Zariski cancellation problem for skew PBW extensions

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Title in English

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Título en español

Problema de cancelación de Zariski para extenciones PBW torcidas.

Abstract: A special question for noncommutative algebras is Zariski cancellation problem. In this thesis we establish cancellation for some special classes of algebras such as skew PBW extensions, some Artin–Schelter regular algebras and universal enveloping algebras of dimension three. In addition, we provide general properties for cancellation and we present a noncommutative analogues of a cancellation theorem for algebras of Gelfand-Kirillov dimension one.

Resumen: Una pregunta especial para álgebras no conmutativas es el problema de cancelación de Zariski. En esta tesis, establecemos cancelación para algunas clases especiales de álgebras como extensiones PBW torcidas, algunas álgebras Artin–Schelter regulares y álgebras envolvente universal de dimensión tres. Adicionalmente, proveemos propiedades generales para cancelación y presentamos un análogo no conmutativo del teorema de cancelación para álgebras de dimensión Gelfand-Kirillov uno.

Keywords: Cancellation, retractable, detectable, Morita cancellation, locally nilpotent derivation, Makar-Limanov invariant, hopfian, discriminant, center, skew PBW extensions, AS-regular algebras.

Palabras clave: Cancelación, retractable, detectable, Morita cancelación, derivación localmente nilpotente, invariante de Makar-Limanov, hopfian, discriminante, centro, extensiones PBW torcidas, álgebras regulares de Artin–Schelter.

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Dedicated to

My parents.

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Introduction

Let k be a field. An algebra A is called *cancellative* if any algebra isomorphism $A[t] \cong B[t]$ of polynomial algebras, for some algebra B, implies that A is isomorphic to B. In the commutative case, the famous Zariski Cancellation Problem (abbreviated as ZCP) asks if the commutative polynomial ring $k[x_1, \ldots, x_n]$ is cancellative for all $n \geq 1$ [Kr, BZ1, Gu3]. It is well-known that $k[x_1]$ is cancellative by a result of Abhyankar-Eakin-Heinzer [AEH], while $k[x_1, x_2]$ is cancellative by Fujita [Fu] and Miyanishi-Sugie [MS] in characteristic zero, and by Russell [Ru] in positive characteristic. The ZCP for $n \geq 3$ has been open for many years. A major breakthrough in this research area is a remarkable result of Gupta in 2014 [Gu1, Gu2] which settled the ZCP negatively in positive characteristic for $n \geq 3$. The ZCP in characteristic zero remains open for $n \geq 3$. Some examples of non-cancellative algebras were given by Hochster [Ho], Danielewski [Da], Gupta [Gu1, Gu2] and Lu, Wu and Zhang [LuWZ].

A noncommutative version of the Zariski Cancellation Problem was investigated in as early as 1970s, see papers by Coleman-Enochs [CE] and Brewer-Rutter [BR], and was re-introduced by Zhang and Bell in 2017 in [BZ1]. In recent years, increased attention has been paid to the noncommutative analogue of the ZCP and invariants related to its study [BZ1, BZ2, BY, CPWZ1, CPWZ2, CYZ1, CYZ2, Gad, GKM, GWY, LY, LeWZ, LuWZ, LMZ, NTY, Ta1, Ta2, WZ]. Recently, ZCP was introduced for commutative Poisson algebras by Gaddis-Wang [GW]. In addition, two new problems has been introduced: The Morita cancellation by Lu, Wu and Zhang in [LuWZ] and the skew cancellation by Bell-Hamidizadeh-Huang and the named author in [BHHV].

An algebra A is called *Morita cancellative* if the statement that $M(A[t])$ is equivalent to $M(B[t])$ for another algebra B implies that $M(A)$ is equivalent to $M(B)$, where $M(A)$ denote the category of right A-modules. This Morita version of the cancellation property is one of the natural generalizations of the original Zariski cancellation property when we study noncommutative algebras. Another generalization involves the skew polynomial extensions. Let σ be an automorphism of the ring R and $\delta : R \to R$ a σ-derivation. We say that R is *skew cancellative* if $R[x; \sigma, \delta] \cong S[x; \sigma', \delta']$ implies $R \cong S$. The derivation type is a special case when σ is the identity, and it was considered in [AKP, Be].

This thesis had been motivated by the following question: What kind of algebras are cancellative? In particular, we are interested in the cancellation property for two groups of important noncommutative algebras: skew PBW extensions (Poincaré-Birkhoff-Witt) and Artin–Schelter regular algebras (abbreviated as AS-regular algebras). Morita cancellation and skew cancellation are considered too.

The skew PBW extensions were introduced by Lezama and Gallego in [GL1] and represent a generalization of PBW (Poincaré-Birkhoff-Witt) extensions defined by Bell and Goodearl [BG]. Many ring-theoretical and homological properties of skew PBW extensions have been studied in the last years (Hilbert's basis theorem, global dimension, Gelfand-Kirillov dimension, Krull dimension, Goldie dimension, prime ideals, Auslander regularity, etc., see [LR], [ACLOV], [ALR], [LV3]), as well as its Gröbner theory (see [GL1], [GL2]). Some prominent algebras that can be covered by the skew PBW extensions are: The algebra of q-differential operators, the algebra of shift operators, the algebra of linear partial q-differential operators, skew polynomial rings of derivation type and Ore extension of derivation type, the additive and multiplicative analogues of the Weyl algebra, the quantum algebra $U'(so(3,K))$, the 3-dimensional skew polynomial algebra, the dispin algebra, the Woronowicz algebra, the q-Heisenberg algebra, the Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2,K)$, algebras diffusion type, some quadratic algebras, among others.

Artin–Schelter regular algebras were defined by Michael Artin and William Schelter in [AS] and their classification in dimension three had been completed by Artin, Tate and Van den Bergh [ATV1, ATV2]. Also, general ring-theoretic properties such as Auslander regularity and Cohen-Macaulay property are well known [BRSSW, Le, LMZ, St]. Cancellation holds for several classes of these algebras. Remarkable results of Bell and Zhang pointed out that algebras with trivial center and algebras 2-dimensional over a base field algebraically closed are cancellative [BZ1, Proposition 1.3, Theorem 0.5]. Also, Lü, Mao and Zhang showed that several classes of AS-regular algebras of global dimension three are cancellative ([LMZ, Corollary 0.9], Example 1.4.7).

In this thesis we follow the ideas of [BZ1, LeWZ, LuWZ] to deal with the ZCP. In particular, we had used the center, the Makar-Limanov invariant and some modifications, and the $\mathcal{P}$-discriminant methods. Besides, we have introduced a noncommutative version of the slice theorem. However, determine new invariants or properties that controls cancellation is an interesting problem.

The thesis is divided into five chapters. In the first chapter, we recall some definitions and fundamental properties about cancellation and Morita cancellation that are necessary in the subsequent sections. Also, we mention some basic properties and key examples of skew PBW extensions whose centers will be studied further. Artin–Schelter regular algebras and some properties are considered in the last section.

The center of an algebra plays a crucial role in whether the cancellation property holds for that algebra. Therefore, in the second chapter we compute the center and some central elements of many noncommutative algebras that can be interpreted as

skew PBW extensions. The question about the center of the total ring of fractions is considered.

We show that, under some natural assumptions on the parameters that define the extension, either the center is trivial, or, it is of polynomial type. In the first section we describe the center of some general subclases of skew PBW extensions, grouped according to some conditions imposed to the parameters that define the extensions. The technique that we use in this part for computing the center is very simple: we use the parameters and relations that define the extension, and, taking an element f of the center of the algebra we deduce some formulae from the basic equation $x_i f = f x_i$, where x_i is any of the variables that define the extension. Thus, One of the most important result of this chapter is Theorem 2.1.5 that establishes trivial center for a wide class of skew PBW extensions, therefore cancellation. Theorem 2.1.6 determines center for operator algebras. In the second section, we treat others algebras that can not be covered by the theorems of the first section. Corollary 2.2.14 summarize the list of cancellative skew PBW extensions.

Finally, in the third section we study the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$, with $Q_r(A)$ the right total ring of fractions of A. This isomorphism became important after the formulation of the Gelfand-Kirillov conjecture: *Let $\mathcal{G}$ be an algebraic Lie algebra of finite dimension over a field k, with $char(k) = 0$. Then, there exist integers $n, l \geq 1$ such that*

$$Q(\mathcal{U}(\mathcal{G})) \cong Q(A_n(k[s_1, \ldots, s_l])),$$

where $\mathcal{U}(\mathcal{G})$ is the enveloping algebra of $\mathcal{G}$ and $A_n(k[s_1, \ldots, s_l])$ is the general Weyl algebra over k. In the investigation of this famous conjecture the isomorphism between the center of the total ring of fractions and the total ring of fractions of the center occupies a special key role. In Theorem 2.3.3 we prove that if A is finitely generated and $\mathrm{GKdim}(A) < \mathrm{GKdim}(Z(A)) + 1$, then $Z(Q_r(A)) \cong Q(Z(A))$. This result can be interpreted as a way of computing the center of $Q_r(A)$. Results of this chapter will appear in [LV1, LV2].

In Chapter 3, we focus on the study of cancellation for algebras with Gelfand-Kirillov dimension one when the characteristic of the base field is zero or positive. An important part of this work is motivated by the question of Lezama, Wang, and Zhang [LeWZ, Question 0.5]: Is every affine prime k-algebra of Gelfand-Kirillov dimension one cancellative? Then we answer this question in the case when the base field has characteristic zero in the domain case. We present a noncommutative analogue of the Abhyankar, Eakin and Heinzer theorem; if A is a finitely generated algebra that is an integral domain of Krull dimension one, then A is strongly cancellative. However we do not know whether this conclusion holds for not-necessarily commutative domains. Thus, Theorem 3.2.4 establishes that affine domains of Gelfand-Kirillov dimension one are cancellative. Since Krull dimension and Gelfand-Kirillov dimension coincide for finitely generated commutative k-algebras, then this result specializes to the classical cancellation results of Abhyankar, Eakin and Heinzer in the characteristic zero base fields when one takes A to be commutative. In addition, Theorem 3.2.6 provides examples that show affine

domains of Gelfand-Kirillov dimension one need not be cancellative when the base field has positive characteristic, giving a counterexample to a conjecture 4.5.5 of Tang, the named author, and Zhang. We noted that well-known non-cancellative algebras examples have dimension at least two ([Ho, Da, Gu1, Gu2]), and our example has dimension one.

We have organized this chapter as follows, in the first section we introduce a new invariant when we work with weak locally nilpotent Hasse-Schmidt derivations. And, we generalize Theorem 1.1.8 and Lemma 1.1.13 in Proposition 3.1.4 when one works with affine prime left Goldie k-algebras of finite Gelfand-Kirillov dimension. Moreover, in Proposition 3.1.6 we prove a result that is related to a conjecture of Makar-Limanov. In the second section, in order to prove the main result of the chapter, Theorem 3.2.4, we introduce a noncommutative version of slice theorem. That is, suppose that the characteristic of k is zero. If δ is a locally nilpotent derivation, and there exist exists $x \in Z(A)$ such that $\delta(x) = 1$, then the sum $\sum_{i\geq 0} A_0 x^i$ is direct and $A = A_0[x]$ where A_0 is the kernel of δ. In commutative case, the slice theorem has been used to decide cancellation [Fr, vdE]. Finally, in the third section we prove a result in a different direction; namely, *skew cancellativity*: that is, if $R[x;\sigma,\delta] \cong S[x;\sigma',\delta']$ when do we necessarily have $R \cong S$? In Theorem 3.3.7, we show that this holds in the two cases: skew polynomial extensions of automorphisms type and derivations type when the coefficient ring R is an affine commutative domain of Krull dimension one. Results of this chapter will appear in [BHHV].

In Chapter 4 our main goal is to study the cancellation for noncommutative noetherian connected graded Artin–Schelter regular algebras of global dimension three. Our main result settles that such algebras are cancellative when they are generated in degree 1 and do not satisfy a polynomial identity (PI for short). This result covers [LMZ, Corollary 0.9] where authors showed that several classes of AS-regular algebras of global dimension three are cancellative. However, as in the commutative case, the PI case remains open. For AS-regular of higher global dimension Theorem 4.1.7 establishes cancellation.

This chapter is organized as follows. In the first section, we show that under certain conditions a k-algebra A with center $Z(A)$ of Gelfand-Kirillov dimension at most one is either k or $k[t]$. In the former case, the algebra is cancellative by Proposition 1.1.4, and if $Z(A) = k[t]$ then we can consider the $\mathcal{P}$-discriminant method. Consequently, we can prove the main results of this section: Lemma 4.1.6 and Theorem 4.1.7. In the next section, we complete some details for proving Theorem 4.2.5. Next, we collect some results that involve cancellation properties. As an application, in Section 3 we establish cancellation for fixed subrings of generic 3-dimensional Sklyanin algebra. In Section 4, Example 4.4.2 establishes cancellation for universal enveloping algebras of 3-dimensional non-abelian Lie algebras. And, in Section 5, we present a result similar to Theorem 4.1.7 for graded isolated singularities which have infinite global dimension. Results of this chapter will appear in [TVZ].

Finally, in Chapter 5, we collect some important issues about cancellation and related subjets. We follow the line of work presented in the main papers [BZ1, LeWZ, LuWZ] and recent advances in [TVZ, BHHV].

In this thesis, k denote a field, non necessarily algebraically closed. All rings are with unity and A, B and R denote arbitrary rings, all modules are right modules and $M(A)$ denote the category of right A-modules. Vector spaces are k-vector spaces, the tensor product $\otimes$ is over k, $\otimes_k$, and algebra always means a k-algebra. We say that A is an affine algebra over k, or simply an affine algebra when the base field is undertood, if A is finitely generated k-algebra. $Z(A)$ or simply Z denote the center of the algebra.

CHAPTER 1

Preliminaries

In this first chapter, we recall some definitions and fundamental properties about cancellation and Morita cancellation from [BZ1, LeWZ, LuWZ] that are necessary in the subsequent sections. Also, we mention skew PBW extensions or σ-PBW extensions introduced by Lezama and Gallego in [GL1]. Artin–Schelter regular algebras are considered.

1.1 Cancellation property

Recall that k denote a field and A, B are k-algebras, that is a ring A with identity which has a copy of k as a subring of its center. Thus, A is also a k-vector space such the multiplication satisfies $(\lambda \cdot a)b = \lambda \cdot (ab) = a(\lambda \cdot b)$ for all $\lambda \in k$, $a, b \in A$.

Definition 1.1.1. [BZ1, Definition 1.1] Let A be an algebra.

(1) We call A *cancellative* if any algebra isomorphism $A[t] \cong B[s]$ for some algebra B implies that $A \cong B$.

(2) We call A *strongly cancellative* if, for each $n \geq 1$, any algebra isomorphism

$$A[t_1, \cdots, t_n] \cong B[s_1, \cdots, s_n]$$

for some algebra B implies that $A \cong B$.

(3) We call A *universally cancellative* if, for any finitely generated commutative domain R such that $R/I \cong k$ for some ideal I of R, and any k-algebra B

$$A \otimes R \cong B \otimes R \Rightarrow A \cong B.$$

It is clear that

Universally cancellative ⇒ strongly cancellative ⇒ cancellative.

But, it is not clear whether any two of them are equivalent.

Example 1.1.2. $k[x_1]$ is cancellative by Abhyankar-Eakin-Heinzer [AEH], (We will discuss this result later), $k[x_1, x_2]$ is cancellative by Fujita [Fu], Miyanishi-Sugie [MS] in characteristic zero and by Russell in positive characteristic [Ru].

Example 1.1.3.
1. Let k be the field of real numbers $\mathbb{R}$. Hochster showed that $k[x, y, z]/(x^2 + y^2 + z^2 - 1)$ is not cancellative [Ho].
2. The Danielewski's surface [Da]. Let $n \geq 1$ and let B_n be the coordinate ring of the surface $x^n y = z^2 - 1$ over $k = \mathbb{C}$. Then $B_i \ncong B_j$ if $i \neq j$, but $B_i[t] \cong B_j[s]$ for all $i, j \geq 1$. Therefore, all the $B_n's$ are not cancellative.
3. Suppose $\operatorname{char} k > 0$. Gupta showed that $k[x_1, \ldots, x_n]$ is not cancellative for every $n \geq 3$ [Gu1],[Gu2]. However the cancellation in characteristic zero remains open for $n \geq 3$.

We noted that these non-cancellative examples all have dimension at least two. An early result for noncommutative algebras is the following proposition.

Proposition 1.1.4. *[BZ1, Proposition 1.3] Let k be a field and A be an algebra with center being k. Then A is universally cancellative.*

Proof. For any algebra A, let $Z(A)$ denote the center of A. Let R be an affine commutative domain such that $R/I \cong k$ for some ideal $I \subset R$ and suppose that $\phi : A \otimes R \to B \otimes R$ is an algebra isomorphism for some algebra B. Since $Z(A) = k$, we have $Z(A \otimes R) = R$. Since $Z(B \otimes R) = Z(B) \otimes R$ and since ϕ induces an isomorphism between the centers, we have

$$R \cong Z(B) \otimes R.$$

Consequently, $Z(B)$ is a commutative domain. By considering the Gelfand-Kirillov dimension of both sides of the last expression, one sees that $\operatorname{GKdim}(Z(B)) = 0$, when regarded as a k-algebra. Hence $Z(B)$ is a field. Since there is an ideal I such that $R/I \cong k$, $Z(B) = k$. Consequently, $Z(B \otimes R) = R$. Now the induced map ϕ is an isomorphism between $Z(A \otimes R) = R$ to $Z(B \otimes R) = R$, so we have $R/\phi(I) = k$. Finally, ϕ induces an automorphism from $A \cong A \otimes R/(I) \cong B \otimes R/(\phi(I)) \cong B$. □

1.1.1 Makar-Limanov invariants

The Makar-Limanov invariant was introduced by Makar-Limanov [Ma1], who called the invariant AK, and they are a powerful tool when dealing with the cancellation problem. The effective computation of these invariants is a difficult task. However, it is possible under certain conditions. In particular, Corollary 1.1.11 is a nice result for classification of cancellative algebras.

Definition 1.1.5. Let k be a field and let A be a k-algebra.

1. We let $\mathrm{Der}(A)$ denote the collection of k-linear derivations of A.
2. We let $\mathrm{LND}(A) = \{\delta \in \mathrm{Der}(A) \mid \delta \text{ is locally nilpotent}\}$, $\delta \in \mathrm{Der}(A)$ is locally nilpotent if given $a \in A$ there exists $n \geq 1$ such that $\delta^n(a) = 0$.
3. A *Hasse-Schmidt derivation* (or *higher derivation*) of A is a sequence of k-linear maps $\partial := \{\partial_i\}_{i\geq 0}$ such that:
$$\partial_0 = \mathrm{id}_A, \text{ and } \partial_n(ab) = \sum_{i=0}^{n} \partial_i(a)\partial_{n-i}(b)$$
for $a, b \in A$ and $n \geq 0$.
4. A Hasse-Schmidt derivation $\partial = (\partial_n)$ is called *locally nilpotent* if
   (a) for each $a \in A$ there exists an integer $N = N(a) \geq 0$ such that $\partial_n(a) = 0$ for all $n \geq N$, and
   (b) the k-algebra homomorphism $A[t] \to A[t]$ given by $t \mapsto t$ and $a \mapsto \sum_{i\geq 0} \partial_i(a)t^i$ is a k-algebra isomorphism.
5. A Hasse-Schmidt derivation $\partial = (\partial_n)$ is called *iterative* if $\partial_i \circ \partial_j = \binom{i+j}{i}\partial_{i+j}$ for all $i, j \geq 0$. The collection of Hasse-Schmidt derivations of an algebra A is denoted $\mathrm{Der}^H(A)$ and the collection of iterative Hasse-Schmidt derivations is denoted $\mathrm{Der}^I(A)$. The collection of locally nilpotent Hasse-Schmidt derivations (resp. iterative Hasse-Schmidt derivations) of A is denoted $\mathrm{LND}^H(A)$ (resp. $\mathrm{LND}^I(A)$).
6. Given $\partial := (\partial_n) \in \mathrm{Der}^H(A)$, the kernel of ∂ is defined to be
$$\ker(\partial) = \bigcap_{i\geq 1} \ker(\partial_i).$$
7. The *Makar-Limanov* invariant* of A is defined to be
$$\mathrm{ML}^*(A) = \bigcap_{\delta \in \mathrm{LND}^*(A)} \ker(\delta).$$
8. The *Makar-Limanov* center* of A is defined to be
$$\mathrm{ML}^*_Z(A) = \mathrm{ML}^*(A) \cap Z(A).$$
9. We say that A is LND^*-*rigid* (respectively *strongly* LND^*-*rigid*) if $\mathrm{ML}^*(A) = A$ (respectively $\mathrm{ML}^*(A[t_1, \ldots, t_d]) = A$, for $d \geq 1$).
10. We say that A is LND^*_Z-*rigid* (respectively *strongly* LND^*_Z-*rigid*) if $\mathrm{ML}^*_Z(A[t])$ is equal to $Z(A)$ (respectively $\mathrm{ML}^*_Z(A[t_1, \ldots, t_d]) = Z(A)$, for $d \geq 1$).

In items (7)–(10), $*$ is either blank, I, H.

Remark 1.1.6. Let k be a field and let A be a k-algebra. We recall some basic facts about derivations and Hasse-Schmidt derivations.

1. If $\partial := (\partial_n)$ is a locally nilpotent Hasse-Schmidt derivation of A then by definition the map $G_{\partial,t} : A[t] \to A[t]$ defined by

$$a \mapsto \sum_{i=0}^{\infty} \partial_i(a)t^i, \text{ for all } a \in A, t \mapsto t \tag{1.1}$$

   extends to a k-algebra automorphism of $A[t]$.

2. Conversely, if one has a k-algebra automorphism (resp. endomorphism) $G : A[t] \to A[t]$ such that $G(t) = t$ and $G(a) - a \in tA[t]$ for $a \in A$, then for $a \in A$ we have

$$G(a) = \sum_{i=0}^{\infty} \partial_i(a)t^i,$$

   and (∂_n) is a locally nilpotent Hasse-Schmidt derivation of A (see [BZ1, Lemma 2.2 (3)]).

3. If the characteristic of k is zero and $\delta : A \to A$ is a k-linear derivation, then the only iterative Hasse-Schmidt derivation (∂_n) of A with $\partial_1 = \delta$ is given by

$$\partial_n = \frac{\delta^n}{n!} \tag{1.2}$$

   for $n \geq 0$. This iterative Hasse-Schmidt derivation is called the *canonical Hasse-Schmidt derivation* associated to δ. If, moreover, δ is locally nilpotent, then by [BZ1, Lemma 2.2(2)], the map $G_{\partial,t}$ defined in item (1.1) is an automorphism and (∂_n) is a locally nilpotent iterative Hasse-Schmidt derivation, and conversely if (∂_n) is locally nilpotent then so is δ. Thus locally nilpotent iterative Hasse-Schmidt derivations correspond naturally to locally nilpotent derivations in the characteristic zero case and so $\mathrm{ML}^I(A) = \mathrm{ML}(A)$ for algebras with characteristic zero base field.

4. [MS, §1.1] If the characteristic of k is a positive integer p, then for an iterative derivation $\partial = \{\partial_i\}_{i\geq 0}$, ∂_i can be explicitly described as

$$\partial_i = \frac{(\partial_1)^{i_0} (\partial_p)^{i_1} \dots (\partial_{p^r})^{i_r}}{(i_0)! (i_1)! \dots (i_r)!},$$

   where $i = i_0 + i_1 p + \dots + i_r p^r$ is the base-p expansion of i. In this case, an iterative Hasse-Schmidt derivation ∂ is completely determined by $\partial_1, \partial_p, \partial_{p^2}, \dots$.

5. Let T be the polynomial ring $A[t_1, \dots, t_d]$ over a k-algebra A. We fix an integer $1 \leq i \leq d$. For each $n \geq 0$, we can define a divided power A-linear

differential operator Δ_i^n as follows:

$$\Delta_i^n : t_1^{m_1} \cdots t_d^{m_d} \longrightarrow \begin{cases} \binom{m_i}{n} t_1^{m_1} \cdots t_i^{m_i - n} \cdots t_d^{m_d} & \text{if } m_i \geq n \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

where $\binom{m_i}{n}$ is defined in $\mathbb{Z}$ or in $\mathbb{Z}/(p)$. Then $\{\Delta_i^n\}_{n=0}^{\infty}$ is a locally nilpotent iterative Hasse-Schmidt derivation of T. We can also extend an element $(\partial_n) \in \mathrm{LND}^H(A)$ to an element of $\mathrm{LND}^H(T)$ by declaring that $t_1, \ldots, t_d$ are in the kernel of (∂_n); moreover, the extension is iterative if the original Hasse-Schmidt derivation is iterative, and it is in $\mathrm{LND}^H(T)$ if the original locally nilpotent Hasse-Schmidt derivation is in $\mathrm{LND}^H(A)$. Combining this observation along with data from the maps Δ_i^n, we see

$$\mathrm{ML}^*(A[t_1, \ldots, t_d]) \subseteq \mathrm{ML}^*(A),$$

where $*$ is either I, H.

Recall that if A is a k-algebra, then the *Gelfand-Kirillov dimension* of A is defined by

$$\mathrm{GKdim}(A) := \sup_V \varinjlim_{n \to \infty} \log_n \dim_K V^n, \tag{1.4}$$

where V ranges over all frames of A and $V^n := {}_k\langle v_1 \cdots v_n | v_i \in V \rangle$ (a *frame* of A is a finite dimensional k-subspace of A such that $1 \in V$; since A is a k-algebra, then $k \hookrightarrow A$, and hence, k is a frame of A of dimension 1). For more information about Gelfand-Kirillov dimension, we refer the reader to the book of Krause and Lenagan [KL].

The following results show the relationship between LND-rigidity and cancellation.

Lemma 1.1.7. *[BZ1, Lemma 3.2] Let $Y := \bigoplus_{i=0}^{\infty} Y_i$ be an $\mathbb{N}$-graded domain. If Z is a subalgebra of Y containing Y_0 such that $\mathrm{GKdim}(Z) = \mathrm{GKdim}(Y_0) < \infty$, then $Z = Y_0$.*

Theorem 1.1.8. *Suppose A is an affine domain of finite Gelfand-Kirillov dimension. Let $*$ be either blank, H, or I. When $*$ is blank we further assume k has characteristic zero.*

1. *If A is strongly LND^*-rigid, then A is strongly cancellative.*
2. *If $\mathrm{ML}^*(A[t]) = A$, then A is cancellative.*

Proof. Let $\phi : A[t_1, \ldots, t_n] \to B[t_1, \ldots, t_d]$ be an isomorphism for some algebra B. By [KL, Proposition 3.11] $\mathrm{GKdim}\, B = \mathrm{GKdim}\, A < \infty$. For each i, let $\partial_i := \frac{\partial}{\partial t_i}$ when $*$ is blank and $\partial_i := \{\Delta_i^n\}_{n=0}^{\infty}$ as in Remark 1.1.6(5) when $*$ is either H or I. We have that $\mathrm{ML}^*(B[t_1, \ldots, t_d])$ is contained in B since ∂_i, for $i = 1, \cdots, d$, are locally nilpotent (higher) derivations of $B[t_1, \ldots, t_d]$ and the intersection of the kernels of these maps is exactly B. On the other hand, we have that $\mathrm{ML}^*(A[t_1, \ldots, t_d]) = A$ by hypothesis.

If ∂ is a locally nilpotent derivation of $B[t_1, \ldots, t_d]$ then $\phi^{-1} \circ \partial \circ \phi$ is a locally nilpotent derivation of $A[t_1, \ldots, t_d]$. Similarly if ∂' is a locally nilpotent derivation of $A[t_1, \ldots, t_d]$ then $\phi \circ \partial' \circ \phi^{-1}$ is a locally nilpotent derivation of $B[t_1, \ldots, t_d]$. Similarly, the higher derivations of $A[t_1, \ldots, t_d]$ and $B[t_1, \ldots, t_d]$ correspond. Thus ϕ induces an algebra isomorphism

$$\mathrm{ML}^*(A[t_1, \ldots, t_d]) \cong \mathrm{ML}^*(B[t_1, \ldots, t_d]).$$

In particular ϕ maps A into B. Let $Y = A[t_1, \ldots, t_d]$ with $\deg t_i = 1$ and $Y_0 = A$ and $Z = \phi^{-1}(B)$. Then Lemma 1.1.7 implies that $\phi^{-1}(B) = A$. So A and B are isomorphic. The result follows. □

The previous Theorem will be used in Chapter 3, where we generalize it for affine prime left Goldie k-algebras.

Lemma 1.1.9. *[BZ1, Lemma 3.5] Let A be a finitely generated Ore domain over k that contains $\mathbb{Z}$. If A is* LND*-rigid, then* $\mathrm{ML}(A[t]) = A$.

Proof. Let $C = \mathrm{ML}(A[x])$. Note that $C \subseteq A$ since differentiation with respect to x gives a locally nilpotent derivation of $A[x]$ and the kernel of this map is exactly A. It suffices to show that $C \supseteq A$. Suppose that there is a locally nilpotent derivation δ of $A[x]$ that does not send A to zero. Suppose $a_1, \ldots, a_s$ generate A as a k-algebra. Then $\delta(A) \subseteq A\delta(a_1)A + \cdots + A\delta(a_s)A$ and so there exists some smallest $m \geq 0$ such that $\delta(A) \subseteq A + Ax + \cdots + Ax^m$. If $m = 0$, then $\delta(A) \subseteq A$. Since A is LND-rigid, $\delta(A) = 0$. This yields a contradiction and therefore $m \geq 1$. We write

$$\delta(a) = \mu(a)x^m + \text{ lower degree terms}$$

for some derivation μ of A. We now consider the following three cases.

Case I: $\delta(x) \in A + Ax + \cdots + Ax^m$. In this case we have $\delta(x^i) \subseteq \sum_{n=0}^{i+m-1} Ax^n$ and $\delta(Ax^i) \subseteq \sum_{n=0}^{i+m} Ax^n$ for all i. Thus for every $a \in A$ we have

$$\delta^2(a) = \mu^2(a)x^{2m} + \text{ lower degree terms.}$$

More generally, we see that

$$\delta^j(a) = \mu^j(a)x^{mj} + \text{ lower degree terms.}$$

Thus μ is a locally nilpotent derivation and so $\mu(A) = 0$, contradicting the minimality of m. Thus $\delta(A) = 0$ in this case.

Case II: $\delta(x) = bx^{m+1}+$ lower degree terms for some $b \neq 0$ in A.

Applying δ to the equation $[x, a] = 0$, one sees that b commutes with every a in A, and so b is in the center of A. Now we define a new derivation δ' to $A[x]$ by declaring that $\delta'(a) = \mu(a)x^m$ for $a \in A$ and $\delta'(x) = bx^{m+1}$. Then we see that δ' sends Ax^i to Ax^{i+m} for every $i \geq 0$. We can view δ' as an associated graded derivation of δ. Since δ is locally nilpotent, δ' is a locally nilpotent derivation

of $A[x]$ [CPWZ2, Lemma 4.11]. Applying [BZ1, Lemma 3.4] to the algebra $A[x]$, $A[x]$ embeds in $E[y;\delta_0]$ where δ_0 is a derivation of E. Moreover, δ' extends to a locally nilpotent derivation of $E[y;\delta_0]$ by declaring that $\delta'(E) = 0$ and $\delta'(y) = 1$. Under this embedding $x = p(y)$ for some nonzero polynomial p. Let d denote the degree of this polynomial. Then bx^{m+1} gets sent to $q(y)p(y)^{m+1}$ for some nonzero polynomial $q(y)$. But since $\delta'(x)$ is nonzero, it has degree exactly $d-1$ and so we have $(m+1)d + \deg q(y) = d-1$, which is impossible.

Case III: $\delta(x) = bx^i +$ lower degree terms for some $b \neq 0$ in A and some $i > m+1$. In this case we see that, for each $n \geq 2$,

$$\delta^n(x) = \left\{ \prod_{s=1}^{n-1} ((i-1)s+1) \right\} b^n x^{(i-1)n+1} + \text{ lower degree terms,}$$

so δ cannot be locally nilpotent, which contradicts the hypothesis.

Combining these cases, we see that $\delta(A) = 0$. The result follows. □

Theorem 1.1.10. *[BZ1, Theorem 3.6] Let A be a finitely generated domain containing $\mathbb{Z}$ and suppose that A has finite* GKdim*. If A is* LND*-rigid, then A is cancellative.*

Proof. Since A is a domain of finite GKdim, it is an Ore domain. By Lemma 1.1.9, $\mathrm{ML}(A[x]) = A$. The assertion follows from Theorem 1.1.8. □

Corollary 1.1.11. *[BZ1, Corollary 3.7] Let A be a domain of* GKdim *two over an algebraically closed field k of characteristic zero.*

1. *If A is PI and not commutative, then A is* LND*-rigid. As a consequence, if we assume in addition that A is finitely generated over k, then A is cancellative.*
2. *If A is not PI, then A is universally cancellative.*

Quantum plane, Jordan plane and the unique universal enveloping algebra of 2-dimensional non-abelian Lie algebra are remarkable cancellative algebras by the previous Corollary.

1.1.2 Retractability and detectability

Retractable property was studied in a paper of Abhyankar-Eakin-Heinzer [AEH], it was called strongly invariant property. But in [LeWZ] it was changed by strongly retractable. Also, we mention Z-retractability and detectability. Hopfian algebras are considered.

Definition 1.1.12. [LeWZ, Definition 2.1, Definition 2.5]

(1) We call A *Z-retractable*, if for any algebra B, any algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $\phi(Z(A)) = Z(B)$. If further $\phi(A) = B$, we just say A is *retractable*.

(2) We call A *strongly Z-retractable*, if for any algebra B and integer $n \geq 1$, any algebra isomorphism $\phi : A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$ implies that $\phi(Z(A)) = Z(B)$. If further $\phi(A) = B$, we just say A is *strongly retractable*.

Every strongly retractable algebra is retractable, and hence cancellative. And (strongly) retractable algebras are (strongly) Z-rectractable. Since LND-rigidity control cancellation, they control retractable property. So with the hypothesis of Theorem 1.1.8, the algebra is retractable. However we copy the following lemma for Z-retractability.

Lemma 1.1.13. *[LeWZ, Lemma 2.6] Let A be an algebra such that the center $Z(A)$ is an affine domain.*

1. *Suppose $\mathrm{ML}_Z(A[t]) = Z(A)$ or $\mathrm{ML}^H_Z(A[t]) = Z(A)$. Then A is Z-retractable.*
2. *Suppose that A is strongly LND_Z-rigid (or strongly LND^H_Z-rigid). Then A is strongly Z-retractable.*

If B is a subring of C and $f_1, \ldots, f_m$ are elements of C, then the subring generated by B and the set $\{f_1, \ldots, f_m\}$ is denoted by $B\{f_1, \ldots, f_m\}$.

Definition 1.1.14. Let A be an algebra.

1. We call A *detectable* if, for any algebra B, an algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $B[s] = B\{\phi(t)\}$, or equivalently, $s \in B\{\phi(t)\}$.
2. We call A *strongly detectable* if, for any algebra B and any integer $n \geq 1$, an algebra isomorphism $\phi : A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$ implies that $B[s_1, \ldots, s_n] = B\{\phi(t_1), \ldots, \phi(t_n)\}$, or equivalently, for each $i = 1, \cdots, n$, $s_i \in B\{\phi(t_1), \ldots, \phi(t_n)\}$.

In the above definition, we do not assume that $\phi(t) = s$. Every strongly detectable algebra is detectable. The polynomial ring $k[x]$ is cancellative, but not detectable.

Lemma 1.1.15. *[LeWZ, Lemma 3.2] If A is Z-retractable (resp. strongly Z-retractable), then it is detectable (resp. strongly detectable).*

Definition 1.1.16. Let A be a k-algebra.

1. We say A is *Hopfian* if every k-algebra epimorphism from A to itself is an automorphism.
2. We say A is *strongly Hopfian* if $A[t_1, \ldots, t_n]$ is Hopfian for every $n \geq 0$.

Luckily, there exists enough algebras with this property.

Lemma 1.1.17. *The following algebras are strongly Hopfian.*

1. *Left or right noetherian algebras.*
2. *Finitely generated locally finite $\mathbb{N}$-graded k-algebras.*
3. *Prime affine k-algebras satisfying a polynomial identity.*
4. *Prime Goldie algebras of finite Gelfand-Kirillov dimension.*

Proof. (1–3) corresponding to [LeWZ, Lemma 3.5] and (4) is a consequence of [KL, Proposition 3.15]. □

Detectability implies cancellation under some conditions.

Lemma 1.1.18. *[LeWZ, Lemma 3.6] Suppose A is strongly Hopfian.*

1. *If A is detectable, then A is cancellative.*
2. *If A is strongly detectable, then A is strongly cancellative.*

Theorem 1.1.19. *[LeWZ, Theorem 4.1] If A is left (or right) artinian, then A is strongly detectable. As a consequence, A is strongly cancellative.*

With the previous definitions we can enunciate the theorem of Abhyankar, Eakin and Heinzer. We will give a noncommutative version of this Theorem in Chapter 3.

Theorem 1.1.20. *[AEH, Theorem 3.3, Corollary 3.4] Let A be an affine domain of GKdimension at most one.*

(1) If $A = k$, then it is trivially strongly retractable and strongly cancellative.

(2) If $A = k[t]$, then it is strongly cancellative.

(3) If $A \ncong k[t]$, then it is strongly retractable, and consequently, strongly cancellative.

1.1.3 Slice theorem

The slice theorem is a powerful tool when working on the Zariski cancellation problem. Roughly speaking, it says that if one has a locally nilpotent derivation δ of a commutative algebra R of characteristic zero and a regular element $r \in R$ such that $\delta(r) = 1$, then $R \cong S[t]$ where S is the kernel of δ. The theorem has been used in commutative case to decide cancellation [Fr, vdE].

Definition 1.1.21. Let R be an k-algebra and $\delta \in \mathrm{Der}(R)$. Any element $r \in R$ with $\delta(r) = 1$ is called a slice for δ.

Theorem 1.1.22. *[Fr, Corollary 1.22] Suppose $\delta \in \mathrm{LND}(R)$ admits a slice $r \in R$, and let $A = \ker \delta$. Then:*

1. *$R = A[r]$ and $\delta = \frac{d}{dr}$.*
2. *If R is affine, then A is affine.*

A close problem to the cancellation in the commutative case is the kernel conjecture:

2-nd Kernel problem: Let δ be a locally nilpotent derivation on $\mathbb{C}[x_1, \ldots, x_n]$ having a slice. Does it follow that $\ker(\delta) = \mathbb{C}[F_1, \ldots, F_{n-1}]$ for some $F_i \in \mathbb{C}[x_1, \ldots, x_n]$ algebraically independent over $\mathbb{C}$?

Proposition 1.1.23. *[vdE, Proposition 3.8] The cancellation problem is equivalent to the 2-nd Kernel problem.*

Proof. Let δ be a locally nilpotent derivation on $\mathbb{C}[x_1, \ldots, x_n]$ with a slice s. Then by slice theorem $\mathbb{C}[x_1, \ldots, x_n] = \ker(\delta)[s]$. So, if the cancellation problem is true, then $\ker(\delta) = \mathbb{C}[F_1, \ldots, F_{n-1}]$ for some $F_i \in \mathbb{C}[x_1, \ldots, x_n]$ algebraically independent over $\mathbb{C}$.

Conversely, let $\mathbb{C}[x_1, \ldots, x_n] = A[T]$, where A is a $\mathbb{C}$-algebra. Then the derivation $\frac{d}{dT}$ is locally nilpotent on $A[T]$ and has a slice, T. Furthermore $\ker(\frac{d}{dT}) = A$. So if the 2-nd Kernel problem is true, then $A = \mathbb{C}[F_1, \ldots, F_{n-1}]$ for some F_i algebraically independent over $\mathbb{C}$, which shows that the cancellation problem is true. □

1.2 Morita cancellation

For any algebra A, let $M(A)$ denote the category of right A-modules. The following Morita version of the cancellation property is a natural generalization of the original Zariski cancellation property when we study noncommutative algebras. The definitions and results are copied of [LuWZ].

Definition 1.2.1. [LuWZ, Definition 1.2] Let A be an algebra.

(1) We call A *m-cancellative* if any equivalence of abelian categories $M(A[t]) \cong M(B[s])$ for some algebra B implies that $M(A) \cong M(B)$.

(2) We call A *strongly m-cancellative* if, for each $n \geq 1$, any equivalence of abelian categories
$$M(A[t_1, \cdots, t_n]) \cong M(B[s_1, \cdots, s_n])$$
for some algebra B implies that $M(A) \cong M(B)$.

The letter m here stands for the word "Morita".

The Makar-Limanov invariant was generalized as following.

Definition 1.2.2. [LuWZ, Definition 1.7] Let A be an algebra and d a nonzero element in A.

1. The *Makar-Limanov$_d^H$ invariant* of A is defined to be

$$\mathrm{ML}_d^H(A) := \bigcap_{\delta \in \mathrm{LND}_d^H(A)} \ker(\delta),$$

with

$$\mathrm{LND}_d^H(A) = \{\partial \in \mathrm{LND}^H(A) \mid d \in \ker(\partial)\}.$$

2. We say that A is LND_d^H*-rigid* if $\mathrm{ML}_d^H(A) = A$.

3. A is called *strongly* LND_d^H*-rigid* if $\mathrm{ML}_d^H(A[t_1, \ldots, t_n]) = A$, for all $n \geq 1$.

4. The *Makar-Limanov$_d^H$ center* of A is defined to be

$$\mathrm{ML}_{d,Z}^H(A) = \mathrm{ML}_d^H(A) \cap Z(A).$$

5. A is called *strongly* $\mathrm{LND}_{d,Z}^H$*-rigid* if $\mathrm{ML}_{d,Z}^H(A[t_1, \ldots, t_n]) = A$, for all $n \geq 1$.

Let Z be a commutative k-algebra , which is usually the center of a noncommutative algebra. We now recall the definition of $\mathcal{P}$-discriminant for a property $\mathcal{P}$. Let $\operatorname{Spec} Z$ denote the prime spectrum of Z and

$$\mathrm{MaxSpec}(Z) := \{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } Z\}$$

is the maximal spectrum of Z. For any $S \subseteq \operatorname{Spec} Z$, $I(S)$ is the ideal of Z vanishing on S, namely,

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

For any algebra A, $A^\times$ denotes the set of invertible elements in A. A property $\mathcal{P}$ considered in the following means a property defined on a class of algebras that is an invariant under algebra isomorphisms.

Definition 1.2.3. [LuWZ, Definition 2.3] Let A be an algebra, $Z := Z(A)$ be the center of A. Let $\mathcal{P}$ be a property defined for k-algebras (not necessarily a Morita invariant).

(1) The $\mathcal{P}$*-locus* of A is defined to be

$$L_{\mathcal{P}}(A) := \{\mathfrak{m} \in \mathrm{MaxSpec}(Z) \mid A/\mathfrak{m}A \text{ has the property } \mathcal{P}\}.$$

(2) The $\mathcal{P}$*-discriminant set* of A is defined to be

$$D_{\mathcal{P}}(A) := \mathrm{MaxSpec}(Z) \setminus L_{\mathcal{P}}(A).$$

(3) The $\mathcal{P}$-*discriminant ideal* of A is defined to be

$$I_{\mathcal{P}}(A) := I(D_{\mathcal{P}}(A)) \subseteq Z.$$

(4) If $I_{\mathcal{P}}(A)$ is a principal ideal of Z generated by $d \in Z$, then d is called the $\mathcal{P}$-*discriminant* of A, denoted by $d_{\mathcal{P}}(A)$. In this case $d_{\mathcal{P}}(A)$ is unique up to an element in $Z^\times$.

(5) Let $\mathcal{C}$ be a class of algebras over k. We say that $\mathcal{P}$ is $\mathcal{C}$-*stable* if for every algebra A in $\mathcal{C}$ and every $n \geq 1$,

$$I_{\mathcal{P}}(A \otimes k[t_1, \cdots, t_n]) = I_{\mathcal{P}}(A) \otimes k[t_1, \cdots, t_n]$$

as an ideal of $Z \otimes k[t_1, \cdots, t_n]$. If $\mathcal{C}$ is a singleton $\{A\}$, we simply call $\mathcal{P}$ A-*stable*. If $\mathcal{C}$ is the whole collection of k-algebras with the center affine over k, we simply call $\mathcal{P}$ *stable*.

In general, neither $L_{\mathcal{P}}(A)$ nor $D_{\mathcal{P}}(A)$ is a subscheme of $\operatorname{Spec} Z(A)$.

Recall from the Morita theory that if $A' := A[t_1, \cdots, t_n]$ is Morita equivalent to $B' := B[s_1, \cdots, s_n]$, then there is an (A', B')-bimodule Ω that is invertible and induces naturally algebra isomorphisms $A' \cong \operatorname{End}(\Omega_{B'})$ and $(B')^{op} \cong \operatorname{End}({}_{A'}\Omega)$ such that

$$Z(A') \cong \operatorname{Hom}_{(A',B')}(\Omega, \Omega) \cong Z(B').$$

The above isomorphism is denoted by

$$\omega : Z(A') \longrightarrow Z(B'). \tag{1.5}$$

Morita equivalences have been studied extensively for decades. For more details about this topic the reader can see the books [AF, We]. Z-retractability and detectability have been generalized to the Morita setting, and they are closely related to the m-cancellative property.

Definition 1.2.4. Let A be an algebra.

(1) [LuWZ, Definition 2.6(3)] We call A *m-Z-retractable* if, for any algebra B, any equivalence of categories $M(A[t]) \cong M(B[s])$ implies that $\omega(Z(A)) = Z(B)$ where $\omega : Z(A)[t] \to Z(B)[s]$ is given as in (1.5).

(2) [LuWZ, Definition 2.6(4)] We call A *strongly m-Z-retractable* if, for any algebra B and any $n \geq 1$, any equivalence of categories $M(A[t_1, \cdots, t_n]) \cong M(B[s_1, \cdots, s_n])$ implies that $\omega(Z(A)) = Z(B)$ where

$$\omega : Z(A)[t_1, \cdots, t_n] \to Z(B)[s_1, \cdots, s_n]$$

is given as in (1.5).

Proposition 1.2.5. *[LuWZ, Proposition 2.7] Let A be an algebra whose center $Z := Z(A)$ is an affine domain. Let $\mathcal{P}$ be a stable Morita invariant property (respectively, stable property) and assume that the $\mathcal{P}$-discriminant of A, denoted by d, exists.*

1. *Suppose* $\mathrm{ML}_d^H(Z[t]) = Z$. *Then* A *is m-*Z*-retractable (respectively,* Z*-retractable).*

2. *Suppose that* Z *is strongly* LND_d^H*-rigid. Then* A *is strongly m-*Z*-retractable (respectively, strongly* Z*-retractable).*

Corollary 1.2.6. *[LuWZ, Corollary 2.11] Let* A *be an algebra such that the center of* A *is* $k[x]$*. Let* $\mathcal{P}$ *be a stable Morita invariant property (respectively, stable property) such that the* $\mathcal{P}$*-discriminant of* A*, denoted by* d *is a nonzero non-invertible element in* $Z(A) = k[x]$*. Then* $Z(A)$ *is strongly* LND_d^H*-rigid and* A *is strongly m-*Z*-retractable (respectively, strongly* Z*-retractable).*

Definition 1.2.7. [LuWZ, Definition 3.3] Let A be an algebra and let ω be the map given in (1.5).

1. We call A *m-detectable* if any equivalence of categories $M(A[t]) \cong M(B[s])$ for some algebra B implies that
$$A[t] = A\{\omega^{-1}(s)\},$$
or equivalently, $t \in A\{\omega^{-1}(s)\}$.

2. We call A *strongly m-detectable* if for each $n \geq 1$ and any equivalence of categories $M(A[t_1, \ldots, t_n]) \cong M(B[s_1, \ldots, s_n])$ for some algebra B implies that
$$A[t_1, \ldots, t_n] = A\{\omega^{-1}(s_1), \ldots, \omega^{-1}(s_n)\},$$
or equivalently, $t_i \in A\{\omega^{-1}(s_1), \ldots, \omega^{-1}(s_n)\}$.

The following result is analogous to Lemma 1.1.15.

Lemma 1.2.8. *[LuWZ, Lemma 3.4] If* A *is m-*Z*-retractable (respectively, strongly m-*Z*-retractable), then it is m-detectable (respectively, strongly m-detectable).*

As in Lemma 1.1.18, m-detectability with Hopfian implies m-cancellative.

Lemma 1.2.9. *[LuWZ, Lemma 3.6] Suppose* A *is strongly Hopfian.*

1. *If* A *is m-detectable, then* A *is m-cancellative and cancellative.*

2. *If* A *is strongly m-detectable, then* A *is strongly m-cancellative and strongly cancellative.*

The following theorem is a consequence of Corollary 1.2.6 and Lemma 1.1.17.

Theorem 1.2.10. *Let* A *be a noetherian algebra such that its center* $Z(A)$ *is* $k[x]$*. Let* $\mathcal{P}$ *be a Morita invariant property (respectively, stable property) such that the* $\mathcal{P}$*-discriminant of* A*, denoted by* d*, is a nonzero non-invertible element in* $Z(A)$*. Then* A *is strongly m-*Z*-retractable (respectively, strongly* Z*-retractable) and strongly m-cancellative (respectively, strongly cancellative).*

Proof. By [LuWZ, Lemma 5.1], $\mathcal{P}$ is stable (when k is algebraically closed). The assertion follows from Corollary 1.2.6 and Lemma 1.1.17. □

In general, when A is noncommutative, it is not clear what are the relationship between both versions of cancellation property.

1.3 Skew PBW extensions

In this section we remembered some definitions and elementary properties of skew PBW extensions. Examples of these algebras are presented. However, in chapter 2, we will extend the list and we determine the center for particular classes of these algebras.

Definition 1.3.1 ([GL1])**.** Let R and A be rings. We say that A is a *skew PBW extension of R* (also called a *σ-PBW* extension of R) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finitely many elements $x_1, \ldots, x_n \in A$ such A is a left R-free module with basis

$$\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}, \text{ with } \mathbb{N} := \{0, 1, 2, \ldots\}.$$

The set $Mon(A)$ is called the set of standard monomials of A.

(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1.6}$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.7}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

Associated to a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$, there are n injective endomorphisms $\sigma_1, \ldots, \sigma_n$ of R and σ_i-derivations, as the following proposition shows.

Proposition 1.3.2. *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [GL1, Proposition 3]. □

A particular case of skew PBW extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective and the constants c_{ij} are invertible. We recall the following definitions.

Definition 1.3.3. Let A be a skew PBW extension.

(a) A is quasi-commutative if the conditions (iii) and (iv) in Definition 1.3.1 are replaced by

(iii') For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r = c_{i,r} x_i. \tag{1.8}$$

(iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.9}$$

(b) A is bijective if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i, j \leq n$.

(c) A is constant if for every $1 \leq i \leq n$ and $r \in R$,

$$x_i r = r x_i, \tag{1.10}$$

i.e., $\sigma_i = i_R$ and $\delta_i = 0$ for every $1 \leq i \leq n$.

If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a skew PBW extension of the ring R, then, as was observed in Proposition 1.3.2, A induces injective endomorphisms $\sigma_k : R \to R$ and σ_k-derivations $\delta_k : R \to R$, $1 \leq k \leq n$. Moreover, from the Definition 1.3.1, there exists a unique finite set of constants $c_{ij}, d_{ij}, a_{ij}^{(k)} \in R$, $c_{ij} \neq 0$, such that

$$x_j x_i = c_{ij} x_i x_j + a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n + d_{ij}, \text{ for every } 1 \leq i < j \leq n. \tag{1.11}$$

Definition 1.3.4. Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension. R, n, $\sigma_k, \delta_k, c_{ij}$, $d_{ij}, a_{ij}^{(k)}$, with $1 \leq i < j \leq n$, $1 \leq k \leq n$, defined as before, are called the parameters of A.

Definition 1.3.5. Let A be a skew PBW extension of R.

(i) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

(ii) For $X = x^\alpha \in Mon(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) Let $0 \neq f \in A$, $t(f)$ is the finite set of terms that conform f, i.e., if $f = c_1 X_1 + \cdots + c_t X_t$, with $X_i \in Mon(A)$ and $c_i \in R - \{0\}$, then $t(f) := \{c_1 X_1, \ldots, c_t X_t\}$.

(iv) Let f be as in (iii), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

The next result is an important property of skew PBW extensions that will be used later.

Theorem 1.3.6 ([LR]). *Let A be an arbitrary skew PBW extension of the ring R. Then, A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A \mid deg(f) \leq m\}, & \text{if } m \geq 1, \end{cases}$$

and the graded ring $Gr(A)$ is a quasi-commutative skew PBW extension of R. If the parameters that define A are as in Definition 1.3.4, then the parameters that define $Gr(A)$ are R, n, σ_k, c_{ij}, with $1 \leq i < j \leq n$, $1 \leq k \leq n$. Moreover, if A is bijective, then $Gr(A)$ is bijective.

Remark 1.3.7. Additionally in this thesis we will assume that if $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a skew PBW extension, then R is k-algebra such that $\sigma_i(c) = c$ and $\delta_i(c) = 0$ for every $c \in k$ and $1 \leq i \leq n$. Therefore, A is also a k-algebra. In particular, if $R = k$, then A is constant.

1.3.1 Examples

In this subsection we review some examples of skew PBW extensions. In addition we mentioned the center of some algebras that has been computed before in the literature. For more details about the precise definition of the next algebras and its homological properties see [LR] and [Re]. Recall that k is a field with $char(k) = 0$.

Example 1.3.8 (PBW **extensions**). Any PBW extension (see [BG]) is a bijective skew PBW extension since in this case $\sigma_i = i_R$ for each $1 \leq i \leq n$, and $c_{i,j} = 1$ for every $1 \leq i, j \leq n$. Thus, for PBW extensions we have $A = i(R)\langle x_1, \ldots, x_n\rangle$. Some examples of PBW extensions are the following:

(a) The *usual polynomial algebra* $A = k[t_1, \ldots, t_n]$, $t_i t_j = t_j t_i$, so $Z(A) = k[t_1, \ldots, t_n]$.

(b) The *Weyl algebra* $A_n(k) := k[t_1, \ldots, t_n][x_1; \partial/\partial t_1] \cdots [x_n; \partial/\partial t_n]$. The *extended Weyl algebra* $B_n(k) := k(t_1, \ldots, t_n)[x_1; \partial/\partial t_1] \cdots [x_n; \partial/\partial t_n]$, where $k(t_1, \ldots, t_n)$ is the field of fractions of $k[t_1, \ldots, t_n]$. It is known that $Z(A_n(k)) = k = Z(B_n(k))$ (see [Di1]).

(c) The *universal enveloping algebra* of a finite dimensional Lie algebra $\mathcal{U}(\mathcal{G})$. In this case, $x_i r - r x_i = 0$ and $x_i x_j - x_j x_i = [x_i, x_j] \in \mathcal{G} = k + kx_1 + \cdots + kx_n$, for any $r \in k$ and $1 \leq i, j \leq n$. The center of some of these algebras was studied in [Lev] and [MR]:

- *The universal enveloping algebra* $\mathcal{U}(\mathfrak{sl}(2,k))$ *of the Lie algebra* $\mathfrak{sl}(2,k)$ is the k-algebra generated by the variables x, y, z subject to the relations

$$[x,y] = z,\ [x,z] = -2x,\ [y,z] = 2y.$$

  Then $Z(\mathcal{U}(\mathfrak{sl}(2,k))) = k[4xy + z^2 - 2z]$. If $\operatorname{char} k = 2$, then $Z(\mathcal{U}(\mathfrak{sl}(2,k))) = k[x^2, y^2, z]$.

- $\mathcal{U}(\mathfrak{so}(3,k))$ is the k-algebra generated by the variables x, y, z subject to the relations

$$[x,y] = z,\ [x,z] = -y,\ [y,z] = x.$$

  So, $Z(\mathcal{U}(\mathfrak{so}(3,k))) = k[x^2 + y^2 + z^2]$.

- Let $\mathcal{G}$ be a three-dimensional completely solvable Lie algebra with basis x, y, z such that

$$[y,x] = y,\ [z,x] = \lambda z,\ [y,z] = 0.$$

  With $\lambda \in k - \{0\}$. Then, if either $\lambda \in k - \mathbb{Q}$, or, $\lambda \in \mathbb{Q}$ and $\lambda > 0$, then $Z(U(\mathcal{G})) = k$. If $\lambda \in \mathbb{Q}$ and $\lambda < 0$ then $Z(U(\mathcal{G})) \supsetneq k$.

Example 1.3.9 (**Ore extensions of bijective type**)**.** Any *skew polynomial ring* $R[x;\sigma,\delta]$ of bijective type, i.e., with σ bijective, is a bijective skew PBW extension. In this case we have $R[x;\sigma,\delta] \cong \sigma(R)\langle x\rangle$. If additionally $\delta = 0$, then $R[x;\sigma]$ is quasi-commutative. More generally, let $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ be an *iterated skew polynomial ring of bijective type*, i.e., the following conditions hold:

- for $1 \leq i \leq n$, σ_i is bijective;
- for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r) \in R$, $\delta_i(r) \in R$;
- for $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c, d \in R$ and c has a left inverse;
- for $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$,

then $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a bijective skew PBW extension. Under those conditions we have

$$R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] \cong \sigma(R)\langle x_1,\ldots,x_n\rangle.$$

Some concrete examples of Ore algebras of bijective type are the following:

(a) *The algebra of q-differential operators* $D_{q,h}[x,y]$: Let $q, h \in k, q \neq 0$; consider $k[y][x;\sigma,\delta]$, $\sigma(y) := qy$ and $\delta(y) := h$. By definition of skew polynomial ring we have $xy = \sigma(y)x + \delta(y) = qyx + h$, and hence $xy - qyx = h$. Therefore, $D_{q,h} \cong \sigma(k[y])\langle x\rangle$. Then, $Z(D_{q,h}) = k$ if q is not a root of unity, and $Z(D_{q,h}) = k[x^n, y^n]$ if q is a primitive n-th root of unity (see [CYZ1]).

(b) *The algebra of shift operators* S_h: Let $h \in k$. The algebra of shift operators is defined by $S_h := k[t][x_h; \sigma_h, \delta_h]$, where $\sigma_h(p(t)) := p(t-h)$, and $\delta_h := 0$. Thus, $S_h \cong \sigma(k[t])\langle x_h\rangle$ and it is easy to check that $Z(S_h) = k$ (see also [Jak]).

(c) The *mixed algebra* D_h: Let $h \in k$. The algebra D_h is defined by $D_h := k[t][x; i_{k[t]}, \frac{d}{dt}][x_h; \sigma_h, \delta_h]$, where $\sigma_h(x) := x$. Then $D_h \cong \sigma(k[t])\langle x, x_h\rangle$ and $Z(D_h) = k$ ([Jak]).

Example 1.3.10 (**Quantum algebras**)**.** Some important examples of quantum algebras are the following:

(a) *Quantum algebra* $\mathcal{U}'(\mathfrak{so}(3,k))$. It is the k-algebra generated by I_1, I_2, I_3 subject to relations

$$I_2I_1 - qI_1I_2 = -q^{1/2}I_3, \quad I_3I_1 - q^{-1}I_1I_3 = q^{-1/2}I_2, \quad I_3I_2 - qI_2I_3 = -q^{1/2}I_1,$$

where $q \in k - \{0\}$. Then, $\mathcal{U}'(\mathfrak{so}(3,k)) \cong \sigma(k)\langle I_1, I_2, I_3\rangle$ and it is known that $Z(\mathcal{U}'(\mathfrak{so}(3,k))) = k[-q^{1/2}(q^2-1)I_1I_2I_3 + q^2I_1^2 + I_2^2 + q^2I_3^2]$ for q generic, and if q is a root of unity, $Z(\mathcal{U}'(\mathfrak{so}(3,k))) = k[-q^{1/2}(q^2-1)I_1I_2I_3 + q^2I_1^2 + I_2^2 + q^2I_3^2, C_n^1, C_n^2, C_n^3]$, where $C_n^l = 2T_p(I_l(q-q^{-1})/2)$ for $l = 1,2,3$ and $T_p(x)$ is a Chebyshev polynomial of the first kind ([Lev]).

(b) *Quantum enveloping algebra of* $\mathfrak{sl}(2,k)$. $\mathcal{U}_q(\mathfrak{sl}(2,k))$ is defined as the algebra generated by x, y, z, z^{-1} with relations

$$zz^{-1} = z^{-1}z = 1, \quad xz = q^{-2}zx, yz = q^2zy, \quad xy - yx = \frac{z - z^{-1}}{q - q^{-1}},$$

where $q \neq 1, -1$. The above relations show that $\mathcal{U}_q(\mathfrak{sl}(2,k)) = \sigma(k[z,z^{-1}])\langle x, y\rangle$. Then for q generic, its center is generated by the Casimir element $C_2 = (q^2-1)^2xy + qz + q^3z^{-1}$, and if $q^n = 1$ for some $n \geq 2$, then $Z(\mathfrak{sl}(2,k)) = k[C_2, x^n, y^n, z^n, z^{-n}]$ ([Lev]).

(c) *Quantum symplectic space* $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$. It is the algebra generated by $x_1, \ldots, x_{2n}$, satisfying the following relations:

$$x_jx_i = qx_ix_j, \quad i < j, \quad i' \neq j, \quad x_{i'}x_i = q^2x_ix_{i'} + (q^2-1)\sum_{1\leq k<i} q^{i-k}x_kx_{k'}, \quad i < i',$$

where $i' = 2n - i + 1$ for $1 \leq i \leq 2n$. From the relations above we have that $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$ is isomorphic to a bijective skew PBW extension,

$$\mathcal{O}_q(\mathfrak{sp}(k^{2n})) \cong \sigma(\sigma(\cdots\sigma(\sigma(k)\langle x_1, x_{2n}\rangle)\langle x_2, x_{2n-1}\rangle)\cdots)\langle x_n, x_{n+1}\rangle.$$

In [Zh], Zhang studied the irreducible representations of $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$, where the main step is the computation of the center, thus in Theorem 4.3 therein is proved that if m is an odd positive integer and if q is a primitive m-th root of unity, then the center is generated by x_i^m for all $i = 1, 2, \ldots, 2n$. However, by [Zh, Lemma 4.2], x_i^m are central elements if q is a m-th root of

unity, and it is possible to show from their relations that if q is generic then $Z(\mathcal{O}_q(\mathfrak{sp}(k^{2n}))) = k$.

(d) *Quantum Weyl algebra of Maltsiniotis* $A_n^{q,\lambda}$. Let $\mathbf{q} = [q_{ij}]$ be a matrix over k such that $q_{ij}q_{ji} = 1$ and $q_{ii} = 1$ for all $1 \leq i, j \leq n$. Fix an element $\lambda := (\lambda_1, \ldots, \lambda_n)$ of $(k^*)^n$. By definition, this algebra is generated by x_i, y_j, $1 \leq i, j \leq n$ subject to the following relations: For any $1 \leq i < j \leq n$, $x_ix_j = \lambda_i q_{ij} x_j x_i$, $y_iy_j = q_{ij}y_jy_i$, $x_iy_j = q_{ji}y_jx_i$, $y_ix_j = \lambda_i^{-1}q_{ji}x_jy_i$. And for any $1 \leq i \leq n$, $x_iy_i - \lambda_i y_i x_i = 1 + \sum_{1\leq j<i}(\lambda_j - 1)y_jx_j$. From these relations we have that $A_n^{\mathbf{q},\lambda}$ is isomorphic (see [LR]) to a bijective skew PBW extension

$$A_n^{\mathbf{q},\lambda} \cong \sigma(\sigma(\cdots\sigma(\sigma(k)\langle x_1, y_1\rangle)\langle x_2, y_2\rangle)\cdots)\langle x_{n-1}, y_{n-1}\rangle)\langle x_n, y_n\rangle.$$

If none of λ_i is a root of unity, the quantum Weyl algebra of Maltisiniotis has trivial center k ([Ta1, Theorem 1.4]).

(e) *Multiparameter quantized Weyl algebra* $A_n^{Q,\Gamma}(k)$. Let $Q := [q_1, \ldots, q_n]$ be a vector in k^n with no zero components, and let $\Gamma = [\gamma_{ij}]$ be a multiplicatively antisymmetric $n \times n$ matrix over k. The multiparameter Weyl algebra $A_n^{Q,\Gamma}(k)$ is defined to be the algebra generated by k and the indeterminates $y_1, \ldots, y_n, x_1, \ldots, x_n$ subject to the relations: $y_iy_j = \gamma_{ij}y_jy_i$, $1 \leq i, j \leq n$, $x_ix_j = q_i\gamma_{ij}x_jx_i$, $x_iy_j = \gamma_{ji}y_jx_i$, $1 \leq i < j \leq n$, $x_iy_j = q_j\gamma_{ji}y_jx_i$, $1 \leq j < i \leq n$, $x_jy_j = q_jy_jx_j + 1 + \sum_{l<j}(q_l - 1)y_lx_l$, $1 \leq j \leq n$. From these we have that $A_n^{Q,\Gamma}(k)$ is isomorphic to a bijective skew PBW extension,

$$A_n^{Q,\Gamma}(k) \cong \sigma(\sigma(\cdots\sigma(\sigma(k)\langle x_1, y_1\rangle)\langle x_2, y_2\rangle)\cdots)\langle x_{n-1}, y_{n-1}\rangle)\langle x_n, y_n\rangle.$$

In [Ta1] was studied the multiparameter quantized Weyl algebra of "symmetric type", where the relations are:

$$\begin{array}{lll} y_iy_j = \gamma_{ij}y_jy_i, & x_ix_j = \gamma_{ij}x_jx_i, & 1 \leq i < j \leq n, \\ x_iy_j = \gamma_{ji}y_jx_i, & x_iy_j = \gamma_{ji}y_jx_i, & 1 \leq j < i \leq n, \\ x_jy_j = q_jy_jx_j + 1 & 1 \leq j \leq n. & \end{array}$$

If none of q_i is a root of unity, then this algebra has a trivial center k ([Ta1], Corollary 1.1).

(f) *n-multiparametric skew quantum space* over R. Let R be a ring with a fixed matrix of parameters $\mathbf{q} := [q_{ij}] \in M_n(R)$, $n \geq 2$, such that $q_{ii} = 1 = q_{ij}q_{ji}$ for every $1 \leq i, j \leq n$, and suppose also that it is given a system $\sigma_1, \ldots, \sigma_n$ of automorphisms of R. The quasi-commutative bijective skew PBW extension $R_{q,\sigma}[x_1, \ldots, s_n]$ defined by

$$x_jx_i = q_{ij}x_ix_j, \quad x_ir = \sigma_i(r)x_i, \quad r \in R, \quad 1 \leq i, j \leq n,$$

is called the *n-multiparametric skew quantum space* over R. When all automorphisms of the extension are trivial, i.e., $\sigma_i = i_R$, $1 \leq i \leq n$, it is called *n-multiparametric quantum space* over R. If $R = k$ is a field, then $k_{q,\sigma}[x_1, \ldots, x_n]$

is called the *n-multiparametric skew quantum space*, and the case particular case when $n = 2$ it is called *skew quantum plane*; for trivial automorphisms we have the *n-multiparametric quantum space* and the *quantum plane* (see [LR]). Shirikov studied the center of the quantum plane; if q is not a root of unity, then its center is k, if q is a root of unity of the degree m, $m \in \mathbb{N}$, then the center is the subalgebra generated by x^m, y^m ([Sh, Theorem 2.2]). A more general case was studied extensively in [CYZ1] assuming that k is a commutative domain and the parameters are roots of the unity, so if $q_{ij} = exp(2\pi\sqrt{-1}k_{ij}/d_{ij})$, where $d_{ij} := o(q_{ij}) < \infty$, $|k_{ij}| < d_{ij}$, and $mcd(k_{ij}, d_{ij}) = 1$, then if $L_i := lcm\{d_{ij} \mid j = 1, \ldots, n\}$ and $P := k[x_1^{L_1}, \ldots, x_n^{L_n}]$, the center $Z(k_q[x_1, \ldots, x_n]) = P$ if and only if $Z(k_q[x_1, \ldots, x_n])$ is a polynomial ring ([CYZ1, Lemma 4.3]).

Example 1.3.11. The *Jordan plane* is the k-algebra defined by $\mathcal{J} = k\{x, y\}/\langle yx - xy - x^2\rangle$; it is a skew PBW extension of $k[x]$. In fact, $\mathcal{J} = \sigma(k[x])\langle y\rangle$, with multiplication given by $yx = xy - x^2$. Then $Z(\mathcal{J}) = k$ (if $char(k) = p > 0$, then $Z(\mathcal{J})$ is the subalgebra generated by x^p, y^p, (see [Sh, Theorem 2.2])).

1.4 Artin–Schelter regular algebras

Artin–Schelter regular algebras are considered as a noncommutative generalization of the commutative polynomial rings. We study Zariski Cancellation problem for these algebras with Gelfand Kirillov dimension three in Chapter 4. In this section we provide the basic background on Artin–Schelter regular algebras. These definitions and properties can be found in [BRSSW, Le, LMZ, St].

Definition 1.4.1. A k-algebra A is graded if it has a k-vector space decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$ such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$. We say that A is positively graded or $\mathbb{N}$-graded if $A_n = (0)$ for all $n < 0$, and, A is connected if $A_0 = k$, then we write $A = k \oplus A_1 \oplus A_2 \oplus \cdots$. An element $x \in A$ is homogeneous if $x \in A_n$ for some n. A right or left ideal I of A is called homogeneous or graded if it is generated by homogeneous elements, or equivalently if $I = \bigoplus_{n\geq 0}(I \cap A_n)$.

For example, the free algebra $A = k\langle x_1, \ldots, x_n\rangle$ is connected $\mathbb{N}$-graded, where A_i is the k-span of all words of length i. For a more general grading, one can put weights $d_i \geq 1$ on the variables x_i and define A_i to be the k-span of all words $x_{i_1}, \ldots, x_{i_m}$ such that $\sum_{j=1}^{m} d_{i_j} = i$.

Definition 1.4.2. Let A be an $\mathbb{N}$-graded k-algebra. A right A-module M is graded if M has a k-space decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ such that $M_iA_j \subseteq M_{i+j}$ for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$.

Given a graded A-module M, we define $M(i)$ to be the graded module which is isomorphic to M as an abstract A-module, but which has degrees shifted so that $M(i)_n = M_{i+n}$. Any such module is called a shift of M.

For graded A-modules M which are finitely generated, it is possible to find set of homogeneous generators of M, say $m_1, \ldots, m_r$ with $m_i \in M_{d_i}$, and thus define a surjective graded right A-module homomorphism $\bigoplus_{i=1}^{r} A(-d_i) \to M$, where the 1 of the i-th summand maps to the generator m_i. This shows that any finitely generated graded A-module M over a finitely graded algebra A has $dim_k M_n < \infty$ for all n and $dim_k M_n = 0$ for $n < 0$, and so the following definition makes sense.

Definition 1.4.3. Let A be finitely graded. If M is a finitely generated graded A-module, then the *Hilbert series* of M is the formal Laurent series

$$h_M(t) = \sum_{n \in \mathbb{Z}} (dim_k M_n) t^n.$$

Example 1.4.4. The Hilbert series of the commutative polynomial ring $k[x]$ is $1/(1-t)$. More generally, if $A = k[x_1, \ldots, x_n]$ then $h_A(t) = 1/(1-t)^m$. The Hilbert series of the quantum plane and Jordan plane (Example 1.3.10(f), 1.3.11 respectively) is the same $h_A(t) = 1/(1-t)^2$.

Now we can define Artin–Schelter regular algebras.

Definition 1.4.5. A finitely generated connected graded algebra A is called *Artin-Schelter Regular* (or, *AS Regular*, for short) if the following conditions hold:

(i) A has (graded) finite global dimension d.

(ii) GKdim$(A) < \infty$; and

(iii) $\operatorname{Ext}^i_A({}_A k, {}_A A) \cong \begin{cases} 0, & \text{if } i \neq d, \\ k(l)_A & \text{if } i = d. \end{cases}$

for some shift $l \in \mathbb{Z}$.

The interpretation of regular algebras is that they are the noncommutative graded algebras most analogues to commutative polynomial ring. The classification is a difficult task. The only AS-regular algebra of global dimension one is the polynomial ring $k[x]$. In dimension two, there are three types of AS-regular algebras, $k[x_1, x_2]$, the Jordan plane $\mathcal{J}$ and the quantum plane ([BRSSW, Theorem 2.2.1]). More generally, the quantum polynomial ring in n variables is AS-regular.

The classification in dimension three was more difficult and included geometric techniques. Thus, every AS-regular algebra surjects onto a twisted homogeneous coordinate ring $\mathcal{B}$. However, according with the next lemma there are only two cases to consider: either A is generated by two elements satisfying two cubic relations, or else by three elements with three quadratic relations. This result will be useful in the proof our main result in Chapter 4, Theorem 4.2.5.

Lemma 1.4.6. *[BRSSW, Lemma 2.3.1] Let A be an algebra of global dimension* 3 *which is generated in degree* 1*. Then exactly one of the following holds:*

1. *$A \cong k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$, where the f_i have degree 2, and $h_A(t) = 1/(1-t)^3$; or*

2. *$A \cong k\langle x_1, x_2\rangle/(f_1, f_2)$, where the f_i have degree 3, and $h_A(t) = 1/(1-t)^2(1-t^2)$.*

The next example presents some AS-regular algebras of dimension three (see [LMZ]).

Example 1.4.7. The following algebras are pair-wise non-isomorphic noetherian AS-regular algebras of global dimension three:

1. $A(2)$ is generated by t_1, t_2, t_3 and subject to the relations

$$t_1t_2 - t_2t_1 = t_1t_3 - t_3t_1 = t_3t_2 - pt_2t_3 - t_1^2 = 0,$$

   where $p \in k^\times$.

2. $A(3)$ is generated by t_1, t_2, t_3 and subject to the relations

$$(t_2 + t_1)t_1 - t_1t_2 = t_3t_1 - qt_1t_3 = t_3t_2 - q(t_2 + t_1)t_3 = 0,$$

   where $q \in k^\times$.

3. $A(4)$ is generated by t_1, t_2, t_3 and subject to the relations

$$(t_2 + t_1)t_1 - t_1t_2 = t_3t_1 - pt_1t_3 = t_3t_2 - pt_2t_3 = 0,$$

   where $p \in k^\times$.

4. $A(6)$ is the graded down-up algebra $A(\alpha, \beta)$ which is generated by x, y and subject to the relations

$$x^2y - \alpha xyx - \beta yx^2 = xy^2 - \alpha yxy - \beta y^2x,$$

   where $\alpha \in k$ and $\beta \in k^\times$.

5. $A(7) := S(p)$ is generated by x, y and subject to the relations

$$x^2y - pyx^2 = xy^2 + py^2x = 0,$$

   where $p \in k^\times$.

By [LMZ, Corollary 0.9], these algebras are cancellative when p, q, β are not roots of unity. We will show a general result in section 4 (Theorem 4.2.5) that covers the Corollary mentioned.

It is well known that AS-regular algebras in low dimension (≤ 4) are domain, noetherian satisfied other good homological properties, such as the Aulander conditions and Cohen-Macaulay property.

Definition 1.4.8. Let A be a ring.

(i) The grade $j(M)$ of a left (or right) $A-$module M is defined by

$$j(M) := min\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}$$

or ∞ if no such i exists.

(ii) A satisfies the *Auslander* condition if for every Noetherian left (or right) $A-$module M and for all $i \geq 0$

$$j(N) \geq i \text{ for all submodules } N \subseteq \mathrm{Ext}^i_A(M, A).$$

(iii) A is *Auslander-Gorenstein* (AG) if A is two-sided Noetherian, satisfies the Auslander condition, $id({}_AA) < \infty$ and $id(A_A) < \infty$.

(iv) A is *Auslander regular* (AR) if it is AG and $gld(A) < \infty$.

(v) Let A be an algebra over a field k, we say that A is *Cohen-Macaulay* (CM) with respect to the classical Gelfand-Kirillov dimension if:

$$\mathrm{GKdim}(A) = j_A(M) + \mathrm{GKdim}(M)$$

for every non-zero noetherian A-module M.

Corollary 1.4.9. *[Le, Corollary 6.2] Let A be an AS-Regular of global dimension three. Then A is Auslander-regular and satisfies the property Cohen-Macaulay.*

A ring A is called *stably free* if, for every finitely generated projective A-module P, there exist integers n and m such that $P \oplus A^{\oplus n} \cong A^{\oplus m}$. Connected graded algebras are automatically stably free [St].

Definition 1.4.10. An Ore domain A is called a *maximal order* if $A \subseteq B$ inside the quotient division ring $Q(A)$ of A for some ring B with the property that $aBb \subseteq A$, for some $a, b \in A - \{0\}$, then $A = B$.

Theorem 1.4.11. *[St, Theorem] Let A be a noetherian algebra that is Auslander regular, Cohen-Macaulay and stably free. Then A is a domain and a maximal order in its quotient division ring $Q(A)$.*

To conclude this section we list two well-known results.

Lemma 1.4.12. *[LPWZ, Lemma 7.6] Let A be a connected graded algebra and t be a central element of degree 1. If $A/(t)$ has finite global dimension, then so does A.*

Lemma 1.4.13. *[SmZ, Corollary 2] Let A be a finitely generated Ore domain that is not PI. Let Z be the center of A. Then*

$$\mathrm{GKdim}\, Z \leq \mathrm{GKdim}\, A - 2.$$

CHAPTER 2

Center of skew PBW extensions

The center of an algebra plays a large role in whether the cancellation property holds for that algebra. Therefore, in this chapter we compute the center and some central elements of many noncommutative algebras that can be interpreted as skew PBW extensions. The question about the center of the total ring of fractions is considered. We show that, under some natural assumptions on the parameters that define the extension, either the center is trivial, or, it is of polynomial type.

In the first section we describe the center of some general subclases of skew PBW extensions, grouped according to some conditions imposed to the parameters that define the extensions. One of the most important result of this chapter is Theorem 2.1.5 that establishes trivial center for a wide class of skew PBW extensions, therefore cancellation. Theorem 2.1.6 determines center for operator algebras. In the second section, we deal others algebras that can not be covered by the theorems of the first section. Corollary 2.2.14 summarize the list of cancellative skew PBW extensions. Finally, in the third section we study the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$, with $Q_r(A)$ the right total ring of fractions of A. The Theorem 2.3.3 can be interpreted as a way of computing the center of $Q_r(A)$. Results of this chapter will appear in [LV1, LV2].

2.1 Centers of some skew PBW extensions

In this section we consider some algebras that can be described as skew PBW extensions and we compute the center of these algebras. We classify these algebras according to the parameters that define the extension. We start with extensions of two variables. Let A be the k-algebra generated by x, y subject to relation

$$yx = q_1xy + q_2x + q_3y + q_4, \tag{2.1}$$

with $q_i \in k$, $i = 1, 2, 3, 4$ and $q_1 \neq 0$. Note that A is a skew PBW extension of k with trivial endomorphisms and derivations (see Remark 1.3.7).

Lemma 2.1.1. *For $n \geq 2$, in A the following rules hold:*

1.

$$\begin{aligned} yx^n =& \sum_{j=0}^{n} \binom{n}{j} q_1^{n-j} q_3^j x^{n-j} y + q_2 \left(\sum_{i=0}^{n-1} q_1^i \right) x^n + \left[q_4 \left(\sum_{i=0}^{n-1} q_1^i \right) + q_2 q_3 \left(\sum_{i=0}^{n-2} \binom{i+1}{1} q_1^i \right) \right] x^{n-1} + \\ & \cdots + \left[q_3^{j-1} q_4 \left(\sum_{i=0}^{n-j} \binom{i+j-1}{j-1} q_1^i \right) + q_2 q_3^j \left(\sum_{i=0}^{n-j-1} \binom{i+j}{j} q_1^i \right) \right] x^{n-j}. \end{aligned}$$

2.

$$\begin{aligned} y^n x =& \sum_{j=0}^{n} \binom{n}{j} q_1^{n-j} q_2^j x y^{n-j} + q_3 \left(\sum_{i=0}^{n-1} q_i^i \right) y^n + \left[q_4 \left(\sum_{i=0}^{n-1} q_1^i \right) + q_2 q_3 \left(\sum_{i=0}^{n-2} \binom{n+1}{1} q_1^i \right) \right] y^{n-1} + \\ & \cdots + \left[q_2^{j-1} q_4 \left(\sum_{i=0}^{n-j} \binom{i+j-1}{j-1} q_1^i \right) + q_2^j q_3 \left(\sum_{i=0}^{n-j-1} \binom{i+j}{j} q_1^i \right) \right] y^{n-j}. \end{aligned}$$

Proof. We proceed by induction on n. For $n = 2$ the equalities hold trivially. Then we assume the formulae for n and we will prove them for $n+1$. To simplify, we can write the product as $yx^n = (b_n x^n + \cdots + b_1 x + b_0) y + (a_n x^n + \cdots + a_1 x + a_0)$, where $b_{n-j} = \binom{n}{j} q_1^{n-j} q_3^j$, $a_n = q_2 \sum_{i=0}^{n-1} q_1^i$, $a_{n-1} = q_4 \left(\sum_{i=0}^{n-1} q_1^i \right) + q_2 q_3 \left(\sum_{i=0}^{n-2} \binom{i+1}{1} q_1^i \right)$, and $a_{n-j} = q_3^{j-1} q_4 \left(\sum_{i=0}^{n-j} \binom{i+j-1}{j-1} q_1^i \right) + q_2 q_3^j \left(\sum_{i=0}^{n-j-1} \binom{i+j}{j} q_1^i \right)$. Then,

$$\begin{aligned} yx^{n+1} =& (yx^n)x \\ =& (b_n x^n + \cdots + b_1 x + b_0) yx + (a_n x^{n+1} + \cdots + a_1 x^2 + a_0 x) \\ =& (b_n x^n + \cdots + b_1 x + b_0)(q_1 xy + q_2 x + q_3 y + q_4) + (a_n x^{n+1} + \cdots + a_1 x^2 + a_0 x) \\ =& \left[b_n q_1 x^{n+1} + (b_{n-1} q_1 + b_n q_3) x^n + \cdots + (b_{n-j} q_1 + b_{n+1-j} q_3) x^{n+1-j} + \cdots + b_0 q_3 \right] y \\ & + (a_n + b_n q_2) x^{n+1} + (a_{n-1} + b_{n-1} q_2 + b_n q_4) x^n + \cdots + \\ & (a_{n-j} + b_{n-j} q_2 + b_{n+1-j} q_4) x^{n+1-j} + \cdots + b_0 q_4. \end{aligned}$$

Now we compare coefficients: For $x^{n+1-j} y$, $b_{n-j} q_1 + b_{n+1-j} q_3 = \binom{n}{j} q_1^{n-j} q_3^j q_1 + \binom{n}{j-1} q_1^{n-(j-1)} q_3^{j-1} q_3 = \left[\binom{n}{j} + \binom{n}{j-1} \right] q_1^{n+1-j} q_3^j = \binom{n+1}{j} q_1^{n+1-j} q_3^j$ is the coefficient of $x^{n+1-j} y$ in the expansion of yx^{n+1}. For x^{n+1-j} we have: $a_{n-j} + b_{n-j} q_2 + b_{n+1-j} q_4 = q_3^{j-1} q_4 \left(\sum_{i=0}^{n-j} \binom{i+j-1}{j-1} q_1^i \right) + q_2 q_3^j \left(\sum_{i=0}^{n-j-1} \binom{i+j}{j} q_1^i \right) + \binom{n}{j} q_1^{n-j} q_3^j q_2 + \binom{n}{j-1} q_1^{n-(j-1)} q_3^{j-1} q_4 = q_3^{j-1} q_4 \left(\sum_{i=0}^{n+1-j} \binom{i+j-1}{j-1} q_1^i \right) + q_2 q_3^j \left(\sum_{i=0}^{n-j} \binom{i+j}{j} q_1^i \right)$. Similarly for x^{n+1} y x^n. So,

$$\begin{aligned} yx^{n+1} =& \sum_{j=0}^{n+1} \binom{n+1}{j} q_1^{n+1-j} q_3^j x^{n+1-j} y + q_2 (\sum_{i=0}^{n} q_1^i) x^{n+1} + \left[q_4 \left(\sum_{i=0}^{n} q_1^i \right) + q_2 q_3 \left(\sum_{i=0}^{n-1} \binom{i+1}{1} q_1^i \right) \right] x^n + \\ & \cdots + \left[q_3^{j-1} q_4 \left(\sum_{i=0}^{n+1-j} \binom{i+j-1}{j-1} q_1^i \right) + q_2 q_3^j \left(\sum_{i=0}^{n-j} \binom{i+j}{j} q_1^i \right) \right] x^{n+1-j}. \end{aligned}$$

This completes the proof of the first equality. The second equality can be proved similarly. □

In particular, if $q_2 = q_3 = 0$, then

$$yx^n = q_1^n x^n y + q_4 \left(\sum_{i=0}^{n-1} q_1^i \right) x^{n-1},$$
$$y^n x = q_1^n x y^n + q_4 \left(\sum_{i=0}^{n-1} q_1^i \right) y^{n-1}.$$

Theorem 2.1.2. *Let $A = \sigma(k)\langle x, y\rangle$ be the skew PBW extension of k defined as in (2.1).*

(i) *If $q_2 = q_3 = 0$, then*
$$Z(A) = \begin{cases} k & \text{if } q_1 \text{ is not a root of unity, or, } q_1 = 1 \text{ and } q_4 \neq 0; \\ k[x^n, y^n] & \text{if } q_1 \text{ is a root of unity of degree } n \geq 2. \end{cases}$$

(ii) *If $q_2 \neq 0$, or, $q_3 \neq 0$, then*
$$Z(A) = \begin{cases} k & \text{if } q_1 \text{ is not a root of unity, or, } q_1 = 1; \\ k[f(x), g(y)] & \text{if } q_1 \text{ is a root of unity of degree } n \geq 2, \\ & \text{with } deg(f) = deg(g) = n. \end{cases}$$

Proof. (i) It is clear that $k \subseteq Z(A)$; let $f = \sum_{i=1}^{t} c_i x^{\alpha_{i1}} y^{\alpha_{i2}} \in Z(A)$, with $c_i \in k$. Since $xf = fx$, then $\sum_{i=1}^{t} c_i x^{\alpha_{i1}+1} y^{\alpha_{i2}} = \sum_{i=1}^{t} c_i x^{\alpha_{i1}} y^{\alpha_{i2}} x = \sum_{i=1}^{t} c_i q_1^{\alpha_{i2}} x^{\alpha_{i1}+1} y^{\alpha_{i2}} + \sum_{i=1}^{t} c_i q_4 \left(\sum_{j=0}^{\alpha_{i2}-1} q_1^j \right) x^{\alpha_{i1}} y^{\alpha_{i2}-1}$.

From this we get that $c_i = c_i q_1^{\alpha_{i2}}$ and $\sum_{i=1}^{t} c_i q_4 \left(\sum_{j=0}^{\alpha_{i2}-1} q_1^j \right) x^{\alpha_{i1}} y^{\alpha_{i2}-1} = 0$, so:

1. If q_1 is not a root of unity, then $\alpha_{i2} = 0$, thus $f = \sum_{i=1}^{t} c_i x^{\alpha_{i1}}$, but $fy = yf$, then
$$\sum_{i=1}^{t} c_i x^{\alpha_{i1}} y = \sum_{i=1}^{t} c_i q_1^{\alpha_{i1}} x^{\alpha_{i1}} y + \sum_{i=1}^{t} c_i q_4 \left(\sum_{k=0}^{\alpha_{i1}-1} q_1^k \right) x^{k-1},$$
whence $c_i = c_i q_1^{\alpha_{i1}}$, and this is valid if $i = 0$, so $f = c_0 \in k$. Therefore, $Z(A) \subseteq k \subseteq Z(A)$.

2. If q_1 is a root of unity of degree $n \geq 2$, $c_i = c_i q_1^{\alpha_{i2}}$ implies that $\alpha_{i2} = nl$ with $l \in \mathbb{N}$, also $\sum_{i=0}^{ln-1} q_1^i = 0$, so $f = \sum_{i=1, l\geq 0}^{t} c_{i,l} x^i y^{nl}$. But $fy = yf$, then

$$\sum_{i=1}^{t} c_{il}x^{\alpha_{i1}}y^{nl+1} = \sum_{i=1}^{t} c_{il}q_1^i x^{\alpha_{i1}}y^{nl+1} + \sum_{i=0}^{t} c_{il}q_4 \left(\sum_{k=0}^{\alpha_{i1}-1} q_1^k\right) x^{k-1}y^{nl}.$$

Now, $c_{il} = c_{il}q_1^i$, thus $i = nr$, with $r \in \mathbb{N}$, in this case $\sum_{i=0}^{nr-1} q_1^i = 0$, and hence $f = \sum c_{rl}(x^n)^r(y^n)^l$, whence $k[x^n, y^n] \subseteq Z(A)$. Finally, since x^n, y^n are central elements of A, then $k[x^n, y^n] = Z(A)$.

(ii) From the basic conditions of commutation $xf = fx, yf = fy$, the proof of this part is similar and it is possible to find the desired polynomials $f(x)$ and $g(y)$. □

Remark 2.1.3. 1. The precise description of polynomials $f(x)$ and $g(y)$ in the previous Theorem it is in general difficult. However, in some particular cases the computation is possible: If q_1 is a root of unity of degree n, $q_3 = 0$ and $q_2 \neq 0$, then $f(x) = x^n$, and if $q_3 \neq 0$ and $q_2 = 0$, then $g(y) = y^n$.

2. Some known results can be obtained as particular cases of the previous Theorem:

   (a) If $q_1 = q_4 = 1$ and $q_2 = q_3 = 0$, then A is the Weyl algebra and $Z(A) = k$.

   (b) If $q_2 = q_3 = 0$ then A is the q_1-quantum Weyl algebra; if q_1 is a primitive n-th roof of unity for some $n \geq 2$, then $Z(A) = k[x^n, y^n]$. In [CYZ1], section 2, it is proved that A is free over $Z(A)$ of rank n^2, with basis $\mathcal{B} = \{x^i y^j \mid 0 \leq i, j \leq n-1\}$.

   (c) If L is a two-dimensional solvable algebra generated by x, y, with $[x, y] = x$, then for its universal enveloping algebra we get $Z(\mathcal{U}(L)) = k$. This particular case was considered in [Ma1].

Now we pass to study the center of extensions with $n \geq 3$ variables. Related to Example 1.3.10 (f), we have the following easy result.

Lemma 2.1.4 (*n-multiparametric quantum space*)**.** *For $q_{ij} = \frac{l_{ij}}{k_{ij}} \in k - \{0\}$, $(l_{ij}, k_{ij}) = 1$, $1 \leq i, j \leq n$ ($q_{ii} = 1$), let A be the k-algebra defined as the quasi-commutative skew PBW extension given by $x_j x_i = q_{ij}x_i x_j$, for all $1 \leq i < j \leq n$. If q_{ij} is not a root of unity and $(l_{ij}, l_{rs}) = (k_{ij}, k_{rs}) = (l_{ij}, k_{rs}) = 1$ for $i, j, r, s \in \{1, 2, \ldots, n\}$, then $Z(A) = k$.*

Proof. In this case we have the rule:

$$x_k^{l_k} x_j^{l_j} x_i^{l_i} = (q_{ij}^{l_i l_j} q_{ik}^{l_i l_k} q_{jk}^{l_j l_k}) x_i^{l_i} x_j^{l_j} x_k^{l_k}.$$

Let $f = \sum_{i=0}^{l} c_i x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}} \in Z(A)$, with $\alpha_{ij} \geq 0$ and $c_i \in k - \{0\}$, for $i = 0, 1, \dots, l$, and $j = 1, 2, \dots, n$. Since $x_1 f = f x_1$, $\dots$, $x_n f = f x_n$, then

$$\begin{aligned} \sum_{i=0}^{l} c_i x_1^{\alpha_{i1}+1} \dots x_n^{\alpha_{in}} &= \sum_{i=0}^{l} c_i q_{1n}^{\alpha_{in}} \dots q_{12}^{\alpha_{i2}} x_1^{\alpha_{i1}+1} \dots x_n^{\alpha_{in}} \\ \vdots &= \vdots \\ \sum_{i=0}^{l} c_i q_{n-1,n}^{\alpha_{i(n-1)}} \dots q_{1n}^{\alpha_{i1}} x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}} &= \sum_{i=0}^{l} c_i x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}+1}. \end{aligned}$$

From this we get that $c_i = c_i q_{1n}^{\alpha_{in}} \dots q_{12}^{\alpha_{i2}}$, $\cdots$, $c_i = c_i q_{1n}^{\alpha_{i1}} \cdots q_{n-1,n}^{\alpha_{i(n-1)}}$, so $1 = q_{1n}^{\alpha_{in}} \dots q_{12}^{\alpha_{i2}}, \dots, 1 = q_{1n}^{\alpha_{in}} \dots q_{12}^{\alpha_{i2}}$ for $i = 0, 1, 2, \dots, l$. Since q_{ij} is not a root of unity and by hypothesis, then necessarily $\alpha_{ij} = 0$ for $i = 0, 1, \dots, l$, and $j = 1, \dots, n$. Thus, $f = c_0 \in k$. Therefore, $Z(A) \subseteq k \subseteq Z(A)$. □

Theorem 2.1.5. *Let $A = \sigma(k)\langle x_1, \dots, x_n\rangle$ be a skew PBW extensions of k defined by*

$$x_j x_i = q_{ij} x_i x_j + a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n + d_{ij},$$

with $1 \leq i < j \leq n$, and $q_{ij} = \frac{l_{ij}}{k_{ij}}$, $(l_{ij}, l_{rs}) = (k_{ij}, k_{rs}) = (l_{ij}, k_{rs}) = 1$ for $i, j, r, s \in \{1, 2, \dots, n\}$. If q_{ij} is not a root of unity, then $Z(A) = k$.

Proof. According to Theorem 1.3.6, $Gr(A)$ is a quasi-commutative skew *PBW* extension with relations $x_j x_i = q_{ij} x_i x_j$, $1 \leq i < j \leq n$ (since A is a k-algebra, $\sigma_i = i_R$ and $\delta_i = 0$, see Remark 1.3.7). Then, by the previous Lemma, $k = Z(Gr(A)) \supseteq Gr(Z(A))$. Thus $Gr(Z(A)) = k$ and therefore $Z(A) = k$. □

The following Theorem allows us to give a wide list of algebras which center probably has not been computed before.

Theorem 2.1.6. *Let $A = \sigma(k[x_1, \dots, x_n])\langle y_1, \dots, y_n\rangle$ be the skew PBW extension defined by*

$$y_i y_j = y_j y_i \quad 1 \leq i < j \leq n, \tag{2.2}$$
$$y_j x_i = x_i y_j \quad i \neq j, \tag{2.3}$$
$$y_i x_i = q_i x_i y_i + d_i y_i + a_i, \quad 1 \leq i \leq n. \tag{2.4}$$

Then,

1. *$Z(A) = k$ in any of the following cases:*
   - *(a) For $1 \leq i \leq n$, q_i is not a root of unity.*
   - *(b) If for some i, $q_i = 1$, then $d_i \neq 0$ or $a_i \neq 0$.*
2. *If for every $1 \leq i \leq n$, q_i is a primitive l_i-th root of unity of degree $l_i \geq 2$, then the elements $y_i^{l_i}$ are central and $k \subsetneq Z(A)$.*

Proof. Step 1. For $k = 1, \ldots, n$, in A the following relations hold for every $m \geq 1$:

$$y_k x_k^m = \sum_{j=0}^{m} \binom{m}{j} q_k^{m-j} d_k^j x_k^{m-j} y_k + a_k \left(\sum_{i=0}^{m-1} q_k^i\right) x_k^{m-1} + \cdots + d_k^{j-1} a_k \left(\sum_{i=0}^{m-j} \binom{i+j-1}{j-1} q_k^i\right) x_k^{m-j},$$

$$y_k^m x_k = q_k^m x_k y_k^m + d_k \left(\sum_{i=0}^{m-1} q_k^i\right) y_k^m + a_k \left(\sum_{i=0}^{m-1} q_k^i\right) y_k^{m-1}.$$

Hence, the elements $y_k^{l_k}$ are central, assuming that q_k is a l_k-th root of unity.

We verify the first equality by induction. The proof of the second identity is similar. The case $m = 1$ is (2.4). Then we assume it is valid for m. So,

$$\begin{aligned}
y_k x_k^{m+1} =& (y_k x_k^m) x_k \\
=& (\sum_{j=0}^{m} \binom{m}{j} q_k^{m-j} d_k^j x_k^{m-j} y_k + a_k \left(\sum_{i=0}^{m-1} q_k^i\right) x_k^{m-1} + \cdots + d_k^{j-1} a_k \left(\sum_{i=0}^{m-j} \binom{i+j-1}{j-1} q_k^i\right) x_k^{m-j}) x \\
=& \sum_{j=0}^{m} \binom{m}{j} q_k^{m-j} d_k^j x_k^{m-j} (q_k x_k y_k + d_k y_k + a_k) + a_k \left(\sum_{i=0}^{m-1} q_k^i\right) x_k^m + \cdots + \\
& d_k^{j-1} a_k \left(\sum_{i=0}^{m-j} \binom{i+j-1}{j-1} q_k^i\right) x_k^{m+1-j}. \\
=& \sum_{j=0}^{m} \binom{m}{j} q_k^{m+1-j} d_k^j x_k^{m+1-j} y_k + \sum_{j=0}^{m} \binom{m}{j} q_k^{m-j} d_k^{j+1} x_k^{m-j} y_k + a_k \sum_{j=0}^{m} \binom{m}{j} q_k^{m-j} d_k^j x_k^{m-j} y_k + \\
& a_k \left(\sum_{i=0}^{m-1} q_k^i\right) x_k^m + \cdots + d_k^{j-1} a_k \left(\sum_{i=0}^{m-j} \binom{i+j-1}{j-1} q_k^i\right) x_k^{m+1-j}. \\
=& \sum_{j=0}^{m+1} \binom{m+1}{j} q_k^{m+1-j} d_k^j x_k^{m+1-j} y_k + a_k \left(\sum_{i=0}^{m} q_k^i\right) x_k^m + \cdots + \\
& d_k^{j-1} a_k \left(\sum_{i=0}^{m+1-j} \binom{i+j-1}{j-1} q_k^i\right) x_k^{m+1-j}. \\
=& y_k x_k^{m+1}.
\end{aligned}$$

Step 2. Let $f = \sum_{i=1}^{t} P_i(X) y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} \in Z(A)$, with $\alpha_{ij} \geq 1$ and $P_i(X) \in k[x_1, \ldots, x_n] - \{0\}$ for $i = 1, \ldots, t$ and $j = 1, 2, \ldots, n$. Since $x_1 f = f x_1$, then

$$\begin{aligned}
\sum_{i=1}^{t} P_i(X) x_1 y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} =& \sum_{i=1}^{t} P_i(X) y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} x_1 \\
=& \sum_{i=1}^{t} P_i(X) y_1^{\alpha_{i1}} x_1 \cdots y_n^{\alpha_{in}} \\
=& \sum_{i=1}^{t} P_i(X) (q_1^{\alpha_{i1}} x_1 y_1^{\alpha_{i1}} + d_1 \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}} + \\
& a_1 \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}-1}) y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}} \\
=& \sum_{i=1}^{t} q_1^{\alpha_{i1}} P_i(X) x_1 y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} + \sum_{i=1}^{t} P_i(X) (d_1 \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}} \\
& + a_1 \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}-1}) y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}}.
\end{aligned}$$

Since $x_1f - fx_1 = 0$ and A is a left $k[x_1, \ldots, x_n]-$free module with canonical basis $\{y_1^{\alpha_{i1}} y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}} \mid (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{N}^n\}$, then $P_i(X) = q_1^{\alpha_{i1}} P_i(X)$, and $\sum_{i=1}^{t} d_1 P_i(X) \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}} y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}} +$
$\sum_{i=1}^{t} a_1 \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}-1} y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}} = 0$. Thus, if q_1 is not root of unity, then the first equality implies that $\alpha_{i1} = 0$ for $1 \leq i \leq t$. If $q_1 = 1$, then we compare the leader monomial of
$\sum_{i=1}^{t} d_1 P_i(X) \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}} y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}}$ if $d_1 \neq 0$, or $\sum_{i=1}^{t} a_1 \left(\sum_{i=0}^{\alpha_{i1}-1} q_1^i\right) y_1^{\alpha_{i1}-1} y_2^{\alpha_{i2}} \ldots y_n^{\alpha_{in}}$ otherwise. Since A is a domain and $char(k) = 0$, then necessarily $\alpha_{i1} = 0$ for $1 \leq i \leq t$, i.e., f does not have powers of y_1. Repeating this procedure with $x_2, \ldots, x_n$ we conclude that $f = P(X) = \sum_{i=1}^{r} k_i x_1^{\beta_{i1}} \ldots x_n^{\beta_{in}} \in k[x_1, \ldots, x_n] - \{0\}$, but $y_1 f = f y_1$, so

$$\begin{aligned}\sum_{i=1}^{r} k_i x_1^{\beta_{i1}} \ldots x_n^{\beta_{in}} y_1 =& y_1 \sum_{i=1}^{r} k_i x_1^{\beta_{i1}} \ldots x_n^{\beta_{in}} \\ &= \sum_{i=1}^{r} k_i \Big(\sum_{j=0}^{\beta_{i1}} \binom{\beta_{i1}}{j} q_1^{\beta_{i1}-j} d_1^j x_1^{\beta_{i1}-j} y_1 + a_1 \left(\sum_{j=0}^{\beta_{i1}-1} q_1^j\right) x_1^{\beta_{i1}-1} + \\ &\quad \cdots + d_1^{j-1} a_1 \left(\sum_{l=0}^{\beta_{i1}-j} \binom{l+j-1}{j-1} q_1^l\right) x_1^{\beta_{i1}-l}\Big) x_2^{\beta_{i2}} \cdots x_n^{\beta_{in}};\end{aligned}$$

then comparing terms, $k_i = k_i q_1^{\beta_{i1}}$ implies $\beta_{i1} = 0$ if q_1 is not root of unity. If $q_1 = 1$ and $d_1 \neq 0$ then $k_i \binom{\beta_{i1}}{j} q_1^{\beta_{i1}-j} d_1^j = 0$, or $a_1 \sum_{j=0}^{\beta_{i1}-1} q_1^i = 0$ if $d_1 = 0$; in any case, it is only possible if $\beta_{i1} = 0$, $1 \leq i \leq r$, i.e., $f = \sum_{i=1}^{r} k_1 x_2^{\beta_{i2}} \cdots x_n^{\beta_{in}}$. Performing the same procedure with $y_2, \ldots, y_n$, we conclude that $f = k_0 \in k$. Therefore $k \subseteq Z(A) \subseteq k$.
In the case when each q_i is a root of unity of degree l_i, then $q_i^{l_i} = 1$ and $\sum_{j=0}^{l_i-1} q_i^j = 0$. Thus the relation $y_i^{l_i} x_i = q_i^{l_i} x_i y_i^{l_i} + d_i \left(\sum_{j=0}^{l_i-1} q_i^j\right) y_i^{l_i} + a_i \left(\sum_{j=0}^{l_i-1} q_i^j\right) y_i^{l_i-1} = x_i y_i^{l_i}$, and since $y_i y_j = y_j y_i$, $y_i x_j = x_j y_i$, then for $1 \leq i \leq n$, $y_i^{l_i}$ is a central element of A. Therefore, $k \subsetneq Z(A)$. □

Remark 2.1.7. In (2.4), if $d_k = 0$, then $y_k x_k^m = q_k^m x_k^m y_k + a_k \left(\sum_{i=0}^{m-1} q_k^i\right) x_k^{m-1}$, and if q_i is a root of unity of degree l_i for $1 \leq i \leq n$, then $x_i^{l_i}$ are central elements. Thus, $Z(A) = k[x_1^{l_1}, \ldots, x_n^{l_n}, y_1^{l_1}, \ldots, y_n^{l_n}]$. Finally, if some q_i is not a root of unity, then we exclude the variables x_i, y_i from the previous center.

Examples 2.1.8. According to Theorem 2.1.6, the following algebras have trivial center:

1. *Operator algebras.* Some important well-known operator algebras can be described as skew PBW extensions of $k[t_1, \ldots, t_n]$.

   (a) *Algebra of linear partial differential operators.* The n-th Weyl algebra $A_n(k)$ over k coincides with the k-algebra of linear partial differential operators with polynomial coefficients $k[t_1, \ldots, t_n]$. The generators of $A_n(k)$ satisfy the following relations $t_i t_j = t_j t_i$, $\partial_i \partial_j = \partial_j \partial_i$, $1 \leq i < j \leq$

n, $\partial_j t_i = t_i\partial_j + \delta_{ij}$, $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker symbol. Thus, $q_i = 1$, $d_i = 0$ and $a_i = 1$ for $1 \leq i \leq n$.

(b) *Algebra of linear partial shift operators.* The k-algebra of linear partial *shift* (*recurrence*) operators with polynomial, respectively with rational coefficients, is $k[t_1, \ldots, t_n][E_1, \ldots, E_n]$, respectively $k(t_1, \ldots, t_n)[E_1, \ldots, E_n]$, subject to the relations: $t_j t_i = t_i t_j$, $E_j E_i = E_i E_j$ for $1 \leq i < j \leq n$, and $E_i t_i = (t_i + 1)E_i = t_i E_i + E_i$, $1 \leq i \leq n$, $E_j t_i = t_i E_j$, $i \neq j$. So $q_i = 1$, $d_i = 1$ and $a_i = 0$ for $1 \leq i \leq n$.

(c) *Algebra of linear partial difference operators.* The k-algebra of linear partial *difference opertors* with polynomial, respectively with rational coefficients, is $k[t_1, \ldots, t_n][\Delta_1, \ldots, \Delta_n]$, respectively $k(t_1, \ldots, t_n)[\Delta_1, \ldots, \Delta_n]$, subject to the relations:

$t_j t_i = t_i t_j$, $1 \leq i < j \leq n$, $\Delta_i t_i = (t_i+1)\Delta_i + 1 = t_i\Delta_i + \Delta_i + 1$, $1 \leq i \leq n$, $\Delta_j t_i = t_i \Delta_j$, $i \neq j$, $\Delta_j \Delta_i = \Delta_i \Delta_j$, $1 \leq i < j \leq n$. Thus $q_i = 1$, $d_i = 1$ and $a_i = 1$ for $1 \leq i \leq n$.

(d) *Algebra of linear partial q-dilation operators.* For a fixed $q \in k - \{0\}$, the k-algebra of linear partial *q-dilation operators* with polynomial coefficients, respectively, with rational coefficients, is $k[t_1, \ldots, t_n][H_1^{(q)}, \ldots, H_n^{(q)}]$, respectively $k(t_1, \ldots, t_n)[H_1^{(q)}, \ldots, H_n^{(q)}]$, subject to the relations: $t_j t_i = t_i t_j$, $1 \leq i < j \leq n$, $H_i^{(q)} t_i = q t_i H_i^{(q)}$, $1 \leq i \leq n$, $H_j^{(q)} t_i = t_i H_j^{(q)}$, $i \neq j$, $H_j^{(q)} H_i^{(q)} = H_i^{(q)} H_j^{(q)}$, $1 \leq i < j \leq n$. Thus $q_i = q$, $d_i = 0$ and $a_i = 0$ for $1 \leq i \leq n$.

(e) *Algebra of linear partial q-differential operators.* For a fixed $q \in k - \{0\}$, the k-algebra of linear partial *q-differential operators* with polynomial coefficients, respectively with rational coefficients is $k[t_1, \ldots, t_n][D_1^{(q)}, \ldots, D_n^{(q)}]$, respectively the ring $k(t_1, \ldots, t_n)[D_1^{(q)}, \ldots, D_n^{(q)}]$, subject to the relations: $t_j t_i = t_i t_j$, $1 \leq i < j \leq n$, $D_i^{(q)} t_i = q t_i D_i^{(q)} + 1$, $1 \leq i \leq n$, $D_j^{(q)} t_i = t_i D_j^{(q)}$, $i \neq j$, $D_j^{(q)} D_i^{(q)} = D_i^{(q)} D_j^{(q)}$, $1 \leq i < j \leq n$. So $q_i = q$, $d_i = 0$ and $a_i = 1$ for $1 \leq i \leq n$.

2. The *algebra D for multidimensional discrete linear systems.* This algebra is a quasi-commutative bijective skew PBW extension of $k[t_1, \ldots, t_n]$, $D = \sigma(k[t_1, \ldots, t_n])\langle x_1, \ldots, x_n\rangle$. In fact, by definition, D is the iterated skew polynomial ring $D := k[t_1, \ldots, t_n][x_1; \sigma_1] \cdots [x_n; \sigma_n]$, where $\sigma_i(p(t_1, \ldots, t_n)) := p(t_1, \ldots, t_{i-1}, t_i + 1, t_{i+1}, \ldots, t_n)$, $\sigma_i(x_i) = x_i$, $1 \leq i \leq n$. Thus, $q_i = 1$, $d_i = 1$ and $a_i = 0$ for $1 \leq i \leq n$.

3. *Additive analogue of the Weyl algebra.* The k-algebra $A_n(q_1, \ldots, q_n)$ is generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ subject to the relations: $x_j x_i = x_i x_j$, $y_j y_i = y_i y_j$, $1 \leq i, j \leq n$, $y_i x_j = x_j y_i$, $i \neq j$, $y_i x_i = q_i x_i y_i + 1$, $1 \leq i \leq n$, where $q_i \in k - \{0\}$. From these relations we have $A_n(q_1, \ldots, q_n) \cong \sigma(k)\langle x_1, \ldots, x_n; y_1, \ldots, y_n\rangle \cong \sigma(k[x_1, \ldots, x_n])\langle y_1, \ldots, y_n\rangle$. So $q_i \neq 0$, $d_i = 0$ and $a_i = 1$ for $1 \leq i \leq n$.

4. The Weyl algebra $A_n(k)$ (Example 1.3.8 (b)), $q_i = 1$, $d_i = 0$ and $a_i = 1$ for $1 \leq i \leq n$.

5. The algebra of q-differential operators $D_{q,h}[x, y]$ (Example 1.3.9(a)), $q_1 = q$, $d_i = 0$ and $a_1 = h$.

6. The algebra of shift operators S_h (Example 1.3.9(b)), $q_1 = 1$, $d_1 = -h$ and $a_i = 0$.

Examples 2.1.9. We have computed the center of the following algebras (our computations of course are agree with Theorem 2.1.6, where the parameters q_i are roots of unity of degree l_i):

1. The algebra of q-differential operators $D_{q,h}[x, y]$, $q_1 = q$, $d_i = 0$ and $a_1 = h$, $Z(A) = k[x^l, y^l]$.

2. Additive analogue of the Weyl algebra, $q_i \neq 0$, $d_i = 0$ and $a_i = 1$ for $1 \leq i \leq n$, $Z(A) = k[x_1^{l_1}, \ldots, x_n^{l_n}, y_1^{l_1}, \ldots, y_n^{l_n}]$.

3. Algebra of linear partial q-dilation operators, $q_i = q$, $d_i = 0$ and $a_i = 0$ for $1 \leq i \leq n$, $Z(A) = k[t_1^l, \ldots, t_n^l, H_1^l, \ldots, H_n^l]$, $l =$degree(q).

4. Algebra of linear partial q-differential operators, $q_i = q$, $d_i = 0$ and $a_i = 1$ for $1 \leq i \leq n$. $Z(A) = k[t_1^l, \ldots, t_n^l, D_1^l, \ldots, D_n^l]$, $l =$degree(q).

2.2 Central subalgebras of other remarkable quantum algebras

In this section we consider other noncommutative algebras of quantum type that can be interpreted as skew PBW extensions, and for them, we compute either the center or some central key subalgebras. These algebras are not covered by Thereom 2.1.6. The proofs of the next theorems are based in some lemmas which proofs we avoid since they can be realized by direct computations.

2.2.1 Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, k))$

This algebra was introduced by Woronowicz in [Wo] and it is generated by x, y, z subject to the relations

$$xz - q^4zx = (1 + q^2)x, \quad xy - q^2yx = qz, \quad zy - q^4yz = (1 + q^2)y,$$

where $q \in k - \{0\}$ is not a root of unity. Then $\mathcal{W}_q(\mathfrak{sl}(2, k)) \cong \sigma(\boldsymbol{k})\langle x, y, z\rangle$.

Lemma 2.2.1. *Let $m \geq 1$, then in $\mathcal{W}_q(\mathfrak{sl}(2, k))$ the following identities hold:*

1. $y^m x = \frac{1}{q^{2m}} xy^m - \frac{\left(\sum_{i=0}^{m-1} q^{2i}\right)}{q} y^{m-1}z - \frac{\left(\sum_{i=1}^{m-1} iq^{2(i-1)} + \sum_{i=m}^{2m-2}((2m-1-i)q^{2(i-1)}\right)}{q^{2m-1}} y^{m-1}$.

2. $yx^m = \frac{1}{q^{2m}}x^m y - \frac{\left(\sum_{i=0}^{m-1} q^{2i}\right)}{q^{4m-3}}x^{m-1}z + \frac{\left(\sum_{i=1}^{m-1} iq^{2(i-1)} + \sum_{i=m}^{2m-2}(2m-1-i)q^{2(i-1)}\right)}{q^{4m-3}}x^{m-1}$.

3. $z^m x = \frac{\sum_{i=0}^{m}(-1)^i\binom{m}{i}(q^2+1)^i x z^{m-i}}{q^{4m}}$.

4. $zx^m = \frac{1}{q^{4m}}x^m z + \frac{1}{q^{4m}}\left(\sum_{i=0}^{2m-1} q^{2i}\right)x^m$.

5. $z^m y = y(q^4 z + (q^2+1))^m$.

6. $zy^m = q^{4m}y^m z + \sum_{i=0}^{2m-1} q^{2i}y^m$.

Theorem 2.2.2. *The Woronowicz algebra has trivial center, $Z(\mathcal{W}_q(\mathfrak{sl}(2,k))) = k$.*

Proof. Let $f(x,y,z) = \sum_{i=0}^{n} C_i x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}} \in Z(\mathcal{W}_q(\mathfrak{sl}(2,k)))$, with $\alpha_{ij} \in \mathbb{N}$, $j = 1,2,3$. Since $zf = fz$, then

$$\sum_{i=0}^{n} C_i x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}+1} = z\sum_{i=0}^{n} C_i x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}} \tag{2.5}$$

$$= \sum_{i=0}^{n} C_i\left(\frac{1}{q^{4\alpha_{i1}}}x^{\alpha_{i1}} z y^{\alpha_{i2}} z^{\alpha_{i3}}\right) + \frac{C_i}{q^{4\alpha_{i1}}}\left(\sum_{j=0}^{2\alpha_{i1}-1} q^{2j}\right)x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}} \tag{2.6}$$

$$= \sum_{i=0}^{n} \frac{C_i}{q^{4\alpha_{i1}}}\left[\left(x^{\alpha_{i1}}\left(q^{4\alpha_{i2}} y^{\alpha_{i2}} z + \sum_{j=0}^{2\alpha_{i2}-1} q^{2j} y^{\alpha_{i2}}\right) z^{\alpha_{i3}}\right) + \left(\sum_{j=0}^{2\alpha_{i1}-1} q^{2j}\right)x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}}\right] \tag{2.7}$$

$$= \sum_{i=0}^{n} C_i \frac{q^{4\alpha_{i2}}}{q^{4\alpha_{i1}}}x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}+1} + \frac{C_i}{q^{4\alpha_{i1}}}\left(\sum_{j=0}^{2\alpha_{i2}-1} q^{2j} + \sum_{j=0}^{2\alpha_{i1}-1} q^{2j}\right)x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}}. \tag{2.8}$$

Equations (2.6) and (2.7) are the relations 4 and 6 in the previous lemma, respectively. From this, $C_i = C_i\frac{q^{4\alpha_{i2}}}{q^{4\alpha_{i1}}}$ and $\frac{C_i}{q^{4\alpha_{i1}}}\left(\sum_{j=0}^{2\alpha_{i2}-1} q^{2j} + \sum_{j=0}^{2\alpha_{i1}-1} q^{2j}\right) = 0$, so $1 = \frac{q^{4\alpha_{i2}}}{q^{4\alpha_{i1}}} = 0$, whence $\alpha_{i1} = \alpha_{i2}$, and since $\text{char}(k) = 0$, $C_i = 0$, or, $\sum_{j=0}^{2\alpha_{i1}-1} q^{2j} = 0$, but q is not a root of unity, then $\sum_{i=0}^{2\alpha_{i1}-1} q^{2j} \neq 0$, thus $C_i = 0$ for $\alpha_{i1}, \alpha_{i2} > 0$, i.e., $f = \sum_{i=0}^{n} C_i z^{\alpha_{i3}}$. But $yf = fy$, so $\sum_{i=0}^{n} C_i y z^{\alpha_{i3}} = \sum_{i=0}^{n} C_i z^{\alpha_{i3}} y = \sum_{i=0}^{n} C_i y(q^4 z + (q^2+1))^{\alpha_{i3}}$, then $C_i = C_i q^{4\alpha_{i3}}$ and $\sum_{j=1}^{\alpha_{i3}}\binom{\alpha_{i3}}{j}q^{4(\alpha_{i3}-j)}(q^2+1)^j = 0$, whence $q^{4\alpha_{i3}} = 1$, thus $\alpha_{i3} = 0$ and therefore $f = C_0 \in k$. Finally, $Z(\mathcal{W}_q(\mathfrak{sl}(2,k))) \subseteq k \subseteq Z(\mathcal{W}_q(\mathfrak{sl}(2,k)))$. □

2.2.2 The algebra U

Let U be the algebra generated over the field $k = \mathbb{C}$ by the set of variables x_i, y_i, z_i, $1 \leq i \leq n$, subject to relations:

$$\begin{aligned}
&x_j x_i = x_i x_j,\ y_j y_i = y_i y_j,\ z_j z_i = z_i z_j,\ 1 \leq i,j \leq n,\\
&y_j x_i = q^{\delta_{ij}} x_i y_j,\ z_j x_i = q^{-\delta_{ij}} x_i z_j,\ 1 \leq i,j \leq n,\\
&z_j y_i = y_i z_j,\ i \neq j,\ z_i y_i = q^2 y_i z_i - q^2 x_i^2,\ 1 \leq i \leq n,
\end{aligned}$$

where $q \in \mathbb{C} - \{0\}$. Note that U is a bijective skew PBW extension of $\mathbb{C}[x_1, \ldots, x_n]$, that is, $U \cong \sigma(\mathbb{C}[x_1, \ldots, x_n])\langle y_1, \ldots, y_n; z_1, \ldots, z_n\rangle$.

Lemma 2.2.3. *For $m \geq 1$ and $i = 1, \ldots, n$, in U the following relations hold:*

1. $z_i y_i^m = q^{2m} y_i^m z_i - mq^2 x_i^2 y_i^{m-1}$,
2. $z_i^m y_i = q^{2m} y_i z_i^m - mq^{2m} x_i^2 z_i^{m-1}$,
3. $y_i^m x_i = q^m x_i y_i^m$,
4. $z_i^m x_i = q^{-m} x_i z_i^m$.

Theorem 2.2.4. *If q is not a root of unity, then the algebra U has a trivial center k. If q is a root of unity of degree $l \geq 2$, $x_1^l, \ldots, x_n^l$ are central elements in U and $k[x_1^l, \ldots, x_n^l] \subseteq Z(U)$.*

Proof. Let $f = \sum_{i=1}^t r_i x_1^{\gamma_{i1}} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} z_1^{\beta_{i1}} \cdots z_n^{\beta_{in}} \in Z(U)$, $\gamma_{ij}, \alpha_{ij}, \beta_{ij} \geq 0$, $j = 1, \ldots, n$. Then $x_1 f = f x_1$, and hence

$$\sum_{i=1}^t r_i x_1^{\gamma_{i1}+1} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} z_1^{\beta_{i1}} \cdots z_n^{\beta_{in}} = \sum_{i=1}^t r_i q^{\alpha_{i1}-\beta_{i1}} x_1^{\gamma_{i1}+1} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} z_1^{\beta_{i1}} \cdots z_n^{\beta_{in}}.$$

Thus, $1 - q^{\alpha_{i1}-\beta_{i1}} = 0$, so $\alpha_{i1} = \beta_{i1}$. Repeating this for $x_2, \ldots, x_n$ we conclude that $\alpha_{ij} = \beta_{ij}$, $j = 1, \ldots, n$. Therefore, $f = \sum_{i=1}^t r_i x_1^{\gamma_{i1}} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} \cdots z_n^{\alpha_{in}}$. But $f z_1 = z_1 f$, then

$$\sum_{i=1}^t r_i x_1^{\gamma_{i1}} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} z_1^{\alpha_{i1}+1} \cdots z_n^{\alpha_{in}} = \sum_{i=1}^t r_i q^{2\alpha_{i1}-\gamma_{i1}} x_1^{\gamma_{i1}} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} z_1^{\alpha_{i1}+1} \cdots z_n^{\alpha_{in}} + \sum_{i=1}^t \alpha_{i1} r_i q^{2\alpha_{i1}-\gamma_{i1}+2} x_1^{\gamma_{i1}+2} \cdots x_n^{\gamma_{in}} y_1^{\alpha_{i1}-1} \cdots y_n^{\alpha_{in}} z_1^{\alpha_{i1}} \cdots z_n^{\alpha_{in}},$$

so necessarily $1 - q^{2\alpha_{i1}-\gamma_{i1}} = 0$ and $\alpha_{i1} r_i q^{2\alpha_{i1}-\gamma_{i1}+2} = 0$, whence $2\alpha_{i1} = \gamma_{i1}$ and $\alpha_{i1} = 0 = \gamma_{i1}$, so $f = \sum_{i=1}^t r_i x_2^{\gamma_{i2}} \cdots x_n^{\gamma_{in}} y_2^{\alpha_{i2}} \cdots y_n^{\alpha_{in}} z_2^{\alpha_{i2}} \cdots z_n^{\alpha_{in}}$; repeating this for $z_2, \ldots, z_n$ then we obtain that $f = r \in k$. Therefore, $k \subseteq Z(U) \subseteq k$.

If $q^l = 1$, then $y_j x_i^l = x_i^l y_j$, $z_j x_i^l = x_i^l z_j$, and $y_i x_i^l = q^l x_i^l y_i = x_i^l y_i$, $z_i x_i^l = q^{-l} x_i^l z_i = x_i^l z_i$, for $1 \leq i \leq n$. Thus, $k[x_1^l, \ldots, x_n^l] \subseteq Z(U)$. □

2.2.3 Dispin algebra $\mathcal{U}(osp(1,2))$

It is generated by x, y, z over k satisfying the relations:

$$yx = xy - x, \qquad zx = -xz + y, \qquad zy = yz - z.$$

Then, $\mathcal{U}(osp(1,2))$ is a skew PBW extension of k, $\mathcal{U}(osp(1,2)) \cong \sigma(k)\langle x, y, z\rangle$. Using the defining relation of the algebra $\mathcal{U}(osp(1,2))$ we get the following preliminary result.

Lemma 2.2.5. *For $n, m \geq 1$,*

1. $yx^n = x^n y - nx^n$.

2. $y^n x = x(y-1)^n$.

3. $y^n x^m = x^m(y-m)^n$.

4. $z^n x = (-1)^n x z^n - \frac{n}{2} z^{n-1} + \left(\frac{1+(-1)^{n+1}}{2}\right) y z^{n-1}$

   4.1 *If* $n = 2l$, $z^{2l}x = xz^{2l} - lz^{2l-1}$.

   4.2 *If* $n = 2l+1$, $z^{2l+1}x = -xz^{2l+1} + yz^{2l} - nz^{2l}$.

5. $zy^n = (y-1)^n z$.

6. $z^n y = (y-n)z^n$.

7. $z^n y^m = (y-n)^m z^n$.

Theorem 2.2.6. *The element* $f = 4x^2z^2 - y^2 - 2xz - y$ *is central in* $\mathcal{U}(osp(1,2))$ *and* $k[f] \subseteq Z(\mathcal{U}(osp(1,2)))$.

Proof. Since $fx = 4x^2z^2x - y^2x - 2xzx - yx = 4x^2(xz^2 - z) - x(y-1)^2 - 2x(-xz+y) - (xy-x) = 4x^3z^2 - 4x^2z - xy^2 + 2xy - x + 2x^2z - 2xy - xy + x = 4x^3z^2 - xy^2 - 2x^2z - xy = xf$. $yf = 4yx^2z^2 - y^3 - 2yxz - y^2 = 4(x^2y - 2x^2)z^2 - y^3 - 2(xy-x)z - y^2 = 4x^2yz^2 - 8x^2z^2 - y^3 - 2xyz - 2xz - y^2$. Whereas $fy = 4x^2z^2y - y^3 - 2xzy - y^2 = 4x^2(yz^2 - 2z^2) - y^3 - 2x(yz - z) - y^2 = 4x^2yz^2 - 8x^2z^2 - y^3 - 2xyz - 2xz - y^2$. Thus, $yf = fy$. $zf = 4zx^2z^2 - zy^2 - 2zxz - zy = 4(x^2z - x)z^2 - (y^2z - 2yz + z) - 2(-xz+y)z - (yz - z) = 4x^2z^3 - 4xz^2 - y^2z + 2yz - z + 2xz^2 - 2yz - yz + z = 4x^2z^3 - y^2z - 2xz^2 - yz = fz$. Therefore, $f \in Z(\mathcal{U}(osp(1,2)))$. □

2.2.4 q-Heisenberg algebra $H_n(q)$

This algebra is generated by $3n$ variables, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $z_1, \ldots, z_n$ with relations:

$$\begin{array}{llll} x_jx_i = x_ix_j, & y_jy_i = y_iy_j, & z_jz_i = z_iz_j, & i \neq j. \\ y_jx_i = x_iy_j, & z_iy_j = y_jz_i, & z_jx_i = x_iz_j & 1 \leq i < j \leq n, \\ y_ix_i = qx_iy_i, & z_iy_i = qy_iz_i, & z_ix_i = q^{-1}x_iz_i + y_i, & \end{array}$$

with $q \in k - \{0\}$. Note that

$$H_n(q) \cong \sigma(k)\langle x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n\rangle \cong \sigma(k[y_1, \ldots, y_n])\langle x_1, \ldots, x_n; z_1, \ldots, z_n\rangle.$$

Theorem 2.2.7. *Let* $C_i = (q^2-1)x_iy_iz_i - y_i^2$, *for* $i = i, \ldots, n$. *Then*

1. *If* q *is not a root of unity, then the elements* C_i *are central,* $1 \leq i \leq n$, *and* $k[C_i]_{1\leq i\leq n} \subseteq Z(H_n(q))$.

2. *If* q *is a root of unity of degree* l, *the following elements are central,* C_i, x_i^l, y_i^l, z_i^l, $i = 1, \ldots, n$, *and* $k[C_i, x_i^l, y_i^l, z_i^l]_{1\leq i\leq n} \subseteq Z(H_n(q))$.

Proof. It is clear that $x_jC_i = C_ix_j$, $y_jC_i = C_iy_j$, $z_jC_i = C_iz_j$, for $i \neq j$. $C_ix_i = (q^2-1)x_iy_iz_ix_i - y_i^2x_i = (q^2-1)x_iy_i(q^{-1}x_1z_i+y_i) - y_iqx_iy_i = q^{-1}(q^2-1)x_i(qx_iy_i)z_i + (q^2-1)x_iy_i^2 - q^2x_iy_i^2 = (q^2-1)x_i^2y_iz_i - x_iy_i^2 = x_iC_i$. $z_iC_i = (q^2-1)z_ix_iy_iz_i - z_iy_i^2 = (q^2-1)(q^{-1}x_iz_i + y_i)y_iz_i - q^2y_i^2z_i = (q^2-1)q^{-1}x_i(qy_iz_i)z_i + (q^2-1)y_i^2z_i - q^2y_iz_i = (q^2-1)x_iy_iz_i^2 - y_i^2z_i = C_iz_i$, and $y_iC_i = (q^2-1)y_ix_iy_iz_i - y_i^3 = (q^2-1)qx_iy_i^2z_i - y_i^3$, whereas $C_iy_i = (q^2-1)x_iy_iz_iy_i - y_i^3 = (q^2-1)qx_iy_i^2z_i - y_i^3$, so $y_iC_i = C_iy_i$, and therefore, C_i, $1 \leq i \leq n$ are central elements in $H_n(q)$. □

2.2.5 The coordinate algebra of the quantum matrix space, $M_q(2)$.

This is the k-algebra generated by the variables x, y, u, v satisfying the relations:

$$xu = qux, \qquad yu = q^{-1}uy, \qquad vu = uv, \tag{2.9}$$

and

$$xv = qvx, \qquad vy = qyv, \qquad yx - xy = -(q-q^{-1})uv, \tag{2.10}$$

with $q \in k - \{0\}$. Thus, $\mathcal{O}(M_q(2)) \cong \sigma(k[u])\langle x, y, v\rangle$. Due to the last relation in (2.10), we remark that it is not possible to consider $\mathcal{O}(M_q(2))$ as a skew PBW extension of k. This algebra can be generalized to n variables, $\mathcal{O}_q(M_n(k))$, and coincides with the *coordinate algebra of the quantum group* $SL_q(2)$, see [BGV] for more details.

Lemma 2.2.8. *In $\mathcal{O}(M_q(2))$ the following relations hold:*

1. $yx^l = x^ly - (q-q^{-1})(q^{2(l-1)} + q^{2(l-2)} + \cdots + q^2 + 1)uvx^{l-1}$. *In particular, if q is a root of unity of degree $l \geq 3$, then $q^{2(l-1)} + q^{2(l-2)} + \cdots + q^2 + 1 = 0$ and $yx^l = x^ly$.*

2. $y^lx = xy^l - (q-q^{-1})(1 + q^{-2} + \cdots + q^{-2(l-2)} + q^{-2(l-1)})uvy^{l-1}$. *Where $1 + q^{-2} + \cdots + q^{-2(l-2)} + q^{-2(l-1)} = 0$ if q is a root of unity of degree l.*

With the previous relations it is possible proof the following theorem.

Theorem 2.2.9. *Let $A = M_q(2)$, then*

1. *$k[xy - quv] \subseteq Z(A)$ if q is not a root of unity.*

2. *$x^l, y^l, xy - quv, u^iv^j$, with $i + j = l$, are central elements if q is a root of unity of degree l, and $k[x^l, y^l, xy - quv, u^iv^j] \subseteq Z(A)$.*

Proof. Let $f = xy - quv$, then $fu = (xy - quv)u = xyu - quvu = xq^{-1}uy - qu^2v = q^-1quxy - qu^2v = uxy - qu^2v = uf$. $vf = v(xy - quv) = vxy - qvuv = vxy - quv^2$, and $fv = (xy-quv)v = xyv-quv^2 = xq^{-1}vy-quv^2 = q^{-1}qvxy-quv^2 = vxy-quv^2$, so $vf = fv$. Also, $xf = x^2y - qxuv = x^2y - q(qux)v = x^2y - q^3uvx$, $fx = xyx - quvx = x(xy - (q-q^{-1})uv) - quvx = x^2y - (q-q^{-1})xuv - quvx = x^2y -$

$(q-q^{-1})q^2uvx = x^2y - q^3uvx + quvx - quvx = x^2y - q^3uvx = xf$; $fy = xy^2 - quvy$, $yf = yxy - qyuv = xy^2 - (q-q^{-1})uvy - q^{-1}uvy = xy^2 - quvy + q^{-1}uvy - q^{-1}uvy = xy^2 - quvy = fy$.
For the second part it is clear from the previous relations that $yx^l = x^ly$ and $y^lx = xy^l$ if q is a root of unity of degree l. Also, $y^lu = q^{-l}uy^l = uy^l$, $y^lv = q^{-l}vy^l = vy^l$ and $x^lu = q^lux^l = ux^l$, $x^lv = q^lvx^l = vx^l$, thus x^l, y^l are central elements in A. For u^iv^j with $i+j = l$ we have, $xu^iv^j = q^iu^ixv^j = q^iq^ju^iv^jx = u^iv^jx$, $yu^iv^j = q^{-i}u^iyv^j = q^{-i}q^{-j}u^iv^jy = u^iv^jy$, so u^iv^j are central for $i+j = l$. □

2.2.6 Quadratic algebras in 3 variables

For quadratic algebras in 3 variables the relations are homogeneous of degree 2. More exactly, a quadratic algebra in 3 variables $\mathcal{A}$ is a k-algebra generated by x, y, z subject to the relations

$$\begin{aligned} yx &= xy + a_1xz + a_2y^2 + a_3yz + \xi_1z^2, \\ zx &= xz + \xi_2y^2 + a_5yz + a_6z^2, \\ zy &= yz + a_4z^2. \end{aligned}$$

If $a_1 = a_4 = 0$ we obtain the relations

$$\begin{aligned} yx &= xy + a_2y^2 + a_3yz + \xi_1z^2, \\ zx &= xz + \xi_2y^2 + a_5yz + a_6z^2, \\ zy &= yz. \end{aligned}$$

One can check that $\mathcal{A}_1 \cong \sigma(k[y,z])\langle x\rangle$. If $a_3 = a_5 = 0$, which implies $a_2 = a_6 = 0$, and thus, there is a family of algebras with relations

$$\begin{aligned} yx &= xy + a_1xz + \xi_1z^2, \\ zx &= xz, \\ zy &= yz + a_4z^2. \end{aligned}$$

These algebras are skew PBW extensions of the form $\mathcal{A}_2 \cong \sigma(k[x,z])\langle y\rangle$. If $a_1 = a_3 = \xi_1 = 0$, then $a_4 = a_5 = a_6 = 0$ and thus there is a family of algebras with relations

$$\begin{aligned} yx &= xy + a_2y^2, \\ zx &= xz + \xi_2y^2, \\ zy &= yz. \end{aligned}$$

These algebras are skew PBW extensions of the form $\mathcal{A}_3 \cong \sigma(k[x,y])\langle z\rangle$.

Lemma 2.2.10. *Assume that in the previous algebra $\mathcal{A}_2$, $a_1 = a$, $\xi_1 = b$ and $a_4 = c$. Then for $m \geq 1$*

1. $yx^m = x^my + max^mz + mbx^{m-1}z^2$.

2. $zy^m = \sum_{i=0}^{m} \dfrac{m!}{(m-i)!} c^i y^{m-i} z^{1+i}$.

3. $z^m y = yz^m + mcz^{m+1}$.

Theorem 2.2.11. *The quadratic algebra $\mathcal{A}_2$ generated by x, y, z with relations $yx = xy + axz + bz^2$, $zx = xz$, $zy = yz + cz^2$, $a, b, c \in k - \{0\}$ and $ac < 0$ has trivial center.*

Proof. Let $f(x,y,z) = \sum_{i=0}^{n} C_i x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}} \in Z(\mathcal{A}_2)$, with $\alpha_{ij} \in \mathbb{N}$, $j = 1,2,3$. Since $zf = fz$, then $\sum_{i=0}^{n} C_i x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}+1} = \sum_{i=0}^{n} C_i x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}+1} + \sum_{i=0}^{n} C_i \sum_{j=1}^{\alpha_{i2}} \frac{(\alpha_{i2})!}{(\alpha_{i2}-j)!} c^j x^{\alpha_{i1}} y^{\alpha_{i2}-j} z^{\alpha_{i3}+1-j}$, so $\sum_{i=0}^{n} C_i \sum_{j=1}^{\alpha_{i2}} \frac{(\alpha_{i2})!}{(\alpha_{i2}-j)!} c^j x^{\alpha_{i1}} y^{\alpha_{i2}-j} z^{\alpha_{i3}+1-j} = 0$, but $\text{char}(k) = 0$, then $C_i = 0$ for $\alpha_{i2} \neq 0$, i.e. $f = \sum_{i=0}^{n} C_i x^{\alpha_{i1}} z^{\alpha_{i3}}$. Also, $yf = fy$, this is $\sum_{i=0}^{n} C_i x^{\alpha_{i1}} yz^{\alpha_{i3}} + C_i \alpha_{i1} a x^{\alpha_{i1}} z^{\alpha_{i3}+1} + C_i \alpha_{i1} b x^{\alpha_{i1}-1} z^{\alpha_{i3}+2} = \sum_{i=0}^{n} C_i x_{i1}^{\alpha} yz^{\alpha_{i3}} + C_i \alpha_{i3} c x^{\alpha_{i1}} z^{\alpha_{i3}+1}$. Thus, $C_i(\alpha_{i1} a - \alpha_{i3} c) x^{\alpha_{i1}} z^{\alpha_{i3}+1} + C_i \alpha_{i1} b x^{\alpha_{i1}-1} z^{\alpha_{i3}+2} = 0$ for $i = 0, 1, \ldots, n$. Since $\alpha_{i1} a - \alpha_{i3} c \leq 0$ or $\alpha_{i1} a - \alpha_{i3} c \geq 0$, then $\alpha_{i1} = 0$, $\alpha_{i3} = 0$ and $f = c_0 \in k$, therefore $k \subseteq Z(\mathcal{A}_2) \subseteq k$. □

For $\mathcal{A}_3$ it is not difficult to prove the following result.

Theorem 2.2.12. *The element $\xi_2 y - a_2 z$ is central in the quadratic algebra $\mathcal{A}_3$.*

2.2.7 Algebra $\mathcal{D}$ of diffusion type

In [IPR] were introduced the *diffusion algebras*; following this notion we define for $n \geq 2$ the algebra $\mathcal{D}$ which is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over k with relations:

$$x_i x_j = x_j x_i, \quad x_i D_j = D_j x_i, \quad 1 \leq i, j \leq n;$$

$$c_{i,j} D_i D_j - c_{j,i} D_j D_i = x_j D_i - x_i D_j, \quad i < j,$$

$c_{i,j}, c_{j,i} \in k^*$. Observe that $A \cong \sigma(k[x_1, \ldots, x_n])\langle D_1, \ldots, D_n \rangle$ is a bijective skew PBW extension of $k[x_1, \ldots, x_n]$, and of course we obtain the following trivial result.

Theorem 2.2.13. $k[x_1, \ldots, x_n] \subseteq Z(\mathcal{D})$.

2.2.8 Tables

In this subsection we summarize in tables the description of the center of some remarkable examples of skew PBW extensions studied in the previous sections. Recall that k is a field with $\text{char}(k) = 0$.

Algebra	$Z(A)$
Weyl algebra $A_n(k)$	k
Extended Weyl algebra $B_n(k)$	k
Universal enveloping algebra of $\mathfrak{sl}(2,k)$	$k[4xy+z^2-2z]$
Universal enveloping algebra of $So(3,k)$	$k[x^2+y^2+z^2]$
Tensor product $R\otimes_k \mathcal{U}(\mathcal{G})$	$Z(R)\otimes_k Z(\mathcal{U}(\mathcal{G}))$
Algebra of q-differential operators $D_{q,h}[x,y]$	k
Algebra of shift operators S_h	k
Mixed algebra D_h	k
Discrete linear systems $k[t_1,\ldots,t_n][x_1,\sigma_1]\cdots[x_n;\sigma_n]$	k
Linear partial shift operators $k[t_1,\ldots,t_n][E_1,\ldots,E_n]$	k
L.P. Differential operators $k[t_1,\ldots,t_n][\partial_1,\ldots,\partial_n]$	k
L. P. Difference operators $k[t_1,\ldots,t_n][\Delta_1,\ldots,\Delta_n]$	k
L. P. q-dilation operators $k[t_1,\ldots,t_n][H_1^{(q)},\ldots,H_m^{(q)}]$	k
L. P. q-differential operators $k[t_1,\ldots,t_n][D_1^{(q)},\ldots,D_m^{(q)}]$	k
Additive analogue of the Weyl algebra $A_n(q_1,\ldots,q_n)$	k
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	k
$n-$multiparametric quantum space	k
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,k))$	$k[-q^{1/2}(q^2-1)I_1I_2I_3+q^2I_1^2+I_2^2+q^2I_3^2]$
Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,))$	k
Algebra $\mathbf{U}$	k
Quantum enveloping algebra of $\mathfrak{sl}(2,k)$, $\mathcal{U}_q(\mathfrak{sl}(2,k))$	$k[(q^2-1)^2EF+qK_e+q^3K_f]$
Differential operators on a quantum space $S_{\mathbf{q}}$, $D_{\mathbf{q}}(S_{\mathbf{q}})$	k
Particular Witten's Deformation of $\mathcal{U}(\mathfrak{sl}(2,k)$	k
Quantum Weyl algebra of Maltsiniotis $A_n^{\mathbf{q},\lambda}$	k
Multiparameter Weyl algebra $A_n^{Q,\Gamma}(k)$	k
Quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$	k
Jordan plane $\mathcal{J}$	k
Quantum plane	k
Quadratic algebras in 3 variables, $\mathcal{A}_2$	k

TABLE 2.1. Center of some bijective skew PBW extensions which parameters q's are not roots of unity.

Algebra	$Z(A)$
Quantum plane	$k[x^n,y^n]$
Algebra of q-differential operators $D_{q,h}[x,y]$	$k[x^n,y^n]$
L. P. q-dilation operators $k[t_1,\ldots,t_n][H_1^{(q)},\ldots,H_m^{(q)}]$	$k[t_1^l,\ldots,t_n^l,H_1^l,\ldots,H_m^l]$
L. P. q-differential operators $k[t_1,\ldots,t_n][D_1^{(q)},\ldots,D_m^{(q)}]$	$k[t_1^l,\ldots,t_n^l,D_1^l,\ldots,D_m^l]$
Additive analogue of the Weyl algebra $A_n(q_1,\ldots,q_n)$	$k[x_1^{\alpha_1},y_1^{\alpha_1},\ldots,x_n^{\alpha_n},y_n^{\alpha_n}]$
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	$k[x_1^{L_1},\ldots,x_n^{L_n}]$
$n-$multiparametric quantum space	$k[x_1^{L_1},\ldots,x_n^{L_n}]$
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,k))$	$k[C,C_n^{(1)},C_n^{(2)},C_n^{(3)}]$
Quantum enveloping algebra of $\mathfrak{sl}(2,k)$, $\mathcal{U}_q(\mathfrak{sl}(2,k))$	$k[(q^2-1)^2EF+qK_e+q^3K_f,E^n,F^n,K_e^n,K_f^n]$
Quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$	$k[x_1^m,\ldots,x_{2n}^m]$

TABLE 2.2. Center of some bijective skew PBW extensions which parameters q's are roots of unity.

Algebra	$\subseteq Z(A)$
Algebra $\mathcal{D}$ of diffusion type	$k[x_1,\cdots,x_n]$
Dispin algebra $\mathcal{U}(osp(1,2))$	$k[4x^2z^2-y^2-2xz-y]$
Algebra $\mathbf{U}$ $(q^l=1)$	$k[x_1^l,\cdots,x_n^l]$
Coordinate algebra of the quantum group $SL_q(2)$	$k[xy-quv]$
Coordinate algebra of the quantum group $SL_q(2)$, $q^l=1$	$k[xy-quv,x^l,y^l,u^iv^j]$, with $i+j=l$
q-Heisenberg algebra $\mathbf{H}_n(q)$	$k[C_i],C_i=(q^2-1)x_iy_iz_i-y_i^2$
q-Heisenberg algebra $\mathbf{H}_n(q)$, $q^l=1$	$k[x_i^l,y_i^l,z_i^l,C_i]$
Quadratic algebras in 3 variables, $\mathcal{A}_3$	$k[\xi_2y-a_2z]$

TABLE 2.3. Central subalgebras of some bijective skew PBW extensions.

According to the conditions presented in the Table 2.1 and Proposition 1.1.4, we have the following Corollary.

Corollary 2.2.14. *The following skew PBW extensions are universally cancellative, and hence, cancellative:*

1. *Weyl algebra $A_n(k)$.*

2. *Extended Weyl algebra* $B_n(k)$.
3. *Algebra of q-differential operators* $D_{q,h}[x,y]$.
4. *Algebra of shift operators* S_h.
5. *Mixed algebra* D_h.
6. *Discrete linear systems* $k[t_1,\dots,t_n][x_1,\sigma_1]\cdots[x_n;\sigma_n]$.
7. *Linear partial shift operators* $k[t_1,\dots,t_n][E_1,\dots,E_n]$.
8. *L.P. Differential operators* $k[t_1,\dots,t_n][\partial_1,\dots,\partial_n]$.
9. *L. P. Difference operators* $k[t_1,\dots,t_n][\Delta_1,\dots,\Delta_n]$.
10. *L. P. q-dilation operators* $k[t_1,\dots,t_n][H_1^{(q)},\dots,H_n^{(q)}]$.
11. *L. P. q-differential operators* $k[t_1,\dots,t_n][D_1^{(q)},\dots,D_n^{(q)}]$.
12. *Additive analogue of the Weyl algebra* $A_n(q_1,\dots,q_n)$.
13. *Multiplicative analogue of the Weyl algebra*$\mathcal{O}_n(\lambda_{ji})$.
14. *n-multiparametric quantum space.*
15. *Woronowicz algebra* $\mathcal{W}_\nu(\mathfrak{sl}(2,k))$.
16. *Algebra* $\boldsymbol{U}$.
17. *Differential operators on a quantum space* $S_{\boldsymbol{q}}$. $D_{\boldsymbol{q}}(S_{\boldsymbol{q}})$.
18. *Particular Witten's Deformation of* $\mathcal{U}(\mathfrak{sl}(2,k)$.
19. *Quantum Weyl algebra of Maltsiniotis.* $A_n^{\boldsymbol{q},\lambda}$.
20. *Multiparameter Weyl algebra* $A_n^{Q,\Gamma}(k)$.
21. *Quantum symplectic space* $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$.
22. *Jordan plane* $\mathcal{J}$.
23. *Quantum plane.*
24. *Quadratic algebras in 3 variables,* $\mathcal{A}_2$.

2.3 The center of the total ring of fractions

Given an Ore domain, it is interesting to know when $Z(Q(A)) \cong Q(Z(A))$, where $Q(A)$ is the total ring of fractions of A. This question became important after the formulation of the Gelfand-Kirillov conjecture in the paper [GK]: *Let $\mathcal{G}$ be an algebraic Lie algebra of finite dimension over a field k, with $char(k) = 0$. Then, there exist integers $n, l \geq 1$ such that*

$$Q(\mathcal{U}(\mathcal{G})) \cong Q(A_n(k[s_1, \ldots, s_l])),$$

where $\mathcal{U}(\mathcal{G})$ is the enveloping algebra of $\mathcal{G}$ and $A_n(k[s_1, \ldots, s_l])$ is the general Weyl algebra over k. In the investigation of this famous conjecture the isomorphism between the center of the total ring of fractions and the total ring of fractions of the center occupies a special key role. In this section we prove that if A is finitely generated and $\text{GKdim}(A) < \text{GKdim}(Z(A)) + 1$, then $Z(Q_r(A)) \cong Q(Z(A))$. Some examples are included.

We start with the following easy proposition.

Proposition 2.3.1. *Let A be a right Ore domain.*

(i) *If $\frac{p}{q} \in Z(Q_r(A))$ then*

  (a) *$pq = qp$.*

  (b) *For every $s \in A - \{0\}$, $psq = qsp$.*

  (c) *$p \in Z(A)$ if and only if $q \in Z(A)$.*

(ii) *Let $p \in A$. Then, $\frac{p}{1} \in Z(Q_r(A))$ if and only if $p \in Z(A)$. Thus, $Z(A) \hookrightarrow Z(Q_r(A))$.*

(iii) *If k is a field and A is a k-algebra such that $Z(Q_r(A)) = k$, then $Q(Z(A)) = k = Z(Q_r(A))$.*

Proof. (i) (a) We have $\frac{p}{q}\frac{q}{1} = \frac{q}{1}\frac{p}{q}$, so $\frac{p}{1} = \frac{qp}{q}$, whence $\frac{p}{1}\frac{q}{1} = \frac{qp}{q}\frac{q}{1}$, i.e., $\frac{pq}{1} = \frac{qp}{1}$, thus $pq = qp$.

(b) Let $s \in A - \{0\}$, then $\frac{p}{q} = \frac{ps}{qs}$, so by (a), $psqs = qsps$, whence $psq = qsp$.

(c) If $\frac{p}{q} = 0$, then $\frac{p}{q} = \frac{0}{1}$ and the claimed trivially holds. We can assume that $p \neq 0$; by (b), for every $s \in A - \{0\}$, $psq = qsp$, hence if $p \in Z(A)$, then $sqp = qsp$, whence $sq = qs$, i.e., $q \in Z(A)$. On the other hand, since $\frac{q}{p} \in Z(Q_r(A))$, then if $q \in Z(A)$ we get $p \in Z(A)$.

(ii) If $\frac{p}{1} \in Z(Q_r(A))$, then by (i), $ps = sp$ for every $s \neq 0$, whence $p \in Z(A)$. Conversely, let $p \in Z(A) - \{0\}$ (for $p = 0$, $\frac{p}{1} \in Z(Q_r(A))$), then for every $\frac{a}{s} \in Q_r(A)$ we have $\frac{p}{1}\frac{a}{s} = \frac{pa}{s}$ and $\frac{a}{s}\frac{p}{1} = \frac{ac}{r} = \frac{acp}{rp}$, where $sc = pr = rp$, with $c, r \neq 0$. From this we get $\frac{acp}{rp} = \frac{apc}{sc} = \frac{ap}{s} = \frac{pa}{s}$, i.e., $\frac{p}{1} \in Z(Q_r(A))$.

(iii) From (ii), $k \subseteq Z(A) \subseteq Z(Q_r(A)) = k$, so $Z(A) = k$, and hence $Q(Z(A)) = k = Z(Q_r(A))$. □

The next example illustrates the part (iii) of Proposition 2.3.1.

Example 2.3.2. We consider the quantum plane $A := k_q[x_1, x_2]$, where q is not a root of unity. We will show that $Z(Q_r(A)) = k$. Let $\frac{p}{s} \in Z(Q_r(A)) - \{0\}$, where $p := \sum_{i=1}^{t} r_i x_1^{\alpha_i} x_2^{\beta_i}$ and $s := \sum_{j=1}^{l} u_j x_1^{\theta_j} x_2^{\gamma_j}$, with $r_i, u_j \in k - \{0\}$. From $px_1s = sx_1p$ and since q is not a root of unity, we get $\beta_i + \beta_i\theta_j = \gamma_j + \gamma_j\alpha_i$ for every $1 \leq i \leq t$ and $1 \leq j \leq l$. Similarly, from $px_2s = sx_2p$ we obtain $\theta_j\beta_i + \theta_j = \alpha_i\gamma_j + \alpha_i$ for all i, j, whence $\beta_i + \alpha_i = \gamma_j + \theta_j$, so fixing i and then fixing j we conclude that p and s are homogeneous of the same degree (this condition is not enough since $\frac{x_1}{x_2} \notin Z(Q_r(A))$). Now,

$$\tfrac{p}{s}\tfrac{x_1}{1} = \tfrac{x_1}{1}\tfrac{p}{s}, \text{ i.e., } \frac{\sum_{i=1}^{t} r_i x_1^{\alpha_i} x_2^{\beta_i}}{\sum_{j=1}^{l} u_j x_1^{\theta_j - 1} x_2^{\gamma_j}} = \frac{\sum_{i=1}^{t} r_i x_1^{\alpha_i + 1} x_2^{\beta_i}}{\sum_{j=1}^{l} u_j x_1^{\theta_j} x_2^{\gamma_j}},$$

hence there exist $c := x_1^m p_m(x_2) + \cdots + p_0(x_2), d := x_1^k q_k(x_2) + \cdots + q_0(x_2) \in A - \{0\}$ such that

$$(\textstyle\sum_{i=1}^{t} r_i x_1^{\alpha_i} x_2^{\beta_i})c = (\sum_{i=1}^{t} r_i x_1^{\alpha_i + 1} x_2^{\beta_i})d,$$
$$(\textstyle\sum_{j=1}^{l} u_j x_1^{\theta_j - 1} x_2^{\gamma_j})c = (\sum_{j=1}^{l} u_j x_1^{\theta_j} x_2^{\gamma_j})d.$$

Since p and q are homogeneous, we can assume $\alpha_1 > \cdots > \alpha_t$ and $\theta_1 > \cdots > \theta_l$, whence $\beta_1 < \cdots < \beta_t$ and $\gamma_1 < \cdots < \gamma_l$. Then,

$$(r_1 x_1^{\alpha_1} x_2^{\beta_1})(x_1^m p_m(x_2)) = r_1 q^{m\beta_1} x_1^{\alpha_1 + m} x_2^{\beta_1} p_m(x_2),$$
$$r_1 x_1^{\alpha_1 + 1} x_2^{\beta_1} x_1^k q_k(x_2) = r_1 q^{k\beta_1} x_1^{\alpha_1 + 1 + k} x_2^{\beta_1} q_k(x_2),$$

whence $\alpha_1 + m = \alpha_1 + 1 + k$, i.e., $m = k + 1$. Moreover, let p_m be the leader coefficient of $p_m(x_2)$ and q_k be the leader coefficient of $q_k(x_2)$, then $q^{\beta_1} p_m = q_k$. Similarly, we can prove that $q^{\gamma_1} p_m = q_k$, but since q is not a root of unity, $\beta_1 = \gamma_1$. From $\alpha_1 + \beta_1 = \theta_1 + \gamma_1$ we get that $\alpha_1 = \theta_1$ (considering instead the identity $\frac{p}{s}\frac{x_2}{1} = \frac{x_2}{1}\frac{p}{s}$ we obtain the same result). Thus, we have

$$\alpha_1 = \theta_1 \text{ and } \beta_1 = \gamma_1.$$

Notice that

$$\tfrac{p}{s} = r_1 u_1^{-1} + \tfrac{p - r_1 u_1^{-1} s}{s}, \text{ with } r_1 u_1^{-1} \in k \subseteq Z(Q_r(A)),$$

hence $\frac{p - r_1 u_1^{-1} s}{s} \in Z(Q_r(A))$. But observe that $\frac{p - r_1 u_1^{-1} s}{s} = 0$, contrary, we could repeat the previous procedure and find that there exists, either $i \geq 2$ such that $\alpha_i = \theta_1 = \alpha_1, \beta_i = \gamma_1 = \beta_1$, or $j \geq 2$ such that $\theta_j = \theta_1, \gamma_j = \gamma_1$, a contradiction. Thus, $\frac{p}{s} = r_1 u_1^{-1} \in k$, and hence, $Z(Q_r(A)) = k$.

The previous example shows that the proof of the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$ by direct computation of the center of the total ring of fractions is tedious. An alternative more practical method is given by the following theorem.

Theorem 2.3.3. *Let k be a field and A be a right Ore domain. If A is a finitely generated k-algebra such that* $\operatorname{GKdim}(A) < \operatorname{GKdim}(Z(A)) + 1$*, then*

$$Z(Q_r(A)) = \{\tfrac{p}{q} \mid p, q \in Z(A), q \neq 0\} \cong Q(Z(A)).$$

Proof. We divide the proof in three steps.

Step 1. As in the proof of Theorem 4.12 in [Kr], we will show that

$$Q_r(A) \cong A(Z(A)_0)^{-1} = \{\tfrac{p}{q} \mid p \in A, q \in Z(A)_0\}, \text{ with } Z(A)_0 := Z(A) - \{0\}.$$

First observe that $Z(A)_0$ is a right Ore set of A, so $A(Z(A)_0)^{-1}$ exists. From the canonical injection $Q(Z(A)) \hookrightarrow A(Z(A)_0)^{-1}$, $\frac{p}{q} \mapsto \frac{p}{q}$, we get that $A(Z(A)_0)^{-1}$ is a vector space over $Q(Z(A))$, moreover, $A(Z(A)_0)^{-1} = AQ(Z(A))$. We will show that the dimension of this vector space is finite. Let V be a frame that generates A. Since $\{V^n\}_{n\geq 0}$ is a filtration of A, then $A = \bigcup_{n\geq 0} V^n$ and $AQ(Z(A)) = \bigcup_{n\geq 0} V^n Q(Z(A))$. Arise two possibilities: Either there exists $n \geq 0$ such that $V^n Q(Z(A)) = V^{n+1}Q(Z(A))$, or else

$$Q(Z(A)) \subsetneq VQ(Z(A)) \subsetneq V^2Q(Z(A)) \subsetneq \cdots$$

In the first case $AQ(Z(A)) = V^nQ(Z(A))$ and we get the claimed. In the second case,

$$\dim_{Q(Z(A))} Q(Z(A)) \lneq \dim_{Q(Z(A))} VQ(Z(A)) \lneq \dim_{Q(Z(A))} V^2Q(Z(A)) \lneq \cdots$$

and we will show that this produces a contradiction. In fact, for every $n \geq 0$,

$$\dim_{Q(Z(A))} V^nQ(Z(A)) \geq n + 1;$$

let $u_1, \ldots, u_{d(n)}$ be a $Q(Z(A))$-basis of $V^nQ(Z(A))$, thus, $d(n) \geq n + 1$; we can assume that $u_i \in V^n$ for every $1 \leq i \leq d(n)$; let W be an arbitrary k-subspace of $Z(A)$ of finite dimension, then

$$(V + W)^{2n} \supseteq V^nW^n \supseteq u_1W^n \oplus \cdots \oplus u_{d(n)}W^n$$

(the sum is direct since the elements u_i are linearly independent over $Z(A)$); from this we get

$$\dim_K(V + W)^{2n} \geq d(n) \dim_k(W^n) \geq (n + 1) \dim_K(W^n),$$

but since $V + W$ is a frame of A, then $\operatorname{GKdim}(A) \geq 1 + \operatorname{GKdim}(Z(A))$, false.

Now we can prove the claimed isomorphism. For this consider the canonical injective homomorphism $g : A \to A(Z(A)_0)^{-1}$, $a \mapsto \frac{a}{1}$. If $a \in A - \{0\}$, then $\frac{a}{1}$ is invertible in $A(Z(A)_0)^{-1}$. In fact, the map $h : A(Z(A)_0)^{-1} \to A(Z(A)_0)^{-1}$, $\frac{p}{q} \mapsto \frac{a}{1}\frac{p}{q}$, is an injective $Q(Z(A))$-homomorphism since A is a domain, but as was observed above, $A(Z(A)_0)^{-1}$ is finite-dimensional over $Q(Z(A))$, therefore h is surjective, whence, there exists $\frac{p}{q} \in A(Z(A)_0)^{-1}$ such that $\frac{a}{1}\frac{p}{q} = \frac{1}{1}$. Observe that $\frac{p}{q}\frac{a}{1} = \frac{1}{1}$: In

fact, since $p \neq 0$, there exists $\frac{p'}{q'}$ in $A(Z(A)_0)^{-1}$ such that $\frac{p}{1}\frac{p'}{q'} = \frac{1}{1}$, and since $\frac{p}{q} = \frac{1}{q}\frac{p}{1}$, then

$$\frac{a}{1}\frac{1}{q}\frac{p}{1}\frac{p'}{q'} = \frac{1}{1}\frac{p'}{q'}, \text{ i.e., } \frac{a}{1}\frac{1}{q} = \frac{p'}{q'}, \text{ so } \frac{a}{1}\frac{1}{q}\frac{q}{1} = \frac{p'}{q'}\frac{q}{1}, \text{ whence } \frac{a}{1} = \frac{p'}{q'}\frac{q}{1}, \text{ so}$$
$$\frac{p}{q}\frac{a}{1} = \frac{p}{q}\frac{p'}{q'}\frac{q}{1} = \frac{1}{q}\frac{p}{1}\frac{p'}{q'}\frac{q}{1} = \frac{1}{q}\frac{q}{1} = \frac{1}{1}.$$

In order to conclude the proof of the isomorphism $A(Z(A)_0)^{-1} \cong Q_r(A)$, observe that any element $\frac{p}{q} \in A(Z(A)_0)^{-1}$ can be written as $\frac{p}{q} = g(p)g(q)^{-1}$.

Step 2. Let $C := \{\frac{p}{q} \mid p, q \in Z(A), q \neq 0\}$. If $\frac{p}{q} \in Z(Q_r(A))$, then by the first step we can assume that $q \in Z(A)_0$, and from the part (i)-(c) of Proposition 2.3.1, we get $p \in Z(A)$. Therefore, $Z(Q_r(A)) \subseteq C$. Conversely, let $\frac{p}{q} \in C$, then $p, q \in Z(A)$, with $q \neq 0$, whence, by the part (ii) of Proposition 2.3.1, $\frac{p}{1}, \frac{q}{1} \in Z(Q_r(A))$, so $\frac{1}{q} \in Z(Q_r(A))$, and hence, $\frac{p}{q} = \frac{p}{1}\frac{1}{q} \in Z(Q_r(A))$. Thus, $C \subseteq Z(Q_r(A))$.

Step 3. According to the part (ii) of Proposition 2.3.1, we have the canonical injective homomorphism $Z(A) \xrightarrow{\iota} Z(Q_r(A))$, $p \mapsto \frac{p}{1}$, that sends invertible elements of $Z(A)$ in invertible elements of $Z(Q_r(A))$, moreover, by the step 2, every element $\frac{p}{q} \in Z(Q_r(A))$ can be written $\frac{p}{q} = \iota(p)\iota(q)^{-1}$. This proves the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$. □

Example 2.3.4. Next we present some k-algebras that satisfy the hypotheses of Theorem 2.3.3.

1. Any commutative finitely generated domain. In particular, k; $k[x_1, \ldots, x_n]/P$, P prime ideal; $k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$, $1 \leq r \leq n$.

2. Any domain A such that $\dim_k A < \infty$. For example, the real algebra $\mathbb{H}$ of quaternions since $\dim_{\mathbb{R}}(\mathbb{H}) = 4$.

3. Consider a skew polynomial ring $A := R[x; \sigma]$ with R a commutative domain that is a k-algebra generated by a subspace V of finite dimension such that $\sigma(V) \subseteq V$, σ is k-linear of finite order m, $R^\sigma = k$ and $\operatorname{GKdim}(R) = 0$. Then, $\operatorname{GKdim}(A) = 1$, and $Z(A) = k[x^m]$, and hence, $\operatorname{GKdim}(Z(A)) = 1$. Thus, $Z(Q_r(A)) \cong Q(Z(A)) = k(x^m)$. A particular case of this general example is $A := \mathbb{C}[x; \sigma]$ as $\mathbb{R}$-algebra, with $\sigma(z) := \overline{z}$, $z \in \mathbb{C}$ (here $\mathbb{C}$ and $\mathbb{R}$ are the fields of complex and real numbers, respectively). In this case the order of σ is two and $\operatorname{GKdim}(\mathbb{C}) = 0$ since $\dim_{\mathbb{R}}(\mathbb{C}) = 2$.

4. Let $char(k) = p > 0$ and $A := S_h := k[t][x_h; \sigma_h]$ be the algebra of shift operators, $\operatorname{GKdim}(A) = 2$. Moreover, for every $n \geq 0$, $\sigma_h^n(t) = t - nh$, then $\sigma_h^p(t) = t$, i.e., the order of σ_h is p, therefore, $Z(A) = k[t]^{\sigma_h}[x_h^p]$. Note that $k[t^p] \subseteq k[t]^{\sigma_h} \subseteq k[t]$, and $k[t]$ is finitely generated over $k[t^p]$, then $\operatorname{GKdim} k[t^p] = \operatorname{GKdim} k[t] = 1$, then $\operatorname{GKdim} k[t]^{\sigma_h} = 1$, so $\operatorname{GKdim}(Z(A)) = 2$. Thus, $Z(Q_r(A)) \cong Q(Z(A)) = Q(k[t]^{\sigma_h})(x_h^p)$.

5. Let $char(k) = p > 0$ and $A := k[t][x; \frac{d}{dt}][x_h; \sigma_h]$ be the algebra of shift differential operators. Then $\operatorname{GKdim}(A) = 3$ and $Z(A) = k[x^p, x_h^p, t^{p^2} - t^p]$. Since $\operatorname{GKdim}(Z(A)) = 3$, then $Z(Q_r(A)) \cong Q(Z(A)) = k(x^p, x_h^p, t^{p^2} - t^p)$.

6. Let $char(k) = p > 0$ and $A := A_n(k)$ be the Weyl algebra. Since $\text{GKdim}(A) = 2n$ and $Z(A) = k[t_1^p, \ldots, t_n^p, x_1^p, \ldots, x_n^p]$ (see [Lev], ejemplo 1.3.), then $\text{GKdim}(Z(A)) = 2n$. Therefore, $Z(Q_r(A)) \cong Q(Z(A)) = k(t_1^p, \ldots, t_n^p, x_1^p, \ldots, x_n^p)$.

7. Let $char(k) = p > 0$ and $A := \mathcal{J} := k\{x, y\}/\langle yx - xy - x^2\rangle$ be the Jordan algebra. Since $\text{GKdim}(\mathcal{J}) = 2$ and $Z(A) = k[x^p, y^p]$ (Example 1.3.11), then $\text{GKdim}(Z(A)) = 2$, whence $Z(Q_r(A)) \cong Q(Z(A)) = k(x^p, y^p)$.

8. Consider the n-multiparametric quantum space $k_{\mathbf{q}}[x_1, \ldots, x_n]$, with matrix of parameters $\mathbf{q} = [q_{ij}]$ and q_{ij} are roots of unity that satisfy the conditions of Example 1.3.10-(f), then $Z(A) = k[x_1^{L_1}, \ldots, x_n^{L_n}]$. Therefore, $\text{GKdim}(Z(A)) = n = \text{GKdim}(k_{\mathbf{q}}[x_1, \ldots, x_n])$ and hence

$$Z(Q_r(A)) \cong Q(Z(A)) = k(x_1^{L_1}, \ldots, x_n^{L_n}).$$

   The particular case when $n = 2$ and q is a root of unity of degree $m \geq 2$, then $Z(Q_r(A)) \cong Q(Z(A)) = k(x_1^m, x_2^m)$.

9. By example 2.1.9, we have the following algebras when the parameters q's are root of unity of degree $l \geq 2$, or $l_i \geq 2$, appropriately:

   (a) Algebra of q-differential operators, then $Z(A) = k[x^l, y^l]$ and $\text{GKdim}(A) = 2$, so $Z(Q_r(A)) \cong Q(Z(A)) = k(x^l, y^l)$.

   (b) Additive analogue of the Weyl algebra, $Z(A) = k[x_1^{l_1}, \ldots, x_n^{l_n}, y_1^{l_1}, \ldots, y_n^{l_n}]$ and $\text{GKdim}(A) = 2n$, so $Z(Q_r(A)) \cong Q(Z(A)) = k(x_1^{l_1}, \ldots, x_n^{l_n}, y_1^{l_1}, \ldots, y_n^{l_n})$.

   (c) Algebra of linear partial q-dilation operators, in this case we have $\text{GKdim}(A) = 2n$ and $Z(A) = k[t_1^l, \ldots, t_n^l, H_1^l, \ldots, H_n^l]$. Therefore, $Z(Q_r(A)) \cong Q(Z(A)) = k(t_1^l, \ldots, t_n^l, H_1^l, \ldots, H_n^l)$.

   (d) Algebra of linear partial q-differential operators, in this case we have $\text{GKdim}(A) = 2n$ and $Z(A) = k[t_1^l, \ldots, t_n^l, D_1^l, \ldots, D_n^l]$. Hence, $Z(Q_r(A)) \cong Q(Z(A)) = k(t_1^l, \ldots, t_n^l, D_1^l, \ldots, D_n^l)$.

10. Let $\mathfrak{sl}(2, k)$ be the Lie algebra of 2×2 matrices with null trace. By example 1.3.8-(c), if $char(k) = 2$, then $Z(\mathcal{U}(\mathfrak{sl}(2, k))) = k[x^2, y^2, z]$. Moreover, $\text{GKdim}(\mathcal{U}(\mathfrak{sl}(2, k))) = 3$. Thus, $Z(Q_r(A)) \cong Q(Z(A)) = k(x^2, y^2, z)$.

Remark 2.3.5. If the hypotheses of Theorem 2.3.3 fail, then the isomorphism $Z(Q_r(A)) \cong Q(Z(A))$ could hold or fail. Thus, the hypotheses are not necessary conditions. For example,

(a) $\mathbb{H}$ is not finitely generated as $\mathbb{Q}$-algebra, however $Q_r(\mathbb{H}) = \mathbb{H}$ and $Z(Q_r(\mathbb{H})) \cong \mathbb{R} \cong Q(Z(\mathbb{H}))$.

(b) Let k be a field with $char(k) = 0$, and let $\mathcal{G}$ be a three-dimensional completely solvable Lie algebra with basis x, y, z such that $[y, x] = y$, $[z, x] = \lambda z$ and $[y, z] = 0$, $\lambda \in k - \{0\}$ (Example 1.3.8-(c)). If $\lambda \in k - \mathbb{Q}$, then $Z(U(\mathcal{G})) = k$

and $\mathrm{GKdim}(\mathcal{U}(\mathcal{G})) = 3$, thus, in this case $\mathrm{GKdim}(\mathcal{U}(\mathcal{G})) \not< \mathrm{GKdim}(Z(\mathcal{U}(\mathcal{G})))+1$ and $\mathrm{GKdim}(\mathcal{U}(\mathcal{G})) \neq \mathrm{GKdim}(Z(\mathcal{U}(\mathcal{G})))$, and 14.4.7 in [MR] shows that $Z(Q_r(\mathcal{U}(\mathcal{G}))) \ncong Q(Z(\mathcal{U}(\mathcal{G})))$.

(c) Let $A := U_q^+(sl_m)$ be the quantum enveloping algebra of the Lie algebra of strictly superior triangular matrices of size $m \times m$ over a field k, where $q \in k - \{0\}$ is not a root of unity. In [AD] was proved that $Z(A)$ is the classical polynomial algebra over k in n variables, with $m = 2n$ or $m = 2n+1$, whence, $\mathrm{GKdim}(Z(A)) = n$. On the other hand, according to [AD], A is an iterated skew polynomial ring of k of $\frac{m(m-1)}{2}$ variables, hence $\mathrm{GKdim}(A) = \frac{m(m-1)}{2}$. Thus, $\mathrm{GKdim}(A) \not< \mathrm{GKdim}(Z(A)) + 1$, however, $Z(Q_r(A)) \cong Q(Z(A))$.

CHAPTER 3

Cancellation for algebras of Gelfand Kirillov dimension one

In this third chapter we consider cancellation for finitely generated (not-necessarily commutative) domains of Gelfand-Kirillov dimension one when the characteristic of the base field is zero or positive. We settled a non-commutative analogues of the result of Abhyankar, Eakin, and Heinzer (Theorem 1.1.20). Their theorem, when one works in the category of commutative algebras, says that if A is a finitely generated algebra that is an integral domain of Krull dimension one, then A is strongly cancellative.

As we saw in Section 1.1.1, the Makar-Limanov invariants help to treat with cancellation problem. Then, in the first section we introduce a new invariant when we work with weak locally nilpotent Hasse-Schmidt derivations. And, we generalize Theorem 1.1.8 and Lemma 1.1.13 in Proposition 3.1.4 when one works with affine prime left Goldie k-algebras of finite Gelfand-Kirillov dimension. Moreover, in Proposition 3.1.6 we prove a result that is related to a conjecture of Makar-Limanov. In the second section, we will introduce a noncommutative version of slice theorem for proving the main result of the chapter, Theorem 3.2.4. In addition, Theorem 3.2.6 provide examples that show affine domains of Gelfand-Kirillov dimension one need not be cancellative when the base field has positive characteristic, giving a counterexample to a conjecture of Tang, the named author, and Zhang. Finally, in the third section we prove a result in a different direction; namely, *skew cancellation*: that is, if $R[x;\sigma,\delta] \cong S[x;\sigma',\delta']$ when do we necessarily have $R \cong S$? We show that this holds in the following two cases: skew polynomial extensions of automorphisms type and derivations type when the coefficient ring R is an affine commutative domain of Krull dimension one. Results of this chapter will appear in [BHHV].

3.1 Weak rigidity

This section is analogous to section 1.1.1 ([BZ1, Section 3]), we introduce a weak locally nilpotent Hasse-Schmidt derivation modifying the condition in Definition 1.1.5-(4) and proving that weak LND-rigidity controls cancellation.

Definition 3.1.1. A Hasse-Schmidt derivation $\partial := (\partial_n)$ is called *weakly locally nilpotent Hasse-Schmidt derivation* if for each $a \in A$ there exists an integer $N = N(a) \geq 0$ such that $\partial_n(a) = 0$ for all $n \geq N$. The collection of weakly locally nilpotent Hasse-Schmidt derivations of an algebra A is denoted $\mathrm{LND}^{H'}(A)$.

The following remark is similar to 1.1.6.

Remark 3.1.2. Let k be a field and let A be a k-algebra.

1. If $\partial := (\partial_n)$ is a weakly locally nilpotent Hasse-Schmidt derivation of A then the map $G_{\partial,t} : A[t] \to A[t]$ defined by

$$a \mapsto \sum_{i=0}^{\infty} \partial_i(a)t^i, \text{ for all } a \in A, t \mapsto t \tag{3.1}$$

   extends to a k-algebra injective endomorphism of $A[t]$.

2. Conversely, if one has a k-algebra endomorphism $G : A[t] \to A[t]$ such that $G(t) = t$ and $G(a) - a \in tA[t]$ for $a \in A$, then for $a \in A$ we have

$$G(a) = \sum_{i=0}^{\infty} \partial_i(a)t^i,$$

   and (∂_n) is a weakly locally nilpotent Hasse-Schmidt derivation of A (see [BZ1, Lemma 2.2 (3)]).

3. As in Remark 1.1.6-(5), $\{\Delta_i^n\}_{n=0}^{\infty}$ is a weak locally nilpotent Hasse-Schmidt derivation, with Δ_i^n differential operator. So we can see

$$\mathrm{ML}^*(A[t_1, \ldots, t_d]) \subseteq \mathrm{ML}^*(A),$$

   where $*$ is either I, H, or H'.

We begin by proving a lemma, which is the counterpart of Lemma 1.1.7.

Lemma 3.1.3. *Let $Y := \bigoplus_{i=0}^{\infty} Y_i$ be an $\mathbb{N}$-graded k-algebra and suppose that $Y_0 y Y_0$ contains a regular element whenever y is a nonzero homogeneous element of Y. If Z is a subalgebra of Y containing Y_0 such that $\mathrm{GKdim}(Z) = \mathrm{GKdim}(Y_0) < \infty$, then $Z = Y_0$.*

Proof. Suppose that Z strictly contains Y_0 as a subalgebra. Since Y is a graded algebra, Z is an $\mathbb{N}$-filtered algebra with $F_0 Z = Y_0$. By [KL, Lemma 6.5],

$\mathrm{GKdim}(Z) \geq \mathrm{GKdim}(\mathrm{gr}(Z))$, where $\mathrm{gr}(Z)$ is the associated graded ring of Z with respect to the filtration induced by the $\mathbb{N}$-grading on Y. Then $\mathrm{gr}(Z)$ is an $\mathbb{N}$-graded subalgebra of Y that strictly contains Y_0 as the degree zero part, and so $\mathrm{gr}(Z)$ contains some nonzero homogeneous element $y \in Y_d$ for some $d \geq 1$. Then it contains the Y_0-Y_0-bimodule $Y_0 y Y_0 \subseteq Y_d$. In particular, there is some regular homogeneous element $a \in Z$ of positive degree and so by considering the grading we have

$$Y_0 + Y_0 a + \cdots$$

is direct and is contained in $\mathrm{gr}(Z)$. From this one can easily show that

$$\mathrm{GKdim}(\mathrm{gr}(Z)) \geq \mathrm{GKdim}((\mathrm{gr}(Z))_0) + 1 \geq \mathrm{GKdim}(Y_0) + 1.$$

Combining these inequalities gives

$$\mathrm{GKdim}(Z) \geq \mathrm{GKdim}(Y_0) + 1,$$

a contradiction. Thus $Z = Y_0$. □

We will use Lemma 3.1.3 in the case when A is a prime left Goldie algebra and $Y = A[t_1, \ldots, t_d]$, where we declare that elements of A have degree 0, and $t_1, \ldots, t_d$ are homogeneous of degree 1. Observe that if $p(t_1, \ldots, t_d)$ is a nonzero homogeneous polynomial of degree m in Y, then we can put a degree lexicographic order on the monomials in $t_1, \ldots, t_d$ by declaring that $t_1 > t_2 > \cdots > t_d$. Then we let $t_1^{i_1} \cdots t_d^{i_d}$ denote the degree lexicographically largest monomial that occurs in $p(t_1, \ldots, t_d)$ with nonzero coefficient and we let $a \in A$ denote this coefficient. Then since $A = Y_0$ is prime Goldie, $Y_0 a Y_0$ contains a regular element, and so $Y_0 p(t_1, \ldots, t_d) Y_0$ contains a nonzero homogeneous polynomial $q = q(t_1, \ldots, t_d)$ such that the degree lexicographically largest monomial that occurs in $q(t_1, \ldots, t_d)$ with nonzero coefficient has the property that this coefficient is regular; moreover, this monomial is again $t_1^{i_1} \cdots t_d^{i_d}$, and we let $c \in A$ denote this coefficient. We now claim that q must be regular. To see this, let h be a nonzero polynomial in Y. Then let $t_1^{j_1} \cdots t_d^{j_d}$ denote the degree lexicographically largest monomial that occurs in h with nonzero coefficient, and let $b \in A$ denote this coefficient. Then by construction the coefficient of $t_1^{i_1+j_1} \cdots t_d^{i_d+j_d}$ in qh is cb and since b is nonzero and c is regular, $qh \neq 0$; similarly, $hq \neq 0$ and so q is regular. In particular, Y satisfies the hypotheses of Lemma 3.1.3, in this case, which we will now apply in the following proposition that is more general that Theorem 1.1.8 and Lemma 1.1.13.

Proposition 3.1.4. *Let A be an affine prime left Goldie k-algebra of finite Gelfand-Kirillov dimension. Let $*$ be either blank, H, H' or I. When $*$ is blank we further assume k has characteristic zero.*

1. *If $\mathrm{ML}^*(A[t_1, \ldots, t_n]) = A$, then A is strongly-retractable and so is strongly cancellative.*

2. *If $\mathrm{ML}^*(A[t]) = A$, then A is retractable and so is cancellative.*

3. *Suppose $Z(A)$ is affine and $\mathrm{ML}_Z(A[t]) = Z(A)$ or $\mathrm{ML}_Z^H(A[t]) = Z(A)$ or $\mathrm{ML}_Z^{H'}(A[t]) = Z(A)$. Then A is Z-retractable.*

4. *Suppose $Z(A)$ is affine and A is strongly LND_Z^*-rigid where $*$ is either blank, H, or H'. Then A is strongly Z-retractable.*

Proof. The proof is identical to the proof given in Theorem 1.1.8, with the one exception being that we invoke Lemma 3.1.3 with $Y = A[t_1, \ldots, t_d]$ (with elements of A having degree 0 and $t_1, \ldots, t_d$ having degree 1) as a replacement for Lemma 3.1.4. We point out that [LeWZ] do not use H' in their paper, but the argument in the H' case goes through in the same manner as it does for H. □

The Makar-Limanov conjecture asks when $\mathrm{ML}(A[t_1, \ldots, t_d]) = \mathrm{ML}(A)$ if A is a commutative domain over a field of characteristic zero (See [Ma1, p. 55]). This is the case when one works with algebras of Gelfand-Kirillov dimension one (remark 3.3.4), so that in Proposition 3.1.6 below, we give a result that is related to this conjecture, which has interesting implications in terms of cancellation. We first need a basic result about vanishing of polynomials in noncommutative rings.

Remark 3.1.5. Let A be a prime ring and let $p(x) \in A[x]$ be a nonzero polynomial of degree d. If there are $d+1$ distinct central elements $z \in A$ such that $p(z) = 0$ then $p(x)$ is the zero polynomial.

Proof. Write $p(x) = a_0 + a_1 x + \cdots + a_d x^d$. Let Z denote the center of A, which is an integral domain since A is prime. Suppose that there exist distinct $z_1, \ldots, z_{d+1} \in Z$ such that $p(z_i) = 0$ for $i = 1, \ldots, d+1$. Let M be the $(d+1) \times (d+1)$ matrix whose (i,j)-entry is z_j^{i-1}. Then considering A^{d+1} as a right $M_{d+1}(Z)$-module, we see $[a_0, a_1, \ldots, a_d]M = 0$. Then right-multiplying by the classical adjoint of M we obtain $a_i \det(M) = 0$ for $i = 0, \ldots, d$. Then M is a Vandermonde matrix and since Z is an integral domain and the z_i are pairwise distinct, $\det(M)$ is a nonzero central element of A, and hence is regular since A is prime. It follows that $a_0 = \cdots = a_d = 0$ and $p(x)$ is the zero polynomial. □

Proposition 3.1.6. *Let A be a prime finitely generated k-algebra with infinite center. Then $\mathrm{ML}^{H'}(A) = \mathrm{ML}^{H'}(A[x_1, x_2, \cdots, x_d])$. In particular, if, in addition, A is left Goldie, has finite Gelfand-Kirillov dimension, $Z(A)$ is affine, and either $\mathrm{ML}_Z^{H'}(A) = Z(A)$ or $\mathrm{ML}^{H'}(A) = A$, then A is strongly cancellative.*

Proof. Remark 3.1.2 gives that $\mathrm{ML}^{H'}(A[x_1, x_2, \ldots, x_d]) \subseteq \mathrm{ML}^{H'}(A) \subseteq A$ for all $d \geq 1$. It thus suffices to show that $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x_1, x_2, \ldots, x_d])$.

We show that $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x])$. Once we have proved this, it will immediately follow by induction that $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x_1, \ldots, x_d])$ and we will obtain the result. Let $\partial := \{\partial_n\}_{n \geq 0}$ be an element of $\mathrm{LND}^{H'}(A[x])$. As in Equation (3.1), we have an induced k-algebra homomorphism $\phi : A[t] \longrightarrow A[x][t]$, given by

$$\phi(a) = \sum_{n \geq 0} \partial_n(a) t^n \quad \text{for } a \in A, \quad \phi(t) = t.$$

In particular, if $a \in \mathrm{ML}^{H'}(A)$, then $\phi(a) = a + tp(x,t)$, for some polynomial $p(x,t) \in A[x,t]$. We now fix $z \in Z(A)$ and consider the map $e_z : A[x,t] \to A[t]$, defined by $e_z(g(x,t)) = g(z,t)$. Then the composition $\phi_z := e_z \circ \phi$ gives a homomorphism from $A[t]$ to $A[t]$ and by construction $\phi_z(a) \equiv a \pmod{(t)}$, and $\phi_z(t) = t$ and so this homomorphism is injective. Thus, there are maps $\mu_j : A \to A$ with $\mu_0 = \mathrm{id}_A$ such that $\phi_z(a) = \sum_{j \geq 0} \mu_j(a) t^j$ for $a \in A$. In particular, for $a \in A$, $\mu_n(a) = 0$ for n sufficiently large, and so (μ_n) is a weakly locally nilpotent Hasse-Schmidt derivation of A. Thus, for $a \in \mathrm{ML}^{H'}(A)$ we have $\mu_i(a) = 0$ for every $i \geq 1$; that is, for $i \geq 1$, $\partial_i(a)|_{x=z} = 0$ for every $z \in Z(A)$. Since $Z(A)$ is infinite and $\partial_n(a)$ is a polynomial in $A[x]$, Remark 3.1.5 gives that $\partial_n(a) = 0$ for $n \geq 1$ and hence $a \in \mathrm{ML}^{H'}(A[x])$. Thus $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x])$ as required.

Now suppose that $Z(A)$ is infinite and affine and that A is prime left Goldie and has finite Gelfand-Kirillov dimension. It follows that if $\mathrm{ML}_Z^{H'}(A) = Z(A)$ then from the above $\mathrm{ML}_Z^{H'}(A[x_1, \ldots, x_d]) = Z(A)$ and so A is strongly $\mathrm{LND}_Z^{H'}$-rigid and hence strongly Z-retractable by Proposition 3.1.4. Thus, by Lemma 1.1.15, A is strongly detectable, and so is strongly cancellative by Lemma 1.1.18, since A is strongly Hopfian by Lemma 1.1.17-(4). On the other hand if $\mathrm{ML}^{H'}(A) = A$ then A is strongly $\mathrm{LND}^{H'}$-rigid and so by Proposition 3.1.4, A is strongly cancellative. □

3.2 The noncommutative slice theorem

We shall prove a noncommutative analogue of slice theorem. We make use of the fact that for a k-algebra A with a locally nilpotent derivation δ, the map δ restricts to a locally nilpotent derivation of the center of A. We first make a basic lemma.

Lemma 3.2.1. *Let k be a field, let A be a prime k-algebra, let $\partial := (\partial_n) \in \mathrm{LND}^I(A)$, and let $B = \ker(\partial)$. Then the following hold:*

1. *If there is $x \in A$ and $m \geq 1$ are such that $\partial_{m+i}(x) = 0$ for $i \geq 1$ and $\partial_m(x)$ is a regular element of A, then the sum*

$$B + Bx + Bx^2 + \cdots$$

   is direct;

2. *If A is a field and A is algebraic over B then $A = B$;*

3. *If $\mathrm{GKdim}(A) < \infty$ and A is an affine domain and $B \neq A$ then $\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A) - 1$;*

4. *If $\mathrm{GKdim}(A) = 1$ and A is an affine domain and $B \neq A$ then B is finite-dimensional.*

Proof. Suppose there exist $x \in A$ and $m \geq 1$ such that $\partial_{m+i}(x) = 0$ for $i \geq 1$ and $\partial_m(x)$ is a regular element of A. A straightforward induction shows that $\partial_{ms}(x^i) =$

0 for $i < s$ and $\partial_{ms}(x^s) = \partial_m(x)^s$. Suppose that there is a non-trivial relation $b_0 + b_1x + \cdots + b_sx^s = 0$ with $b_0, \ldots, b_s \in B$ and b_s nonzero. Then applying ∂_{ms} to this dependence gives $b_s\partial_m(x)^s = 0$, which is impossible as $b_s \neq 0$ and $\partial_m(x)$ is regular. This establishes (1). Next, to prove (2), observe that if A is a field, then B is a subfield of A. We have just shown that for $x \in A \setminus B$, the sum $B + Bx + \cdots$ is direct, and so if A is algebraic over B then we must have $A = B$.

We next prove (3). Suppose that A is a domain of finite Gelfand-Kirillov dimension and that $B \neq A$ and that

$$\mathrm{GKdim}(B) > \mathrm{GKdim}(A) - 1.$$

Then there exists $\alpha > \mathrm{GKdim}(A) - 1$ and a finite-dimensional k-vector subspace W of B that contains 1 and such that $\dim(W^n) \geq n^\alpha$ for n in an infinite set $\mathcal{T}$ of natural numbers. Pick $x \in A \setminus B$. Then by (1), $B + Bx + Bx^2 + \cdots$ is direct. Now let $V = W + kx$. Then $V^{2n} \supseteq W^n + W^nx + \cdots + W^nx^n$ and so $\dim(V^{2n}) \geq (n+1)n^\alpha \geq n^{\alpha+1}$ for $n \in \mathcal{T}$. Thus $\mathrm{GKdim}(A) \geq \alpha + 1$, a contradiction. Thus we obtain (3).

We now prove (4). Suppose that A is an affine domain of Gelfand-Kirillov dimension one and that $B \neq A$. We claim that $\dim_k(B) < \infty$. Pick $z \in A \setminus B$. By part (1), the sum $B + Bz + Bz^2 + \cdots$ is direct. Now suppose towards a contradiction that $\dim_k(B)$ is infinite and let V be a finite-dimensional subspace of A that contains 1 and z and which generates A as a k-algebra. Then since $\bigcup_{i\geq 0} V^i \supseteq B$, we have $W_n := V^n \cap B$ has the property that $\dim(W_n) \to \infty$ as $n \to \infty$. Since A has Gelfand-Kirillov dimension one, by a result of Bergman (see the proof of [KL, Theorem 2.5]) there is some positive constant C such that $\dim(V^n) \leq Cn$ for n sufficiently large. On the other hand, for each $d \geq 1$ we have

$$\dim(V^{n+d}) \geq \dim(W_dV^n) \geq \dim(W_d + W_dz + \cdots + W_dz^n) = \dim(W_d)(n+1).$$

Thus $\dim(W_d) \leq C(n+d)/(n+1)$ for all n sufficiently large and so $\dim(W_d) \leq C$ for every $d \geq 1$, a contradiction. Thus B is finite-dimensional. □

This following theorem is an extension of the slice theorem for a (not necessarily commutative) k-algebra.

Theorem 3.2.2. *(Noncommutative slice theorem) Let k be a field and let A be a k-algebra. Then the following statements hold.*

1. *Suppose that the characteristic of k is zero and $\delta \in \mathrm{LND}(A)$. If there exists $x \in Z(A)$ such that $\delta(x) = 1$, and if A_0 is the kernel of δ, then the sum $\sum_{i\geq 0} A_0x^i$ is direct and $A = A_0[x]$.*

2. *Suppose that $\partial := \{\partial_n\}_{n\geq 0} \in \mathrm{LND}^I(A)$. If there exists $x \in Z(A)$ such that $\partial_1(x) = 1$ and $\partial_i(x) = 0$ for $i \geq 2$, and if A_0 is the kernel of ∂, then the sum $\sum_{i\geq 0} A_0x^i$ is direct and $A = A_0[x]$.*

Proof. It suffices to prove part (2) by Remark 1.1.6. We let

$$A_0 = \{a \in A \mid \partial_n(a) = 0 \text{ for } n \geq 1\}.$$

We claim that $A = A_0[x]$. By Lemma 3.2.1, $\sum A_0 x^i$ is direct. Thus, A_0 and x generate a polynomial ring and $A \supseteq A_0[x]$. We next claim that $A \subseteq A_0[x]$. To see this, suppose that there exists some $a \in A \setminus A_0[x]$. Then there is some largest $m \geq 1$ such that $\partial_m(a) \neq 0$. Among all $a \in A \setminus A_0[x]$, we choose one with this m minimal. Since $\partial_i(\partial_m(a)) = \binom{i+m}{m}\partial_{i+m}(a) = 0$ for $i \geq 1$, $\partial_m(a)$ is in the kernel of ∂ and hence in A_0. Let $c = \partial_m(a) \in A_0$ and consider $a' = a - cx^m$. Observe that $\partial_j(a') = 0$ for $j > m$ and $\partial_m(a') = 0$ by construction. Thus, by minimality of m, $a' \in A_0[x]$ and hence so is a, a contradiction. The result follows. □

Proposition 3.2.3. *Let k be a field of characteristic zero and let A be a prime finitely generated k-algebra of finite Gelfand-Kirillov dimension, and suppose that $Z(A)$ is an affine domain of Gelfand-Kirillov dimension at most 1. Then one of the following alternatives must hold:*

1. $\mathrm{ML}_Z(A) = Z(A)$; *or*
2. *there is a prime k-subalgebra A_0 of A such that $A \cong A_0[t]$.*

Proof. If $\mathrm{ML}_Z(A) \neq Z(A)$, then there is some $\delta \in \mathrm{LND}(A)$ and some $z \in Z(A)$ such that $\delta(z) \neq 0$. We now pick the largest j such that $\delta^j(z) \neq 0$ and we replace z by $\delta^{j-1}(z)$. By construction, $\delta^i(z) = 0$ for $i \geq 2$ and $c := \delta(z) \neq 0$. Then $c \in A_0 \cap Z(A)$. Now $A_0 \cap Z(A)$ is a subalgebra of $Z(A)$ and since $Z(A)$ has Krull dimension one and $A_0 \subsetneq Z(A)$, A_0 is finite-dimensional by Lemma 3.2.1 and thus is a field. Thus c is a unit and so if we replace z by $c^{-1}z$ then we have $\delta(z) = 1$ and we may invoke Theorem 3.2.2 to get that $A \cong A_0[t]$. Since A is prime, A_0 is necessarily prime too. □

In general, the proof of Proposition 3.2.3 shows that if A is an affine prime k-algebra of finite Gelfand-Kirillov dimension over a field k of characteristic zero, then either $\mathrm{ML}_Z(A) = Z(A)$ or there is a prime subalgebra A_0 of A and some $c \in Z(A) \cap A_0$ such that $A[c^{-1}] \cong A_0[c^{-1}][t]$. In the case, that $Z(A)$ is affine of Gelfand-Kirillov dimension one we are able to deduce that c is invertible in the proof, which gives us part (2) in the dichotomy occurring in Proposition 3.2.3.

The following theorem answers a question of Lezama, Wang, and Zhang about if every affine prime k-algebra of Gelfand-Kirillov dimension one is cancellative? We consider the domain case when the base field has characteristic zero. Moreover, Theorem 3.2.4 specializes to the classical cancellation result of Abhyankar-Eakin-Heinzer in the case of characteristic zero base fields when one takes R to be commutative. Therefore, the result can be considered as a noncommutative version of the Theorem 1.1.20.

Theorem 3.2.4. *Let k be a field of characteristic zero and let A be an affine domain over k of Gelfand-Kirillov dimension one. Then A is cancellative.*

Proof. We recall that affine prime algebras of Gelfand Kirillov dimension one are noetherian and hence left Goldie [SW]. If $\mathrm{ML}(A) = A$ then $\mathrm{ML}(A[x]) = \mathrm{ML}(A)$ by Lemma 1.1.9, then A is cancellative by Proposition 3.1.4. If $\mathrm{ML}(A) \neq A$, then there is a nonzero locally nilpotent derivation δ of A. Let A_0 denote the kernel of δ. Then by Lemma 3.2.1 A_0 is finite-dimensional and since it is a domain, it is a division ring.

In particular, $Z(A) \not\subseteq A_0$ since $Z(A)$ has Gelfand-Kirillov dimension one [SW]. We let $E = A_0 \cap Z(A)$. Then E is a commutative integral domain that is finite-dimensional over k and hence E is a field. Since δ is not identically zero on $Z(A)$ and is locally nilpotent on A, there exists some $z \in Z(A)$ such that $z \notin E$ and $c := \delta(z) \in E \setminus \{0\}$. As E is a field and is contained in the kernel of δ, $x := c^{-1}z \in Z(A)$ satisfies $\delta(x) = 1$ and so by the noncommutative slice theorem, we get $A \cong A_0[x]$. Then by the same analysis as above if $A[t] \cong B[t]$ then we necessarily have $\mathrm{ML}(B) \neq B$ and so $B \cong B_0[x]$ for some finite-dimensional division ring B_0. Since A_0 is a finite-dimensional division algebra, it follows from Theorem 1.1.19 that A_0 is strongly cancellative, therefore $A_0 \cong B_0$ and hence A is cancellative. Thus we obtain the result in this case. □

Remark 3.2.5. We point out that Lezama, Wang, and Zhang [LeWZ, Theorem 0.6] proved that for algebraically closed base fields k, affine prime k-algebras of Gelfand-Kirillov dimension one are cancellative. The algebraically closed property is needed, because the authors invoke Tsen's Theorem at one point in their proof. However, our Theorem is somewhat orthogonal to this result, since domains of Gelfand-Kirillov dimension one over algebraically closed fields are commutative by an application of Tsen's Theorem to a result of Small and Warfield [SW] and hence the only part of the previous theorem covered by [LeWZ, Theorem 0.6] is the commutative case, which was previously known from the result of Abhyankar-Eakin-Heinzer.

Now we consider the positive characteristic case, the following Theorem gives a family of counterexamples to a conjecture 4.5.5 ([TVZ, Conjecture 0.3]).

Theorem 3.2.6. *Let p be prime. Then there exists a field k of characteristic p and an affine domain A of Gelfand-Kirillov dimension one that is not cancellative.*

Proof. Let p be a prime, and let $K = \mathbb{F}_p(x_1, \ldots, x_{p^2-1})$. We let $k = \mathbb{F}_p(x_1^p, \ldots, x_{p^2-1}^p)$ and we let δ be the k-linear derivation of K given by $\delta(x_i) = x_{i+1}$ for $i = 1, \ldots, p^2-1$, where we take $x_{p^2} = x_1$. Since k has characteristic $p > 0$, we have δ^{p^i} is a k-linear derivation for every $i \geq 0$, and since $\delta^{p^2}(x_i) = \delta(x_i) = x_{i+1}$ for $i = 1, \ldots, p^2 - 1$, then $\delta^{p^{j+2}} = \delta^{p^j}$ for every $j \geq 0$. We let $\delta' := \delta^p$, which as we have just remarked is a k-linear derivation of K. We let $A = K[x; \delta]$ and we let $B = K[x'; \delta']$. Since $\mathrm{ad}_u^p = \mathrm{ad}_{u^p}$ for u in a ring of characteristic p, we have $z := x^{p^2} - x$ and $z' := (x')^{p^2} - x'$ are central by the above remarks. We claim that A and B have Gelfand-Kirillov dimension one, $A \ncong B$, and $A[t] \cong B[t']$.

Since $[K : k] < \infty$, A and B are finitely generated k-algebras of Gelfand-Kirillov dimension one [KL, Proposition 3.5]. We construct an isomorphism $\Phi : A[t] \to B[t']$ as follows. We define $\Phi(\alpha) = \alpha$ for $\alpha \in K$, $\Phi(x) = (x')^p + t'$ and $\Phi(t) = (x')^{p^2} -$

$x' + (t')^p$. Then to show that Φ extends to a k-algebra homomorphism from $A[t]$ to $B[t']$, it suffices to show that

$$\delta(\alpha) = \Phi([x, \alpha]) = [\Phi(x), \alpha]$$

for $\alpha \in K$ and that $\Phi(t)$ is central. For $\alpha \in K$,

$$[\Phi(t), \alpha] = [(x')^{p^2} - (x'), \alpha] = (\delta')^{p^2}(\alpha) - \delta'(\alpha) = 0$$

and since $\Phi(t)$ also commutes with x', $\Phi(t)$ is central. To show that

$$\delta(\alpha) = \Phi([x, \alpha]) = [\Phi(x), \alpha]$$

for $\alpha \in K$, observe that $\Phi([x, \alpha]) = \Phi(\delta(\alpha)) = \delta(\alpha)$ and

$$[\Phi(x), \Phi(\alpha)] = [(x')^p + t', \alpha] = (\delta')^p(\alpha) = \delta^{p^2}(\alpha) = \delta(\alpha).$$

Thus, Φ induces a homomorphism from $A[t]$ to $B[t']$. We claim that Φ is onto. We have $\Phi(z) = (z')^p + (t')^{p^2} - t'$ and $\Phi(t) = z' + (t')^p$. In particular,

$$\Phi(z - t^p) = (z')^p + (t')^{p^2} - t' - (z')^p - (t')^{p^2} = -t'$$

and so $\Phi(t + (z - t^p)^p) = z'$. Thus, K, t' and z' are in the image of Φ. Since $\Phi(x) = (x')^p + t'$ we also have $(x')^p \in \mathrm{Im}(\Phi)$. Finally, observe that $z' = (x')^{p^2} - x'$ and since z' and $(x')^p$ are in the image of Φ, so is

$$x' = (x')^{p^2} - z' = ((x')^p)^p - z'.$$

Hence x', z', t' and K are in the image of Φ and so Φ is onto. Let I denote the kernel of Φ. Then since $\Phi : A[t] \to B[t]$ is onto, we have $A[t]/I \cong B[t]$. But $A[t]$ and $B[t]$ are both affine domains of Gelfand-Kirillov dimension two, and so I is necessarily zero [KL, Proposition 3.15]. Thus Φ is an isomorphism and so $A[t] \cong B[t]$.

Therefore, it only remains to show that $A \ncong B$ as k-algebras. To see this, suppose that $\Psi : A \to B$ is a k-algebra isomorphism. Then since the units group of A and B are both K^*, Ψ induces a k-algebra automorphism of K; furthermore, every $\alpha \in K$ satisfies $\alpha^p \in k$ and for $\beta \in k$ there is a unique $\alpha \in K$ such that $\alpha^p = \beta$. Since Ψ is the identity on k, Ψ is the identity on K. Thus $\Psi(x) = p(x')$ for some $p(x') \in K[x'; \delta'] \setminus K$. Let $d \geq 1$ denote the degree of $p(x')$ as a polynomial in x'. If $d > 1$, it is straightforward to show that Ψ cannot be onto, as every element in the image of Ψ necessarily then has degree in x' equal to a multiple of d. Since $\Psi(x) \notin K$, we see $\Psi(x) = \alpha x' + \beta$ with $\alpha \in K^*$ and $\beta \in K$. Since Ψ is an isomorphism, for $\zeta \in K$ we have

$$\delta(\zeta) = \Psi(\delta(\zeta)) = \Psi([x, \zeta]) = [\Psi(x), \Psi(\zeta)] = [\alpha x' + \beta, \zeta] = \alpha[x', \zeta] = \alpha\delta'(\zeta).$$

But by construction, $\delta(x_1) = x_2$ and $\delta'(x_1) = x_{p+1}$ and so $\alpha = x_2/x_{p+1}$. We also have $\delta(x_2) = x_3$ and $\delta'(x_2) = x_{p+2}$, and so $\alpha = x_3/x_{p+2}$, which gives $x_2x_{p+2} = x_3x_{p+1}$, where we take $x_{p+2} = x_1$ when $p = 2$. This is a contradiction. Thus $A \ncong B$. □

3.3 Skew cancellation

We now consider the case when an isomorphism of skew polynomial extensions $R[x; \sigma, \delta] \cong S[x; \sigma'; \delta']$ implies that R and S are isomorphic. We consider the case when R and S are finitely generated commutative integral domains of Krull dimension one over a field. We observe that when σ, σ' are the identity maps and δ, δ' are zero, the question reduces to the classical cancellation problem for affine curves, answered by Abhyankar, Eakin, and Heinzer (Theorem 1.1.20). To prove the main Theorem, we must consider two types of extensions: skew extensions of automorphism type and skew extensions of derivation type. We first look at the automorphism type case, in which the analysis is more straightforward.

Lemma 3.3.1. *Let k be a field, R be an affine commutative domain over k of Krull dimension one, and σ be a k-algebra automorphism of R that is not the identity. If A is a commutative domain of Krull dimension one that is a homomorphic image of $R[x; \sigma]$, then either $A \cong R$ or $A \cong K[x]$ for some finite extension K of k; moreover R occurs as a homomorphic image of $R[x; \sigma]$.*

Proof. We consider prime commutative homomorphic images of $T := R[x; \sigma]$ of Krull dimension one. Observe that if P is a prime ideal of T such that T/P is commutative, then since T/P is an integral domain and $R/(P \cap R)$ embeds in T/P, $R/(P \cap R)$ is also an integral domain. Since R is an integral domain of Krull dimension one, either $P \cap R = (0)$ or $P \cap R = I$, with I a maximal ideal of R. In the former case, observe that since $xr = \sigma(r)x \equiv x\sigma(r) \pmod P$, we have $x(r - \sigma(r)) \in P$. Moreover, since σ is not the identity and P is completely prime, we necessarily have $x \in P$. Thus T/P is a homomorphic image of $R[x; \sigma]/(x) \cong R$. Since T/P and R are both integral domains of Krull dimension one, we then have $T/P \cong R$ in this case. In the case where $P \cap R = I$, with I a maximal ideal of R, we claim that $I = I^\sigma$. To see this, suppose that this is not the case. Then since I is maximal, $I + \sigma(I) = R$. In particular, there are $a, b \in I$ such that $a + \sigma(b) = 1$. Then $ax, xb \in P$ and so $ax + xb \in P$. But $ax + xb = (a + \sigma(b))x = x$ and so $x \in P$. Thus T/P is a homomorphic image of R/I, which contradicts the assumption that T/P has Krull dimension one. Hence $I = \sigma(I)$. Then by the Nullstellensatz $K := R/I$ is a finite extension of k and σ induces a k-algebra automorphism of K. We next claim that σ is the identity on K; if not, there is some $\lambda \in K$ such that $\lambda \not\equiv \sigma(\lambda) \pmod P$. But since $[\lambda, x] = (\lambda - \sigma(\lambda))x \in P$ and since P is completely prime, we again have $x \in P$, which gives $T/P \cong K$, a contradiction. Thus σ induces the identity map on $R/I = K$ and so $T/IT \cong K[x]$. Since P contains IT, we then see that T/P is a homomorphic image of $K[x]$ and since T/P has Krull dimension one, we have $T/P \cong K[x]$. The result follows. □

Proposition 3.3.2. *Let k be a field and let R be an affine commutative domain over k of Krull dimension one. If $R[x;\sigma] \cong S[x;\sigma']$, with S an affine commutative domain of Krull dimension one, then $R \cong S$.*

Proof. If σ is the identity then both $R[x;\sigma]$ and $S[x;\sigma']$ are commutative and so σ' is also the identity and the result follows from Theorem 1.1.20. Hence we may assume that σ and σ' are not the identity maps on their respective domains. By Lemma 3.3.1, the set of isomorphism classes of prime commutative images of $R[x;\sigma]$ of Krull dimension one is contained in $\{K[x]\colon [K:k] < \infty\} \cup \{R\}$, with R occurring on the list. Similarly, the set of isomorphism classes of prime commutative images of $S[x;\sigma']$ of Krull dimension one is contained in $\{K[x]\colon [K:k] < \infty\} \cup \{S\}$, with S occurring on the list. It follows that either $R \cong S$ or $R \cong K[x]$ for some finite extension K of k. Similarly, either $S \cong R$ or $S \cong K'[x]$ for some finite extension K' of k. Thus we may assume without loss of generality that $R = K[t]$ and $S \cong K'[t]$ with K, K' finite extensions of k. Then the k-algebra isomorphism $R[x;\sigma] \to S[x;\sigma']$ restricts to an isomorphism of the units groups. Since the units groups of $R[x;\sigma] = K^*$ and the units group of $S[x;\sigma']$ is $(K')^*$, we see the isomorphism restricts to a k-algebra isomorphism between K and K'. Thus $K \cong K'$ and so $R \cong S$. □

We now prove a lemma, which is a straightforward extension of Lemma 1.1.9.

Lemma 3.3.3. *Let k be a field of characteristic zero and let A be a finitely generated Ore domain over k. If $\mathrm{ML}(A) = A$ then $\mathrm{ML}(A[x;\delta]) = \mathrm{ML}(A)$.*

Proof. Let μ be a locally nilpotent derivation of A. Then μ extends to a locally nilpotent derivation of $A[x;\delta]$ by declaring that $\mu(x) = 1$. Then the kernel of this extension of μ is equal to $\ker(\mu|_A)$ and hence $\mathrm{ML}(A[x;\delta]) \subseteq \mathrm{ML}(A)$. Now we show that the reverse containment holds. Let μ be a locally nilpotent derivation of $A[x;\delta]$ and suppose that μ is not identically zero on $\mathrm{ML}(A)$. Since A is finitely generated there is some largest $m \geq 0$ such that for $r \in A$ we have

$$\mu(r) = \partial(r)x^m + \text{lower degree terms},$$

with ∂ a derivation of A that is not identically zero on $\mathrm{ML}(A)$. If $m = 0$ then ∂ is a locally nilpotent derivation of A and hence vanishes on $\mathrm{ML}(A)$, a contradiction. Thus we may assume that $m > 0$. We now argue as in the three cases given in Lemma 1.1.9. □

The following result is due to Crachiola and Makar-Limanov [CM, Lemma 2.3].

Remark 3.3.4. Let k be a field of characteristic zero and let R be an affine commutative domain of Krull dimension one. Then, either $\mathrm{ML}(R) = R$ or $R \cong k'[t]$ for some finite field extension k' of k.

Proof. Suppose that $\mathrm{ML}(R) \neq R$. Then there is a locally nilpotent derivation δ of R that is not identically zero on R. In particular, the kernel of δ is a subalgebra R_0 of R. By Lemma 3.2.1, R_0 is finite-dimensional and hence a finite extension k'

of k. Then pick $x \in R$ such that $\delta(x) \neq 0$ and $\delta^2(x) = 0$. Then $\delta(x) \in (k')^*$ and so we may rescale and assume that $\delta(x) = 1$. Then by Lemma 3.2.2, $R \cong k'[x]$, as required. □

Combining Lemma 3.3.3 and Remark 3.3.4, we have the following result.

Corollary 3.3.5. *Let k be a field of characteristic zero, let R be a finitely generated k-algebra that is a commutative domain of Krull dimension one, and let δ be a k-linear derivation of R. Then either $R \cong k'[t]$ for some finite extension k' of k or $\mathrm{ML}(R[x;\delta]) = R$.*

Proposition 3.3.6. *Let k be a field of characteristic zero and let R and S be affine commutative domains over k of Krull dimension one. If δ and δ' are respectively k-linear derivations of R and S and $R[x;\delta] \cong S[x;\delta']$, then $R \cong S$.*

Proof. If neither R nor S is isomorphic to an algebra of the form $k_0[t]$, with k_0 a finite extension of k, then by Corollary 3.3.5,

$$R = \mathrm{ML}(R[x;\delta]) \cong \mathrm{ML}(S[x;\delta') = S,$$

as desired. If precisely one of R and S is isomorphic to an algebra of the form $k_0[t]$, then without loss of generality we may assume that $R = k_0[t]$. Then again by Corollary 3.3.5

$$S \cong \mathrm{ML}(S[x;\delta']) \cong \mathrm{ML}(R[x;\delta]) = k_0',$$

which is impossible. Thus we may assume that $R = k_0[t]$ and $S = k_0'[t]$, where k_0 and k_0' are finite extensions of k. But now the units group of $R[x;\delta]$ is $(k_0)^*$ and the units group of $S[x;\delta']$ is $(k_0')^*$ and so the isomorphism from $R[x;\delta]$ to $S[x;\delta']$ restricts to a k-algebra isomorphism between k_0 and k_0' and so $R \cong S$ in this case. □

We do not know whether Proposition 3.3.6 is true when the base field k has positive characteristic. We compare the examples from Theorem 3.2.6 with the positive characteristic case of Proposition 3.3.6. In positive characteristic, there exists a field k and a finite extension K of k and k-linear derivations δ, δ' of K such that $K[t;\delta][x] \cong K[t;\delta'][x]$ but $K[t;\delta] \ncong K[t;\delta']$. But we can extend δ and δ' to $K[x]$ by declaring that $\delta(x) = \delta'(x) = 0$ and we have

$$K[t;\delta][x] \cong K[x][t;\delta] \cong K[x][t;\delta'] \cong K[t;\delta'][x].$$

So the algebra $K[t;\delta][x] \cong K[x][t;\delta]$ is cancellative with respect to the variable t but not with respect to the variable x. Thus these examples do not give rise to counterexamples to the positive characteristic of Proposition 3.3.6.

Theorem 3.3.7. *Let k be a field, let A and B be affine commutative integral domains of Krull dimension one, and let σ, σ' be k-algebra automorphisms of A and B respectively and let δ, δ' be k-linear derivations of A and B respectively. If $A[x;\sigma] \cong B[x';\sigma']$ then $A \cong B$. If, in addition, k has characteristic zero and if $A[x;\delta] \cong B[x';\delta']$ then $A \cong B$.*

Proof. This follows immediately from Propositions 3.3.2 and 3.3.6. □

We do not know whether cancellation holds for skew polynomial extensions of mixed type with coefficient rings being domains of Krull dimension one.

Remark 3.3.8.

1. A special case of Theorem 3.3.7 was proved by Bergen [Be] in the derivation case. Specifically, he proved that if k is a field of characteristic zero and $R[t;\delta]$ is isomorphic to $k[x][y;\delta']$, with $\delta'(x) \in k[x]$ having degree at least one, then $R \cong k[x]$.

2. In [AKP], the ring A is called *Ore invariant* whenever $A[x;\delta_1] \cong B[y;\delta_2]$ implies $A \cong B$. If, furthermore, the isomorphism σ carries A onto B then A is said to be *strongly Ore invariant*. In this paper, the authors established that regular self-injective rings with a polynomial identity and no $\mathbb{Z}$-torsion are Ore invariant. Also, they proved that Abelian regular rings are strongly Ore invariant.

3. ([AKP, Example 11]) Let A be the ring $\mathbb{Z}_2[x]/(x^2)$ with the derivation δ such that $\delta(\overline{x}) = 1$ where $\overline{x} = x + (x^2)$. Consider the Ore extensions $A[y;\delta] = (\mathbb{Z}_2[x]/(x^2))[y;\delta]$. If we set $e_{11} = \overline{x}y,\ e_{12} = \overline{x},\ e_{21} = \overline{x}y^2 + y,\ e_{22} = 1 + \overline{x}y$, then they form a system of matrix units in $A[y;\delta]$. Now the centralizer of these matrix units in $A[y;\delta]$ is $\mathbb{Z}_2[y^2]$. Therefore, $A[y;\delta] \cong M_2(\mathbb{Z}_2[y^2]) \cong M_2(\mathbb{Z}_2)[t]$. But $M_2(\mathbb{Z}_2) \ncong \mathbb{Z}_2[x]/(x^2)$. Hence $M_2(\mathbb{Z}_2)$ is a simple Artinian ring of characteristic 2 which is not Ore invariant. Therefore, this example and the constructions given in Theorem 3.2.6 show that skew cancellation and cancellation can behave strangely with skew extensions of derivation type in positive characteristic.

CHAPTER 4

Cancellation for AS-Regular algebras of dimension three

In this chapter we study cancellation and Morita cancellation for noncommutative noetherian connected graded Artin–Schelter regular algebras of global dimension three. AS-regular algebras are considered as a noncommutative analogue of the commutative polynomial rings. It is well-known that the only AS-regular algebra of global dimension one is the polynomial ring $k[x]$, which is cancellative by Theorem 1.1.20. Combining Example 1.1.2 with Corollary 1.1.11, every AS-regular algebra of global dimension two (over a base field of characteristic zero) is cancellative. On the other hand, by the Gupta's results not every AS-regular algebra of global dimension three or higher are cancellative when the base field has positive characteristic. However, in [LMZ] the authors showed that several classes of AS-regular algebras of global dimension three are cancellative. Therefore it is natural to ask which AS-regular algebras of global dimension three are cancellative.

Algebras with center trivial were considered in the previous chapters. In the first section, we show that under certain conditions a k-algebra A with center $Z(A)$ of Gelfand-Kirillov dimension at most one is either k or $k[t]$. Next, considering the $\mathcal{P}$-discriminant we prove one of the most important result of this chapter: Lemma 4.1.6. Then we can establish cancellation for AS-regular algebras of higher global dimension in Theorem 4.1.7. In the second section, we show that connected graded Artin-Schelter regular algebras of global dimension three that not satisfies polynomial identity are cancellative.

Next, we collect some results that involve cancellation properties. As an application, in Section 3 we establish cancellation for fixed subrings of generic 3-dimensional Sklyanin algebra. In Section 4, Example 4.4.2 establishes cancellation for universal enveloping algebras of 3-dimensional non-abelian Lie algebras. And, in Section 5, we present a result similar to Theorem 4.1.7 for graded isolated singularities which have infinite global dimension. In this chapter let k be a base field that is algebraically closed. Results of this chapter will appear in [TVZ].

4.1 Center $Z(A) = k[t]$

By Proposition 1.1.4, algebras with trivial center are cancellative. Now we begin to consider algebras with center $k[t]$.

Lemma 4.1.1. *Let Z be a connected graded domain of GKdimension one.*

(1) It is noetherian and finitely generated over k.

(2) If Z is normal, then Z is isomorphic to $k[t]$.

Proof. Note that every domain of GKdimension one is commutative (See [SW]).

(1) Since Z is connected graded and k is algebraically closed, Z is a subring of $k[t]$ where $\deg t = 1$. Now, let $p(t)$ be a polynomial of smallest degree in Z, we denote $n := deg(p(t)) \geq 1$ and let $k[p(t)]$ denote the subalgebra generated by $p(t)$ and k in $k[t]$. We claim that $k[t]$ is a module over $k[p(t)]$ with generators $1, t, \ldots, t^{n-1}$. To see this, let $q \in k[t]$, then by induction on the degree of q, we show that $q = \sum_{i=0}^{n-1} r_i t^i$, with $r_i \in k[p(t)]$. Since by the Euclidean algorithm $q = ap + \sum_{i=0}^{n-1} \alpha_i t^i$ and $\alpha_1 \in k$, and by induction hypothesis $a = \sum_{i=0}^{n-1} r_i t^i$ with $r_i \in k[p(t)]$, so $q = \sum_{i=0}^{n-1} (r_i p + \alpha_i) t^i$. Now $k[p(t)]$, being an algebra on one generator over k is a principal ideal ring. Thus, $k[t]$ is a module with n generators over $k[p(t)]$, and, since Z is a $k[p(t)]$-submodule of $k[t]$, it follows from theorem on modules over a principal ideal ring that Z has a set of generators, $g_1, \ldots, g_m$ with $m \leq n$. Thus, every element of Z can be written in the form $\sum_{i=0}^{m} r_i g_i$ where $r_i \in k[p(t)]$, and hence Z is generated by the elements $p, g_1, \ldots, g_m$. The second part is clear (for more details see [Ga, Theorem], [No, Theorem 2.1]).

(2) First of all, $\operatorname{Kdim} Z = \operatorname{GKdim} Z = 1$. By part (1), Z is noetherian. Every noetherian normal domain Z of Krull dimension one or zero is regular (namely, has finite global dimension). So Z is regular of global dimension no more than one. Since Z is connected graded, its graded maximal ideal is principal, which implies that $Z \cong k[t]$. □

Note that a noetherian commutative maximal order is a normal domain.

Lemma 4.1.2. *Let A be a domain that is a maximal order.*

(1) Its center $Z(A)$ is a maximal order in the field of fractions $Q(Z(A))$.

(2) If A is connected graded and $\operatorname{GKdim} Z(A) \leq 1$, then $Z(A)$ is either k or $k[t]$.

Proof. (1) Let B be a subring of $Q(Z(A))$ containing $Z(A)$ such that $aBb \subseteq Z(A)$ for some $a, b \in Z(A)$. Let $C = AB$ the subring generated by A and B. Then $aCb \subseteq AZ(A) = A$. Since A is a maximal order, $C = A$. As a consequence, $B = Z(A)$. The assertion follows.

(2) The assertion follows from part (1) and Lemma 4.1.1. □

Note that if $\operatorname{GKdim} Z(A) = 2$, then $Z(A)$ may not be regular. For example let $A = k_{p_{ij}}[x_1, x_2, x_3, x_4]$ where

$$p_{ij} = \begin{cases} 1 & (i,j) = (1,2), \\ -1 & (i,j) = (1,3), (2,3), (1,4), (2,4), \\ q & (i,j) = (3,4) \end{cases}$$

where q is not a root of unity. Then it is easy to see that $Z(A)$ is the second Veronese subring $k[x_1, x_2]^{(2)}$ of the commutative polynomial ring. Hence $Z(A)$ is not regular.

Proposition 4.1.3. *Let A be a noetherian connected graded Auslander regular Cohen-Macaulay algebra. If $\operatorname{GKdim} Z(A) \leq 1$, then $Z(A)$ is either k or $k[t]$.*

Proof. By Theorem 1.4.11 A is a domain that is a maximal order in its quotient division ring $Q(A)$. The result follows from Lemma 4.1.1. □

Lemma 4.1.4. *Let A be a connected graded domain and t be a central element in A of positive degree d.*

(1) For every $\alpha \in k^*$, $A/(t-\alpha)$ contains $(A[t^{-1}])_0$ as a subalgebra.

(2) Suppose that A is generated in degree 1 and that $d \neq 0$ in k. Then

$$\operatorname{gldim} A/(t-\alpha) = \operatorname{gldim}(A[t^{-1}])_0.$$

(3) Suppose that A is generated in degree 1 and that $d \neq 0$ in k. If A has finite global dimension, then so does $A/(t-\alpha)$ for all $\alpha \in k^*$.

Proof. (1) Let T denote the dth Veronese subalgebra of A where $d = \deg t$. So, in T, t can be treated as an element of degree 1. Now

$$T/(t-\alpha) \cong T/(\alpha^{-1}t - 1) \cong (T[(\alpha^{-1}t)^{-1}])_0 = (T[t^{-1}])_0 \cong (A[t^{-1}])_0 \tag{4.1}$$

where the second $\cong$ is due to [RSS, Lemma 2.1].

Note that $A/(t-\alpha)$ is a $\mathbb{Z}/(d)$-graded algebra with the degree 0 component being $T/(t-\alpha)$. By (4.1) $A/(t-\alpha)$ contains $(A[t^{-1}])_0$ as a subalgebra.

(2) Since A is generated in degree 1, $A/(t-\alpha)$ is a strongly $\mathbb{Z}/(d)$-graded algebra with the degree 0 component being $(A[t^{-1}])_0$. Since we assume $d \neq 0$ in k, by [Yi, Lemma 2.2(iii)],

$$\operatorname{gldim} A/(t-\alpha) = \operatorname{gldim}(A[t^{-1}])_0.$$

(3) By part (2) it suffices to show that $(A[t^{-1}])_0$ has finite global dimension. Since A has finite global dimension, A has finite graded global dimension. Then $A[t^{-1}]$ has finite graded global dimension. As a consequence, $(A[t^{-1}])_0$ has finite global dimension as required. □

Lemma 4.1.5. *Let A be a noetherian domain with $Z(A) = k[t]$. Let $\mathcal{P}$ be a property such that the $\mathcal{P}$-discriminant is t.*

(1) A is strongly Z-retractable and strongly cancellative.

(2) If $\mathcal{P}$ is a Morita invariant, then A is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.

Proof. We only prove (2).

t is an effective element in $k[t]$ by [LeWZ, Example 2.8]. By [LuWZ, Theorem 2.10], Z is strongly LND_t^H-rigid. By Proposition 1.2.5(2), A is both strongly Z-retractable and strongly m-Z-retractable. Since A is noetherian, it is Hopfian by Lemma 1.1.17. By Lemmas 1.2.8 and 1.2.9, A is strongly cancellative and strongly m-cancellative. □

Next we consider the connected graded case, this lemma is important for the main theorems.

Lemma 4.1.6. *Let A be a noetherian connected graded domain.*

(1) If $Z(A)$ has GKdimension ≤ 1 and $Z(A)$ is not isomorphic to $k[t]$, then A is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.

For the following parts, we assume that A is generated in degree 1, that $Z(A) \cong k[t]$ and that $\operatorname{char} k = 0$.

(2) If $\operatorname{gldim} A/(t) = \infty$ *and* $\operatorname{gldim} A/(t-1) < \infty$, *then A is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.*

(3) Suppose the global dimension of A is finite and $\operatorname{gldim} A/(t) = \infty$. *Then A is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.*

(4) Suppose A is AS-regular and $\operatorname{gldim} A/(t) = \infty$. *Then A is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.*

Proof. (1) By Lemma 4.1.1(1), Z is an affine domain. By Lemma 1.1.20, Z is strongly retractable. By taking $\mathcal{P}$ to be a trivial property, say being an algebra, the $\mathcal{P}$-discriminant is 1. By [LeWZ, Remark 3.7(6)], Z is strongly LND_1^H-rigid. By Proposition 1.2.5(2), A is both strongly Z-retractable and strongly m-Z-retractable. Since A is noetherian, it is Hopfian by Lemma 1.1.17. By Lemmas 1.2.8 and 1.2.9, A is strongly cancellative and strongly m-cancellative.

(2) Let $\mathcal{P}$ be the property of having finite global dimension. By Lemma 4.1.4, for every $0 \neq \alpha \in k$,

$$\operatorname{gldim} A/(t-\alpha) = \operatorname{gldim}(A[t^{-1}])_0 = \operatorname{gldim} A/(t-1) < \infty$$

and by the hypothesis, we have that

$$\operatorname{gldim} A/(t) = \infty.$$

Hence $\mathcal{P}$-discriminant is t. The assertion follows from Lemma 4.1.5(2).

(3) By Part (2), it suffices to show that $\operatorname{gldim} A/(t-1) < \infty$. Since A has finite global dimension, so does $A[t^{-1}]$. Then $A[t^{-1}]$ has finite graded global dimension. Since A is generated in degree 1, $A[t^{-1}]$ is strongly $\mathbb{Z}$-graded. Hence $\operatorname{gldim}(A[t^{-1}])_0$ is finite. By Lemma 4.1.4,

$$\operatorname{gldim} A/(t-1) = \operatorname{gldim}(A[t^{-1}])_0 < \infty$$

as required.

(4) The assertion follows from part (3) and the fact that an AS-regular algebra has finite global dimension. □

The following theorem establish cancellation for AS-regular algebras of higher global dimension.

Theorem 4.1.7. *Suppose* $\operatorname{char} k = 0$. *Let A be a noetherian connected graded domain of finite global dimension that is generated in degree* 1. *Suppose that*

(a) $\operatorname{GKdim} Z(A) \leq 1$, *and*

(b) $\operatorname{gldim} A/(t) = \infty$ *for every homogeneous central element t in $Z(A)$ of positive degree.*

Then A is cancellative.

Proof. If $Z(A)$ is not isomorphic to $k[t]$, the assertion follows from Lemma 4.1.6(1). If $Z(A)$ is isomorphic to $k[t]$, the assertion follows from Lemma 4.1.6(3). □

4.2 Cancellation for AS-Regular algebras of dimension three

In this section we complete the study of cancellation for AS-regular algebras of dimension three as a particular case of Theorem 4.1.7. However, we need some previous results.

The following lemma concerns cancellation properties for a tensor product $A \otimes R$ where R is commutative. An opposite result was proved in [LuWZ, Lemma 1.4].

Lemma 4.2.1. *Let A be an algebra with trivial center and let R be a commutative algebra that is cancellative* (*respectively, strongly cancellative*). *Then the tensor product $A \otimes R$ is both cancellative* (*respectively, strongly cancellative*) *and m-cancellative* (*respectively, strongly m-cancellative*).

Proof. The proofs for the assertions without the word "strongly" are similar by taking $n = 1$ in the following proof. So we only prove the "strongly" version.

First we show that $A \otimes R$ is strongly cancellative assuming that R is strongly cancellative. Let B be an algebra such that

$$\phi : (A \otimes R)[t_1, \cdots, t_n] \xrightarrow{\cong} B[s_1, \cdots, s_n]$$

is an isomorphism of algebras. Taking the center on both sides, we obtain an isomorphism

$$\phi_Z : R[t_1, \cdots, t_n] \xrightarrow{\cong} Z(B)[s_1, \cdots, s_n]$$

where ϕ_Z is a restriction of ϕ on the centers. Since R is strongly cancellative, $R \cong Z(B)$. Let

$$f_i = \phi_Z^{-1}(s_i) = \phi^{-1}(s_i) \in R[t_1, \cdots, t_n]$$

for $i = 1, \cdots, n$. Let I be the ideal of $Z(B)[s_1, \cdots, s_n]$ generated by $\{s_i\}_{i=1}^n$. Then $J := \phi^{-1}(I)$ is an ideal of $R[t_1, \cdots, t_n]$ and

$$A \otimes (R[t_1, \cdots, t_n]/J) \cong B \otimes (k[s_1, \cdots, s_n]/I) \cong B.$$

Taking the center on both sides of the above isomorphism and using the fact that $Z(A) = k$, we have

$$R[t_1, \cdots, t_n]/J \cong Z(B) \cong R.$$

Therefore

$$B \cong A \otimes (R[t_1, \cdots, t_n]/J) \cong A \otimes R$$

as required.

Next we show that if R is strongly cancellative, then $A \otimes R$ is strongly m-cancellative. Let B be an algebra such that

$$A' := (A \otimes R)[t_1, \cdots, t_n] \quad \text{is Morita equivalent to} \quad B[s_1, \cdots, s_n] =: B'.$$

By [LuWZ, Lemma 2.1(3)], there is an invertible (A', B')-bimodule Ω and an isomorphism

$$\omega : Z(A') = R[t_1, \cdots, t_n] \xrightarrow{\cong} Z(B)[s_1, \cdots, s_n] = Z(B')$$

such that the left action of $x \in Z(A')$ on Ω agrees with the right action of $\omega(x) \in Z(B')$ on Ω. Since R is strongly cancellative, $R \cong Z(B)$. Let

$$f_i = \omega^{-1}(s_i) \in R[t_1, \cdots, t_n]$$

for $i = 1, \cdots, n$. Let I be the ideal of $Z(B)[s_1, \cdots, s_n]$ generated by $\{s_i\}_{i=1}^n$. Then $J := \omega^{-1}(I)$ is an ideal of $R[t_1, \cdots, t_n]$, and by [LuWZ, Lemma 2.1(5)],

$$A \otimes (R[t_1, \cdots, t_n]/J) \quad \text{is Morita equivalent to} \quad B \otimes (k[s_1, \cdots, s_n]/I) \cong B.$$

Taking the center of the above Morita equivalence and using the fact that $Z(A) = k$, we have

$$R[t_1, \cdots, t_n]/J \cong Z(B) \cong R.$$

Hence

$$A \otimes (R[t_1, \cdots, t_n]/J) \cong A \otimes R.$$

Therefore

$$A \otimes R \quad \text{is Morita equivalent to} \quad B$$

as required. □

Corollary 4.2.2. *Let k be of characteristic zero and A be a commutative algebra. Let δ be a locally nilpotent derivation of A with $\delta(y) = 1$ for some $y \in A$. Suppose that* $\ker(\delta)$ *is cancellative (respectively, strongly cancellative). Then $A[x; \delta]$ is cancellative (respectively, strongly cancellative).*

Proof. Let $C = \ker(\delta)$. By [MR, Lemma 14.6.4] $A[x; \delta] \cong C \otimes A_1(k)$. By hypothesis, then C is cancellative and $Z(A_1(k)) = k$. The assertion follows from Lemma 4.2.1. □

With a slight modification to the previous Lemma we can consider the case in which R is a (noncommutative) Z-retractable algebra.

Lemma 4.2.3. *Let A be an algebra with trivial center and let R be a Z-retractable algebra (respectively, strongly Z-rectractable). Then the tensor product $A \otimes R$ is Z-rectractable (respectively, strongly Z-retractable).*

The proof of Lemma 4.2.3 is similar to the proof of Lemma 4.2.1, so it is omitted. We call an algebra R *universally right noetherian* if $A \otimes R$ is right noetherian for every right noetherian k-algebra A. This property was studied in [ASZ]. Thus, if in the previous Lemma, we suppose that A is right noetherian algebra and R is universally right noetherian, then $A \otimes R$ is Hopfian, and by Lemma 1.1.18 $A \otimes R$ is cancellative (respectively, strongly cancellative).

Lemma 4.2.4. *Let k be a field of characteristic zero and $q \neq 1$ be a nonzero scalar in k. The following hold.*

(1) Algebras $k\langle x, y\rangle/(xy - yx)$, $k\langle x, y\rangle/(xy - yx - 1)$ and $k\langle x, y\rangle/(xy - yx - x)$ are pairwise not Morita equivalent.

(2) If q is not a root of unity, $k\langle x, y\rangle/(xy - qyx)$ is not Morita equivalent to $k\langle x, y\rangle/(xy - qyx - 1)$.

(3) The Jordan algebra $k\langle x, y\rangle/(xy - yx + x^2)$ is not Morita equivalent to $k\langle x, y\rangle/(xy - yx + x^2 - 1)$.

(4) The Jordan algebra $k\langle x, y\rangle/(xy - yx + x^2)$ is not Morita equivalent to $k\langle x, y\rangle/(xy - yx + x^2 - x)$.

(5) The Jordan algebra $k\langle x,y\rangle/(xy-yx+x^2)$ *is not Morita equivalent to* $k\langle x,y\rangle/(xy-yx+x^2-y)$.

Proof. (1) First of all $k\langle x,y\rangle/(xy-yx)$ and $k\langle x,y\rangle/(xy-yx-x)$ have global dimension two while $k\langle x,y\rangle/(xy-yx-1)$ has global dimension one. So either the algebra $k\langle x,y\rangle/(xy-yx)$ or the algebra $k\langle x,y\rangle/(xy-yx-x)$ is not Morita equivalent to $k\langle x,y\rangle/(xy-yx-1)$. Second, the centers of $k\langle x,y\rangle/(xy-yx)$ and $k\langle x,y\rangle/(xy-yx-x)$ are non-isomorphic, so these algebras are not Morita equivalent.

(2) Let $A := k\langle x,y\rangle/(xy-qyx)$ and $B := k\langle x,y\rangle/(xy-qyx-1)$. Suppose on the contrary that A is Morita equivalent to B. Let J be the height one prime ideal of B generated by $(1-q)xy-1$ such that $B/J = k[x^{\pm 1}]$ (with the image of y being $(1-q)^{-1}x^{-1}$). Since A is Morita equivalent to B, there is an ideal I of A such that A/I is Morita equivalent to $k[x^{\pm 1}]$. Since every projective module over $k[x^{\pm 1}]$ is free, A/I is a matrix algebra over $k[x^{\pm 1}]$. When q is not a root of unity, the only height one prime ideals I of A are (x) or (y) [BrG, Example II.1.2]. In both cases, A/I is isomorphic to $k[t]$, which is not a matrix algebra over $k[x^{\pm 1}]$. This yields a contradiction and therefore A is not Morita equivalent to B.

(3) Let $A := k\langle x,y\rangle/(xy-yx+x^2)$ and $B := k\langle x,y\rangle/(xy-yx+x^2-1)$ by recycling notation from the proof of part (2) and suppose on the contrary that A is Morita equivalent to B. Let $J_\pm$ be the height one prime ideals of B generated by $(xy-yx, x\pm 1)$. Since A is Morita equivalent to B, there is an ideal $I_\pm$ of A such that $A/I_\pm$ is Morita equivalent to $B/J_\pm$. Since $\operatorname{char} k = 0$, A has only a single height one prime that is (x) [Sh, Theorem 2.4]. This yields a contradiction. Therefore A is not Morita equivalent to B.

(4) The assertion follows from part (3) because $k\langle x,y\rangle/(xy-yx+x^2-1) \cong k\langle x,y\rangle/(xy-yx+x^2-x)$.

(5) Let $A := k\langle x,y\rangle/(xy-yx+x^2)$ and $B := k\langle x,y\rangle/(xy-yx+x^2-y)$ by recycling notation from the proof of part (2) and suppose on the contrary that A is Morita equivalent to B. Let $y' = y-x^2$. Then the relation in B becomes $xy'-y'x-y' = 0$. Exchanging x and y', one sees that B is isomorphic to $k\langle x,y\rangle/(xy-yx+x)$. Let I be the unique height one prime ideal of B generated by x. Then $B/I \cong k[y]$. Since A and B are Morita equivalent, there is a height one prime J of A. Since the only height one prime of A is (x). Let $J = (x)$, then I^2 corresponds to J^2. This implies that $B/I^2 \cong k\langle x,y\rangle/(xy-yx+x,x^2)$ is Morita equivalent to $A/J^2 \cong k[x,y]/(x^2)$. Since the center is preserved by Morita equivalence,

$$k[x,y]/(x^2) \cong Z(k\langle x,y\rangle/(xy-yx+x,x^2)) \cong k$$

yielding a contradiction. Therefore A and B are not Morita equivalent. □

The central result of this section. Thus we establish cancellation for special AS-regular algebras of dimension three that are non PI,

Theorem 4.2.5. *Suppose* $\operatorname{char} k = 0$. *Let A be an AS-regular algebra of global dimension three that is generated in degree* 1. *If A is not PI , then it is cancellative.*

Proof. Note that every AS-regular algebra has finite global dimension. If A is not PI, then by Lemma 1.4.13,

$$\operatorname{GKdim} Z \leq \operatorname{GKdim} A - 2 = 3 - 2 = 1.$$

If $Z(A)$ is not isomorphic to $k[t]$, the assertion follows from Lemma 4.1.6(1). If $Z(A)$ is isomorphic to $k[t]$ and if $\operatorname{gldim} A/(t) = \infty$, the assertion follows from Lemma 4.1.6(3). For the rest of the proof we assume that $Z(A) = k[t]$ and $A/(t)$ has finite global dimension.

By Rees lemma, $\operatorname{gldim} A/(t) \leq 2$. By the Hilbert series computation, we obtain that $\operatorname{GKdim} A/(t) = 2$. This implies that $A/(t)$ is AS-regular of global dimension two. Since we assume that k is algebraically closed, $A/(t)$ is either $k_q[x, y]$ or $k_J[x, y]$. In particular, the Hilbert series of $A/(t)$ is $\frac{1}{(1-s)^2}$. Since A is AS-regular of global dimension three, it is generated by either 3 elements or 2 elements. Next we consider these two cases.

Case 1: A is generated by two elements. Then the Hilbert series of A is $\frac{1}{(1-s)^2(1-s^2)}$. It forces that $\deg t = 2$. If $A/(t) = k_q[x, y]$, then $t = xy - qyx$ and $A/(t-1) = k\langle x, y\rangle/(xy - qyx - 1)$. If $q = 1$, let $\mathcal{P}$ be the property of **not** being Morita equivalent to $A/(t)$. Then the $\mathcal{P}$-discriminant is t by Lemma 4.2.4(1). Now the assertion follows from Lemma 4.1.5(2).

If $q \neq 1$, we claim that q is not a root of unity. If q is a root of unity, then $A/(t-1)$ is PI. By Lemma 4.1.4(1), $(A[t^{-1}])_0$ is PI. Note that $A^{(2)}[t^{-1}] = (A[t^{-1}])_0[t^{\pm 1}]$. So $A^{(2)}[t^{-1}]$ is PI. Consequently, $A^{(2)}$ is PI and whence A is PI, a contradiction. Then by the argument with Lemma 4.2.4(2) being replaced by Lemma 4.2.4(1), one sees that A is strongly cancellative and strongly m-cancellative.

If $A/(t) = k_J[x, y]$, then $t = xy - yx + x^2$ and $A/(t-1) = k\langle x, y\rangle/(xy - yx + x^2 - 1)$. Let $\mathcal{P}$ be the property of being Morita equivalent to $A/(t-1)$. Then the $\mathcal{P}$-discriminant is t by Lemma 4.2.4(3). By Lemma 4.1.5(2), A is strongly cancellative and strongly m-cancellative.

Case 2: A is generated by three elements. Then the Hilbert series of A is $\frac{1}{(1-s)^3}$. If A is isomorphic to $A' \otimes k[t]$ for some algebra A', then $Z(A')$ is trivial and the assertion follows from Lemma 4.2.1. So we further assume that A is not a tensor product of two nontrivial algebras. In this case A is generated by x, y, t subject to three relations

$$\begin{aligned} xt - tx &= 0, \\ yt - ty &= 0, \\ xy - qyx = ft + \epsilon t^2, \quad &\text{or} \quad xy - yx + x^2 = ft + \epsilon t^2 \end{aligned}$$

where ϵ is either 0 or 1 and f is a linear combination of x and y.

Again we have three cases. If $q = 1$, using Lemmas 4.2.4(1) and 4.1.5(2), one sees that A is strongly cancellative and strongly m-cancellative. If $q \neq 1$, then we can assume that $f = 0$ and $\epsilon = 1$ after a base change. Since A is not PI, q is not a root of unity. Then we use Lemma 4.2.4(2) instead of Lemma 4.2.4(1). Otherwise

we have the relation

$$xy - yx + x^2 - ft - \epsilon t^2 = 0.$$

Up to a base change, we may assume that $\epsilon = 0$. Then we have either $xy - yx + x^2 - xt = 0$ or $xy - yx + x^2 - yt = 0$. In the case of $xy - yx + x^2 - xt = 0$, using Lemmas 4.2.4(4) and 4.1.5(2), one sees that A is strongly cancellative and strongly m-cancellative. In the case of $xy - yx + x^2 - yt = 0$, using Lemmas 4.2.4(5) and 4.1.5(2), one sees that A is strongly cancellative and strongly m-cancellative. □

Remark 4.2.6. In [LMZ, Corollary 0.9] (Example 1.4.7) and [BZ1, Example 4.8], the authors showed that several classes of AS-regular algebras of global dimension three are cancellative. However, Theorem 4.2.5 covers these results.

4.3 Cancellation for 3-dimensional Sklyanin algebra

By Lemma 4.1.6(1), the case of $\operatorname{GKdim} Z(A) = 1$ is covered except for $Z(A) = k[t]$.

Lemma 4.3.1. *Let A be a noetherian connected graded algebra.*

(1) Suppose $Z(A) = k[t]$ for some homogeneous element t of positive degree. If $(A[t^{-1}])_0$ does not have any nonzero finite dimensional left module, then A is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.

(2) Suppose B is a connected graded subalgebra of A satisfying

- *(i) $Z(B) = k[t]$ for some homogeneous element $t \in B$ of positive degree.*
- *(ii) $Z(A) \cap Z(B) \neq k$.*
- *(iii) $(A[t^{-d}])_0$ does not have any nonzero finite dimensional left module for some $t^d \in Z(B) \cap Z(A)$, where d is a positive integer.*
- *(iv) A_B is finitely generated and B is noetherian.*
- *(v) $A = B \oplus C$ as a right B-module.*

Then B is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative.

Proof. (1) Let $\mathcal{P}$ be the property of not having nonzero finite dimensional left module over an algebra. Since A is connected graded, $\mathcal{P}$ fails for $A/(t)$. We claim that $\mathcal{P}$ holds for $A/(t - \alpha)$ for all $\alpha \in k^*$. By the hypothesis, $(A[t^{-1}])_0$ does not have any nonzero finite dimensional left module. Since $A/(t-\alpha)$ contains $(A[t^{-1}])_0$ by Lemma 4.1.4(1), $A/(t - \alpha)$ does not have any nonzero finite dimensional left module. So the claim holds. Therefore the $\mathcal{P}$-discriminant of A is t. Now the assertion follows from Lemma 4.1.5(2).

(2) By part (1) it suffices to show that $(B[t^{-1}])_0$ does not have any nonzero finite dimensional left module. Note that $B[t^{-1}] = B[t^{-d}]$. So it is equivalent to show that $(B[t^{-d}])_0$ does not have any nonzero finite dimensional left module. We prove this claim by contradiction. Suppose otherwise M is a nonzero finite dimensional left $(B[t^{-d}])_0$-module. By hypotheses (iii)-(iv) and by inverting t^d, $A[t^{-d}] = C[t^{-d}] \oplus B[t^{-d}]$ and $A[t^{-d}]$ is a finitely generated right $B[t^{-d}]$-module. Then $(A[t^{-d}])_0 = (C[t^{-d}])_0 \oplus (B[t^{-d}])_0$ and $(A[t^{-d}])_0$ is a finitely generated right $(B[t^{-d}])_0$-module. Hence $(A[t^{-d}])_0 \otimes_{(B[t^{-d}])_0} M$ is a nonzero finite dimensional left $(A[t^{-d}])_0$-module. This yields a contradiction. At this point, we have proved that $(B[t^{-1}])_0$ does not have any nonzero finite dimensional left module. The assertion follows from part (1). □

Lemma 4.3.1 can be applied to many examples. Here is an easy one.

Example 4.3.2. Suppose $\operatorname{char} k = 0$. Let A be a generic 3-dimensional Sklyanin algebra generated by $\{x, y, z\}$, see [GKMW, Introduction] for the relations. Then $Z(A) = k[g]$ where g is a homogeneous element of degree three. Let G be any finite group of graded algebra automorphisms of A. Let B be the fixed subring A^G. Then we claim that B is strongly Z-retractable, strongly m-Z-retractable, strongly cancellative and strongly m-cancellative. It is easy to see that hypotheses (ii), (iii), (iv) and (v) in Lemma 4.3.1(2) hold. If hypothesis (i) in Lemma 4.3.1(2) fails, then the claim follows by Lemma 4.1.6(1). If hypothesis (i) in Lemma 4.3.1(2) holds, then the claim follows by Lemma 4.3.1(2).

4.4 Cancellation for universal enveloping algebras of dimension three

The following lemma is useful in some situations.

Lemma 4.4.1. *Let A be a noetherian algebra such that*

(i) its center $Z(A)$ is the commutative polynomial ring $k[t]$ for some $t \in A$, and

(ii) t is in the ideal $[A, A]$ of A generated by the commutators and $[A, A] \neq A$.

Then A is strongly Z-retractable and strongly cancellative.

Proof. Let $\mathcal{P}$ be the property that the commutators generate the whole algebra. By (ii) the property $\mathcal{P}$ fails for the maximal ideal (t) in $k[t]$ since $[\overline{A}, \overline{A}] \neq \overline{A}$ where $\overline{A} = A/(t)$. By (ii) again, $(t - \alpha) + [A, A] = A$ for all $\alpha \neq 0$. This means that the property $\mathcal{P}$ holds for the maximal ideal $(t - \alpha)$ in $k[t]$ for all $\alpha \neq 0$. Thus the $\mathcal{P}$-discriminant of A is t. The assertion follows from Theorem 1.2.10. □

To conclude this section we give some examples of non connected graded algebras that are cancellative.

Example 4.4.2. Suppose $\operatorname{char} k = 0$. Let A be the universal enveloping algebra $U(\mathfrak{g})$ where $\mathfrak{g}$ is a 3-dimensional non-abelian Lie algebra. One can use Bianchi classification to list all 3-dimensional non-abelian Lie algebras [Ja, Section 1.4] as follows (Example 1.3.8-(c)).

(1) $\mathfrak{g} = sl_2$.

(2) $\mathfrak{g}$ is the Heisenberg Lie algebra

(3) $\mathfrak{g} = L \oplus kz$ where L is the 2-dimensional non-abelian Lie algebra.

(4) $\mathfrak{g}$ has a basis $\{e, f, g\}$ and subject to the following relations

$$[e, f] = 0, \qquad [e, g] = e, \qquad [f, g] = \alpha f$$

where $\alpha \neq 0$.

(5) $\mathfrak{g}$ has a basis $\{e, f, g\}$ and subject to the following relations

$$[e, f] = 0, \qquad [e, g] = e + \beta f, \qquad [f, g] = f$$

where $\beta \neq 0$.

For each class, one can verify that A is strongly cancellative.

(1) See [LuWZ, Example 5.11].

(2) The universal enveloping algebra of the Heisenberg Lie algebra has the center $Z = k[t]$ with t in the ideal generated by the commutators, then the assertion follows from Lemma 4.4.1.

(3) In this case $U(\mathfrak{g}) = U(L) \otimes k[z]$ with $Z(U(L)) = k$. The assertion follows from Lemma 4.2.1.

(4,5) In both cases, one can write $A := U(\mathfrak{g})$ as an Ore extension $k[e, f][g; \delta]$ for some derivation δ of the commutative polynomial ring $k[e, f]$. If the center of A is trivial, then A is strongly cancellative by Proposition 1.1.4. For the rest of proof we assume that $Z(A) \neq k$. Note that the derivation δ of $k[e, f]$ is determined by

$$\delta : \quad e \longrightarrow -e, \quad f \longrightarrow -\alpha f \tag{4.2}$$

in part (4), and by

$$\delta : \quad e \longrightarrow -(e + \beta f), \quad f \longrightarrow -f \tag{4.3}$$

in part (5). By an easy calculation, one sees that

$$Z(A) = \{x \in k[e, f] \mid \delta(x) = 0\}$$

which is a graded subring of $k[e, f]$ (which is inside A). Since A contains $U(L)$ as a subalgebra where L is the 2-dimensional non-abelian Lie algebra, A is not PI. By Lemma 1.4.13, $\operatorname{GKdim} Z(A) \leq 1$. Since $Z(A) \neq k$ and k is algebraically closed, $\operatorname{GKdim} Z(A) \geq 1$. Thus $\operatorname{GKdim} Z(A) = 1$. By Lemma 4.1.1, $Z(A)$ is a domain

that is finitely generated over k. If $Z(A)$ is not isomorphic to $k[t]$, then $Z(A)$ is strongly retractable by Lemma 1.1.20(3). (The next few lines are copied from the proof of Lemma 4.1.6(1).) By taking $\mathcal{P}$ to be a trivial property, say being an algebra, the $\mathcal{P}$-discriminant is 1. By [LeWZ, Remark 3.7(6)], Z is strongly LND_1^H-rigid. By Proposition 1.2.5(2), A is both strongly Z-retractable and strongly m-Z-retractable. Since A is noetherian, it is Hopfian by Lemma 1.1.17. By Lemmas 1.2.8 and 1.2.9, A is strongly cancellative and strongly m-cancellative. For the rest, we assume that $Z(A) \cong k[t]$ for some homogeneous element t in $k[e, f]$. By the form of δ in (4.2)-(4.3), the degree of t is at least 2. This implies that $k[e, f]/(t)$ has infinite global dimension. On the other hand, if $\alpha \neq 0$, then $k[e, f]/(t - \alpha)$ has finite global dimension (applying Lemma 4.1.4 to the algebra $k[e, f]$). Therefore $A/(t) = (k[e, f]/(t))[g; \delta]$ has infinite dimensional dimension and $A/(t - \alpha) = (k[e, f]/(t - \alpha))[g; \delta]$ has finite global dimension. Then the argument similar to the proof of Lemma 4.1.6(2) shows that A is strongly cancellative. By the way, the center of $U(\mathfrak{g})$ can explicitly be worked out, see [MR, Example 14.4.2] for some hints.

4.5 Cancellation for graded isolated singularities

We study cancellation property for some graded isolated singularities. In noncommutative algebraic geometry, Ueyama gave the following definition of a graded isolated singularity.

Definition 4.5.1. [Ue, Definition 2.2] Let A be a noetherian connected graded algebra. Then A is called a *graded isolated singularity* if

(1) $\operatorname{gldim} A$ is infinite.

(2) The associated noncommutative projective scheme Proj A (in the sense of [AZ]) has finite global dimension.

Examples of graded isolated singularities are given in [CYZ3, GKMW, MU1, MU2, Ue]. One nice example of graded isolated singularities is the fixed subring of the generic Sklyanin algebra under the cyclic permutation action [GKMW, Theorem 5.2] which is cancellative by Example 4.3.2.

Lemma 4.5.2. *Suppose* $\operatorname{char} k = 0$. *Let A be a graded isolated singularity generated in degree one and $t \in B$ be a central regular element of positive degree. Then $A/(t - \alpha)$, for every $0 \neq \alpha \in k$, has finite global dimension.*

Proof. By Lemma 4.1.4(2) it suffices to show that $(A[t^{-1}])_0$ has finite global dimension. Since there is a localizing functor from Proj A to GrMod $A[t^{-1}]$,

$$\operatorname{gldim} \operatorname{GrMod} A[t^{-1}] \leq \operatorname{gldim} \operatorname{Proj} A < \infty.$$

It is well-known that

$$\operatorname{gldim}(A[t^{-1}])_0 = \operatorname{gldim} \operatorname{GrMod} A[t^{-1}]$$

as $A[t^{-1}]$ is strongly $\mathbb{Z}$-graded. The assertion follows. □

The following result is similar to Theorem 4.1.7 for graded isolated singularities which have infinite global dimension.

Theorem 4.5.3. *Let A be a noetherian connected graded domain generated in degree 1. Suppose*

(1) $\operatorname{char} k = 0$, *and*

(2) $\operatorname{GKdim} Z(A) \leq 1$, *and*

(3) A is a graded isolated singularity.

Then A is strongly cancellative and strongly m-cancellative.

Proof. If $Z(A) \ncong k[t]$, then the assertion follows from Lemma 4.1.6(1). Now we assume that $Z(A) \cong k[t]$ where t can be chosen to be a homogeneous element of positive degree. Since A has infinite global dimension, so does $A/(t)$ by Lemma 1.4.12. For every $0 \neq \alpha \in k$, by Lemma 4.5.2, $A/(t-\alpha)$ has finite global dimension. The assertion follows from Lemma 4.1.6(2). □

Now we are ready to prove the following Corollary.

Corollary 4.5.4. *Suppose* $\operatorname{char} k = 0$. *Let A be a noetherian connected graded AS-regular algebra of global dimension three that is generated in degree 1. If A is not PI , then the dth Veronese subring $A^{(d)}$ of A is cancellative for every $d \geq 1$.*

Proof. If $d = 1$, then it follows from Theorem 4.2.5.

Next we assume that $d > 1$. Note that the Hilbert series of A is either $\frac{1}{(1-s)^3}$ or $\frac{1}{(1-s)^2(1-s^2)}$. By an easy computation, the Hilbert series of $A^{(d)}$ can not be of the form $\frac{1}{f(s)}$ for some polynomial $f(s)$. By [StZ, Theorem 2.4] and the argument before it, $A^{(d)}$ does not have finite global dimension. By [AZ, Proposition 5.10(3)], $A^{(d)}$ is a graded isolated singularity. Since A is not PI, $\operatorname{GKdim} Z(A) \leq 1$ by Lemma 1.4.13. The assertion follows from Theorem 4.5.3. □

Finally, we have the following conjecture for a general noncommutative algebra. Theorem 3.2.6 established that it is false when the characteristic of the base field is positive. However, it remains open for characteristic zero.

Conjecture 4.5.5. *[TVZ, Conjecture 0.3] Let A be a noetherian finitely generated prime algebra.*

(1) If $\operatorname{GKdim} Z(A) \leq 1$, *then A is cancellative.*

(2) If $\operatorname{GKdim} A = 3$ *and A is not PI, then A is cancellative.*

CHAPTER 5

Future work

Following the line of work presented in the main papers [BZ1, LeWZ, LuWZ] and recent advances in [TVZ, BHHV], we collect some important issues about cancellation and related subjets.

In Conjecture 4.5.5 we posed for algebras with center $Z(A) = k[t]$ and good properties these algebras are cancellative. Also, in Theorem 3.2.6 we showed that this affirmation is false when the base field has positive characteristic, so it shows that under certain conditions the center of an algebra is sufficiently rigid then the algebra is strongly cancellative. Then a new version of this conjecture suggests that over nice base fields the center completely determine cancellation.

Question 5.0.1. *[BHHV, Conjecture 2.10] Let k be an uncountable algebraically closed field of characteristic zero and let A be an affine noetherian domain over k. Suppose that $Z(A)$ is affine and cancellative (respectively strongly cancellative). Then A is cancellative (resp. strongly cancellative).*

Following this line, the next question is obvious.

Question 5.0.2. *[TVZ, Question 0.6] Let A be a noetherian connected graded Auslander-regular algebra. If $GKdim(Z(A)) = 2$, what can we say about the center $Z(A)$? For example, is $Z(A)$ always noetherian in this case?*

As in the commutative case, it is usually difficult to determine whether or not an AS-regular algebra is cancellative. In particular the PI case remains open, even for the following algebra.

Question 5.0.3. *[TVZ, Question 0.7] Let $q \in k \setminus \{0, 1\}$ be a root of unity. Is the skew polynomial ring of three variables $k_q[x_1, x_2, x_3]$ (or of odd number variables $k_q[x_1, x_2, \cdots, x_{2n+3}]$) cancellative?*

When $q = 1$ and $\operatorname{char} k = 0$, the above question is the classical Zariski Cancellation Problem which has been open for many years. Note that if $q = 1$ and $\operatorname{char} k > 0$, then $k[x_1, \ldots, x_n]$ for $n \geq 3$ is not cancellative by [Gu1, Gu2].

We have been studying cancellation for different kind of algebras of polynomial type. By Corollary 1.1.11 every noncommutative domain of Gelfand-Kirillov dimension two is cancellative. In particular, the unique universal enveloping algebra of a 2-dimensional non-abelian Lie algebra. And, by Example 4.4.2 this happens for 3-dimensional. So the next question is natural.

Question 5.0.4. *Is every universal enveloping algebras of a 4-dimensional non-abelian Lie algebra cancellative?*

In light of the Zariski cancellation problem, in section 3.3 we studied skew cancellation, that is, if $R[x;\sigma,\delta] \cong S[x';\sigma',\delta']$ implies $R \cong S$. And we proved that this holds in the derivation and automorphisms case when the coefficient ring is an affine conmutative domain of krull dimension one. But the mixed case is more difficult, even for usual ZCP. So we posed the following question.

Question 5.0.5. *[BHHV, Question 5.7] Let k be a field, let R be an affine commutative domain over k of Krull dimension one, and let σ and δ be respectively a k-algebra automorphism and a k-linear σ-derivation of R. Is $R[x;\sigma,\delta]$ cancellative?*

Several new methods were introduced to deal with the noncommutative versions of the ZCP. For example, we worked with center, retractability and detectability, the Makar-Limanov invariants, Slice theorem, Azumaya locus and $\mathcal{P}$-discriminant. In addition, [LMZ] used the Nakayama automorphisms and the discriminant method was studied in [BZ1, CPWZ1, CPWZ2]. Thus one should continue to look for new invariants and methods to handle with different kind of algebras. So the following question is sufficient valid.

Question 5.0.6. *Determine new invariants or properties that controls cancellation?*

In [LuWZ] Morita cancellation was studied. Another generalization involves the derived category of modules. Let $D(A)$ denote the derived category of right A-modules for an algebra A.

Definition 5.0.7. [LuWZ, Definition 0.2] An algebra A is called *derived cancellative* if the statement that

$$D(A[t]) \text{ is triangulated equivalent to } D(B[t]) \text{ for an algebra } B$$

implies that

$$D(A) \text{ is triangulated equivalent to } D(B).$$

The authors proved that if Z is a commutative domain, then Z is Morita cancellative if and only if Z is cancellative if and only if Z is derived cancellative. However, when A is noncommutative, it is not clear what are the relationship between these three different versions of cancellation property. An interesting problem is if the following question is true:

Question 5.0.8. *Let A be a noetherian domain k-algebra that is cancellative. Is cancellation equivalent to Morita cancellation and derived cancellation?*

Others interesting questions include to study cancellation for especial kind of algebras such as: Yang Mills, Hopf algebras, Koszul, skew Calabi Yau and Calabi Yau or even the invariant ring.

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