# CONSTRUCCIÓN HAMILTONIANA DE SISTEMAS DE VLÁSOV-MAXWELL POR REDUCCIÓN SIMPLÉCTICA

DIEGO ALEJANDRO RODRÍGUEZ TORRES


UNIVERSIDAD NACIONAL DE COLOMBIA
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS
BOGOTÁ, D.C., COLOMBIA
ABRIL DE 2025

# Hamiltonian construction of Vlásov-Maxwell systems by symplectic reduction

Diego Alejandro Rodríguez Torres
Physicist

Thesis Work to Obtain the Degree of
Master of Science in Mathematics

Advisor
Nicolas Martínez Alba, Ph.D.
Associate Professor


Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, D.C., Colombia
Abril de 2025

**TITLE**

HAMILTONIAN CONSTRUCTION OF VLÁSOV-MAXWELL SYSTEMS

**TÍTULO**

CONSTRUCCIÓN HAMILTONIANA DE SISTEMAS DE VLÁSOV-MAXWELL POR REDUCCIÓN SIMPLÉCTICA

**ABSTRACT:** In this project, we study the methodology to transform the well-known Vlásov-Maxwell equations into a Hamiltonian system relative to a Poisson bracket. First, we introduce the Vlasov-Maxwell equations as a model to describe physical interactions in a plasma. After that, we will review the mathematical formalism of classical mechanics, specifically Hamiltonian systems, we will start with a motivation of the importance of Hamiltonian systems in physics and mathematics, and following that we will present some basic results of symplectic geometry used to describe Hamiltonian systems, the concept of Poisson bracket will naturally appear in this description. Subsequently, we will deal with the problem of constructing new symplectic structures from a symplectic manifold, which is not always possible in general, the Marsden-Weinstein reduction theorem will tell us under which circumstances it is possible. Once symplectic reduction is addressed, we will see how this theorem can help to reduce mechanical systems with symmetry. In the next step, all the symplectic geometry and reduction machinery will be applied to construct a physically adequate Poisson bracket for Vlásov-Maxwell equations. Finally, we will make the necessary adjustments to extend these results for the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra, we will get a new set of equations that have a similar structure to the Vlásov Maxwell equations.

**RESUMEN:** En este proyecto estudiamos la metodología para convertir las conocidas ecuaciones de Vlásov-Maxwell en un sistema Hamiltoniano relativo a un corchete de Poisson. Primero, introducimos las ecuaciones de Vlasov-Maxwell como un modelo para describir las interacciones físicas en un plasma. Luego, revisaremos el formalismo matemático de la mecánica clásica, específicamente el de los sistemas Hamiltonianos. Comenzaremos con una motivación sobre la importancia de los sistemas Hamiltonianos tanto en física como en matemáticas, y a continuación usarémos algunos resultados básicos de la geometría simpléctica para describir sistemas Hamiltonianos, allí el concepto de corchete de Poisson aparecerá de manera natural. Posteriormente, abordaremos el problema de construir nuevas estructuras simplécticas a partir de una variedad simpléctica, lo cual no siempre es posible en general; el teorema de reducción de Marsden-Weinstein nos indicará bajo qué circunstancias es factible. Una vez sea discutido el teorema de reducción simpléctica, exploraremos cómo este resultado puede ayudar a reducir sistemas mecánicos con simetría. En el siguiente paso, toda la maquinaria de la geometría simpléctica y la reducción se aplicará para construir un corchete de Poisson, físicamente adecuado, para las ecuaciones de Vlásov-Maxwell. Finalmente, haremos los ajustes necesarios para extender estos resultados al álgebra de Lie $\mathfrak{sl}(2,\mathbb{R})$, obteniendo un nuevo conjunto de ecuaciones que tiene una estructura similar a las ecuaciones de Vlásov-Maxwell.

**KEYWORDS:** *Vlásov-Maxwell equations, Hamiltonian system, Poisson bracket, symplectic geometry, Marsden-Weinstein reduction.*

**PALABRAS CLAVE:** *Ecuaciones de Vlásov-Maxwell, sistema Hamiltoniano, corchete de Poisson, geometría simpléctica, reducción Marsden-Weinstein.*

# Dedicatory

*To my parents, Alba and Miguel.*
*To my cousin Gabriela .*

# Acknowledgments

Recuerdo que hace algunos años tenía entre mis planes estudiar una maestría en matemáticas. Sentía mucho temor de tomar esa decisión, pues sabía que, al no ser matemático de profesión, enfrentaría numerosos desafíos en el camino. Quiero comenzar agradeciendo a mi yo de aquel entonces por haberse atrevido a dar el paso, a pesar del miedo y la incertidumbre que sentía en ese momento. Haber tenido la oportunidad de estudiar matemáticas en la Universidad Nacional de Colombia, sin más obligación que aprender sobre lo que verdaderamente me apasiona, ha sido un privilegio inmenso que agradezco profundamente a la Universidad y a Dios. Durante este tiempo, crecí no solo en el ámbito académico, sino también como persona.

Por otro lado, quiero agradecer profundamente a quienes siempre estuvieron allí de manera incondicional en los momentos más difíciles: mis padres. A mi Madre, quien siempre me motivó y creyó firmemente en mí, incluso cuando yo dudaba de mis capacidades. A mi Padre, quien, a pesar de su enfermedad, nunca dejó de extenderme su mano y su apoyo cuando más lo necesité.

Finalmente, quiero expresar mi agradecimiento y destacar especialmente la labor de mi asesor ,el profesor Nicolás Martínez, quién desde nuestras primeras conversaciones me ayudó a entender las conexiones entre física y geometría, guiandome a explorarlas y con ello a desarrollar el trabajo que presento en esta tesis. Le agradezco profundamente su paciencia, dedicación e interés genuino en mi aprendizaje y en el desarrollo de este proyecto, así como por esas enriquecedoras conversaciones acompañadas de un café, que siempre me motivaron a seguir aprendiendo y esforzándome. También valoro enormemente su apoyo en la gestión administrativa, gracias a la cuál pude participar en múltiples congresos de geometría, tanto a nivel nacional como internacional.

# CONTENTS

# INTRODUCTION

The word *geometry* is derived from two greek words: The word $g\bar{e}$ ($\gamma\eta$) which means Earth, and the word *metron* ($\mu\varepsilon\tau\rho o\nu$), which means measure. This etimology reflects the deep connection between these two branches of knowledge from their very origin. This relationship was strengthened through the systematic works on optics and geometry of several thinkers such as Euclid and Descartes. However, the actual scientific revolution began in the 17th century when Galileo Galilei introduced *mechanics* into the picture through his numerous discoveries, such as the law of the pendulum. These underlying principles continued to be studied by Sir Isaac Newton, who published his seminal work *Principia*, where he introduced the principles of mechanics of particles, changing the history of science forever. A better mathematical description of Newton's laws of motion appeared in the 18th century with the work of Leonhard Euler and Joseph-Louis Lagrange, leading to the *Euler-Lagrange equations*, which are completely equivalent to Newton's second law, and the *least action principle*. In the 19th century, Sir Willian Rowan Hamilton established a new perspective to study mechanics, introducing the idea of *phase space* and *Hamilton's equations*, which are also equivalent to Newton's second law. In the same period, Sophus Lie was studying symmetries and groups of continuous transformations, an important tool for mechanics that would later play a significant role. At this point, mechanics was incorporating some ideas that were already familiar in *differential geometry*, for instance, integral curves, vector fields, and manifolds. Therefore, in the 20th century, Henri Poincaré developed the geometric foundation for the description of Lagrangian and Hamiltonian mechanics, which opened the door for the advancement of new branches of differential geometry, in particular, *symplectic geometry, Poisson geometry* and *Dirac geometry*. Other notable 20th-century mathematicians, such as Vládimir Arnold, Jerrold Marsden, and Alan Weinstein, made significant contributions to the development of *geometric mechanics*. The work of these mathematicians, along with the beautiful connection between geometry and physics inspired me to write this thesis on geometric mechanics applied to the formulation of Vlásov-Maxwell equations as a Hamiltonian system.

In classical mechanics, particularly in Hamiltonian mechanics, physical systems with symmetries and, therefore with conserved quantities can be described using a *Poisson bracket*, which allow us to derive the equations of motion of the system. Such systems are known as *Hamiltonian systems*. Their study is of great interest to physicists and mathematicians because, due to their properties, they simplify the stability calculations for partial differential equations, making it easier to understand the long-term behaviour of the system. Another useful property of Hamiltonian systems is in studying perturbations of a given system that may lead to chaotic

motions. This makes them crucial for the study of highly interactive and complex systems, including fluid dynamics and plasma physics. Additionally, Hamiltonian systems are a bridge between classical and quantum mechanics.

In the case of plasma physics, the paradigmatic model to study a collisionless plasma is called the Vlásov-Maxwell system, which consists of a set of partial differential equations that couple electromagnetic fields with matter and a kinetic equation, the Vlásov equation. This equation describes the plasma dynamics through the evolution of a distribution density function. A natural question arises: Can a Hamiltonian description of the Vlásov-Maxwell equations be formulated relative to a Poisson bracket? The first answer to this question was provided by Phillip J. Morrison [Mor80], indeed, he provided a Poisson bracket for the Vlásov-Maxwell system. However, Morrison's bracket did not satisfy Jacobi's identity, something that, as we may see later, is crucial for the geometric description of the system and the concept of constant of motion. On the other hand, Jerrold Marsden and Alan Weinstein developed the symplectic reduction theorem [MW74], and a perfect application of this theory was used to derive a Poisson bracket for the Vlásov-Maxwell system. Thus, by combining Morrison's work with their reduction theorem, Marsden and Weinstein were able to find a Poisson bracket that satisfies Jacobi's identity for the Vlásov-Maxwell equations [MW82, MWR+83]. The first objective of this dissertation is to reproduce the construction of the Poisson bracket for the Vlásov-Maxwell, proposed by Marsden and Weinstein.

Moreover, once we understand the methodology used by Marsden and Weinstein to construct the Poisson bracket, for the second objective of this work, we will extend this approach to the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra in order to get a new bracket for an extended Vlásov-Maxwell system. For that, we will first examine the properties of $\mathfrak{sl}(2,\mathbb{R})$ to identify the adequate representation of this Lie algebra for our context. Then, we will adapt the Marsden-Weinstein procedure for this representation, resulting not only in a new bracket but also in a new set of partial differential equations. At this point, having established that the approach is working in the Lie algebra of $\mathbb{R}^3$ and in $\mathfrak{sl}(2,\mathbb{R})$, it is reasonable to ask whether there exists other three-dimensional Lie algebras for which this methodology is also applicable. The third objective of this thesis is to address this question precisely.

This document is organized as follows: In Chapter 1, we briefly summarize the physical background of the Vlásov-Maxwell equations and plasma physics. We give a precise definition of what a plasma is and its main characteristics, along with the introduction of the distribution density and all the variables that play a role in the system. In Chapter 2, we cover some of the necessary background material to understand the geometric approach of the problem, namely symplectic and Poisson structures, Lie group actions, the momentum map, and the Marsden-Weinstein-Meyer reduction theorem. Finally, in Chapter 3, first we summarize the step-by-step procedure to find the Poisson bracket for the Vlásov-Maxwell system, and after that we extend the result to $\mathfrak{sl}(2,\mathbb{R})$. Furthermore, we study the three-dimensional Lie algebra classification to find another possible candidate to apply the same approach, finding some interesting results.

# CHAPTER 1

# VLASOV-MAXWELL SYSTEM

In this chapter, we will briefly explore the physical phenomena that the Vlasov-Maxwell equations describe. Additionally, we will provide a heuristic derivation of the equations based on physical ideas rather than mathematical formalism. We highly recommend the following references [Che84, BS03, N.17] for a complete understanding of the topics treated in this section.

The Vlasov-Maxwell equations are fundamental in the study of Plasma Physics. These equations provide a physical description of a plasma based on kinetic theory. Unlike macroscopic models such as Magnetohydrodynamics (MHD), which treat the plasma as an electrically charged field, the Vlasov-Maxwell equations capture the dynamics of the plasma in the physical phase space. While the MHD model offers a significant description of the plasma's macroscopic behavior and possesses a rich geometric structure in the context of semi-direct products [MRW84], our focus in this work will center on the kinetic-based Vlasov-Maxwell framework. First, we will provide a definition of what a plasma is, and describe some of its main characteristics.

## 1.1 WHAT IS A PLASMA?

Plasma, often referred to as the fourth state of matter alongside solids, liquids, and gases, is rarely observed on Earth in phenomena such as Aurora Borealis or a lighting-bolt, although it is the form of matter more abundant in the Universe such as galaxies, nebulae, and stars. The reason for this is related to the temperature: as we heat a solid, the molecules within it will acquire energy and start separating from each other until they become a liquid, something similar happens when we transition from liquid to gas, and if we increase the temperature of the gas enough, the thermal energy will surpass the binding energy, and some electrons will escape from their molecules, this process is called ionization, so roughly speaking a plasma is an ionized gas. Therefore, the air on Earth would cool the plasma, causing the ions to recombine with electrons and return to their neutral molecular state.

Defining a plasma as an ionized gas is not entirely precise. As highlighted in reference [N.17], the idea of a "state of matter" can be quite ambiguous, for instance, the tectonic plates

which formed the continents, are clearly in the solid state. However, when we consider the broader context and observe the behavior of tectonic plates from the Carboniferous Period, 335 million years ago, to the present, it becomes evident that tectonic plates exhibit motion—flowing, traveling, and merging—characteristic of a fluid rather than a solid. The moral is that the "state of matter" notion depends on length, time scales, and other relevant scales under consideration.

A key length scale in the context of plasma physics is the *Debye length* $\lambda_D$. Let's suppose that we introduce two charged balls, one positive and one negative, connected to a battery inside a plasma. The positive ball will attract the electrons and the negative ball will attract the ions, outside the balls a sphere-shaped cloud will be formed, such phenomenon is known as *Debye shielding*. In a completely cold plasma, where the particles are stationary, the amount of charge in the clouds would be the same as in the balls, therefore any electric field outside the cloud is canceled out or shielded, leaving no electric field in the plasma outside the shield. Nevertheless, if we increase the temperature enough for the particles at the edge of the sphere to escape the cloud, then the shielding would not be complete, hence there would be finite electric fields in the plasma outside the clouds. The Debye length $\lambda_D$ is the characteristic distance from the center of the charged object (the ball in this case) to the edge of the shielding region, beyond which the electric field decreases exponentially but remains finite. This length can be computed as:

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{n_e e^2}} \tag{1.1}$$

where $\epsilon_0$ is permitivity of free space, $k_B$ is Boltzmann's constant, $T_e$ the electron temperature, $n_e$ the electron number density.

Other important quantities include the Debye sphere, which is a spherical region with a radius equal to the Debye length ($\lambda_D$), and the number of particles within the Debye sphere ($N_D$), which is calculated as:

$$N_D = \frac{4}{3}\pi\lambda_D^3 n_e \tag{1.2}$$

In the case of a plasma, a definition that takes into account the relevant scales that we have just defined, is: *A plasma is a quasineutral gas of charged and neutral particles that exhibits collective behavior.* We will now explain what this means:

- Collective behavior: A plasma contains charged particles moving around some region, if an amount of electrons moves into a certain region and the number of electrons surpasses that of the ions, then there is going to be an excess of negative charge, this imbalance will generate an electric field. Of course, the motion of electric charges will provoke the appearance of magnetic fields. By Coulomb's law, two regions of the plasma that are separated by a distance $d$ will generate a force proportional to $1/d^2$, therefore if the regions are too distant the force will be negligible. However, considering the plasma as a continuum, the number of particles between the distant regions increases with the volume that separates them, in other words, the number of particles is proportional to $d^3$, so this effect compensates that of the distance, as a consequence, distant regions of the plasma will experiment electromagnetic interactions. The previous conclusion illustrates

what we mean by *collective behavior*: the motion of a particle at point $A$ is influenced by the electromagnetic forces generated by a particle at point $B$, regardless of whether the distance between the two points is large or small. Hence, the motion of particles in a given region depends not only on local conditions but also on the state of the plasma in distant areas. Rigorously speaking, we can define collective behavior in terms of the following conditions:

(i) Let's say that the size of the system is $L$. Then, the following must hold:

$$L >>> \lambda_D$$

(ii) The number of particles in a Debye sphere must satisfy:

$$N_D >>> 1$$

- Quasineutral: The word "quasi" suggests that the plasma appears neutral but it is not entirely so. At a macroscopic level the plasma's net charge is neutral. However, at a microscopic scale there are charge imbalances, resulting in local regions where the net charge is not neutral. The quasineutral property is fulfilled, if the following equation holds at distances much larger than Debye length:

$$n_e = \sum_q Z n_s \tag{1.3}$$

where $n_s$ is the ion density of the species $s$ and $n_e$ is the electron density, $Zs$ is the charge number of the species $s$.

## 1.2 KINETIC THEORY AND THE VLASOV EQUATION

Considering the collective behavior described in the previous section, a kinetic theory approach offers an effective framework for accurately describing some plasmas. For that, we need a *distribution density function* $f(\mathbf{r},\mathbf{v},t)$ [1], in the six-dimensional phase space with coordinates $(\mathbf{r},\mathbf{v})$ at time $t$. The meaning of $f(\mathbf{r},\mathbf{v},t)$ is that the number of particles per $m^3$ at position $\mathbf{r}$ and time $t$ with velocity components between $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$ is given by:

$$f(x,y,z,v_x,v_y,v_z)dv_x dv_y dv_z$$

The function $f(\mathbf{r},\mathbf{v},t)$ gives the particle density of a certain particle type in the 6-dimensional phase space with coordinates $(\mathbf{r},t)$ at time $t$. Therefore, $f(\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v}$ is the total number of particles in the infinitesimal volume $d\mathbf{r}d\mathbf{v}$ ($d\mathbf{r}$ and $d\mathbf{v}$ are not vectors, they are just the element of volume in the 3-dimensional position and velocity space) at point $(\mathbf{r},\mathbf{v})$ and time $t$.

The goal of kinetic theory is to find equations that describe the time evolution of $f(\mathbf{r},\mathbf{v},t)$,

[1] We will have a density function for each species of particles, that is one for electrons $f_e$ and one for the ions $f_i$, and with this we will have a kinetic equation for each type of density function.

for that let's compute its total time derivative:

$$\frac{d}{dt}f(\mathbf{r},\mathbf{v},t) = \frac{\partial f}{\partial \mathbf{r}}\dot{\mathbf{r}} + \frac{\partial f}{\partial \mathbf{v}}\dot{\mathbf{v}} + \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_c \tag{1.4}$$

Where $\frac{\partial}{\partial \mathbf{r}}$ is just the spatial gradient $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ and $\frac{\partial}{\partial \mathbf{v}}$ is the velocity space gradient $\nabla_{\mathbf{v}} = (\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z})$. The term $\left(\frac{\partial f}{\partial t}\right)_c$ is is describing the collisions between particles; if there are no collisions, then this term must be zero. By Newton's second law, we can take $\dot{\mathbf{v}} = \mathbf{a} = \frac{\mathbf{F}}{m}$, where $\mathbf{F}$ is the net force generated by the electromagnetic fields felt by the particles, this can be taken as the Lorentz force:

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.5}$$

Kinetic equation 1.4 is known as the *Boltzmann equation*, and it states that the total change of $f(\mathbf{r},\mathbf{v},t)$ is zero if and only if the particles have no collisions. In a sufficient hot plasma, where particles are moving at velocities in which they spend less time near each other, therefore collisions are very infrequent, we can consider the effect of collisions as negligible, so that the term $\left(\frac{\partial f}{\partial t}\right)_c = 0$. The collisionless approach together with equation 1.5 leads to the well known *Vlasov equation*:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \frac{q}{m}\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0 \tag{1.6}$$

For capturing the complete dynamics of the plasma, it's important to consider the electric and magnetic fields, which play a fundamental role in the collective behavior. The electromagnetic fields are generated by the particles in motion, affecting other regions of the plasma, generating more particles to move, This self-consistent coupling between particles and fields gives rise to complex dynamics and a variety of novel physical phenomena. In this framework, the evolution of the electromagnetic fields is governed by Maxwell equations, while the particle dynamics is described by the collisionless Boltzmann equation. Together, these form the Vlásov–Maxwell system, whose behavior is represented by the following set of partial differential equations:

$$\frac{\partial f_e}{\partial t} + \mathbf{v}_e \cdot \nabla_{\mathbf{r}} f_e - \frac{e}{m_e}\,(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_e = 0 \tag{1.7}$$

$$\frac{\partial f_i}{\partial t} + \mathbf{v}_i \cdot \nabla_{\mathbf{r}} f_i + \frac{Z_i e}{m_i}\,(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_i = 0 \tag{1.8}$$

Where $f_e$ is the distribution function of the electrons and $f_i$ is the distribution function for the ions.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{1.9}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.10}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.11}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.12}$$

with the charge and current densities defined as

$$\rho(\mathbf{r},t) = e\int \left(Z_i f_i - f_e\right) d^3 v, \tag{1.13}$$

$$\mathbf{J}(\mathbf{r},t) = e\int \left(Z_i f_i \mathbf{v}_i - f_e \mathbf{v}_e\right) d^3 v. \tag{1.14}$$

# CHAPTER 2

## GEOMETRIC MECHANICS IN A NUTSHELL

Physicists and mathematicians have largely studied mechanical systems with symmetry throughout history, applying and developing tools in all branches of mathematics, such as group theory, partial differential equations, and geometry. In particular, *reduction theory* has emerged as a fundamental method for analyzing such systems. Broadly speaking, this approach takes advantage of the physical symmetries of the system to reduce its degrees of freedom, leading to simplified equations of motion. This method has undergone significant development and has been rigorously formulated within the framework of symplectic and Poisson geometry, primarily through the pioneering work of Jerrold Marsden and Alan Weinstein in the 1970s.

In the context of this thesis, reduction theory is applied to the Vlasov-Maxwell system of plasma physics. To achieve this, we will discuss two fundamental results of reduction theory:

- The *Symplectic Reduction Theorem*, which provides a framework for reducing any cotangent bundle, and
- *Lie-Poisson Reduction*, which enables the reduction of the dual of a Lie algebra.

These results play a crucial role in the development of the next chapter, as they form the geometrical foundation for the analysis of the system. In this section, we mean to quickly review such results, but for a more comprehensive and detailed treatment, the following references are recommended:

- For smooth and topological manifolds [Lee13].
- For an introduction to symplectic geometry [CdS01] chapter I, and for Hamiltonian mechanics and a concise summary on reduction see chapters VII, VIII and IX. For Poisson geometry [CFMat21].
- For geometric mechanics [AM78, Hol11, MR94]
- For the Poisson structure on Lie algebras [MWR+83].

In this chapter, we will first introduce basic concepts regarding mechanical systems in the language of symplectic geometry. After that, we will introduce the necessary geometric framework needed to properly state the Marsden-Weinstein-Meyer reduction theorem, starting with Lie group actions, then proceed to infinitesimal generators, the momentum map, and other fundamental concepts. All the definitions are taken from the references cited above.

## 2.1 HAMILTONIAN SYSTEMS

Let us start our discussion with Newton's second law of motion, which describes the dynamics of classical systems:

$$F = \ddot{q} \tag{2.1}$$

Here $q \in \mathbb{R}^n$ denotes the position of some particle, and $F$ is the total force exerted on the particle. Note that equation 2.1 is a second-order differential equation, which we wish to simplify to a system of first-order differential equations. If the force $F$ is conservative [2] then we can write Newton's second law in terms of a physical potential $V(q)$:

$$\ddot{q} = -\frac{\partial V(q)}{\partial q} \tag{2.2}$$

Introducing linear momentum $p = \dot{q}$ [3], we can write Newton's second law as a system of two first-order differential equations:

$$\begin{aligned} p &= \dot{q} \\ -\frac{\partial V(q)}{\partial q} &= \dot{p} \end{aligned} \tag{2.3}$$

The total energy is a conserved quantity for any isolated classical system, and does not depend explicitly on time. The energy will be called the Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$, which in this case is simply the sum of kinetic and potential energy:

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q) \tag{2.4}$$

Note that if we take the derivatives of $H$ with respect to both $p$ and $q$, we get the following system:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned} \tag{2.5}$$

[1] Here, we are taking $m = 1$

[2] This means that the work is independent on the path and the time, and it only depends on the initial and end points

[3] In the presence of a magnetic field, momentum will be affected, and this equation won't hold.

which are known as the *Hamilton equations*. To see the usefulness of having a Hamiltonian function associated with an Ordinary Differential Equations (EDO) system, notice that if we suppose that $\phi(t) = (q(t), p(t))$ is a solution of the system 2.5, then

$$\frac{d}{dt}H(\phi(t)) = \nabla H(\phi(t)) \cdot \dot{\phi}(t) = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = 0.$$

This tells us that the solutions of the equations of motion 2.3 are contained in the level sets of the Hamiltonian, that is, all the points $c \in \mathbb{R}$ such that $H(q,p) = c$. To illustrate this property, we consider the example of a simple pendulum.

**EXAMPLE 2.1.** (Pendulum) The motion of a pendulum with length $L$, can be described in terms of an angular position $\theta$, the angular velocity $\omega = \dot{\theta}$, both as functions of time $t$, and the acceleration due to gravity $g$:

$$\begin{cases} \ddot{\theta}(t) = -\frac{g}{L}\sin(\theta(t)) \\ \dot{\theta}(t) = \omega(t) \end{cases}$$

The Hamiltonian function associated to this system is:

$$H = \frac{\omega^2 L^2}{2} + gL\cos(\theta)$$

An analytical solution of the system comes up with elliptic functions, we can also study the level curves of the Hamiltonian $H$ to have a qualitative description of the dynamics. The equation that gives the level curves of the Hamiltonian is:

$$c = \omega^2 + 2\frac{g}{L}\cos(\theta)$$

In figure 2.1, we can see the periodic orbits that represent the low-energy oscillations, as energy increases, the orbits stop being periodic which reflects a more complex dynamics.

Notice that we can write the system 2.5 using matrix notation as follows. Let $z = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$, then:

$$\dot{z} = -J_0 \nabla H(z) = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} \tag{2.6}$$

With $J_0$ the matrix:

$$J_0 = \begin{pmatrix} 0 & \mathbf{I_n} \\ -\mathbf{I_n} & 0 \end{pmatrix}$$

and $\mathbf{I_n}$ the $n \times n$ identity matrix. We can also write the equation 2.6 in terms of the "Hamiltonian vector field":

$$\dot{z} = X_H(z)$$

A geometric interpretation of $J_0$ can be given if we take $n = 1$:

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Note that equation 2.6 this is a $\frac{\pi}{2}$ rotation of the gradient $\nabla H(z)$. This rotation makes the gradient

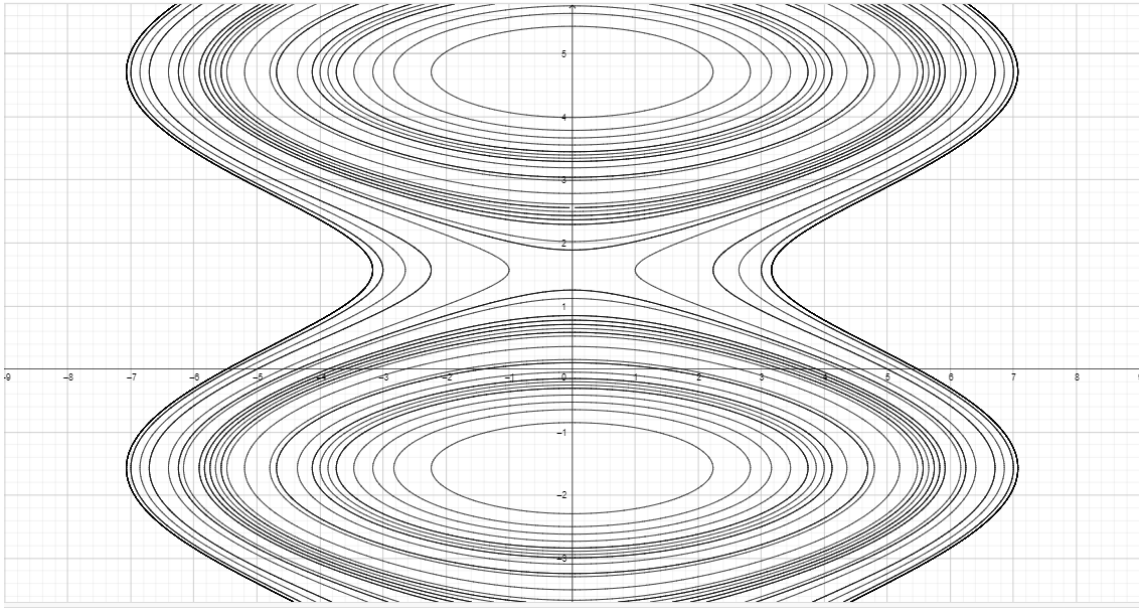

Figure 2.1: Phase diagram of the pendulum.

point in the direction of the level curves of the Hamiltonian.

Matrix $J_0$ being skew-symmetric suggests the relation of classical mechanics to symplectic forms, which will be addressed in subsequent sections.

Another important ingredient of the Hamiltonian formulation of classical mechanics is the Poisson bracket operation between functions:

$$\{f,g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2.7}$$

The evolution of any function in the system can be determined with the Hamiltonian function $H$ and the Poisson bracket $\{,\}$ with the relation:

$$\dot{f} = \{f, H\} \tag{2.8}$$

The Poisson bracket defined above can be used to determine the dynamics of smooth functions $f \in C^\infty(\mathbb{R}^{2n})$, such as the Hamiltonian $H$. This is because the phase space of our initial system is $\mathbb{R}^{2n}$. If we wanted to study the equations of motion of a system on a manifold $M$ we need to understand the theory and structure behind the Poisson bracket and the Hamiltonian formulation. We will do that in the next section. Now, we will give examples of Hamiltonian systems with different bracket operations.

**EXAMPLE 2.2.** (Euler's top) The motion of a top in the absence of gravity, moving around its

center of mass is defined by Euler's equations:

$$\begin{cases} \dot{L_1} = \dfrac{I_2 - I_3}{I_2 I_3} L_2 L_3, \\ \dot{L_2} = \dfrac{I_3 - I_1}{I_1 I_3} L_1 L_3, \\ \dot{L_3} = \dfrac{I_1 - I_2}{I_2 I_1} L_2 L_1, \end{cases}$$

where $\mathbf{L}(t) \in \mathbb{R}^3$ is the angular momentum and $I_1, I_2, I_3$ are the moments of inertia. This is a Hamiltonian system related to the Hamiltonian:

$$H(L) = \frac{1}{2} \sum_{1=1}^{3} \frac{L_i^2}{I_i}$$

and to the Poisson bracket operation:

$$\{f, g\}(L) = \langle \nabla_L f \times \nabla_L g, L \rangle$$

To study rigid motion with more detail, check [AM78, MR94].

The Hamiltonian systems are not uniquely defined for finite-dimensional configuration spaces, but we can also define them in the infinite-dimensional case. Here we present some examples but we refer to [KW09] for more details.

**EXAMPLE 2.3.** (Incompressible fluid) The famous Euler equations give the physical description of a homogenous, incompressible fluid, with velocity $v$ and pressure $p$:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p$$

If we consider the kinetic energy of the fluid, as its Hamiltonian function:

$$H = \frac{1}{2} \int_D \|v\|^2 d^3 x$$

with $D \subseteq \mathbb{R}^3$ some region in which the fluid is contained. Euler equations can be obtained as $\dot{x} = \{x, H\}$ relative to the next bracket [MR94]:

$$\{F, G\}(v) = \int_D v \cdot \left[ \frac{\delta F}{\delta v}, \frac{\delta G}{\delta v} \right] d^3 x$$

Where $[\cdot, \cdot]$ is the Lie-Jacobi bracket of vector fields and $\cdot$ is the usual dot product in $\mathbb{R}^3$.The functional derivatives can be computed as the usual Frèchet derivative and the bracket $[\cdot, \cdot]$ is the Jacobi-Lie bracket for vector fields.

### 2.1.1 THE POISSON BRACKET

Reflecting on all the Poisson bracket operations defined above, some evident properties of the bracket can be realized. For example, this operation is defined on smooth functions, it takes two functions $f, g$ and returns a new smooth function $\{f, g\}$. If we analyze the structure of the canonical Poisson bracket 2.7, it is quite noticeable that it is skew-symmetric and $\mathbb{R}$-bilinear. Nevertheless, two properties don't appear at first glance, the first one is related to the fact that the bracket is a bi-derivation, which means that it satisfies the Leibniz identity (product rule), in both components:

$$\begin{aligned} \{f, g \cdot h\} &= \{f, g\} \cdot h + g \cdot \{f, h\} \\ \{f \cdot g, h\} &= \{f, h\} \cdot g + f \cdot \{g, h\} \end{aligned} \tag{2.9}$$

And the other one is the Jacobi identity:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{2.10}$$

It seems that this identity appeared out of nowhere. However, to motivate this identity. Suppose that some function $f$ satisfies that $\{H, f\} = 0$, where $H$ is a Hamiltonian function, we will call any function that satisfies this property a "constant of motion". Now, if we have two constants of motion $f, g$ then Jacobi identity ensures that $\{f, g\}$ is also a constant of motion, that is:

$$\{H, \{f, g\}\} = 0$$

**DEFINITION 2.1.** A *Poisson Manifold* is a manifold $M$ equipped with a Poisson bracket operation $\{,\}$ defined on the space of smooth functions $C^\infty(M)$.

$$\{\cdot, \cdot\} \colon C^\infty(M) \times C^\infty(M) \mapsto C^\infty(M)$$

that gives $C^\infty(M)$ a Lie algebra structure, that satisfies Leibniz identity:

$$\{f, g \cdot h\} = g \cdot \{f, h\} + \{f, g\} \cdot h$$

for all $f, g, h \in C^\infty(M)$.

The fact that $\{,\}$ gives $C^\infty(M)$ a Lie algebra structure means that the bracket also satisfies the next properties:

- $\mathbb{R}-$ *Bilinearity:* $\{\lambda f + \mu g, h\} = \lambda\{f, h\} + \mu\{g, h\}$ para $\lambda, \mu \in \mathbb{R}$
- *Skew-Symmetry:* $\{f, g\} = -\{g, f\}$
- *Jacobi's identity:* $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$

We can have a local structure in coordinates for the bracket for any Poisson manifold $(M, \{,\})$. If $U$ is an open subset of $M$, and $(U, x_1, \ldots, x_n)$ is a local chart, then for some functions $\pi_{ij} \in C^\infty(M)$:

$$\{f, g\}|_U = \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

Leibniz's identity suggests that the Poisson bracket is a derivation, a classical result from differential geometry is that there is a 1-1 correspondence between derivations and vector fields, that motivates the next definition.

**DEFINITION 2.2.** Let $f \in C^\infty(M)$, there exists a vector field $X_f \in \mathfrak{X}(M)$ such that:

$$X_f(g) = \{f, g\}$$

for all $g \in C^\infty(M)$. $X_f$ is called the *Hamiltonian vector field* of $f$.

**EXAMPLE 2.4.** If we consider our Poisson manifold to be $M = \mathbb{R}^{2n}$, i.e the canonical phase space, given a function $f \in C^\infty(M)$, its Hamiltonian vector field $X_f \in \mathfrak{X}(\mathbb{R}^{2n})$ is:

$$X_f = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} \right)$$

In particular, if we take $H \in C^\infty(M)$ as the Hamiltonian function, then we have that its Hamiltonian vector field will generate the time evolution for another function $f \in C^\infty(M)$, that is:

$$X_H(f) = \{H, f\} = \frac{df}{dt}$$

**DEFINITION 2.3.** A *constant of motion, integral of motion* or a *first integral* is a function $f \in C^\infty(M)$ such that $\{H, f\} = 0$, with $H$ the Hamiltonian function. That is to say, $f$ is constant along the integral curves of $X_H$.

As we already mentioned, Jacobi's identity guarantees that the next property holds:

**PROPOSITION 2.1.** *If $f, g \in C^\infty(M)$ are constants of motion, then the function $\{f, g\}$ is also a constant of motion* [4].

Thanks to Jacobi's identity we can also see that there is a Lie algebra homomorphism between $C^\infty(M)$ and the set of Hamiltonian vector fields $\mathfrak{X}(M)$.

**PROPOSITION 2.2.** $[X_f, X_g] = X_{\{f,g\}}$ *is a Lie algebra isomorphism between $C^\infty(M)$ and $\mathfrak{X}(M)$.*

## 2.2 SYMPLECTIC MANIFOLDS

This section will focus on the geometry underlying the Hamiltonian systems. In the previous section, we expressed Hamilton's equations 2.5 using a skew-symmetric matrix $J_0$, which is the matrix representation of what is known as a symplectic form on a vector space. For a more general description of classical mechanics, we must extend the idea of symplectic structure to the non-linear case, namely manifolds

A symplectic manifold is a pair $(M, \omega)$ of an even manifold $M$ and a closed 2-form $\omega$ such

[4] This statement can be found in the literature as *Poisson's theorem*, and it is an immediate consequence of Jacobi's identity [Arn78]. However, historically, Poisson proved this property before Jacobi's identity was proved.

that the induced bundle map $\omega^\sharp : TM \to T^*M$ is pointwise non-degenerated, i.e

$$\omega_p^\sharp : T_pM \to T_p^*M, \qquad X \mapsto \iota_X \omega_x$$

has trivial kernel.

**REMARK 1.** An important fact about symplectic manifolds is that they are always even-dimensional, that is $\dim(M) = \dim(T_pM) = 2n$ for $n \in \mathbb{N}$.

Note that the fact that as $\omega$ is smooth 2-form and non-degenerated, for each function (observable) $f$, there exists a unique vector field $X_f$ so that

$$\iota_{X_f}\omega = df$$

The relation with a Poisson bracket is given by

$$\{f, g\} = \omega(X_f, X_g)$$

This definition satisfies all the properties of a Poisson bracket, by definition $\omega$ is skew-symmetric and bilinear, since $X_f$ and $X_g$ are derivations, the bracket satisfies Leibniz's identity. The geometry of symplectic structures is indeed equivalent to the Jacobi identity of the bracket, for this just note that:

$$d\omega(X_f, X_g, X_h) = 2\big(\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}\big) = 0$$

**EXAMPLE 2.5.** If $M$ is any manifold, then the cotangent bundle $T^*M$ is a symplectic manifold. This example has a critical application in physics because the configuration space of a physical system is usually defined as a manifold $M$ and its cotangent bundle $T^*M$ is known as the phase space. Because it is symplectic, classical mechanics can be naturally studied. This construction can be extended to the **poly-symplectic** and **multi-symplectic** case in the setting of *classical field theory* in [Got91, Gün87].

**EXAMPLE 2.6.** Let's consider $M = \mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, then the canonical 2-form:

$$\omega_{\text{can}} = \sum_{i=1}^{n} dq_i \wedge dp_i \tag{2.11}$$

is symplectic.

**EXAMPLE 2.7.** $S^2$ is a symplectic manifold and, in fact, it is the only sphere that admits a symplectic structure. This is not a particular conditon of the sphere, but of the dimension. Indeed, any surface (i.e two dimensional manifold) that is oriented, the volume form is a symplectic 2-form on the surface. The remaining spheres lack of topological conditions to guarantee the existence of symplectic form.

**THEOREM 2.3 (DARBOUX).** *Let $(M, \omega)$ be any $2n-$dimensional symplectic manifold, then for any*

*$q \in M$ there exists a local coordinate chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at $q$ such that:*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

As a consequence, all symplectic structures are locally identical.

## 2.3 LIE GROUP ACTIONS

We start by providing some basic definitions and examples regarding the actions of a Lie group $G$ on a manifold $M$. Remember that a Lie group is a manifold $G$ equipped with a group structure, where the group operation and the inverse map are smooth. Lie groups and their actions capture the symmetries of the configuration space, therefore they play a fundamental role in the symplectic reduction procedure.

**DEFINITION 2.4.** An action of a Lie group $G$ on a manifold $M$ is a smooth map $\Psi : G \times M \mapsto M$ which is also a group action. A left action is denoted as $\Psi(g, x) = g \cdot x$ (or $\Psi_g(x) = g \cdot x$) and the right action as $\Psi(g, x) = x \cdot g$.

**EXAMPLE 2.8.** For example, $SO(2, \mathbb{R})$ acts on $\mathbb{R}^2$ by matrix multiplication:

$$R_\theta \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Notice that this action preserves the euclidean norm, as $\|R_\theta x\| = \|x\|$

**DEFINITION 2.5.** Given an action $\Psi$ of a Lie group $G$ on a manifold $M$, the orbit of $G$ through $x \in M$ is the set:

$$\mathcal{O}_x := G \cdot x = \{g \cdot x \colon\ g \in G\}$$

And the stabilizer (or isotropy group) of $x \in M$ is the subgroup:

$$G_x := \{g \in G \colon\ \Psi(g, x) = x\}$$

**DEFINITION 2.6.** The action $\Psi$ of a Lie group $G$ on $M$ is called:

- *Transitive* if for any $x, y \in M$, there is an element $g \in G$ such that $\Psi(g, x) = y$. This means that there is just one orbit and all points of $M$ can be connected by the action of $G$.
- *Free* if all the stabilizers are trivial, that is $G_x = \{e\}$ for all $x \in M$
- *Proper* if the map $\hat{\Psi} \colon\ G \times M \mapsto M \times M \colon\ (g, x) \mapsto (\Psi(g, x), x)$ is proper. [5]

[5] A map $f : X \mapsto Y$ between topological spaces is called proper if the preimage of every compact set in $Y$ is compact in $X$.

### 2.3.1 ONE PARAMETER SUBGROUP AND THE EXPONENTIAL MAP

The Lie algebra $\mathfrak{g}$ of a Lie group $G$ is considered to be a "linearized" version of the Lie group, capturing its local structure near the identity. Indeed, the Lie algebra structure is given by the vector space $\mathfrak{g} \cong T_eG$, (where $e$ is the identity element of $G$) and the bracket on $\mathfrak{g}$ is induced by the Lie brackets of right-invariant vector fields on the Lie group. One way to connect a Lie algebra with its Lie group is via the exponential map.

One way to see this connection is through left-invariant vector fields, which are vector fields that remain unchanged by left translations by elements of the group, and therefore they are completely defined by the identity $e$. Let $X_\xi$ be the left-invariant vector field associated to $\xi \in \mathfrak{g}$, then there is a unique integral curve $\gamma_\xi : \mathbb{R} \to G$, starting at $e$ at $t = 0$, that is:

$$\gamma_\xi(0) = e, \quad \text{with } \frac{d\gamma_\xi(t)}{dt} = X_\xi(\gamma_\xi(t))$$

The family of curves $\{\gamma_\xi(t) : t \in \mathbb{R}\}$ satisfying $\gamma_\xi(s+t) = \gamma_\xi(s)\gamma_\xi(t)$ is known as a *one-parameter subgroup* of diffeomorphisms generated by $X$.

**DEFINITION 2.7.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g} = T_eG$. The *exponential map* is the function $\exp : \mathfrak{g} \to G$ defined as:

$$\exp(\xi) = \gamma_\xi(1)$$

**PROPOSITION 2.4.** *The exponential map satisfies* $exp(\xi s) = \gamma_\xi(s)$

*Proof.* At $t = 0$, the curve $\gamma_\xi(ts)$ passes through the identity $e$, and it satisfies the differential equation:

$$\frac{d\gamma_\xi(ts)}{dt} = s\gamma'_\xi(st) = X_{s\xi}\left(\gamma_\xi(st)\right)$$

Due to uniqueness, it follows that $\gamma_\xi(ts) = \gamma_{s\xi}(t)$, and by setting $t = 1$ we have the result. □

### 2.3.2 INFINITESIMAL GENERATORS OF AN ACTION

We have defined how an action $\Psi$ of a Lie group $G$ moves points on a manifold $M$. The exponential map allows us to associate elements $\xi \in \mathfrak{g}$ to smooth curves on $G$. By applying the action $\Psi$ to these curves and computing their time derivatives, we obtain a notion of velocity, which provides a foundation for describing the dynamics induced by the group action. The previous description motivates the idea of *Infinitesimal generators of an action.*

Let $\Psi : G \times M \mapsto M$ be an action of a Lie group on a manifold. Let $\xi \in T_eG$, let's define the following action:

$$\Psi^\xi : \mathbb{R} \times M \to M$$

$$(t, x) \mapsto \Psi(\exp(t\xi), x)$$

**DEFINITION 2.8.** The infinitesimal generator corresponding to the action $\Psi^\xi$ denoted by $\xi_M$ is

defined as follows:

$$\xi_M(x) := \frac{d}{dt}\Big|_{t=0} \Psi\left(\exp(t\xi), x\right) \tag{2.12}$$

**EXAMPLE 2.9.** The infinitesimal generator of the left translation of $G$ on $G$ is given by $\xi_G = \xi^L \in \mathfrak{G}$ as the left-invariant vector field defined by $\xi \in \mathfrak{g}$.

### 2.3.3 THE ADJOINT AND COADJOINT REPRESENTATION

We want to study properties that remain unchanged in the group, which are the ones that encrypt the symmetries. For this purpose, we should study how a Lie group acts on itself by conjugation.

**DEFINITION 2.9.** Let $G$ be a Lie group, the action of $G$ on itself by conjugation is defined as:

$$I : G \times G \to G$$

$$(g, h) \mapsto ghg^{-1}$$

The map $I_g$ is a Lie group homomorphism for each $g \in G$.

**DEFINITION 2.10.** Let $G$ be a Lie group. The adjoint action to $g \in G$ is the Lie algebra homomorphism $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$:

$$\mathrm{Ad}_g(\xi) = \frac{d}{dt}\Big|_{t=0} \left(g\exp(t\xi)g^{-1}\right)$$

By fixing an element of the Lie algebra, we obtain a morphism between the Lie group and its Lie algebra, by taking it derivative on $e$, we produce the adjoint representation of the Lie algebra, and it is possible to prove that it is given by

$$\mathrm{ad}_\xi(\cdot) = [\xi, \cdot]$$

**PROPOSITION 2.5.** *For all $\xi \in \mathfrak{g}$, we have that $\xi_{\mathfrak{g}} = ad_\xi$*

We also have a dual version of the adjoint representation, called the coadjoint action, and is defined through the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, defined as follows:

$$\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$$

$$\langle \alpha, \xi \rangle \mapsto \alpha(\xi)$$

Given $\xi \in \mathfrak{g}^*$, we define $\mathrm{Ad}^*_{\mathfrak{g}}\xi$ as:

$$\langle \mathrm{Ad}^*_g \alpha, \xi \rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}}\xi \rangle \tag{2.13}$$

The set of all maps $\mathrm{Ad}^*_{\mathfrak{g}}$ forms the coadjoint representation of $G$ in $\mathfrak{g}$.

**EXAMPLE 2.10.** For the adjoint and coadjoint action of $G$ on $\mathfrak{g}$ and $\mathfrak{g}^*$ (respectively), we have the infinitesimal generators as:

$$\xi_{\mathfrak{g}} = [\xi, \cdot] \qquad \text{and} \qquad \xi_{\mathfrak{g}^*}(\mu) = -\mu([\xi, \cdot]).$$

### 2.3.4 POISSON BRACKETS ON DUALS OF LIE ALGEBRAS

Although there are several and rich examples of Poisson structures, for the purpose of our construction, we will only deal with the cases that will be exposed in this section. However, for a better geometric approach to these structures and to visualize more diverse examples, we highly recommend the following references [Bur22, CMMR24].

Lie [Lie88] and Berezin [Ber67] discovered a method to define a Poisson structure in the dual of a Lie algebra. Such construction is relevant in several physical applications, such as in rigid motion, fluid dynamics, as well as in the context of the Vlasov equation. We start with a Lie group $G$, its Lie algebra $\mathfrak{g}$, and the respective dual space $\mathfrak{g}^*$, and the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The goal is to construct a Poisson bracket $\{F,G\}$ for functions $F,G\colon \mathfrak{g}^* \to \mathbb{R}$. For that, we can proceed in three different ways, the whole construction is developed at [GS80], but we are following the notation and treatment of [MW82] :

- **Direct method:** We write down the following formula for functional derivative [Ber67]:

$$DF(\mu)\cdot\nu = \left\langle \nu, \frac{\delta F}{\delta \mu}\right\rangle \quad \text{for all } \mu \in \mathfrak{g}^*$$

  We are taking $\frac{\delta F}{\delta \mu}$ as an element of $\mathfrak{g}$, and under the identification $\mathfrak{g}^{**} \cong \mathfrak{g}$, we have that $DF(\mu) \in \mathfrak{g}$. Then, the Poisson structure is given by:

$$\{F,G\}(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle \tag{2.14}$$

  The bracket $[\cdot,\cdot]$ is the Lie bracket of $\mathfrak{g}$. This bracket is known as the *Lie-Poisson bracket* and it defines a Poisson structure on $\mathfrak{g}^*$.

- **Extension method:** Extend the functions $F,G$ to maps $\hat{F},\hat{G}\colon \to \mathbb{R}$. Identifying $\mathfrak{g}^*$ with $T^*_e G$, we can construct the bracket using the canonical symplectic structure defined on $T^*G$, and then we restrict the bracket to $\mathfrak{g}^*$ as follows:

$$\{F,G\}(\mu) = \{\hat{F},\hat{G}\}\Big|_{\mathfrak{g}^*} \tag{2.15}$$

There is a third way to construct such structure. This approach comes by the coadjoint orbits on $\mathfrak{g}^*$. This procedure explores an additional geometric consequence of the Hamiltonian vector field, called symplectic foliation, that in this case coincides with the orbits of the coadjoint action:

- **Restriction method:** Given a Lie group $G$ acting on its Lie algebra through the adjoint action, we have the coadjoint representation on $\mathfrak{g}^*$ provided by Equation 2.13. In this case, symplectic reduction leads to a reduced space which is isomorphic to the orbit of $\mu \in \mathfrak{g}^*$, therefore $\mathfrak{g}^*$ is the disjoint union of symplectic manifolds, each corresponding to a coadjoint orbit [Arn78]. This result is precisely the statement of the *Kirillov-Konstant-Soriau (KKS) Theorem*. Thus, the Poisson structure is defined by:

$$\{F,G\}(\mu) = \{F|_{\mathcal{O}_\mu}, G|_{\mathcal{O}_\mu}\}_\mu(\mu) \tag{2.16}$$

Where $\{\cdot,\cdot\}_\mu$ is the Poisson bracket on $\mathcal{O}_\mu$ defined by the KKS symplectic form $\omega_\mu$.

We focus on the dynamics of the system, which can be described using the Poisson bracket. To conclude this section, we present a key result concerning the evolution equations on Lie algebras. This result will play a crucial role in the next section.

**THEOREM 2.6.** *[MWR+83] If $H\colon T^*G \to \mathbb{R}$ is a left-invariant Hamiltonian (that is, invariant under the action of a Lie group by left) and $H_-\colon \mathfrak{g}^* \to \mathbb{R}$ is the induced function on $\mathfrak{g}^*$, then the Hamiltonian evolution equations for $H$ on $T^*G$ induce the Lie-Poisson evolution equations*

$$\dot{F} = \{F, H_-\}_- \quad \text{on } \mathfrak{g}^*$$

*which are equivalent to*

$$\dot{\mu} = X_{H_-}(\mu) = ad^*_{\frac{\delta H_-}{\delta \mu}}\mu$$

*The coadjoint orbits in $\mathfrak{g}^*$ are invariant under this evolution. For right invariant systems, interchange the $-$ with a $+$.*

## 2.4 THE MOMENTUM MAP

In classical mechanics, Noether's theorem establishes a connection between conserved quantities in a Hamiltonian system and the symmetries of the physical system. The momentum map, together with the action of a Lie group, provides a rigorous mathematical framework to extend this idea to arbitrary phase spaces.

**DEFINITION 2.11.** A Lie group action $\Psi : G \times M \to M$ on a symplectic manifold $(M,\omega)$ is called symplectic if $\Psi_g^*\omega = \omega$ for all $g \in G$.

**DEFINITION 2.12.** Let $\Psi : G \times M \to M$ be a symplectic action on a symplectic manifold $M$:

(i) A *momentum map* for the action $\Psi$ is a smooth map

$$J\colon M \to \mathfrak{g}^*$$

such that for each $\xi \in \mathfrak{g}$, the associated map

$$\hat{J}(\xi)\colon M \to \mathbb{R}$$

$$x \mapsto \langle J(x), \xi\rangle$$

satisfies

$$d(\hat{J}(\xi)) = \iota_{\xi_M}\omega \tag{2.17}$$

That is to say, $J$ is a momentum map provided that for all $\xi \in \mathfrak{g}$:

$$X_{\hat{J}(\xi)} = \xi_M$$

(ii) Furthermore, we say that $J$ is an *$Ad^*$ - equivariant momentum map*, if the following holds:

$$J(\Psi(g,x)) = \mathrm{Ad}^*_{g^{-1}}(J(x)), \quad \text{for all } x \in M \text{ and } g \in G$$

(iii) Additionally, a *Hamiltonian action* is a symplectic action with an $\mathrm{Ad}^*$-equivariant momentum map

The following theorem will explain how the momentum map is related to conserved quantities.

**THEOREM 2.7.** *Let $\Psi$ be a symplectic action of $G$ on a symplectic manifold $(M,\omega)$. Let $H : M \to \mathbb{R}$ be a smooth map, and suppose $H$ is invariant under the action of $\Psi$, that is $H(\Psi(g,x)) = H(x)$ for all $x \in M$ and all $g \in G$. Then $\hat{J}(\xi)$ is a constant of motion for the dynamics generated by $H$*

*Proof.* We know that $H$ is invariant, then $H(\Psi_{\exp(t\xi)}(x)) = H(x)$ , for all $\xi \in \mathfrak{g}$ and for all $t \in \mathbb{R}$. Taking the derivative with respect to time at $t = 0$, by the chain rule, we have that:

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} H(x) \\ &= \frac{d}{dt}\Big|_{t=0} H(\Psi(\exp(t\xi), x)) \\ &= dH(x) \cdot \frac{d}{dt}\Big|_{t=0} \Psi(\exp(t\xi), x) \\ &= -X_H \hat{J}(\xi) \end{aligned}$$

Then $\hat{J}(\xi)$ is a constant of motion. □

As we know, in Hamiltonian mechanics, the phase space $T^*Q$ is always a symplectic manifold. Therefore, it would be useful to construct momentum maps on the contingent bundle of a manifold $Q$.

**THEOREM 2.8 (COTANGENT LIFT THEOREM).** *Given two manifolds $S$ and $Q$, a diffeomorphism $\varphi : T^*S \to T^*Q$ preserves the canonical one-forms $\theta_Q$ on $T^*Q$ and $\theta_S$ on $T^*S$, respectively, if and only if $\varphi = T^*f$ for some diffeomorphism $f : Q \to S$.*

Therefore, given an action $\Psi : G \times Q \to Q$, By fixing a $g \in G$, we can lift that action to the cotangent level, and we would get a symplectic action $\Psi_g : T^*Q \to T^*Q$. Then, repeating this for all the values $g \in G$ and gluing the results, we would get the *lifted action* $\Psi^{T^*}$, which is a symplectic action invariant under the action of $G$, defined as:

$$\begin{aligned} \Psi^{T^*} : T^*Q &\to T^*Q \\ (g,\alpha) &\mapsto \Psi^*_{g^{-1}}\alpha \end{aligned}$$

**THEOREM 2.9.** *Let $\Psi$ be the action of $G$ on $Q$ and let $\Psi^{T^*}$ be its corresponding lifted action on*

*$M = T^*Q$. Then, this action is Hamiltonian, with the following momentum map $J\colon M \to \mathfrak{g}^*$*

$$\hat{J}(\xi)(\alpha_q) = \alpha_q \cdot \xi_Q(q) \tag{2.18}$$

### 2.4.1 LINEAR AND ANGULAR MOMENTUM

Angular and linear momentum are well-known physical quantities that are conserved under rotational and translational symmetries, respectively, in classical mechanics. The term "momentum map" arises as a natural generalization of these conserved quantities, in this section, we will explore those examples.

#### 2.4.1.1 LINEAR MOMENTUM

Following the construction of the previous section, we start with the following ingredients:

- We start with the Lie group $G = \mathbb{R}^n$, and it's going to act on itself, so the configuration space is $Q = \mathbb{R}^n$ (configuration space of a free particle). The Lie algebra of $\mathbb{R}^n$ is simply $(\mathbb{R}^n)^* = \mathbb{R}^n$ with the trivial Lie bracket. The pairing $\langle \cdot, \cdot \rangle$ between $G$ and $\mathfrak{g}$ is the dot product.
- The symplectic manifold is simply $M = T^*Q \cong \mathbb{R}^{2n}$, with coordinates $(q, p)$ (position-momenta) and the symplectic form $\omega = \sum_i dq_i \wedge dp_i$.
- $G$ is going to act on $Q$ by translation, that is:

$$\begin{aligned} \Psi\colon \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \\ (s, q) &\mapsto s + q \end{aligned}$$

- The exponential map is given by $\exp(t\xi) = t\xi$, so the infinitesimal generator of the action is:

$$\xi_{\mathbb{R}^n} = \frac{d}{dt}\Big|_{t=0} (p + t\xi, q) = \xi$$

- Using Theorem 2.9 we can compute the momentum map $J\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ as:

$$\hat{J}(p, q)(\xi) = p \cdot \xi$$

  And finally, we obtain that $J(q) = p$, the linear momentum. While it was already known from physics that linear momentum is conserved due to translational symmetry, we have now expressed this fundamental principle in the elegant language of geometry.

#### 2.4.1.2 ANGULAR MOMENTUM

- We consider the action of $SO(3,\mathbb{R})$ on $\mathbb{R}^3$ by rotations, that is:

$$\begin{aligned}\Phi\colon\ & S0(3,\mathbb{R})\times\mathbb{R}^3\to\mathbb{R}^3\\ & (A,x)\mapsto Ax\end{aligned}$$

  Where $A\in SO(3,\mathbb{R})$ is a rotation matrix of the form:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{pmatrix}$$

  Let's recall the fact that we can identify the Lie algebra of $SO(3,\mathbb{R})$, $\mathfrak{so}(3,\mathbb{R})$ (algebra of $3\times 3$ orthogonal matrices) with $\mathbb{R}^3$ and the vector cross product by the following isomorphism:

$$\begin{aligned}\hat{\cdot}\colon\ & \mathbb{R}^3\to\mathfrak{so}(3,\mathbb{R})\\ & x=(x_1,x_2,x_3)\mapsto\hat{x}=\begin{pmatrix}0 & -x_3 & x_2\\ x_3 & 0 & -x_1\\ -x_2 & x_1 & 0\end{pmatrix}\end{aligned}$$

  And, we have that $\hat{x}y=x\times y$

- The exponential map is $\exp(t\hat{\xi})=\gamma(t)$, where [MR94]:

$$\gamma(t)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(t\|w\|) & -\sin(t\|w\|)\\ 0 & \sin(t\|w\|) & \cos(t\|w\|)\end{pmatrix}$$

  And we have that $c'(t)=c(t)\hat{w}$

- Computing the infinitesimal generator of the action, we get:

$$\begin{aligned}\hat{\xi}_{\mathbb{R}}(q)&=\frac{d}{dt}\Big|_{t=0}\exp(t\hat{\xi})q\\ &=\hat{\xi}q\\ &=\xi\times q\end{aligned}$$

- Finally, we compute the momentum map $J\colon T^*\mathbb{R}^3\to\mathfrak{so}(3,\mathbb{R})^*$ (Identifying $\mathfrak{so}(3,\mathbb{R})$ with $\mathbb{R}^3$), and the pairing is given by the euclidean inner product.

$$\begin{aligned}\hat{J}(\xi)(q,p)&=\langle q,\xi\times p\rangle\\ &=\langle q\times p,\xi\rangle\end{aligned}$$

  Therefore, we have that $J(q,p)=q\times p$, which is the classical angular moment.

**REMARK 2.** For any system with a momentum map $J$ for the symplectic action $\Psi$ of a Lie group $G$ on $(M,\omega)$, the momentum map is a constant of motion for any invariant Hamiltonian $H$ under

the action ($H(x) = H(\Psi(g,x))$); that is: if $F_t$ is the flow of $H$, then:

$$J(F_t(x)) = J(x), \quad \text{for all } x \in M$$

This is a direct consequence of the definition of a momentum map and the Theorem 2.7, providing precisely the geometric formulation of the well-known Noether's Theorem.

## 2.5 Reduction Theorem

We now have the necessary tools to understand the statement of the symplectic reduction theorem. This theorem will be used multiple times throughout the development of this work.

**Theorem 2.10 (Marsden-Weinstein-Meyer symplectic reduction Theorem [MW74, AM78, Mey73]).** *Let $(M,\omega)$ be a symplectic manifold on which the Lie group $G$ acts simplectically and let $J\colon M \to \mathfrak{g}^*$ be an $Ad^*$-equivariant momentum map for this action. Assume $\mu \in \mathfrak{g}^*$ is a regular value of $J$ and that the isotropy group $G_\mu$ under the $Ad^*$ action on $\mathfrak{g}^*$ acts freely and properly on $J^{-1}(\mu)$. Then $M_\mu = J^{-1}(\mu)\big/G_\mu$ has a unique symplectic form $\omega_\mu$ with the property:*

$$\pi_\mu^*\omega_\mu = i_\mu^*\omega \tag{2.19}$$

*where $\pi_\mu\colon J^{-1}(\mu) \to M_\mu$ is the canonical projection and $i_\mu\colon J^{-1}(\mu) \to M$ is the inclusion.*

We refer to the proof in the following references [CdS01, AM78]. The key point of the proof is the following: under the smoothness assumption of the level set, there always exists a closed 2-form that satisfies the relation 2.19, the pullbacks of the coordinates in the quotient manifold. The non-degeneracy of the reduced form holds from the particular condition that for an isotropic subspace $I$ ( that is $\omega|_I = 0$) of a symplectic vector space $(V,\omega)$ the condition $(I^\perp)^\perp = I$ holds, then $\omega$ induces a canonical symplectic form $\Omega$ on $I^\omega\big/I$.

In addition, by considering the case $\mu = 0$, we can omit the condition requiring 0 to be a regular value of the level set, relying only on the condition for the action to be free. In particular, we obtain the following corollary:

**Corollary 2.11.** *Let $(M,\omega)$ be a symplectic manifold on which the Lie group $G$ acts simplectically and let $J\colon M \to \mathfrak{g}^*$ be an $Ad^*$-equivariant momentum map for this action. If the action is free, and the group is compact, then $M_{red} = J^{-1}(0)\big/G$ has a unique symplectic form $\omega_{red}$ uniquely determined by relation (2.19) at $\mu = 0$.*

Furthermore, given a Hamiltonian system $(M,\omega,H)$ with momentum map $J$, the Hamiltonian function itself will be reduced, resulting in a reduction of the dynamics of the system. More precisely, this result is formalized in the following theorem:

**Theorem 2.12.** *Under the assumptions of the Marsden-Weinstein-Meyer reduction Theorem, let $H\colon M \mapsto \mathbb{R}$ be invariant under the action of $G$. Then we have:*

*(i) The flow $F$ of $X_H$ leaves $J^{-1}(\mu)$ invariant,*

*(ii) It commutes with the action of $G_\mu$ on $J^{-1}(\mu)$, and thus induces canonically a flow $H_t$ on*

*$M_\mu$ satisfying $\pi_\mu \circ F_t = H_t \circ \pi_\mu$.*

*(iii) The flow on $M_\mu$ with Hamiltonian $H_\mu$ defined by $H_\mu \circ \pi_\mu = H \circ i_\mu$. We call it the reduced Hammiltonian.*

The reduction theorem 2.10 discussed in this section applies only to symplectic manifolds. However, there exist physical scenarios in which the configuration space of the system is more naturally described as a Poisson manifold rather than a symplectic one. In such cases, the Marsden-Weinstein-Meyer reduction theorem can be extended to Poisson manifolds. A detailed construction of this generalization can be found in Theorem 1.7.9 of [DZ05]; here, we briefly summarize the main result.

A symmetry[6] by a Lie group $G$ on a Poisson manifold $(M, \{\cdot,\cdot\})$ is descried by

$$\varphi_g^*\{f,h\} = \{\varphi_g^* f, \varphi_g^* h\}.$$

Where $g \in G$, $f, g \in C^\infty(M)$. A moment map for the action is an equivariant map $J : M \to \mathfrak{g}^*$ such that

$$\{J(u),\cdot\} = u_M$$

for all $u \in \mathfrak{g}$. By using this map, it is possible to consider the submanifold $J^{-1}(0)$ and define a bracket $\{\cdot,\cdot\}_0$ (which is not necessarily a Poisson bracket) via

$$\{f,g\}_0 = i^*\{\tilde{f},\tilde{g}\}$$

where the tildes means any extension of the function on the 0−level set to $M$. The key point here, is that any two different extensions define the same bracket. The equivariance of the moment map $J$ gives us the invariance of $\{\cdot,\cdot\}_0$ and yields a reduced bracket via the quotient map $q : J^{-1}(0) \to M_0 = J^{-1}(0)/G$, i.e $q^*\{f_r, g_r\}_{red} = \{q^* f_r, q^* g_r\}_0$, in particular we get

$$q^*\{f_r, g_r\}_{red} = i^*\{\widetilde{q^* f_r}, \widetilde{q^* g_r}\}.$$

As $q$ is submersion, then the Jacobi condition on $\{\cdot,\cdot\}_{red}$ holds from the same condition in the initial bracket on $M$.

[6] We consider again a free and proper action $\varphi$

# CHAPTER 3

## VLÁSOV-MAXWELL SYSTEM: HAMILTONIAN STRUCTURE AND EXTENSION TO $\mathfrak{sl}(2,\mathbb{R})$ LIE ALGEBRA

In addition to its interesting physical properties, the Vlasov-Maxwell (VM) system studied in previous sections can be regarded as a Hamiltonian system related to a Poisson bracket, this was first brilliantly noticed by Morrison [Mor80]. Although its construction worked, since the VM equations could be obtained using the bracket and a Hamiltonian function, this bracket did not satisfy Jacobi's identity, as noted and corrected in [MW82], however, a question that remains unanswered is whether Morrison's bracket has a *Jacobi structure* i.e, the bracket is defined by a bilinear, bidifferential operator. Marsden and Weinstein's approach consisted of constructing the bracket through symplectic reduction, taking advantage of the physical symmetries of the system, including the gauge symmetry of electromagnetism. In this chapter, we will first briefly describe Marsden and Weinstein's geometric construction of the bracket, and then we will extend this result to the $\mathfrak{sl}(2,\mathbb{R})$ Lie Algebra.

The main goal of this chapter is to study how the Vlasov-Maxwell (VM) system will be affected when we introduce a different inner and cross product related to a different algebra. First, we start with the Maxwell system which as we know, its Poisson structure is constructed through symplectic reduction choosing the configuration space $\mathbb{A} = \mathfrak{X}(\mathbb{R}^3)$ of vector potentials, and the corresponding phase space is the cotangent bundle $T^*\mathbb{A}$, that is identified with the pairs $(\mathbf{A},\mathbf{Y})$ with $\mathbf{A} \in \mathbb{A}$ and $\mathbf{Y} \in \mathbb{A}^*$ ($Y$ is a vector field density on $\mathbb{R}^3$), that is to say, $T^*\mathbb{A} \cong \mathbb{A} \times \mathbb{A}^*$. The usual Hamiltonian for variables $(\mathbf{A},\mathbf{Y}) \in T^*\mathbb{A}$ is as follows :

$$H(\mathbf{A},\mathbf{Y}) = \frac{1}{2}\left(\int \|\mathbf{Y}\|^2 + \|\mathrm{curl}(\mathbf{A})\|^2 d\mathbf{x}\right)$$

It is possible to observe that the Hamiltonian depends on the Euclidean norm and the curl, which is defined by the regular formula $\mathrm{curl}(\mathbf{A}) = \nabla \times \mathbf{A}$, and this depends on the vector cross product. A very known fact is that $(\mathbb{R}^3, \times)$ is a Lie algebra that is isomorphic to $\mathfrak{so}(3,\mathbb{R})$. The previous statement motivates us to consider different Lie algebra operations in the Hamiltonian, and then construct the new Maxwell's equations through symplectic reduction. We chose $\mathfrak{sl}(2,\mathbb{R})$, which is the algebra of $2 \times 2$ matrices with zero trace, as the candidate to make this experiment.

In the first section of this chapter, we will discuss the appropriate representation of $\mathfrak{sl}(2,\mathbb{R})$ that we will introduce in the Hamiltonian, and in the subsequent sections, we will show our results.

## 3.1 THE HAMILTONIAN VERSION OF THE VLÁSOV-MAXWELL SYSTEM

In this section, we will outline the main aspects of the geometric construction of the Poisson bracket for the VM system. The whole picture can be seen in [MW82] and [MWR+83]. The VM system is a continuous Hamiltonian system, in terms of the non-canonical variables $f, \mathbf{E}, \mathbf{B}$, where $f$ is the plasma density, which is a distribution density on $\mathbb{R}^6$, and $\mathbf{E}, \mathbf{B} \in \mathfrak{X}(\mathbb{R}^3)$ are the electric and magnetic fields. Its Hamiltonian function is:

$$H(f,\mathbf{B},\mathbf{E}) = \frac{1}{2}\left(\int \|\mathbf{E}\|^2 + \|\mathbf{B}\|^2 d\mathbf{x}\right) + \frac{1}{2}\int \|\mathbf{v}\|^2 f(\mathbf{x},\mathbf{v},t)\, d\mathbf{x}d\mathbf{v}$$

The first term is the potential energy and the second one is the kinetic energy. Now the Hamiltonian function, together with the Morrison-Marsden-Weinstein bracket:

$$\begin{aligned}\{F,G\}_{MW}(f,\mathbf{B},\mathbf{E}) = &\int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\} d\mathbf{x}d\mathbf{v} + \int\left(\frac{\delta F}{\delta \mathbf{E}}\cdot \operatorname{curl}\left(\frac{\delta G}{\delta \mathbf{B}}\right) - \frac{\delta G}{\delta \mathbf{E}}\cdot\operatorname{curl}\left(\frac{\delta F}{\delta \mathbf{B}}\right)\right) d\mathbf{x} \\ &+ \int\left(\frac{\delta F}{\delta \mathbf{E}}\cdot\frac{\partial f}{\partial \mathbf{v}}\frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{E}}\cdot\frac{\partial f}{\partial \mathbf{v}}\frac{\delta F}{\delta \mathbf{B}}\right) d\mathbf{x}d\mathbf{v} + \int f\mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\times\frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right) d\mathbf{x}d\mathbf{v}\end{aligned} \tag{3.1}$$

make the VM system into a Hamiltonian system.

The bracket 3.1 arises from symplectic reduction [MW82], and each term on this bracket represents the different interactions that appear in the VM system (1.1). In this section, we will explain the information within each of the terms of the bracket.

1. **The Vlasov term:** This term comes from constructing a Poisson bracket $\{\cdot,\cdot\}$ on the group of canonical transformations through the dual of its Lie algebra. As we may see, this group provides the appropriate framework for describing the density variables $f$. The non-relativistic limit of Equation 1.6 is known as the Poisson-Vlasov equation:

$$\frac{\partial f}{\partial t} + v\cdot\nabla f - \frac{e}{m}\nabla\varphi_f\cdot\frac{\partial f}{\partial v} = 0 \tag{3.2}$$

This equation arises from the following evolution equation:

$$\frac{\partial f}{\partial t} = \{\mathscr{H}_f, f\} \tag{3.3}$$

In the absence of a magnetic field, we can write the single particle energy as:

$$\mathscr{H}_f = \frac{1}{2}mv^2 + q\varphi_f \tag{3.4}$$

where $\{\cdot,\cdot\}$ is the usual bracket and $\varphi_f$ is the scalar potential. Equation 3.3 suggests that

$f$ is evolving by an infinitesimal canonical transformation so that the Lie group for the density variables is the group of canonical transformations of $\mathbb{R}^6$, which we may call $S$ [MWR+83]. The Lie algebra $\mathfrak{s}$ of $S$ is the algebra of Hamiltonian vector fields on $\mathbb{R}^6$ with polynomial growth in the momentum direction, which can be identified with their generating functions, that is $\mathfrak{s} = C^\infty(\mathbb{R}^6)$ which has a Lie algebra structure provided by the usual Poisson bracket on phase space. Now, from the dual space of the Lie algebra $\mathfrak{s}^*$, which may be identified with the distribution densities on $\mathbb{R}^6$, we can build a Poisson structure by the standard Lie-Poisson construction described in Section 2.3.4, specifically by using Equation 2.14. In our case, the pairing between an element $g \in \mathfrak{s}$ and an element $f \in \mathfrak{s}^*$ is given by integration. All the integrals in this chapter are defined on $\mathbb{R}^3$:

$$\langle g, f \rangle = \int_{\mathbb{R}^3} g f dx dp$$

And the Lie bracket is given by Poisson structure of $\mathfrak{s}$ which is given by the usual Poisson bracket on $C^\infty(\mathbb{R}^6)$. By plugging the inner product defined above and the Poisson bracket on $C^\infty(\mathbb{R}^6)$, we get the Poisson structure for $s^*$:

$$\{F, G\}(f) = \int_{\mathbb{R}^3} f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx dp \tag{3.5}$$

This Poisson structure leads to the first term of the bracket. Note that all of the groups used in this work are infinite-dimensional structures ; this should be treated with more carefully due to functional analysis problems that may appear, and for that, we suggest [RS81, EM70].

2. **The Maxwell term:** The second component of the bracket 3.1 is related to the Poisson description of Maxwell's equations, meaning that we can obtain two of Maxwell's equations merely by this term, the other two equations arise due to the gauge invariance of electromagnetism and the theory of reduction. First, we start with the configuration space for field variables which is $\mathscr{U} = \mathfrak{X}(\mathbb{R}^3)$ the space of vector fields on $\mathbb{R}^3$. The corresponding phase space is $T^*\mathscr{U}$ with elements of the form $(A, Y)$ where $A \in \mathscr{U}$ and $Y$ is a vector field density on $\mathbb{R}^3$. The space $T^*\mathscr{U}$ possess a canonical weakly-symplectic[1] structure with the symplectic form:

$$\omega((A_1, Y_1), (A_2, Y_2)) = \int (Y_2 \cdot A_1 - Y_1 \cdot A_2) dx \tag{3.6}$$

with its associated Poisson bracket:

$$\{F, G\} = \int \left( \frac{\delta F}{\delta A} \cdot \frac{\delta G}{\delta Y} - \frac{\delta F}{\delta Y} \cdot \frac{\delta G}{\delta A} \right) dx \tag{3.7}$$

From this symplectic structure, we can construct a new bracket that generates Maxwell's equations through symplectic reduction. To that end, we begin with a "pre-reduced

[1] This is not an usual symplectic by dimensional issues, but it stil has a good behaviour in the sense that the induced morphism $\omega^\sharp$ is injective

Hamiltonian" in the canonical variables $(A, Y)$.

$$H(A,Y) = \frac{1}{2}\int \|Y\|^2 dx + \frac{1}{2}\int \|\text{curl}(A)\|^2 dx \tag{3.8}$$

**REMARK 3.** Hamilton's equations can be computed from Hamiltonian 3.8 as:

$$\frac{\delta H}{\delta Y} = \frac{\partial A}{\partial t} \qquad\qquad \frac{\delta H}{\delta A} = -\frac{\partial Y}{\partial t}$$

Using the Poisson bracket 3.7 we can calculate the time derivatives, and we'll get the following expression:

$$\frac{\partial A}{\partial t} = Y \qquad\qquad \frac{\partial Y}{\partial t} = -\text{curl}(\text{curl}(A))$$

Setting $Y = -E$ and $B = \text{curl(A)}$, we obtain Faraday's and Ampere's laws:

$$\dot{E} = \text{curl}(B) \qquad\qquad \dot{B} = -\text{curl}(E)$$

For the other two equations, note that the Hamiltonian 3.8 is gauge invariant, that is: it is invariant under the action:

$$(A, Y) \mapsto (A + \nabla f, Y) \tag{3.9}$$

of the group $G = C^\infty(\mathbb{R}^3)$ with the addition. We can reduce the system's degrees of freedom by taking advantage of the gauge symmetry. To reduce the system we take into account the momentum map $J : T^*\mathscr{U} \mapsto \mathfrak{g}^*$ which can be computed using the exact symplectic moment map, given by Equation 2.18, which also works on the special case of an infinite dimensional space.

$$J(A,Y) = -\text{div}(Y). \tag{3.10}$$

To get Gauss Law, we take an element of $J^{-1}(\rho) = \{(A,Y) \in T^*\mathscr{U} : \text{div}(Y) = -\rho\}$ where $\rho \in \mathfrak{g}^*$ is a density. Again, by setting $E = -Y$ we have $\text{div}(E) = \rho$. The reduced phase space can be constructed following the theory of reduction on $J^{-1}(\rho)/G$. This space can be identified with $\mathfrak{M} = \{(E,B) : \text{div}(E) = \rho \quad \text{div}(B) = 0\}$, therefore we have all of Maxwell's Equations. Finally, this space inherits the Poisson structure given by the second term on 3.1:

$$\int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \text{curl}\left(\frac{\delta G}{\delta \mathbf{B}}\right) - \frac{\delta G}{\delta \mathbf{E}} \cdot \text{curl}\left(\frac{\delta F}{\delta \mathbf{B}}\right)\right) d\mathbf{x}$$

3. **The coupling terms:** These final terms portray the coupling between electromagnetic fields and matter, for this reason from a geometric point of view they arise as the reduction of the interacting system $\mathfrak{s}^* \times T^*\mathscr{U}$. On the pre-reduced phase of the construction, we start with variables $(f_{\text{mom}}, A, Y)$, where $f_{\text{mom}}$ is the distribution density function described in chapter 1, but in this case it is written on the $(x,p)-$ space, and $p = v + A$ is the canonical momentum, that is $f(x,v) = f_{\text{mom}}(x,p)$. Additionally, we have the Hamiltonian written in

these variables:

$$H(f_{\text{mom}}, A, Y) = \frac{1}{2}\int \|p - A(x)\|^2 f_{\text{mom}}(x,p)dxdp + \frac{1}{2}\int \left(\|Y\|^2 + \|\text{curl}(A)\|^2\right)dx \tag{3.11}$$

and the Poisson bracket on $\mathfrak{s}^* \times T^*\mathscr{U}$:

$$\{F,G\}_{\mathfrak{s}^*\times T^*\mathscr{U}}(f_{mom}, A, Y) = \int f_{\text{mom}}\left\{\frac{\delta F}{\delta f_{\text{m}}}, \frac{\delta G}{\delta f_{\text{m}}}\right\}dxdp + \int\left(\frac{\delta F}{\delta A}\cdot\frac{\delta G}{\delta Y} - \frac{\delta F}{\delta Y}\cdot\frac{\delta G}{\delta A}\right)dx \tag{3.12}$$

Observe that the first term of the previous Hamiltonian suggests a coupling between fields and particles. Intending to obtain a Poisson structure in terms of the Vlásov-Maxwell variables $(f, E, B)$ we need to perform the reduction procedure to $\mathfrak{s}^* \times T^*\mathscr{U}$. Let's recall that we obtained the Poisson structure of the density variables $f$ employing the Lie-Poisson structure on $\mathfrak{s}^*$ while for the electromagnetic fields $(E,B)$, the Poisson structure was obtained using symplectic reduction, due to gauge symmetry, on the cotangent bundle $T^*\mathscr{U}$. Now, we need to combine both of these results for $\mathfrak{s}^* \times T^*\mathscr{U}$. However, this could be quite challenging because we know that $T^*\mathscr{U}$ is a symplectic space so its reduction is straightforward, nevertheless that is not the case for $\mathfrak{s}^*$. Despite of that, we know that $\mathfrak{s}^*$ is a Poisson manifold, therefore $\mathfrak{s}^* \times T^*\mathscr{U}$ is also a Poisson manifold with the product structure. We can reduce this manifold through an action on $\mathfrak{s}^* \times T^*\mathscr{U}$ and a momentum map by using the methodology described in section 2.5 to reduce Poisson structures.

We need the action of a Lie group $G$ on $s^* \times T^*\mathscr{U}$, a good candidate for this group could be $G = C^\infty(\mathbb{R}^3)$ because we already know that the action 3.9 leaves the Hamiltonian 3.8 invariant, and we can use the momentum translation map as an action on $\mathfrak{s}^*$

$$f_{\text{mom}} \mapsto f_{\text{mom}} \circ \tau_{-\nabla\psi} \tag{3.13}$$

$$\tau_{-\nabla\psi}(x,p) = (x, p - \nabla\psi) \tag{3.14}$$

such action is a canonical transformation, so it preserves the symplectic form. From this action we can get the following momentum map:

$$J(f_{\text{mom}}A, Y) = -\text{div}(Y) - \int f_{\text{mom}}(x,p)dp$$

Hence, we can get a reduced space setting $J^{-1}(0)/G = M_0$. Finally, to get the Poisson bracket on $M_0$, we consider 3.12 and compute all of the derivatives under the change of variables $E = -Y$, $B = \text{curl(A)}$ and $f(x,v) = f_{\text{mom}}(x, v + A(x))$. The result will be the bracket 3.1, and the third and fourth terms of it are a result of the coupling between matter and fields, which is geometrically understood as the reduction of $\mathfrak{s}^* \times T^*\mathscr{U}$.

## 3.2 THE $\mathfrak{sl}(2,\mathbb{R})$ LIE-ALGEBRA

It is important to note that all the previous construction works for the usual gradient, curl and divergence which are differential operators in the Euclidean space. Those operators can also be comprehended as differential operators on the Lie algebra $(\mathbb{R}^3, \times)$. We want to see if

this construction works for a different three-dimensional Lie algebra, and how to extend the Vlasov-Maxwell system to it. For that, we need to construct a new set of differential operators that, together with the Lie bracket and a certain inner product, satisfy the properties required to perform the same setup. We will begin with $\mathfrak{sl}(2,\mathbb{R})$ that can be adapted to the Vlasov-Maxwell framework. In this section, we will provide a detailed description of how to adapt $\mathfrak{sl}(2,\mathbb{R})$ to the Vlásov-Maxwell setup.

### 3.2.1 $\mathfrak{sl}(2,\mathbb{R})$ AS A LORENTZIAN ALGEBRA

We begin this section talking about Lorentz-Minkowski space $\mathbb{L}^3$, that is the vector space $\mathbb{R}^3$ with the usual sum and scalar product equipped with the Lorentz inner product, that is, let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{L}^3$, the inner product is defined as:

$$\langle \mathbf{x}, \mathbf{y} \rangle_L = x_1 y_1 + x_2 y_2 - x_3 y_3 \tag{3.15}$$

By notation we will use $\mathbb{L}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_L)$.

We can define " $\times_L$", an adequate version of the cross product for $\mathbb{L}^3$. To define this cross product, let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis of $\mathbb{L}^3$, we can redefine the right hand rule like this;

$$\mathbf{e_1} \times_L \mathbf{e_2} = -\mathbf{e_3} \qquad \mathbf{e_2} \times_L \mathbf{e_3} = \mathbf{e_1} \qquad \mathbf{e_3} \times_L \mathbf{e_1} = \mathbf{e_2} \tag{3.16}$$

With these relations, the cross product for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{L}^3$ is:

$$\mathbf{x} \times_L \mathbf{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_2 y_1 - x_1 y_2) \tag{3.17}$$

For future use, we will consider the following notation:

**DEFINITION 3.1.** Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ , we define the flat operation as the reflection:

$$\begin{aligned} (\cdot)^\flat \colon \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \mathbf{x} &\mapsto \mathbf{x}^\flat = (x_1, x_2, -x_3). \end{aligned}$$

It is also extended to vector fields, in the same way.

$$(\cdot)^\flat \colon \mathfrak{X}(\mathbb{R}^3) \longrightarrow \mathfrak{X}(\mathbb{R}^3)$$

Note that the operation $(\cdot)^\flat$ is a change of orientation on the $z-$ axis. In particular, on a vector field it provides the following change:

$$\frac{\partial}{\partial z} \mapsto -\frac{\partial}{\partial z}$$

It is noticeable that the only difference between the Euclidean cross product and the Lorentz

cross product 3.17 is the sign of the third component, this also happens with the inner products. We can explicitly write the relation between these two products using flat notation.

We only need to prove one more lemma to derive the triple cross-product identity.

**LEMMA 3.1.** *Let* $\mathbf{a},\mathbf{b} \in \mathbb{R}^3$*, then the following properties hold:*

*(i)* $\mathbf{x} \times_L \mathbf{y} = (\mathbf{x} \times \mathbf{y})^\flat$

*(ii)* $\langle \mathbf{x},\mathbf{y}\rangle_L = \langle \mathbf{x}^\flat, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}^\flat\rangle$

*(iii)* $(\mathbf{x}^\flat)^\flat = \mathbf{x}$

*(iv)* $\mathbf{b} \times \mathbf{a}^\flat = (\mathbf{a} \times \mathbf{b}^\flat)^\flat$

*Proof.* The proof of properties (i) to (iii) is straightforward, so we'll only focus on property (iv).To demonstrate this statement we can directly compute the cross-product:

$$\begin{aligned}\mathbf{a}^\flat \times \mathbf{b} &= (-a_2b_3 - a_3b_2, b_3a_1 + a_3b_1, a_1b_2 - a_2b_1)\\ &= -(a_2b_3 + a_3b_2, -a_1b_3 - a_2b_1, a_1b_2 - a_2b_1)\\ &= -(\mathbf{a} \times \mathbf{b}^\flat)^\flat\end{aligned}$$

By skew-symmetry of the cross-product, we have the result. □

**REMARK 4.** $(\mathbb{R}^3, \times)$ is a Lie algebra, and the Jacobi identity of $\times$ is a direct consequence of the following identity:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \langle \mathbf{x},\mathbf{z}\rangle \mathbf{y} - \langle \mathbf{x},\mathbf{y}\rangle \mathbf{z}$$

with $\langle , \rangle$ being the usual Euclidean inner product. We want to resemble this relation for the Lorentz triple cross product in $\mathbb{L}^3$.

Just as $(\mathbb{R}^3, \times)$ forms a Lie algebra, the following theorem establishes that $(\mathbb{L}^3, \times_L)$ also satisfies the structure of a Lie algebra. While this is a well-known result, we will present a novel proof using the flat operators.

**THEOREM 3.2.** *Let* $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{L}^3$*. Then the following hold*

1. *The triple product identity:*

$$\mathbf{x} \times_L (\mathbf{y} \times_L \mathbf{z}) = -\langle \mathbf{x},\mathbf{z}\rangle_L \mathbf{y} + \langle \mathbf{x},\mathbf{y}\rangle_L \mathbf{z} \tag{3.18}$$

2. $(\mathbb{L}^3, \times_L)$ *is a Lie algebra.*

*Proof.* 1. This equality follows from a straightforward calculation using the Lemma 3.1 and

the usual triple cross product identity for $\mathbb{R}^3$ :

$$\begin{aligned}\mathbf{x}\times_L(\mathbf{y}\times_L\mathbf{z}) &= \mathbf{x}\times_L(\mathbf{y}\times\mathbf{z})^\flat = \Big(\mathbf{x}\times(\mathbf{y}\times\mathbf{z})^\flat\Big)^\flat \\ &= \Big((\mathbf{y}\times\mathbf{z})\times\mathbf{x}^\flat\Big) = -\Big(\langle\mathbf{x}^\flat,\mathbf{z}\rangle\mathbf{y}-\langle\mathbf{x}^\flat,\mathbf{y}\rangle\mathbf{z}\Big) \\ &= -\langle\mathbf{x},\mathbf{z}\rangle_L\mathbf{y}+\langle\mathbf{x},\mathbf{y}\rangle_L\mathbf{z}\end{aligned}$$

2. Let $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{L}^3$ and $\lambda,\mu\in\mathbb{R}$.

   (a) **Bilinearity:** We have that:

   $$(\lambda\mathbf{x}+\mu\mathbf{y})\times_L\mathbf{z} = \big((\lambda\mathbf{x}+\mu\mathbf{y})\times\mathbf{z}\big)^\flat$$

   and since the Euclidean cross product is bilinear and the flat operation is also linear, we have that:

   $$(\lambda\mathbf{x}+\mu\mathbf{y})\times_L\mathbf{z} = \lambda\mathbf{x}\times_L\mathbf{y}+\mu\mathbf{x}\times_L\mathbf{z}$$

   (b) **Skew-symmetry:** This property follows immediately due to the skew-symmetry of Euclidean cross product.

   $$\mathbf{x}\times_L\mathbf{y} = (\mathbf{x}\times\mathbf{y})^\flat = (-\mathbf{y}\times\mathbf{x})^\flat = -\mathbf{y}\times_L\mathbf{x}$$

   (c) **Jacobi identity:** This identity follows from the previous identity:

   $$\begin{aligned}&\mathbf{x}\times_L(\mathbf{y}\times_L\mathbf{z})+\mathbf{y}\times_L(\mathbf{z}\times_L\mathbf{x})+\mathbf{z}\times_L(\mathbf{x}\times_L\mathbf{y}) \\ &= -\langle\mathbf{x},\mathbf{z}\rangle_L\mathbf{y}+\langle\mathbf{x},\mathbf{y}\rangle_L\mathbf{z}-\langle\mathbf{y},\mathbf{x}\rangle_L\mathbf{z}+\langle\mathbf{y},\mathbf{z}\rangle_L\mathbf{x}-\langle\mathbf{z},\mathbf{y}\rangle_L\mathbf{x}+\langle\mathbf{z},\mathbf{x}\rangle_L\mathbf{y} = 0\end{aligned}$$

□

The section aimed to provide an adequate representation of $\mathfrak{sl}(2,\mathbb{R})$ for the Vlásov-Maxwell context, therefore we finish the section with that result.

**THEOREM 3.3.** *The Lie algebra* $(\mathbb{L}^3,\times_L)$ *is isomorphic to* $\mathfrak{sl}(2,\mathbb{R})$.

*Proof.* Let $\{e_i\}_{i=1}^3$ be the standard basis for $\mathbb{L}^3$, and lets define $\{E_i\}_{i=1}^3$ a basis for $\mathfrak{sl}(2,\mathbb{R})$ as:

$$E_1=\begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad E_2=\begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad E_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

It is widely known that the commutation relations in $\mathfrak{sl}(2,\mathbb{R})$ for these matrices are:

$$[E_1,E_2]=E_3 \qquad [E_1,E_3]=-2E_1 \qquad [E_2,E_3]=2E_2.$$

We can go further and write the specific isomorphism.

$$\psi : \mathbb{L}^3 \mapsto \mathfrak{sl}(2,\mathbb{R})$$
$$\mathbf{x} = (x_1, x_2, x_3) \mapsto \psi(x) = \frac{1}{2}\begin{pmatrix} x_1 & x_2 - x_3 \\ x_2 + x_3 & -x_1 \end{pmatrix} = \sum_{i=1}^{3} x_i E_i$$

This isomorphism can be constructed using the relations in 3.16, in the following way:

$$E_3 \mapsto 2e_1, \qquad E_2 \mapsto e_2 + e_3 \qquad \text{and } E_1 \mapsto e_2 - e_3.$$

Hence, from now on we will identify $\mathfrak{sl}(2,\mathbb{R})$ with the algebra $(\mathbb{L}^3, \times_L)$ and we will change the notation for the inner and cross product to $\langle , \rangle_{\mathfrak{sl}(2)}$ and $\times_{\mathfrak{sl}(2)}$. □

### 3.2.2 GRADIENT AND CURL IN $\mathfrak{sl}(2,\mathbb{R})$

Once we have defined the appropriate inner and cross products, the next ingredients needed for our analysis are the gradient, divergence, and curl. In this section, the correct version of these operators will be introduced.

#### GRADIENT

The notion of gradient depends on the inner product of the space, for example, the Euclidean gradient of a function $f \in C^\infty(\mathbb{R}^n)$ is the vector field $\nabla f$ associated to the differential $df$ via the inner product:

$$\langle \nabla f, \cdot \rangle = df(\cdot) = \left( \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \right)(\cdot)$$

Likewise, the $\mathfrak{sl}(2,\mathbb{R})$ gradient of a function $f \in C^\infty(\mathbb{L}^3)$ is the vector field $\nabla_{\mathfrak{sl}(2)}$ associated to the differential $df$ via the inner product $\langle , \rangle_{\mathfrak{sl}(2)}$. On the grounds of this, the correct notion of gradient is similar to Euclidean one but with a sign change in the third component.

$$\nabla_{\mathfrak{sl}(2)} f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -\frac{\partial f}{\partial z} \right) \tag{3.19}$$

And we can denote $\nabla_{\mathfrak{sl}(2)} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)$.

#### CURL

In $\mathbb{R}^3$ the curl of a vector field $\mathbf{F} = (F_x, F_y, F_z)$ is defined using the cross product by:

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

In particular, we know that the curl of a gradient is always the vector field zero, that is:

$$\text{curl}(\nabla f) = \mathbf{0}$$

For this property to remain true, which as we will see later is very important, we will define the curl for $\mathfrak{sl}(2,\mathbb{R})$ as:

$$\operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{F}) = \nabla_{\mathfrak{sl}(2)} \times_{\mathfrak{sl}(2)} \mathbf{F} = \left( \frac{\partial F_z}{\partial y} + \frac{\partial F_y}{\partial z}, -\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) \tag{3.20}$$

Using the flat notation, the next properties can be easily checked:

**PROPOSITION 3.4.** *For any function $f \in C^\infty(\mathbb{L}^3)$ and any smooth vector field $F \in \mathfrak{X}(\mathbb{L}^3)$ the following properties hold:*

*(i)* $\nabla_{\mathfrak{sl}(2)} f = (\nabla f)^\flat$

*(ii)* $curl_{\mathfrak{sl}(2)}(\mathbf{F}) = -curl(\mathbf{F}^\flat)$

*(iii)* $curl_{\mathfrak{sl}(2)}\left(\nabla_{\mathfrak{sl}(2)} f\right) = \mathbf{0}$. *That is, the $\mathfrak{sl}(2)$ curl of any $\mathfrak{sl}(2)$ gradient is the vector zero.*

*Proof.* The first two properties are straightforward from replacing the flat operators. We can prove the third property using Equations 3.20 and 3.19, but it is easier if we use properties (i) and (ii).

$$\operatorname{curl}_{\mathfrak{sl}(2)}\left(\nabla_{\mathfrak{sl}(2)} f\right) = -\operatorname{curl}\left((\nabla_{\mathfrak{sl}(2)} f)^\flat\right) = -\operatorname{curl}\left(\nabla f\right) = \mathbf{0}$$

□

Proposition 3.4 will allow us to have gauge symmetry, which is fundamental in the symplectic reduction procedure for the Maxwell variables.

## 3.3 EXTENSION TO $\mathfrak{sl}(2,\mathbb{R})$

In this section, we will replicate the construction outlined in Section 3.1 for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, taking advantage of the representation and tools introduced in Section 3.2. Following the structure of Section 3.1, we will begin by describing the Poisson structure of the density variables. Next, we will perform a symplectic reduction to derive a new set of Maxwell equations. Finally, we will couple these systems to formulate the new Vlasov-Maxwell equations.

We recall that the Vlasov-Maxwell system has three components: the electromagnetic part given by fields $E, B$, the density part in which the plasma density $f$ plays a fundamental role, and the coupling between these two previous components. Due to the coupling component, we need to make adequate changes in the Poisson structure of the density variables so that our results are consistent with the electromagnetic component.

This section is organized as follows: In Subsection 3.3.1, we will specify the modifications on the canonical symplectic structure for it to be consistent with our work. After that, the subsequent Subsections 3.3.2, 3.3.3 , and 3.3.4 will mimic the order of items 1(*The Vlasov term*), 2(*The Maxwell term*) and 3(*The couplig term*) in Section 3.1. Following this structure, we will derive the $\mathfrak{sl}(2,\mathbb{R})$ Vlásov-Maxwell system.

### 3.3.1 MODIFIED CANONICAL SYMPLECTIC STRUCTURE

In this section, the changes performed on the canonical structure, and therefore on the canonical Poisson bracket, will be briefly explained.

As a starting point, let's consider $\mathbb{R}^6$ as our phase space, in the absence of a magnetic field and taking the mass as $m = 1$ we can identify velocity and momentum so the local coordinates are $(x, y, z, p_x, p_y, p_z)$. Ordinarily, the canonical symplectic structure for the phase space is $\omega_{\text{can}} = \sum dx_i \wedge dp_i$, as was explained in the previous paragraph, we will change this structure with the aim of not having trouble when we couple the density variables with the electromagnetic component. Recall that, the magnetic field is provided by the change of variables $\mathbf{B} = \text{curl}_{\mathfrak{sl}(2)}(\mathbf{A})$, which is given by 3.20, if we use the usual symplectic structure there will be some problems with the signs on 3.20, to ensure that doesn't happen, the symplectic form that we will use is as follows:

$$\omega = dx \wedge dp_x + dy \wedge dp_y - dz \wedge dp_z = -d\left(p_x dx + p_y dy - p_z dz\right) \tag{3.21}$$

If we consider the Liouville 1-form as $\alpha = p_x dx + p_y dy + p_z dz$, and by extending the flat operation 3.1 defined in the previous section as:

$$(\cdot)^\flat \colon \Omega^1(\mathbb{R}^3) \mapsto \Omega^1(\mathbb{R}^3)$$
$$f_1 dx + f_2 dy + f_3 dz \mapsto f_1 dx + f_2 dy + f_3(-dz)$$

we can write the following relation:

$$\omega = -d(\alpha^\flat) \tag{3.22}$$

Remarkably, this result is consistent with the treatment that we had given to the fields $E, B$, since the only modification we are making to the canonical symplectic structure, is a sign change in the third component.

Thus, the Poisson bracket for functions $h, k \in C^\infty(\mathbb{R}^6)$ as defined by the equation $\{h, k\}_{\mathfrak{sl}(2)} = \omega(X_h, X_k)$ is:

$$\{h, k\}_{\mathfrak{sl}(2)} = \left(\frac{\partial h}{\partial x}\frac{\partial k}{\partial p_x} - \frac{\partial h}{\partial p_x}\frac{\partial k}{\partial x}\right) + \left(\frac{\partial h}{\partial y}\frac{\partial k}{\partial p_y} - \frac{\partial h}{\partial p_y}\frac{\partial k}{\partial y}\right) - \left(\frac{\partial h}{\partial z}\frac{\partial k}{\partial p_z} - \frac{\partial h}{\partial p_z}\frac{\partial k}{\partial z}\right). \tag{3.23}$$

Here we use the subscript $\mathfrak{sl}(2)$ to be consistent with the same notation used in the previous section. Again, observe how this result differs from the canonical bracket on a sign in the third term.

### 3.3.2 THE VLASOV TERM AND THE POISSON STRUCTURE OF DENSITY VARIABLES

Let $\mathfrak{S}$ be the group of canonical transformations of $\mathbb{R}^6$ (that is, the transformations that left invariant the Poisson bracket) and let $\mathfrak{s}$ be its Lie algebra, this Lie algebra consists on Hamiltonian vector fields on $\mathbb{R}^6$ and it can be identified with $C^\infty(\mathbb{R}^6)$.In our analysis, we might consider the Lie bracket of $\mathfrak{s}$ as $[g, f] = -\{f, g\}_{\mathfrak{sl}(2)}$. The dual Lie algebra $\mathfrak{s}^*$ can be identified with distribution densities on $\mathbb{R}^6$ and the dual pairing between elements $h \in \mathfrak{s}$ and $k \in \mathfrak{s}^*$ is determined by

integration:

$$\langle h,k\rangle_{\mathfrak{s}^*} = \int hk d\mathbf{x}d\mathbf{v} \tag{3.24}$$

The dual of any Lie algebra $\mathfrak{s}^*$ has a natural Poisson structure defined by the so-called Lie-Poisson bracket 2.14 described in the previous section. Applying this equation with the Lie-Poisson bracket as 3.23 and the pairing 3.24, we have the next Poisson bracket:

$$\{F,G\}_{\mathfrak{s}^*}(f) = \int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}_{\mathfrak{sl}(2)} d\mathbf{x}d\mathbf{v} \tag{3.25}$$

To find an analogous version of the usual Poisson-Vlasov equation, but for our context, let's start with the Hamiltonian

$$H = \frac{1}{2}\int \|\mathbf{v}\|^2_{\mathfrak{sl}(2)} f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} + \frac{1}{2}\int \phi_f(\mathbf{x}) f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} \tag{3.26}$$

where $\phi_f(\mathbf{x})$ is the classical electric potential, defined by:

$$\mathbf{E} = -\nabla\phi_f(x) \tag{3.27}$$

And $f(\mathbf{x},\mathbf{v},t)$ is the plasma density. Using the bracket 3.25 the Hamiltonian 3.26 and the test functional:

$$F(f) = \int f(\tilde{\mathbf{x}},\tilde{\mathbf{v}},t)\delta(\mathbf{x}-\tilde{\mathbf{x}})\delta(\mathbf{v}-\tilde{\mathbf{v}})d\tilde{\mathbf{x}}d\tilde{\mathbf{v}}$$

Poisson-Vlasov equation can be computed as $\{F,H\}_{\mathfrak{s}^*}$, with $\frac{\delta H}{\delta f} = \frac{1}{2}\|\mathbf{v}\|^2_{\mathfrak{sl}(2)} + \phi_f(\mathbf{x})$, and the result will be:

$$\frac{\partial f}{\partial t} + \left\langle \mathbf{v}, \frac{\partial f}{\partial \mathbf{x}}\right\rangle - \left\langle \nabla\phi_f(\mathbf{x}), \frac{\partial f}{\partial \mathbf{v}}\right\rangle_{\mathfrak{sl}(2)} = 0 \tag{3.28}$$

### 3.3.3 $\mathfrak{sl}(2,\mathbb{R})$ MAXWELL'S EQUATIONS AND REDUCTION

We will find the $\mathfrak{sl}(2,\mathbb{R})$ Maxwell's equations via the same methodology in which Maxwell's equations are described in the standard case, that is to construct the appropriate Poisson bracket for the electric and magnetic fields by reduction, once we do this we will couple the Vlasov equation.

For $\mathfrak{sl}(2,\mathbb{R})$ Maxwell's equations we will consider $\mathfrak{A} = \mathfrak{X}(\mathbb{L}^3)$ as the configuration space. The corresponding phase space is $T^*\mathfrak{A}$, which is identified with pairs $(\mathbf{A},\mathbf{Y})$, where $\mathbf{A}\in\mathfrak{A}$ and $\mathbf{Y}$ is a vector field density on $\mathbb{L}^3$, in other words is $T^*\mathfrak{A}\cong\mathfrak{A}\times\mathfrak{A}^*$. We define the dual pairing between $\mathbf{A}\in\mathfrak{A}$ and $\mathbf{Y}\in\mathfrak{A}^*$ by integration. From now on, this integral, and all the integrals, are calculated in $\mathbb{R}^3$:

$$\langle \mathbf{A},\mathbf{Y}\rangle_{T^*\mathfrak{A}} = \int_{\mathbb{R}^3}\langle A(\mathbf{x}),Y(\mathbf{x})\rangle_{\mathfrak{sl}(2)}d\mathbf{x} \tag{3.29}$$

so that the symplectic structure on $T^*\mathfrak{A}$ is given by the form:

$$\omega((A_1,Y_1),(A_2,Y_2)) = \int \left(\langle Y_2,A_1\rangle_{\mathfrak{sl}(2)} - \langle Y_1,A_2\rangle_{\mathfrak{sl}(2)}\right)d\mathbf{x} \tag{3.30}$$

The Poisson bracket for functionals $F, G\colon T^*\mathfrak{A} \mapsto \mathbb{R}$ is given by the relation:

$$\{F, G\} = \omega(X_F, X_G)$$

where $X_F$ and $X_G$ are the Hamiltonian vector fields of $F$ and $G$ respectively, to be specific

$$X_F = \frac{\delta F}{\delta \mathbf{Y}}\frac{\partial}{\partial \mathbf{A}} - \frac{\delta F}{\delta \mathbf{A}}\frac{\partial}{\partial \mathbf{Y}}$$
$$X_G = \frac{\delta G}{\delta \mathbf{Y}}\frac{\partial}{\partial \mathbf{A}} - \frac{\delta G}{\delta \mathbf{A}}\frac{\partial}{\partial \mathbf{Y}}$$

consequently, the Poisson bracket is given as follows:

$$\{F, G\}(\mathbf{A}, \mathbf{Y}) = \int \left( \left\langle \frac{\delta F}{\delta \mathbf{A}}, \frac{\delta G}{\delta \mathbf{Y}} \right\rangle_{\mathfrak{sl}(2)} - \left\langle \frac{\delta F}{\delta \mathbf{Y}}, \frac{\delta G}{\delta \mathbf{A}} \right\rangle_{\mathfrak{sl}(2)} \right) d\mathbf{x}. \tag{3.31}$$

With the Hamiltonian,

$$H(\mathbf{A}, \mathbf{Y}) = \frac{1}{2}\left( \int \|\mathbf{Y}\|^2_{\mathfrak{sl}(2)} + \|\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A})\|^2_{\mathfrak{sl}(2)}\, d\mathbf{x} \right), \tag{3.32}$$

where $\|X\|^2_{\mathfrak{sl}(2)} = \langle X, X\rangle_{\mathfrak{sl}(2)}$. To compute Hamilton's equations

$$\frac{\delta H}{\delta \mathbf{Y}} = \frac{\partial \mathbf{A}}{\partial t}$$
$$-\frac{\delta H}{\delta \mathbf{A}} = \frac{\partial \mathbf{Y}}{\partial t},$$

we need to calculate the functional derivative, which is given by:

$$D_{\mathbf{A}} F \cdot \delta\mathbf{A} = \int_{\mathbb{R}^3} \left\langle \frac{\delta F}{\delta \mathbf{A}}, \delta\mathbf{A} \right\rangle_{\mathfrak{sl}(2)} d\mathbf{x} = \lim_{\varepsilon \to 0} \frac{F(\mathbf{A} + \varepsilon\delta\mathbf{A}) - F(\mathbf{A}, \mathbf{Y})}{\varepsilon}. \tag{3.33}$$

This follows from a general formula for functional derivatives on the dual of a Lie algebra [MW82, MWR+83] and the Fréchet derivative. Then, $\frac{\delta H}{\delta \mathbf{Y}}$ is calculated like this:

$$\int \left\langle \frac{\delta H}{\delta \mathbf{Y}}, \delta\mathbf{Y} \right\rangle_{\mathfrak{sl}(2)} d\mathbf{x} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}\left( \int \|\mathbf{Y} + \varepsilon\delta\mathbf{Y}\|^2_{\mathfrak{sl}(2)} - \|\mathbf{Y}\|^2_{\mathfrak{sl}(2)}\, d\mathbf{x} \right) = \int \langle \mathbf{Y}, \delta\mathbf{Y}\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x},$$

and the same to calculate $\frac{\delta H}{\delta \mathbf{A}}$:

$$\begin{aligned} \int \left\langle \frac{\delta H}{\delta \mathbf{A}}, \delta\mathbf{A} \right\rangle_{\mathfrak{sl}(2)} d\mathbf{x} &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}\left( \int \|\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A} + \epsilon\delta\mathbf{A})\|^2_{\mathfrak{sl}(2)} - \|\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A})\|^2_{\mathfrak{sl}(2)}\, d\mathbf{x} \right) \\ &= \int \langle \mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A}), \mathrm{curl}_{\mathfrak{sl}(2)}(\delta\mathbf{A})\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x}. \end{aligned} \tag{3.34}$$

To simplify the previous expression, let's recall an useful identity from Vector Calculus:

**LEMMA 3.5.** *Let* $\mathbf{a}, \mathbf{b} \in \mathfrak{X}(\mathbb{R}^3)$*, then, the following identity holds:*

$$\int_{\mathbb{R}^3} \langle curl(\mathbf{a}), \mathbf{b}\rangle\, d\mathbf{x} = \int_{\mathbb{R}^3} \langle \mathbf{a}, curl(\mathbf{b})\rangle\, d\mathbf{x} \tag{3.35}$$

*Proof.* We start with the classical identity for the divergence of a cross product in $\mathbb{R}^3$ [MT03]:

$$\operatorname{div}(\mathbf{a}\times\mathbf{b}) = \langle \operatorname{curl}(\mathbf{a}), \mathbf{b}\rangle - \langle \mathbf{a}, \operatorname{curl}(\mathbf{b})\rangle. \tag{3.36}$$

Integrating on both sides, we get the following:

$$\int_{\mathbb{R}^3} \operatorname{div}(\mathbf{a}\times\mathbf{b})\, d\mathbf{x} = \int_{\mathbb{R}^3} \left(\langle \operatorname{curl}(\mathbf{a}), \mathbf{b}\rangle - \langle \mathbf{a}, \operatorname{curl}(\mathbf{b})\rangle\right) d\mathbf{x}.$$

By divergence theorem, we know that the following holds:

$$\int_{\mathbb{R}^3} \operatorname{div}(\mathbf{a}\times\mathbf{b})\, d\mathbf{x} = \int_{\partial\mathbb{R}^3} \mathbf{a}\times\mathbf{b}\, dS.$$

The boundary $\partial\mathbb{R}^3 = \emptyset$, so the previous equation is identically zero. Therefore, we have that:

$$\int_{\mathbb{R}^3} \left(\langle \operatorname{curl}(\mathbf{a}), \mathbf{b}\rangle - \langle \mathbf{a}, \operatorname{curl}(\mathbf{b})\rangle\right) d\mathbf{x} = 0$$

And with that, we finish the proof. □

Using Proposition 3.4, Lemma 3.1, and Lemma 3.5, we have the next relation:

$$\int \langle \operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{a}^\flat), \mathbf{b}^\flat\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x} = \int \langle \operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{b}^\flat), \mathbf{a}^\flat\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x} \tag{3.37}$$

Taking $\mathbf{a}^\flat = \delta\mathbf{A}$ and $\mathbf{b}^\flat = \operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{A})$, and plugging that in Equation 3.34, we get that:

$$\int \left\langle \frac{\delta H}{\delta \mathbf{A}}, \delta\mathbf{A}\right\rangle_{\mathfrak{sl}(2)} d\mathbf{x} = \int \langle \operatorname{curl}_{\mathfrak{sl}(2)}(\operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{A})), \delta\mathbf{A}\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x}.$$

Therefore, Hamilton's equations are:

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial t} &= \mathbf{Y} \\ \frac{\partial \mathbf{Y}}{\partial t} &= -\operatorname{curl}_{\mathfrak{sl}(2)}\left(\operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{A})\right) \end{aligned} \tag{3.38}$$

If we take $\mathbf{E} = -\mathbf{Y}$ and $\mathbf{B} = \operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{A})$ we will get two of the $\mathfrak{sl}(2)$ Maxwell's equations, namely:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{E}) \\ \frac{\partial \mathbf{E}}{\partial t} &= \operatorname{curl}_{\mathfrak{sl}(2)}(\mathbf{B}) \end{aligned} \tag{3.39}$$

The Poisson bracket and the two remaining Maxwell's will come into sight through reduction, for that we will make use of the gauge symmetry of the Hamiltonian given by Proposition 3.4. Consider $\mathfrak{G} = C^\infty(\mathbb{L}^3)$ with addition as group operation. Let $\mathfrak{G} \curvearrowright T^*\mathfrak{A}$ ( $\mathfrak{G} \curvearrowright \mathfrak{A}$) by the next rule:

$$\begin{aligned} \Phi_\psi(\mathbf{A}, \mathbf{Y}) &= (\mathbf{A} + \nabla_{\mathfrak{sl}(2)}\psi, \mathbf{Y}) \\ \Theta_\psi(\mathbf{A}) &= \mathbf{A} + \nabla_{\mathfrak{sl}(2)}\psi \end{aligned} \tag{3.40}$$

with $\psi \in \mathfrak{G}$.It is easy to prove that the Hamiltonian 3.32 remains invariant under the action of $\mathfrak{G}$, that is:

$$H(\mathbf{A},\mathbf{Y}) = H(\Phi_\psi(\mathbf{A},\mathbf{Y})) \tag{3.41}$$

Therefore, we can use the gauge symmetry to reduce our system. The action has a momentum map $J\colon T^*\mathfrak{A} \mapsto \mathfrak{g}^*$, with $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G})$, which is identified with $\mathfrak{G} = C^\infty(\mathbb{L}^3)$, and $\mathfrak{g}^*$ is its dual space (densities on $\mathbb{L}^3$). The momentum map can be computed, using Equation 2.18:

$$\langle J(\mathbf{A},\mathbf{Y}), \varphi\rangle_{\mathfrak{g}} = \langle \mathbf{Y}, \varphi_{\mathfrak{A}}(\mathbf{A})\rangle_{T^*\mathfrak{A}},$$

where the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ between elements of $\mathfrak{g}^*$ and $\mathfrak{g}$ is given by integration. With an element $\varphi \in \mathfrak{g}$ and $\varphi_{\mathfrak{A}}(\mathbf{A})$ the infinitesimal generator of the action $\Theta_\varphi$ on $\mathfrak{A}$. Using equation 2.12 one determines that $\varphi_{\mathfrak{A}}(\mathbf{A}) = \nabla_{\mathfrak{sl}(2)}\varphi$. Taking this into account, we get:

$$\langle J(\mathbf{A},\mathbf{Y}), \varphi\rangle_{\mathfrak{g}} = \langle \mathbf{Y}, \nabla_{\mathfrak{sl}(2)}\varphi\rangle_{T^*\mathfrak{A}} = \int \langle \mathbf{Y}, \nabla_{\mathfrak{sl}(2)}\varphi\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x}.$$

To work on this, we use the divergence product rule and integration on both sides:

$$\int_{\mathbb{R}^3} \mathrm{div}(\varphi\mathbf{Y})\, d\mathbf{x} = \int_{\mathbb{R}^3} \left(\varphi \mathrm{div}(\mathbf{Y}) + \langle \mathbf{Y}, \nabla\varphi\rangle\right) d\mathbf{x} = 0.^{2}$$

Therefore, we have:

$$\int \mathrm{div}(\mathbf{Y})\varphi\, d\mathbf{x} = -\int \langle \nabla\varphi, \mathbf{Y}\rangle\, d\mathbf{x} = -\int \langle \nabla_{\mathfrak{sl}(2)}\varphi, \mathbf{Y}\rangle_{\mathfrak{sl}(2)}\, d\mathbf{x}.$$

From this, we get that:

$$\langle J(\mathbf{A},\mathbf{Y}), \varphi\rangle_{\mathfrak{g}} = -\int \mathrm{div}(\mathbf{Y})\varphi\, d\mathbf{x}$$

Out of the last equation, we conclude that the momentum map is:

$$J(\mathbf{A},\mathbf{Y}) = -\mathrm{div}(\mathbf{Y}). \tag{3.42}$$

Let $\rho \in \mathfrak{g}^*$ be a density, consider the set $J^{-1}(\rho) = \{(\mathbf{A},\mathbf{Y}) \in T^*\mathfrak{A}\colon -\mathrm{div}(\mathbf{Y}) = \rho\}$. Notice that if we take $\mathbf{Y} = -\mathbf{E}$, the condition $\mathrm{div}(\mathbf{Y}) = -\rho$ gives us another Maxwell's equation, the Gauss Law.

$$\mathrm{div}(\mathbf{E}) = \rho \tag{3.43}$$

This is quite a remarkable observation; the Gauss law did not change under this procedure. That's why we understand $\rho$ as a charge density.

Now, we are ready to reduce our system and find the Poisson Bracket. As we know, for Marsden-Weinstein reduction Theorem, $J^{-1}(\rho)/\mathfrak{G}$ is manifold which inherits the symplectic structure of $T^*\mathfrak{A}$.The next theorem will provide an identification of $J^{-1}(\rho)/\mathfrak{G}$ relevant to our discussion; in other words, we will give an identification in terms of $E$ and $B$, the electric and magnetic fields.

**THEOREM 3.6.** *$J^{-1}(\rho)/\mathfrak{G}$ can be identified with $\mathfrak{M}_{\mathfrak{sl}(2)} = \{(\mathbf{E},\mathbf{B})\colon div(\mathbf{E}) = \rho; div(\mathbf{B}) = 0\}$*

[2] This is zero due to the same argument used in the proof of lemma 3.5

*Proof.* To start with, $J^{-1}(\rho)/\mathfrak{G}$ defines the next equivalent relation, two elements $(\mathbf{A_1},\mathbf{Y_1}) \sim (\mathbf{A_2},\mathbf{Y_2})$ if there exists $\psi \in \mathfrak{G}$ such that

$$(\mathbf{A_2},\mathbf{Y_2}) = \Phi_\psi(\mathbf{A_1},\mathbf{Y_1})$$
$$(\mathbf{A_2},\mathbf{Y_2}) = (\mathbf{A_1} + \nabla_{\mathfrak{sl}(2)}\psi, \mathbf{Y_2})$$

From this, we need that $\mathbf{A_2} - \mathbf{A_1} = \nabla_{\mathfrak{sl}(2)}\psi$. If we take curl on both sides, we find that:

$$\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A_2} - \mathbf{A_1}) = \mathrm{curl}_{\mathfrak{sl}(2)}(\nabla_{\mathfrak{sl}(2)}\psi) = -\mathrm{curl}(\nabla\psi) = \mathbf{0}$$

So, we get that $(A_1, Y_1) \sim (A_2, Y_2)$ if and only if $\mathrm{curl}_{\mathfrak{sl}(2)}(A_1) = \mathrm{curl}_{\mathfrak{sl}(2)}(A_2)$. The relation $\mathrm{div}(B) = 0$ comes from both the magnetic Gauss law and from the fact that

$$\mathrm{div}(\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A})) = -\mathrm{div}(\mathrm{curl}(\mathbf{A}^\flat)) = 0$$

Accordingly, associating $\mathbf{B} = \mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A})$ and $\mathbf{E} = -\mathbf{Y}$ we got the remaining two Maxwell's equations, and the equivalence relation follows. Moreover, this association gives a 1-1 correspondence between the two manifolds in question. □

An interesting result from the previous theorem is that the reduced space $\mathfrak{M}_{\mathfrak{sl}}$ coincides with the reduced space $\mathfrak{M}$ described in Section 3.1. The only difference is that, in the $\mathfrak{sl}(2,\mathbb{R})$ case, we have $B = \mathrm{curl}_{\mathfrak{sl}(2)}(A)$ instead of $B = \mathrm{curl}(A)$. Consequently, both the electric and magnetic Gauss laws remain unchanged compared to the standard construction.

We finish this discussion with the Poisson bracket for the manifold $\mathfrak{M}_{\mathfrak{sl}(2)}$, for that we take two functionals $F, G$ on $\mathfrak{M}_{\mathfrak{sl}(2)}$ and pull them back to $J^{-1}(\rho)$, extend them to $T^*\mathfrak{A}$ and take the Poisson bracket 3.31 and the using the chain rule we must push the result down to $\mathfrak{M}_{\mathfrak{sl}(2)}$ again. In other words:

$$\{F, G\}_{\mathfrak{M}_{\mathfrak{sl}(2)}} = \{\hat{F}, \hat{G}\}|_{T^*\mathfrak{A}}$$

With $\hat{F}, \hat{G}$ the pullback of $F$ and $G$ respectively, given by:

$$\hat{F}(\mathbf{A},\mathbf{Y}) = F(\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A}), -\mathbf{Y}) = F(\mathbf{B},\mathbf{E})$$

By chain rule, we have the next relations:

$$D_{\mathbf{A}}\hat{F}(\mathbf{A},\mathbf{Y})\cdot\mathbf{A}' = D_{\mathbf{B}}F(\mathbf{B},\mathbf{E})\cdot\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A}')$$
$$D_{\mathbf{Y}}\hat{F}(\mathbf{A},\mathbf{Y})\cdot\mathbf{Y}' = -D_{\mathbf{E}}F(\mathbf{B},\mathbf{E})\cdot\mathbf{E}'$$

Out of the second relation, we get that:

$$\frac{\delta\hat{F}}{\delta\mathbf{Y}} = -\frac{\delta F}{\delta\mathbf{E}} \tag{3.44}$$

And from the first one:

$$\int\left\langle\frac{\delta\hat{F}}{\delta\mathbf{A}},\mathbf{A}'\right\rangle_{\mathfrak{sl}(2)}d\mathbf{x} = \int\left\langle\frac{\delta F}{\delta\mathbf{B}},\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A}')\right\rangle_{\mathfrak{sl}(2)}d\mathbf{x} = \int\left\langle\mathbf{A}',\mathrm{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta F}{\delta\mathbf{B}}\right)\right\rangle_{\mathfrak{sl}(2)}d\mathbf{x}$$

the last equality comes from 3.37. From this, we find that:

$$\frac{\delta \hat{F}}{\delta \mathbf{A}} = \text{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta F}{\delta \mathbf{B}}\right) \tag{3.45}$$

Now, using the bracket 3.31 on $T^*\mathfrak{A}$ for $\hat{F}$ and $\hat{G}$:

$$\{\hat{F},\hat{G}\} = \int \left(\left\langle \frac{\delta \hat{F}}{\delta \mathbf{A}}, \frac{\delta \hat{G}}{\delta \mathbf{Y}}\right\rangle_{\mathfrak{sl}(2)} - \left\langle \frac{\delta \hat{F}}{\delta \mathbf{Y}}, \frac{\delta \hat{G}}{\delta \mathbf{A}}\right\rangle_{\mathfrak{sl}(2)}\right) d\mathbf{x} = \int \left(\left\langle \frac{\delta F}{\delta \mathbf{E}}, \frac{\delta \hat{G}}{\delta \mathbf{A}}\right\rangle_{\mathfrak{sl}(2)} - \left\langle \frac{\delta \hat{F}}{\delta \mathbf{A}}, \frac{\delta G}{\delta \mathbf{E}}\right\rangle_{\mathfrak{sl}(2)}\right) d\mathbf{x}$$

And finally, using 3.45 we get the Poisson bracket for functionals on $\mathfrak{M}_{\mathfrak{sl}(2)}$:

$$\{F,G\}_{\mathfrak{M}} = \int \left(\left\langle \frac{\delta F}{\delta \mathbf{E}}, \text{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta G}{\delta \mathbf{B}}\right)\right\rangle_{\mathfrak{sl}(2)} - \left\langle \text{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta F}{\delta \mathbf{B}}\right), \frac{\delta G}{\delta \mathbf{E}}\right\rangle_{\mathfrak{sl}(2)}\right) d\mathbf{x} \tag{3.46}$$

The $\mathfrak{sl}(2)$ Maxwell's equations can be computed from Hamiltonian 3.32 and Poisson bracket 3.46. That is to say, let $F$ be a functional on $\mathfrak{sl}(2)$, Hamilton´s equation for this functional are:

$$\dot{F} = \{F,H\}_{\mathfrak{M}_{\mathfrak{sl}(2)}}$$

For example, let's consider the evaluation functionals:

$$E^i(x)(E,B) = \int \delta(y-x)\langle \mathbf{e}_i, \mathbf{E}\rangle_{\mathfrak{sl}(2)} d\mathbf{y}, \qquad \dot{E}^i(x)(E,B) = \int \delta(y-x)\langle \mathbf{e}_i, \dot{\mathbf{E}}\rangle_{\mathfrak{sl}(2)} d\mathbf{y}$$

Then, the Maxwell equation is:

$$\dot{E}^i(x)(E,B) = \{E^i(x),H\}_{\mathfrak{M}_{\mathfrak{sl}(2)}} = \int \left(\left\langle \frac{\delta E^i}{\delta \mathbf{E}}, \text{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta H}{\delta \mathbf{B}}\right)\right\rangle_{\mathfrak{sl}(2)} - \left\langle \text{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta E^i}{\delta \mathbf{B}}\right), \frac{\delta H}{\delta \mathbf{E}}\right\rangle_{\mathfrak{sl}(2)}\right) d\mathbf{x}$$

We can get all the terms needed using the formula for functional derivative 3.33, namely:

$$\frac{\delta H}{\delta \mathbf{E}} = \mathbf{E} \qquad \frac{\delta E^i(x)}{\delta \mathbf{E}} = \delta(y-x)\mathbf{e}_i$$

$$\frac{\delta H}{\delta \mathbf{B}} = \mathbf{B} \qquad \frac{\delta E^i(x)}{\delta \mathbf{B}} = 0$$

Thus, by replacing all these terms, we get that:

$$\dot{\mathbf{E}} = \text{curl}_{\mathfrak{sl}(\mathbf{2})}(\mathbf{B})$$

We can implement a similar procedure for the magnetic field, and we will get the next equation:

$$\dot{\mathbf{B}} = -\text{curl}_{\mathfrak{sl}(\mathbf{2})}(\mathbf{E})$$

### 3.3.4 THE COUPLING TERMS AND THE $\mathfrak{sl}(2,\mathbb{R})$ VLÁSOV-MAXWELL EQUATIONS

We will now couple the Maxwell equations with the Poisson-Vlásov system, using the results of the two previous sections, to obtain the Vlásov-Maxwell system. We start with the pre-reduction

phase, where there is no coupling between the density variables and the electromagnetic fields. For that, we choose density variables $f_m(\mathbf{x},\mathbf{p})$ defined in the momentum phase space $(\mathbf{x},\mathbf{p})$ rather than the velocity phase space $(\mathbf{x},\mathbf{v})$, these two are linked via the physical relation $\mathbf{p}=\mathbf{v}+\mathbf{A}(\mathbf{x})$, that is to say $f_m(\mathbf{x},\mathbf{p})=f_m(\mathbf{x},\mathbf{v}+\mathbf{A})=f(\mathbf{x},\mathbf{v})$. For the fields variables, we choose $(\mathbf{A},\mathbf{Y})\in T^*\mathfrak{A}$ as in Section 3.3.3. The configuration space is now $\mathfrak{s}^*\times T^*\mathfrak{A}$ and the Poisson bracket is the sum of the bracket in $\mathfrak{s}^*$ and the one in $T^*\mathfrak{A}$, specifically, for functionals $\tilde{F},\tilde{G}\colon \mathfrak{s}^*\times T^*\mathfrak{A}\mapsto\mathbb{R}$:

$$\{\tilde{F},\tilde{G}\}=\int f_m\left\{\frac{\delta\tilde{F}}{\delta f_m},\frac{\delta\tilde{G}}{\delta f_m}\right\}^{xp}_{\mathfrak{sl}(2)}d\mathbf{x}d\mathbf{p}+\int\left(\left\langle\frac{\delta\tilde{F}}{\delta\mathbf{A}},\frac{\delta\tilde{G}}{\delta\mathbf{Y}}\right\rangle_{\mathfrak{sl}(2)}-\left\langle\frac{\delta\tilde{F}}{\delta\mathbf{Y}},\frac{\delta\tilde{G}}{\delta\mathbf{A}}\right\rangle_{\mathfrak{sl}(2)}\right)d\mathbf{x} \tag{3.47}$$

As expected, the Hamiltonian will be the sum of the Poisson-Vlasov Hamiltonian 3.26 and the Maxwell Hamiltonian 3.32.

$$\tilde{H}(f_m,\mathbf{A},\mathbf{Y})=\frac{1}{2}\int\|\mathbf{p}-\mathbf{A}(\mathbf{x})\|^2_{\mathfrak{sl}(2)}f_m(\mathbf{x},\mathbf{p})d\mathbf{x}d\mathbf{p}+\frac{1}{2}\left(\int\|\mathbf{Y}\|^2_{\mathfrak{sl}(2)}+\|\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A})\|^2_{\mathfrak{sl}(2)}d\mathbf{x}\right) \tag{3.48}$$

To reduce the dynamics, we need to indicate an action of $\mathfrak{G}$ on $\mathfrak{s}^*\times T^*\mathfrak{A}$ such that it leaves the Hamiltonian 3.48 invariant. We already know that the action 3.40 leaves invariant the electromagnetic part of the Hamiltonian. An action of $\mathfrak{G}$ on $\mathfrak{s}^*$ can be the momentum translation map $\tau_{-\nabla_{\mathfrak{sl}(2)}\psi}\colon\mathbb{R}^6\mapsto\mathbb{R}^6$, provided by:

$$\tau_{-\nabla_{\mathfrak{sl}(2)}\psi}(\mathbf{x},\mathbf{p})=(\mathbf{x},\mathbf{p}-\nabla_{\mathfrak{sl}(2)}\psi) \tag{3.49}$$

Since $\tau^*_{-\nabla_{\mathfrak{sl}(2)}\psi}\omega=\omega$, with $\omega$ the symplectic form defined by 3.21, the momentum translation map is a canonical transformation, therefore it preserves the Poisson structure. Hence the action of $\mathfrak{G}$ on $\mathfrak{s}^*\times T^*\mathfrak{A}$ will be:

$$\Psi_\psi(f_m,\mathbf{A},\mathbf{Y})=(f_m\circ\tau_{-\nabla_{\mathfrak{sl}(2)}\psi}(\mathbf{x},\mathbf{p}),\mathbf{A}+\nabla_{\mathfrak{sl}(2)}\psi,\mathbf{Y}) \tag{3.50}$$

Now, we need to compute the momentum map generated by this action.

**PROPOSITION 3.7.** *The action of $\mathfrak{G}$ on $\mathfrak{s}^*\times T^*\mathfrak{A}$ defined by 3.50 has a moment map $J\colon\mathfrak{s}^*\times T^*\mathfrak{U}\mapsto\mathfrak{g}^*$ given by:*

$$J(f_m,\mathbf{A},\mathbf{Y})=-\int f_m(\mathbf{x},\mathbf{p})d\mathbf{p}-div(\mathbf{Y})$$

*Proof.* Notice that the second term is the same obtained in 3.42 for the Maxwell action on $T^*\mathfrak{A}$, so to complete the proof we only need to compute the moment map for the action 3.49. Let's recall that the moment map must satisfy the next property:

$$X_{\langle J_{\mathfrak{s}^*},\psi\rangle}(f_m)=\psi_{\mathfrak{s}^*}(f_m)$$

Where $\psi_{\mathfrak{s}^*}:\mathfrak{s}^*\mapsto\mathfrak{s}^*$ is the infinitesimal generator of the action, which is given by:

$$\psi_{\mathfrak{s}^*}(f_m)=\frac{d}{dt}\Big|_{t=0}\Psi_{\exp(t\psi)}(f_m(x,p))=\frac{d}{dt}\Big|_{t=0}f_m(x,p-t\nabla_{\mathfrak{sl}(2)}\psi)=\{f_m,\psi\}_{\mathfrak{sl}(2)}$$

Let us assume for a second that $J_{s^*} = -\int f_m(x,p)dp$ , then for theorem 2.6 we have that:

$$X_{\langle J_{s^*},\psi\rangle}(f_m) = -\mathrm{ad}^*_{\frac{\delta\langle J_{s^*},\psi\rangle}{\delta f_m}}(f_m) = \mathrm{ad}^*_{\psi}(f_m)$$

We can get $\mathrm{ad}^*_{\psi}(f_m)$ by:

$$\begin{aligned}\langle \mathrm{ad}^*_{\psi}(f_m), g\rangle &= \langle f_m, \mathrm{ad}_{-\psi}(g)\rangle \\ &= \langle f, \{\psi, g\}_{\mathfrak{sl}(2)}\rangle \\ &= \langle g, \{f_m, \psi\}_{\mathfrak{sl}(2)}\rangle\end{aligned}$$

Here we used the identity $\langle g, \{h,k\}_{\mathfrak{sl}(2)}\rangle = \langle k, \{g,h\}_{\mathfrak{sl}(2)}\rangle$ that can be easily proven using integration by parts. So, we have that $X_{\langle J_{s^*},\psi\rangle}(f_m) = \psi_{\mathfrak{s}^*}(f_m)$. □

**PROPOSITION 3.8.** *The reduced manifold* $(\mathfrak{s}^* \times T^*\mathfrak{A})_0 = J^{-1}(0)/\mathfrak{G}$ *may be identified with the* $\mathfrak{sl}(2)-$*Vlásov-Maxwell phase-space:*

$$\mathfrak{VM}_{\mathfrak{sl}(2)} = \left\{(f, \mathbf{B}, \mathbf{E}) :\ div(\mathbf{B}) = 0, div(\mathbf{E}) = \int f(\mathbf{x},\mathbf{v})d\mathbf{v}\right\}$$

*Proof.* There is a one-to-one correspondence between any $(f_m, \mathbf{A}, \mathbf{Y}) \in J^{-1}(0)$ and the triple $(f, \mathbf{B}, \mathbf{E}) \in \mathfrak{MV}_{\mathfrak{sl}(2)}$:

$$\begin{aligned} f(\mathbf{x},\mathbf{v}) &= f_m(\mathbf{x}, \mathbf{v} + \mathbf{A}(\mathbf{x})) \\ \mathbf{B} &= \mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A}) \\ \mathbf{E} &= -\mathbf{Y} \end{aligned}$$

As we saw in the proof of the Theorem 3.6, if two elements on $J^{-1}(0)$ are in the same equivalence class, then they are related by a gauge transformation. Also, taking the pre-image at 0 of the momentum mapping, we have the result. □

The manifold $\mathfrak{VM}_{\mathfrak{sl}(2)}$ inherits a Poisson structure from $\mathfrak{s}^* \times T^*\mathfrak{A}$ by a general theory of reduction. As the Hamiltonian 3.48 is invariant under the action 3.50, then the $\mathfrak{sl}(2)$ Maxwell-Vlasov system is Hamiltonian with respect to the inherited Poisson structure. Now, using a similar approach as in Section 3.3.3, we are going to perform the calculation of the Poisson bracket for $F, G$ on $\mathfrak{VM}_{\mathfrak{sl}(2)}$.

**LEMMA 3.9.** *For* $\overline{F}(f_m, A, Y) = F(f, B, E)$*, we have:*

$$\begin{aligned} \frac{\delta \overline{F}}{\delta A} &= curl_{\mathfrak{sl}(2)}\left(\frac{\delta F}{\delta \mathbf{B}}\right) + \frac{\delta F}{\delta f}\frac{\partial f}{\partial \mathbf{v}} \\ \frac{\delta \overline{F}}{\delta Y} &= -\frac{\delta F}{\delta \mathbf{E}} \end{aligned}$$

*Proof.* Due to the definition of functional derivative, we have that:

$$D_{\mathbf{A}}\overline{F}(f_m, \mathbf{A}, \mathbf{Y})\cdot \mathbf{A}' = \int \left\langle \frac{\delta \overline{F}}{\delta \mathbf{A}}, \mathbf{A}' \right\rangle_{\mathfrak{sl}(2)} d\mathbf{x}$$

Now, by the chain rule:

$$D_{\mathbf{A}}\overline{F}(f_m,\mathbf{A},\mathbf{Y})\cdot\mathbf{A}' = D_f F(f,\mathbf{B},\mathbf{E})\cdot D_{\mathbf{A}}f\cdot\mathbf{A}' + D_{\mathbf{B}}F(f,\mathbf{B},\mathbf{E})\cdot \mathrm{curl}_{\mathfrak{sl}(2)}(A')$$

But:

$$D_{\mathbf{A}}f\cdot\mathbf{A}' = \frac{\partial f_m}{\partial \mathbf{p}}\cdot\mathbf{A}' = \frac{\partial f}{\partial \mathbf{v}}\cdot\mathbf{A}'$$

Now, combining these results with the definition of functional derivative, we get:

$$\int\left\langle\frac{\delta\overline{F}}{\delta\mathbf{A}},\mathbf{A}'\right\rangle_{\mathfrak{sl}(2)} = \int\left(\left\langle\frac{\delta F}{\delta f}\frac{\partial f}{\partial\mathbf{v}},\mathbf{A}'\right\rangle_{\mathfrak{sl}(2)} + \left\langle\frac{\delta F}{\delta\mathbf{B}},\mathrm{curl}_{\mathfrak{sl}(2)}(\mathbf{A}')\right\rangle\right)d\mathbf{x}$$

And using Equation 3.37, we have that:

$$\int\left\langle\frac{\delta\overline{F}}{\delta\mathbf{A}},\mathbf{A}'\right\rangle_{\mathfrak{sl}(2)} = \int\left(\left\langle\frac{\delta F}{\delta f}\frac{\partial f}{\partial\mathbf{v}},\mathbf{A}'\right\rangle_{\mathfrak{sl}(2)} + \left\langle\mathrm{curl}_{\mathfrak{sl}(2)}\left(\frac{\delta F}{\delta\mathbf{B}}\right),\mathbf{A}'\right\rangle_{\mathfrak{sl}(2)}\right)d\mathbf{x}$$

□

**LEMMA 3.10.** *Let* $\mathbf{B} = curl_{\mathfrak{sl}(2)}(\mathbf{A})$ *,*$h(\mathbf{x},\mathbf{v}) = h_m(\mathbf{x},\mathbf{v}+\mathbf{A}(\mathbf{x}))$ *and* $k(\mathbf{x},\mathbf{v}) = k_m(\mathbf{x},\mathbf{v}+\mathbf{A}(\mathbf{x}))$*.Then:*

$$\{h_m,k_m\}^{xp}_{\mathfrak{sl}(2)} = \{h,k\}^{xv}_{\mathfrak{sl}(2)} + \mathbf{B}\cdot\left(\frac{\partial h}{\partial v}\times_{\mathfrak{sl}(2)}\frac{\partial k}{\partial v}\right)$$

*Proof.* Using the equation for the Poisson bracket under a change of variables $v_i = p_i + A_i(\mathbf{x})$ [CFMat21]:

$$\{h_m,k_m\}^{xp}_{\mathfrak{sl}(2)} = \sum_{ij}^{3}\{z_i,z_j\}_{\mathfrak{sl}(2)}\frac{\partial h_m}{\partial z_i}\frac{\partial k_m}{\partial z_j}$$

This can be reduced by applying the following relations:

$$\{x,p_x\}_{\mathfrak{sl}(2)} = \{y,p_y\}_{\mathfrak{sl}(2)} = 1$$
$$\{z,p_z\}_{\mathfrak{sl}(2)} = -1$$

By skew-symmetry of the bracket,and considering that $\frac{\partial h_m}{\partial p_i} = \frac{\partial h}{\partial v_i}$ we have that:

$$\{h_m,k_m\}^{xp}_{\mathfrak{sl}(2)} = \{h,k\}^{xv}_{\mathfrak{sl}(2)} + \sum_{ij}^{3}\frac{\partial h}{\partial v_i}\{v_i,v_j\}^{xp}_{\mathfrak{sl}(2)}\frac{\partial k}{\partial v_j}$$

Now, by using the canonical momentum relation, we can write:

$$\{v_i,v_j\}^{xp}_{\mathfrak{sl}(2)} = -\{p_i,A_j(\mathbf{x})\}^{xp}_{\mathfrak{sl}(2)} - \{A_i(\mathbf{x}),p_j\}^{xp}_{\mathfrak{sl}(2)}$$

By a simple calculation, we can get the following property for any function $f(\mathbf{x})$:

$$\{p_\alpha,f(\mathbf{x})\}^{xp}_{\mathfrak{sl}(2)} = \begin{cases}-\frac{\partial f(\mathbf{x})}{\partial x_\alpha} & \text{if } \alpha = x,y\\ \frac{\partial f(\mathbf{x})}{\partial z} & \text{if } \alpha = z\end{cases}$$

Inspired by the Levi-Civita symbols used in electrodynamics to write in a simple but elegant way

the cross product and the curl of a vector field [3], let's define the following symbols $\epsilon^L_{ijk}$:

$$\begin{aligned} \epsilon^L_{231} &= 1 & \epsilon^L_{321} &= -1 \\ \epsilon^L_{312} &= 1 & \epsilon^L_{132} &= -1 \\ \epsilon^L_{213} &= 1 & \epsilon^L_{123} &= -1 \end{aligned}$$

And $\epsilon_{ijk} = 0$ in any other case. We can finally write:

$$\{v_i, v_j\}^{xp}_{\mathfrak{sl}(2)} = \epsilon^L_{ijk} B_k.$$

Combining this with our previous result, we have:

$$\{h_m, k_m\}^{xp}_{\mathfrak{sl}(2)} = \{h,k\}^{xv}_{\mathfrak{sl}(2)} + \sum_{ij}^{3} \epsilon^L_{ijk} B_k \frac{\partial h}{\partial v_i}\frac{\partial k}{\partial v_j}.$$

The last term in this equation coincides with the next triple product:

$$\sum_{ij}^{3} \epsilon^L_{ijk} B_k \frac{\partial h}{\partial v_i}\frac{\partial k}{\partial v_j} = \mathbf{B}\cdot\left(\frac{\partial h}{\partial v} \times_{\mathfrak{sl}(2)} \frac{\partial k}{\partial v}\right)$$

Which concludes the proof. □

**THEOREM 3.11.** *Let $F, G\colon \mathfrak{VM}_{\mathfrak{sl}(2)} \mapsto \mathbb{R}$, the Poisson bracket is given by:*

$$\begin{aligned} \{F,G\}_{\mathfrak{VM}} = &\int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}^{xv}_{\mathfrak{sl}(2)} d\mathbf{x}d\mathbf{v} + \int \left(\left\langle \frac{\delta F}{\delta \mathbf{E}}, curl_{\mathfrak{sl}(2)}\left(\frac{\delta G}{\delta \mathbf{B}}\right)\right\rangle_{\mathfrak{sl}(2)} - \left\langle curl_{\mathfrak{sl}(2)}\left(\frac{\delta F}{\delta \mathbf{B}}\right), \frac{\delta G}{\delta \mathbf{E}}\right\rangle_{\mathfrak{sl}(2)}\right) d\mathbf{x} \\ &+ \int \left(\left\langle \frac{\delta F}{\delta \mathbf{E}}, \frac{\partial f}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right\rangle_{\mathfrak{sl}(2)} - \left\langle \frac{\delta G}{\delta \mathbf{E}}, \frac{\partial f}{\partial \mathbf{v}}\frac{\delta F}{\delta f}\right\rangle_{\mathfrak{sl}(2)}\right) d\mathbf{x}d\mathbf{v} + \int f\mathbf{B}\cdot\left(\frac{\partial}{\partial \mathbf{v}}\frac{\delta F}{\delta f} \times_{\mathfrak{sl}(2)} \frac{\partial}{\partial \mathbf{v}}\frac{\delta G}{\delta f}\right) d\mathbf{x}d\mathbf{v} \end{aligned}$$

*Proof.* This follows from a straightforward calculation using the Poisson bracket defined for $\overline{F}, \overline{G}$ on $\mathfrak{s}^* \times T^*\mathfrak{A}$ given by Equation 3.47, and then with the help of Lemmas 3.9,3.10 expressing the result in terms of the variables $(f, \mathbf{E}, \mathbf{B})$. □

Finally, we wish to use the Poisson structure defined by Theorem 3.11 to establish a new Vlasov-Maxwell, that we may call "$\mathfrak{sl}(2)(\mathbb{R})$ *Vlásov-Maxwell equations*". Such a system is Hamiltonian, regarding:

$$H(f, \mathbf{E}, \mathbf{B}) = \frac{1}{2}\int \|\mathbf{v}\|^2_{\mathfrak{sl}(2)} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x}d\mathbf{v} + \frac{1}{2}\left(\int \|\mathbf{E}\|^2_{\mathfrak{sl}(2)} + \|\mathbf{B}\|^2_{\mathfrak{sl}(2)} d\mathbf{x}\right) \tag{3.51}$$

[3] The triple scalar product in $\mathbb{R}^3$ is expressed as $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \varepsilon_{ijk} a^i b^j c^k$, where $\varepsilon_{ijk}$ are the well-known Levi-Civita symbols. However, we are defining a new set of Levi-Civita symbols, that help us to write a similar relation, but for $\mathbf{a}\cdot(\mathbf{b}\times_L\mathbf{c})$, making our calculations easier.

And we can get the equations by $\dot{F} = \{F, H\}_{\mathfrak{VM}}$, with $F = f, E, B$. These equations are:

$$\begin{aligned}
\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \left\langle (\mathbf{E} - \mathbf{v}\times\mathbf{B}), \frac{\partial f}{\partial \mathbf{v}} \right\rangle_{\mathfrak{sl}(2)} &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} &= -\mathrm{curl}_{\mathfrak{sl}(2)}\mathbf{E} \\
\frac{\partial \mathbf{E}}{\partial t} &= \mathrm{curl}_{\mathfrak{sl}(2)}B - \int \mathbf{v}^{\flat} f(\mathbf{x},\mathbf{v},t)\,d\mathbf{v} \\
\nabla\cdot\mathbf{E} &= \int f(\mathbf{x},\mathbf{v},t)\,d\mathbf{v} \\
\nabla\cdot\mathbf{B} &= 0.
\end{aligned}$$

## 3.4 Construction in other 3-dimensional Lie-Algebras

After extending the results of [MW82] to the 3-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ the natural question that arises is if it is possible to extend this construction to any other 3-dimensional Lie algebras given additional symmetries to the configuration space. For that purpose we are going to follow the 3-dimensional Lie algebra classification found in [EW06, Bow05], although the original classification was proposed by Bianchi [Bia98].

We already know that the construction is possible for $\mathfrak{so}(3,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$, and after reviewing the details on the construction on Sections 3.1, and 3.2 and the properties of the algebra on 3.2 we can conclude some specific conditions that yield desired the geometric construction:

(i) We need to properly define the gradient and the divergence operators. These operators arise from a *non-degenerate inner product*. Specifically, properties 3.35, 3.37, and the divergence theorem must hold. It is essential for the inner product to be non-degenerate, as we define the symplectic structure of the variables $E$ and $B$ through this inner product (see 3.30). Without the non-degenerate condition, the reduction process would not work.

(ii) There must exist a relation between the Lie bracket on $\mathbb{R}^3$ and the Lie bracket of the Lie algebra. Additionaly, a corresponding relation should hold between the inner product of $\mathbb{R}^3$ and the inner product of the Lie algebra. In other words, an analogue of Lemma 3.1 must be satisfied. This property is used several times throughout this work, for example, it was used in conjunction with Lemma 3.5 to obtain the Equation 3.37, which is a previous step to get Maxwell's equations. Moreover, Lemma 3.1 implies Lemma 3.4, which is the gauge symmetry condition, crucial for the reduction procedure.

(iii) The inner product and the Lie bracket must be related in some way. For that, we can ask for equation 3.18 of Theorem 3.2 to be fulfilled in a general case. That is, let $(\mathfrak{g}, [\cdot,\cdot])$ be a 3-dimensional Lie algebra, then the following relation must hold:

$$[x,[y,z]] = \lambda_1\langle x,z\rangle_{\mathfrak{g}} y + \lambda_2\langle x,y\rangle_{\mathfrak{g}} z, \tag{3.52}$$

where $\lambda_1, \lambda_2$ are nonzero real coefficients. If we can guarantee that the 3-dimensional Lie algebra $\mathfrak{g}$ satisfies the conditions above, then we can have a similar system of PDE as the Vlásov-Maxwell system for the Lie algebra.

### 3.4.1 CLASSIFICATION OF 3-DIMENSIONAL LIE ALGEBRAS

In this section, we present the classification (up to isomorphism) all 3-dimensional Lie algebras. While we present the key results relevant to our development, a more comprehensive discussion and detailed classification can be found in [Bow05].

**DEFINITION 3.2.** Let $\mathfrak{g}$ be a Lie algebra. The *derived algebra* of $\mathfrak{g}$ is the set of all linear combinations of Lie brackets in $\mathfrak{g}$, it is denoted as $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}'$.The *rank* of a Lie bracket is the dimension of the derived algebra $\mathfrak{g}'$

We will use the following basic result in Lie algebra theory to simplify the classification

**PROPOSITION 3.12.** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two finite dimensional Lie algebras. Suppose each algebra has a basis in which the structure constants are the same. Then $\mathfrak{g}_1 \cong \mathfrak{g}_2$*

We now classify Lie algebras (up to isomorphism) of rank 0 to 3. Throughout this discussion, $\{e_1, e_2, e_3\}$ denotes the canonical basis vectors of the Lie algebra $\mathfrak{g}$.

#### 3.4.1.1 RANK 0 AND 1

- **Rank 0:** We will call this algebra $L(3,0)$. If the rank is zero, then $[\mathfrak{g},\mathfrak{g}] = 0$ which means that all of the Lie brackets are zero, therefore the Lie algebra is abelian. The relation $[x,[y,z]] = 0$ implies that condition 3.52 is only satisfied by a vanishing inner product, so we can't perform the construction in this Lie algebra.
- **Rank 1:** For dimension 1, at least one of the Lie brackets in the derived algebra is not zero. We have the following two cases:

  (i) Heisenberg Lie algebra or $L(3,1)$: The commutation relations are:

  $$[e_1,e_2] = 0$$
  $$[e_1,e_3] = 0$$
  $$[e_2,e_3] = e_1$$

  For simplicity, we will refer to this algebra as $\mathfrak{h}$.

  (ii) $L(3,-1)$[4]: The second Lie algebra of rank 0 is characterized by the following relations:

  $$[e_1,e_2] = 0$$
  $$[e_1,e_3] = 0$$
  $$[e_2,e_3] = e_3$$

#### 3.4.1.2 RANK 2

In this case, let $\{Y.Z\}$ be a basis of the derived algebra $\mathfrak{g}'$ such that $\{X,Y,Z\}$ it can be proved that $\mathrm{ad}_X : \mathfrak{g}' \mapsto \mathfrak{g}'$ is an isomorphism [EW06, Bow05]. Therefore, the matrix representation of $\mathrm{ad}_X$

[4]The negative sign is used to indicate Lie algebras that can be decomposed into a direct sum of other Lie algebras

must have two non-zero eigenvalues $\lambda_1$ and $\lambda_2$. We can classify the rank 2 case in terms of the eigenvalues

(i) $L(3,2,x)$: In this case $\mathrm{ad}_X$ has two real eigenvalues, $\lambda_1 \neq \lambda_2$, and let's take $x = \frac{\lambda_2}{\lambda_1}$

$$\begin{aligned}[e_1, e_2] &= 0\\ [e_1, e_3] &= e_1\\ [e_2, e_3] &= x e_2\end{aligned}$$

These structure equations describe a parametrized family of continuously varying Lie algebras, depending on $x \in \mathbb{R}$ factor.

(ii) $L(3,3)$: The matrix $\mathrm{ad}_X$ has only one real eigenvalue of multiplicity 2. Then $x = 1$

$$\begin{aligned}[e_1, e_2] &= 0\\ [e_1, e_3] &= e_1\\ [e_2, e_3] &= e_1 + e_2\end{aligned}$$

(iii) $L(3,4,x)$: In this case we have two distinct complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \bar{\lambda}_1$. Let $x = \frac{\alpha}{\beta}$, we can always assume that $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)$, so that $x \geq 0$.

$$\begin{aligned}[e_1, e_2] &= 0\\ [e_1, e_3] &= x e_1 - e_2\\ [e_2, e_3] &= e_1 + x e_2\end{aligned}$$

As in the first case, these structure relations also describe a continuously varying family of Lie algebras.

#### 3.4.1.3 RANK 3

When the derived algebra $\mathfrak{g}'$ has full rank, we have the most familiar cases, on which we already have the Vlasov-Maxwell development :

(i) $L(3,5) = \mathfrak{sl}(2,\mathbb{R})$ with the structure relations:

$$\begin{aligned}[e_1, e_2] &= -e_3\\ [e_1, e_3] &= -e_2\\ [e_2, e_3] &= e_1\end{aligned}$$

(ii) $L(3,6) = \mathfrak{so}(3,\mathbb{R})$: For this algebra, the structure relations become:

$$\begin{aligned}[e_1, e_2] &= e_3\\ [e_1, e_3] &= -e_2\\ [e_2, e_3] &= e_1\end{aligned}$$

We already have the classification of 3-dimensional Lie algebras by its rank. Note that from rank 0 to rank 2, there is (at least) a bracket of generators that vanishes.

**PROPOSITION 3.13.** *Let any Lie algebra of dimension 3 so that its rank belongs to* $\{0,1,2\}$*, then it does not exist a non-degenerate inner product for which the condition 3.52 holds.*

*Proof.* We will assume (without loss of generality) that $[e_1,e_2]=0$ in any of each rank varying from 0 to 2, and also that Equation 3.52 holds for some inner product.

- **Rank 2:** Condition 3.52 in particular yields

$$\lambda_1\langle e_3,e_1\rangle_{\mathfrak{g}}e_2+\lambda_2\langle e_3,e_2\rangle_{\mathfrak{g}}e_1=[e_3,[e_1,e_2]]=0,$$
$$\lambda_1\langle e_2,e_1\rangle_{\mathfrak{g}}e_2+\lambda_2\langle e_2,e_2\rangle_{\mathfrak{g}}e_1=[e_2,[e_1,e_2]]=0,$$

  which imply that $e_2$ belongs to the kernel of the inner product.

- **Rank 1:** Condition 3.52 in particular yields

$$\lambda_1\langle e_3,e_1\rangle_{\mathfrak{g}}e_2+\lambda_2\langle e_3,e_2\rangle_{\mathfrak{g}}e_1=[e_3,[e_1,e_2]]=0,$$
$$\lambda_1\langle e_3,e_1\rangle_{\mathfrak{g}}e_3+\lambda_2\langle e_3,e_3\rangle_{\mathfrak{g}}e_1=[e_3,[e_1,e_3]]=0,$$

  which imply that $e_3$ belongs to the kernel of the inner product.

- **Rank 0:** Similar argument as the first item, gives that $e_2$ belongs to the kernel of the inner product.

As a conclusion from these three cases, we get that any inner product satisfying 3.52 is non-degenerated unless the rank is 3. □

As a direct consequence, we cannot obtain, by this geometric method, a Hamiltonian system for a Vlasov-Maxwell type of system of PDE´s with a Lie algebra of rank 0,1, or 2 as the underlying symmetry.

### 3.4.2 AN INTERESTING CONNECTION WITH THE KILLING FORM

The Killing form of $\mathfrak{so}(3,\mathbb{R})$ is given by $\kappa(x,y)=-2\langle x,y\rangle$, while for $\mathfrak{sl}(2,\mathbb{R})$ it is $\kappa(x,y)=2\langle x,y\rangle_L$. This suggests a remarkable relationship between the Killing form of a Lie algebra and the existence of a non-degenerate inner product that satisfies the condition given in equation 3.18. In this section, we will compute the Killing form of all 3-dimensional Lie algebras and explore such relationships.

**DEFINITION 3.3.** Let $\mathfrak{g}$ be a Lie algebra. The *Killing form* is the symmetric bilinear form defined by:

$$\kappa(x,y):=\operatorname{tr}\big(\operatorname{ad}(x)\circ\operatorname{ad}(y)\big) \tag{3.53}$$

The Killing form plays a fundamental role in the classification of Lie algebras. From a geometric perspective, when the Killing form is non-degenerate, it induces a (pseudo-)Riemannian

metric on the Lie algebra. This property is crucial for determining whether the Vlasov-Maxwell reduction procedure can be consistently applied within the Lie algebraic framework.

In particular, when the Killing form is non-degenerate, we observe a relationship between the expression

$$[\mathbf{u},[\mathbf{v},\mathbf{w}]]$$

for vectors $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3$, and a corresponding inner product structure.

The following table summarizes key results for the 3-dimensional Lie algebras discussed in Section 3.4.1. Specifically, we present:

1. The **matrix representation** $B$ of $\mathrm{ad}(x)\circ\mathrm{ad}(y)$,
2. The **Killing form** $\kappa(\mathbf{u},\mathbf{v})$, and
3. The expression $[\mathbf{u},[\mathbf{v},\mathbf{w}]]$

| Lie algebra | $B$ | $\kappa(\mathbf{u},\mathbf{v})$ | $[\mathbf{u},[\mathbf{v},\mathbf{w}]]$ |
|---|---|---|---|
| $L(3,0)$ | $[0]_{3\times3}$ | 0 | 0 |
| $\mathfrak{h}$ | $[0]_{3\times3}$ | 0 | 0 |
| $L(3,-1)$ | $\mathrm{diag}(0,1,0)$ | $u_2v_2$ | $u_2v_2(w_1,w_2,0)-u_2w_2(v_1,v_2,0)$ |
| $L(3,2,x)$ | $\mathrm{diag}(0,0,1+x^2)$ | $u_3v_3(1+x^2)$ | $u_3v_3(w_1,x^2w_2,w_3)-u_3w_3(v_1,x^2v_2,v_3)$ |
| $L(3,3)$ | $\mathrm{diag}(0,0,2)$ | $2u_3v_3$ | $u_3v_3\mathbf{w}-u_3w_3\mathbf{v}-2u_3(w_3v_2-v_3w_2,0,0)$ |
| $L(3,4,x)$ | $\mathrm{diag}(0,0,2(x^2-1))$ | $2(x^2-1)u_3v_3$ | $x^2(u_3v_3\mathbf{w}-u_3w_3\mathbf{v})+u_3v_3(w_1,-w_2,w_3)$ $-u_3w_3(v_1,-v_2,v_3)+2x(0,v_1w_3-v_3w_1,0)$ |
| $\mathfrak{sl}(2,\mathbb{R})$ | $\mathrm{diag}(2,2,-2)$ | $2\langle\mathbf{u},\mathbf{v}\rangle_{\mathfrak{sl}(2)}$ | $-\langle\mathbf{u},\mathbf{w}\rangle_L\mathbf{v}+\langle\mathbf{u},\mathbf{v}\rangle_L\mathbf{w}$ |
| $\mathfrak{so}(3,\mathbb{R})$ | $-2I$ | $-2\langle\mathbf{u},\mathbf{v}\rangle$ | $\langle\mathbf{u},\mathbf{w}\rangle\mathbf{v}-\langle\mathbf{u},\mathbf{v}\rangle\mathbf{w}$ |

The Killing forms of 3-dimensional Lie algebras were previously computed in [NP06], and our results are in full agreement with their findings.

As observed in the table, the Killing form of $L(3,0)$ and $\mathfrak{h}$ is identically zero, as is the double bracket identity. Consequently, no inner product can be associated with these algebras, and the result cannot be extended to them. For the remaining rank-1 Lie algebra, these quantities are nonzero; however, the Killing form remains degenerate, and the double bracket identity does not exhibit any clear relation to an inner product.

The cases $L(3,2,x)$, $L(3,3)$, and $L(3,4,x)$, which correspond to rank-2 Lie algebras, have a non-trivial kernel of the Killing forms. Moreover, their double bracket identities do not resemble the identity given in 3.52. Consequently, there is no apparent relationship between the bracket and an inner product, implying that the results cannot be extended to these algebras.

The rank-3 Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3,\mathbb{R})$ are the only cases where the Killing form is non-degenerate, thereby inducing an inner product on the Lie algebra. The original Vlasov-Maxwell system was formulated for $\mathfrak{so}(3,\mathbb{R})$, while our work extends this framework to $\mathfrak{sl}(2,\mathbb{R})$. Furthermore, our results demonstrate that this extension is not possible for any other 3-dimensional Lie algebra.

To conclude this section, we state the following theorem that summarizes the results of our work:

**THEOREM 3.14.** *In dimension 3, there are only 2 Lie algebras with a non-degenerate Killing form:* $\mathfrak{sl}(2,\mathbb{R})$ *and* $\mathfrak{so}(3,\mathbb{R})$. *Therefore, the Vlásov-Maxwell PDE's system can only be extended to these two Lie algebras.*

# CONCLUSIONS AND FUTURE WORK

A guiding purpose of this work was to develop the necessary concepts of symplectic and Poisson geometry in order to apply the Marsden-Weinstein-Meyer reduction theorem in a physical context, such as plasma physics.

In Chapter 1, we presented some basic preliminaries of plasma physics. Nevertheless, plasma physics is a field in its own right, with many topics that can be reviewed. The purpose of this summary was to provide context for the Vlásov-Maxwell system in a simplified manner, allowing the reader to become familiar with the underlying physical concepts.

The unscripted goal of Chapter 2 was to develop all the background theory needed to state and understand all the ingredients needed to find the Poisson bracket for the Vlásov-Maxwell system. We started with several examples of Hamiltonian systems, and after that, we defined symplectic and Poisson structures. We also treated Lie groups actions, momentum mapping, and finally some reduction theory for both symplectic and Poisson structures.

Once all the ingredients were defined in Chapter 3, we proceeded to explain, step-by-step, the procedure carried out by Marsden and Weinstein to establish the Hamiltonian structure of the Vlásov-Maxwell system. With the aim of extending this approach to a different three-dimensional Lie algebra, we studied a suitable representation of $\mathfrak{sl}(2,\mathbb{R})$ in this context. We found that the appropriate choice was the Lorentz space $\mathbb{L}^3$, defined as the vector space $\mathbb{R}^3$ equipped with a pseudo-Riemannian metric. Additionally, we provided all the detailed calculations required to extend the approach to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, obtaining analogous relations to those in the original case, along with a new bracket and a corresponding set of differential equations. Finally, seeking to generalize the methodology to arbitrary three-dimensional Lie algebras, we studied the classification of such algebras, and identified the required properties to extend the result. Our analysis revealed that the methodology cannot be extended beyond the cases of $\mathbb{R}^3$ and $\mathfrak{sl}(2,\mathbb{R})$.

Finally, some questions and possibilities for future research came up during the realization of this work:

(i) The Vlásov-Maxwell system is not the only model used in plasma physics. Geometric formulations have already been developed for other models, such as magnetohydrodynamics (MHD) and multi-species fluid electrodynamics [MWR+83], using the framework of semi-direct products. To study the geometric structures of these models and the possibility to extend the results, such as the extension we performed in this work, to the different

plasma physics models, is a question worth pursuing.

(ii) There are numerous computational methods used to solve the Vlásov-Maxwell system [GHS22, CEF15, HQS+15]. Would they work to solve the $\mathfrak{sl}(2,\mathbb{R})$ Vlásov-Maxwell system?

(iii) In this work, we studied the Hamiltonian structure of the Vlásov-Maxwell equations. However, there also a Lagrangian formulation of the system, whose geometric structure is worth exploring [CHHM98]. Moreover, it is natural to ask whether it is possible to do similar extensions to the Lagrangian formulation, as the one we did in this thesis.

(iv) The study of the conservative properties of Vlásov-Maxwell system, for instance: energy and momentum conservation and Casimir functions, as discussed in Section 3 of [GHS22]. An interesting question arises: what happens to these conserved properties when the system is extended to the $\mathfrak{sl}(2,\mathbb{R})$ setting?

# BIBLIOGRAPHY


[AM78] Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, MA, second edition, 1978. With the assistance of Tudor Raţiu and Richard Cushman. 6, 10, 22

[Arn78] V. I. Arnold. *Mathematical methods of classical mechanics. Translated by K. Vogtman and A. Weinstein*, volume 60 of *Grad. Texts Math.* Springer, Cham, 1978. 12, 17

[Ber67] Felix Alexandrovich Berezin. Some remarks about the associated envelope of a lie algebra. *Functional Analysis and Its Applications*, 1:91–102, 1967. 17

[Bia98] Luigi Bianchi. Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti. *Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Terza*, 11:267–352, 1898. 45

[Bow05] Adam Bowers. Classification of three-dimensional real lie algebras, 2005. Revised March 2010, coursework at the University of Connecticut under Maria Gordina. 45, 46

[BS03] Thomas J. M. Boyd and Jeffrey J. Sanderson. *The physics of plasmas*. Cambridge: Cambridge University Press, 2003. 1

[Bur22] Henrique Bursztyn. A brief introduction to Poisson geometry, 2022. https://w3.impa.br/~henrique/papers/Poisson22CRM.pdf. 17

[CdS01] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. 6, 22

[CEF15] Nicolas Crouseilles, Lukas Einkemmer, and Erwan Faou. Hamiltonian splitting for the vlasov–maxwell equations. *Journal of Computational Physics*, 283:224–240, 2015. 52

[CFMat21] Marius Crainic, Rui Loja Fernandes, and Ioan M˘ arcuţ. *Lectures on Poisson geometry*, volume 217 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, [2021] ©2021. 6, 43

[Che84] F F Chen. *Introduction to plasma physics and controlled fusion. Second edition. Volume 1: Plasma physics*. Plenum Press, New York, NY (United States), 12, 1984. 1

[CHHM98] Hernán Cendra, Darryl D Holm, Mark JW Hoyle, and Jerrold E Marsden. The maxwell-vlasov equations in euler–poincaré form. *Journal of Mathematical Physics*, 39(6):3138–3157, 1998. 52

[CMMR24] Ivan Contreras, Diego Martinez, Nicolas Martinez, and Diego Rodriguez. Symplectic geometry, poisson geometry, and beyond. Submitted for publication to *Boletín de Matemáticas de la Universidad Nacional de Colombia*, 2024. https://arxiv.org/abs/2411.12551. 17

[DZ05] Jean-Paul Dufour and Nguyen Tien Zung. *Poisson Structures and Their Normal Forms*, volume 242 of *Progress in Mathematics*. Birkhäuser, Basel, 2005. 23

[EM70] David Ebin and Jerrold Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Annals of Mathematics*, 92, 07 1970. 26

[EW06] K. Erdmann and Mark J. Wildon. *Introduction to Lie Algebras*. Springer Undergraduate Mathematics Series. Springer London, London, 1st ed. 2006. edition, 2006. 45, 46

[GHS22] Anjiao Gu, Yang He, and Yajuan Sun. Hamiltonian particle-in-cell methods for vlasov–poisson equations. *Journal of Computational Physics*, 467:111472, 2022. 52

[Got91] Mark J Gotay. A multisymplectic framework for classical field theory and the calculus of variations: I. covariant hamiltonian formalism. In *Mechanics, analysis and geometry: 200 years after Lagrange*, pages 203–235. Elsevier, 1991. 13

[GS80] Victor Guillemin and Shlomo Sternberg. The moment map and collective motion. *Ann. Phys.*, 127:220–253, 1980. 17

[Gün87] Christian Günther. The polysymplectic hamiltonian formalism in field theory and calculus of variations. i. the local case. *Journal of differential geometry*, 25(1):23–53, 1987. 13

[Hol11] Darryl D. Holm. *Geometric mechanics. Part I: Dynamics and symmetry*. London: Imperial College Press, 2nd ed. edition, 2011. 6

[HQS+15] Yang He, Hong Qin, Yajuan Sun, Jianyuan Xiao, Ruili Zhang, and Jian Liu. Hamiltonian time integrators for vlasov-maxwell equations. *Physics of Plasmas*, 22(12), 2015. 52

[KW09] Boris Khesin and Robert Wendt. *The Geometry of Infinite-Dimensional Groups*, volume 51 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, Heidelberg, 2009. 10

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Grad. Texts Math.* New York, NY: Springer, 2nd revised ed edition, 2013. 6

[Lie88] Sophus Lie. *Theorie der Transformationsgruppen Abschn. 1*. Teubner, 1888. 17

[Mey73] Kenneth R. Meyer. Symmetries and integrals in mechanics. In M.M. PEIXOTO, editor, *Dynamical Systems*, pages 259–272. Academic Press, 1973. 22

[Mor80] Philip J. Morrison. The maxwell-vlasov equations as a continuous hamiltonian system. *Physics Letters A*, 80(5-6):383–386, 1980. IV, 24

[MR94] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1994. A basic exposition of classical mechanical systems. 6, 10, 21

[MRW84] Jerrold E. Marsden, Tudor Ratiu, and Alan Weinstein. Semidirect products and reduction in mechanics. *Transactions of the American Mathematical Society*, 281(1):147–177, 1984. 1

[MT03] Jerrold E Marsden and Anthony Tromba. *Vector calculus*. Macmillan, 2003. 37

[MW74] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rept. Math. Phys.*, 5(1):121–130, 1974. IV, 22

[MW82] Jerrold E. Marsden and Alan Weinstein. The Hamiltonian structure of the Maxwell-Vlasov equations. *Phys. D*, 4(3):394–406, 1981/82. IV, 17, 24, 25, 36, 45

[MWR+83] Jerrold Marsden, Alan Weinstein, Tudor Ratiu, Rudolf Schmid, and R. Spencer. Hamiltonian systems with symmetry, coadjoint orbits and plasma physics. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, 117, 01 1983. IV, 6, 18, 25, 26, 36, 51

[N.17] McGravy N. General plasma physics lecture notes. https://nickmcgreivy.scholar.princeton.edu/sites/g/files/toruqf5041/files/documents/general-plasma-physics-notes-McGreivy_0.pdf, 2017. 1

[NP06] Maryna Nesterenko and Roman Popovych. Contractions of low-dimensional Lie algebras. *J. Math. Phys.*, 47(12):123515, 45, 2006. 49

[RS81] Tudor Ratiu and Rudolf Schmid. The differentiable structure of three remarkable diffeomorphism groups. *Mathematische Zeitschrift*, 177, 03 1981. 26