written as $\alpha = \widetilde{\alpha} + (0, ..., 0, 1, 0, ..., 0)$ where the 1 is at position j and $|\widetilde{\alpha}| = k$. Using induction hypothesis we calculate

$$\partial^\alpha(fg) = \partial_j \partial^{\widetilde{\alpha}}(fg) = \sum_{0 \leq \beta \leq \widetilde{\alpha}} \binom{\widetilde{\alpha}}{\beta} (\partial_j \partial^\beta f)(\partial^{\widetilde{\alpha}-\beta} g) + (\partial^\beta f)(\partial_j \partial^{\widetilde{\alpha}-\beta} g) = (\dagger).$$

Adding summands with multi-indices β such that

$$\binom{\widetilde{\alpha}}{\beta} = \prod_{1 \leq i \neq j \leq n} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha_i}{\alpha_i + 1},$$

next making an index shift $(0 \leq \beta_j \leq \widetilde{\alpha}_j = \alpha_j - 1) \mapsto (1 \leq \beta_j \leq \alpha_j)$, and adding summands where $\beta_j = 0$ and thus

$$\binom{\alpha_j - 1}{\beta_j - 1} = 0$$

we have that

$$\begin{aligned}
(\dagger) &= \sum_{i \neq j; 0 \leq \beta_i \leq \widetilde{\alpha}_i; 0 \leq \beta_j \leq \widetilde{\alpha}_j} \prod_{i=1, i \neq j}^{n} \binom{\widetilde{\alpha}_i}{\beta_i} \binom{\widetilde{\alpha}_j}{\beta_j} (\partial_j \partial^\beta f)(\partial^{\widetilde{\alpha}-\beta} g) + \sum_{0 \leq \beta \leq \alpha} \binom{\widetilde{\alpha}}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{i \neq j; 0 \leq \beta_i \leq \alpha_i; 0 \leq \beta_j \leq \alpha_j} \prod_{i=1, i \neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j - 1} (\partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \leq \beta \leq \alpha} \binom{\widetilde{\alpha}}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \leq \beta \leq \alpha} \prod_{i=1, i \neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j - 1} (\partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \leq \beta \leq \alpha} \prod_{i=1, i \neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \leq \beta \leq \alpha} \left(\binom{\alpha_j - 1}{\beta_j - 1} \binom{\alpha_j - 1}{\beta_j} \right) \prod_{i=1, i \neq j} \binom{\alpha_i}{\beta_i} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \leq \beta \leq \alpha} \binom{\alpha_j}{\beta_j} \prod_{i=1, i \neq j} \binom{\alpha_i}{\beta_i} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).
\end{aligned}$$

□

Definition 6.1. A **differential operator** of order at most $k \in \mathbb{N}$ on open set $U \subset \mathbb{R}^n$ is an endomorphism $P \in \mathrm{End}(C^\infty(U))$ of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha(x) \partial^\alpha,$$

with coefficients $c_\alpha \in C^\infty(U)$ for all α. The linear space of differential operators on U of order at most k is denoted by $\mathcal{D}_k(U)$. The union of these, for $k \in \mathbb{N}$, is denoted by $\mathcal{D}(U)$.

Remark 6.1. Let $P \in \mathcal{D}_k(U)$ and $Q \in \mathcal{D}_l(U)$ be differential operators of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha(x)\partial^\alpha \ , \quad Q = \sum_{|\beta| \leq l} c_\beta(x)\partial^\beta$$

then applying the Leibniz' rule (Lemma 1), the composition of P and Q is given by

$$(P \circ Q)(f)(x) = \sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} c_\alpha(x)(\partial^\gamma c_\beta)(x)(\partial^{\beta+\alpha-\gamma} f)(x),$$

then $P \circ Q \in \mathcal{D}_{k+l}(U)$. Therefore, the set $\mathcal{D}(U)$ of differential operators is an algebra with unit.

In order to obtain a definition of differential operator on a smooth manifold M, let $h : U \subset \mathbb{R}^n \to U' \subset \mathbb{R}^n$ be a diffeomorphism, then h induces by pullback the bijection $h^* : C^\infty(U') \to C^\infty(U)$ given by $h^* f(x) = f(h(x))$ for any map $f \in C^\infty(U')$. Thus we have an induced map $h_* : \mathrm{End}(C^\infty(U)) \to \mathrm{End}(C^\infty(U'))$ given by $h_*(F) = (h^*)^{-1} \circ F \circ h^*$.

Lemma 6.2. The map h_* maps $\mathcal{D}(U)$ bijectively onto $\mathcal{D}(U')$.

Proof. Is an immediate consequence of the Leibniz rule. □

In addition we need a local condition for the operators that will be under consideration. Given a smooth function $f \in C^\infty(M)$, the **support** of f, denoted by $\mathrm{supp}(f)$, is the closure of the subset of M where f is non-zero.

Definition 6.2. Let M be a smooth manifold. A linear operator $P \in \mathrm{End}(C^\infty(M))$ is called **local** if for all $f \in C^\infty(M)$ we have that $\mathrm{supp}(P(f)) \subset \mathrm{supp}(f)$.

Since the complement of the support of a function is the largest open on which the function vanishes, the condition in the previous definition is equivalent to have that for any open set $U \subset M$ and $f \in C^\infty(M)$, if $f|_U = 0$, then $P(f)|_U = 0$.

Lemma 6.3. There is a unique way to associate to any local operator $P \in \mathrm{End}(C^\infty(M))$ and any open $U \subset M$, a restricted operator

$$P_U = P|_U \in \mathrm{End}(C^\infty(U))$$

such that if $V \subset U$. then $(P|_U)|_V = P|_V$.

Proof. For $f \in C^\infty(U)$ and a point $x \in U$, we choose a function $f_x \in C^\infty(M)$ which coincides with f in an open neighborhood $V_x \subset U$ of x. To satisfy the condition of the lemma, we are forced to define

$$P_U(f)(x) := P(f_x)(x).$$

for checking the well definition of P_U, let g_x be another another election, then $f_x - g_x$ vanishes on a neighborhood of x, since P is local, we deduce that $P(f_x) - P(g_x) = P(f_x - g_x)$ vanishes on that neighborhood, therefore also at x. □

Let $\{(U, \varphi)\}$ be a family of coordinate charts whose domain cover M, then $P|_U$ can be moved to $\varphi(U)$ using the pull-back map $\varphi^* : C^\infty(\varphi(U)) \to C^\infty(U)$, to obtain an operator $P_\varphi : C^\infty(\varphi(U)) \to C^\infty(\varphi(U))$ given by $P_\varphi = \varphi_*(P|_U) = (\varphi^*)^{-1} \circ P|_U \circ \varphi^*$.

Definition 6.3. Let M be a smooth manifold. A **differential operator** of order at most k on M is a local operator $P \in \mathrm{End}(C^\infty(M))$ such that for any coordinate chart (U, φ) of a family of coordinate charts whose domain cover M, $P_\varphi \in \mathcal{D}_k(\varphi(U))$, i.e., for a coordinate chart $(U, \varphi = (x_1, ..., x_n))$, we have that

$$P|_U = \sum_{|\alpha| \leq k} c_\alpha(x) \partial_\varphi^\alpha,$$

with $c_\alpha \in C^\infty(U)$ and ∂_φ^α act on $C^\infty(U)$ and are defined analogous to ∂^α but using the derivative along the vector $\partial/\partial x_j$ induce by the chart.

Next, we discuss the symbol of differential operators.

Definition 6.4. Let $U \subset \mathbb{R}^n$ be an open set and let $P \in \mathcal{D}_k(U)$ be of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha(x) \partial^\alpha.$$

The **full symbol** of the operator P is the function $\sigma(P) : U \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$\sigma(P)(x, \xi) = \sum_{|\alpha| \leq k} c_\alpha(x) (\mathrm{i}\xi)^\alpha.$$

The **principal symbol** of order k of P is the function $\sigma_k(P) : U \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$\sigma_k(P)(x, \xi) = \sum_{|\alpha| = k} c_\alpha(x) (\mathrm{i}\xi)^\alpha.$$

Where we must remember that

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}.$$

Remark 6.2. We can view the variable $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ as a set of variables ξ_i in the dual of $\mathbb{R}^n$, where by abuse of notation $\xi_i : \mathbb{R}^n \to R$ is given by $(\xi_1, ..., \xi_n) \mapsto \xi_i$. Then we identify $U \times \mathbb{R}^n$ with the cotangent bundle T^*U, where given $(x, \xi) \in U \times \mathbb{R}^n$ we associate the element $\xi_1(\mathrm{d}x_1)_x + \cdots + \xi_n(\mathrm{d}x_n)_x$. In other words, the principal symbol should be viewed as the function $\sigma_k(P) : \mathrm{T}^*U \to \mathbb{C}$ given by

$$\xi_1(\mathrm{d}x_1)_x + \cdots + (\xi_n \mathrm{d}x_n)_x \mapsto \sum_{|\alpha| = k} c_\alpha(x) (\mathrm{i}\xi)^\alpha.$$

Now we show that with this interpretation in hand, the principal symbol can be characterized without reference to the coordinates.

Lemma 6.4. Let $U \subset \mathbb{R}^n$, $P \in \mathcal{D}_k(U)$. For $\xi_x := (x, \xi) \in \mathrm{T}_x^*\mathbb{R}^n$, with $x \in U$, choose $f \in C^\infty(U)$ such that $(\mathrm{d}f)_x = \xi_x$. Then

$$\sigma_k(P)(x, \xi) = \lim_{t \to \infty} t^{-k} \mathrm{e}^{-\mathrm{i}tf(x)} P(\mathrm{e}^{\mathrm{i}tf})(x).$$

Proof. Given $P \in \mathcal{D}_k(U)$ of the form

$$P = \sum_{|\alpha| \leq k} c_\alpha(x) \partial^\alpha$$

applying the Leibniz' rule we have

$$\begin{aligned}\lim_{t\to\infty} t^{-k}\mathrm{e}^{-\mathrm{i}tf(x)}P(\mathrm{e}^{\mathrm{i}tf})(x) &= \lim_{t\to\infty} t^{-k}\mathrm{e}^{-\mathrm{i}tf(x)}\sum_{|\alpha|\le k} c_\alpha(x)\partial^\alpha(\mathrm{e}^{\mathrm{i}tf(x)})\\ &= \lim_{t\to\infty} t^{-k}\mathrm{e}^{-\mathrm{i}tf(x)}\sum_{|\alpha|\le k} c_\alpha(x)(\mathrm{i}t)^\alpha(\xi_{x_1})^{\alpha_1}\cdots(\xi_{x_n})^{\alpha_n}\mathrm{e}^{\mathrm{i}tf(x)}\\ &= \sum_{|\alpha|\le k} c_\alpha(x)\lim_{t\to\infty} t^{-k}t^\alpha(\mathrm{i}\xi)^\alpha\\ &= \sum_{|\alpha|=k} c_\alpha(x)(\mathrm{i}\xi)^\alpha.\end{aligned}$$

□

Now let h be a diffeomorphism from $U\subset\mathbb{R}^n$ onto a second open subset $U'\subset\mathbb{R}^n$. It induces the map $\mathrm{T}^*h:\mathrm{T}^*U\to\mathrm{T}^*U'$ given by $\mathrm{T}^*h(x,\xi)=(h(x),\xi\circ\mathrm{T}_xh^{-1})$. Therefore we have the map $h_*:C^\infty(\mathrm{T}^*U)\to C^\infty(\mathrm{T}^*U')$ given by $h_*\sigma=\sigma\circ(\mathrm{T}^*h)^{-1}$. Thus,

$$h_*\sigma(x,\xi)=\sigma(h^{-1}(x),\xi\circ\mathrm{T}_xh).$$

Lemma 6.5. For all $P\in\mathcal{D}_k(U)$, we have that

$$\sigma_k(h_*(P))=h_*(\sigma_k(P)).$$

Proof. Fix $(x,\xi)\in U\times\mathbb{R}^n\cong\mathrm{T}^*U$. Put $y=h(x)$ and $\eta=\xi\circ\mathrm{T}_xh^{-1}$. Select $f\in C^\infty(U')$ with $\mathrm{d}f(y)=\eta$. Then by previous lemma

$$\begin{aligned}\sigma_k(h_*(P))(y,\eta) &= \lim_{t\to\infty} t^{-k}\mathrm{e}^{-\mathrm{i}tf(y)}(h_*P)(\mathrm{e}^{\mathrm{i}tf})(y)\\ &= \lim_{t\to\infty} t^{-k}\mathrm{e}^{-\mathrm{i}th^*f(x)}P(\mathrm{e}^{\mathrm{i}th^*f})(x)\\ &=\sigma_k(P)(x,\xi).\end{aligned}$$

□

So we have the immediate consequence:

Corollary 6.1. Let M be a manifold and $P\in\mathcal{D}_k(M)$. Then there is a well-defined smooth function

$$\sigma_k(P):\mathrm{T}^*M\to\mathbb{C}$$

such that, for any coordinate chart $(U,\varphi=(x_1,...,x_n))$ of a family of coordinate charts whose domain cover M, $\sigma_k(P)$ is given by

$$\xi_1(\mathrm{d}x_1)_x+\cdots+\xi_n(\mathrm{d}x_n)_x\mapsto\sigma_k(P_\varphi)(x,\xi_1,...,\xi_n).$$

Definition 6.5. Let $P\in\mathcal{D}_k(M)$. The function $\sigma_k(P):\mathrm{T}^*M\to\mathbb{C}$ from the previous corollary is called the **principal symbol** of order k of the operator P.

2 Differential operators on spaces of sections of vector bundles

Now we extend the notion of a differential operator acting at the level of sections of smooth vector bundles E and F. We begin with degree zero differential operators. Given a section $C \in \Gamma(\mathrm{Hom}(E,F))$, i.e., a smooth map $C : M \to \mathrm{Hom}(E,F)$ given by $x \mapsto C_x \in \mathrm{Hom}(E_x, F_x)$, we define a $C^\infty(M)$-linear map $C : \Gamma(E) \to \Gamma(F)$ given by $s \mapsto C(s)(x) = C_x(s(x))$. This construction identifies sections of $\mathrm{Hom}(E,F)$ with $C^\infty(M)$-linear maps $\Gamma(E) \to \Gamma(F)$.

As before we will begin in the situation in that $M = U$ is the domain of a coordinate chart $(U, \varphi = (x_1, ..., x_n))$ and $E \cong U \times \mathbb{C}^r$ is a trivial bundle with associated frame $s_1, ..., s_l$ (see Lemma 2.2). Then in this case we have higher order derivatives operators $\partial^\alpha_\varphi : \Gamma(E) \to \Gamma(E)$ given by $s = f_1 s_1 + \cdots + f_l s_l \mapsto \partial^\alpha_\varphi(f_1)s_1 + \cdots + \partial^\alpha_\varphi(f_l)s_l$ (see Lemma 2.3).

Definition 6.6. A **differential operator** of order at most k from E to F is a linear map $P : \Gamma(U,E) \to \Gamma(U,F)$ of the form

$$P = \sum_{|\alpha| \le k} C_\alpha \circ \partial^\alpha,$$

with $C_\alpha \in \Gamma(U, \mathrm{Hom}(E,F))$. The space of such differential operators is denoted by $\mathcal{D}_k(U,E,F)$.

More explicitly, $P : \Gamma(U,E) \to \Gamma(U,F)$ is given by

$$s \mapsto P(s) = \sum_{|\alpha| \le k} \partial^\alpha(f_1(x)) c_\alpha(s_1(x)) + \cdots + \partial^\alpha(f_l(x)) c_\alpha(s_l(x)).$$

Remark 6.3. Notice that $E = F = U \times \mathbb{R}$, the sections of the bundle E are precisely the continuous functions on U, so in this case we have the particular case of differential operators acting on spaces of functions on an open set U. More generally note that if also F is trivialized, with trivializing frame $\{s'_1, ..., s'_m\}$, then each C_α is uniquely determined by a matrix of smooth functions on U, $c(\alpha)^i_j \in C^\infty(U)$, characterized by $C_\alpha(s_i) = \sum_j c(\alpha)^j_i s'_j$. With respect to the identification $\Gamma(U,E) \cong C^\infty(U,\mathbb{C})^r$, given by $(f_1 s_1 + \cdots + f_l s_l) \mapsto (f_1, ..., f_l)$, and similarly for F, P becomes

$$P(f_1, ..., f_l) = (\sum_{\alpha, j} \partial^\alpha_\varphi(f_j) c(\alpha)^1_j, ..., \partial^\alpha_\varphi(f_j) c(\alpha)^l_j).$$

Now we will show that $\mathcal{D}_k(U,E,F)$ does not depend on the trivialization of E. Another trivialization over U is associated with a vector bundle isomorphism $\varphi : E \to E$ given by $(x,v) \mapsto (x, g(x)v)$, where $g : U \to \mathrm{GL}(\mathbb{C}^r)$ is a smooth map. The map φ induces a linear isomorphism φ_* of $\Gamma(U,E)$ onto itself given by $\varphi_*(s) = \varphi \circ s$. Accordingly, we have an induced linear isomorphism $\widetilde{\varphi}_*$ from $\mathrm{Hom}(\Gamma(U,E), \Gamma(U,F))$ onto itself given by $\widetilde{\varphi}_*(T) = T \circ \varphi_*^{-1}$. Applying Leibniz' formula, we see that the map $\widetilde{\varphi}_*$ maps $\mathcal{D}_k(U,E,F)$ bijectively onto $\mathcal{D}_k(U,E,F)$. Hence the space $\mathcal{D}_k(U,E,F)$ defined above not depend on the trivialization of E. Furthermore, we can already proceed as in the previous section and show that it does not depend on the choice of the coordinates on U either. Therefore the space $\mathcal{D}_k(U,E,F)$ only depends on the fact that E is trivializable.

As before, we have the following definition

Definition 6.7. A linear operator between spaces of sections of two vector bundles E and F over a manifold M, $P : \Gamma(E) \to \Gamma(F)$, is **local** if, for any $s \in \Gamma(E)$, $\mathrm{supp}(P(s)) \subset \mathrm{supp}(s)$.

By the same arguments as before, any such local operator can be restricted to arbitrary opens $U \subset M$, obtaining new operators $P|U$ from $E|_U$ to $F|_U$. Also, for a coordinate chart (U, φ), we obtain an operator P_φ acting on the resulting bundles over $\varphi(U)$, from $E_\varphi := \varphi_*(E|_U) = (\varphi^{-1})^*(E|_U)$ to f_φ defined similarly.

Definition 6.8. Let E, F be smooth vector bundles over a smooth manifold M. A **differential operator of order at most** k **from** E **to** F is a linear local operator $P : \Gamma(E) \to \Gamma(F)$ with the property that, for any coordinate chart (U, φ) with the property that $E|_U$ is trivializable, $P_\varphi \in \mathcal{D}_k(E_\varphi, F_\varphi)$. The space of differential operators of order at most k from E to F is denoted by $\mathcal{D}_k(E, F)$.

We extend the definition of principal symbol as follows. Let $\pi : T^*M \to M$ the canonical projection. For a vector bundle $p : E \to M$, let π^*E be the pullback of E to T^*M via the map π, where the fibers of π^*E are E_x for $\xi_x \in T^*_xM$. For two vector bundles E and F on M, we consider the hom-bundle $\mathrm{Hom}(E, F)$ on M with fibers $\mathrm{Hom}(E_x, F_x)$, for $x \in M$, and its pullback to T^*M obtaining the bundle $\pi^*\mathrm{Hom}(E, F) \cong \mathrm{Hom}(\pi^*E, \pi^*F)$ with fiber $\mathrm{Hom}(E_x, F_x)$ for $\xi_x \in T^*_xM$.

Lemma 6.6. Let E and F be smooth vector bundles on M and let $P \in \mathcal{D}_k(E, F)$. There exists a unique section $\sigma_k(P)$ of $\pi^*\mathrm{Hom}(E, F)$, i.e., a smooth map $\sigma_k(P) : T^*M \to \mathrm{Hom}(\pi^*E, \pi^*F)$ given by $T^*_xM \ni \xi_x \mapsto \sigma_k(P)(\xi_x) \in \mathrm{Hom}(E_x, F_x)$, with the property that for each $x_0 \in M$ and all $s \in \Gamma(E)$ and $\varphi \in C^\infty(M)$,

$$\sigma_k(P)((\mathrm{d}\varphi)_{x_0})(s(x_0)) = \lim_{t\to\infty} t^{-k} \mathrm{e}^{-\mathrm{i}t\varphi(x_0)} P(\mathrm{e}^{\mathrm{i}t\varphi} s)(x_0).$$

Moreover, for each $x \in M$ the function $\xi \mapsto \sigma_k(P)(x, \xi)$ is a degree k homogeneous polynomial function $T^*_xM \to \mathrm{Hom}(E_x, F_x)$.

Proof. Uniqueness follows from the fact that for every $(x, \xi) \in T^*M$ and $v \in E_x$ there exist a smooth section $s \in \Gamma(E)$ such that $s(x) = v$ and a function $\varphi \in C^\infty(M)$ such that $\mathrm{d}\varphi(x) = \xi$. Let $x_0 \in M$ be given and select a coordinate patch U with $x_0 \in U$ on which E and F admit trivializations $\phi_E : E|_U \to U \times E_0$ and $\phi_F : F|_U \to U \times F_0$. Let $(x_1, ..., x_n)$ be a system of local coordinates on U, then

$$\phi_*(P) = \sum_{|\alpha|\le k} C_\alpha \partial^\alpha,$$

with $C_\alpha \in C^\infty(U, \mathrm{Hom}(E_0, F_0))$. It follows by application of Lemma 6.4 that

$$\lim_{t\to\infty} t^{-k} \mathrm{e}^{-\mathrm{i}t\varphi(x_0)} P(\mathrm{e}^{\mathrm{i}t\varphi} s)(x_0) = \sum_{|\alpha|\le k} [\phi_F^{-1}] \circ C_\alpha(x_0) \phi_E[s(x_0)] \eta^\alpha.$$

Here we have used the multi-index notation with $\eta_j = \partial_j \varphi(x_0)$, $\partial_1, ..., \partial_n$ being the derivations induced by the choice of coordinates. It follows that the limit on the left in the previous equality depends on s and ϕ through the values $s(x_0)$ and $\mathrm{d}\phi(x_0)$. This implies the existence of a section $\sigma_k(P)$ of $\mathrm{Hom}(\pi^*E, \pi^*F)$. $\square$

For $P \in \mathcal{D}_k(E_1, E_2)$, with E_1, E_2 smooth vector bundles on M, by the Remark 2.7 the symbol $\sigma_k(P) \in \Gamma(\mathrm{Hom}(\pi^*E_1, \pi^*E_2))$ may be viewed as a homomorphism of bundles from the bundle π^*E_1 to π^*E_2. Thus, if E_3 is a third vector bundle and $Q \in \mathcal{D}_l(E_2, E_3)$ then the composition $\sigma_l(Q) \circ \sigma_k(P)$ is a vector bundle homomorphism from E_1 and E_3.

Lemma 6.7. Let E_1, E_2, E_3 be smooth vector bundles on M. Let $P \in \mathcal{D}_k(E_1, E_2)$ and $Q \in \mathcal{D}_l(E_2, E_3)$. Then the composition $Q \circ P$ belongs to $\mathcal{D}_{k+l}(E_1, E_3)$ and

$$\sigma_{k+l}(Q \circ P) = \sigma_l(Q) \circ \sigma_k(P).$$

Definition 6.9. A **differential complex** over a manifold M,

$$\epsilon : \Gamma(E^0) \xrightarrow{P_0} \Gamma(E^1) \xrightarrow{P_1} \Gamma(E^2) \xrightarrow{P_2} \to \cdot$$

consist of:

1. For each $k \geq 0$, a vector bundle E_k over M, with $E_k = 0$ for k large enough.
2. For each $k \geq 0$, a differential operator P_k from E_k to E_{k+1}, of some order d independent of k such that, for all k, $P_{k+1} \circ P_k = 0$.

For a differential complex ϵ, we define $Z^k(\epsilon) = \mathrm{Ker}(P_k), B^k(\epsilon) = \mathrm{Im}(P_{k+1})$ and the k-th **cohomology groups**

$$H^k(\epsilon) := Z^k(\epsilon)/B^k(\epsilon).$$

Note that, by the Lemma 6.7 it follows that for a complex of differential operators as above, the associated sequence $\sigma_{d_k}(P_k)$ of principal symbols is a complex of homomorphisms of the vector bundles $\pi^* E_k$ on T^*M, i.e., for any $\xi_x \in T^*_x M$, the sequence

$$E^0_x \xrightarrow{\sigma_d(P_0)(\xi_x)} E^1_x \xrightarrow{\sigma_d(P_1)(\xi_x)} E^2_x \xrightarrow{\sigma_d(P_2)(\xi_x)} \cdots$$

is a complex of vector spaces. In turn, this means that the composition of any two consecutive maps in this sequence is zero. Equivalently,

$$\mathrm{Im}(\sigma_d(P_k)(\xi_x)) \subset \mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)).$$

Definition 6.10. Let $P \in \mathcal{D}_k(E, F)$ be a differential operator between two vector bundles E ad F over a manifold M. We say that P is an **elliptic operator** of order k if, for any $\xi_x \in T^*_x M$ non-zero,

$$\sigma_k(P)(\xi_x) : E_x \to F_x$$

is an isomorphism.

Definition 6.11. A differential complex ϵ is called an **elliptic complex** if, for any $\xi_x \in T^*_x M$ non-zero, the sequence

$$E^0_x \xrightarrow{\sigma_d(P_0)(\xi_x)} E^1_x \xrightarrow{\sigma_d(P_1)(\xi_x)} E^2_x \xrightarrow{\sigma_d(P_2)(\xi_x)} \cdots$$

is exact, i.e.,

$$\mathrm{Ker}(\sigma_d(P_{k+1})(\xi_x)) = \mathrm{Im}(\sigma_d(P_k)(\xi_x)).$$

So, given an elliptic operator $P : \Gamma(E) \to \Gamma(F)$ of order k over a compact manifold M, by the Atiyah-Bott-Shapiro construction over compactly supported K-theory its principal symbol $\sigma_k(P) : \pi^* E \to \pi^* F$, where $\pi : T^*M \to M$ is the projection, will induce an element

$$[\pi^* E, \pi^* F, \sigma_k(P)] \in K_{\mathrm{cpt}}(T^*M).$$

Definition 6.12. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order k over a compact manifold M. The **topological index** of P is the integer

$$\mathrm{Top} - \mathrm{Ind}(P) := g_! f_! \sigma_k(P),$$

where $g_! f_! : K_{\mathrm{cpt}}(TX) \to \mathbb{Z}$ (see Equation 3.8).

Remark 6.4. By the Remark 3.2, the topological index is well defined.

Definition 6.13. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator over a compact manifold M. The **analytical index** of P is defined by

$$\mathrm{Ind}(P) := \dim(\mathrm{Ker}(P)) - \dim(\mathrm{Coker}(P)).$$

More generally, given an elliptic compex

$$\epsilon : \Gamma(E^0) \xrightarrow{P_0} \Gamma(E^1) \xrightarrow{P_1} \Gamma(E^2) \xrightarrow{P_2} \to \cdot$$

its analytical index is the alternating sum

$$\mathrm{Ind}(P) := \sum_i (-1)^i \dim(Ker(P_i)).$$

Remark 6.5. Now, we show that the analytical index is well defined, i.e., $\dim(\mathrm{Ker}(P))$ and $\dim(\mathrm{Coker}(P))$ are finite dimensional, and depend only on the principal symbol $\sigma_k(P)$.

Now we can state the Atiyah-Singer index theorem:

Theorem 6.1. [Atiyah-Singer index theorem]
For an elliptic differential operator $P : \Gamma(E) \to \Gamma(F)$ of order d, its analytical index $\mathrm{Ind}(P)$ is equal to the topological index $\mathrm{Top} - \mathrm{Ind}(P)$, i.e.,

$$\mathrm{Ind}(P) = \mathrm{Top} - \mathrm{Ind}(P).$$

For a proof of this theorem see the book [Law89].

The principal goal of this thesis is to prove the cohomological form of the Atiyah-Singer index theorem, that states that for an elliptic differential operator $P : \Gamma(E) \to \Gamma(F)$ of order d, its topological index $\mathrm{Top} - \mathrm{Ind}(P)$ can be given in terms of cohomological data, i.e.,

$$\mathrm{Ind}(P) = \mathrm{Top} - \mathrm{Ind}(P) = (-1)^n \{\mathrm{ch}\sigma(P) \cdot \hat{\mathbf{A}}(X)^2\}[TX].$$

3 Well definition of the analytical index

In this section we will discuss a sketch of proof of the well definition of the analytical index. for a detailed discussion see [Ban-Crainic]. More precisely, if M is a compact smooth manifold and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator on M, then we will show that $\mathrm{Ker}(P)$ and $\mathrm{Coker}(P)$ are finite dimensional, and thus the analytical index $\mathrm{Ind}(P) := \dim(\mathrm{Ker}(P)) - \dim(\mathrm{Coker}(P))$ is well defined.

The general strategy to prove this is to replace the differential elliptic operator $P \in D_k(M, E)$ by a general operator defined on certain Banach space of sections denoted by $H_r(M, E)$ on which the general operators acts, and then using the theory of Fredholm operators on Banach spaces we can prove that these general operators are Fredholm. Such spaces $H_r(M,E)$ are Sobolev spaces and are defined such that these are subspaces of the distributional space, denoted by $\mathcal{D}'(M,E)$, and contained the space of compactly supported smooth sections, denoted by $\mathcal{D}(M,E)$, as a dense subspace.

In order to define the spaces of particular interest we start with a discussion of topological vector spaces.

Definition 6.14. A **topological vector space** is a vector space V over $\mathbb{C}$ together with a topology τ, such that the vector space operations $V\times V\to V$ and $\mathbb{C}\times V\to V$ given by $(u,w)\mapsto v+w$ and $(\lambda,v)\mapsto\lambda v$ respectively are continuous maps.

By continuity of the translations $\tau_x : V\to V$ given by $y\mapsto y+x$, the topology τ is uniquely determined by $\tau(0):=\{O\in\tau \mid 0\in O\}$, i.e., for $O\subset V$, $O\in\tau$ if and only if for all $x\in O$ there is $B\in\tau(0)$ such that $x+B\subset O$. Thus if we know a basis of neighborhoods $\mathcal{B}(0)\subset\tau(0)$ of $0\in V$, then we know the topology τ.

Definition 6.15. A **locally convex vector space** is a topological vector space (V,τ) such that

$$\tau_{\mathbf{convex}}(0):=\{C\in\tau(0)\mid C \textit{ is convex}\}$$

is a basis of $\tau(0)$.

Definition 6.16. A **seminorm** on a vector space V is a map $p : V\to[0,\infty)$ satisfying for all $v,w\in V$ and $\lambda\in\mathbb{C}$

$$p(v+w)\le p(v)+p(w),\ p(\lambda v)=|\lambda|p(v).$$

Notice that it is possible for $p(v)=0$ for nonzero v.

The key observation is that locally convex vector spaces are associated to families of seminorms, namely, given a family of seminorms $P=\{p_i\}_{i\in I}$ on a vector space V, one has the notion of balls

$$B^r_{i_1,\ldots,i_n}:=\{v\in V\mid p_{i_k}(v)<r, 1\le k\le n\},$$

defined for all $r>0$ and $i_1,\ldots,i_n\in I$. The collection of all such balls form a family $\mathcal{B}(0)$, which will induce a locally convex topology τ_P on V.

Theorem 6.2. A topological vector space (V,τ) is a locally convex vector space if and only if there exist a family of seminorms P such that $\tau=\tau_P$.

Remark 6.6. Notice that if $P_0\subset P$ is a smaller family of seminorms such that for any $p\in P$ there exist $p_0\in P_0$ such that $p_0(v)\le p(v)$ for all $v\in V$, then P_0 and P defines the same locally convex vector space.

Next, we have a criteria for continuity of linear maps between locally convex vector spaces in terms of the seminorms.

Proposition 6.1. Let (V,P) and (W,Q) be two locally convex vector spaces and let $A : V\to W$ be a linear map. Then A is continuous if and only if, for any $q\in Q$, there exist $p_1,\ldots,p_n\in P$ and a constant $C>0$ such that

$$q(A(v))\le C\cdot\max\{p_1(v),\ldots,p_n(v)\},$$

for all $v\in V$.

Definition 6.17. A **Frechet space** is a topological vector space V such that it is complete, and its topology is induced by a countable family of seminorms $\{p_1, p_2, ...\}$.

Notice that the topology of a Frechet space V is induced by a complete metric

$$d(v,w) := \sum_{n\geq 1} \frac{1}{2^n} \frac{p_n(v-w)}{1+p_n(v-w)}.$$

The important examples for us are the following.

Example 6.1. Any Hilbert space or Banach space is a locally convex vector space. In particular we have the L^p-spaces on an open $\Omega \subset \mathbb{R}^n$

$$L^p(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is measurable}, \int_\Omega |f| < \infty\},$$

with the norm

$$||f||_{L^p} = \left(\int_\Omega |f|^p\right)^{1/p}.$$

Example 6.2. For an open $\Omega \subset \mathbb{R}^n$, $r \in \mathbb{Z}_+$ and $K \subset \Omega$ compact, we consider the space

$$C^r_K(\Omega) := \{\phi : \Omega \to \mathbb{C} \mid \phi \text{ is of class } C^r, \mathrm{supp}(\phi) \subset K\}.$$

This space becomes a Banach space with the norm

$$||\phi||_{r,K} := \sup\{|\partial^\alpha \phi(x)| \mid x \in K, |\alpha| \leq r\}.$$

In the case in that $r = \infty$, then $C^\infty_K(\Omega)$ should be considered with the family of seminorms $\{|| \cdot ||_{K,r}\}_{r\in\mathbb{Z}_+}$.

In the other hand, we can induce a topology on the set of all smooth functions $C^\infty(\Omega)$ by the family of seminorms

$$\{|| \cdot ||_{K,r} \mid K \subset \Omega \text{ compact}, r \in \mathbb{Z}_+\}.$$

It can be seen that this family of seminorms can be replaced by a contable one, namely

$$\{|| \cdot ||_{K_n,r} \mid K_n \subset \Omega \text{ compact}, n, r \in \mathbb{Z}_+\},$$

and using the Remark 6.6 we have that both seminorms define the same topology on $C^\infty(\Omega)$.

For the following example, recall that a subset $B \subset V$ is called **bounded** if, for any neighborhood of the origin, there exist $\lambda > 0$ such that $B \subset \lambda V$. Thus, if the topology of V is generated by a family of seminorms P, this means that for any $p \in P$ there exist $\lambda_p > 0$ such that

$$B \subset B_p(r_p) = \{v \in V \mid p(v) < r_p\}.$$

Therefore by the Proposition 6.1 this implies that for any continuous linear functional $u \in V^*$,

$$p_B(u) := \sup\{|u(v)| \mid v \in B\} < \infty,$$

obtaining a family of seminorms $\{p_B\}_B$, indexed by all the bounded sets B.

Example 6.3. Let V be a topological vector space. We can consider two topologies on the set

$$V^* := \{u : V \to \mathbb{R} \mid u \text{ is linear and continuous}\}.$$

The **weak topology** on V^*, τ_s, induced by the family of seminorms $\{p_v\}_{v\in V}$ with $p_v : V^* \to \mathbb{R}$ given by $p_v(u) = |u(v)|$. The another topology, the **strong topology**, τ_b, is the topology induced by the family of seminorms $\{p_B\}_B$.

Remark 6.7. We will be interested in the dual V^* of a particular locally convex vector spaces V, in which cases if we consider the convergence or continuity aspects on the topologies τ_s or τ_b on V^*, it does not make a difference, i.e., the aspects of convergence and continuity are the same for the two topologies τ_s and τ_b.

In the previous examples, by the Theorem 6.2 locally convex vector spaces come with naturally associated seminorms and the topology is just the induced one. But we will be interested in a locally convex vector space in which the topology comes first. We now discuss the general construction of locally convex topologies known as the **inductive limit**.

Let X be a vector space and $X_\alpha \subset X$ vector subspaces such that $X = \bigcup_\alpha X_\alpha$, where α runs in an indexing set I. Moreover, assume that for each $\alpha \in I$, we have given a locally convex topology τ_α on X_α. We want to associate to this data a topology τ on X, namely, this is induced by the basic neighborhoods

$$\mathcal{B}(0) := \{B \subset X \mid B \textit{ is convex such that } B \cap X_\alpha \in \tau_\alpha(0) \textit{ for all } \alpha \in I\}.$$

This topology makes X a locally convex vector space, all inclusions $i_\alpha : X_\alpha \to X$ become continuous and this topology is the largest one with this property.

Proposition 6.2. Let X be endowed with the inductive limit topology τ, let Y be another locally convex vector space and let $A : X \to Y$ be a linear map. Then A is continuous if and only if each $A_\alpha := A|_{X_\alpha} : X_\alpha \to Y$ is continuous.

For the following result we set I the set $\mathbb{N}$ of positive integers and we have the strict restrictions $X_1 \subset X_2 \subset \cdots$ with X_n closed in X_{n+1} and $\tau_n = \tau_{n+1}|_{X_n}$.

Theorem 6.3. In the case above, a sequence $(x_n)_{n\geq 1}$ of elements in X converges to $x \in X$ in the inductive limit topology if and only if the following two conditions hold:

- There exists n_0 such that $x, x_m \in X_{n_0}$ for all m.
- $x_m \to x$ in X_{n_0}.

With the previous discussions and examples we now define the main functional spaces on any open set $\Omega \subset \mathbb{R}^n$, and summarize the main properties of these spaces.

1. We define $\mathcal{E}(\Omega) := C^\infty(\Omega)$, endowed with the locally convex topology induced by the family of seminorms $\{|| \cdot ||_{K,r}\}_{K\subset\Omega, r\in\mathbb{Z}_+}$, with K compact (see the Example 6.2). The properties of this space are the following:
    - The convergence means that $f_n \to f$ if and only if for each multi-index α and each compact $K \subset \Omega$, $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on K.
    - As a locally convex vector space, $\mathcal{E}(\Omega)$ is a Frechet space.
    - The continuous operation of multiplication of functions, makes this space an algebra over $\mathbb{C}$.
2. Let Ω be a open set of $\mathbb{R}^n$. For each $K \subset \Omega$ we consider $\mathcal{E}_K(\Omega) := C_K^\infty(\Omega)$ the space of smooth functions with support inside K, endowed with the topology induced by the family

of seminorms $\{||\cdot||_{K,r}\}_{r\in\mathbb{Z}_+}$. The **test functions** or **compactly supported smooth functions** are defined by $\mathcal{D}(\Omega) := C_c^\infty$, i.e., the space of smooth functions with compact support, or equivalently

$$\mathcal{D}(\Omega) := C_c^\infty = \bigcup_K \mathcal{E}_K(\Omega),$$

where the union is taken over all compacts $K \subset \Omega$. Then we consider the topology on $\mathcal{D}(\Omega)$ given by inductive limit topology. We have the following properties

- By the Theorem 6.3, convergence means that $f_n \to f$ in $\mathcal{D}(\Omega)$ if and only if there exists a compact K such that $f_n \in \mathcal{E}_K$ for all n, and $f_n \to f$ in $\mathcal{E}_K$.
- As a locally convex vector space, $\mathcal{D}(\Omega)$ is complete but it is not Frechet, since a space (X,τ) with the inductive limit topology cannot be metrizable, but any Frechet space are metrizable.
- With respect to pointwise multiplication, $\mathcal{D}(\Omega)$ is also an algebra over $\mathbb{C}$. Moreover, since the product between a compactly supported smooth function and an arbitrary smooth function is again compactly supported, then $\mathcal{D}(\Omega)$ is an ideal in $\mathcal{E}(\Omega)$.

3. The space of **distributions** on an open subset $\Omega \subset \mathbb{R}^n$ is defined as the topological dual of the space of test functions (see the Example 6.3) $\mathcal{D}'(\Omega) := (\mathcal{D}(\Omega))^*$, with topology given by the strong topology. An element on this space is called a **distribution** on Ω. The properties are the following.

   - Equivalently, a distribution on Ω is a linear map $u : C_c^\infty(\Omega) \to \mathbb{C}$ such that for any compact $K \subset \Omega$, there exist $C = C_K > 0$, $r = r_K \in \mathbb{N}$ such that

     $$|u(\phi)| \le C||\phi||_{K,r},$$

     for all $\phi \in C_K^\infty(\Omega)$.
   - Convergence of sequences (u_n) of distributions in the strong topology is equivalent to simple pointwise convergence.
   - Any smooth function f induces a continuous inclusion $i : \mathcal{E}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ given by $\phi \mapsto \int_{\mathbb{R}^n} f\phi$. Thus any smooth function f induces a distribution u_f. For this reason, distributions are often called **generalized distributions**.
   - The multiplication on $\mathcal{E}(\Omega)$ extends to a $\mathcal{E}(\Omega)$-module structure on $\mathcal{D}'(\Omega)$, $\mathcal{E}(\Omega) \times \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ given by $(f,u) \mapsto fu$, where $(fu)(\phi) = u(f\phi)$.

4. The space of **compactly supported distributions** on Ω is defined as the topological dual (see the Example 6.3) of the space of all smooth functions $\mathcal{E}'(\Omega) := (\mathcal{E}(\Omega))^*$, with the strong topology. We have the following properties.

   - Equivalently, a compactly supported distribution on Ω is a linear map $u : C^\infty(\Omega) \to \mathbb{C}$ such that there exist a compact $K \subset \Omega$, there exist $C > 0$, $r \in \mathbb{N}$ such that

     $$|u(\phi)| \le C||\phi||_{K,r},$$

     for all $\phi \in C_K^\infty(\Omega)$.

- The dual of the inclusion $\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$ induces a continuous inclusion $\mathcal{E}'(\Omega) \hookrightarrow \mathcal{E}(\Omega)$. Hence the four distributional spaces fit into a diagram

$$\begin{tikzcd}
\mathcal{D}(\Omega) \arrow[r] \arrow[d] & \mathcal{E}(\Omega) \arrow[d] \\
\mathcal{E}'(\Omega) \arrow[r] & \mathcal{D}'(\Omega)
\end{tikzcd}$$

  in which all the arrows are continuous inclusions, and the spaces on the left are topologically the compactly supported version of the spaces on the right.

The principal property of distributions is that one can talk about the partial derivatives of any distribution.

Lemma 6.8. Let $f \in C^\infty(\Omega)$ and let u_f be the associated distribution. Let $\partial^\alpha f \in C^\infty(\Omega)$ be the higher derivative of f associated to a multi-index α, and let $u_{\partial^\alpha f}$ the associated distribution. Then $u_{\partial^\alpha f}$ can be expressed in terms of u_f by

$$u_{\partial^\alpha f}(\phi) = (-1)^{|\alpha|} u_f(\partial^\alpha \phi).$$

This motivates the following definition:

Definition 6.18. For a distribution u on Ω and a multi-index α, one defines the new distribution $\partial^\alpha u$ on Ω by

$$(\partial^\alpha u)(\phi) = (-1)^\alpha u(\partial^\alpha \phi),$$

for all $\phi \in C_c^\infty(\Omega)$.

Definition 6.19. For any $r \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, we define the **Sobolev space** on Ω of order r as

$$H_r(\Omega) := \{u \in \mathcal{D}'(\Omega) \mid \partial^\alpha(u) \in L^2(\Omega), |\alpha| \leq r\},$$

endowed with the inner product

$$\langle u, u' \rangle_{H_r} := \sum_{|\alpha| \leq r} \langle \partial^\alpha(u), \partial^\alpha(u') \rangle,$$

thus $H_r(\Omega)$ becomes a Hilbert space.

Now we extend the local definitions of the locally convex vector spaces $\mathcal{E}(\Omega), \mathcal{E}'(\Omega), \mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ to the arbitrary smooth manifold M, and more generally to arbitrary vector bundles E over a smooth manifold M. Let M be an n-dimensional smooth manifold and let E be a complex vector bundle over M of rank p.

1. We define the **smooth sections** as $\mathcal{E}(M; E) := \Gamma(E)$, endowed with the following local convex topology: Let $\{U_i, \varphi_i\}$ be a trivialization cover of the bundle E, with (U_i, ϕ_i) local charts of M. Then the trivializations $\phi_i : E|_{U_i} \to U_i \times \mathbb{C}^p$ induce an isomorphisms of vector spaces $\psi_i : \Gamma(E|_{U_i}) \to C^\infty(\phi_i(U_i))^p$. Restricting sections of E to the various U_i's, these define an injection

$$\gamma : \Gamma(E) \to \prod_i C^\infty(\phi_i(U_i))^p = \prod_i \mathcal{E}(\phi_i(U_i))^p.$$

   Endowing the right hand side with the product topology, the topology on $\Gamma(E)$ is the induced topology via these inclusions. The space $\mathcal{E}(M; E)$ has the following properties.

- $\mathcal{E}(M;E)$ is a Frechet space.
- A sequence $(s_m)_{m\geq 1}$ converges to s if and only if, for any open U which is the domain of a local chart ϕ for M and of a local frame $\{s_1,...,s_p\}$ for E, and for any compact $K\subset U$, writing $s_m=(f_m^1,...,f_m^p)$, $s=(f^1,..,f^p)$ with respect to the frame, all the derivatives $\partial_\phi^\alpha(f_m^i)$ converge uniformly on K to $\partial_\phi^\alpha(f^i)$ when $m\to\infty$.
- When E is the trivial bundle over M, we use the notation $\mathcal{E}(M)$. $\mathcal{E}(M;E)$ is an algebra with continuous multiplication and the multiplication of sections by functions make $\mathcal{E}(M;E)$ into a module over $\mathcal{E}(M)$.

2. The **space of compactly supported smooth sections** is defined by $\mathcal{D}(M;E):=\Gamma_c(E)$, i.e., the space of all compactly supported smooth sections endowed with the topology as in the local case. More precisely, $\mathcal{D}(M;E)=\bigcup_K \mathcal{E}_K(M;E)$, where the union is over all compacts $K\subset M$, and $\mathcal{E}_K(M;E)\subset\mathcal{E}_K(M;E)$ is the space of smooth sections supported in K, endowed with the topology induced from $\mathcal{E}_K(M;E)$: on $\mathcal{D}_K(M;E)$ we consider the inductive limit topology. When E is the trivial bundle we simplify the notation to $\mathcal{D}(M)$.

3. Let V be an n-dimensional vector space, we define the **space of r-densities** of V, $D_r(V)$, as the set

$$D_r(V):=\{\omega:\bigwedge^n V\to\mathbb{C}\mid \omega(\lambda\xi)=|\lambda|^r\omega(\xi), for\ all\ \xi\in\bigwedge^n V\},$$

or equivalently

$$D_r(V):=\{\omega:\mathrm{Fr}(V)\to\mathbb{C}\mid \omega(A(e))=|\det|^r\omega(e), for\ all\ e\in\mathrm{Fr}(V)\ and\ A\in\mathrm{Gl}_n(\mathbb{C})\}.$$

Intuitively, each frame e determines a hypercube, so we can think about a r-density of V as some rule of computing volumes of the hypercubes.

For a manifold M, we apply this construction to all the tangent spaces of M, TM, to obtain a line bundle $D_r(M)$ over M, whose fiber at $x\in M$ is $D_r(T_xM)$. In particular, for $r=1$, $D_1(M)$ is simply denoted by D or D_M. The compactly supported sections can be integrated over M without any further choice, i.e., there is an integral

$$\int_M:\Gamma_c(D)\to\mathbb{C}.$$

Now we consider the bundle $E^\vee:=E^*\otimes D=\mathrm{Hom}(E,D)$ with pairing $\langle -,-\rangle:\Gamma(E^\vee)\times\Gamma(E)\to\Gamma(D)$ given by evaluation, and its respective version with supports. Then using the integration of sections of D, we get canonical pairings $[-,-]:\Gamma_c(E^\vee)\times\Gamma(E)\to\mathbb{C}$ given by $(s_1,s_2)\mapsto\int_M\langle s_1,s_2\rangle$. Thus we define the space of **generalized sections** of E by $\mathcal{D}'(M,E):=(\mathcal{D}(M,E^\vee))^*$, endowed with the strong topology. When E is the trivial bundle, then we simplify the notation to $\mathcal{D}'(M)$. Some properties of the space $\mathcal{D}'(M,E)$.

- We have the canonical inclusions $\mathcal{E}(M;E)\hookrightarrow\mathcal{D}'(M;E)$, sending a section s to the functional $u_s:=\langle\cdot,s\rangle$. As before, we identify s with the induced distribution u_s.
- $\mathcal{D}'(M;E)$ is a module over $\mathcal{E}(M)$, with continuous multiplication $\mathcal{E}(M)\times\mathcal{D}'(M;E)\to\mathcal{D}'(M;E)$ defined by $(fu)(s)=u(fs)$.

4. The **compactly supported generalized sections** of E is defined as $\mathcal{E}'(M;E) := (\mathcal{E}(M,E^\vee))^*$. By the same pairing as before, one obtains an inclusion $\mathcal{D}(M;E) \hookrightarrow \mathcal{E}'(M,E)$. Therefore we obtain a diagram of inclusions

$$\begin{tikzcd} \mathcal{D}(M,E) \arrow[r] \arrow[d] & \mathcal{E}(M,E) \arrow[d] \\ \mathcal{E}'(M,E) \arrow[r] & \mathcal{D}'(M,E). \end{tikzcd}$$

Given two vector bundles $E \to M$ and $F \to N$, over smooth manifolds M and N, an isomorphism $h : E \to F$ of vector bundles induces isomorphisms between the four functional spaces $\mathcal{E}, \mathcal{E}', \mathcal{D}$ and $\mathcal{D}'$ of E and those of F, denoted by the same letter h_*.
With this space in hand we discuss the definition of operators on these spaces.

Definition 6.20. If E is a vector bundle over M and F is a vector bundle over N, a **general operator** from E to F is a linear continuous map

$$P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F).$$

A **smoothing operator** is a general operator $P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F)$ which

- P take values in $\mathcal{E}(N,F)$.
- P extends to a continuous linear map from $\mathcal{E}'(M,E)$ to $\mathcal{E}(N,F)$.

The set of smoothing operators is noted by $\Psi^{-\infty}(E,F)$.

Remark 6.8. The extension of the general operator $P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F)$ of a continuous linear map from $\mathcal{E}'(M,E)$ to $\mathcal{E}(N,F)$ is unique, since we have the inclusions $\mathcal{D}(M,E) \subset \mathcal{E}'(M,E)$, $\mathcal{E}(N,F) \subset \mathcal{D}'(N,F)$ and $\mathcal{D}(M,E)$ is dense in all other functional spaces that we have discussed.

Now replacing the Sobolev space $H_r(\mathbb{R}^n)$ defined in the Definition 6.19 by their local versions denoted by $H_{r,\mathrm{loc}}$, we extend the definition of the Sobolev space $H_r(\mathbb{R}^n)$ on smooth manifolds or on vector bundles over smooth manifolds. These local versions satisfy certain axioms (namely, invariance under changes of coordinates) can be extended to all vector bundles over a manifold. The resulting Sobolev space on vector bundles on smooth manifolds, denoted by $H_r(M,E)$, is the Banach space of sections that we want to obtain.

Next we explain in more detail the construction of the Sobolev space $H_r(\mathbb{R}^n)$.

Definition 6.21. A **functional space** on $\mathbb{R}^n$ is a locally convex vector space $\mathcal{F}$ satisfying:

- $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{F} \subset \mathcal{D}'(\mathbb{R}^n)$, where the inclusions are continuous linear maps.
- For all $\phi \in \mathcal{D}$, multiplication by ϕ defines a continuous map $m_\phi : \mathcal{F} \to \mathcal{F}$.

Similarly, given a vector bundle E over a smooth manifold M, we can define a functional space on M with coefficients in E.

Notice that the space $\mathcal{D}(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ are functional spaces.

Given a functional space $\mathcal{F}$ on $\mathbb{R}^n$, we define the following functional spaces:

- For any compact $K \subset \mathbb{R}^n$ the set

$$\mathcal{F}_K := \{u \in \mathcal{F} \mid \operatorname{supp}(u) \subset K\},$$

endowed with the topology induced from $\mathcal{F}$, and we also define

$$\mathcal{F}_{\text{comp}} := \cup_K \mathcal{F}_K,$$

where the union is taken over all compacts in $\mathbb{R}^n$, endowed with the inductive limit topology.

- The space

$$\mathcal{F}_{\text{loc}} := \{u \in \mathcal{D}'(\mathbb{R}^n) \mid \phi u \in \mathcal{F},\ \textit{for all}\ \phi \in C_c^\infty(\mathbb{R}^n)\},$$

with locally convex topology induced by the family $\{q_{p,\phi} \mid p \in P,\ \phi \in C_c^\infty(\mathbb{R}^n)\}$, where P is a family of seminorms defining the locally convex topology on $\mathcal{F}$ and the seminorm $q_{p,\phi}$ on $\mathcal{F}_{\text{loc}}$ is given by $q_{p,\phi}(u) = p(\phi u)$.

Now we list the axioms that permit us to pass from the Sobolev space $H_r(\mathbb{R}^n)$ to Sobolev spaces on vector bundles over smooth manifolds:

1). **Banach axiom**: Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

- $\mathcal{F}$ is **Banach** if the topology of $\mathcal{F}$ is a Banach topology.
- $\mathcal{F}$ is **locally Banach** if, for each compact $K \subset \mathbb{R}^n$, the topology of $\mathcal{F}_K$ is a Banach topology.

The same axiom applies for **Frechet**, **locally Frechet**, **Hilbert** and **locally Hilbert** functional spaces on $\mathbb{R}^n$, or more generally, on a functional space on M with coefficients in E. Notice that if $\mathcal{F}$ is Banach (Frechet, Hilbert) space, then it is also locally Banach (Frechet, Hilbert) space. However, the converse is not true. For manifolds, we need this axiom in order to preserve the property of $H_r(\mathbb{R}^n)$ to be a Hilbert space when we pass to manifolds.

2). **Invariance axiom**: Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$, we say that:

- $\mathcal{F}$ is **invariant** if for any diffeomorphism $\chi : \mathbb{R}^n \to \mathbb{R}^n$, we have a topological isomorphism $\chi_* : \mathcal{F} \xrightarrow{\cong} \mathcal{F}$.
- $\mathcal{F}$ is **locally invariant** if for any diffeomorphism $\chi : \mathbb{R}^n \to \mathbb{R}^n$ and any compact $K \subset \mathbb{R}^n$, we have a topological isomorphism $\chi_* : \mathcal{F}_K \xrightarrow{\cong} \mathcal{F}_{\chi(K)}$.

Similarly, we talk about invariance and local invariance of functional spaces on vector bundles over smooth manifolds. Notice that invariance implies locally invariant, but the converse is not true. We need invariance of $\mathcal{F}$ under changes of coordinates in order to pass to manifolds.

3). **Density axiom**: Let $\mathcal{F}$ be a functional space on $\mathbb{R}^n$. We say that:

- $\mathcal{F}$ is **normal** if $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{F}$.
- $\mathcal{F}$ is **locally normal** if for any compact $K \subset \mathbb{R}^n$, $\mathcal{F}_K$ is contained in the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{F}$.

Similarly, we talk about normal and locally normal functional spaces on vector bundles over manifolds. Notice that normal implies locally normal.

4). **Locality axiom**: We say that a functional space $\mathcal{F}$ is **local** if, as locally convex vector space $\mathcal{F} = \mathcal{F}_{\text{loc}}$. Similarly we talk about local functional spaces on vector bundles over smooth manifolds. This axiom allow us to pass to open sets $\Omega \subset \mathbb{R}^n$ without loosing the properties of the functional space, since for any functional space $\mathcal{F}$ and for any compact $K \subset \mathbb{R}^n$ we have that $(\mathcal{F}_{\text{loc}})_K = \mathcal{F}_K$, as locally convex vector space, thus $\mathcal{F}$ is locally Banach (or locally Frechet, or locally Hilbert), or locally invariant, or locally normal if and only if $\mathcal{F}_{\text{loc}}$ is.

Therefore, if $\mathcal{F}$ is a local functional space on $\mathbb{R}^n$, then we have the following equivalences:

- $\mathcal{F}$ is locally Frechet if and only if it is Frechet.
- $\mathcal{F}$ is locally invariant if and only if it is invariant.
- $\mathcal{F}$ is locally normal if and only if it is normal.

A similar statement is not obtained for locally Banach spaces, since local spaces cannot be Banach. In order to maintain the property of being a Banach space when we pass to manifolds, we restrict to opens $\Omega \subset \mathbb{R}^n$.

As any compactly supported distribution on an open set $\Omega \subset \mathbb{R}^n$ can be seen as a compactly supported distribution on $\mathbb{R}^n$, we have the canonical inclusions $\mathcal{E}'(\Omega) \subset \mathcal{E}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. On the other hand we have that $\phi u \in \mathcal{E}'(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$. Hence the following makes sense.

Definition 6.22. Given a local functional space $\mathcal{F}$, for any open $\Omega \subset \mathbb{R}^n$, we define

$$\mathcal{F}(\Omega) := \{u \in \mathcal{D}'(\Omega) \mid \phi u \in \mathcal{F}, for\ all\ \phi \in C_c^\infty(\Omega)\},$$

endowed with the same topology as in the case of the space $\mathcal{F}_{\text{loc}}$.

Next we have the theorem that assures that the previous axioms are still fulfilled when we pass to opens.

Theorem 6.4. For any local functional space $\mathcal{F}$ and any open $\Omega \subset \mathbb{R}^n$, $\mathcal{F}(\Omega)$ is a local functional space on Ω and, as such,

- $\mathcal{F}(\Omega)$ is locally Banach (or Hilbert, or Frechet) if $\mathcal{F}$ is.
- $\mathcal{F}(\Omega)$ is invariant if $\mathcal{F}$ is.
- $\mathcal{F}(\Omega)$ is normal if $\mathcal{F}$ is.

Corollary 6.2. Let $\mathcal{F}$ be a local functional space. If $\mathcal{F}$ is invariant then, for any diffeomorphism $\chi : \Omega_1 \to \Omega_2$ between two opens in $\mathbb{R}^n$, we have a topological isomorphism $\chi_* : \mathcal{F}(\Omega_1) \to \mathcal{F}(\Omega_2)$.

Now we explain how to induce the functional space $\mathcal{F}(M, E)$, with $\mathcal{F}$ a local, invariant functional space on $\mathbb{R}^n$, for any vector bundle E over an n-dimensional smooth manifold M.

Given a vector bundle $E \to M$ of rank p, we say that the triple (U, φ, τ) is a **total trivialization** for E over U if $U \subset M$ is an open set such that (U, φ) is a coordinate chart for M and $\tau : E|_U \to U \times \mathbb{C}^p$ is a local trivialization for E. Thus in particular, the data (U, φ, τ) defines an isomorphism h between the vector bundle $E|_U$ over U and the trivial bundle $\varphi(U) \times \mathbb{C}^p$ given by $h(e_x) = (\varphi(x), \tau(e_x))$, hence any total trivialization (U, φ, τ) induces isomorphisms

$$h_{\varphi,\tau} : \mathcal{D}'(U, E|_U) \to \mathcal{D}'(\Omega_\phi)^p,\ \Omega_\phi := \varphi(U) \subset \mathbb{R}^n$$

and the same for the other functional spaces.

Definition 6.23. We define the **functional space of generalized sections of E of type $\mathcal{F}$**, denoted by $\mathcal{F}(M,E)$, as the space of all $u \in \mathcal{D}'(M,E)$ with the property that for any domain U of a total trivialization of E, $h_{\varphi,\tau}(u|_U) \in \mathcal{F}(\Omega_\phi)^p$.

The topology of the functional space $\mathcal{F}(M,E)$ is given in the following way: Let $\{U_i\}_i \in I$ be a open cover of the smooth manifold M, such that these are domains of total trivialization charts (U_i, ϕ_i, τ_i). Then we have the inclusions

$$\gamma : \mathcal{F}(M,E) \to \prod_{i\in I} \mathcal{F}(\Omega_{\phi_i})^p,$$

thus $\mathcal{F}(M,E)$ is endowed with the induced topology.

The key theorem is the following.

Theorem 6.5. For any vector bundle $E \to M$ over an n-dimensional smooth manifold M, $\mathcal{F}(M,E)$ is a local functional space on (M,E). Moreover, if $\mathcal{F}$ is locally Banach, or locally Hilbert, or locally Frechet, or locally normal, then so is $\mathcal{F}(M,E)$. Finally, if F is a vector bundle over a another n-dimensional smooth manifold N and $h : E \to F$ is an isomorphism of vector bundles, then $h_* : \mathcal{D}'(M,E) \to \mathcal{D}'(N,F)$ restricts to an isomorphism of locally convex vector spaces $h_* : \mathcal{F}(M,E) \to \mathcal{F}(N,F)$.

Corollary 6.3. If $\mathcal{F}$ is locally Banach (or Hilbert) and normal then, for any vector bundle E over a compact n-dimensional smooth manifold M, $\mathcal{F}(M,E)$ is a Banach (or Hilbert) space which contains $\mathcal{D}(M,E)$ as a dense subspace.

Definition 6.24. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local, invariant functional spaces on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and assume that $\mathcal{F}_1$ is normal. Let M be an m-dimensional manifold and N an n-dimensional one, and let $E \to M$ and $F \to N$ be vector bundles. We say that a general operator $P : \mathcal{D}(M,E) \to \mathcal{D}'(N,F)$ is of type $(\mathcal{F}_1, \mathcal{F}_2)$ if it takes values in $\mathcal{F}_2$ and extends to a continuous linear operator

$$P_{\mathcal{F}_1,\mathcal{F}_2} : \mathcal{F}_1(M,E) \to \mathcal{F}_2(N,F).$$

We apply the previous constructions to the Sobolev space on $\mathbb{R}^n$

$$H_r(\mathbb{R}^n) := \{u \in \mathcal{D}'(\mathbb{R}^n) \mid \partial^\alpha(u) \in L^2(\mathbb{R}^n), |\alpha| \leq r\}.$$

These spaces are Hilbert spaces, $\mathcal{D}$ is dense in H_r and using pseudo-differential operators we can prove that the H_r are locally invariant (see reference [Ban-Crainic]). $H_r(\mathbb{R}^n)$ is not local, but in general for any functional space $\mathcal{F}$, $\mathcal{F}_{\text{loc}}$ is local. Thus we consider the associated local spaces

$$H_{r,\text{loc}} := \{u \in \mathcal{D}' \mid \phi u \in H_r,\ \textit{for all } \phi \in \mathcal{D}\},$$

and the theory that we have developed implies that these spaces are a functional spaces which is locally Hilbert, invariant and normal. Hence these spaces extend to manifolds.

Definition 6.25. For a vector bundle E over an n-dimensional smooth manifold M.

- The resulting functional space $H_{r,\text{loc}}(M,E)$ are called the **local r-Sobolev spaces** of E.
- For $K \subset M$ compact, the resulting K-supported spaces are denoted $H_{r,K}(M,E)$.
- The resulting compactly supported spaces are denoted

$$H_{r,\text{comp}}(M,E) := \bigcup_{K\ \textit{compact}} H_{r,K}(M,E),$$

with the inductive limit topology.

If M is compact, we define the r**-Sobolev space** of E as

$$H_r(M,E) := H_{r,\mathrm{loc}}(M,E) = H_{r,\mathrm{comp}}(M,E).$$

Definition 6.26. A linear map $T : V \to W$ between vector spaces is said **compact** if for any bounded sequence $\{x_n\}$ in V, $\{T(x_n)\}$ has a convergent subsequence.

Corollary 6.4. For any vector bundle $E \to M$ over a smooth manifold M,

- $H_{r,\mathrm{loc}}(M,E)$ are Frechet spaces.
- $\mathcal{D}(M,E)$ is dense in $H_{r,\mathrm{loc}}(M,E)$.
- For $K \subset M$ compact, $H_{r,K}(M,E)$ has a Hilbert topology and, for $r > s$, the inclusion $H_{r,K}(M,E) \hookrightarrow H_{s,K}(M,E)$ is compact.

Corollary 6.5. For any vector bundle $E \to M$ over a compact smooth manifold M, $H_r(M,E)$ has a Hilbert topology, contains $\mathcal{D}(M,E)$ as a dense subspace,

$$\bigcap_r H_{r,\mathrm{loc}}(M,E) = \Gamma(E),$$

and for $r > s$, the inclusion $H_r(M,E) \hookrightarrow H_s(M,E)$ is compact.

Finally we have the following propositions. The first one says that differential operators of order k are also operators of type (H_r, H_{r-k}).

Proposition 6.3. For $r \geq k \geq 0$, given a differential operator $P \in \mathcal{D}_k(E,F)$ between two vector bundles over M, the operator

$$P : \mathcal{D}(M,E) \to \mathcal{D}(M,F)$$

admits a unique extension to a continuous linear operator

$$P_r : H_{r,\mathrm{loc}}(M,E) \to H_{r-k,\mathrm{loc}}(M,F).$$

And the second one says that smoothing operators on compact manifolds, viewed as operators of type (H_r, H_s), are compact.

Proposition 6.4. Let E and F be two vector bundles over a compact manifolds M and consider a smoothing operator $P \in \Psi^{-\infty}(E,F)$ (see Definition 6.20). Then for any r and s, P viewed as an operator

$$P : H_r(M,E) \to H_s(M,F)$$

is compact.

Remember that the principal goal in this discussion is to show the general strategy of proving that the analytical index of an elliptic differential operator $P \in \mathcal{D}_k(E,F)$ between two vector bundles E and F over a compact smooth manifold M is well-defined. We have finally produced our Banach spaces of sections on which our operator will act, namely

$$P_r : H_r(M,E) \to H_{r-k}(M,F).$$

The plan is to use the theory of Fredholm operators between Banach spaces.

Definition 6.27. Consider $\mathbb{E}$ and $\mathbb{F}$ Banach spaces and let $\mathcal{L}(\mathbb{E}, \mathbb{F})$ be the space of bounded operators from $\mathbb{E}$ to $\mathbb{F}$, i.e., continuous linear operators. Let

$$\mathcal{F}(\mathbb{E}, \mathbb{F}) := \{A \in \mathcal{L}(\mathbb{E}, \mathbb{F}) |\dim(\ker A) < \infty, \dim(\operatorname{coker} A) < \infty\},$$

denoted by the space of **Fredholm operators**. The **index** of a Fredholm operator A is defined by

$$\operatorname{Index}(A) := \dim(\operatorname{Ker}(A)) - \dim(\operatorname{Coker}(A)).$$

Proposition 6.5. Let $\mathbb{E}, \mathbb{F}, \mathbb{G}$ be Banach spaces and $B : \mathbb{E} \to \mathbb{F}$, $A : \mathbb{F} \to \mathbb{G}$ bounded operators, then:

1. If any two of A, B, $A \circ B$ are Fredholm, then the third one is also Fredholm and

$$\operatorname{Index}\ (A \circ B) = \operatorname{ind}\ (A) + \operatorname{ind}(B).$$

2. $\mathcal{F}(\mathbb{E}, \mathbb{F})$ is open in $\mathcal{L}(\mathbb{E}, \mathbb{F})$ and $\operatorname{Index} : \mathcal{F}(\mathbb{E}, \mathbb{F}) \to \mathbb{Z}$ is continuous (i.e. locally constant).

The relation between compact operators and Fredholm operators are given by the Atkinson characterization theorem for Fredholm operators.

Theorem 6.6. Given a bounded operator $A : \mathbb{E} \to \mathbb{F}$, the following are equivalent:

- A is Fredholm.
- A is invertible modulo compact operators, i.e., there exist an operator $B \in \mathcal{L}(\mathbb{F}, \mathbb{E})$ and compact operators K_1 and K_2 such that

$$BA = \operatorname{Id} + K_1,\ AB = \operatorname{Id} + K_2.$$

Thus to prove that $P_r : H_r(M, E) \to H_{r-k}(M, F)$ is Fredholm, using the previous theorem and the fact that all smoothing operators are compact, we would need some kind of inverse of P_r modulo smoothing operators, i.e., some kind of operator of order $-k$, $Q : H_{r-k}(M, F) \to H_r(M, E)$, going backwards, such that $PQ - \operatorname{Id}$ and $QP - \operatorname{Id}$ are smoothing operators. Such operators Q cannot be differential, so this brings us to pseudo-differential operators. For more details on this topics see [Ban-Crainic].

Now the principal theorem.

Theorem 6.7. Let E and F be vector bundles over a compact smooth manifold M. Let $d \in \mathbb{R}$ and let $P : \mathcal{D}(M, E) \to \mathcal{D}(M, F)$ be elliptic. For all $s \in \mathbb{R}$ the operator $P_s : H_s(M, E) \to H_{s-d}(M, F)$ is Fredholm and their index is independent of s. In particular, the operator P has a well defined index.

Chapter 7

Atiyah-Singer index theorem and applications

The commutativity of the right side of the diagram 1.1 will be discussed in this chapter, giving the cohomological formulation of the Atiyah-Singer index theorem. Finally we discuss some instances of the Atiyah-Singer index theorem, namely, the Gauss-Bonnet's theorem, the Chern-Gauss-Bonnet's theorem, the Hirzebruch's signature theorem and the Hirzebruch-Riemann-Roch's theorem.

1 The cohomological formula for the index

First we need some results on De Rham cohomology. For details and proofs the canonical reference is [Bott-Tu].

Let X be an oriented n-dimensional manifold which is not necessarily compact. Remember that a cochain c has compact support if there is a compact subset $K \subset X$ such that $c(\sigma) = 0$ for any chain σ which does not meet K. Let $H^*_{cpt}(X)$ denote the cohomology of the complex of rational singular cochains with compact support on X. We have Poincaré Duality given by

$$D_X : H^p_{\mathrm{cpt}}(X) \xrightarrow{\cong} H_{n-p}(X).$$

Let Y be another m-dimensional manifold and $f : Y \to X$ be a continuous map. For each $p \geq m - n$ we define the **integration over the fiber**

$$f_! : H^p_{\mathrm{cpt}}(Y) \to H^{p-(m-n)}_{\mathrm{cpt}}(X)$$

given by $f_! = D_X^{-1} f_* D_Y$, where f_* is the usual map induced on homology.

Let $\pi : E \to X$ be the oriented vector bundle of rank k given by the projection and denote by $i : X \to E$ the zero section. As π and i are homotopy equivalences, then π_* and i_* induce isomorphisms on homology with $\pi_* i_* = \mathrm{Id}$. Therefore both $\pi_! : H^{p+k}_{\mathrm{cpt}}(E) \to H^p_{\mathrm{cpt}}(X)$ and $i_! : H^p_{\mathrm{cpt}}(X) \to H^{p+k}_{\mathrm{cpt}}(E)$ are isomorphisms for all p. Using De Rham forms we remark that $\pi_!$ really is the integration over the fiber. We have $\pi_* i_* = i_* \pi_* = \mathrm{id}$ and

$$\pi_! i_! = D_X^{-1} \pi_* D_E D_E^{-1} i_* D_X = \mathrm{id} = i_! \pi_!$$

and so $\pi_! = (i_!)^{-1}$.

Definition 7.1. The isomorphism $i_! : H^p_{\text{cpt}}(X) \to H^{p+k}_{\text{cpt}}(E)$ is called the **Thom isomorphism** of E for compactly supported cohomology.

We also have the following result.

Lemma 7.1. If X is compact, then for all $u \in H^*(X) = H_{\text{cpt}}(X)$, one has

$$i^* i_!(u) = \chi(E) \cdot u,$$

where $\chi(E)$ is the Euler class of E.

We have defined the Chern character $\text{ch} : K(X) \to H^{\text{even}}(X)$. This homomorphism has a direct extension $\text{ch} : K_{\text{cpt}}(X) \to H^{\text{even}}_{\text{cpt}}(X)$ to the case of compact supports. For any given complex vector bundle $\pi : E \to X$, we have also defined Thom isomorphisms given by the Corollary 3.5 and the Definition 7.1

$$K_{\text{cpt}}(X) \xrightarrow{i_!} K_{\text{cpt}}(E)$$

and

$$H_{\text{cpt}}(X) \xrightarrow{i_!} H_{\text{cpt}}(E).$$

It is natural to ask whether $i_!\text{ch} = \text{ch}i_!$, i.e., whether the following diagram commutes

$$\begin{tikzcd} K_{\text{cpt}}(X) \arrow[r, "i_!"] \arrow[d, "\text{ch}"'] & K_{\text{cpt}}(E) \arrow[d, "\text{ch}"] \\ H^{\text{even}}_{\text{cpt}}(X) \arrow[r, "i_!"] & H^{\text{even}}_{\text{cpt}}(E). \end{tikzcd}$$

This is not true in general and the resulting correction term is of basic importance. To each complex vector bundle $\pi : E \to X$, with X a compact manifold, we associated the class

$$\mathfrak{I}(E) = \pi_!\text{ch } i_!(1).$$

For each $\xi \in K_{cpt}(X)$ we have that

$$\pi_!\text{ch } i_!\xi = \pi_!\text{ch}(i_!(1) \cdot \pi^*\xi) = \pi_![\text{ch } i_!(1) \cdot \text{ch}\pi^*\xi] = [\pi_!\text{ch } i_!(1)]\text{ch}\xi,$$

and so, since $\pi_! = (i_!)^{-1}$ on $H^*_{\text{cpt}}(E)$, we have that

$$(i_!)^{-1}\text{ch}(i_!\xi) = \mathfrak{I}(E)\text{ch}\xi, \tag{7.1}$$

so, $\mathfrak{I}(E)$ is just the commutativity defect mentioned above. Notice that $\mathfrak{I}(E)$ is natural, i.e., $f^*\mathfrak{I}(E) = \mathfrak{I}(f^*(E))$ for the pull-back on any continuous map between manifolds. Moreover we have that

$$i_!\mathfrak{I}(E) = i_!\pi_!\text{ch } i_!(1) = \text{ch}\lambda_E$$

i.e., $i_!\mathfrak{I}(E) = \text{ch}\lambda_E$, where λ_E is the Bott element (see Definition 3.19). With this in hand and the Propositions 7.1 and the Proposition 3.11

$$\chi(E)\mathfrak{I}(E) = i^*i_!\mathfrak{I}(E) = i^*\text{ch}\lambda_E = \text{ch } i^*\lambda_E = \text{ch } i^*i_!(1) = \text{ch } \lambda_{-1}(E).$$

Therefore

$$\mathfrak{I}(E) = \frac{\text{ch } \lambda_{-1}(E)}{\chi(E)}.$$

If we split E as sum of line bundles, then $c(E) = \prod(1 + x_j)$, and we find from the Equation 4.1 and using the fact that $\chi(E) = x_1 \cdots x_n$, where $x_k := \chi(E_k)$, for all k,

$$\mathfrak{I}(E) = \prod_{k=1}^{n} \frac{1 - e^{x_k}}{x_k}$$

So, from the Equation 5.10 and the Remark 4.1 we can rewrite this as

$$\mathfrak{I}(E) = (-1)^n \mathbf{Td}_{\mathbb{C}}(\overline{E})^{-1}. \tag{7.2}$$

Let X be a manifold, then the tangent bundle TX is canonically an almost complex manifold since $T(TX) = \pi^*TX \oplus \pi^*TX \cong \pi^*TX \otimes \mathbb{C}$. This gives TX a canonical orientation as a manifold. A positively oriented basis of $T(TX)$ is of the form $(e_1, Je_1, e_2, Je_2, ..., e_n, Je_n)$ where $e_1, ..., e_n$ is a basis of π^*TX and J carries the horizontal to the vertical factor. With this orientation we can evaluate any element $u \in H^{2n}_{\text{cpt}}(TX)$ on the fundamental class $[TX]$ of the manifold. The result is denoted by $u[TX]$.

Theorem 7.1. Let P be an elliptic operator over a compact manifold X of dimension n. Then

$$\text{ind}P = (-1)^n \{\text{ch}\sigma(P) \cdot \hat{\mathbf{A}}(X)^2\}[TX],$$

where $\hat{\mathbf{A}}(X)$ denotes the total $\hat{A}$-class of X pulled back to TX.

Proof. As the construction of the pullback in K-theory, we take the scrunch map $g : T\mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{C}^N \to \{*\}$ viewed as a complex bundle and consider the the inclusion map $i : \{*\} \hookrightarrow \mathbb{C}^N$ as the origin. Fix an element $\xi \in K_{\text{cpt}}(\mathbb{C}^N)$ and we apply the defect formula given by the Equation 7.1 with $u = i_!\xi$. Recalling that $g_! = (i_!)^{-1}$, notice that $\mathfrak{I}(\mathbb{C}^N) = 1$, and that $\text{ch} : K(\{*\}) \to H^0(\{*\})$ is an isomorphism, we have that $g_!\text{ch } u = g_!u$. Since $g_!$ on H^*_{cpt} is just the integration over the fiber, we find that

$$g_!u = \text{ch}u[T\mathbb{R}^N]. \tag{7.3}$$

Consider now a real vector bundle $p : \nu \to X$ and let $i : X \to \nu$ denote the inclusion as the zero section. Taking derivatives gives the bundle $\pi := dp : T\nu \to TX$ with zero section $i : TX \to T\nu$. Notice that the bundle $p : T\nu \to TX$ is equivalent to $\pi^*\nu \oplus \pi^*\nu = \pi^*\nu \otimes \mathbb{C}$. While TX is not compact, the map i is proper and the Equation 7.1 can be shown to hold for elements of $K_{\text{cpt}}(TX)$. Thus for any $\sigma \in K_{\text{cpt}}(TX)$ we have

$$p_!\text{ch } i_!\sigma = \mathfrak{I}(\nu \otimes \mathbb{C})\text{ch } \sigma.$$

As $p_!$ is integration over the fiber, evaluating on the fundamental class gives the formula

$$(\text{ch } i_!\sigma)[T\nu] = \{\mathfrak{I}(\nu \otimes \mathbb{C})\text{ch } \sigma\}[TX]. \tag{7.4}$$

Consider now an embedding $f : X \hookrightarrow \mathbb{R}^N$ with normal bundle ν. We identify ν with an open tubular neighborhood of $f(X)$ in $\mathbb{R}^N$. Similarly we have an open embedding $T\nu \subset T\mathbb{R}^N$ as a tubular neighborhood of $f(TX)$. Given any $\sigma \in K_{\text{cpt}}(TX)$, the class $i_!\sigma$ has compact support in $T\nu$ and naturally extends to $T\mathbb{R}^N$ under the open inclusion $T\nu \subset T\mathbb{R}^N$. This extended class is, by definition, the element $f_!\sigma \in K_{\text{cpt}}(T\mathbb{R}^N)$ given in the Equation 3.7. In particular we have

$$\text{ch}(i_!\sigma)[T\nu] = \text{ch}(f_!\sigma)[T\mathbb{R}^N].$$

Since $T\nu \subset T\mathbb{R}^N$, by the Equations 7.3 and 7.4, by the Definition 6.12 and Theorem 6.1 we have that

$$\mathrm{ind}P = \{\mathfrak{I}(\nu \otimes \mathbb{C})\mathrm{ch}\ \sigma(P)\}[TX].$$

It remains to identify $\mathfrak{I}(\nu \otimes \mathbb{C})$. For this we note that since ν is the normal bundle to X, we have $TX \oplus \nu$ is the trivial bundle. Since $\mathfrak{I}$ is multiplicative, this means that $\nu \otimes \mathbb{C} = \mathfrak{I}(TX \otimes \mathbb{C})^{-1}$. Applying Formula 7.2 we see that

$$\mathfrak{I}(TX \otimes \mathbb{C})^{-1} = (-1)^n \mathbf{Td}_{\mathbb{C}}(TX \otimes \mathbb{C}).$$

We have used here that $TX \otimes \mathbb{C}$ is self-conjugate. Finally by the Proposition 5.4 we have

$$\mathrm{ind}P = (-1)^n \{\mathrm{ch}\ \sigma(P) \cdot \hat{\mathbf{A}}(X)^2\}[TX].$$

□

2 Gauss-Bonnet theorem

We will now see how the Chern classes are related to the Euler characteristic of surfaces. Since Chern classes are defined for complex vector bundles, we first have to see how to define the Chern class of the tangent bundle to a surface.

Lemma 7.2. The classification of complex line bundles over a space X is equivalent to the classification of oriented real bundles of rank 2. More precisely, the function:

$$F : \{\text{Iso classes of complex line bundles overX}\} \to \{\text{Iso classes of rank 2 oriented bundles overX,}\}$$

which sends a complex line bundle to the underlying real bundle with the orientation such that (v, iv) is an oriented basis, is bijective.

Proof. First, we need to prove that the function F is well defined, that is, that the rule that (v, iv) is an oriented basis is well defined. Suppose that we choose a different $v' = z \cdot v = (a + bi)v = av + b(iv)$ then, $iv' = -bv + a(iv)$. We see that the determinant of the change of basis matrix is $z\overline{z} > 0$, so the orientation is well defined. Let us now construct an inverse G to F. Let E be an oriented rank two vector bundle over X and choose a riemannian metric $g = \langle , \rangle$ on E. Then, g determines a complex structure on E by setting:

$$iv := \text{ the unique vector such that:} |v| = |iv|, \langle v, iv\rangle = 0, \text{and}\, (v, iv) \text{ is oriented.}$$

This construction gives E the structure of a complex vector bundle. We need to prove that the isomorphism class of this bundle is independent of the metric. Suppose there are two metrics g and g', and denote by g_t the metric:

$$g_t = (1 - t)g + tg'.$$

Now consider the complex vector bundle $\pi^*(E)$ over $X \times I$ whose fiber over a point (x, t) is the real vector bundle E_x with complex structure given by the metric g_t. We have proved before that in this situation the restriction of the vector bundle to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic. We conclude that the complex bundles associated to g and g' are isomorphic. □

By the previous lemma we can define the first Chern class of an oriented rank 2 vector bundle to be the first Chern class of the corresponding complex line bundle.

Theorem 7.2 (Gauss-Bonnet)**.** Let Σ be a compact oriented surface. Then:

$$\int_\Sigma c_1(T\Sigma) = \chi(\Sigma).$$

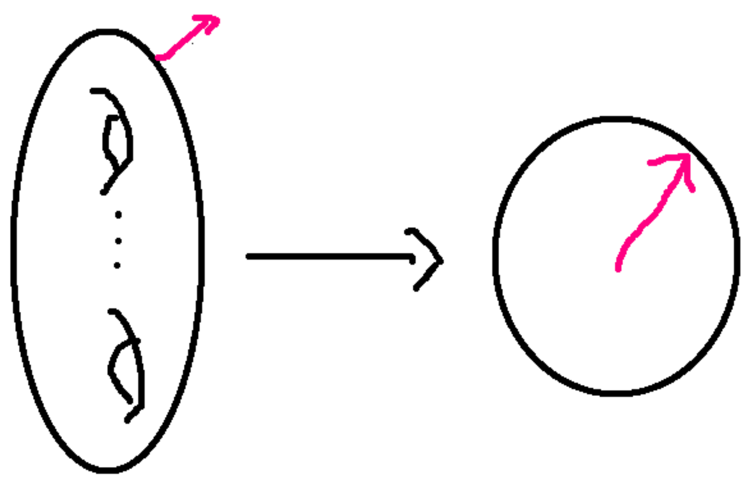
Proof. By the classification of surfaces [Gallier-Xu], we know that $\Sigma = \Sigma_g$ the surface of genus g for some $g \geq 0$. This surface can be embedded into $\mathbb{R}^3$, let us consider an arbitrary embedding $\iota : \Sigma \to \mathbb{R}^3$. This embedding determines a function $f_\iota : \Sigma \to \mathbb{S}^2$ defined by the condition that $f_\iota(p)$ is the unique vector which is normal to $T\Sigma_p$, has norm one and (x, y, v) is an oriented basis of $\mathbb{R}^3$ whenever (a, b) is an oriented basis of $T\Sigma$. Clearly, this map has the property that:

$$f_\iota^*(T\mathbb{S}^2) = T\Sigma,$$

and therefore:

$$\int_\Sigma c_1(T\Sigma) = \int_\Sigma f_\iota^*(c_1(T\mathbb{S}^2)), \tag{7.5}$$

in particular, the right hand side is independent of the embedding ι. In order to compute this number we may choose a particularly simple embedding of Σ. Let us choose the embedding in the following picture:



Let us denote by N the north pole of the sphere. Then $f_\iota^{-1}(N)$ consists of $g+1$ points: the highest point in Σ and the low sadle points in each of the wholes. Let us denote these points by $p_0, \ldots, p_g$. The function f_ι is a local diffeomorphism around each of the points p_i, therefore we can choose neihgborhoods U_i of p_i such that $f_\iota : U_i \to U$ is a diffeomorphism to some neighborhood U of N. Now in order to compute (Equation 7.5) we may use any differential form ω representing the Chern class of $T\mathbb{S}^2$. We have that:

$$\int_{\mathbb{S}^2} c_1(T\mathbb{S}^2) = \int_{\mathbb{CP}^2} c_1(T\mathbb{CP}^2) = 2.$$

Thus, we can choose ω to be any form whose integral over the sphere is 2. Let us choose ω to be supported on U. Then:

$$\int_\Sigma f_\iota^*(c_1(T\mathbb{S}^2)) = \int_\Sigma f_\iota^*(\omega) = \sum_{i=0}^{g} \int_{U_i} f_\iota^*(\omega) = \sum_{i=0}^{g} \tau_i \int_U \omega = 2\sum_{i=0}^{g} \tau_i,$$

where τ_i is plus or minus one depending on whether $D_p(f_\iota)$ preserves of reverses the orientation. On the other hand, it is clear that the orientation is only preserved at p_0 and therefore:

$$2\sum_{i=0}^{g} \tau_i = 2 - 2g = \chi(\Sigma).$$

□

Now we connect the previous theorem to the Atiyah-Singer index theorem. Let M be closed and oriented m-dimensional manifold. Choose a Riemannian metric $\langle , \rangle$, g. There is a volume form Vol associated to g. Now, we define the $*-$ Hodge operator as a map $* : \Omega^i(M) \to \Omega^{m-i}$ satisfying that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{Vol}$. Let $d : \Omega^i(M) \to \Omega^{i+1}(M)$ the deRham operator. The adjoint of d is defined as

$$d^* = \pm * d *,$$

where $d^* : \Omega^i(M) \to \Omega^{i-1}(M)$. The Hodge- Laplacian operator is defined as

$$\Delta = (d + d^*)^2 = dd^* + d^*d.$$

The Hodge theorem states the relation between the cohomology of the manifold and harmonic forms (the kernel of the Hodge-Laplacian).

Theorem 7.3. (Hodge).

- $\Omega^i(M) = \mathcal{H}^i \oplus im\, d \oplus im\, d^*$, where $\mathcal{H}^i$ is the kernel of Δ.
- $\mathcal{H}^i = \ker(\Delta) = \ker d \cap \ker d^*$.
- $\mathcal{H}^i$ and $H^i(M)$ are isomorphic.

Theorem 7.4 (Gauss-Bonnet's theorem)**.** Let Σ be an oriented, closed surface and let g be a Riemannian metric. Let $\chi(M) = \sum_i (-1)^i \dim(H^i(M))$. Consider the map $d + d^* : \Omega^{even} \to \Omega^{odd}$. then

$$\mathrm{ind}(d + d^*) = \chi(M) = \int_\Sigma K\, Vol,$$

where K denotes the Gaussian curvature of the surface. In local coordinates, being x_1, x_2 coordinates for Σ and $e_i := \frac{\partial}{\partial x_i}, \nabla_i := \nabla_{e_i}$ with ∇ the Levi-Civita connection associated to g and $g_{ij} = \langle e_i, e_j \rangle$,

$$K = \frac{(\nabla_2 \nabla_1 - \nabla_1 \nabla_2) e_1, e_2}{\det(g_{12})}$$

Proof. By Hodge's theorem,

$$\begin{aligned} \mathrm{ind}(d + d^*) &= \dim(\ker(d + d^*)) - \dim(coker(d + d^*)) \\ &= \sum_{\text{even}} \dim(\mathcal{H}^i) - \sum_{\text{odd}} \dim(\mathcal{H}^i) \\ &= \chi(M). \quad \square \end{aligned}$$

□

3 Hirzebruch's signature theorem

We will now describe how the Pontryagin classes can be used to construct topological invariants of manifolds.

Definition 7.2. A genus φ is a function:

$$\varphi : \{\text{Isomorphism classes of closed oriented manifolds}\} \to \mathbb{Q},$$

satisfying the following properties:

1. $\varphi(*) = 1$
2. $\varphi(\emptyset) = 0$
3. $\varphi(M \coprod N) = \varphi(M) + \varphi(N)$
4. $\varphi(M \times N) = \varphi(M)\varphi(N)$
5. $\varphi(\partial M) = 0$

Proposition 7.1. Let $f(x)$ be a power series. Then the function

$$\overline{F} : \{\text{Isomorphism classes of closed oriented manifolds}\} \to \mathbb{Q},$$

defined by:

$$\overline{F}(M) := \int_M F(TM)$$

is a genus.

Proof. The first three properties are obvious. In order to prove 4 we compute:

$$\overline{F}(M \times N) := \int_{M\times N} F(T(M \times N)) = \int_{M\times N} F(TM)F(TN) = \left(\int_M F(TM)\right) \times \left(\int_N F(TN)\right)$$

Let us now prove property 5:

$$\int_{\partial M} F(T(\partial M)) = \int_{\partial M} F(T(\partial M) \oplus \mathbb{R}) = \int_{\partial M} F(TM) = \int_M d(F(TM)) = \int_M 0 = 0.$$

□

Example 7.1. The signature function:

$$M \mapsto \sigma(M)$$

is a genus.

There is an important theorem of R. Thom that gives a criterium for comparing genera.

Theorem 7.5 (Thom)**.** Given any sequence of rational numbers $\{a_n\}_{n\geq 1}$ there is a unique genus φ such that:

$$\varphi(\mathbb{CP}^{2n}) = a_n.$$

Assuming Thom's theorem, we can now prove the signature theorem.

Theorem 7.6 (Hirzebruch)**.** Let M be a closed oriented manifold. Then:

$$\sigma(M) = L(M).$$

Here the left hand side is the signature of M, and the right hand side is the L-genus associated to the power series

$$f(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})}.$$

Proof. Since both σ and L are genera, it suffices to show that they coincide on $\mathbb{CP}^{2n}$. First note that the cohomology of degree $2n$ is $\mathbb{CP}^{2n}$ is generated by u^n and

$$\int_{\mathbb{CP}^{2n}} u^n \wedge u^n = 1,$$

so we conclude that $\sigma(M) = 1$.
Let us now compute $L(\mathbb{CP}^{2n})$. We have already computed the total Pontryagin class of $\mathbb{CP}^{2n}$:

$$P(\mathbb{CP}^{2n}) = 1 + p_1(\mathbb{CP}^{2n}) + p_2(\mathbb{CP}^{2n}) + \cdots = (1+u^2)^{2n+1}.$$

By the multiplicative property of the sequence L_k we know that:

$$L(T(\mathbb{CP}^{2n})) = \sum_{k\geq 0} L_k(p_1(T(\mathbb{CP}^{2n})), \ldots, p_k(T(\mathbb{CP}^{2n}))) = \left(\sum_{k\geq 0} L_k(u^2, 0, \ldots, 0)\right)^{2n+1}.$$

Using the Remark 5.1, we conclude that:

$$L(T(\mathbb{CP}^{2n})) = \left(\frac{u}{\tanh(u)}\right)^{2n+1},$$

and therefore the value

$$L(\mathbb{CP}^{2n}) := \int_{\mathbb{CP}^{2n}} \left(\frac{u}{\tanh(u)}\right)^{2n+1}$$

is equal to the coefficient of z^{2n} in the power series $\left(\frac{x}{\tanh x}\right)^{2n+1}$. By Cauchy's theorem, this number is equal to:

$$\oint \frac{dz}{(\tanh z)^{2n+1}}.$$

We use the change of variables $y = \tanh z$ to obtain:

$$\oint \frac{dz}{(\tanh z)^{2n+1}} = \oint \frac{dy(1+y^2+y^4+\ldots)}{y^{2n+1}} = 1.$$

□

Now we connect the previous theorem to the Atiyah-Singer index theorem. Let M be a closed, oriented manifold of dimension $4k$. The following symmetric form defined in cohomology

$$\begin{aligned} H^{2k}(M) \otimes H^{2k}(M) &\to \mathbb{R} \\ \alpha \times \beta &\to \int_M \alpha \wedge \beta \end{aligned}$$

by Poincare duality is non-degenerate; therefore H^{2k} admits the decomposition

$$H^{2k} = H^{2k}_{+}(M) \oplus H^{2k}_{-}(M),$$

where

$$H^{2k}_{\pm} = \{\alpha \in H^{2k}(M) | \pm \int_M \alpha \wedge \alpha > 0\}.$$

Now fix a Riemannian metric g on M. Define

$$\sigma(M) = \dim(H_{+}) - \dim(H_{-})$$

and a map $\tau := (-1)^{i(i+1)/2+k} * : \Omega^i(M) \to \Omega^{m-i}(M)$, satisfying the property that $\tau^2 = id$. Consider now

$$\Omega_{\mathbb{C}} = \Gamma(\wedge^i(T^*M \otimes \mathcal{C})) = E_{+} \oplus E_{-},$$

where E_i are the eigenspaces of τ.

Observation 1. The following identities hold

$$\begin{aligned} \tau(d + d^*) &= \tau(d - \mathbb{I} d\,\tau) = \mathbb{I} d - d\tau \\ (d + d^*)\tau &= d\,\tau - \tau\, d = -\tau(d + d^*) \end{aligned}$$

therefore $d + d^*$ can be understood as an operator from E_{+} to E_{-}.

Now, we prove the following

Proposition 7.2.

$$\mathrm{Ind}(d + d^*) = \sigma(M).$$

Proof: Observe that

$$\begin{aligned} \mathrm{ind}(d + d^*) &= \dim(\ker(d + d^*)) - \dim(coker(d + d^*)) \\ &= \dim(\ker(\Delta) \cap E_{+}) - \dim(\ker(\Delta) \cap E_{-}) \\ &= \dim(E_{+} \cap H^{2k}) - \dim(E_{+} \cap \mathcal{H}^{\dagger}) \\ &= \dim(E_{+} \cap H^{2k}) - \dim(E_{-} \cap H^{2k}) \\ &= \sigma(M). \end{aligned}$$

Here, $\mathcal{H}^{\dagger}$ is the space of harmonic forms that are not in the middle dimension component.

Observation 2. The component coming from harmonic forms outside the middle dimension part vanishes. This follows from the fact that, defining $\mathcal{H}^L := \bigoplus_{i<2k} \mathcal{H}^i$, then the map

$$\begin{aligned} \epsilon : \mathcal{H} &\to E_{\pm} \cap \mathcal{H}^{\pm} \\ \alpha &\mapsto 1/2(\alpha \pm \tau\,\alpha) \end{aligned}$$

is an isomorphism that makes the contribution vanish.

Theorem 7.7. (Hirzebruch)

$$\sigma(M) = \mathrm{Ind}(d + d^*) = \int_M \frac{\sqrt{x}}{\tanh(\sqrt{x})}(\Omega).$$

Finally we discuss briefly the Hirzebruch-Riemann-Roch's theorem. Let M be a complex compact manifold and E a holomporphic vector bundle over M. Define

$$\chi(M,E) = \sum_i (-1)^i \dim(H^i(M,E)).$$

Consider the operator

$$\bar{\partial} + \bar{\partial}^* : \Omega_{\mathbb{C}}^{0,even} \to \Omega_{\mathbb{C}}^{0,odd}.$$

Then,

$$\mathrm{ind}(\bar{\partial} + \bar{\partial}^*) = \chi(M,E) = \int_M \mathrm{Ch}(E)\mathrm{Todd}(M))$$

where Ch and Todd are the Chern and Todd class respectively.

4 The Euler class and Chern-Gauss-Bonnet

We will now prove a version of the Chern-Gauss-Bonnet theorem, which expresses the Euler charactesistic of a compact oriented manifold as an integral.

Definition 7.3. A good cover on a manifold M is a cover $\{U_\alpha\}$ such that all finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ are diffeomorphic to $\mathbb{R}^n$. A manifold is of finite type if it admits a finite good cover.

In the remaining part of this section we will assume all manifolds to be of finite type. Compact manifolds are of finite type and the total space of a vector bundle over a manifold of finite type is of finite type. By the Mayer-Vietoris principle, the cohomology ring of a manifold of finite type is finite dimensional.

Definition 7.4. We will denote by $\Omega_c(M)$ the space of compactly supported differential forms on a manifold M. The compactly supported cohomology is denoted by $H_c(M)$.

The fundamental fact about compactly supported cohomology is:

Proposition 7.3 (Poincaré Lemma)**.** The natural pairing:

$$\int : H(\mathbb{R}^n) \otimes H_c(\mathbb{R}^n) \to \mathbb{R},$$

given by:

$$\alpha \otimes \beta \mapsto \int_{\mathbb{R}^n} \alpha \wedge \beta,$$

is a perfect pairing.

Proposition 7.4. Let M be an oriented manifold of finite type, then the integration pairing:

$$\int : H(M) \otimes H_c(\mathbb{R}^n) \to \mathbb{R},$$

is perfect.

Proof. This can be proved by induction on the size of the finite good cover using the Poincaré Lemma above. □

The identification given by the previous proposition makes compactly supported cohomology a covariant functor in the category of oriented manifolds of finite type. If $f : M \to N$ is a smooth function between manifolds of finite type then the induced map in compactly supported cohomology is denoted $f_* : H_c(M) \to H_c(N)$. The pull-back of a compactly supported form is not necessarily compactly supported, so compactly supported cohomology is not a contravariant functor in general. Recall that a map $f : M \to N$ is called proper if $f^{-1}(K)$ is compact for all compact $K \subset N$. Since $\mathsf{Sup}(f^*(\alpha)) = f^{-1}(\mathsf{Sup}(\alpha))$, the pull-back of a compactly supported form along a proper map is compactly supported. Thus, compactly supported cohomology is also a contravariant functor with respect to proper maps.

Remark 7.1. Suppose that $f : M \to N$ is a proper map between oriented manifolds of finite type. For any $\alpha \in H(M)$ the push-forward of α denoted $f_*(\alpha)$ is the differential form characterized by the condition that for any compactly supported class $\beta \in H_c(N)$:

$$\int_N f_*(\alpha) \wedge \beta = \int_M f^*(\beta).$$

This formula makes cohomology a covariant functor in the category of oriented manifolds of finite type and proper maps. The push forward map $f_* : H(M) \to H(N)$ has degree equal to $n - m$, where m and n are the dimensions of M and N.

Definition 7.5. Let M be an oriented manifold of finite type and N an oriented submanifold of finite type which is closed as a subspace of M. The poincare dual class of N, denoted $P_M(N) \in H(M)$ is the unique cohomology class such that for any closed compactly supported form α in M:

$$\int_N \iota^*(\alpha) = \int_M P_M(N) \wedge \alpha.$$

Note that the proposition above guaranties that this is well defined.

Lemma 7.3. Let M be an oriented manifold of finite type and N an oriented submanifold of finite type which is closed as a subspace of M. Then:

1. The inclusion map $\iota : N \to M$ is proper and
$$P_M(N) = \iota_*(1).$$
2. Given any neighborhood U of N in M, denote by ι' and ι'' the inclussions: $\iota' : N \to U$ and $\iota'' : U \to M$. Then ι' is proper and
$$(\iota'')^* \iota_*(\alpha) = \iota'_*(\alpha).$$

Proof. Suppose that $K \subset M$ is compact, since M is Hausforff then K is closed in M and therefore so is $N \cap K = \iota^{-1}K$. Since a closed subspace of a compact space is compact we conclude $K \cap N$ is compact. For the other part of the first claim, we choose a closed $\beta \in \Omega_c(M)$ and compute:

$$\int_M F_M(N) \wedge \beta = \int_N \iota^*(\beta) = \int_N \iota_*(1) \wedge \beta.$$

For the second claim we choose a closed compactly supported form $\beta \in \Omega_c(U)$. Since β is compactly supported it can be extended by 0 outside of U and we denote by β' the extension.

Then:

$$\int_U (\iota'')^*(\iota_*(\alpha)) \wedge \beta = \int_M (\iota'')^*(\iota_*(\alpha) \wedge \beta') = \int_N \alpha \wedge \iota^*(\beta') = \int_N \alpha \wedge (\iota')^*(\beta).$$

□

Lemma 7.4. Let $p : E \to M$ be an oriented vector bundle over an oriented manifold M. The total space of E is an oriented manifold.

Proof. There is a natural short exact sequence of vector bundles over E:

$$0 \to p^*(E) \to TE \to p^*(TM) \to 0.$$

On the other hand, it is a general fact that for any short exact sequence of vector bundles:

$$0 \to A \to B \to C \to 0,$$

there is a natural isomorphism:

$$\Lambda^{top} B \cong \Lambda^{top}(A) \otimes \Lambda^{top}(C),$$

Since the bundles E and TM are oriented, TE gets an orientation in this way. □

Remark 7.2. The usual convention is that the induced orientation on the total space of E differs from the one described above by a sign equal to $(-1)^{em}$, where e is the rank of E and m is the dimension of M. This convention corresponds to the fact that one usually writes the trivial bundle over M as $M \times \mathbb{R}^n$ and not $\mathbb{R}^n \times M$.

Definition 7.6. Let E be an oriented rank k vector bundle over an oriented manifold of finite type M of dimension m. The Euler class of E is defined by:

$$e(E) := s^* s_*(1),$$

where s denotes the zero section of E.

Lemma 7.5. Let $\iota : N \to M$ be an embedding of compact manifolds and denote by E the normal bundle to N. Then:

$$\iota^* \iota_*(1) = e(E).$$

Proof. By the tubular neighborhood theorem there exists an open neighborhood U of N which is diffeomorphic to E. Let us write ι' and ι'' for the inclusions $\iota' : N \to U$ and $\iota'' : U \to M$. Using Lemma 7.3 we compute:

$$\iota^* \iota_*(1) = (\iota')^* \circ (\iota'')^*(\iota_*)(1) = (\iota')\iota'_*(1) = e(E).$$

□

Theorem 7.8 (Chern-Gauss-Bonnet)**.** Let M be a closed oriented manifold and denote by $\Delta : M \to M \times M$ the diagonal map. Then:

$$\int_M (e(TM)) = \int_M \Delta^* \Delta_*(1) = \chi(M).$$

Proof. Since the normal bundle to the diagonal is the tangent bundle, the previous lemma guaranties that
$$e(TM) = \Delta^*\Delta_*(1).$$
It remains to prove that $\int_M \Delta^*\Delta_*(1) = \chi(M)$.

Let $\alpha_1, \dots, \alpha_k$ be a basis for the cohomology of M and $\beta_1, \dots, \beta_k$ the dual basis in the sense that:
$$\int_M \alpha_i \wedge \beta_j = \delta_{ij}.$$
We use Kunneth's theorem to identify $H^*(M \times M) \cong H^*(M) \otimes H^*(M)$. We claim that under this identification:
$$\Delta_*(1) = \sum_i (-1)^{|\alpha_i|} \alpha_i \otimes \beta_i.$$
In order to prove this we compute:
$$\begin{aligned}
\int_{M\times M} \left(\sum_i (-1)^{|\alpha_i|}\alpha_i \otimes \beta_i\right)(\beta_r \otimes \alpha_s) &= \int_{M\times M} \sum_i (-1)^{|\alpha_i|+|\beta_i||\beta_r|}(\alpha_i \wedge \beta_r) \otimes (\beta_i \wedge \alpha_s) \\
&= \sum_i (-1)^{|\alpha_i|+|\beta_i||\beta_r|+dim(M)} \left(\int_M (\alpha_i \wedge \beta_r)\right) \\
&\times \left(\int_M (\beta_i \wedge \alpha_s)\right) \\
&= \sum_i (-1)^{|\alpha_i|+|\beta_i||\beta_r|+|\beta_i||\alpha_s|+dim(M)} \delta_{ir}\delta_{is} \\
&= (-1)^{|\alpha_s||\beta_s|}\delta_{rs} \\
&= \int_M \Delta^*(\beta_k \otimes \alpha_s).
\end{aligned}$$
Thus:
$$\int_M \Delta^*\Delta_*(1) = \int_M \Delta^*\left(\sum_i (-1)^{|\alpha_i|}\alpha_i \otimes \beta_i\right) = \sum_i \int_M (-1)^{|\alpha_i|}\alpha_i \wedge \beta_i = \sum_i (-1)^{|\alpha_i|} = \chi(M).$$
□

Chapter 8

Heat kernel and minimal morse functions

1 The Heat Equation

Our main objective in this section will be to define some basic concepts necessary to formulate the heat equation in an arbitrary Rimannian Manifold.

1.1 Some Motivation: The heat equation over $\mathbb{R}^n$

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we can define the Laplace operator

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Then, the heat equation over $\mathbb{R}^n$ takes the form

$$\Delta f = \frac{\partial f}{\partial t}$$

A solution of this equation is a function $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ where for each $t \in (0, \infty)$, $f_t(x) = f(x, t)$ is a smooth function.

In order to formulate this equation over an arbitrary Rimannian manifold, one needs to properly define the Laplace operator. The following discussion sheds some light on how this can be done.

Consider the de Rham complex of $\mathbb{R}^n$

$$0 \to \Omega^0(\mathbb{R}^n) \overset{d_0}{\to} \Omega^1(\mathbb{R}^n) \overset{d_1}{\to} \cdots \overset{d_{n-2}}{\to} \Omega^{n-1}(\mathbb{R}^n) \overset{d_{n-1}}{\to} \Omega^n(\mathbb{R}^n) \to 0$$

We define operators $\star_0 : \Omega^0(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n)$ and $\star_1 : \Omega^1(\mathbb{R}^n) \to \Omega^{n-1}(\mathbb{R}^n)$ in the following manner

$$\star_0(f) = (f)dx_1 \wedge \cdots \wedge dx_n$$

$$\star_1 \left(\sum_{i=1}^{n} (f_i) dx_i \right) = \sum_{i=1}^{n} (-1)^{n-i} (f_i) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

Operators $\star_0$ and $\star_1$ are easily seen to be isomorphisms, therefore we will denote their inverses by $\star_n$ and $\star_{n-1}$ respectively. With this operators defined it is readily checked that the Laplace operator can be obtained as the composition

$$\Omega^0(\mathbb{R}^n) \xrightarrow{d_0} \Omega^1(\mathbb{R}^n) \xrightarrow{\star_1} \Omega^{n-1}(\mathbb{R}^n) \xrightarrow{d_{n-1}} \Omega^n(\mathbb{R}^n) \xrightarrow{\star_n} \Omega^0(\mathbb{R}^n)$$

Indeed

$$\begin{aligned}
(\star_n d_{n-1} \star_1 d_0)(f) &= (\star_n d_{n-1} \star_1) \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \right) \\
&= (\star_n d_{n-1}) \left(\sum_{i=1}^n (-1)^{n-i} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \\
&= (\star_n) \left(\left(\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} \right) dx_1 \wedge \cdots \wedge dx_n \right) \\
&= \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} = \Delta f
\end{aligned}$$

The star operators we just defined are called Hodge star operators. We will define this operators over the terms of the de Rham complex of a Riemannian Manifold and its definition will depend on the choice of metric for the manifold.

1.2 Riemannian Manifolds and the Hodge Star Operator

Definition 8.1. An inner product on a real vector space V means a symmetric bilinear map $\mu : V \times V \to \mathbb{R}$. It is said to be positive definite if $\mu(v, v) \geq 0$, for all $v \in V$, and if $\mu(v, v) = 0$ implies that $v = 0$.

Remark 8.1. Whenever we say an inner product from now on, we shall assume that it is positive definite.

Definition 8.2. Let M be a smooth manifold of dimension n. If for each point $p \in M$ the tangent space T_pM is provided with an inner product

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

in such a way that g_p is of class C^∞ in p, i.e, for all differentiable vector fields X and Y on M, we have that $p \mapsto g_p(X(p), Y(p))$ defines a smooth function $M \to \mathbb{R}$. We say that the family $g = \{g_p\}_{p \in M}$ is a Riemannian metric on M. We also say that the pair (M, g) is a Riemannian manifold.

Now, given a Riemannian manifold (M, g) of dimension n, we aim to define the Hodge star operators $\star_k : \Omega^k(M) \to \Omega^{n-k}(M)$. For this purpose we will state some results from linear algebra.

Proposition 8.1. Let V be a real vector space of dimension n with an inner product $\langle -, - \rangle$. V is isomorphic to its dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ via the application $v \mapsto \alpha_v = \langle v, - \rangle$.

Given this map we can define an inner product on V^* in a natural way $\langle \alpha_v, \alpha_w \rangle = \langle v, w \rangle$. This inner product may be extended to $\bigwedge V^* = \bigoplus_{k=0}^n \bigwedge^k V^*$ as follows:

If η and ω are 0-forms or have different grades, then $\langle \eta, \omega \rangle = 0$. If both η and ω are k-forms, lets say $\eta = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\omega = \beta_1 \wedge \cdots \wedge \beta_k$, then $\langle \eta, \omega \rangle = \det(\langle \alpha_i, \beta_j \rangle)$.

Now, given an orthonormal basis for V^*, one easily obtains an orthonormal basis for $\bigwedge^k V^*$ as stated in the next result.

Proposition 8.2. If $\theta_1, \cdots, \theta_n$ is an orthonormal basis for V^*, then

$$\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \big| 1 \leq i_1 < \cdots < i_k \leq n\}$$

is an orthonormal basis for $\bigwedge^k V^*$.

Proof. Clearly we have that $\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}, \theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \rangle = \det(\mathbb{I}_k) = 1$. Furthermore, let $\eta = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ and $\omega = \theta_{j_1} \wedge \cdots \wedge \theta_{j_k}$ where $\eta \neq \omega$. Then there is one i_l which differs from all j's, leading to the matrix $(\langle \theta_{i_m}, \theta_{j_n} \rangle)$ to have its l-th row equal to zero. So, we get that $\langle \eta, \omega \rangle = \det(\langle \theta_{i_m}, \theta_{j_n} \rangle) = 0$. □

Having fixed the orthonormal basis $\theta_1, \cdots, \theta_n$ for V^*, we define the star operator $\star_k : \bigwedge^k V^* \to \bigwedge^{n-k} V^*$ on the basic elements as follows

$$\star_k(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}) = \operatorname{sgn}(I, J) \theta_{j_1} \wedge \cdots \wedge \theta_{j_{n-k}}$$

where $I = \{i_1, \cdots, i_k\}$, $J = \{1, \cdots n\} - I$, and (I, J) denotes de permutation

$$(i_1, \cdots, i_k, j_1 \cdots j_{n-k}).$$

Now we are ready to define the star operator over a Riemannian manifold (M, g) of dimension n. Recall that for every open set $U \subset M$ we have

$$\Omega^k(U) = \left\{ \omega : U \to \bigsqcup_{p \in U} \bigwedge^k T_p^*(M) \quad : \quad \omega \text{ is locally coherent} \right\}.$$

Local coherence of ω means that for every point $p \in U$ there is a coordinate neighbourhood $(U_p, x^1, \cdots, x^n)$ and smooth functions $f_{(i_1, \cdots, i_k)}$ such that $\omega|_{U_p} = \sum_{(i_1, \cdots, i_k)} f_{(i_1, \cdots, i_k)} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

Notice that for each point $p \in U$ we have a star operator $\star_{k,p} : \bigwedge^k T_p^*(M) \to \bigwedge^{n-k} T_p^*(M)$. Then we can define Hodge's star operator $\star_k : \Omega^k(U) \to \Omega^{n-k}(U)$ as $\big(\star_k(\omega)\big)(p) = \star_{k,p}(\omega(p))$.

Let us check that $\star_k(\omega)$ is indeed locally coherent. We know that for each $p \in U$ we have $\omega|_{U_p} = \sum_{(i_1, \cdots, i_k)} f_{(i_1, \cdots, i_k)} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then $\star_k(\omega) = \sum_{(j_1, \cdots, j_{n-k})} f_{(i_1, \cdots, i_k)} \operatorname{sgn}(I, J) dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}$ where $I = \{i_1, \cdots, i_k\}$ and $J = \{1, \cdots, n\} - I$. Then it is clear that $\star_k(\omega) \in \Omega^{n-k}(U)$ and we have a sheaf morphism $\star_k : \Omega^k \to \Omega^{n-k}$.

1.3 The Laplace Operator and the Heat Equation

Definition 8.3. Let (M, g) be a Riemannian manifold of dimension n. Consider the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \to 0.$$

We define the Laplace operator $\Delta_g : \Omega^0(M) \to \Omega^0(M)$ to be the composition $\Delta_g = \star_n d \star_1 d$.

At this point it is convenient to write Δ_g as the composition of other two operators which we are already familiar with, the gradient and divergence operators. Let us denote the space of smooth vector fields over M by $\mathrm{Vect}(M)$. We can identify $\mathrm{Vect}(M)$ with $\Omega^1(M)$ using the metric g in the following manner: we know that for each point $p \in M$ we have an isomorphism $\hat{g}_p : T_p(M) \to T_p^*(M)$. Now, given a smooth vector field $x \in \mathrm{Vect}(M)$, we may define the function $\omega : M \to \bigsqcup_{p\in M} \bigwedge^1 T_p^*(M)$ given by $\omega(p) = \hat{g}_p(x_p)$. This gives the correspondence $\hat{g} : \mathrm{Vect}(M) \to \Omega^1(M)$.

Definition 8.4. Let (M, g) be a Riemannian manifold of dimension n. The gradient operator $\nabla : \Omega^0(M) \to \mathrm{Vect}(M)$ is defined as the composition

$$\Omega^0(M) \overset{d}{\to} \Omega^1(M) \overset{\hat{g}^{-1}}{\to} \mathrm{Vect}(M)$$

and the divergence operator $\mathrm{div} : \mathrm{Vect}(M) \to \Omega^0(M)$ is defined as the composition

$$\mathrm{Vect}(M) \overset{\hat{g}}{\to} \Omega^1(M) \overset{\star_1}{\to} \Omega^{n-1}(M) \overset{d}{\to} \Omega^n(M) \overset{\star_n}{\to} \Omega^0(M).$$

Notice that $\Delta_g = \mathrm{div}(\nabla)$.

Now we are ready to formulate the heat equation.

Definition 8.5. Let (M, g) be a Riemannian n-manifold which is oriented, compact, connected and without border. The heat equation over M is the equation

$$\Delta_g f = \frac{\partial f}{\partial t}$$

where $f : M \times (0, \infty) \to \mathbb{R}$ has the property that for every $t \in (0, \infty)$, $f_t(x) = f(x, t)$ is a smooth function.

1.4 Initial Conditions for the Heat Equation

Notice that the heat equation defined in the previous section may have multiple solutions. In order to get a unique solution (if any), we wish to set an initial condition for the heat equation which will lie in the space $\mathcal{L}^2(M)$. This space is the completion of $C^\infty(M)$ with an appropriate norm. To define this norm we will need to integrate functions over M with respect to a certain form called *the volume form*. First let us see how this form rises naturally in the case $M = \mathbb{R}^n$.

Recall that 3 given linearly independent vectors in $\mathbb{R}^3$ define a solid called a parallelepiped, and the volume of this solid may be calculated using the determinant function.

In a little more general setting, given $v_1, \cdots, v_n \in \mathbb{R}^n$, we denote

$$\mathrm{box}(v_1, \cdots, v_n) = \left\{ \sum_{i=1}^n t_i v_i \middle| 0 \le t_i \le 1 \right\}.$$

The volume of a box is $|\det(A)|$ where $A = [v_1, \cdots, v_n]$ is the matrix with columns $v_1, \cdots, v_n$. We say that $\{v_1, \cdots, v_n\}$ is a *positive basis* for $\mathbb{R}^n$ if $\det(A) > 0$ and, if $\det(A) < 0$, we say it is a *negative basis*. Now, a volume form for boxes in $\mathbb{R}^n$ is a multilinear, alternating function $\mathrm{Vol} : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ such that

$$\mathrm{Vol}(\mathrm{box}(v_1, \cdots, v_n)) = \begin{cases} 0 & \text{if } \ v_1, \cdots, v_n \text{ is learly dependent} \\ < 0 & \text{if } \ v_1, \cdots, v_n \text{ is a negative basis} \\ > 0 & \text{if } \ v_1, \cdots, v_n \text{ is a positive basis} \end{cases}$$

Now we aim to define a volume form in an arbitrary Riemannian manifold.

Definition 8.6. Let V be real vector space of dimension n. An orientation for V is a choice of basis $\beta = \{v_1, \cdots, v_n\}$ for V. We say that another basis $\{w_1, \cdots, w_n\}$ for V is positive basis with respect to β if the linear transformation $v_i \to w_i$ has a positive determinant, otherwise we say that $\{w_1, \cdots, w_n\}$ is a negative basis with respect to β.

Proposition 8.3. Let $(V, \langle -, - \rangle)$ be a finite dimensional vector space with a positive definite, bilinear product. Let $\beta = \{e_1, \cdots, e_n\}$ be a fixed orthonormal basis for V. There exist a unique multilinear alternating form

$$\mathrm{Vol}_{\langle -,- \rangle} : V^n \to \mathbb{R}$$

such that $\mathrm{Vol}_{\langle -,- \rangle}(q_1, \cdots, q_n) = 1$ for every orthonormal basis $\{q_1 \cdots, q_n\}$ which is positive with respect to β.

Proof. Recall that the inner product $\langle -, - \rangle$ defined in V induces an inner product in $\bigwedge^n V$ which we will also denote $\langle -, - \rangle$. Let us define

$$\mathrm{Vol}_{\langle -,- \rangle} : V^n \to \mathbb{R}$$

by the formula $\mathrm{Vol}_{\langle -,- \rangle}(v_1, \cdots, v_n) = \pm\sqrt{\langle v_1 \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_n \rangle}$. The sign on the formula is $+$ if $\{v_1, \cdots, v_n\}$ is a positive basis with respect to β and $-$ otherwise. We know that if $\{q_1 \cdots, q_n\}$ is an orthonormal basis for V, then $q_1 \wedge \cdots \wedge q_n$ is an orthonormal basis for $\bigwedge^n V$, therefore $\mathrm{Vol}_{\langle -,- \rangle}(q_1, \cdots, q_n) = \pm 1$ depending on the sign of the basis with respect to β. □

1.5 Finally, the Heat Equation and its Solution

Let (M, g) be a Riemannian manifold of dimension n which is oriented, connected, compact and closed. The heat equation on M with respect to the metric g is

$$\Delta_g f = \frac{\partial f}{\partial t} \qquad f : M \times (0, \infty) \to \mathbb{R}$$

where $f_t(x) = f(x, t) \in C^\infty(M)$ for each $t \in (0, \infty)$. Also we have an initial condition

$$f(x, 0) = f_0(x) \in \mathcal{L}^2(M).$$

For this equation we have a solution as stated in the following theorem.

Theorem 8.1. The Laplace operator $\Delta_g : \mathcal{L}^2(M) \to \mathcal{L}^2(M)$ has a discrete spectrum of the form

$$\lambda_0 < \lambda_1 < \cdots < \lambda_j < \cdots$$

Each eigenspace E_j associated to λ_j is finite dimensional, say $\dim(E_j) = r_j$. For each E_j an orthonormal basis may be chosen $E_j = \langle \phi_{i,j} \big| i = 1, \cdots, r_j \rangle$ such that $B = \bigcup_{j=0}^\infty \{\phi_{i,j} \big| i = 1, \cdots, r_j\}$ is an orthonormal basis for $\mathcal{L}^2(M)$. Then, the solution to the heat equation may be written in the form

$$f_t = \sum_{j=0}^\infty \left(e^{-\lambda_j t} \sum_{i=1}^{r_j} \langle f_0, \phi_{i,j} \rangle \phi_{i,j} \right).$$

2 A note on Group Actions and Quotient Spaces

Most of the spaces we are interested in can be obtained as quotients of $\mathbb{R}^n$ by the action of a group. So, in this section, we will study under what conditions a group action $G \times M \to M$ may be used to define a Riemannian manifold structure over the quotient M/G.

Definition 8.7. Let M be a manifold and $G \leq \mathrm{Diff}(M)$ a discrete subgroup. G acts on M in the natural way $(g,p) \mapsto g(p)$. The action of G over M is called properly discontinuous if for every $p \in M$ there is a compact space $K_p \subset M$ containing p such that $g(K_p) \cap K_p = \emptyset$ for almost every $g \in G$.

Proposition 8.4. If the action $G \times M \to M$ is properly discontinuous, then the quotient M/G has a smooth manifold structure.

Proof. See [Lee]. □

3 Searching for Morse Functions

Now we head for the really interesting part. We want to use the heat equation to find minimal Morse functions over Riemannian manifolds. One of the main tools for doing so is *Mather's Stability Theorem.*

Theorem 8.2. Let M be a compact, smooth n-manifold and $h : M \to \mathbb{R}$ a Morse function. There exist $\epsilon > 0$ and $r > 0$ such that, if $f \in C^\infty(M)$ with $\|f - h\|_r < \epsilon$, then f is a Morse function with the same number of critical points as h.

Lets see how this theorem may be used in some special cases.

3.1 Example: The n-Torus

Consider the group action $\mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(z,x) \mapsto x + 2\pi z$. The n-torus, which we denote T^n, is the quotient space $\mathbb{R}^n/\mathbb{Z}^n$. Notice that a fundamental domain for this action is the n-cube $[0, 2\pi)^n$. The next proposition gives us a big deal of information about the n-torus.

Proposition 8.5. The following are facts regarding T^n.

1. T^n is a smooth n-manifold.

2. T^n is diffeomorphic to $(S^1)^n$.

3. The usual metric on $\mathbb{R}^n$ induces a metric in T^n called the flat metric of the torus.

4. If $h \in C^\infty(T^n)$, then h is the projeción of a certain $f \in C^\infty(\mathbb{R}^n)$ that is invariant under the action of $\mathbb{Z}^n$, this is, for every $z \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$, $f(x + 2\pi z) = f(x)$. We denote $h = \bar{f}$.

5. The critical points of $\bar{f} : T^n \to \mathbb{R}$ are the projections of the critical points of $f : \mathbb{R}^n \to \mathbb{R}$. Also, the non-degenerate critical points of $\bar{f}$ are the projections of the non-degenerate critical points of f.

6. The Morse functions over T^n are the projections of Morse functions over $\mathbb{R}^n$.

Now, the spectrum of the Laplace operator over the torus is $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_j < \cdots$ where λ_j is the j-th integer that can be written as

$$\lambda_j = k_1^2 + \cdots + k_n^2$$

with $k_i \in \mathbb{Z}$. The eigenspaces are $E_j = \langle \alpha_{(k),j}, \beta_{(k),j} \rangle$ where (k) denotes an n-tuple $(k_1, \cdots, k_n)$ with $\sum_{i=1}^n k_i^2 = \lambda_j$ and

$$\alpha_{(k),j} = \cos(k_1 x_1 + \cdots + k_n x_n) \qquad \beta_{(k),j} = \sin(k_1 x_1 + \cdots + k_n x_n). \tag{8.1}$$

Notice that for $j = 1$ we have that $\lambda_1 = 1$ and

$$E_1 = \langle \cos(x_1), \sin(x_1), \cdots, \cos(x_n), \sin(x_n) \rangle.$$

Lemma 8.1. A function $h = \sum_{i=1}^n (a_i \cos(x_i) + b_i \sin(x_i)) \in E_1$ is a Morse function if and only if $a_i^2 + b_i^2 \neq 0$ for $i = 1, \cdots, n$.

Proof. First we note that the partial derivatives

$$\frac{\partial h}{\partial x_i} = -a_i \sin(x_i) + b_i \cos(x_i)$$

may be written in the form $A_i \sin(x_i + \theta_i)$ where $A_i = \sqrt{a_i^2 + b_i^2}$ and $\cos(\theta_i) = -a_i/A_i$, $\sin(\theta_i) = b_i/A_i$ whenever $A_i \neq 0$.
Also, note that $\mathrm{Hess}(h)(x_1, \cdots, x_n) = \left[\partial^2 h/\partial x_j \partial x_i\right]$ is a diagonal matrix with $\partial^2 h/\partial x_i \partial x_i = -(a_i \cos(x_i) + b_i \sin(x_i))$. Therefore,

$$\det(\mathrm{Hess}(h))(x_1, \cdots, x_n) = (-1)^n \prod_{i=1}^n (a_i \cos(x_i) + b_i \sin(x_i)).$$

For h to be a Morse function we need that every solution $(x_1, \cdots, x_n)$ of the system $A_i \sin(x_i + \theta_i) = 0$, $i = 1, \cdots, n$, satisfies $\det(\mathrm{Hess}(h))(x_1, \cdots, x_n) \neq 0$. If $A_l = 0$ for some $l \in \{1, \cdots, n\}$, then the determinant of the Hessian matrix is identically zero and every possible critical point is degenerate. On the other hand, if $A_i \neq 0$ for every i, then the critical points are easily seen to be projections of points of the form $(-\theta_1, \cdots, -\theta_n) + z\pi$ with $z \in \mathbb{Z}^n$. It is readily checked that these points are non-degenerate. Restricting ourselves to the fundamental domain for T^n we find that, if h is a Morse function, then it has exactly 2^n critical points. Note that the set of Morse functions provided is dense in E_1. $\square$

The previous lemma together with the next result allows us to state that the morse functions found in E_1 are minimal.

Lemma 8.2. The minimal amount of critical points for a Morse function in T^n is 2^n.

Proof. From Morse theory we know that the number of critical points for a Morse function is bounded bellow by $\sum_{i=0}^n \dim_{\mathbb{R}} H^i(M, \mathbb{R})$. Using Künneth's formula we have that

$$H^i(T^n, \mathbb{R}) = \sum_{j_1 + \cdots + j_n = i} \left(H^{j_1}(S^1, \mathbb{R}) \otimes \cdots \otimes H^{j_n}(S^1, \mathbb{R})\right).$$

Since the cohomology $H^j(S^1, \mathbb{R}) = \mathbb{R}$ for $j = 0, 1$ and is zero for any other j, we find that

$\dim_{\mathbb{R}}(H^i(T^n,\mathbb{R})) = \binom{n}{i}$. So we get

$$\sum_{i=0}^{n} \dim_{\mathbb{R}} H^i(M,\mathbb{R}) = \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

$\square$

The main result for this example, and the one we want to generalize (if possible) is this:

Theorem 8.3. There exists an open, dense subset $S \subset \mathcal{L}^2(T)^n$ such that for every initial condition $f_0 \in S$ and its corresponding solution to the heat equation f there is $T_{f_0} \in (0,\infty)$ with the property that if $t > T_{f_0}$, then f_t is a minimal Morse function.

Proof. We know that $f_t = \sum_{j=0}^{\infty} e^{-\lambda_j t} h_j$ where h_j is the projection of f_0 over E_j. If h_1 is a minimal Morse function, then we will show that $e^{\lambda_1 t}(f_t - h_0)$ is a minimal Morse function for large enough t, it follows that f_t is also a minimal Morse function. For this purpose we will use Theorem 8.2.

Note that $e^{\lambda_1 t}(f_t - h_0) - h_1 = \sum_{j=2}^{\infty} e^{(\lambda_1-\lambda_j)t} h_j$, then

$$\|e^{\lambda_1 t}(f_t - h_0) - h_1\|_r = \left\|\sum_{j=2}^{\infty} e^{(\lambda_1-\lambda_j)t} h_j\right\|_r = e^{(\lambda_1-\lambda_2)t}\left\|\sum_{j=2}^{\infty} e^{(\lambda_2-\lambda_j)t} h_j\right\|_r .$$

For each j we have $h_j = \sum_{(k)} (a_{(k)}\alpha_{(k),j} + b_{(k)}\beta_{(k),j})$ where $\alpha_{(k),j}, \beta_{(k),j}$ are as in Equation 8.1. Then, depending on r, we get one of these possibilities

$$\frac{\partial^r h_j}{\partial x_{i_r}\cdots\partial x_{i_1}} = \sum_{(k)} k_{i_1}\cdots k_{i_r}(\pm a_{(k)}\alpha_{(k),j} \pm b_{(k)}\beta_{(k),j}),$$

$$\frac{\partial^r h_j}{\partial x_{i_r}\cdots\partial x_{i_1}} = \sum_{(k)} k_{i_1}\cdots k_{i_r}(\pm a_{(k)}\beta_{(k),j} \pm b_{(k)}\alpha_{(k),j}).$$

Notice that $|k_{i_1}\cdots k_{i_r}| \leq \lambda_j^r$. We find an upper bound for the partial derivative in either one of the previous cases

$$\left|\frac{\partial^r h_j}{\partial x_{i_r}\cdots\partial x_{i_1}}\right| \leq \sum_{(k)} |k_{i_1}\cdots k_{i_r}(\pm a_{(k)}\alpha_{(k),j} \pm b_{(k)}\beta_{(k),j})|,$$

$$\left|\frac{\partial^r h_j}{\partial x_{i_r}\cdots\partial x_{i_1}}\right| \leq \lambda_j^r\left(\sum_{(k)} |a_{(k)}\alpha_{(k),j}| + \sum_{(k)} |b_{(k)}\beta_{(k),j}|\right). \tag{8.2}$$

Using Cauchy-Schwarz inequality bearing in mind that the number of n-tuples $(k) = (k_1,\cdots,k_n)$ such that $\sum k_i^2 = \lambda_j$ is bounded above by λ_j^n, we get

$$\sum_{(k)} |a_{(k)}\alpha_{(k),j}| \leq \lambda_j^{n/2}\left(\sum_{(k)} a_{(k)}^2\right)^{1/2}, \quad \sum_{(k)} |b_{(k)}\beta_{(k),j}| \leq \lambda_j^{n/2}\left(\sum_{(k)} b_{(k)}^2\right)^{1/2}. \tag{8.3}$$

Now, since $a_{(k)}$ and $b_{(k)}$ are the coefficients of the linear combination that gives the projection of f_0 onto E_j, we have that

$$\left(\sum_{(k)} a_{(k)}^2\right)^{1/2} \leq \|f_0\|_{\mathcal{L}^2(T^n)}, \quad \left(\sum_{(k)} b_{(k)}^2\right)^{1/2} \leq \|f_0\|_{\mathcal{L}^2(T^n)}. \tag{8.4}$$

Combining Equations 8.2, 8.3 and 8.4 we find the upper bound

$$\left|\frac{\partial^r h_j}{\partial x_{i_r} \cdots \partial x_{i_1}}\right| \leq 2\lambda_j^{r+n/2} \|f_0\|_{\mathcal{L}^2(T^n)}.$$

Then we can estimate

$$e^{(\lambda_1-\lambda_2)t} \left\| \sum_{j=2}^{\infty} e^{(\lambda_2-\lambda_j)t} h_j \right\|_r \leq 2e^{(\lambda_1-\lambda_2)t} \|f_0\|_{\mathcal{L}^2(T^n)} \sum_{j=2}^{\infty} \lambda_j^{r+n/2} e^{(\lambda_2-\lambda_j)t}. \tag{8.5}$$

Since the term $\lambda_j^{r+n/2} e^{(\lambda_2-\lambda_j)t}$ decays exponentially as $j \to \infty$, the expression at the right side of Equation 8.5 is convergent and can be made arbitrarily small by taking t large enough.

Now we define $S = \{f_0 \in \mathcal{L}^2(T^n) \big| \operatorname{proj}_{E_1} f_0 \text{ is Morse}\}$. We have seen that this set of initial conditions for the heat equation provide minimal Morse functions for large enough t. Let us now see that S is dense. Take $g \in \mathcal{L}^2(T^n)$ with orthogonal decomposition $g = \sum_{j=0}^{\infty} g_j$, $g_i \in E_i$. We can find some $h_1 \in E_1$ that is Morse minimal and is "close enough" to g_1. Then the function $g_0 + h_1 + \sum_{j=2}^{\infty} g_i \in S$ and is "close enough" to g. $\square$

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