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Grothendieck topologies, fibered categories and descent theory: An introduction to the stack theory

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Introduction

The notion of *stack* grew out of attempts to parameterize geometric objects varying in families. Such families are classically know as *Moduli* and their understanding is a central theme in Algebraic Geometry. The formalism of stacks was introduced by A. Grothendieck and M. Artin as the natural context in which moduli problems of objects with symmetries can be tackled. The usual spaces used in geometry– Manifolds, Varieties, Schemes– are inadequate for the parametrization of geometry objects that are self similar. Instead one needs a procedure that will not only encode the way things vary in families but will also remembers the intrinsic symmetries of each object in the family.
As an example, let us consider two families of ellipses parametrized by a circle: The left-hand family is trivial, all the ellipses in this one are positioned in exactly the same way. The ellipses in the right-hand family are all the same but they are rotated in their planes. Families of this kind are called *isotrivial*. The usual parameter spaces do not capture the distinction between trivial and isotrivial families.

A moduli problem is about grouping mathematical objects into equivalence classes. The equivalence relation may vary; for example we can classify triangles up to similarity, or use the even finer classification, where triangles belong to the same family if they are congruent. But a moduli problem also requires that the classifying objects themselves have rich mathematical structure.

One of the first appearances of the stacks was the study of moduli problem of curves. To solve moduli problem for curves we need to find a geometric object, denoted *the universal family*, a smaller geometric object, called *the moduli space*, and a "good" mapping from the universal family into the moduli space. The moduli space is constructed such that every point in the space corresponds to a certain class of curves, and vice versa, every class of curves is represented by a point in the moduli space. The universal family contains all the curves and the map from the universal family to the moduli space maps a curve into the point representing the curve. The moduli space is the classifying space of all curves, and the existence of a universal family makes sure that the classification remembers the structure of the curves. Continuity of the map from the universal family into the moduli space guarantees that curves which are almost similar will correspond to points in the moduli space which are close.

Deligne and Mumford proved that the moduli problem of curves in question gives rise to what is today called a *Deligne-Mumford stack* [**DM69**], and they used this fact to prove irreducibility properties of certain moduli spaces of curves. Later, M. Artin generalized Deligne-Mumford's work by introducing *Artin stacks* [**MA74**], which have ever since proved to be a vital tool in algebraic geometry, especially in the study of moduli problems, and also in the study of quotient spaces; usually the stacks are used for the study of homological properties of quotient singularities in topology and geometry: any singular space which is locally obtained as a quotient of a manifold by a finite group of symmetries can be "effectly" replaced by a special kind of a *smooth stack* called an *Orbifold* [**Thur79**]. Orbifolds are stacks that look generically like spaces but have non-trivial groups of symmetries attached to some special points. Every scheme is a Deligne-Mumford stack, and every Deligne-Mumford stack is an Artin stack. The concept of stack is merely a categorical concept. To do geometry we have to add some conditions, and then we get the concept of *algebraic stack*.
Even if we can find a moduli space it is not at all obvious that there exists a universal family. Consider the following example: A rather coarse classification of the real numbers is to split them into to classes, 0 and different from 0. The first class contains one element (0), and the second contains the rest of the real numbers $(\neq 0)$. The moduli space consists of two distinct points, and a candidate for the universal family is the real numbers itself. But in this case the map from the universal family into the moduli space fails to be continuous, since there are points different from 0, but infinitesimal close to 0. The conclusion is that there is a moduli space, but it is not possible to construct a universal family on top of it.
In a nutshell, stacks are a new breed of spaces for algebraic geometers to work with, providing them with greater flexibility for performing constructions hitherto impossible in the category of schemes, while being manageable enough to allow the entire machinery of scheme theory to be applicable to them. The fact that the Descent theory relates to quantum cohomology and mirror symmetry, modern theoretical physics, geometric Langlands program and geometric representation theory, modern homotopy theory, symplectic geometry, perverse sheaves and intersection theory makes it a desirable subject to learn for many mathematicians and theoretical physicicist. The broad range of each of these areas clearly indicates that the field of stacks has gathered a huge momentum and is bound to become one of the main tools for the working mathematician.

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The question you raise "how can such a formulation lead to computations" doesn't bother me in the least! Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding of what was going on. Thus I never bothered about whether what would come out would be suitable for this or that, but just tried to understand - and it always turned out that understanding was all that mattered.

—Alexander Grothendieck (in a letter to Ronnie Brown, 12.04.1983)

CHAPTER 1

Preliminary notions

1.1. Category theory

A *category* $\mathcal{C}$ is defined by the following data:

1. A collection of objects denoted by $\mathrm{Ob}(\mathcal{C})$.
2. For any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ there is a set denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and referred to as the set of morphisms from A to B.
3. For any three objects A, B and C there is a rule of composition for morphisms, that is to say, a mapping

$$\begin{array}{ccc} Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) & \longrightarrow & Hom_{\mathcal{C}}(A,C) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

This data is to satisfy the following conditions:

i) For each object A in $\mathcal{C}$ there is a morphism $\mathrm{id_A} \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{A}, \mathrm{A})$, referred to as the identity morphism on A, such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we have $f \circ \mathrm{id_A} = \mathrm{f}$, and for all $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ we have $\mathrm{id_A} \circ \mathrm{g} = \mathrm{g}$.

ii) Composition of morphisms is associative, in the sense that whenever one side in the below equality is defined, so is the other and equality holds:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an isomorphism of there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{id_A}$ and $f \circ g = \mathrm{id_B}$

EXAMPLE 1.1. Let R be a ring and denoted $\mathrm{Mod(R)}$ the set of all R-modules (to right). If $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is $\mathrm{Hom}_R(A, B)$, the set of all R-homomorphism from A to B, then is easy to check that $\mathrm{Mod(R)}$ is a category.

EXAMPLE 1.2. Let $\mathcal{S}_R$ be the class of all short exact sequences of R-modules. A morphism between two short exact sequences is a 3-tuple (f, g, h) such that the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

It is easy to see that the composition the 3-tuples (f, g, h) and (f', g', h') defined $(f' \circ f, g' \circ g, h' \circ h)$ satisfies the axioms of a category.

If $\mathcal{A}$ and $\mathcal{B}$ are categories. A *covariant functor* between $\mathcal{A}$ and $\mathcal{B}$ is a function $F : \mathcal{A} \to \mathcal{B}$ such that for each object A in the category $\mathcal{A}$, FA is an object in the category $\mathcal{B}$, and each morphism $h \in \mathrm{Hom}_{\mathcal{A}}(C, D)$, $Fh \in \mathrm{Hom}_{\mathcal{B}}(FC, FD)$ such that:

i) $F(\mathrm{id_A}) = \mathrm{id_{F(A)}}$.

ii) $F(g \circ h) = Fg \circ Fh$, for all $h \in \mathrm{Hom}_{\mathcal{A}}(C, D)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(D, E)$.

Given a category $\mathcal{C}$, the opposite category $\mathcal{C}^{op}$ is the category with the same object as $\mathcal{C}$ but all morphisms reversed. In other words $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$ and if $f : B \to C$ and $g : A \to B$ are in $\mathcal{C}^{op}$ then $f \circ_{op} g := g \circ f$.
So, if $\mathcal{A}$ and $\mathcal{B}$ are categories, a *contravariant functor* F from $\mathcal{A}$ to $\mathcal{B}$ is a functor $\mathcal{A}^{op} \to \mathcal{B}$.

EXAMPLE 1.3. Let $\mathrm{Mod_R}$ be the set of all R-modules (to right), let M be a R-module. Consider the following morphisms:

$$\begin{array}{rcl} H_M : \mathrm{Mod(R)} & \longrightarrow & \mathrm{Mod(R)} \\ N & \longmapsto & \mathrm{Hom}_R(M, N) \end{array}$$

$$\begin{array}{rcl} H_M(h) : \mathrm{Hom}_R(M, N) & \longrightarrow & \mathrm{Hom}_R(M, L) \\ f & \longmapsto & h \circ f \end{array}$$

where $h : N \to L$ is a $R-$homomorphism. Then is easy to check that H_M is a covariant functor from $\mathrm{Mod(R)}$ to $\mathrm{Mod(R)}$. Similarly, consider

$$\begin{array}{rcl} G_M : \mathrm{Mod(R)} & \longrightarrow & \mathrm{Mod(R)} \\ N & \longmapsto & \mathrm{Hom}_R(N, M) \end{array}$$

$$\begin{array}{rcl} G_M(h) : \mathrm{Hom}_R(L, M) & \longrightarrow & \mathrm{Hom}_R(N, M) \\ f & \longmapsto & f \circ h \end{array}$$

then G_M is a contravariant functor of the category $\mathrm{Mod(R)}$ in itself.

$$\begin{tikzcd} N \arrow[r, "H_M"] \arrow[d, "h"'] & \mathrm{Hom}_R(M, N) \arrow[d, "H_M(h)"] \\ L \arrow[r, "H_M"'] & \mathrm{Hom}_R(M, L) \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "G_M"] & \mathrm{Hom}_R(N, M) \arrow[d, "G_M(h)"] \\ L \arrow[u, "h"] \arrow[r, "G_M"'] & \mathrm{Hom}_R(L, M) \end{tikzcd}$$

A category $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$, if: $\mathrm{Ob}(\mathcal{C}')$ is in $\mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$ for any A, B in $\mathcal{C}'$, the composition in $\mathcal{C}'$ is induced by the composition in $\mathcal{C}$ and the identity morphisms in $\mathcal{C}'$ are identity morphisms in $\mathcal{C}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all A, $B \in \mathcal{C}'$.

If $F : \mathcal{A} \to \mathcal{B}$ is a functor, F is called *fully faithful* when for any two objects A and A' of $\mathcal{A}$, the function

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$$

defined by F is a bijection. F is called *essentially surjective* if every object of $\mathcal{B}$ is isomorphic to the image of an object of $\mathcal{A}$.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be two functors. A natural transformation between F and G is a family of morphisms $\eta = \{\eta_A\}_{A \in \mathrm{Ob}(\mathcal{A})}$ with $\eta_A \in \mathrm{Hom}_{\mathcal{B}}(FA, GA)$ such that for all morphisms $f : A \to A'$ in $\mathcal{A}$ the following diagram commute

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow{\eta_{A'}} & GA' \end{array}$$

If, for every object A in $\mathcal{A}$, the morphism η_A is an isomorphism in $\mathcal{B}$, then η is said to be a *natural isomorphism* (or sometimes *natural equivalence* or *isomorphism of functors*). Two functors F and G are called *naturally isomorphic* or simply *isomorphic* if there exists a natural isomorphism from F to G. The functor F is called an *equivalence* when there exists a functor $G : \mathcal{B} \to \mathcal{A}$, such that the composite $GF : \mathcal{A} \to \mathcal{A}$ is isomorphic to $\mathrm{id}_{\mathcal{A}}$, and $FG : \mathcal{B} \to \mathcal{B}$ is isomorphic to $\mathrm{id}_{\mathcal{B}}$.

PROPOSITION 1.4. *(*[**KSHA**]*, Proposition 1.3.13) A functor is a equivalence if and only if it is both fully faithful and essentially surjective.*

REMARK 1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and let $F, G, H : \mathcal{A} \to \mathcal{B}$ be functors. Let $\eta = \{\eta_A\}_{A \in \mathrm{Ob}(\mathcal{A})}$ and $\tau = \{\tau_A\}_{A \in \mathrm{Ob}(\mathcal{A})}$ natural transformations $\eta : F \to G$ and $\tau : G \to H$ respectively. Then there is a composition $\tau \circ \eta := \{\tau_A \circ \eta_A\}_{A \in \mathrm{Ob}(\mathcal{A})}$. Thus the set of functors, denoted by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is equipped with the structure of a category, where the morphisms are natural transformations between functors.

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. If $F : \mathcal{A}' \to \mathcal{A}$ is a functor, then there is an induced functor

$$F^* : \mathrm{Hom}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Hom}(\mathcal{A}', \mathcal{B})$$

defined at the level of objects by the natural rule $F^*\Phi := \Phi \circ F : \mathcal{A}' \longrightarrow \mathcal{B}$, for any functor $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$. At the level of arrows F^* is defined by the formula $(F^*\eta)_{A'} := \eta_{FA'} : \Phi FA' \longrightarrow \Psi FA'$, for any natural transformation $\eta : \Phi \to \Psi$. Analogously, for any functor $F : \mathcal{B} \to \mathcal{B}'$ we get an induced functor

$$F_* : \mathrm{Hom}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathcal{B}')$$

An object $T \in \mathrm{Ob}(\mathcal{C})$ is called a *terminal object* (or *final object*), if for all $A \in \mathrm{Ob}(\mathcal{C})$ there exists one and only one morphism $t : A \to T$. An object $I \in \mathrm{Ob}(\mathcal{C})$ is called an *initial object*, if for all $A \in \mathrm{Ob}(\mathcal{C})$ there exists one and only one morphism $i : I \to A$.

For any category $\mathcal{C}$ and any object X of $\mathcal{C}$ we denote by $(\mathcal{C}/X)$ the *comma category*, whose objects are arrows $U \to X$ in $\mathcal{C}$, and whose arrows are commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

REMARK 1.6. For any comma category $\mathcal{C}/X$, there exists always a forgetful functor $F : \mathcal{C}/X \to \mathcal{C}$ such that for any arrow $U \to X$, $F(U \to X) = U$ and for any morphism f between $U \to X$ and $V \to X$ we have $Ff = U \to V$ such that the following diagram commutes

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

If T is a terminal object in $\mathcal{C}$, then $\mathcal{C}/T$ and $\mathcal{C}$ are isomophic as categories.

In many applications, it is necessary to lift objects over some Y along a morphism $f : X \to Y$ to objects over X. This is called a *base change* or a *pullback*. The tool to accomplish it is the construction of *fibre products*. A commutative diagram

$$\begin{tikzcd} T' \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & T \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is called *cartesian* if it induces all other commutative diagrams with the same lower-right corner; that is, for any other commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "q"'] & T \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

there is a unique morphism $h : U \to T'$ such that $q = p' \circ h$ and $g = \bar{f} \circ h$.

REMARK 1.7. Another way to say the previous is to say that T' is the *fiber product* of S' and T over S, and to write $T' = S' \times_S T$. One can also say that p' is the *base change* of p induced by the morphism f.

Whenever we have a fibered product $X_1 \times_Y X_2$ in a category, we denote by $pr_1 : X_1 \times_Y X_2 \to X_1$ and $pr_1 : X_1 \times_Y X_2 \to X_2$ the two projections. We will also a similar notation for the product of more objects.

Let A, B and C be sets, and suppose that we are given a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

We say that the diagram is an *equalizer* if f is injective, and maps A surjectively onto the subset $\{b \in B : g(b) = h(b)\} \subset B$. Equivalently, the diagram is an equalizer if $g \circ f = h \circ f$, and every function $p : D \to B$ such that $g \circ p = h \circ p$ factors uniquely through A.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \\ D \arrow[u] \arrow[ur, "p"'] & & \end{tikzcd}$$

Finally, we denote by (Set) the category of sets, by (Top) the category of topological spaces, (Grp) the category of groups, and by (Sch/S) the category of schemes over a fixed base scheme S.

1.2. Algebraic Geometry

We will assume that the reader is acquainted with the language of sheaves and schemes at the level Hartshorne's book **[Har77]**. All rings and algebras will be commutative. Let us remember some basic results:

PROPOSITION 1.8. *(**[Har77]**-II, Proposition 2.3) If $\varphi : B \to A$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces $(f, f^\sharp)$: $(\text{Spec A}, \mathcal{O}_{\text{SpecA}}) \to (\text{Spec B}, \mathcal{O}_{\text{SpecB}})$.*

PROPOSITION 1.9. *(**[Har77]**-II, Proposition 2.3) If A and B are rings, then any morphism of locally ringed spaces $(f, f^\sharp) : (\text{Spec A}, \mathcal{O}_{\text{SpecA}}) \to (\text{Spec B}, \mathcal{O}_{\text{SpecB}})$ is induced by a homomorphism of rings $\varphi : B \to A$*

PROPOSITION 1.10. *Let B be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \text{Spec B}$ we have an associated map of sheaves $f^\sharp : \mathcal{O}_{\text{SpecB}} \to f_*\mathcal{O}_X$. Taking global sections we obtain we obtain an homomorphism $\varphi : B \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map*

$$\text{Hom}_{Sch}(X, \text{Spec}(B)) \longrightarrow \text{Hom}_{Ring}(B, \Gamma(X, \mathcal{O}_X))$$

Such α is bijective.

PROOF. Let $\varphi : B \to \Gamma(X, \mathcal{O}_X)$ be a homomorphism of rings. For each $p \in X$ there exists an affine neighborhood $(U, \mathcal{O}_X|_U) \simeq (\text{Spec}\, A, \mathcal{O}_{\text{Spec}\, A})$ for some ring A, thus $B \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_{UX}} \Gamma(U, \mathcal{O}_X) \simeq A$ induces a morphism of schemes

$$(f_U, f_U^\sharp) : (U, \mathcal{O}_X|_U) \simeq (\text{Spec}\, A, \mathcal{O}_{\text{Spec}\, A}) \longrightarrow (\text{Spec}\, B, \mathcal{O}_{\text{Spec}\, B})$$

If $(V, \mathcal{O}_X|_V)$ is another affine neighborhood of p, then for any affine open set $(W, \mathcal{O}_X|_W)$ with $W \subset U \cap V$, $(f_U, f_U^\sharp)\Big|_W = (f_V, f_V^\sharp)\Big|_W$ since the following diagram commute

$$\begin{tikzcd}
 & & \Gamma(U,\mathcal{O}_X) \arrow[dr, "\rho_{WU}"] & \\
B \arrow[r, "\varphi"] & \Gamma(X,\mathcal{O}_X) \arrow[ur, "\rho_{UX}"] \arrow[dr, "\rho_{VX}"'] & & \Gamma(W,\mathcal{O}_X) \\
 & & \Gamma(V,\mathcal{O}_X) \arrow[ur, "\rho_{WV}"'] &
\end{tikzcd}$$

Thus the collection of morphisms $\{(f_U, f_U^\sharp)\}_{U \subset X}$ are compatible with the intersections, we defined $(f, f^\sharp) : X \longrightarrow \text{Spec B}$ as $(f, f^\sharp)\Big|_U := (f_U, f_U^\sharp)$ with U an affine neighborhood. Since for each affine open set $(U, \mathcal{O}_X|_U)$ the homomorphism $(f, f^\sharp)\Big|_U$ is the associated homomorphism $B \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_{UX}} \Gamma(U, \mathcal{O}_X) \simeq A$ then α is surjective. And the other hand, let $f, g \in \text{Hom}_{Sch}(X, \text{Spec}\, B)$ be such that $f^\sharp_{\text{Spec}\, B} = g^\sharp_{\text{Spec}\, B} = \varphi : B \to \Gamma(X, \mathcal{O}_X)$. For each affine open set $(U, \mathcal{O}_X|_U)$ the homomorphism induced by f and g

$$B \to \Gamma(U, \mathcal{O}_X) \simeq A$$

are the same, by proposition 1.9 $f = g$ thus α is injective. ♠

PROPOSITION 1.11 (Glueing morphisms). *Let X and Y be schemes, and let $\{U_i\}_i$ be a open cover of X. Suppose morphisms $f_i : U_i \to Y$ such that the restriction of f_i and f_j on $U_i \cap U_j$ are the same for any i, j, then there exists a unique morphism of schemes $f : X \to Y$ such that $f|_{U_i} = f_i$*

PROOF. Let $f : X \to Y$ defined as: for any $x \in X$, there exists i such that $x \in U_i$, then let $f(x) = f_i(x)$ (In this point f is trivially well defined). Now, for any $V \subset Y$, since $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ are the same for any i, j, we have the following commutative diagram

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "f_i"] \\ U_j \arrow[r, "f_j"] & Y \end{tikzcd}$$

which induces the following commutative diagram

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "f^\sharp_{iV}"] \arrow[d, "f^\sharp_{jV}"'] & \mathcal{O}_X(f^{-1}(V) \cap U_i) \arrow[d, "\rho"] \\ \mathcal{O}_X(f^{-1}(V) \cap U_j) \arrow[r, "\rho"] & \mathcal{O}_X(f^{-1}(V) \cap U_i \cap U_j) \end{tikzcd}$$

Then for any element $s \in \mathcal{O}_Y(V)$ the set $\{f^\sharp_{iV}(s) \in \mathcal{O}_X(f^{-1}(V) \cap U_i)\}_i$ is a family of sections that coincides on $f^{-1}(V) \cap U_i \cap U_j$. Hence there exists a unique element $f^\sharp_V(s) \in \mathcal{O}_X(f^{-1}(V))$ and therefore we have a homomorphism $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ such that the following diagram commute

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "f^\sharp_V"] \arrow[dr, "f^\sharp_i v"'] & \mathcal{O}_X(f^{-1}(V)) \arrow[d] \\ & \mathcal{O}_X(f^{-1}(V) \cap U_i) = \mathcal{O}_X(f_i^{-1}(V)) \end{tikzcd}$$

Thus $(f, f^\sharp) : X \to Y$ is the desired morphism. ♠

Recall that an A-module M is *flat* if every short exact sequence of A-modules

$$0 \to N' \to N \to N'' \to 0$$

tensoring with M gives an exact sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

Equivalent $M \otimes_A -$ preserve injectivity or also $\mathrm{Tor}_1^A(M,N) = 0$ for all A-modules N.

A morphism of commutative rings $f : A \to B$ is *flat* if the induced A-module structure on B makes a flat A-module. A morphism of schemes $f : X \to Y$ is flat if for every $x \in X$, the stalks $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module. A morphism of schemes is *faithfully flat* if it is flat and surjective. A homomorphism of rings $A \to B$ is *faithfully flat* if the corresponding morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat. With some additional finiteness conditions, a flat morphism behaves well topologically. If B is an algebra over the ring A, we say that B is *finitely presented* if it is the quotient of a polynomial ring $A[x_1, \dots, x_n]$ over A by a finitely generated ideal of $A[x_1, \dots, x_n]$. It is clear that if A is noetherian, every finitely generated algebra is finitely presented.

PROPOSITION 1.12. *(**[EGAIV-1]**, Proposition 1.4.4) If B is finitely presented over A, whenever we write $B = A[x_1, \dots, x_n]/I$, I is always finitely generated in $A[x_1, \dots, x_n]$*

A morphism of schemes $f : X \to Y$ is *locally of finite presentation* if for any $x \in X$ there are affine neighborhoods U of x in X and V of $f(x)$ in Y such that $f(U) \subset V$ and $\mathcal{O}(U)$ is finitely presented over $\mathcal{O}(V)$.

PROPOSITION 1.13. *(**[EGAIV-1]**, Proposition 1.4.3)*

i) Any local isomomorphism is locally of finite presentation.
ii) The composite of morphisms locally of finite presentation is locally of finite presentation.
iii) Given a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

if $X \to Y$ is locally of finite presentation, so is $X' \to Y'$.

PROPOSITION 1.14. *(**[EGAIV-2]**, Proposition 2.1.2) Let $f : X \to Y$ be a morphism of schemes. Then the following are equivalent:*

i) f is flat.
ii) For any $x \in X$, there are affine neighborhoods U of x and V of $f(x)$ in Y such that $f(U) \subset V$, and $\mathcal{O}(U)$ is flat over $\mathcal{O}(V)$.
iii) For any open affine subsets U in X and V in Y such that $f(U) \subset V$, $\mathcal{O}(U)$ is flat over $\mathcal{O}(V)$.

PROPOSITION 1.15. *(**[EGAIV-2]**, Proposition 2.1.4)*

i) The composite of flat morphisms is flat.
ii) Given a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

if $X \to Y$ is flat, so is $X' \to Y'$.

PROPOSITION 1.16. *(**[EGAIV-2]**, Proposition 2.4.6) A flat morphism that is locally of finite presentation is open.*

PROPOSITION 1.17. *Let R be a ring, then* Spec R *is quasi-compact, i.e. every open cover has a finite subcover.*

PROOF. Is enough to see that if $\operatorname{Spec} R = \bigcup_\alpha D(f_\alpha)$ then there exist finites $f_{\alpha_1}, \ldots, f_{\alpha_n}$ such that $\operatorname{Spec} R = D(f_{\alpha_1}) \bigcup \ldots \bigcup D(f_{\alpha_n})$. We compute

$$\begin{aligned}\operatorname{Spec} R &= \bigcup_\alpha D(f_\alpha)\\ &= \bigcup_\alpha \operatorname{Spec} R - V(f_\alpha)\\ &= \bigcup_\alpha \operatorname{Spec} R - V(I_\alpha)\\ &= \operatorname{Spec} R - \bigcap_\alpha V(I_\alpha)\\ &= \operatorname{Spec} R - V(\sum_\alpha I_\alpha)\end{aligned}$$

where $I_\alpha = \langle f_\alpha \rangle$. Thus $V(\sum_\alpha I_\alpha) = 0$. By "weak Nullstellesantz" $\sum_\alpha I_\alpha = R$, since $1 \in R = \sum_\alpha I_\alpha$ then there exist $i_{\alpha_j} \in I_{\alpha_j}$ with $j = 1, \ldots, n$ such that $1 = i_{\alpha_1} + \ldots + i_{\alpha_n}$, so for each $r \in R$ we have $r = r.1 = \sum_{j=1}^{n} I_{\alpha_j}$ and therefore $\sum_{j=1}^{n} I_{\alpha_j} = R$. Hence $V(\sum_{j=1}^{n} I_{\alpha_j}) = \emptyset$ and

$$\begin{aligned}\operatorname{Spec} R &= \operatorname{Spec} R - V(\sum_{j=1}^{n} I_{\alpha_j})\\ &= \operatorname{Spec} R - \bigcap_{j=1}^{n} V(I_{\alpha_j})\\ &= \bigcup_{j=1}^{n} \operatorname{Spec} R - V(f_{\alpha_j})\\ &= \bigcup_{j=1}^{n} D(f_{\alpha_j})\end{aligned}$$

Thus $\operatorname{Spec} R$ is quasi-compact. ♠

A morphism of schemes $X \to Y$ is *quasi-compact* if the inverse image in X of a quasi-compact open subset of Y is quasi-compact.

LEMMA 1.18. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent.*

i) There is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i.
ii) For every open affine subset $V \subset Y$, $f^{-1}(V)$ is quasi-compact.

PROOF. Let $V \subset Y$ be an open affine, and V_i be an open covering by affines of Y such that $f^{-1}(V_i)$ is quasi-compact for all i. Let $U_\alpha \subset X$ be a cover of $f^{-1}(V)$. Since V is quasi-compact and V_i cover to V, there is a finite set I such that $\bigcup_{i\in I} V_i = Y$. Now, because the $U_\alpha's$ cover $f^{-1}(V_i)$, they must cover each $f^{-1}(V_i)$ and as $f^{-1}(V_i)$ is quasi-compact, we can find a finite set J such that $\bigcup_{\alpha\in J} U_{\alpha,i} = f^{-1}(V_i)$. Further

$$\begin{aligned}\bigcup_{i\in I}\bigcup_{\alpha\in J} U_{\alpha,i} &= \bigcup_{i\in I} f^{-1}(V_i)\\ &== f^{-1}(\bigcup_{i\in I} V_i)\\ &= f^{-1}(Y)\end{aligned}$$

so, $f^{-1}(Y)$ is quasi-compact. The reciprocal is obvious. ♠

Hence, it is easy to prove the following.

PROPOSITION 1.19. *([**EGAI**], 6.6]) Let $f : X \to Y$ be a morphism of schemes. The following are equivalent.*

i) f is quasi-compact.
ii) The inverse image of an open affine subscheme of Y is quasi-compact.
iii) There exists a covering $Y = \bigcup_i V_i$ by open affine subschemes, such that the inverse image in X of each V_i is quasi-compact.

In particular, a morphism from a quasi-compact scheme to an affine scheme is quasi-compact.

PROPOSITION 1.20. *([**EGAI**], Proposition 6.6.4])*

i) The composite of quasi-compact morphisms is quasi-compact.
ii) Given a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

if $X \to Y$ is quasi-compact, so is $X' \to Y'$.

This propositions we will serve to define two important ”topologies”.

CHAPTER 2

Representable functors and Grothendieck topologies

2.1. Representable functors and the Yoneda Lemma

Let $\mathcal{C}$ be a category. For any object X in $\mathcal{C}$ there is a functor

$$h_X : \mathcal{C}^{op} \to (Set)$$

which sends an object U of $\mathcal{C}$ to the set $h_X U = \mathrm{Hom}_{\mathcal{C}}(U, X)$. If $\alpha : U' \to U$ is an arrow in $\mathcal{C}$, then $h_X \alpha : h_X U \to h_X U'$ is defined to be composition with α.

$$\begin{tikzcd}
U \arrow[r, dash] & h_X U \arrow[d, "h_X\alpha"] \\
U' \arrow[u, "\alpha"] \arrow[r, dash] & h_X U'
\end{tikzcd}$$

Clearly, if $\alpha' : U'' \to U'$ is another morphism in $\mathcal{C}$ then $h_X(\alpha \circ \alpha') = h_X\alpha' \circ h_X\alpha$. When $\mathcal{C}$ is the category of schemes over a fixed base scheme, h_X is often called the *functor of points* of X. From Grothendieck's point of view, if X and Y are schemes, then an element of $h_X U$ can be considered as an Y-valued point of X.
The functor of points induces the so-called *Yoneda's functor*. An arrow $f : X \to Y$ yields a function $h_f U : h_X U \to h_Y U$ for each object U of $\mathcal{C}$, obtained by composition with f. It is

$$\begin{array}{rcl} h_f U : h_X U & \longrightarrow & h_Y U \\ g & \longmapsto & f \circ g \end{array}$$

This define a morphism $h_X \to h_Y$ such that for all arrows $\alpha : U' \to U$ the following diagram commutes

$$\begin{tikzcd}
U & h_X U \arrow[r, "h_f U"] \arrow[d, "h_X\alpha"'] & h_Y U \arrow[d, "h_Y\alpha"] \\
U' \arrow[u, "\alpha"] & h_X U' \arrow[r, "h_f U'"] & h_Y U'
\end{tikzcd}$$

Sending each object X of $\mathcal{C}$ to h_X, and each arrow $f : X \to Y$ of $\mathcal{C}$ to $h_f : h_X \to h_Y$ defines a functor $\mathcal{C} \xrightarrow{Yon} \mathrm{Hom}(\mathcal{C}^{op}, (Set))$, the *Yoneda's functor*.

PROPOSITION 2.1. *(Weak version of Yoneda's lemma) Let $\mathcal{C}$ be a category. Let X and Y be objects of $\mathcal{C}$. The function*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathrm{Hom}(h_X, h_Y) \\ f & \longmapsto & h_f \end{array}$$

is bijective. In other words, the functor $\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{op}, (Set))$ *is fully faithful.*

PROOF. Let $\phi \in \mathrm{Hom}(h_X, h_Y)$. Then there exists a morphism

$$\phi_X : h_X X \to h_Y X$$

.

Let $f := \phi_X(\mathrm{id}_X) : X \to Y$. Thus for each object U in $\mathcal{C}$

$$h_f U(g) = f \circ g = \phi_X(\mathrm{id}_X)(g)$$

for all $g \in \mathrm{Hom}_{\mathcal{C}}(U, X)$. But $\phi_X(\mathrm{id}_X)(g) = \phi_U(g)$ since ϕ is a natural transformation

$$\begin{tikzcd} X & h_X X \arrow[r, "\phi_X"] \arrow[d, "h_X g"'] & h_Y X \arrow[d, "h_Y g"] \\ U \arrow[u, "g"] & h_X U \arrow[r, "\phi_U"] & h_Y U \end{tikzcd}$$

Thus $h_f = \phi$.

Conversely, for any morphism $f : X \to Y$ we have $h_f : h_X \to h_Y$ in $\mathrm{Hom}(\mathcal{C}, (Set))$. Then $h_f X : h_X X \to h_Y X$ and $h_f X(\mathrm{id}_X) = f \circ \mathrm{id}_X = f$. Hence the assignment $f \mapsto h_f$ is injective.

♠

DEFINITION 2.2. A *representable* functor on the category $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{op} \to (Set)$$

which is isomorphic to a functor of the form h_X for some object X of $\mathcal{C}$. If this happens, we say that F is *represented* by X.

REMARK 2.3. Given two isomorphisms $F \simeq h_X$ and $F \simeq h_Y$, we have that the resulting isomorphism $h_X \simeq h_Y$ comes from a unique isomorphism $X \simeq Y$ in $\mathcal{C}$, because of the weak form of Yoneda's lemma. Hence *two objects representing the same functor are canonically isomorphic*.

PROPOSITION 2.4. *(Strong version of Yoneda's lemma) Let* $F : \mathcal{C}^{op} \to (Set)$ *be a functor, then for all object* X *in* $\mathcal{C}$, $\mathrm{Hom}(h_X, F) \simeq FX$.

PROOF. Given a natural transformation $\tau : h_X \to F$, one gets an element $\xi \in FX$, defined as the image of the identity map $\mathrm{id}_X \in h_X X$ via the function $\tau_X : h_X X \to FX$. This construction defines a function $\mathrm{Hom}(h_X, F) \to FX$.
Conversely, given an element $\xi \in FX$, one can define a morphism $\tau : h_X \to F$ as follows. Given an object U of $\mathcal{C}$, an element of $h_X U$ is an arrow $f : U \to X$; this arrow induces a function $Ff : FX \to FU$. We define a function

$$\begin{array}{rcl} \tau_U : h_X U & \longrightarrow & FU \\ f & \longmapsto & Ff(\xi) \end{array}$$

It is straightforward to check that the τ is a natural transformation, in this way we have defined functions

$$\mathrm{Hom}(h_X, F) \longrightarrow FX$$

and

$$FX \longrightarrow \mathrm{Hom}(h_X, F)$$

These two functions are inverse to each other. Let $\xi \in FX$, then ξ define $\tau : h_X \to F$ such that for each object U in $\mathcal{C}$

$$\begin{array}{rcl} \tau_U : h_X U & \longrightarrow & FU \\ f & \longmapsto & Ff(\xi) \end{array}$$

Applying the function $\mathrm{Hom}(h_X, F) \to FX$ to τ we have

$$\tau(\mathrm{id}_X) = Fid_X(\xi) = \mathrm{id}_{FX}(\xi) = \xi$$

And the other hand, let $\tau : h_X \to F \in \mathrm{Hom}(h_X, F)$, then applying $\mathrm{Hom}(h_X, F) \to FX$ to τ we have $\xi = \tau_X(\mathrm{id}_X) \in FX$. Applying $FX \to \mathrm{Hom}(h_X, F)$ to ξ we have a morphism $\psi : h_X \to F$ such that for each object U in $\mathcal{C}$

$$\begin{array}{rcl} \psi_U : h_X U & \longrightarrow & FU \\ f & \longmapsto & Ff(\xi) = Ff(\tau_X \mathrm{id}_X) \end{array}$$

Since the following diagram commutes

$$\begin{tikzcd} h_X X \arrow[r, "\tau_X"] \arrow[d, "h_X f"'] & FX \arrow[d, "Ff"] \\ h_X U \arrow[r, "\tau_U"] & FU \end{tikzcd}$$

then $Ff(\tau_X(\mathrm{id}_X)) = \tau_U(f)$, thus $\psi = \tau$. Therefore $\mathrm{Hom}(h_X, F) \longrightarrow FX$ and $FX \longrightarrow \mathrm{Hom}(h_X, F)$ are inverse to each other. Hence, $\mathrm{Hom}(h_X, F) \simeq FX$.

♠

We can use Yoneda's Lemma to give a very important characterization of representable functors.

DEFINITION 2.5. Let $F : \mathcal{C}^{op} \to (Set)$ be a functor. A *universal object* for F is a pair (X, ξ) consisting of an object X of $\mathcal{C}$, and an element $\xi \in FX$, with the property that for each object U of $\mathcal{C}$ and each $\sigma \in FU$, there is a unique arrow $f : U \to X$ such that $Ff(\xi) = \sigma$.

$$\begin{tikzcd} X \arrow[r, dash] & FX \arrow[d, "Ff"] & .\xi \\ U \arrow[u, "f"] \arrow[r, dash] & FU & .\sigma \end{tikzcd}$$

Note that the pair (X, ξ) is an universal object if the morphism $h_X \to F$ defined by ξ is a isomorphism. Since every natural transformation $h_X \to F$ is defined by some object $\xi \in FX$, we get the following proposition.

PROPOSITION 2.6. *A functor $F : \mathcal{C}^{op} \to (Set)$ is representable if and only if it has a universal object*

Also, if F has a universal object (X, ξ), then F is represented by X. Yoneda's lemma ensures that the natural functor (Yoneda's functor) $\mathcal{C} \to \operatorname{Hom}(\mathcal{C}^{op}, (Set))$ which sends an object X to the functor h_X is an equivalence of $\mathcal{C}$ with the category of representable functors.

EXAMPLES 2.7. Representable and non-representable functors.

i) Consider the functor

$$\begin{array}{rcl} P : (Set)^{op} & \longrightarrow & (Set) \\ S & \longmapsto & P(S) \end{array}$$

where $P(S)$ is the set of subsets of S. If $f : S \to T$ is a function, then $Pf : P(T) \to P(S)$ is defined by $Pf(\tau) = f^{-1}\tau$ for all $\tau \subset T$.
Given a subset $\sigma \subset S$, there is a unique function $\chi_\sigma : S \to \{0,1\}$ such that $\chi_\sigma^{-1}(\{1\}) = \sigma$, namely the *characteristic function*, defined by

$$\chi_\sigma(s) = \begin{cases} 1 & \text{if } s \in \sigma \\ 0 & \text{if } s \notin \sigma. \end{cases}$$

Hence the pair $(\{0,1\}, \{1\})$ is a universal object, and the functor P is represented by $\{0,1\}$.

ii) Consider the category (Top) of all topological spaces, with the arrows being given by continuous functions. Consider the functor

$$\begin{array}{rcl} F : (Top)^{op} & \longrightarrow & (Set) \\ S & \longmapsto & F(S) \end{array}$$

where $F(S)$ is the set of all open subspaces of S. Endow $\{0,1\}$ with the coarsest topology in which the subset $\{1\} \subset \{0,1\}$ is open; the open subsets in this topology are $\emptyset, \{1\}$ and $\{0,1\}$. Note that a function $S \to \{0,1\}$ is continuous if and only if $f^{-1}(\{1\})$ is open in S. Thus reasoning as before one sees that the pair $(\{0,1\}, \{1\})$ is a universal object for this functor. The space $\{0,1\}$ is called *the Sierpinsky space*.

iii) Consider the category (Grp) of all groups, and consider the functor

$$Sgr : (Grp)^{op} \to (Set)$$

that associates with each group G the set of all its subgroups. If $f : G \to H$ is a group homomorphism, we take $Sgrf : SgrH \to SgrG$ to be the function associating with each subgroup of H its inverse image in G.
This is not representable: there does not exist a group Γ, together with a subgroup $\Gamma_1 \subset \Gamma$, with the property that for all groups G with a subgroup $G_1 \subset G$, there is a unique homomorphism $f : G \to \Gamma$ such that $f^{-1}\Gamma_1 = G_1$; for example, if we take the subgroup $\{0\} \subset \mathbb{Z}$, there should be a unique homomorphism $f : \mathbb{Z} \to \Gamma$ such that $f^{-1}\Gamma_1 = \{0\}$. But given one such f , then the homomorphism

$$\begin{array}{rcl} \mathbb{Z} & \longrightarrow & \Gamma \\ n & \longmapsto & f(2n) \end{array}$$

also has this property, and is different, so this contradicts uniqueness.

iv) Recall the Proposition 1.10. Let $S = \operatorname{Spec} R$. Consider the affine line $\mathbb{A}^1_S = \operatorname{Spec} R[x]$. We have a functor

$$\mathcal{O} : (Sch/S)^{op} \longrightarrow (Set)$$

that sends each S-scheme U to the ring of global sections $\mathcal{O}(U)$. If $f : U \to V$ is a morphism of schemes, the corresponding function $\mathcal{O}(V) \to \mathcal{O}(U)$ is that induced by $f^\sharp : \mathcal{O}_V \to f_*\mathcal{O}_U$. Then is clear that $x \in \mathcal{O}(\mathbb{A}^1_S)$, and given a scheme U over S, and an element $f \in \mathcal{O}(U)$, there is a unique morphism $U \to \mathbb{A}^1_s$ such that the pullback of x to U is precisely f. This means that the functor $\mathcal{O}$ is represented by $\mathbb{A}^1_S$, and the pair $(\mathbb{A}^1_S, x)$ is a universal object.

2.2. Sheaves in Grothendieck topologies

2.2.1. Grothendieck topologies. When algebraic geometers work over the complex numbers, they get two topologies to work with, namely, the classical topology, which has the sort of open sets analysts and differential geometers use all the time, and the Zariski topology, which has the property that every open set is dense. It is beloved by algebraists for the simple reason that it captures a lot of the algebraic information. For instance, dimension can be defined by using irreducible subsets in the Zariski topology, which just correspond to irreducible polynomials. There is a very strong algebra-geometry dictionary for this topology.

Unfortunately, when algebraic geometers decide to work in positive characteristic, say in the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime number p, there is only the Zariski topology. So we need some way to generalize the notion of topology so that all varieties have small open sets.

We are in position to see that a presheaf on a topological spaces X can be considered as a functor. A presheaf of sets on X is a functor $X_{cl} \to (Set)$, where X_{cl} is the category in which the objects are the open subsets of X, and the arrows are given by inclusions; and this is a sheaf when it satisfies appropriate gluing conditions.

There is a very general notion of sheaf in a Grothendieck topology. In a Grothendieck topology the "open sets" of a spaces are maps into this space; instead of intersections we have to look at fibered products, while unions play no role. The axioms do not describe the "open sets", but the coverings of a space.

DEFINITION 2.8. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is the assignment to each object U of $\mathcal{C}$ a collection of sets of arrows $\{U_i \to U\}$, called *coverings* of U, such that the following conditions are satisfied:

i) If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.
ii) If $\{U_i \to U\}$ is a covering and $V \to U$ is any arrow, then the fibered products $\{U_i \times_U V\}$ exist, , and the collections of projections $\{U_i \times_U V \to V\}$ is a covering (of V).
iii) If $\{U_i \to U\}$ is a covering, and for each index i we have a covering $\{V_{ij} \to U_i\}$ (here j varies on a set depending on i), the collection of composites $\{V_{ij} \to U_i \to U\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

REMARK 2.9. If $\{U_i \to U\}$ and $\{V_j \to U\}$ are two coverings of the same object, then $\{U_i \times_U V_j \to U\}$ is also a covering (from ii) and iii)).

DEFINITION 2.10. A set $\{U_i \to U\}$ of functions, or morphisms of schemes, is called *jointly surjective* when the set-theoric union of their images equals U.

EXAMPLE 2.11 (The site of a topological space). Let X be a topological space. Then we get a Grothendieck topology on X_{cl} (defined above) by associating with each open subset $U \subset X$ a covering given by the set of all open injections of the open subsets in an open covering in the usual sense:

$$\{U_i \to U\} \text{ is a covering if and only if } U = \bigcup_i U_i$$

In this case, if $U_1 \to U$ and $U_2 \to U$ are arrows, the fibered product $U_1 \times_U U_2$ is the intersection $U_1 \cap U_2$.

EXAMPLE 2.12 (The small Zariski site). The small Zariski site of a scheme X is the full subcategory of Sch/X of objects $U \to X$ that are open immersions equipped with a Grothendieck topology by defining a cover $\{U_i \to X\}$ to be a jointly surjective set of open embeddings, this gives something that can be thought of as the standard Zariski topology. Here, an open embedding is a morphism $V \to X$ that gives an isomorphism of V with a open subvariety of X and not simply as the embedding of an open subvariety.

EXAMPLE 2.13 (The big Zariski site). The big Zariski site of a scheme X is the category Sch/X equipped with a Grothendieck topology by defining a cover $\{U_i \to U\}$ to be a jointly surjective set of open embeddings.

EXAMPLE 2.14. Let $\mathcal{C}$ be a site, S be an object of $\mathcal{C}$. We define the *comma topology* on the comma category $(\mathcal{C}/S)$ as the topology in which a covering of an object $U \to S$ of $(\mathcal{C}/S)$ is a collection of arrows $\{U_i \to S\}$ with

$$\begin{tikzcd} U_i \arrow[rr, "f_i"] \arrow[dr] & & U \arrow[dl] \\ & S & \end{tikzcd}$$

such that the collection $\{f_i : U_i \to U\}$ is a covering in $\mathcal{C}$. In other words, the coverings of $U \to S$ are simply the coverings of U.

2.2.2. Sheaves. The classical condition that F be a sheaf can easily be generalized to any site.

DEFINITION 2.15. Let $\mathcal{C}$ be a site, $F : \mathcal{C}^{op} \to (Set)$ a functor.

i) F is *separated* if, given a covering $\{U_i \to U\}$ and two sections a and b in FU whose pullbacks to each FU_i coincide, it follows that $a = b$.
ii) F is a *sheaf* if the following condition is satisfied: Suppose that we are given a covering $\{U_i \to U\}$ in $\mathcal{C}$, and a set of elements $a_i \in FU_i$. Let $\mathrm{pr}_1 : U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2 : U_i \times_U U_j \to U_j$ the first and second projection respectively, and assume that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in F(U_i \times_U U_j)$ for all i and j. Then there is a unique section $a \in FU$ whose pullback to FU_i is a_i for all i.
iii) If F and G are sheaves on a site $\mathcal{C}$, a morphism of sheaves $F \to G$ is a natural transformation of functors.

REMARK 2.16. A sheaf on a site is clearly separated.
One can also define sheaves of groups, rings, and so on, as usual: a functor from $\mathcal{C}^{op}$ to the category of groups, or rings, is a sheaf if its composite with the forgetful functor to the category of sets is a sheaf.
A equivalent way to say that F is a sheaf is the following: Take a functor $F : \mathcal{C}^{op} \to (Set)$ and a covering $\{U_i \to U\}$ in $\mathcal{C}$. There is a diagram

$$FU \longrightarrow \prod_i FU_i \underset{\mathrm{Pr}_2^*}{\overset{\mathrm{Pr}_1^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j) \quad (\bigstar)$$

where the function $FU \to \prod_i FU_i$ is induced by the restrictions $FU \to FU_i$, while

$$\mathrm{Pr}_1^* : \prod_{\mathrm{i}} \mathrm{FU_i} \longrightarrow \prod_{\mathrm{i,j}} \mathrm{F(U_i \times_U U_j)}$$

sends an element $(a_i) \in \prod_i FU_i$ to the element $\mathrm{Pr}_1^*(\mathrm{a_i}) \in \prod_{\mathrm{i,j}} \mathrm{F(U_i \times_U U_j)}$ whose component in $F(U_i \times_U U_j)$ is the pullback $\mathrm{pr}_1^* a_i$ of a_i along the first projection $U_i \times_U U_j \to U_i$. Analogously we define

$$\mathrm{Pr}_2^* : \prod_{\mathrm{i}} \mathrm{FU_i} \longrightarrow \prod_{\mathrm{i,j}} \mathrm{F(U_i \times_U U_j)}$$

Then F is a sheaf if and only if the diagram $(\bigstar)$ is an equalizer for all coverings $\{U_i \to U\}$ in $\mathcal{C}$. To see this, suppose that F is a sheaf. Then, clearly the function $FU \to \prod_i FU_i$ is injective because F is separated. Let $(a_i) \in \prod_i FU_i$ such that $\mathrm{Pr}_1^*(\mathrm{a_i}) = \mathrm{Pr}_2^*(\mathrm{a_i})$, then $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_i$ in $F(U_i \times_U U_j)$. Since F is a sheaf there exists a unique section $a \in FU$ whose the pullback to FU_i is a_i for all i, hence the function $FU \to \prod_i FU_i$ is surjective onto the subset $\{(a_i) \in \prod_i FU_i : \mathrm{Pr}_1^*(\mathrm{a_i}) = \mathrm{Pr}_2^*(\mathrm{a_i})\}$.
Recyprocally, if the diagram $(\bigstar)$ is an equalizer, suppose that for any covering $\{U_i \to U\}$ in $\mathcal{C}$ and any set of elements $a_i \in FU_i$, $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_i$ in $F(U_i \times_U U_j)$ for all i and j. Since $FU \to \prod_i FU_i$ is surjective onto the subset $\{(a_i) \in \prod_i FU_i : \mathrm{Pr}_1^*(\mathrm{a_i}) = \mathrm{Pr}_2^*(\mathrm{a_i})\}$ and injective, there exists a unique $a \in FU$ whose pullback to FU_i is a_i for all i. Thus F is a sheaf.

REMARK 2.17. There is always a bit of confusion as to whether it is necessary to say something about the set of sections of a sheaf on a topological spaces over the empty set $\emptyset$. The definition right. The existence of an empty function from $\emptyset$ to $\emptyset$ is required to make the category (Set) a category (each object needs to have an "identity morphism"). The existence of a unique empty function from $\emptyset$ into each set A means that the empty set is an initial object in the category (Set). Now, since

$$\prod_{i \in I} X_i = \left\{ g : I \to \bigcup_{i \in I} X_i : \text{for all } i, g(i) \in X_i \right\}$$

if I is empty, the only such g is the empty function which is the unique subset of $\emptyset \times \emptyset$ that is a function $\emptyset \to \emptyset$, thus

$$\prod_{\emptyset} = \{\emptyset \to \emptyset\} = \{\emptyset\}$$

Now, for any index set I we can consider the covering of $U = \emptyset$ by sets $U_i = \emptyset$ for $i \in I$. This gives us that

$$FU \xrightarrow{f} \prod_i FU_i \underset{\mathrm{Pr}_2^*}{\overset{\mathrm{Pr}_1^*}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer. For $I = \emptyset$ we have

$$FU \xrightarrow{f} T \underset{\mathrm{Pr}_2^*}{\overset{\mathrm{Pr}_1^*}{\rightrightarrows}} T$$

where T is a terminal object and the two arrows on the right are id_T. The equalizer condition says that for any object D in (Set) and morphism $p : D \to T$, if $\mathrm{id}_T \circ p = \mathrm{id}_T \circ p$ then there exists a unique morphism $q : D \to FU$ such that $f \circ q = p$. There exists always a unique morphism from D to T and the equality of the two paths is always given. On the other hand, any morphism $D \to FU$ make commutes the diagram

$$\begin{tikzcd} FU \arrow[r, "f"] & T \arrow[r, shift left, "\mathrm{id}_T"] \arrow[r, shift right, "\mathrm{id}_T"'] & T \\ D \arrow[u, "q"] \arrow[ur, "p"'] & & \end{tikzcd}$$

Therefore FU is an object such that for any D there exists a unique morphism $D \to FU$. In other words, $F\emptyset$ is a terminal object.
Thus in the definition of presheaf of groups or rings is not necessary the condition that $F\emptyset = 0$, because in this case 0 is a terminal object. (compare the definition of presheaf in [**Har77**]).

2.2.3. Sieves. Let U be an object of a subcategory $\mathcal{C}$. A *sieve* on U is a subfunctor of $h_U : \mathcal{C}^{op} \to (Set)$.
Given an object U in a category $\mathcal{C}$ and a set of arrows $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, we define a subfunctor $h_{\mathcal{U}}$ of h_U, by taking $h_{\mathcal{U}}(T)$ to be the set of arrows $\{T \to U\}$ with the property that for some i there is a factorization $T \to U_i \to U$, i.e. $h_{\mathcal{U}}$ is the *sieve* associated with the covering $\mathcal{U}$.
Now, let $\mathcal{U} = \{U_i \to U\}$ be a set of arrows, $F : \mathcal{C}^{op} \to (Set)$ a functor. We define $F\mathcal{U}$ to be the set of elements of $\prod_i FU_i$ whose images in $\prod_{i,j} F(U_i \times_U U_j)$ are equal. Then the restrictions $FU \to FU_i$ induce a function $FU \to F\mathcal{U}$; by definition, a sheaf is a functor F such that $FU \to F\mathcal{U}$ is a bijection for all coverings $\mathcal{U} = \{U_i \to U\}$.

PROPOSITION 2.18. *There is a canonical bijection* $R : \mathrm{Hom}(h_{\mathcal{U}}, F) \simeq F\mathcal{U}$ *such that the diagram*

$$\begin{tikzcd} \mathrm{Hom}(h_U, F) \arrow[r] \arrow[d] & FU \arrow[d] \\ \mathrm{Hom}(h_{\mathcal{U}}, F) \arrow[r, "R"] & F\mathcal{U} \end{tikzcd} \qquad (\bigstar)$$

in which the top row is the Yoneda isomorphism, the left hand column is the restriction function induced by the embedding of $h_{\mathcal{U}}$ in h_U and the right hand column is induced by the restriction functions $FU \to FU_i$, commutes.

PROOF. Take a natural transformation $\phi : h_{\mathcal{U}} \to F$. Note that for each i, the arrow $U_i \to U$ is an object of $h_{\mathcal{U}} U_i$, thus we get an element

$$R\phi := (\phi_{U_i}(U_i \to U)) \in \prod_i FU_i$$

Since ϕ is a natural transformation and the following diagram commutes

$$\begin{tikzcd} U_i \times_U U_j \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & U_j \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd}$$

the pullbacks $\mathrm{pr}_1^* \phi_{U_i}(U_i \to U)$ and $\mathrm{pr}_2^* \phi_{U_j}(U_j \to U)$ to $F(U_i \times_U U_j)$ both coincide with $\phi_{U_i \times_U U_j}(U_i \times_U U_j \to U)$ (is obvious that any arrow $U_i \times_U U_j \to U$ is an object of $h_{\mathcal{U}}(U_i \times_U U_j)$), hence $R\phi$ is an element of $F\mathcal{U}$. This defines a function

$$R : \mathrm{Hom}(h_{\mathcal{U}}, F) \longrightarrow F\mathcal{U}$$

Since the following diagram commutes

$$\begin{tikzcd} h_U U \arrow[r, "\phi_U"] \arrow[d] & FU \arrow[d] \\ h_U U_i \arrow[r, "\phi_{U_i}"] & FU_i \end{tikzcd}$$

the commutativity of the diagram ($\bigstar$) is clear. Let us see that R is injective. Take two natural transformations $\phi, \psi : h_{\mathcal{U}} \to F$ such that $R\phi = R\psi$. Consider an element $T \to U$ of some $h_{\mathcal{U}} T$; by definition, this factors as $T \xrightarrow{f} U_i \to U$ for some arrow $f : T \to U_i$. By definition of a natural transformation

$$\begin{tikzcd} U_i & h_{\mathcal{U}} U_i \arrow[r, "\phi_{U_i}"] \arrow[d, "h_{\mathcal{U}} f"'] & FU_i \arrow[d, "f^*"] \\ T \arrow[u, "f"] & h_{\mathcal{U}} T \arrow[r, "\phi_T"] & FT \end{tikzcd}$$

we have $\phi_T(T \to U) = f^* \phi_{U_i}(U_i \to U) = f^* \psi_{U_i}(U_i \to U) = \psi_T(T \to U)$. Hence $\phi = \psi$.

Now let us show that R is surjective. Take an element $(\xi_i) \in F\mathcal{U}$; we need to define a natural transformation $h_{\mathcal{U}} \to F$. If $T \to U$ is an element of $h_{\mathcal{U}} T$, choose a factorization $T \xrightarrow{f} U_i \to U$. This define an element $f^* \xi_i$ of FT. This element is independent of the factorization: two factorizations $T \xrightarrow{f} U_i \to U$ and $T \xrightarrow{g} U_j \to$

U give an arrow $T \to U_i \times_U U_j$, whose composites with $\mathrm{pr}_1 : U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2 : U_i \times_U U_j \to U_j$ are equal to f and g respectively. Since $\mathrm{pr}_1^* \xi_i = \mathrm{pr}_1^* \xi_j$, we see that $f^* \xi_i = g^* \xi_j$. This defines a function $h_{\mathcal{U}} T \to FT$ for each T. Since $\alpha^* \phi_T(T \to U) := \alpha^* f^* \xi_i := \phi_S(S \xrightarrow{\alpha} T \xrightarrow{f} U_i \to U) = S \xrightarrow{\alpha} T \to U := h_{\mathcal{U}} \alpha(T \to U)$, then the following diagram commutes

$$\begin{array}{ccc} T & & \\ \uparrow{\scriptstyle\alpha} & & \\ S & & \end{array} \qquad \begin{array}{ccc} h_{\mathcal{U}} T & \xrightarrow{\phi_T} & FT \\ {\scriptstyle h_{\mathcal{U}}\alpha}\downarrow & & \downarrow{\scriptstyle\alpha^*} \\ h_{\mathcal{U}} S & \xrightarrow{\phi_S} & FS \end{array}$$

Hence, this defines a natural transformation $\phi : h_{\mathcal{U}} \to F$. Since $R\phi = (\phi_{U_i}(U_i \to U))$ and $U_i \xrightarrow{\mathrm{id}_{U_i}} U_i \to U$, then $\mathrm{id}_{U_i}^* \xi_i = \xi_i$, and therefore $R\phi = (\xi_i)$. ♠

COROLLARY 2.19. *A functor $F : \mathcal{C}^{op} \to (Set)$ is a sheaf if and only if for any covering $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, the induced function*

$$FU \simeq \mathrm{Hom}(h_U, F) \to \mathrm{Hom}(h_{\mathcal{U}}, F)$$

is bijective. Furthermore, F is separated if and only if this function is always injective.

This characterization can be sharpened with the following concept.

DEFINITION 2.20. Let $\mathcal{T}$ be a Grothendieck topology on a category $\mathcal{C}$. A sieve $S \subset h_U$ on an object U of $\mathcal{C}$ is said to *belong to* $\mathcal{T}$ if there exists a covering $\mathcal{U}$ of U such that $h_{\mathcal{U}} \subset S$. If $\mathcal{C}$ is a site, we will talk about the *sieves* of $\mathcal{C}$ to mean the sieves belonging to the topology of $\mathcal{C}$.

PROPOSITION 2.21. *A functor $F : \mathcal{C}^{op} \to (Set)$ is a sheaf in a topology $\mathcal{T}$ if and only if for any sieve S belonging to $\mathcal{T}$ the induced function*

$$FU \simeq \mathrm{Hom}(h_U, F) \longrightarrow \mathrm{Hom}(S, F)$$

is bijective.

PROOF. The fact that this condition implies that F is a sheaf is a consequence of Corollary 2.19. Reciprocally, let F be a sheaf, $S \subset h_U$ a sieve belonging to $\mathcal{T}$, and choose a covering $\mathcal{U}$ of U with $h_{\mathcal{U}} \subset S$. By Corollary 2.19, the natural composite

$$\mathrm{Hom}(h_U, F) \longrightarrow \mathrm{Hom}(S, F) \longrightarrow \mathrm{Hom}(h_{\mathcal{U}}, F)$$

is a bijection. We claim that, if F is separated, the restriction function

$$\mathrm{Hom}(S, F) \longrightarrow \mathrm{Hom}(h_{\mathcal{U}}, F)$$

is injective. To see this, let us take two natural transformations $\phi, \psi : S \to F$ with the same image in $\mathrm{Hom}(h_{\mathcal{U}}, F)$, and element $T \to U$. Let $\mathcal{U} = \{U_i \to U\}$, and consider the fibered products $T \times_U U_i$ with their projections $p_i : T \times_U U_i \to T$. Since $T \times_U U_i \to U$ is in $h_{\mathcal{U}}(T \times_U U_i)$ we have

$$p_i^* \phi_T(T \to U) = \phi_T(T \times_U U_i) = \psi_T(T \times_U U_i) = p_i^* \psi_T(T \to U)$$

Since $\{p_i : T \times_U U_i \to T\}$ is a covering and F is a separated presheaf, then we conclude that $\phi_T(T \to U) = \psi_T(T \to U)$ in FT. Hence $\phi = \psi$, as desired. We then conclude that $\mathrm{Hom}(h_U, F) \longrightarrow \mathrm{Hom}(S, F)$ is bijective. ♠

REMARK 2.22. Suppose that $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{V} = \{V_j \to U\}$ are coverings. Then we know that $\mathcal{U} \times_U \mathcal{V} := \{U_i \times_U V_j \to U\}$ is a covering. As an arrow $T \to U$ factors through $U_i \times_U V_j$ if and only if factors through U_i and through V_j, we have the following proposition.

PROPOSITION 2.23.

(1) *If $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{V} = \{V_j \to U\}$ are coverings, then*
$$h_{\mathcal{U} \times_U \mathcal{V}} = h_{\mathcal{U}} \cap h_{\mathcal{V}} \subset h_U$$
(2) *If S_1 and S_2 are sieves on U belonging to $\mathcal{T}$, the intersection $S_1 \cap S_2 \subset h_U$ also belongs to $\mathcal{T}$*

2.2.4. Equivalence of Grothendieck topologies. Sometimes two different topologies on the same category define the same sheaf.

DEFINITION 2.24. Let $\mathcal{C}$ be a category, $\{U_i \to U\}_{i \in I}$ a set of arrows. A *refinement* $\{V_j \to U\}_{j \in J}$ is a set of arrows such that for each index $j \in J$ there is some index $i \in I$ such that $V_j \to U$ factors through $U_i \to U$

This relation between sets of arrows is most easily expressed in terms of sieves. The following fact is immediate.

PROPOSITION 2.25. *Let there be given two sets of arrows $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{V} = \{V_j \to U\}$. Then $\mathcal{V}$ is a refinement of $\mathcal{U}$ if and only if $h_{\mathcal{V}} \subset h_{\mathcal{U}}$*

REMARK 2.26. A refinement of a refinement is obviously a refinement. Also, any covering is a refinement of itself. Thus, the relation of being a refinement is a pre-order on the set of coverings of an object U.

DEFINITION 2.27. Let $\mathcal{C}$ be a category, $\mathcal{T}$ and $\mathcal{T}'$ two topologies on $\mathcal{C}$. We say that $\mathcal{T}$ is *subordinate* to $\mathcal{T}'$, and write $\mathcal{T} \prec \mathcal{T}'$, if every covering in $\mathcal{T}$ has a refinement that is a covering in $\mathcal{T}'$. If $\mathcal{T} \prec \mathcal{T}'$ and $\mathcal{T}' \prec \mathcal{T}$, we say that $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent*, and write $\mathcal{T} \equiv \mathcal{T}'$.

REMARK 2.28. Being a refinement is a relation between sets of arrows into U that is transitive and reflexive. Therefore being subordinate is a transitive and reflexive relation between topologies on $\mathcal{C}$, and being equivalent is an equivalence relation.

This relation between topologies is naturally expressed in terms of sieves. The following proposition conveys this and is clear from Proposition 2.25.

PROPOSITION 2.29. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a category $\mathcal{C}$. Then $\mathcal{T} \prec \mathcal{T}'$ if and only if every sieve belonging to $\mathcal{T}$ also belongs to $\mathcal{T}'$. In particular, two topologies are equivalent if and only if they have the same sieves.*

The following proposition follows from Proposition 2.21 and 2.29.

PROPOSITION 2.30. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two Grothendieck topologies on the same category $\mathcal{C}$. If $\mathcal{T}$ is subordinate to $\mathcal{T}'$, then every sheaf in $\mathcal{T}'$ is also a sheaf in $\mathcal{T}$. In particular, two equivalent topologies have the same sheaves.*

Let us define two Grothendieck topologies on the same category:

EXAMPLE 2.31. Let (Top) be the category of topological spaces. If U is a topological space, then a covering of U will be a jointly surjective collection of open embeddings $U_i \to U$, where "open embedding" means open continuous injective map $V \to U$. This topology is called *the global classical topology*.

EXAMPLE 2.32. Let (Top) be the category of topological spaces. If U is a topological space, then a covering of U will be a jointly surjective collection of local homeomorphisms $U_i \to U$. This topology is called *the global étale topology for topological spaces*. This two topologies are equivalents.

Now, we can give conditions on representable functors to know when it is a sheaf.

PROPOSITION 2.33. *A representable functor $(Top)^{op} \to (Set)$ is a sheaf in the global classical topology.*

PROOF. Since $(Top)^{op} \to (Set)$ is representable, the claim is amounts to saying that, given two topological spaces U and X, an open covering $\{U_i \to U\}$, and continuous functions $f_i : U_i \to X$, with the property that the restriction of f_i and f_j to $U_i \cap U_j$ coincide for all i and j, there exists a unique continuous function $U \to X$ whose restriction $U_i \to X$ is f_i. But this is obvious (it boils down to the fact that, for a function, the property of being continuous is local on the domain). ♠

The following proposition is immediate from Proposition 1.11 (Glueing morphisms).

PROPOSITION 2.34. *A representable functor $(Sch/S)^{op} \to (Set)$ is a sheaf in the Zariski topology.*

On the other hand the following is highly nontrivial: a scheme is a topological space, together with a sheaf of rings in the Zariski topology. A priori, there does not seem to be a reason why we should be able to glue morphisms of schemes in a finer topology than the Zariski topology.

2.2.5. The fpqc topology. It is sometimes useful to consider coverings that are not locally finitely presented. One can define a topology on (Sch/S) simply by taking all collections of morphisms $\{U_i \to U\}$ such that the resulting morphism $\coprod_i U_i \to U$ is faithfully flat. Unfortunately, this topology is not well behaved. One needs some finiteness condition in order to get a reasonable topology.

PROPOSITION 2.35. *(*[**Vis05**]*, Proposition 2.33) Let $f : X \to Y$ be a surjective morphism of schemes. Then the following properties are equivalents*

i) Every queasi-compact open subset of Y is the image of a quasi-compact open subset of X.
ii) There exists a covering $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.
iii) Given a point $x \in X$, there exists an open neighborhood U of x in X, such that the image fU is open in Y, and the restriction $U \to fU$ of f is quasi-compact.
iv) Given a point $x \in X$, there exists a quasi-compact open neighborhood U of x in X, such that the image fU is open and affine in Y.

The definition of the fpqc topology that follows, suggested by Steve Kleiman, gives the correct sheaf theory.

DEFINITION 2.36. An *fpqc morphism of schemes* is a faithfully flat morphism that satisfies the equivalent conditions of Proposition 2.35.

The abbreviation fpqc stands for "fidèlement plat et quasi-compact".

REMARK 2.37. Paul Goerss [Goe] has a different (but equivalent) definition. He defines an fpqc cover of an affine scheme U to be a finite collection $\{U_i \to U\}$ of at morphisms with $\coprod_i U_i \to U$ surjective, and in general an fpqc cover of a scheme X to be a finite collection of morphisms $\{V_i \to X\}$ such that for every open affine $U \subset X$ the pullback $\{V_i \times_X U \to U\}$ is an fpqc cover of U.

DEFINITION 2.38. The fpqc topology on the category (Sch/S) is the topology in which the coverings $\{U_i \to U\}$ are collections of morphisms, such that the induced morphism $\coprod_i U_i \to U$ is fpqc.

The fact that this is indeed a topology it follows that the following proposition

PROPOSITION 2.39. *(**[Vis05]**, Proposition 2.35)*

i) The composite of fpqc morphisms is fpqc.
ii) If $f : X \to Y$ is a morphism of schemes, and there is an open covering V_i of Y, such that the restriction $f^{-1}V_i \to V_i$ is fpqc, then f is fpqc.
iii) An open faithfully flat morphism is fpqc.
iv) A faithfully flat morphism that is locally of finite presentation is fpqc.
v) A morphism obtained by base change from an fpqc morphism is fpqc.
vi) If $f : X \to Y$ is an fpqc morphism, a subset of Y is open in Y if and only if its inverse image is open in X.

DEFINITION 2.40. The *fppf* topology on Sch/S is defined as follows: the coverings of $X \to S$ are collections $U_i \to X$ of flat maps locally of finite presentation over S such that $\coprod_i U_i \to X$ is surjective as a map of sets.

REMARK 2.41. Note that by (iv) above, fppf covers are fpqc covers and as an fppf cover is a Zariski cover if each $X_i \to X$ is an open immersion, it follows that the fpqc topology is finer than all of these..

THEOREM 2.42 (Grothendieck). *(**[Vis05]**, Theorem 2.55) A representable functor on (Sch/S) is a sheaf in the fpqc topology.*

In other words, if we have a fpqc cover $\{U_i \to U\}$ of a scheme U and morphisms $U_i \to X$ that glue on the fibered products, then there is a unique morphism $U \to X$ that the pullback to each of these.

CHAPTER 3

Fibered Categories

The notion of category generalizes the notion of a set. A set can be thought of as a category in which every arrow is an identity. Furthermore functors between sets are simply functions. In the same way, the *fibered categories* are generalizations of functors.
Following to A. Vistoli, fix a category $\mathcal{C}$. Let $\mathcal{F}$ be a category over $\mathcal{C}$, that is, a category $\mathcal{F}$ equipped with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$. An arrow going from an object ξ of $\mathcal{F}$ to an object U of $\mathcal{C}$ will be of type "$\xi \mapsto U$", and will mean that $p_{\mathcal{F}}\xi = U$. Furthermore the commutativity of the diagram

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

will mean that $p_{\mathcal{F}}\phi = f$.

DEFINITION 3.1. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any arrow $\psi : \varsigma \to \eta$ in $\mathcal{F}$ and any arrow $h : p_{\mathcal{F}}\varsigma \to p_{\mathcal{F}}\eta$ in $\mathcal{C}$ with $p_{\mathcal{F}}\phi \circ h = p_{\mathcal{F}}\psi$, there exists a unique arrow $\theta : \varsigma \to \xi$ with $p_{\mathcal{F}}\theta = h$ and $\phi \circ \theta = \psi$ as in the commutative diagram

$$\begin{tikzcd}
\varsigma \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
p_{\mathcal{F}}\varsigma \arrow[dr, "h"'] \arrow[drr, bend left] & & \\
& p_{\mathcal{F}}\xi \arrow[r] & p_{\mathcal{F}}\eta
\end{tikzcd}$$

If $\xi \to \eta$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we also say that ξ is a *pullback* of η to U.

REMARK 3.2. Given two pullbacks $\phi : \xi \to \eta$ and $\varphi : \zeta \to \eta$ of η to U, the unique arrow $\theta : \zeta \to \xi$ that fits into the diagram

$$
\begin{tikzcd}
\zeta \arrow[rrd, bend left, "\psi"] \arrow[rd, dashed, "\theta"'] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
U \arrow[rrd, bend left] \arrow[rd, equal] & & \\
& U \arrow[r] & V
\end{tikzcd}
$$

is an isomorphism; the inverse is the arrow $\xi \to \zeta$ obtained by exchanging ξ and ζ in the diagram above.
In conclusion, a pullback is unique, up to a unique isomorphism.

PROPOSITION 3.3.

i) If $\mathcal{F}$ is a category over $\mathcal{C}$, the composite of cartesian arrows in $\mathcal{F}$ is cartesian.
ii) If $\xi \to \eta$, $\eta \to \zeta$ are arrows in $\mathcal{F}$ and $\eta \to \zeta$ is cartesian, then $\xi \to \eta$ is cartesian if and only if the composite $\xi \to \zeta$ is cartesian.
iii) An arrow in $\mathcal{F}$ whose image in $\mathcal{C}$ is an isomorphism is cartesian if and only if it is an isomorphism.
iv) Let $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{C}$ and $F : \mathcal{F} \to \mathcal{G}$ be functors, $\phi : \xi \to \eta$ an arrow in $\mathcal{F}$. If ϕ is cartesian over its image $F\phi : F\xi \to F\eta$ in $\mathcal{G}$ and $F\phi$ is cartesian over its image $p_{\mathcal{G}}F\phi : p_{\mathcal{G}}F\xi \to p_{\mathcal{G}}F\eta$ in $\mathcal{C}$, then ϕ is cartesian over its image $p_{\mathcal{G}}F\phi$ in $\mathcal{C}$.

PROOF. i) Suppose that we have the following diagram

$$
\begin{tikzcd}
\varsigma \arrow[rrrd, bend left, "\varphi"] \arrow[rd, dashed, "\theta"'] \arrow[dd, mapsto] & & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd, mapsto] & \eta \arrow[r, "\psi"'] \arrow[dd, mapsto] & \zeta \arrow[dd, mapsto] \\
p_{\mathcal{F}}\varsigma \arrow[rrrd, bend left] \arrow[rd, "h"'] & & & \\
& p_{\mathcal{F}}\xi \arrow[r] & p_{\mathcal{F}}\eta \arrow[r] & p_{\mathcal{F}}\zeta
\end{tikzcd}
$$

where the squares commute and $\xi \to \eta$, $\eta \to \zeta$ are cartesian arrows in $\mathcal{F}$. We need to find an arrow $\theta : \varsigma \to \xi$ in $\mathcal{F}$ such that the diagram above commutes, $p_{\mathcal{F}}\theta = h$ and $(\psi \circ \phi) \circ \theta = \varphi$. Since $\eta \to \zeta$ is cartesian, there exists a unique arrow $\vartheta : \varsigma \to \eta$ in $\mathcal{F}$ such that $\psi \circ \vartheta = \varphi$, $p_{\mathcal{F}}\vartheta = p_{\mathcal{F}}\phi \circ h$, and the following diagram commutes

$$
\begin{tikzcd}
\varsigma \arrow[rrd, bend left, "\varphi"] \arrow[rd, dashed, "\vartheta"'] \arrow[dd, mapsto] & & \\
& \eta \arrow[r, "\psi"'] \arrow[dd, mapsto] & \zeta \arrow[dd, mapsto] \\
p_{\mathcal{F}}\varsigma \arrow[rrd, bend left] \arrow[rd, "p_{\mathcal{F}}\phi \circ h"'] & & \\
& p_{\mathcal{F}}\eta \arrow[r] & p_{\mathcal{F}}\zeta
\end{tikzcd}
$$

Now, since $\xi \to \eta$ is cartesian, there exists a unique arrow $\theta : \varsigma \to \xi$ such that $\phi \circ \theta = \vartheta$, $p_{\mathcal{F}}\theta = h$, and the following diagram commutes

$$\begin{tikzcd}
\varsigma \arrow[rrd, bend left, "\vartheta"] \arrow[rd, dashed, "\theta"'] \arrow[dd] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd] & \eta \arrow[dd] \\
p_{\mathcal{F}}\varsigma \arrow[rd, "h"'] \arrow[rrd, bend left] & & \\
& p_{\mathcal{F}}\xi \arrow[r] & p_{\mathcal{F}}\eta
\end{tikzcd}$$

Since $(\psi \circ \phi) \circ \theta = \psi \circ (\phi \circ \theta) = \psi \circ \vartheta = \varphi$, so θ is as desired.
ii) Following the above notation. Suppose that $\xi \to \zeta$ is cartesian, and let us see that $\xi \to \eta$ is cartesian. Let $\vartheta : \varsigma \to \eta$, $h : p_{\mathcal{F}}\varsigma \to p_{\mathcal{F}}\xi$ be arrows such that $p_{\mathcal{F}}\phi \circ h = p_{\mathcal{F}}\vartheta$. Since $\xi \to \zeta$ is cartesian and taken the arrow $\psi \circ \vartheta : \varsigma \to \zeta$, there exists a unique $\theta : \varsigma \to \xi$ such that $p_{\mathcal{F}}\theta = h$ and $(\psi \circ \phi) \circ \theta = \psi \circ \vartheta$, as θ is unique, $\phi \circ \theta = \vartheta$. Reciprocally is the literal *i*).
iii) Suppose that $\phi : \xi \to \eta$ is cartesian in $\mathcal{F}$ and its image in $\mathcal{C}$ is an isomorphism. By definition, there exists a unique $\theta : \eta \to \xi$ such that $\phi \circ \theta = id_\eta$, $p_{\mathcal{F}}\theta = h$ and the following diagram commutes

$$\begin{tikzcd}
\eta \arrow[rrd, bend left, "id_\eta"] \arrow[rd, dashed, "\theta"'] \arrow[dd] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd] & \eta \arrow[dd] \\
p_{\mathcal{F}}\eta \arrow[rd, "h"'] \arrow[rrd, bend left] & & \\
& p_{\mathcal{F}}\xi \arrow[r] & p_{\mathcal{F}}\eta
\end{tikzcd}$$

where h is the inverse morphism of the image in $\mathcal{C}$ of ϕ. Since $\theta \circ \phi = \theta \circ \mathrm{id}_\eta \circ \phi = (\theta \circ \phi) \circ (\theta \circ \phi)$, we conclude that $\theta \circ \phi \simeq \mathrm{id}_\xi$. Thus ϕ is an isomorphism. Reciprocally is straightforward.

iv) Let $\psi : \zeta \to \eta$ and $h : p_{\mathcal{G}}F\zeta \to p_{\mathcal{G}}F\xi$ be arrows in $\mathcal{F}$ and $\mathcal{C}$ like in the following diagram

$$\begin{tikzcd}
\zeta \arrow[rrd, bend left, "\psi"] \arrow[rd, dashed, "\theta"'] \arrow[dd] & & \\
& \xi \arrow[r, "\phi"'] \arrow[dd] & \eta \arrow[dd] \\
F\zeta \arrow[rd, dashed, "\vartheta"'] \arrow[rrd, bend left] \arrow[dd] & & \\
& p_{\mathcal{F}}\xi \arrow[r, "F\phi"'] \arrow[dd] & p_{\mathcal{F}}\eta \arrow[dd] \\
p_{\mathcal{G}}F\zeta \arrow[rd, "h"'] \arrow[rrd, bend left] & & \\
& p_{\mathcal{G}}F\xi \arrow[r, "p_{\mathcal{G}}F\phi"'] & \eta
\end{tikzcd}$$

Since $F\phi$ is cartesian over its image in $\mathcal{C}$, there exists a unique $\vartheta : F\zeta \to F\xi$ such that $p_{\mathcal{G}}\vartheta = h$ and $F\phi \circ \vartheta = F\psi$. Again, as $\phi : \xi \to \eta$ is cartesian over its image in $\mathcal{G}$, there exists a unique $\theta : \zeta \to \xi$ such that $F\theta = \vartheta$ and $\phi \circ \theta = \psi$. Note that $p_{\mathcal{G}}F\theta = h$. Hence ϕ is cartesian over its image $p_{\mathcal{G}}F\phi$ in $\mathcal{C}$. ♠

DEFINITION 3.4. A *fibered category over $\mathcal{C}$* is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f : U \to V$ in $\mathcal{C}$ and an object η of $\mathcal{F}$ mapping to V, there is a cartesian arrow $\phi : \xi \to \eta$ with $p_{\mathcal{F}}\phi = f$.

DEFINITION 3.5. If $\mathcal{F}$ and $\mathcal{G}$ are fibered categories over $\mathcal{C}$, then a *morphism of fibered categories* $F : \mathcal{F} \to \mathcal{G}$ is a functor such that:

(i) F is *base-preserving*, that is, $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$.
(ii) F sends cartesian arrows to cartesian arrows.

The following proposition it follows from Proposition 3.3.

PROPOSITION 3.6. *Given two functors $\mathcal{F} \to \mathcal{G}$ and $\mathcal{G} \to \mathcal{C}$. If $\mathcal{F}$ is fibered over $\mathcal{G}$ and $\mathcal{G}$ is fibered over $\mathcal{C}$, then $\mathcal{F}$ is fibered over $\mathcal{C}$.*

DEFINITION 3.7. Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$. Given an object U of $\mathcal{C}$, the *fiber* $\mathcal{F}(U)$ of $\mathcal{F}$ over U is the subcategory of $\mathcal{F}$ whose objects are the objects ξ of $\mathcal{F}$ with $p_{\mathcal{F}}\xi = U$, and whose arrows are arrows ϕ in $\mathcal{F}$ with $p_{\mathcal{F}}\phi = \mathrm{id}_U$.

REMARK 3.8. By definition, if $F : \mathcal{F} \to \mathcal{G}$ is a morphism of fibered categories over $\mathcal{C}$ and U is an obeject of $\mathcal{C}$, the functor F sends $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so we have a restriction functor $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

We could give the same definition of a fiber for any functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$, without assuming that $\mathcal{F}$ is fibered over $\mathcal{C}$. However, we would end up with a useless notion. For example, it may very well happen that we have two objects U and V of $\mathcal{C}$ which are isomorphic, but such that $\mathcal{F}(U)$ is empty while $\mathcal{F}(V)$ is not. This kind of pathology does not arise for fibered categories:

Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$, and $f : U \to V$ an arrow in $\mathcal{C}$. For each object η over V, we choose a pullback $\phi_\eta : f^*\eta \to \eta$ of η to U. We define a functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ by sending each object η of $\mathcal{F}(V)$ to $f^*\eta$, and each arrow $\beta : \eta \to \eta'$ of $\mathcal{F}(V)$ to the unique arrow $f^*\beta : f^*\eta \to f^*\eta'$ in $\mathcal{F}(U)$ making the following diagram commute

$$\begin{tikzcd}
f^*\eta \arrow[rrd, bend left, "\beta\circ\phi_\eta"] \arrow[rd, dashed, "f^*\beta"'] \arrow[dd] & & \\
& f^*\eta' \arrow[r, "\phi_{\eta'}"'] \arrow[dd] & \eta' \arrow[dd] \\
U \arrow[rrd, bend left] \arrow[rd, equal] & & \\
& U \arrow[r, "f"'] & V
\end{tikzcd}$$

DEFINITION 3.9. A *cleavage* of a fibered category $\mathcal{F} \to \mathcal{C}$ consist of a class K of cartesian arrows in $\mathcal{F}$ such that for each arrow $f : U \to V$ in $\mathcal{C}$ and each object η in $\mathcal{F}(V)$ there exists a unique arrow in K with target η mapping to f in $\mathcal{C}$.

By the axiom of choice, every fibered category has a cleavage.
Given a fibered category $\mathcal{F} \to \mathcal{C}$ with a cleavage, we associate with each object U of C a category $\mathcal{F}(U)$, and to each arrow $f : U \to V$ a functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$, constructed as above. It is very tempting to believe that in this way we have defined a functor from $\mathcal{C}$ to the category of categories; however, this is not quite correct. First of all, pullbacks $\mathrm{id}_U^* : \mathcal{F}(U) \to \mathcal{F}(U)$ are not necessarily identities. Of course we could just choose all pullbacks along identities to be identities on

the fiber categories: this would certainly work, but it is not very natural, as there are often natural defined pullbacks where this does not happen (Example 3.11 (i)). What happens in general is that, when U is an object of $\mathcal{C}$ and ξ an object of $\mathcal{F}(U)$, we have the pullback $\epsilon_U(\xi) : \mathrm{id}_U^*\xi \to \xi$ is an isomorphism, because of Proposition 3.3 (iii), and this defines an isomorphism of functors $\epsilon_U : \mathrm{id}_U^* \simeq \mathrm{id}_{\mathcal{F}(U)}$.

A serious problem is the following: Suppose that we have two arrows $f : U \to V$ and $g : V \to W$ in $\mathcal{C}$, and an object ζ of $\mathcal{F}$ over W. Then $f^*g^*\zeta$ is a pullback of ζ to U; however, pullbacks are not unique, so there is no reason why $f^*g^*\zeta$ should coincide with $(gf)^*\zeta$. However, there is a canonical isomorphism $f^*g^*\zeta \simeq (gf)^*\zeta$ in $\mathcal{F}(U)$, because both are pullbacks, and this gives an isomorphism $f^*g^* \simeq (gf)^*$ of functors $\mathcal{F}(W) \to \mathcal{F}(U)$.
So, after choosing a cleavage a fibered category almost gives a functor from $\mathcal{C}$ to the category of categories, but not quite. The point is that the category of categories is not just a category, but what is known as a 2-category; that is, its arrows are functors, but two functors between the same two categories in turn form a category, the arrows being natural transformations of functors. Thus there are 1-arrows (functors) between objects (categories), but there are also 2-arrows (natural transformations) between 1-arrows.
For resolve the above problem there is a concept called *pseudo-functor*, or, in a more modern terminology, a *lax 2-functor*.

DEFINITION 3.10. A *pseudo-functor* Φ on $\mathcal{C}$ consist of the following data.

(i) For each object U of $\mathcal{C}$ a category ΦU.
(ii) For each arrow $f : U \to V$ in $\mathcal{C}$, a functor $f^* : \Phi V \to \Phi U$.
(iii) For each object U of $\mathcal{C}$ an isomorphism $\epsilon_U : \mathrm{id}_U^* \simeq \mathrm{id}_{\Phi U}$ of functors $\Phi U \to \Phi U$.
(iv) For each pair of arrows $U \xrightarrow{f} V \xrightarrow{g} W$ an isomorphism

$$\alpha_{f,g} : f^*g^* \simeq (gf)^* : \Phi W \longrightarrow \Phi U$$

of functors $\Phi W \to \Phi U$.

These data are required to satisfy the following conditions.

a) If $f : U \to V$ is an arrow in $\mathcal{C}$ and η is an object of ΦV, we have

$$\alpha_{\mathrm{id}_U,f}(\eta) = \epsilon_U(f^*\eta) : \mathrm{id}_U^* f^*\eta \longrightarrow f^*\eta$$

and

$$\alpha_{f,\mathrm{id}_V}(\eta) = f^*\epsilon_V(\eta) : f^*\mathrm{id}_V^*\eta \longrightarrow f^*\eta$$

b) Whenever we have arrows $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ and an object ζ of $\Phi(T)$, the diagram

$$\begin{tikzcd}
f^*g^*h^*\zeta \arrow[r, "\alpha_{f,g}(h^*\zeta)"] \arrow[d, "f^*\alpha_{g,h}(\zeta)"'] & (gf)^*h^*\zeta \arrow[d, "\alpha_{gf,h}(\zeta)"] \\
f^*(hg)^*\zeta \arrow[r, "\alpha_{f,hg}(\zeta)"] & (hgf)^*\zeta
\end{tikzcd}$$

commutes.

We can see that if we star with a pseudo-functor, we construct the associated fibered category with a cleavage, and then we take the associated pseudo-functor, this is isomorphic to the original pseudo-functor. Conversely, if we start from a fibered category with a cleavage, construct the associated pseudo-functor, and then take the associated fibered category with a cleavage, we get something isomorphic to the original fibered category with a cleavage ([**Vis05**], pp. 44-48). So really giving a pseudo-functor is the same as giving a fibered category with a cleavage.
On the other hand, since cartesian pullbacks are unique up to a unique isomorphism (Remark 3.2), also cleavages are unique up to a unique isomorphism. This means that, in a sense that one could make precise, the theory of fibered categories is equivalent to the theory of pseudo-functors.

EXAMPLE 3.11. Assume that $\mathcal{C}$ has fibered produts. Let $\operatorname{Arr}\mathcal{C}$ be the category of arrows in $\mathcal{C}$; its objects are the arrows in $\mathcal{C}$, while an arrow from $f : X \to U$ to $g : Y \to V$ is a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[d, "g"] \\
U \arrow[r] & V
\end{tikzcd}$$

The functor $p_{\operatorname{Arr}\mathcal{C}} : \operatorname{Arr}\mathcal{C} \to \mathcal{C}$ sends each arrow $S \to U$ to its codomain U, and each commutative diagram to its bottom row. Let us see that $\operatorname{Arr}\mathcal{C}$ is a fibered category over $\mathcal{C}$. Suppose that we have a arrow $U \xrightarrow{f} V$ in $\mathcal{C}$ and an object $\eta \to V$ of $\operatorname{Arr}\mathcal{C}$ such that $p_{\operatorname{Arr}\mathcal{C}}(\eta \to V) = V$. To find an arrow ϕ from $\xi \to U$ to $\eta \to V$, for some ξ in $\mathcal{C}$; it is equivalent to find a commutative diagram

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \eta \arrow[d] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

Since $\mathcal{C}$ has fibered products, with $\xi = U \times_V \eta$ and the natural projections we have the above diagram. To check that ϕ is cartesian is the same to find a unique θ such that $p_{\operatorname{Arr}\mathcal{C}}\theta = h$ and the following diagram commutes

$$\begin{tikzcd}
\zeta \to W \arrow[drr, bend left, "\psi"] \arrow[dr, dashed, "\theta"'] \arrow[dd] & & \\
& \xi \to U \arrow[r, "\phi"'] \arrow[dd] & \eta \to V \arrow[dd] \\
W \arrow[dr, "h"'] \arrow[drr, bend left] & & \\
& U \arrow[r, "f"'] & V
\end{tikzcd}$$

for any h and ψ. But the above diagram say that the following diagrams commute

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \eta \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd} \qquad \begin{tikzcd} \zeta \arrow[r, "\sigma"] \arrow[d, "\tau"] & \eta \arrow[d] \\ W \arrow[r] & V \end{tikzcd}$$

Joining the diagrams and by the definition of fiber product there exist a unique $\varphi : \zeta \to \xi$ such that the following diagram commutes

$$\begin{tikzcd} \zeta \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\sigma"] \arrow[ddr, bend right, "h \circ \tau"'] & & \\ & \xi \arrow[r] \arrow[d] & \eta \arrow[d] \\ & U \arrow[r, "f"'] & V \end{tikzcd}$$

Thus we have a unique $\varphi : \zeta \to \xi$ such that

$$\begin{tikzcd} \xi \arrow[r, "\varphi"] \arrow[d, "\tau"] & \xi \arrow[d] \\ W \arrow[r, "h"] & U \end{tikzcd}$$

commutes. Thus we have θ as desired. Therefore $\mathrm{Arr}\,\mathcal{C}$ is a fibered category over $\mathcal{C}$.

EXAMPLE 3.12. Let $\mathcal{C}$ be a site, $\mathcal{T}$ its topology. We will refer to a sheaf in the category $(\mathcal{C}/X)$, endowed with the comma topology (Example 2.14) as a sheaf on X, and denote the category of sheaves on X by ShX. If $f : X \to Y$ is an arrow in $\mathcal{C}$, there is a corresponding restriction functor $f^* : ShY \to ShX$, defined as follows. If F is a sheaf on Y and $U \to X$ is an object of $(\mathcal{C}/X)$, we define $f^*F(U \to X) := F(U \to Y)$, where $U \to Y$ is the composite of $U \to X$ with f. If $U \to X$ and $V \to X$ are objects of $(\mathcal{C}/X)$ and $\phi : U \to V$ is an arrow in $(\mathcal{C}/X)$, then ϕ is also an arrow from $U \to Y$ to $V \to Y$, hence it induces a function $f^* : f^*F(U \to X) = F(U \to Y) \to F(V \to Y) = f^*F(V \to X)$. This gives f^*F the structure of a functor $(\mathcal{C}/X)^{op} \to (Set)$. Let us check that f^*F is a sheaf on X. Let $\{(U_i \to X) \longrightarrow (U \to X)\}$ be a covering in $(\mathcal{C}/X)$ and take $a_i \in f^*F(U_i \to X)$. Suppose that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j$ in $f^*F(U_i \times_U U_j \to X)$. Since F is a sheaf on Y there exists a unique section $a \in f^*F(U \to X)$ whose pullback to $f^*F(U_i \to X)$ is a_i; so f^*F is a sheaf. On the other hand, if $\phi : F \to G$ is a natural transformation of sheaves on $(\mathcal{C}/Y)$, there is an induced natural transformation $f^*\phi : f^*F \to f^*G$ of sheaves on $(\mathcal{C}/X)$, defined in the obvious way. This defines a functor $f^* : ShY \to ShX$. It is immediate to check that, if $f : X \to Y$ and $g : Y \to Z$ are arrows in $\mathcal{C}$, we have an equality of functors $(gf)^* = f^*g^* : ShZ \to ShX$ and that $\mathrm{id}_X^* : ShX \to ShX$ is the identity. This means that we have defined a functor from $\mathcal{C}$ to the category of categories, sending an object X into the category of sheaves on $(\mathcal{C}/X)$.

3.1. Equivalence of fibered categories

3.1.1. Natural transformations of functors. The fact that fibered categories are categories, and not functors, has strong implications, and does cause difficulties. As usual, the main problem is that functors between categories can be isomorphic without being equal; in other words, functors between two fixed categories form a category, the arrows being given by natural transformations.

DEFINITION 3.13. Let $\mathcal{F}$ and $\mathcal{G}$ be two categories fibered over $\mathcal{C}$ and $F, G : F \to G$ two morphisms. A *base-preserving natural transformation* $\alpha : F \to G$ is a natural transformation such that for any object ξ of $\mathcal{F}$, the arrow $\alpha_\xi : F\xi \to G\xi$ is in $\mathcal{G}(U)$, where $U := p_{\mathcal{F}}\xi = p_{\mathcal{G}}(F\xi) = p_{\mathcal{G}}(G\xi)$. An *isomorphism* of F with G is a base-preserving natural transformation $F \to G$ which is an isomorphism of functors.

It is immediate to check that the inverse of a base-preserving isomorphism is also base-preserving. Hence, there is a category whose objects are the morphism from $\mathcal{F}$ to $\mathcal{G}$, and the arrows are base-preserving natural transformations; we denote it by $Hom_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$.

3.1.2. Equivalences.

DEFINITION 3.14. Let $\mathcal{F}$ and $\mathcal{G}$ be two fibered categories over $\mathcal{C}$. An *equivalence*, of $\mathcal{F}$ with $\mathcal{G}$ is a morphism $F : \mathcal{F} \to \mathcal{G}$, such that there exists another morphism $G : \mathcal{G} \to \mathcal{F}$, together with isomorphisms of $G \circ F$ with $\mathrm{id}_{\mathcal{F}}$ and of $F \circ G$ with $\mathrm{id}_{\mathcal{G}}$. We call G simply an *inverse* to F.

For give a criterion when a morphism of fibered categories is an equivalence we need to prove the following lemma.

LEMMA 3.15. *Let $F : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories such that every restriction $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful. Then the functor F is fully faithful.*

PROOF. Let ξ', η' be two objects of $\mathcal{F}$ and let $\phi : \xi \to \eta$ in $\mathcal{G}$ with $\xi := F\xi'$ and $\eta := F\eta'$. Call $U \to V$ the image of ϕ in $\mathcal{C}$. Since $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$ then $p_{\mathcal{F}}\eta' = V$; let $\eta_1' \to \eta'$ be a pullback of η' to U and call η_1 to $F\eta_1'$. Then the image $\eta_1 \to \eta$ is cartesian, thus the arrow $\xi \to \eta$ factors uniquely as $\xi \to \eta_1 \to \eta$, where the arrow $\xi \to \eta_1$ is $\mathcal{G}(U)$

$$\begin{tikzcd}
\xi \arrow[dr, dashed] \arrow[drr, bend left, "\phi"] \arrow[dd, mapsto] & & \\
& \eta_1 \arrow[r] \arrow[dd, mapsto] & \eta \arrow[dd, mapsto] \\
U \arrow[dr, equal] \arrow[drr, bend left] & & \\
& U \arrow[r] & V
\end{tikzcd}$$

Since $\xi \to \eta_1$ is in $\mathcal{G}(U)$, this lifts uniquely to an arrow $\xi' \to \eta_1'$ in $\mathcal{F}(U)$. Hence we have a unique arrow $\xi' \to \eta_1' \to \eta'$ such that $F(\xi' \to \eta_1' \to \eta') = \phi$, because every arrow $\xi' \to \eta'$ factor uniquely through η_1'. ♠

PROPOSITION 3.16. *Let $F : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories. Then F is an equivalence if and only if the restriction $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$.*

PROOF. Suppose that $G : \mathcal{G} \to \mathcal{F}$ is an inverse to F; the two isomorphisms $F \circ G \simeq \mathrm{id}_{\mathcal{G}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{F}}$ restrict to isomorphisms $F_U \circ G_U \simeq \mathrm{id}_{\mathcal{G}(U)}$ and $G_U \circ F_U \simeq \mathrm{id}_{\mathcal{F}(U)}$, so G_U is an inverse to F_U.
Conversely, we assume that $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object U of $\mathcal{C}$, and construct an inverse $G : \mathcal{G} \to \mathcal{F}$. Since $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence there exists $G_U : \mathcal{G}(U) \to \mathcal{F}(U)$ with $F_U \circ G_U \simeq \mathrm{id}_{\mathcal{G}(U)}$ and $G_U \circ F_U \simeq \mathrm{id}_{\mathcal{F}(U)}$. Let ξ be an object of $\mathcal{G}$ and $U = p_{\mathcal{G}}\xi$ an object in $\mathcal{C}$, then $\xi \in \mathcal{G}(U)$ and $\alpha_\xi : \xi \simeq F(G\xi)$ in $\mathcal{G}(U)$. Let $\phi : \xi \to \eta$ is an arrow in $\mathcal{G}$, by the above lemma there is a unique arrow $G\phi : G\xi \to G\eta$ such that the following diagram commutes

$$\begin{tikzcd} G_U\xi \arrow[d, "G\phi"] & F(G\xi) \arrow[r, "\alpha_\xi"] \arrow[d, "F(G\phi)"] & \xi \arrow[d, "\phi"] \\ G_U\eta & F(G\eta) \arrow[r, "\alpha_\eta"] & \eta \end{tikzcd}$$

These operations define a functor $G : \mathcal{G} \to \mathcal{F}$ and is clear that sending each object ξ to the isomorphism $\alpha_\xi : \xi \simeq F(G\xi)$ we define an isomorphism of functors $\mathrm{id}_{\mathcal{G}} \simeq F \circ G : \mathcal{G} \to \mathcal{G}$. Now, let ξ' in $\mathcal{F}$ and $U = p_{\mathcal{F}}\xi'$; then $\alpha_{F\xi'} : F\xi' \simeq F(G(F\xi'))$ in $\mathcal{G}(U)$. Since F_U is fully faithful there is a unique isomorphism $\beta_{\xi'} : \xi' \simeq G(F\xi')$ in $\mathcal{F}(U)$ such that $F\beta_{\xi'} = \alpha_{F\xi'}$; once again, is clear that this operations define an isomorphism of functors $\beta : G \circ F \simeq \mathrm{id}_{\mathcal{G}}$.

♠

3.2. The functors of arrows of a fibered category

Let us define a functor of arrows. Suppose that $F \to C$ is a fibered category; if U is an object in $\mathcal{C}$ and ξ, η are objects of $\mathcal{F}(U)$, we denote by $\mathrm{Hom}_U(\xi, \eta)$ the set of arrows from ξ to η in $F(U)$. Let ξ and η be two objects of $\mathcal{F}$ over the same object S of $\mathcal{C}$. Let $u_1 : U_1 \to S$ and $u_2 : U_2 \to S$ be arrows in $\mathcal{C}$; these are objects of the comma category $(\mathcal{C}/S)$. Suppose that $\xi_i \to \xi$ and $\eta_i \to \eta$ are pullbacks along $u_i : U_i \to S$ for $i = 1, 2$. For each arrow $f : U_1 \to U_2$ in $(\mathcal{C}/S)$, by definition of pullback there are two arrows, each unique, $\alpha_f : \xi_1 \to \xi_2$ and $\beta_f : \eta_1 \to \eta_2$, such that the two diagrams

$$\begin{tikzcd} \xi_1 \arrow[rr, "\alpha_f"] \arrow[dr] & & \xi_2 \arrow[dl] \\ & \xi & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \eta_1 \arrow[rr, "\beta_f"] \arrow[dr] & & \eta_2 \arrow[dl] \\ & \eta & \end{tikzcd}$$

commute. By proposition 3.3 (ii) the arrows α_f and β_f are cartesian; we define a pullback function

$$f^* : \mathrm{Hom}_{U_2}(\xi_2, \eta_2) \longrightarrow \mathrm{Hom}_{U_1}(\xi_1, \eta_1)$$

in which $f^*\phi$ is the unique arrow $f^*\phi : \xi_1 \to \eta_1$ in $\mathcal{F}(U_1)$ making the diagram

$$\begin{tikzcd} \xi_1 \arrow[r, "f^*\phi"] \arrow[d, "\alpha_f"] & \eta_1 \arrow[d, "\beta_f"] \\ \xi_2 \arrow[r, "\phi"] & \eta_2 \end{tikzcd}$$

commute. Given a third arrow $g : U_2 \to U_3$ in $(\mathcal{C}/S)$ with pullbacks $\xi_3 \to \xi$ and $\eta_3 \to \eta$, we have arrows $\alpha_g : \xi_2 \to \xi_3$ and $\beta_g : \eta_2 \to \eta_3$. Thus is clear that

$$\alpha_{gf} = \alpha_g \circ \alpha_f : \xi_1 \longrightarrow \xi_3 \quad \textit{and} \quad \beta_{gf} = \beta_g \circ \beta_f : \eta_1 \longrightarrow \eta_3$$

and this implies that

$$(gf)^* = f^* g^* : \mathrm{Hom}_{U_3}(\xi_3, \eta_3) \longrightarrow \mathrm{Hom}_{U_1}(\xi_1, \eta_1)$$

Now, is time for the definition of the functor of arrows of a fibered category. Let us choose a cleavage for $\mathcal{F}$ and define a functor

$$\underline{\mathrm{Hom}}_S(\xi, \eta) : (\mathcal{C}/S)^{op} \longrightarrow (Set)$$

by sending each object $u : U \to S$ into the set $\mathrm{Hom}_U(u^*\xi, u^*\eta)$ of arrows in the category $\mathcal{F}(U)$. An arrow $f : U_1 \to U_2$ from $u_1 : U_1 \to S$ to $u_2 : U_2 \to S$ yields a function

$$f^* : \mathrm{Hom}_{U_2}(u_2^*\xi, u_2^*\eta) \longrightarrow \mathrm{Hom}_{U_1}(u_1^*\xi, u_1^*\eta)$$

and this defines the effect of $\underline{\mathrm{Hom}}_S(\xi, \eta)$ on arrows.

Note that the functor $\underline{\mathrm{Hom}}_S(\xi, \eta)$ is independent of the choice of a cleavage, in the sense that cleavage give a canonically isomorphism functors: Suppose that we have chosen for each $U \to S$ and for each object η in $\mathcal{F}(S)$ another pullback $u^\vee\eta \to \eta$, then there is a canonical isomorphism $u^*\eta \simeq u^\vee\eta$ in $\mathcal{F}(U)$ for each arrow $u : U \to S$, analogously we have $u^*\xi \simeq u^\vee\xi$. This gives a bijective correspondence

$$\mathrm{Hom}_U(u^*\xi, u^*\eta) \simeq \mathrm{Hom}_U(u^\vee\xi, u^\vee\eta)$$

yielding an isomorphism of the functors of arrows defined by the two pullbacks.

CHAPTER 4

Descent theory and Stacks

Following A. Vistoli, we start with the archetypal example of descent.

4.1. continuous maps and topological spaces

Take $(Cont)$ to be the category of continuous maps, that is, the category of arrows in (Top), as in Example 3.11; this category is fibered over (Top) via the functor $p_{(Cont)} : (Cont) \to (Top)$ sending each continuous map to its codomain. Now, suppose that $f : X \to U$ and $g : Y \to U$ are two objects of $(Cont)$ mapping to the same object U in (Top); we want to construct a continuous map $\phi : X \to Y$ over U, that is, an arrow in $(Cont)(U) = (Top/U)$. Suppose that we are given an open covering $\{U_i\}$ of U, and continuous maps $\phi_i : f^{-1}U_i \to g^{-1}U_i$ over U_i; assume furthermore that the restriction of ϕ_i and ϕ_j to $f^{-1}(U_i \cap U_j) \to g^{-1}(U_i \cap U_j)$ coincide. Then there is a unique continuous map $\phi : X \to Y$ over U whose restriction to each $f^{-1}U_i$ coincides with ϕ_i. This can be written as follows. The category $(Cont)$ is fibered over (Top), and if $f : V \to U$ is a continuous map, $X \to U$ an object of $(Cont)(U) = (Top/U)$, then a pullback of $X \to U$ to V is given by the projection $V \times_U X \to V$. The functor $f^* : (Cont)(U) \to (Cont)(V)$ sends each object $X \to U$ to $V \times_U X \to V$, and each arrow in (Top/U), given by continuous function $\phi : X \to Y$ over U, to the continuous function $f^*\phi = id_V \times_U f : V \times_U X \to V \times_U Y$.

$$\begin{tikzcd}
V \times_U X \arrow[dr, dashed] \arrow[drr, bend left, "\phi \circ \mathrm{pr}_2"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & \\
& V \times_U Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& V \arrow[r, "f"'] & U
\end{tikzcd}$$

where $\mathrm{pr}_1 : V \times_U X \to V$ and $\mathrm{pr}_2 : V \times_U X \to X$ are the usual projections. Suppose that we are given two topological spaces X and Y with continuous maps $X \to S$ and $Y \to S$. Consider the functor

$$\underline{\mathrm{Hom}}_S(X, Y) : (Top/S)^{op} \longrightarrow (Set)$$

from the category of topological spaces over S, defined in Section lalala. This sends each arrow $U \to S$ to the set of continuous maps $\mathrm{Hom}_U(U \times_S X, U \times_S Y)$ over U. The action on arrows is obtained as follows: given a continuous function $f : V \to U$ over S, we send each continuous function $\phi : U \times_S X \to U \times_S Y$ to the function

$$f^*\phi = id_V \times_S \phi : V \times_S X = V \times_U (U \times_S X) \longrightarrow V \times_U (U \times_S Y) = V \times_S Y.$$

Then the fact that continuous functions can be constructed locally and then glued together can be expressed by saying that the functor

$$\underline{\mathrm{Hom}}_S(X,Y) : (Top/S)^{op} \longrightarrow (Set)$$

is a sheaf in the classical topology of (Top).
We do not only can we construct continuous functions locally: we can also do this for spaces, although this is more complicated.

PROPOSITION 4.1. *Suppose that we are given a topological space U with an open covering $\{U_i\}$; for each triple of indices i, j and k choose fibered products $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$. Assume that for each i we have a continuous map $u_i : X_i \to U_i$, and that for each pair of indices i and j we have a homeomorphism $\phi_{ij} : u_j^{-1}U_{ij} \simeq u_i^{-1}U_{ij}$ over U_{ij}, satisfying the cocycle condition*

$$\phi_{ik} = \phi_{ij} \circ \phi_{jk} : u_k^{-1}U_{ijk} \longrightarrow u_j^{-1}U_{ijk} \longrightarrow u_i^{-1}U_{ijk}$$

Then there exists a continuous map $u : X \to U$, together with isomorphisms $\phi_i : u^{-1}U_i \simeq X_i$, such that $\phi_{ij} = \phi_i \circ \phi_j^{-1} : u_j^{-1}U_{ij} \to u^{-1}U_{ij} \to u_i^{-1}U_{ij}$ for all i and j.

PROOF. Consider the disjoint union V of the U_i (take the union of the $U_i \times \{i\}$); the fibered product $V \times_U V$ is the disjoint union of the U_{ij}. The disjoint union Y of the X_i, maps to V; consider the subset $R \subset Y \times Y$ consisting of pairs $(x_i, x_j) \in X_i \times X_j \subset Y \times Y$ such that $x_i = \phi_{ij}x_j$. R is an equivalence relation in Y: Notice that the cocycle condition $\phi_{ii} = \phi_{ii} \circ \phi_{ii}$ implies that ϕ_{ii} is the identity on X_i, and this shows that the equivalence relation is reflexive. The fact that $\phi_{ii} = \phi_{ij} \circ \phi_{ji}$, and therefore $\phi_{ji} = \phi_{ij}^{-1}$, prove that it is symmetric; and transitivity follows directly from the general cocycle condition. We define X to be the quotient Y/R. If two points of Y are equivalent, then their images in U coincide; so there is an induced continuous map $u : X \to U$. The restriction to $X_i \subset Y$ of the projection $Y \to X$ gives a continuous map $\phi_i : Xi \to u^{-1}U_i$, which is a homeomorphism. One also sees that $\phi_{ij} = \phi_i \circ \phi_j^{-1}$, and this completes the proof. ♠

The fact that we can glue continuous maps and topological spaces says that $(Cont)$ is a *stack* over (Top).

In the following section we will see a more complicated example of stack

4.2. Gluing sheaves and topological spaces

Before we prove the main result we need the following lemma.

LEMMA 4.2. *Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of abelian groups on X. For any open set $U \subset X$, the set $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group and the presheaf $U \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is called the sheaf* Hom *and is denoted by* $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$.

PROOF. For two morphism $\varphi, \psi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$, we define $\varphi + \psi$ to be the map such that for every open set $V \subset U$

$$\begin{array}{rcl}(\varphi+\psi)_V : (\mathcal{F}|_U)(V) & \longrightarrow & (\mathcal{G}|_U)(V) \\ s & \longmapsto & \varphi_V(s) + \psi_V(s)\end{array}$$

Since φ and ψ are morphisms of sheaves, for $W \subset V \subset U$ and $s \in \mathcal{F}|_U(V)$ we have

$$
\begin{aligned}
(\varphi+\psi)_W(s|_W) &:= \varphi_W(s|_W) + \psi_W(s|_W) \\
&= \varphi_V(s)|_W + \psi_V(s)|_W \\
&= \varphi_V(s) + \psi_v(s)|_W \\
&:= (\varphi+\psi)_V(s)|_W
\end{aligned}
$$

This means that the following diagram commute

$$
\begin{array}{ccc}
\mathcal{F}|_U(V) & \xrightarrow{(\varphi+\psi)_V} & \mathcal{G}|_U(V) \\
\rho \big\downarrow & & \big\downarrow \rho' \\
\mathcal{F}|_U(W) & \xrightarrow{(\varphi+\psi)_W} & \mathcal{G}|_U(W)
\end{array}
$$

Hence $\varphi+\psi : \mathcal{F}|_U \to \mathcal{G}|_U$ is a morphism of sheaves. For each $\psi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ we define $-\psi : \mathcal{F}|_U \to \mathcal{G}|_U$ as $(-\psi)(V) := -\psi_V$ for all $V \subset U$. The morphism 0 is such that 0_V is the zero map for all $V \subset U$. Therefore $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ together with the restrictions as follow. If $\psi : \mathcal{F}|_U \to \mathcal{G}|_U$, $U' \subset U$,

$$
\begin{array}{rcl}
\psi_{U'} : \mathcal{F}|_U(U') & \longrightarrow & \mathcal{G}|_U(U') \\
s & \longmapsto & \psi_{U'}(s)
\end{array}
$$

then for every open set $V \subset U$

$$
\begin{array}{rcl}
\rho_{VU}(\psi) := \psi_V : \mathcal{F}|_V(U' \cap V) & \longrightarrow & \mathcal{G}|_V(U' \cap V) \\
s & \longmapsto & \psi_{U'}(s)|_{U' \cap V}
\end{array}
$$

Becames a presheaf.

Let $U \subset X$ be an open set and $\{U_i\}_i$ a covering of U. Let $\varphi_i \in \mathrm{Hom}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, let us build $\varphi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ such that $\varphi|_{U_i} = \varphi_i$. Let $V \subset U$, $s \in \mathcal{F}|_U(V)$ and $t_i := \varphi_i v(s|_{V \cap U_i}) \in \mathcal{G}|_U(V \cap U_i)$, then

$$
\begin{aligned}
t_i|_{U_i \cap U_j} &= \varphi_i v(s|_{V \cap U_i})\Big|_{U_i \cap U_j} \\
&= \varphi_i u_i \cap u_j(s|_{U_i \cap U_j \cap V}) \\
&= \varphi_j u_i \cap u_j(s|_{U_i \cap U_j \cap V}) \\
&= \varphi_j v(s|_{V \cap U_j})\Big|_{U_i \cap U_j} \\
&= t_j|_{U_i \cap U_j}
\end{aligned}
$$

The above equalities are obtained from the commutativity of

$$\begin{tikzcd}
\mathcal{F}|_V\,(U_i \cap V) \arrow[r, "\varphi_i v"] \arrow[d] & \mathcal{G}|_V\,(U_i \cap V) \arrow[d] \\
\mathcal{F}|_{U_i \cap U_j}\,(U_i \cap U_j \cap V) \arrow[r, "\varphi_i u_j \cap v"] & \mathcal{G}|_{U_i \cap U_j}\,(U_i \cap U_j \cap V)
\end{tikzcd}$$

Now, since $\{U_i \cap V\}_i$ is an open cover of V and $\mathcal{G}|_U$ is a sheaf there exists a unique $t \in \mathcal{G}|_U\,(V)$ such that $t|_{V \cap U_i} = t_i = \varphi_i v(s|_{V \cap U_i})$. Let $\varphi_V(s) := t$. Then $\varphi|_{U_i} = \varphi_i$. Now, for open sets $W \subset V \subset U$ and $s \in \mathcal{F}|_U\,(V)$, then

$$\begin{aligned}
\varphi_V(s)|_W|_{W \cap U_i} &= t|_W|_{W \cap U_i} \\
&= t|_{W \cap U_i} \\
&= t|_{V \cap U_i}\Big|_{W \cap U_i} \\
&= t|_{W \cap U_i} \\
&= \varphi_i v(s|_{V \cap U_i})\Big|_{W \cap U_i} \\
&= \varphi_i w(\,s|_{V \cap U_i}\Big|_{W \cap U_i}) \\
&= \varphi_i w(s|_{W \cap U_i}) \\
&= \varphi_W(s|_W)|_{W \cap U_i}
\end{aligned}$$

The last equality is by definition and by $\varphi|_{U_i} = \varphi_i$ and the others by the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{F}|_U\,(V) \arrow[d] & \\
\mathcal{F}|_V\,(V \cap U_i) \arrow[r, "\varphi_i v"] \arrow[d] & \mathcal{G}|_V\,(V \cap U_i) \arrow[d] \\
\mathcal{F}|_W\,(W \cap U_i) \arrow[r, "\varphi_i w"] & \mathcal{G}|_{W \cap U_i}\,(W \cap U_i)
\end{tikzcd}$$

Since $\{W \cap U_i\}$ is an open cover for W and $\mathcal{G}|_U$ is a sheaf then $\varphi_V(s)|_W = \varphi_W(s|_W)$, therefore $\varphi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ and $\varphi|_{U_i} = \varphi_i$ as desired. In conclusion $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is a sheaf. ♠

PROPOSITION 4.3. *Let X be a topological space , let $\{U_i\}$ be an open cover of X and suppose we are given for each i a sheaf of abelian groups $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ji} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ such that*

(1) *for each i, φ_{ii} is the identity.*
(2) *for each i, j, k, $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ on $U_i \cap U_j \cap U_k$*

Then there exist a unique sheaf of abelian groups (up to isomorphism) $\mathcal{F}$ on X, together with isomorphism $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{ji} \circ \psi_i$ on $U_i \cap U_j$.

PROOF. Let $U \subset X$ be an open set and define

$$\mathcal{F}(U) := \left\{ (s_i)_i \in \prod_i \mathcal{F}_i(U \cap U_i) \ : \varphi_{ji}(s_i|_{U \cap U_i \cap U_j}) = s_j|_{U \cap U_i \cap U_j} \ \textit{on } U \cap U_i \cap U_j \right\}$$

If $V \subset U$ the morphisms $\mathcal{F}_i(U \cap U_i) \xrightarrow{\rho^i} \mathcal{F}_i(V \cap U_i)$ induce restriction maps

$$\begin{array}{rcl} \rho_{VU} : \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) \\ (s_i)_i & \longmapsto & (\rho^i(s_i))_i := (s_i|_V)_i \end{array}$$

We check that ρ_{VU} is well defined. If $(w_i)_i = (s_i|_V)_i \in \prod_i \mathcal{F}_i(V \cap U_i)$, then

$$\begin{aligned} w_j|_{V \cap U_i \cap U_j} &= s_j|_{V \cap U_j}\Big|_{V \cap U_i \cap U_j} \\ &= s_j|_{V \cap U_i \cap U_j} \\ &= s_j|_{U \cap U_i \cap U_j}\Big|_{V \cap U_i \cap U_j}, \end{aligned}$$

and as φ_{ji} is an isomorphism

$$\begin{aligned} w_j|_{V \cap U_i \cap U_j} &= s_j|_{U \cap U_i \cap U_j}\Big|_{V \cap U_i \cap U_j} \\ &= \varphi_{ji}(s_i|_{U \cap U_i \cap U_j})\Big|_{V \cap U_i \cap U_j} \end{aligned}$$

and since the following diagram commute

$$\begin{array}{ccccc} \mathcal{F}_i(U \cap U_i) & \longrightarrow & \mathcal{F}_i(U \cap U_i \cap U_j) & \longrightarrow & \mathcal{F}_i(V \cap U_i \cap U_j) \\ & & \big\downarrow \varphi_{ji} & & \big\downarrow \varphi_{ji} \\ & & \mathcal{F}_j(U \cap U_i \cap U_j) & \longrightarrow & \mathcal{F}_j(V \cap U_i \cap U_j) \end{array}$$

we have

$$\begin{aligned} w_j|_{V \cap U_i \cap U_j} &= \varphi_{ji}(s_i|_{U \cap U_i \cap U_j})\Big|_{V \cap U_i \cap U_j} \\ &= \varphi_{ji}\left(s_i|_{U \cap U_i \cap U_j}\Big|_{V \cap U_i \cap U_j}\right) \\ &= \varphi_{ji}(w_i|_{V \cap U_i \cap U_j}) \end{aligned}$$

this implies $(w_i)_i \in \mathcal{F}(V)$, hence ρ_{VU} is well defined. Is clear $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$ for all open sets $W \subset V \subset U$ and $\rho_{VU}(\emptyset) = 0$, therefore $\mathcal{F}$ is a presheaf.
Let us now show that $\mathcal{F}$ is a sheaf. Let $\{V_\alpha\}_\alpha$ be a covering of U, let $s_\alpha = (s_{i,\alpha})_i \in \mathcal{F}(V_\alpha)$ (where $s_{i,\alpha} \in \mathcal{F}_i(V_\alpha \cap U_i)$) such that $s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$, this implies $\rho_{(V_\alpha \cap V_\beta)V_\alpha}(s_\alpha) = \rho_{(V_\alpha \cap V_\beta)V_\beta}(s_\beta)$, hence for each i, $s_{i,\alpha}|_{V_\alpha \cap V_\beta \cap U_i} = s_{j,\alpha}|_{V_\alpha \cap V_\beta \cap U_i}$. Since $\{V_\alpha \cap U_i\}_\alpha$ is an open cover for $U \cap U_i$ and $\mathcal{F}_i$ is a sheaf there exists a unique

$s_i \in \mathcal{F}_i(U \cap U_i)$ such that $s_i|_{V_\alpha \cap U_i} = s_{i,\alpha}$. Let $s = (s_i)_i$, thus $s|_{V_\alpha} = (s_i|_{V_\alpha \cap U_i})_i = (s_{i,\alpha})_i = s_\alpha$. Let us see that $s \in \mathcal{F}(U)$. It is obvious that $s \in \prod_i \mathcal{F}_i(U \cap U_i)$. Now,

$$\begin{aligned}
s_j|_{U\cap U_i \cap U_j}\Big|_{V_\alpha \cap U_i \cap U_j} &= s_i|_{V_\alpha \cap U_i \cap U_j} \\
&= s_j|_{V_\alpha \cap U_j}\Big|_{V_\alpha \cap U_i \cap U_j} \\
&= s_{j,\alpha}|_{V_\alpha \cap U_i \cap U_j} \\
&= \varphi_{ji}(s_{i,\alpha}|_{V_\alpha \cap U_i \cap U_j}) \\
&= \varphi_{ji}(s_{i,\alpha}|_{V_\alpha \cap U_i}\Big|_{V_\alpha \cap U_i \cap U_j}) \\
&= \varphi_{ji}(s_i|_{V_\alpha \cap U_i \cap U_j}) \\
&= \varphi_{ji}(s_i|_{U \cap U_i \cap U_j}\Big|_{V_\alpha \cap U_i \cap U_j}) \\
&= \varphi_{ji}(s_i|_{U_\alpha \cap U_i \cap U_j})\Big|_{V_\alpha \cap U_i \cap U_j}.
\end{aligned}$$

Since $\mathcal{F}_j$ is a sheaf and $\{V_\alpha \cap U_i \cap U_j\}_\alpha$ is an open cover for $U \cap U_i \cap U_j$, then $s_i|_{U\cap U_i \cap U_j} = \varphi_{ji}(s_i|_{U \cap U_i \cap U_j})$, therefore $s \in \mathcal{F}(U)$. In conclusion, $\mathcal{F}$ is a sheaf. Now, let us fix i, let $V \subset U_i$ be an open set. Define the isomorphism $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ as follows. Noticing that $(\mathcal{F}|_{U_i})(V) = \mathcal{F}(V)$, let

$$\begin{aligned}
\psi_i v : (\mathcal{F}|_{U_i})(V) &\longrightarrow \mathcal{F}_i(V) \\
s = (s_i)_i &\longmapsto s_i
\end{aligned}$$

be the natural projection. If $t \in (\mathcal{F}|_{U_i})(V) = \mathcal{F}(V)$ is such that $\psi_i v(t) = \psi_i v(s)$, that is to say, $t_i = s_i$, then

$$s_i|_{V\cap U_i \cap U_j} = t_i|_{V \cap U_i \cap U_j} = \varphi_{ij}(t_j|_{V \cap U_i \cap U_j})$$

Since φ_{ij} is an isomorphism, we have $s_j|_{V\cap U_i \cap U_j} = t_j|_{V \cap U_i \cap U_j}$ for all j, which in turn implies $s = t$, as $\mathcal{F}_j$ is a sheaf. Therefore, $\psi_i v$ is injective. Let us check that $\psi_i v$ is also surjective. To this end, let $s_i \in \mathcal{F}_i(V)$. Notice that $(\varphi_{ji}(s_i|_{V \cap U_j}))_j \in \mathcal{F}(V)$. Indeed, $\varphi_{ji}(s_i|_{V\cap U_j}) \in \mathcal{F}_j(V \cap U_j)$ because $V \subset U_i$ and φ_{ji} is an isomorphism. Next, by the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}_i(V) \arrow[r] & \mathcal{F}_i(V \cap U_j) \arrow[r] \arrow[d, "\varphi_{ji}"] & \mathcal{F}_i(V \cap U_j \cap U_k) \arrow[d, "\varphi_{ji}"] \\
& \mathcal{F}_j(V \cap U_j) \arrow[r] & \mathcal{F}_j(V \cap U_j \cap U_k)
\end{tikzcd}$$

we have

$$\begin{aligned}
\varphi_{ki}(s_i|_{V\cap U_k})\Big|_{V\cap U_j\cap U_k} &= \varphi_{ki}(\,s_i|_{V\cap U_k}\Big|_{V\cap U_j\cap U_k}) \\
&= \varphi_{ki}(s_i|_{V\cap U_j\cap U_k}) \\
&= \varphi_{kj}(\varphi_{ji}(s_i|_{V\cap U_j\cap U_k})) \\
&= \varphi_{kj}(\varphi_{ji}(\,s_i|_{V\cap U_j}\Big|_{V\cap U_j\cap U_k})) \\
&= \varphi_{kj}(\,\varphi_{ji}(s_i|_{V\cap U_j})\Big|_{V\cap U_j\cap U_k})
\end{aligned}$$

Here, the third equality is obtained from condition (2). Finally, from condition (1), we have that $\psi_i v\left[(\varphi_{ji}(s_i|_{V\cap U_j}))_j\right] = s_i$ and therefore $\psi_i v$ is surjective.
In conclusion, $\psi_i v$ is an isomorphism for each $V \subset U_i$ and is clear that the following diagram commute for all open sets $W \subset V \subset U_i$

$$\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\psi_i v} & \mathcal{F}_i(V_p) \\
\Big\downarrow \rho & & \Big\downarrow \rho^i \\
\mathcal{F}(W) & \xrightarrow{\psi_i w} & \mathcal{F}_i(W)
\end{array}$$

Therefore, ψ_i is an isomorphism. Let us now check the gluing condition. For each $s = (s_i)_i \in \mathcal{F}(U)$, we have $s_j\big|_{U\cap U_i\cap U_j} = \varphi_{ji}(s_i|_{U\cap U_i\cap U_j})$. It follows immediately that $\psi_j = \varphi_{ji} \circ \psi_i$ on $U_i \cap U_j$.
Finally, let us show the uniqueness of $\mathcal{F}$. Suppose that there is another sheaf $\mathcal{G}$ on X such that there are isomorphisms $\widetilde{\psi}_i : \mathcal{G}|_{U_i} \to \mathcal{F}_i$ satisfying $\widetilde{\psi}_j = \varphi_{ji} \circ \widetilde{\psi}_i$ on $U_i \cap U_j$. This gives a isomorphism $\mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ via $\widetilde{\psi}_i^{-1} \circ \psi$, by the above claim $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is a sheaf and as $\{U_i\}_i$ is a covering of U there is a unique morphism $\theta : \mathcal{F} \to \mathcal{G}$ such that $\theta|_{U_i} = \widetilde{\psi}_i^{-1} \circ \psi_i$. Analogously we have a unique morphism $\widetilde{\theta} : \mathcal{G} \to \mathcal{F}$ such that $\widetilde{\theta}\Big|_{U_i} = \psi_i^{-1} \circ \widetilde{\psi}_i$, since $(\theta \circ \widetilde{\theta})\Big|_{U_i}$ is the identity map, so $\theta \circ \widetilde{\theta}$ is the identity. Similarly $\widetilde{\theta} \circ \theta\Big|_{U_i}$ is the identity. This implies that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, so $\mathcal{F}$ is unique up isomophisms.

♠

The above result says that the category of sheaves of abelian groups is a *stack* for the open immersion topology on (Top).

As we shall now see, we can generalize the above procedure to any fibered category.

4.3. The category of descent data

Let $\mathcal{C}$ be a site. We have seen that a fibered category over $\mathcal{C}$ should be thought of as a functor from $\mathcal{C}$ to the category of categories, that is, as a presheaf of categories over $\mathcal{C}$. A *stack* is, morally, a sheaf of categories over $\mathcal{C}$.
Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$ and fix a cleavage. Given a covering $\{U_i \to U\}$,

we set $U_{ij} := U_i \times_U U_j$ for each pair of indices i and j, and $U_{ijk} := U_i \times_U U_j \times_U U_k$ for each triple of indices i, j and k.

DEFINITION 4.4. Let $\mathcal{U} = \{U_i \to U\}$ be a covering in $\mathcal{C}$. An *object with descent data* $(\{\xi_i\}\{\phi_{ij}\})$ on $\mathcal{U}$, is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\xi_j \simeq \mathrm{pr}_1^*\xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that the following cocycle condition is satisfied. For any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk} : \mathrm{pr}_3^*\xi_k \longrightarrow \mathrm{pr}_1^*\xi_i$$

where the pr_{ab} and pr_a are the projections on the a^{th} and b^{th} factor, or the a^{th} factor respectively. The isomorphisms ϕ_{ij} are called *transition isomorphisms* of the object with descent data.
An arrow between objects with descent data

$$\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \longrightarrow (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i : \xi_i \to \eta_i$ in $\mathcal{F}(U)$, with the property that for each pair of indices i, j the diagram

$$\begin{tikzcd}
\mathrm{pr}_2^*\xi_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"] \arrow[d, "\phi_{ij}"] & \mathrm{pr}_2^*\eta_j \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_1^*\xi_i \arrow[r, "\mathrm{pr}_1^*\xi_i"] & \mathrm{pr}_1^*\eta_i
\end{tikzcd}$$

commutes.

REMARK 4.5. For each object ξ of $\mathcal{F}(U)$ we can construct an object with descent data on a covering $\{\sigma_i : U_i \to U\}$ as follows. The objects are the pullbacks $\sigma_i^*\xi$; the isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\sigma_j^*\xi \simeq \mathrm{pr}_1^*\sigma_i^*\xi$ are the isomorphisms that come from the fact that both $\phi_{ij} : \mathrm{pr}_2^*\sigma_j^*\xi$ and $\phi_{ij} : \mathrm{pr}_1^*\sigma_i^*\xi$ are pullbacks of ξ to U_{ij}. If we identify $\phi_{ij} : \mathrm{pr}_2^*\sigma_j^*\xi$ with $\phi_{ij} : \mathrm{pr}_1^*\sigma_i^*\xi$, as is commonly done, then the ϕ_{ij} are identities. Given an arrow $\alpha : \xi \to \eta$ in $\mathcal{F}(U)$, we get arrows $\sigma_i^*\alpha : \sigma_i^*\xi \to \sigma_i^*\eta$, yielding an arrow from the object with descent associated with ξ to the one associated with η. This defines a functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$. It is important to notice that these constructions do not depend on the choice of a cleavage, in the following sense: Given a different cleavage, for each covering $\{U_i \to U\}$ there is a canonical isomorphism of the resulting categories $\mathcal{F}(\{U_i \to U\})$; and the functors $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ commute with these equivalences.
Here is a definition of the category of descent data that does not depend on choosing of a cleavage. Let $\{U_i \to U\}$ be a covering. We define an object with descent data to be a triple of sets

$$(\{\xi_i\}_{i \in I}, \{\xi_{ij}\}_{i,j \in I}, \{\xi_{ijk}\}_{i,j,k \in I})$$

where each ξ is an object of $\mathcal{F}(U_\alpha)$, plus, for each triple of indices i, j and k, a commutative diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr, crossing over] \arrow[dd] & & \xi_j & \\
& \xi_{jk} \arrow[rr] \arrow[dl] & & \xi_k \\
\xi_i & & &
\end{tikzcd}$$

in which every arrow is cartesian, and such that when applying $p_{\mathcal{F}}$ every arrow maps to the appropriate projection in the following diagram.

$$\begin{tikzcd}[row sep=small, column sep=small]
& U_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & U_{jk} \arrow[dl] \arrow[dd] \\
U_{ij} \arrow[rr, crossing over] \arrow[dd] & & U_j & \\
& U_{jk} \arrow[rr] \arrow[dl] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U \arrow[from=uu, crossing over] &
\end{tikzcd}$$

These form the objects of a category $\mathcal{F}_{desc}(\{U_i \to U\})$. An arrow

$$\{\phi_i\}_{i \in I} : (\{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\}) \longrightarrow (\{\eta_i\}, \{\eta_{ij}\}, \{\eta_{ijk}\})$$

consists of set of arrows with $\phi_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$, such that for every pair of indices i and j we have

$$\mathrm{pr}_1^* \phi_i = \mathrm{pr}_2^* \phi_j : \xi_{ij} \longrightarrow \eta_{ij}.$$

There are others equivalence definitions of objects with descent data, even without the choice of a cleavage. The tool necessary for to see such equivalences is a version 2-categorical of the Yoneda lemma (**[Vis05]** pp.62 and pp.73-74).

4.4. Stack

DEFINITION 4.6. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category on a site $\mathcal{C}$.

(i) $\mathcal{F}$ is a *prestack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\{U_i \to U\})$$

is fully faithful.

(ii) $\mathcal{F}$ is a *stack* over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$, the functor

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\{U_i \to U\})$$

is an equivalence of categories.

Concretely, for $\mathcal{F}$ to be a prestack means the following. Let U be an object of $\mathcal{C}$, ξ and η objects of $\mathcal{F}(U)$, $\{U_i \to U\}$ a covering, ξ_i and η_i pullbacks of ξ and η to U_i, ξ_{ij} and η_{ij} pullbacks of ξ and η to U_{ij}. Suppose that there are arrows $\alpha_i : \xi_i \to \eta_i$ in $\mathcal{F}(Ui)$, such that $\mathrm{pr}_1^* \alpha_i = \mathrm{pr}_2^* \alpha_j : \xi_{ij} \to \eta_{ij}$ for all i and j. Then there is a unique arrow $\alpha : \xi \to \eta$ in $\mathcal{F}(U)$, whose pullback to $\xi_i \to \eta_i$ is α_i for all i.
This condition can be restated using the functor of arrows, and the comma topology on the category (C/S).

PROPOSITION 4.7. *(**[Vis05]**, Proposition 4.7) Let $\mathcal{F}$ be a fibered category over a site $\mathcal{C}$. Then $\mathcal{F}$ is a prestack if and only if for any object S of $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(S)$, the functor $\underline{\mathrm{Hom}}_S(\xi,\eta) : (C/S)^{op} \to (Set)$ is a sheaf in the comma topology.*

DEFINITION 4.8. An object with descent data $(\{\xi_i\},\{\phi_{ij}\})$ in $\mathcal{F}(\{U_i \to U\})$ is effective if it is isomorphic to the image of an object of $\mathcal{F}(U)$.

So, $\mathcal{F}$ is a stack if and only if it is a prestack and all objects with descent data in $\mathcal{F}$ are effective.
Stacks are the correct generalization of sheaves, and give the right notion of "sheaf of categories". We should of course prove the following statement.

PROPOSITION 4.9. *(**[Vis05]**, Proposition 4.9) Let $\mathcal{C}$ be a site, $F : \mathcal{C}^{op} \to (Set)$ a functor.*

(i) *F is a prestack if and only if it is a separated functor.*
(ii) *F is a stack if and only if it is a sheaf.*

EXAMPLE 4.10. Let $\mathcal{C}$ be a site. We can see that the functor in the Example 3.12 (ii) is equivalent to a functor $(Sh/\mathcal{C}) \to \mathcal{C}$, whose fiber over X is ShX , this functor is a stack. To see this, let F and G be two sheaves on an object X of $\mathcal{C}$. To show that $(Sh/\mathcal{C})$ is a prestack we need to show that $\underline{\mathrm{Hom}}_X(F,G) : (C/X)^{op} \to (Set)$ is a sheaf. For each arrow $U \to X$, let us denote by F_U and G_U the restrictions of F and G to U. Let $\{U_i \to U\}$ be a covering, $\phi_i : F_{U_i} \to G_{U_i}$ a morphism of sheaves on $(\mathcal{C}/U_i)$, such that the restrictions of ϕ_i and ϕ_j to $(\mathcal{C}/U_{ij})$ coincide. Denote by $\phi_{ij} : F_{U_{ij}} \to G_{U_{ij}}$ this restriction. If $T \to U$ is an arrow, set $T_i = U_i \times_U T$, and consider the covering $\{T_i \to T\}$. Each $T_i \to U$ factors through U_i, so ϕ_i defines a function $\phi_i : FT_i \to GT_i$, and analogously ϕ_{ij} defines functions $\phi_{ij} : FT_{ij} \to GT_{ij}$. There is commutative diagram of sets with rows that are equalizers

$$\begin{tikzcd} FT \arrow[r] \arrow[d, dashed] & \prod_i FT_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\prod_i \phi_i"] & \prod_{ij} FT_{ij} \arrow[d, "\prod_{ij} \phi_{ij}"] \\ GT \arrow[r] & \prod_i GT_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{ij} GT_{ij} \end{tikzcd}$$

There is a unique function $\phi_T : FT \to GT$ that one can insert in the diagram while keeping it commutative. This proves uniqueness. Also, it is easy to check that the collection of the ϕ_T defines a natural transformation $\phi_U : F_U \to G_U$, whose restriction $F_{U_i} \to G_{U_i}$ is ϕ_i. Now let us show that every object with descent data $(\{F_i\},\{\phi_{ij}\})$ is effective. Notice first that F_i is a sheaf on $(\mathcal{C}/U_i)$, and ϕ_{ij} is an isomorphism of sheaves on $(\mathcal{C}/U_{ij})$ between the restrictions $(F_j)_{U_{ij}} \simeq (F_i)_{U_{ij}}$. For each object $T \to U$ of $(\mathcal{C}/U)$, set $T_i = U_i \times_U T$ as before, and define FT to be the subset of $\prod_i F_iT_i$ consisting of objects $(s_i) \in \prod_i FT_i$, with the property that ϕ_{ij} carries the restriction $(s_j)_{T_{ij}}$ to $(s_i)_{T_{ij}}$. In other words, FT is the equalizer of two functions $\prod_i F_iT_i \to \prod_{ij} F_iT_{ij}$, where the first sends (s_i) to the collections of restrictions $((s_i)_{T_{ij}})$, and the second sends it to $(\phi_{ij}(s_j)_{T_{ij}})$. For any arrow $T' \to T$ in $(\mathcal{C}/U)$, the product of the restriction functions $\prod_i F_iT_i \to \prod_i F_iT_i'$ carries FT to FT'; this gives F the structure of a functor $(\mathcal{C}/U)^{op} \to (Set)$. Clearly F is a sheaf.
Now we have to show that the image of F into $(Sh/\mathcal{C})(\{U_i \to U\})$ is isomorphic to $(\{F_i\},\{\phi_{ij}\})$. For each index k let us construct an isomorphism of the restriction

F_{U_k} with F_k as sheaves on $(\mathcal{C}/U_k)$. Let $T \to U_k$ be an object of $(\mathcal{C}/U_k)$, s an element of $F_k T$. Each T_i maps into U_{ik}, so we produce an element $(\phi_{ik}(s_{T_i})) \in \prod_i F_i T_i$; the cocycle condition ensures that this is an element of FT. This defines a natural transformation $F_k \to F_{U_k}$. In the other direction, let $T \to U_k$ be an object of $(\mathcal{C}/U_k)$, (s_i) an element of $FT \subset \prod_i F_i T_i$. Factor each $T_i \to U_i$ through the projection $U_{ki} \to U_i$. Then $\phi_{ki}(si)$ is an element of $F_k T_i$, and the cocycle condition implies that the restrictions of $\phi_{ki}(s_i)$ and $\phi_{kj}(s_j)$ to $F_k T_{ij}$ coincide. Hence there a unique element of $F_k T$ that restricts to $\phi_{ki}(s_i) \in F_k T_i$ for each i. This construction defines a function $FT \to F_k T$ for each T, which is easily seen to give a natural transformation $F_{U_k} \to F_k$. It is easy to check that these two natural transformations are inverse to each other, so they define an isomorphism of sheaves $F_{U_k} \simeq F_k$, and that this collection of isomorphisms constitutes an isomorphism in $(Sh/S)(\{U_i \to U\})$ between the object associated with F and the given object $(\{F_i\}, \{\phi_{ij}\})$, which is therefore effective.

4.5. Moduli spaces

Moduli spaces are geometric spaces which parametrise something - each point represents one of the objects being parametrised, such as the solution of a particular equation, or a geometric structure on some other object.
Suppose that we have a category $\mathcal{C}$ of algebraic objects that we want to parametrize (elliptic curves, vector spaces,...). Let $\mathcal{F}$ be a category. A *moduli functor* is a functor $F : \mathcal{F}^{op} \to (Set)$ which sends an object X (for example a scheme, topological spaces, variety,...) to the set of families of objects of $\mathcal{C}$ parametrized by X modulo an equivalence relation, such as *isomorphism classes* (a collection of mathematical objects isomorphic to each other). A morphism $f : X \to Y$ of spaces is send to the set map induced by pulling back families via f. A *fine moduli space* is an object M of $\mathcal{F}$ representing the moduli functor. If F be a representable functor, the object of FM corresponding to the element id_M of $\mathrm{Hom}_{\mathcal{F}}(M, M)$, to say $\pi : \mathcal{U} \to M$, is called the *universal family*. Thus for each S in $\mathcal{F}$, every family over S is the pullback of the universal family of a unique map.

Here a trivial example.

EXAMPLE 4.11. To describe a circle in $\mathbb{R}^2$ we need the center (x_0, y_0) and the radius r. Therefore, the moduli space of circles M_C in $\mathbb{R}^2$ is

$$M_C = \{(x_0, y_0, r) \in \mathbb{R}^3 : r > 0\}$$

Note that M_C is an open set of $\mathbb{R}^3$. Set

$$U_C = \{(x, y, x_0, y_0, r) \in \mathbb{R}^5 : (x - x_0)^2 + (y - y_0)^2 = r^2, r > 0\}$$

There is a map $\pi : U_C \to M_C$ defined by $\pi(x, y, x_0, y_0, r) = (x_0, y_0, r)$. Thus, for example, for the point $(1, 2, 3)$, we have

$$\pi^{-1}(1, 2, 3) = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y - 2)^2 = 9\}$$

Hence U_C contains all the circles in $\mathbb{R}^2$ and is the universal family of M_C.

A surprising number of stacky phenomena can be seen in such simple cases. Michael Artin, is famously reputed to have said that one need only understand the stack of triangles to understand stacks.

4.5.1. Triangles. Let us construct the moduli space of plane triangles, which we denote by T. The objects in this case are triangles up to isometry. We set the category $(MetricSpace)$ of locally compact metric spaces with the morphism being the maps $f : (S,d) \to (S';d')$ such that $d(x,y) \leq d'(f(x), f(y))$. Let us try first the case of ordered triangles. We set a family of ordered triangles as being a proper morphism of metric spaces $\varphi : X \to S$, such that $\varphi^{-1}(s)$ is isometric to a plane triangle, together with a triple of sections α, β, γ that marks the vertices of the triangle in X; also, the morphism are required to be compatible with the sections. As a set, T consists of triples (a,b,c) of side lengths, satisfying (strictly) the triangle inequalities, up to reordering. As a space, T is a quotient of a subset of Euclidean space. Indeed, consider the open cone in $\mathbb{R}^3$

$$\overline{T} = \{(a,b,c) \in \mathbb{R}^3_+ : a+b > c,\ a+c > b,\ b+c > a\}$$

Then we have a map $\overline{T} \to T$, and T inherits a topology from $\overline{T}$, the quotient topology. Note that $\overline{T}$ is a fine moduli space for the ordered triangles (since an object M of an category is a fine moduli space if and only if there exists a universal family over M) if we consider as a universal family the map $\pi : \mathcal{U} \to \overline{T}$, with $\mathcal{U} \subset \overline{T} \times \mathbb{R}^2$ which we associate to each point (a,b,c) in $\overline{T}$ the triangle with sides a,b,c in $\mathbb{R}^2$. Indeed, given any family $\varphi : X \to S$, consider the map $f : S \to \overline{T}$ given by

$$f(s) = (d(\alpha(s), \beta(s)), d(\alpha(s), \gamma(s)), d(\beta(s), \gamma(s)))$$

where d stands for the metric in X. Then the family $\varphi : X \to S$ is the pullback of the universal family via f, and any other map that makes the diagram commute necessarily agrees with f in every point of S.

REMARK 4.12. Let $\mathfrak{S}$ be the category of schemes. An object X in $\mathfrak{S}$ determines a category $\mathfrak{T}$, whose objects are pairs (S, f), where S is an object in $\mathfrak{S}$ and $f : S \to X$ is a morphism. A morphism from (S', f') to (S, f) in $\mathfrak{T}$ is given by a morphism $g : S' \to S$ such that $f \circ g = f'$. The functor $\mathfrak{T} \to \mathfrak{S}$ takes an object (S, f) to S, and takes a morphism from (S', f') to (S, f) to the underlying morphism from S' to S. By Yoneda lemma this category $\mathfrak{X}$ determines X up to canonical isomorphism. This category will be denoted by $\underline{X}$.

In the categorical language the moduli problem of triangles is as follow: we take S to be the category of topological spaces, and define a category $\mathfrak{T}$ whose objects are families of triangles $X \to S$. A morphism in $\mathfrak{T}$ from one family $X' \to S'$ to another family $X \to S$ is given by a pair of (continuous) maps $X' \to X$ and $S' \to S$ such that the following diagram commutes

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

so, that the induced maps on the fibers are isometries. The functor from $\mathfrak{T}$ to S takes an object $X \to S$ to S, and takes a morphism from $X' \to S'$ to $X \to S$ to the underlying morphism from S' to S. Say that $\overline{T}$ is a fine moduli space in the language of stacks, means that there is an isomorphism of stacks (equivalence) between $\mathfrak{T}$ and $\underline{\overline{T}}$.

If we consider now the case of unordered triangles, however, the situations is more complicated. The symmetric group S_3 acts (on the right) on $\overline{T}$ by permuting the coordinates, and the quotient space $T = \overline{T}/S_3$ is the candidate for a moduli space of triangles. Its points, at least, do correspond to triangles up to isometry. The group S_3 also acts on $\mathcal{U}$, compatibly with its projection to $\overline{T}$. Thus we can construct $Y = \mathcal{U}/S_3$, with an induced map $Y \to T$. This would be a first guess.
Any family of triangles$X \to S$ will determine a map from S to T, but the family may not be isomorphic (uniquely) to the pullback of $Y \to T$. In classical language, then, this moduli space T is a *coarse*, but not a fine moduli space, for $\mathfrak{T}$. For example, when S is the circle S^1 and $X \to S$ is a family of equilateral triangles that rotates the triangle by $120°$ in one revolution around the circle, then this is not a constant family even though the corresponding map from S to T is constant. For an isosceles triangle (taking S to be a point), say with sides of lengths $1, 2$, and 2, corresponding to a point t in T, there are three points $(1, 2, 2)$, $(2, 1, 2)$, and $(2, 2, 1)$ in $\overline{T}$ lying over t; the action of the group includes flips over the altitude, and the fiber of Y over t is the quotient of the triangle by this flip **[BPR14]**. For an equilateral triangle, there is only one point of $\overline{T}$ over the point t in T, and the fiber of Y over t is the quotient of the triangle by the action of S_3; In fact, $Y \to T$ fails to satisfy the definition of family of triangles **[BPR14]**. The problems with $Y \to T$ arise from triangles with nontrivial automorphisms.

It is easy to show that if $Y^\circ \to T^\circ$ be the restriction of $Y \to T$ to the locus T° of triangles with sides of distinct lengths (equilateral triangles). Then $Y^\circ \to T^\circ$ is a fiber bundle and gives a universal family, that is T° is a fine moduli space for such triangles **[BPR14]**.

DEFINITION 4.13. Let $\mathfrak{S} = (Top)$ be the category of the topological spaces and let G be a topological group. A *G-torsor*, or *principal G-bundle*, is a continuous map $E \to S$, with a continuous action of G on E, which we take to be a right action; one requires that it be locally trivial, in the sense that S has an open covering $\{U_\alpha\}$ such that the restriction $E|_{U_\alpha}$ is isomorphic to the trivial bundle $U_\alpha \times G \to U_\alpha$. One has a category, which we denote by BG, whose objects are the G-torsors $E \to S$. A morphism from $E' \to S'$ to $E \to S$ is given by a pair of maps $E' \to E$ and $S' \to S$ with the map $E' \to E$ being equivariant (commuting with the action of G) and the induced diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & T \end{tikzcd}$$

being cartesian. The functor from BG to $\mathfrak{S}$ takes an object $E \to S$ of BG to the object S of $\mathfrak{S}$, and a morphism from $E' \to S'$ to $E \to S$ to the underlying map $S' \to S$. If G acts on the right on a space X, one defines a category, denoted $[X/G]$, whose objects are G-torsors $E \to S$, together with an equivariant map from E to X. A morphism from $E' \to S'$, $E' \to X$ to $E \to S$, $E \to X$ is given by a pair of maps $E'E$ and $S' \to S$ giving a map of torsors as above, but, in addition, the composite $E' \to E \to X$ is required to be equal to the given map from E' to X.

Note that $[X/\{1\}]$, where $\{1\}$ denotes the group with one element is the same as $\underline{X}$.

Now, given any family $X \to S$ of unordered triangles, let $\overline{S}$ be the space of pairs $(s, \textit{ordering of the edges of } X_s)$. Here X_s denoted the fiber. Then $\overline{S} \to S$ is a 6-sheeted covering space. If $\overline{X} \to \overline{S}$ is the pullback of the given family $X \to S$ by the covering map $\overline{S} \to S$, we have the following commutative diagram

$$\begin{array}{ccc} \overline{S} & \longrightarrow & \overline{T} \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

and the map $\overline{S} \to \overline{T}$ commutes with the action of S_3. This is exactly the data for an object of the stack $[\overline{T}/S_3]$, the stack $\mathfrak{T}$ is isomorphic (equivalence of categories) to the quotient stack $[\overline{T}/S_3]$.

4.5.2. Elliptic curves. Recall that an elliptic curve is a Riemann surface of genus 1 together with a choice of origin for the group structure. Thus an elliptic curve can be expressed as a two-sheeted cover of the Riemann sphere branched at the set $\{0, 1, \infty, \lambda\}$ with $\lambda \in \mathbb{C} - \{0, 1\}$. Consider the j-invariant $j(\lambda) = \dfrac{2^8(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$. Thus the map from $\mathbb{C} - \{0, 1\}$ to $\mathbb{C}$ given by $\lambda \mapsto j(\lambda)$ is a surjective map that is generically a 6:1 covering. Moreover, two elliptic curves are isomorphic if and only if they have the same j-invariant, so the isomorphism class of an elliptic curve is determined by a single complex number **[Sil09]**. Thus we can think that $\mathbb{C}$ is the fine moduli space, but $\mathbb{C}$ lacks an important additional property and cannot be considered the true moduli space of elliptic curves:
A family of elliptic curves over a base space B is a fibration $\chi : X \to B$ with a section $O : B \to X$ such that for every $b \in B$ the fiber $\chi^{-1}(b)$ is an elliptic curve with origin $O(b)$. Given a family of elliptic curves $\chi : X \to B$, define a classifying map $j_B : B \to \mathbb{C}$ by $b \mapsto j(\chi^1(b))$. Because $\mathbb{C}$ is the coarse moduli space, $j_B(b) = j_B(b')$ if and only if the fibers $\chi^{-1}(b)$ and $\chi^{-1}(b')$ are isomorphic elliptic curves. Since every elliptic curve has an involution **[Sil09]**, there are nontrivial families of elliptic curves $\chi : X \to B$ such that $\chi^1(b) \simeq E_0$ for all $b \in B$, where E_0 is a fixed elliptic curve (such a family is called *isotrivial*) The classifying map $j_B : B \to \mathbb{C}$ is the constant map $b \mapsto j(E_0)$. This contradicts the existence of a universal family, because the classifying map $B \to \mathbb{C}$ associated to the trivial family $E_0 \times B \to B$ is also constant. Thus we defined the stack of elliptic curves $\mathcal{M}$. Its objects are families of elliptic curves, and a morphism from $X' \to B'$ to $X \to B$ is a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

The subcategory of $\mathcal{M}$ corresponding to families over a fixed base B is called the fiber over B. That the diagram to be commutative says that the fibers of $\mathcal{M}$ are *groupoids*, that is, categories where all morphisms are isomorphisms.

Let $\mathcal{C}$ be the category whose objects are families of elliptic curves with a (nowhere zero) section and morphisms defined as for $\mathcal{M}$. Forgetting the section defines a functor $\mathcal{C} \to \mathcal{M}$. For any family of elliptic curves $X \to B$, the pullback of $\mathcal{C}$ via the corresponding map $\underline{B} \to \mathcal{M}$ is $\underline{X}$. Thus, $\mathcal{M}$ is the moduli space of elliptic curves and $\mathcal{C} \to \mathcal{M}$ is the universal family (to see [**Dan00**]).

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