UNIVERSIDAD NACIONAL DE COLOMBIA

Contagion Modeling in Credit Risk

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A mi madre, Olga.

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Resumen

Entre las herramientas matemáticas y estadísticas para modelar problemas o fenómenos financieros, son pocas aquellas que permiten modelar escenarios de contagio y agrupamiento de sucesos, es decir, aquellas situaciones en las que encontramos que hay ocurrencia de sucesos de manera que, si las organizamos en el tiempo, se evidencian agrupamientos. En este trabajo en particular, abordaremos un método basado en procesos estocásticos para modelar situaciones de agrupamiento, el cual generaliza los procesos de Hawkes y los procesos doblemente estocásticos con intensidad de *shot noise*, conocido como el proceso de contagio dinámico [Dassios and Zhao, 2011]. Este proceso considera factores endógenos y exógenos que pueden potencialmente tener un impacto en el sistema que se está estudiando y los cuales son llamados los saltos auto-excitados y externamente excitados respectivamente. Usando este proceso se modelará la crisis bancaria colombiana de 1998 y se evaluará la pertinencia de esta herramienta en este tipo de situaciones de crisis económica y financiera. Finalmente, una medida distinta de riesgo financiero es introducida a partir de estos modelos.

Palabras clave: Proceso de contagio dinámico; Proceso de Hawkes; Proceso doblemente estocástico; Proceso puntual de Cluster; Crisis hipotecaria Colombia.

Abstract

Among the mathematical and statistical tools for modeling financial problems, few are those that allow modeling contagion scenarios and event clustering, that is to say, when events happen and we organize them chronologically we can see groups of events in certain periods of time. In this work, we will address in particular a method based on stochastic processes that models clustering situations and generalizes the Hawkes processes and the doubly stochastic processes with shot noise intensity, known as the dynamic contagion process [Dassios and Zhao, 2011]. This process takes into consideration endogenous and exogenous factors that potentially could have an impact in the underlying system and which are called self-excited and externally excited jumps respectively. By using this process, we will model the Colombian banking crisis of 1998 and we will evaluate the relevance of this tool in this kind of economic and financial crisis situations. Finally, a different financial risk measure is introduced on the basis of these models.

Keywords: Dynamic contagion process; Hawkes process; Doubly stochastic process; Cluster point process, Colombian mortgage crisis.

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1. Introduction

In the study of financial risk, models in continuous time of credit risk and credit derivatives are of great importance, the latter as an effective financial instrument of managing risk. Credit derivatives have had major development and growth in the financial market caused by its two main uses: as an adequate mechanism to protect oneself against credit risk and as a financial speculation instrument. The second one was the cause of the global financial crisis in 2008 from where they have gained significance due to uncertainty regarding suitable tools for pricing these kind of instruments.

In this regard, the reduced-form models are quite popular in practice because of the resulting formulas which are tractable and explain the credit-risky securities in terms of a default intensity process that quantifies the likeliness of default per unit time and depends on economic covariates, these formulas ease the estimation and there exists wide literature in the area, one of the most known is [McNeil et al., 2005]. Despite this facility, [Jarrow and Yu, 2001] stated that default intensity models that depend linearly on macroeconomic variables might not be appropriate to account for the contagion phenomenon observed in economic recessions such as after the collapse of Lehman Brothers in September 2008 or any other kind of jumps, bankruptcies, crises and catastrophes in finance, in which disturbance in one financial institution or industry can affect other institutions or sectors. Contagion phenomenon is evidenced in clustering defaults around an economic recession that could be explained by the linkages or even dependency relationships between institutions. Contagion problem has also been studied in other areas. The first application of a model of this kind is in biology or medicine in infectious diseases transmission, they have been modeled as systems of coupled partial differential equations and there exist several numerical methods to solve this problem from this point of view.

Defaultable securities and credit derivatives have mainly been modeled with different mathematical tools including, among others, point processes such as Poisson processes, Markov chains and Cox Processes (also called doubly stochastic Poisson processes defined in [Lando, 1998] as a generalization of a Poisson process); on the other hand, copula models have been widely studied and used for explaining default in credit risk. For example, in order to incorporate what [Jarrow and Yu, 2001] called *counterparty risk*, they worked with Lando's reduced-form model [Lando, 1998] that uses a Cox process, and included a jump process in the set of economy-wide state variables used to construct the default intensity process.

Also, [Davis, 2011] concentrated in reduced-form models based on Markov chains using the piecewise deterministic Markov process theory to model contagion between the obligors, but he mentions that 'contagion' is in some sense already built into the copula concept. This idea is brought up in [McNeil et al., 2005] where they stated that "default contagion in factor copula models can be attributed to the fact that information about default history alters the conditional distribution of an unobservable factor vector $\mathbf{V}$", in other words, copula model absorbs the 'internal' factors affecting default intensity of the underlying system. Conversely, [Duffie and Garleanu, 2001] supposed a default intensity that could be disturbed by a state process X_c that represents common economic factors in an industry, sector, or currency region which means they considered certain 'external' factors of the market.

However, in the literature before [Dassios and Zhao, 2011], there is no consideration of endogenous and exogenous aspects jointly concerning the studied system. For this reason, in order to analyze in a more complete manner the problem of contagion, [Dassios and Zhao, 2011] defined the dynamic contagion process, by generalizing the Hawkes process ([Hawkes, 1971]) and the Cox process with shot noise intensity ([Dassios and Jang, 2003]) where they model self-excited and externally excited jumps, respectively. Besides, it introduces a constant reversion level and it considers an exponential decay which both make the model to stabilize as time passes and is consistent with the reality of the contagion phenomenon observed in reality.

Within this framework, the problem to be studied will be dynamic contagion modeling and we will focus on the piecewise deterministic Markov processes with the principal guide being the paper of [Dassios and Zhao, 2011], *A Dynamic Contagion Process*, and the text book of [McNeil et al., 2005], *Quantitative Risk Management: Concepts, Techniques and Tools*. The aim is to be able to study Colombian banking crisis caused by mortgages in 1998 where the contagion among financial institutions was evident since a considerable amount of them went bankrupt. We would like to emphasize here that, in finance, having clustering of a certain financial event in a period of time is enough to assume contagion [López Reina et al., 2006].

2. Background

2.1. Stochastic processes

The main objects that we are going to use in this work are *stochastic processes*, in particular, Markov Processes, and because of this, we make in this section the following summary of definitions and results which are all taken from [Eberle, 2015]. Let us consider I as the time indexes set that can be continuous, for instance $I = \mathbb{R}$ or discrete, for example $I = \mathbb{Z}_+$. From here, we have the following definitions.

Definition 1. *([Eberle, 2015], page 6) Let I be the time indexes set and (Ω, A, P) be a probability space. If $(S, \mathcal{B})$ is a measurable space then a* ***stochastic process*** *with state space S is a collection $(X_t)_{t\in I}$ of random variables $X_t : \Omega \to S$.*

In particular, in this work we will always consider $S = \mathbb{R}$.

Definition 2. *([Eberle, 2015], page 6) Let I be the time indexes set and (Ω, A, P) be a probability space. An increasing collection $(\mathcal{F}_t)_{t\in I}$ of σ-algebras $\mathcal{F}_t \subseteq A$ is a* ***filtration*** *on (Ω, A, P).*

Definition 3. *([Eberle, 2015], page 6) Let I be the time indexes set and (Ω, A, P) be a probability space. A stochastic process $(X_t)_{t\in I}$ is* ***adapted*** *with respect to a filtration $(\mathcal{F}_t)_{t\in I}$ iff X_t is $\mathcal{F}_t$-measurable for any $t \in I$.*

Definition 4. *([Eberle, 2015], page 7) Let I be the time indexes set and (Ω, A, P) be a probability space. A stochastic process $X = (X_t)_{t\in I}$ on (Ω, A, P) with state space $(S, \mathcal{B})$ is called an $(\mathcal{F}_t)$* ***Markov process*** *iff (X_t) is adapted with respect to the filtration $(\mathcal{F}_t)_{t\in I}$, and*

$$P[X_t \in B|\mathcal{F}_s] = P[X_t \in B|X_s] \quad P\text{-a.s. for any } B \in \mathcal{B} \text{ and } s, t \in I \text{ with } s \leq t.$$

Definition 5. *([Eberle, 2015], page 6) Let I be the time indexes set and (Ω, A, P) be a probability space. The* ***filtration generated by X*** *or the history of X up to time t is*

$$(\mathcal{F}_t^X) = \sigma(X_s : s \in I, s \leq t), \quad t \in I,$$

[Eberle, 2015] stated the result: " Any $(\mathcal{F}_t)$-Markov process, $X = (X_t)$, is also a Markov process with respect to the filtration $(\mathcal{F}_t^X)$ generated by the process. Hence an (F_t^X) Markov process will be called simply a Markov process".

2.1.1. Generators

Definition 6. *([Eberle, 2015], page 12) Let I be the continuous time indexes set, (Ω, A, P) be a probability space and $(X_t)_{t\geq 0}$ a Markov process. Also, let $\mathcal{F}_b(\mathbb{R})$ be the collection of bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ and $t \geq 0$, then the* ***generator at time t****, $(\mathcal{A}_t)$, is defined as:*

$$\mathcal{A}_t : \mathcal{F}_b(\mathbb{R}) \to \mathcal{F}_b(\mathbb{R}),$$

$$(\mathcal{A}_t f)(x) = \lim_{h\downarrow 0} \frac{1}{h} E[f(X_{t+h}) - f(X_t)|X_t = x].$$

[Eberle, 2015] clarifies that this limit may not exist for all bounded functions f but one way to handle this partially is by the martingale problem defined as follows.

Definition 7. *([Eberle, 2015], page 13) Let L be a linear space of bounded measurable functions on $(\mathbb{R}, \mathcal{B})$. A stochastic process $((X_t)_{t\in\mathbb{R}_+}, P)$ that is adapted to a filtration $(\mathcal{F}_t)$ is said to be a solution of the* ***martingale problem*** *for $((\mathcal{A}_t)_{t\in\mathbb{R}_+}, L)$ iff the real valued process*

$$\mathcal{M}_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds$$

is a $(\mathcal{F}_t)$ martingale for all funtions $f \in L$.

This definition builds a relationship to the generator on a class of nice funtions. It is important to see that, for $h > 0$ sufficiently small, we have the following approximation

$$f(x) + (t + h)(\mathcal{A}_t f)(x) \approx E[f(X_t)|X_t = x].$$

2.2. Financial derivatives

We consider now a financial market consisting of only two assets: a stock with the price process S, and a risk free asset with price process B just as in [Björk, 2009]. Let us assume that the dynamic of S, the stock price, is given by

$$dS(t) = S(t)\alpha(t, S(t))dt + S(t)\sigma(t, S(t))dW(t),$$

where W is a Wiener process, α is the **local mean rate of return** and σ is the **volatility**. α and σ are given deterministic functions.

Definition 8. *([Björk, 2009], page 92) An asset is* ***risk free*** *if its price process B has the dynamics*

$$dB(t) = r(t)B(t)dt,$$

where r is any adapted process.

Therefore, we see that the property what makes an asset risk free is not having driving dW-term. That said, if we focus our attention on financial derivatives, we have the following definition.

Definition 9. *([Björk, 2009], page 94) Consider a financial market with vector price process S. A **financial derivative** (or contingent claim) with **date of maturity** (exercise date) T, also called a T-claim, is any stochastic variable $\chi \in \mathcal{F}_T^S$ (given in Definition 5). A contingent claim χ is called a **simple claim** if it is of the form $\chi = \Phi(S(T))$. The function Φ is called the **contract function**.*

This definition can be understood as follows: there is a contract (contingent claim) that establishes that, at the time of maturity T, its holder will obtain χ (which can be positive or negative). Also, when demanded that $\chi \in \mathcal{F}_T^S$, it means that, with the information generated by S on the interval $[0, T]$, we can determine how much money has to be paid out.

In this context, we construct models whose trading strategies can not profit without cost or risk, such models are known as ***arbitrage free*** and, thanks to the First Fundamental Theorem of Asset Pricing (Theorem 3.8 [Björk, 2009], page 33), we know that a model for security prices is arbitrage free if, and only if, it admits at least one equivalent martingale measure $\mathcal{Q}$ where we can price credit derivatives as mathematical expectations of the discounted pay-off. Furthermore, the Second Fundamental Theorem of Asset Pricing (Proposition 3.14 [Björk, 2009], page 36) shows that there is exactly one equivalent martingale measure $\mathcal{Q}$ if, and only if, the market is arbitrage free and complete which means that in a complete market the most important thing for pricing derivative securities is the $\mathcal{Q}$-dynamics of the traded underlying assets. Thus, under the measure Q we are in a risk neutral measure and we obtain the general pricing formula for financial derivatives in the following theorem.

Theorem 1. *([Björk, 2009], page 152)* ***(General Pricing Formula)*** *The arbitrage free price process for the T-claim X is given by*

$$\Pi(t; X) = B(t)E^Q\left[\frac{\chi}{B(T)}\Big|\mathcal{F}_t\right],$$

where Q is the (not necessarily unique) martingale measure for the a priori given asset of the risk-free market $B, S_1, \ldots, S_N$ with B as the numeraire.

2.3. Credit risk

The intuitive notion of *Risk* is heavily related to uncertainty and randomness. However, there was no formal definition until 1933 when the Russian mathematician A. N. Kolmogorov defined a probabilistic model described by $(\Omega, \mathcal{F}, P)$ where Ω represents the set of all realizations of an experiment, $\mathcal{F}$ is the set of all events and $P(A)$ is the probability that an event A occurs.

More particularly, in financial context, when performing theoretical pricing of financial instruments it is assumed that every part of the contract will fully and punctually meet their obligations. Unfortunately, this is not always the case specially in the over-the-counter markets and then we can talk about the risk of loans and bonds borrower's default which can derive in not receiving promised repayments on pending investments, this is called **credit risk**. From this context, the need to be able to estimate the probability of default arises.

Independent agencies such as Standard & Poor's, Moody's and Fitch Ratings, among others, give credit ratings to medium or long term corporate bonds through an impartial opinion. Usually the highest credit quality (ex: AAA rating) is assigned to debt securities guaranteed or issued by the government [Venegas-Martínez, 2008]. For example, the following table (taken from [Venegas-Martínez, 2008]) shows the Standard & Poor's credit ratings to corporate bonds in some country

Rating	Description
AAA	The highest credit quality
AA	Very high credit quality
A	High credit quality
BBB	Good credit quality
BB	Speculative quality
B	Highly speculative quality
CCC	High risk of default

Table 2-1.: Credit ratings' description

Since our main goal is to estimate the probability of default, we are going to consider as a demonstrative example (developed in [Venegas-Martínez, 2008], page 758) when we have the possibility of partially recover the face value N of a zero coupon corporate bond (which has credit risk) with maturity T and find its risk neutral probability of default in t, $Q(t,T)$. Let $B_c(t,T)$ be the price of the bond in t, $B_g(t,T)$ be the price in t of a zero coupon government bond (without credit risk) with maturity T and $R_g(t,T)$ be the yield of the government bond. Now, because of the definition of a zero coupon bond we have only two options for its price as shown in the following diagram:

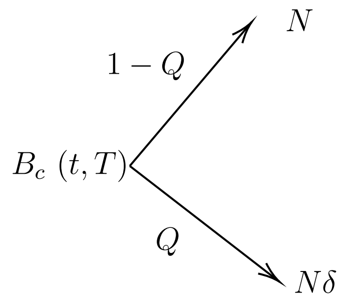

Figure 2-1.: Probability of default tree

Consequently, under the assumption that the expected value of the corporate bond nominal is equal to the future value of its price, calculated with the credit risk free rate of return, the risk neutral probability of default $\mathcal{Q}(t,T)$ satisfies that

$$B_c(t,T)e^{R_g(t,T)(T-t)} = (1 - Q(t,T))N + Q(t,T)N\delta,$$

where $0 \leq \delta < 1$ is the recovery rate. Then,

$$Q(t,T) = \left(\frac{1}{1-\delta}\right)\frac{B_g(t,T) - B_c(t,T)}{B_g(t,T)}.$$

Likewise, we can write the probability of default with a Bernoulli variable $X_{t,T}$ which is defined as

$$P(X_{t,T} = 1) = 1 - Q(t,T) \quad \text{and} \quad P(X_{t,T} = \delta) = Q(t,T),$$

whereby the probability of default is determined by

$$B_c(t,T)e^{R_g(t,T)(T-t)} = E[NX_{t,T}] = (1 - Q(t,T))N + Q(t,T)N\delta.$$

Besides, [Venegas-Martínez, 2008] makes the following development of the instantaneous rate of default risk $v(t,T)$ associated with a continuous random variable $\tau > t$ with density function $f_\tau(T)$ and distribution function $F_\tau(T)$ which is defined as

$$v(t,T) = \frac{f_\tau(T)}{1 - F_\tau(T)}.$$

One way of interpreting the previous is as following. Note first that

$$\begin{aligned} v(t,T)dT &= \frac{f_\tau(T)}{1 - F_\tau(T)} \\ &= \frac{dF_\tau(T)}{1 - F_\tau(T)} \\ &\approx \frac{P(t \leq \tau \leq T + dT)}{P(\tau > T)} \\ &= P(t \leq \tau \leq T + dT | \tau > t). \end{aligned}$$

Then, $v(t,T)dT$ represents the probability of default arriving in the interval $[T, T+dT]$ given that until T it has not defaulted. Besides, also notice that

$$\begin{aligned}\Lambda(t,T) &:= \int_t^T v(t,s)ds \\ &= \int_t^T \frac{dF_\tau(T)}{1-F_\tau(T)} \\ &= -ln(1-F_\tau(T)),\end{aligned}$$

which implies that

$$F_\tau(T) = 1 - e^{-\Lambda(t,T)}.$$

In the particular case of the Poisson process, with constant intensity parameter, the non-probability of default satisfies

$$\frac{d}{dt}P(\tau > T) = \lambda P(\tau > T),$$

which means that the non-probability of default percentage change, per unit time, is constant and equal to λ.

2.4. Point process

Point processes are the main tool that we are going to use to model contagion in economic recessions, then, in this section we will have the basic definitions needed.

To understand the notion of a point process, [Mikosch, 2009] (page 216) first considered "a sequence $(X_n)_{n\geq 1}$ of random vectors in the state space E, which is a Borel subset of a finite-dimensional Euclidean space. If $A \subset E$,

$$N(A) = \#\{i \geq 1 : X_i \in A\},$$

i.e. $N(A)$ is the number of X_i's falling in A. From this notion, we see that for a set A, $N(A) = N(A,\omega)$ is random and when ω is fixed, $N(\cdot,\omega)$ defines an ordinary counting measure with atoms $X_i(\omega)$ on the σ-field $\mathcal{E} = \mathcal{B}(E)$ of the Borel sets generated by the open sets of E".

For convenience, [Mikosch, 2009] (page 216) also wrote $N(\cdot,\omega)$ with the Dirac measure

$$\epsilon_x(A) = I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases},$$

considering that, for a given sequence $(x_i)_{i\geq 1}$ in E

$$m(A) = \sum_{i=1}^{\infty} \epsilon_{x_i}(A) = \#\{i \geq 1 : x_i \in A\} = N(A, \omega), \quad for A \in \mathcal{E},$$

where $(x_i) = (X_i(\omega))$. m defines a **counting measure** on $\mathcal{E}$ which is called a **point measure** if $m(K) < \infty$ for all compact set $K \subset E$.

Definition 10. *([Mikosch, 2009], page 216) A **point process** N with state space E is a measurable map:*

$$N : (\Omega, \mathcal{F}) \to (M_p(E), \mathcal{M}_p(E)),$$

where

- *$\mathcal{F}$ is a σ-field of Ω,*
- *$M_p(E)$ is the set of all point measures on E,*
- *$\mathcal{M}_p(E)$ is the smallest σ-field containing all sets of the form $\{m \in M_p(E) : m(A) \in B\}$ for any $A \in \mathcal{E}$ and any $B \subset [0, \infty]$.*

Note ([Mikosch, 2009], page 216): $\mathcal{M}_p(E)$ is the smallest σ-field making the map $m \to m(A)$ measurable $\forall A \in \mathcal{E}$.

2.4.1. Counting Process

In particular, when $E = [0, \infty)$ N_t, defined as before, is a counting process in the sense of Definition 11.

Definition 11. *([Swishchuk and Huffman, 2020], page 3) A **counting process** is a stochastic process $N(t)$ with $t \geq 0$, where $N(t)$ takes non-negative integer values and satisfies $N(0) = 0$. It is almost surely finite and its paths are right-continuous step functions with increments of size $+1$.*

As mentioned in Section 2.1, we have the filtration $\mathcal{F}_t^N$ which is the history of the process N and in the case when it is a counting process, $\mathcal{F}_t^N$ is the history of the arrival of events up to time t. [Swishchuk and Huffman, 2020] explained that "a counting process can be interpreted as a cumulative count of the number of arrivals into a system up to time t and it can be also characterized by the sequence of random arrival times $(T_1, T_2, \dots)$ at which the counting process N_t has jumped".

With this characterization, [Swishchuk and Huffman, 2020] defined "a point process as a sequence of random variables $(T_1, T_2, \dots)$ taking values in $[0, \infty)$ with $P(0 \geq T_1 \geq T_2 \geq \dots) = 1$ and the number of points in a bounded region almost sure finite". This definition is closest to the intuitive meaning of a point process that we are going to use in this work.

2.4.2. Conditional Intensity Function

The following definition introduces the concept of conditional intensity function. Intuitively, it determines how the point process present outcome is affected by its own history, that is to say, the likelihood of an event arriving at time t given the previous arrivals.

Definition 12. *([Swishchuk and Huffman, 2020],page 3) Consider a counting process N_t with associated histories $\mathcal{F}_t^N$, $t \geq 0$. If a non-negative function λ_t exists such that*

$$\lambda_t = \lim_{h \to 0} \frac{E[N_{t+h} - N_t | \mathcal{F}_t^N]}{h},$$

*then it is called the **conditional intensity function** of N_t.*

2.4.3. Hawkes Process

The main process that we are going to use is a generalization of this particular kind of point processes: the Hawkes Processes. This family of models were introduced in [Hawkes, 1971], they were originally called "self-exciting and mutually exciting point processes" and their basic motivation was to represent the fact that an event has the potential to generate new events in the future. This effect has been observed in empirical data were clustering of events is evident, in which case, Poisson processes are not accurate because they do not have history dependence.

As highlighted in [Hawkes, 2018], it was not near the new millennium that these processes had some impact and began to find their way in several areas with various applications. Among them, epidemiology, neuroscience, ecology, medicine, crime prediction, terrorism, e-marketing, etc.

Specifically in finance, [Bacry et al., 2013] applies mutually excited processes to high frequency data of asset price changes and [Azizpour et al., 2018] addressed the problem of clustering of corporate default events in the US economy by finding in financial, legal or business relationships between firms a statistically and economically significant contagion effect, that is to say, these links may spread the risk of default. For example, the buyer of protection could be exposed when the protection seller in a credit swap has defaulted which increases the default risk undertaken by the protection buyer's other counter-parties. From here, we can see the importance of studying this family of models.

Definition 13. *([Hawkes, 1971], [Swishchuk and Huffman, 2020]]) The **one-dimensional Hawkes process** is a point process N_t which is characterized by its intensity λ_t with respect to its natural filtration:*

$$\lambda_t = \lambda_0 + \int_0^t \mu(t-s)dN(s),$$

where $\lambda_0 > 0$*, and the response function* $\mu(t)$ *is a positive function that satisfies* $\int_0^\infty \mu(s)ds < 1$.

The function $\mu(t)$ is sometimes called the excitation function and the constant λ_0 is known the background intensity. The integral can be understood as: when there are changes in N, that affects the intensity of the process and then the occurrence probability of events.

Additionally, [Swishchuk and Huffman, 2020] showed that from Definitions 12 and 13, it follows that

$$P(N_{t+h} - N_t = m|\mathcal{F}_t^N) = \begin{cases} \lambda_t h + o(h) & m = 1 \\ o(h) & m > 1 \\ 1 - \lambda_t h + o(h) & m = 0 \end{cases}.$$

Let us notice that this may look similar to the usual non-homogeneus Poisson process, but in this case the intensity varies stochastically.

Finally, [Swishchuk and Huffman, 2020] also gave a way of interpreting the definition of a Hawkes process. Basically, events occur in accordance with the background or initial intensity λ_0. When each new event happens, intensity increases by $\mu(0)$ and then it tends to decay back to the background intensity value according to the evolution of the function $\mu(t)$.

When $\mu(0) > 0$, there is a shock in the intensity every time an event arrives which is called the **self-exciting** feature. This means that if an event occurs, then the probability of further events occurring increases.

It is usual to choose $\mu(t) = \alpha e^{-\beta t}$ with $\alpha, \beta > 0$ parameters, in which case, N_t is a Hawkes process with exponentially decaying intensity. For this process, the conditional intensity function becomes

$$\lambda_t = \lambda + \int_0^t \alpha e^{-\beta(t-s)} dN(s).$$

In this particular case, the process (N_t, λ_t) is a continuous-time Markov process, when for a general excitation function $\mu(t)$ might not be the case.

2.4.4. The general cluster model

The previous intuitive idea of counting is now generalized to cluster point processes. So far, we have addressed the issue of counting and likelihood of occurrence of an event, now we are going to consider that an event arrival is going to trigger a cluster of activities.

Definition 14. *([Mikosch, 2009], page 363) The* ***general cluster model*** *satisfies the following conditions:*

- *Bad events arrive at random instants of time* $0 < T_1 < T_2 < \cdots$,
- *The i-th bad event causes random stream or cluster of bankruptcies or failures in the system. The j-th bankrupt for the i-th bad event is a positive random variable* X_{ij} *executed at time*

$$T_{ij} = T_i + \sum_{k=1}^{j} Y_{ik}, \quad 1 \leq j \leq K_i,$$

where $(Y_{ik})_{k\geq 1}$ *is a sequence of positive random variables and* K_i *is a positive integer-valued random variable.*

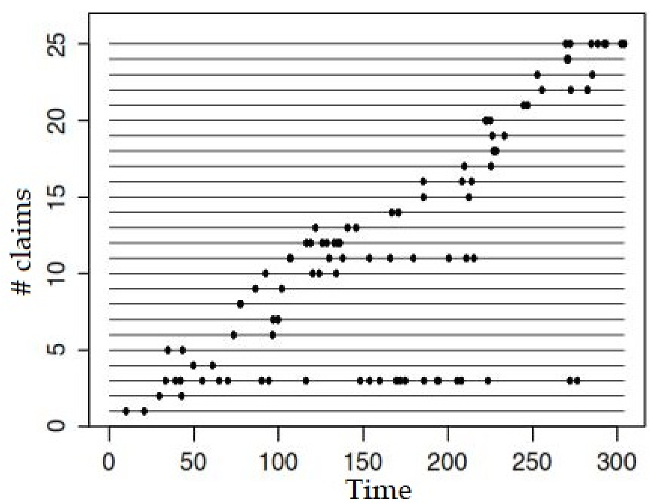

Figure 2-2.: Graphical representation of cluster point process

In order to understand the previous definition we take the actuarial interpretation of it [1]. In Figure **2-2** each claim is assigned to a line. When a claim arrives at time T_i a dot is drawn at the left of the line and if the claim generates payments more dots are drawn in times T_{ij}. This means that a single claim triggers several payments and each of them might be payed latter. Now, in the contagion context each point will represent a defaulted company and the first company to be bankrupt will be the dot at the left whereas the other dots will be the infected ones. Finally, we obtain a **cluster point process** when in a cluster process we count how many dots there are before a time t in all of the lines generated by that process.

[1] Taken from [Mikosch, 2009].

3. Dynamic Contagion Problem

So as to be able to define the Dynamic Contagion Problem, we first need an understanding on Cox processes with shot noise intensity. [Dassios and Jang, 2003] used the Cox process (also known as doubly stochastic Poisson process) to model the claim arrival process for catastrophic events seeing that it is more flexible than other models since it lets the intensity depend on time and be stochastic. The shot noise process is used for the intensity function within the Cox process.

Definition 15. *([Dassios and Jang, 2003], page 3) Let (Ω, F, P) be a probability space with information structure given by $F = \{\mathcal{F}_t, t \in [0, T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non-negative process adapted to F such that*

$$\int_0^t \lambda_s ds < \infty \qquad \textit{almost surely (no explosions).}$$

If for all $0 \leq t_1 \leq t_2$, $u \in \mathcal{R}$ and

$$E\left\{e^{iu(N_{t_2}-N_{t_1})}|\mathcal{F}_{t_2}^{\lambda}\right\} = exp\left\{(e^{iu}-1)\int_{t_1}^{t_2} \lambda_s ds\right\}.$$

*Then N_t is called a $\mathcal{F}_t$**-doubly stochastic Poisson process** with intensity λ_t where $\mathcal{F}_t = \sigma\{\lambda_s; s \leq t\}$.*

In particular, we can use the shot noise process which can be used to measure the impact of catastrophic events since it considers the frequency, magnitud and time period needed to determine the effect of those events ([Dassios and Jang, 2003]).

Definition 16. *([Dassios and Jang, 2003], page 4) The **shot noise process** is defined as*

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{all\ i \\ s_i \neq t}} y_i e^{-\delta(t-s_i)},$$

where

- *$\lambda_0 > 0$ is the initial value of λ,*
- *y_i is the jump size of catastrophe i where $E(y_i) < \infty$ (i.e. magnitude of contribution of catastrophe i to intensity),*

- *s_i is the time at which catastrophe i occurs, where $s_i < t < \infty$,*
- *δ is the exponential decay,*
- *ρ is the rate of catastrophe jump arrival.*

Finally, using the previous definitions, [Dassios and Zhao, 2011] define the Dynamic Contagion Process as follows.

Definition 17. *([Dassios and Zhao, 2011], page 815) The **dynamic contagion process** is a cluster point process $\mathbb{D}$ on $\mathbb{R}_+$: The number of points in the time interval $(0, t]$ is defined by $N_t = N_{\mathbb{D}(0,t]}$. The cluster centers of $\mathbb{D}$ are the particular points called **immigrants**, the other points are called **offspring**. They have the following structure:*

(a) The immigrants are distributed according to a Cox process A with points $\{D_m\}_{m=1,2,\ldots} \in (0, \infty)$ and shot noise stochastic intensity process

$$a + (\lambda_0 - a)e^{-\delta t} + \sum_{i \geq 1} Y_i e^{-\delta(t - T_i^{(1)})} I \left\{ T_i^{(1)} \leq t \right\},$$

where

- *$a \geq 0$ is the constant reversion level,*
- *$\lambda_0 > 0$ is a constant as the initial value of the stochastic intensity process (defined later by (3-1)),*
- *$\delta > 0$ is the constant rate of exponential decay,*
- *$\{Y_i\}_{i=1,2,\ldots}$ is a sequence of independent identical distributed positive (externally excited) jumps with distribution function $H(y), y > 0$, at the corresponding random times $\left\{ T_i^{(1)} \right\}_{i=1,2,\ldots}$ following a homogeneous Poisson process M_t with constant intensity $\rho > 0$,*
- *I is the indicator function.*

(b) Each immigrant D_m generates a cluster $C_m = C_{D_m}$, which is the random set formed by the points of generations $0, 1, 2, \ldots$ with the following branching structure:

the immigrant D_m is said to be of generation 0. Given generations $0, 1, \ldots, j$ in C_m, each point $T^{(2)} \in C_m$ of generation j generates a Cox process on $(T^{(2)}, \infty)$ of offspring of generation $j+1$ with the stochastic intensity $Ze^{-\delta(\cdot - T^{(2)})}$ where Z is a positive (self-excited) jump at time $T^{(2)}$ with distribution function $G(z), z > 0$, independent of the points of generation $0, 1, \ldots, j$.

(c) Given the immigrants, the centered clusters

$$C_m - D_m = \left\{T^{(2)} - D_m : T^{(2)} \in C_m\right\}, \quad D_m \in A,$$

are independent identical distributed, and independent of A.

(d) $\mathbb{D}$ consists of the union of all clusters, i.e.

$$\mathbb{D} = \bigcup_{m=1}^{\infty} C_{D_m}.$$

Therefore, the dynamic contagion process can also be defined as a point process N_t on $\mathbb{R}_+$, with the non-negative $\mathcal{F}_t$-stochastic intensity process λ_t following the piecewise deterministic dynamics with positive jumps, i.e.

$$\lambda_t = a + (\lambda_0 - a)e^{-\delta t} + \sum_{i\geq 1} Y_i e^{-\delta(t-T_i^{(1)})} I\left\{T_i^{(1)} \leq t\right\} + \sum_{k\geq 1} Z_k e^{-\delta(t-T_k^{(2)})} I\left\{T_k^{(2)} \leq t\right\}, \tag{3-1}$$

where

- $\{\mathcal{F}_t\}_{t\geq 0}$ *is a history of the process N_t, with respect to which $\{\lambda_t\}_{t\geq 0}$ is adapted,*
- $\{Z_k\}_{k=1,2,\dots}$ *is a sequence of independent identical distributed positive (self-excited) jumps with distribution function $G(z), z > 0$, at the corresponding random times $\left\{T_k^{(2)}\right\}_{k=1,2,\dots}$,*
- *the sequences $\{Y_i\}_{i=1,2,\dots}$, $\left\{T_i^{(1)}\right\}_{i=1,2,\dots}$ y $\{Z_k\}_{k=1,2,\dots}$ are assumed to be independent of each other.*

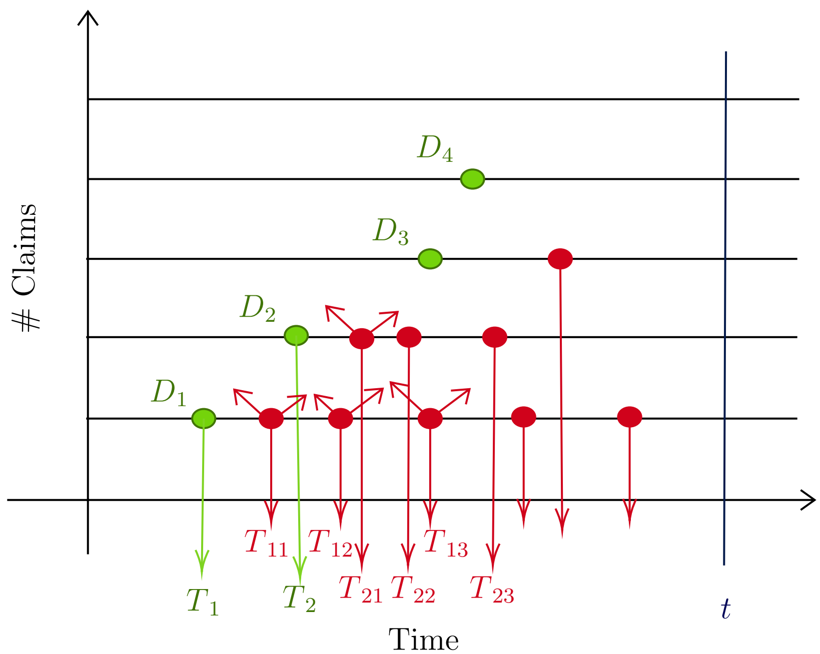

Figure 3-1.: Visual representation of the dynamic contagion process

Let us now review equation (3-1) of the previous definition. First of all, we see that when $t = 0$ we obtain the initial value λ_0 as in the definition of shot noise intensity but [Dassios and Zhao, 2011] included a reversion level in which the model will stabilize to a when $t \to \infty$. Besides, it considers intensity decay of the contagion phenomenon which is consistent with the reality where if there is an economic recession, its force of contagion tends to decrease; in particular, it is assumed an exponential decay with constant rate δ.

It is of crucial importance to emphasize that the arriving of immigrants, with rate ρ, is what represents the external factors affecting the market such as rate of exchange fluctuation or rise in oil's prices and their effects are given by the jump size Y_i with distribution function H. Similarly, the internal factors are represented by the offspring and their impact is given by the random variable Z_k with distribution function G and rate λ_t. In Figure **3-1** we see the immigrants in green arriving in times $T_m = T_m^{(1)}$. Each of them infects others (red) and then we have the first generation with arrival times $T_{m\ell}$. As we have a branching structure of contagion, each offspring of generation one can infect others, we represent this phenomenon with red arrows emerging from the red points. Now, if we have a random number J of contagion generations, the times of arrival are $T_{(i_1,\ldots,i_J)}$ where the subindexes i_j might be zeros from some index hereafter. To visualize this, suppose we already had the following structure of contagion.

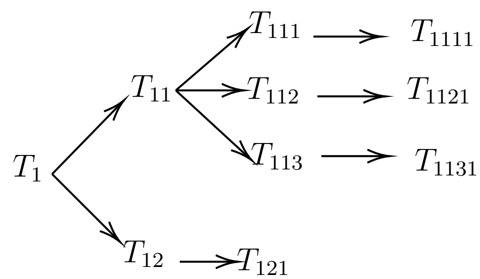

Figure 3-2.: Example of branching structure

Then, for example T_{113} is short for $T_{(1,1,3,0)}$ or T_{12} is short for $T_{(1,2,0,0)}$. We want to clarify that this notation is used just to explain the branching structure of the dynamic contagion problem, since in the definition every time of arrival of contagion is ordered, so the set $\{T_{i_1,\dots,i_J}\} = \{T_k^{(2)}\}_{k\geq 1}$ where the superindex means a time of default of generation $j \geq 1$.

Therefore, [Dassios and Zhao, 2011] stated that "from Definition 17 and because of the exponential decay, we can see that λ_t is a Markov process"from where, [Dassios and Zhao, 2011] derived the infinitesimal generator of the dynamic contagion process (λ_t, N_t, t).

$$\mathcal{A}f(\lambda, n, t) = \frac{\partial f}{\partial t} - \delta(\lambda - a)\frac{\partial a}{\partial \lambda} + \rho\left(\int_0^\infty f(\lambda + y, n, t)dH(y) - f(\lambda, n, t)\right) \tag{3-2}$$

$$+ \lambda\left(\int_0^\infty f(\lambda + z, n + 1, t)dG(z) - f(\lambda, n, t)\right). \tag{3-3}$$

From here, [Dassios and Zhao, 2011] derive the joint Laplace transform - probability generating function (λ_T, N_T) for a fixed time T, the conditional Laplace transform of λ_T given λ_0, the probability generating function of N_T given λ_0 and $N_0 = 0$ which they used to calculate the moments of λ_t y N_t.

Now recall that, for a continuous random variable X, the Laplace transform is defined as

$$\mathcal{L}_v(X) := E[e^{-vX}],$$

and if X is a discrete random variable, then we have, for $0 \geq \theta \geq 1$, the moment generating function as

$$M_X(\theta) = E[\theta^X].$$

Taking into account that (λ_T, N_T) is a random vector since T is fixed, [Dassios and Zhao, 2011] derived the following result for the counting process at time T.

Theorem 2. *([Dassios and Zhao, 2011], page 824) The conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by*

$$E\left[\theta^{N_T}|\lambda_0\right] = exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{a\delta u + \rho[1-\hat{h}(u)]}{1-\delta u - \theta\hat{g}(u)}du\right) \times exp\left(-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0\right),$$

where

$$\mathcal{G}_{0,\theta}(L) := \int_0^L \frac{du}{1-\delta u - \theta\hat{g}(u)}, \quad 0 \leq \theta < 1. \tag{3-4}$$

The condition $\delta > \mu_{1_G}$ is essential since if the rate of decay is greater than the contagion self-excited effects, then the phenomenon tends to stop, the contagion process will indeed tend to the reversion level and clustering effects are going to pass (they are not going to maintain in time since the limit when $t \to \infty$ of λ_t is a), which coincide with reality. It is also fundamental to obtain properties of the asymptotic distribution.

3.1. Moments of λ_t and N_t

Theorem 3. *([Dassios and Zhao, 2011], page 829) The conditional expectation of the process λ_t given λ_0 at time $t = 0$, is given by*

$$E[\lambda_t|\lambda_0] = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} + \left(\lambda_0 - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right)e^{-(\delta-\mu_{1_G})t}, \quad \textit{for } \delta \neq \mu_{1_G},$$

$$E[\lambda_t|\lambda_0] = \lambda_0 + (\mu_{1_H}\rho + a\delta)t, \quad \textit{for } \delta = \mu_{1_G},$$

where

$$\mu_{1_H} := \int_0^\infty y dH(y).$$

Again, for the asymptotic distributions for λ_t and N_t we have the following results.

Corollary 1. *([Dassios and Zhao, 2011], page 831) Assume $\delta > \mu_{1_G}$, then the first and second moments and the variance of the stationary distribution of the process λ_t are given by*

$$E[\lambda_t] = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}},$$

$$E[\lambda_t^2] = \frac{(2(\mu_{1_H}\rho + a\delta) + \mu_{2_G})(\mu_{1_H}\rho + a\delta)}{2(\delta - \mu_{1_G})^2} + \frac{\mu_{2_H}\rho}{2(\delta - \mu_{1_G})},$$

$$Var[\lambda_t] = \frac{1}{2(\delta - \mu_{1_G})}\left(\mu_{2_G}\rho + \frac{\mu_{2_G}(\mu_{1_H}\rho + a\delta)}{\delta - \mu_{1_G}}\right).$$

Theorem 4. *([Dassios and Zhao, 2011], page 831) For the stationary distribution of the process, given the condition $\delta > \mu_{1_G}$ and $N_0 = 0$, the expectation of the point process N_t is given by*

$$E[N_t] = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} t.$$

Lemma 1. *([Dassios and Zhao, 2011], page 831) For the stationary distribution of the process λ_t, given the condition $\delta > \mu_{1_G}$ nad $N_0 = 0$, we have*

$$E[\lambda_t N_t] = \bar{k}\left(1 - e^{-(\delta-\mu_{1_G})t}\right) + \left(\frac{\mu_{1_G}\rho + a\delta}{\delta - \mu_{1_G}}\right)^2 t,$$

where

$$\bar{k} := \frac{2\mu_{1_G}(\mu_{1_H}\rho + a\delta) + \mu_{2_H}\rho}{2(\delta - \mu_{1_G})^2} + \frac{\mu_{2_G}(\mu_{1_H}\rho + a\delta)}{2(\delta - \mu_{1_G})^3}.$$

Theorem 5. *([Dassios and Zhao, 2011], page 832) For the stationary distribution of the process λ_t, given the condition $\delta > \mu_{1_G}$ and $N_0 = 0$, the second moment and the variance of the point process N_t are given by*

$$E[N_t^2] = \frac{2}{\delta - \mu_{1_G}}\left(e^{-(\delta-\mu_{1_G})t} - 1\right) + 2\bar{k}t + \left(\frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\right)^2 t^2,$$

$$Var[N_t] = \frac{2}{\delta - \mu_{1_G}}\left(e^{-(\delta-\mu_{1_G})t} - 1\right) + 2\bar{k}t,$$

where constant $\bar{k}$ is the same as in Lemma 1.

3.2. Particular Case: Jumps with Exponential Distributions

Following the results from [Dassios and Zhao, 2011], in the case where the density functions of externally excited and self-excited jumps are exponential, i.e.

$$h(y) = \alpha e^{-\alpha y}; \quad g(z) = \beta e^{-\beta z}, \quad \text{where } y, z; \alpha, \beta > 0, \tag{3-5}$$

the Laplace transforms have the explicit forms

$$\hat{h}(u) = \frac{\alpha}{\alpha + u}; \quad \hat{g}(u) = \frac{\beta}{\beta + u}.$$

Theorem 6. *([Dassios and Zhao, 2011], page 836) If both the externally excited and self-excited jumps follow exponential distributions, i.e. the density functions are specified as (3-5), then the conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta\beta > 1$, is given by*

$$E[\theta^{N_T}|\lambda_0] = e^{-\left(\mathcal{C}_2(\mathcal{G}_{0,\theta}^{-1}(T)) - \mathcal{C}_2(0)\right)} e^{-\mathcal{G}_{0\theta}^{-1}(T)\lambda_0}, \qquad \alpha \neq -v_-^*,$$

where

$$\begin{aligned}\mathcal{C}_2(u) :=& - au + \frac{\alpha(\beta-\alpha)\rho}{\delta(\alpha+v_-^*)(\alpha+v^*)} ln(u+\alpha) \\ &+ \frac{1}{\delta(v^*-v_-^*)}\left\{\left[a(v_-^* + (1-\theta)\beta) + \rho v_-^* \frac{\beta+v_-^*}{\alpha+v_-^*}\right] ln(u-v_-^*)\right. \\ &\left. - \left[a(v^* + (1-\theta)\beta)\rho v^* \frac{\beta+v^*}{\alpha+v^*}\right] ln(v^*-u)\right\},\end{aligned}$$

and

$$\mathcal{G}_{0,\theta}(L) = K(L) - K(0), \quad 0 \leq L < v^*,$$

where

$$K(u) := -\frac{1}{\delta(v^*-v_-^*)}\left[(v^*+\beta)ln(v^*-u) - (v_-^*+\beta)ln(u-v_-^*)\right], \quad 0 \leq u < v^*,$$

$$v^* = \frac{\sqrt{\Delta} - \delta\beta - 1}{2\delta} > 0;$$

$$-\beta \leq v_-^* = -\frac{\sqrt{\Delta} + (\delta\beta - 1)}{2\delta} < 0;$$

$$\Delta = (\delta\beta+1)^2 - 4\theta\delta\beta > 0, \quad 0 \leq \theta < 1.$$

3.3. Simulation Procedure

Moreover, in order to simulate one sample path of the general dynamic contagion process (N_t, λ_t), with m jump times $\{T_1^*, T_2^*, \ldots, T_m^*\}$ in the process λ_t, [Dassios and Zhao, 2011] (page 844) proposed the following algorithm:

Set the initial conditions $T_0^* = 0, \lambda_{T_0^{*+}} = \lambda_0 > a$ and $i \in \{0, 1, 2, \ldots, m-1\}$.

1. Simulate the $(i+1)^{th}$ externally excited jump waiting time E_{i+1}^* by

$$E_{i+1}^* = -\frac{1}{\rho} lnU, \quad U \sim U[0,1].$$

2. Simulate the $(i+1)^{th}$ self-excited jump waiting time S^*_{i+1} by

$$S^*_{i+1} = \begin{cases} S^{*(1)}_{i+1} \wedge S^{*(2)}_{i+1} & d_{i+1} > 0 \\ S^{*(2)}_{i+1} & d_{i+1} < 0 \end{cases},$$

where

$$d_{i+1} = 1 + \frac{\delta ln(U_1)}{\lambda_{T_i^{+*}} - a}, \quad U_1 \sim U[0,1]$$

and

$$S^{*(1)}_{i+1} = -\frac{1}{\delta} ln(d_{i+1}); \quad S^{*(2)}_{i+1} = -\frac{1}{a} ln(U_2), \quad U_2 \sim U[0,1].$$

3. Simulate the $(i+1)^{th}$ jump time T^*_{i+1} in the process λ_t by

$$T^*_{i+1} = T^*_i + S^*_{i+1} \wedge E^*_{i+1}.$$

4. The change at the jump time T^*_{i+1} in the process λ_t is given by

$$\lambda_{T^{*+}_{i+1}} = \begin{cases} \lambda_{T^{*-}_{i+1}} + Z_{i+1}, & Z_{i+1} \sim G(z) \quad S^*_{i+1} \wedge E^*_{i+1} = S^*_{i+1} \\ \lambda_{T^{*-}_{i+1}} + Y_{i+1}, & Y_{i+1} \sim H(y) \quad S^*_{i+1} \wedge E^*_{i+1} = E^*_{i+1} \end{cases},$$

where

$$T^{*-}_{i+1} = \left(T^{*+}_i - a\right) e^{-\delta(T^*_{i+1} - T^*_i)} + a.$$

5. The change at the jump time T^*_{i+1} in the point process N_t is given by

$$N_{T^{*+}_{i+1}} = \begin{cases} N_{T^{*-}_{i+1}} + 1 & S^*_{i+1} \wedge E^*_{i+1} = S^*_{i+1} \\ N_{T^{*-}_{i+1}} & S^*_{i+1} \wedge E^*_{i+1} = E^*_{i+1} \end{cases}.$$

It is important to mention that the previous simulation procedure does not assume any particular distribution but it lets the externally and self-excited jumps distribution be $H(y)$ and $G(z)$ respectively.

Also, it is possible to estimate the system's survival probability up to a maturity time T by assuming that when a bad event arrives, the probability that the event causes an economic collapse is d, $0 < d \leq 1$, and so $P_s(T) = E[(1-d)^{N_T} | \lambda_0]$, where d would be interpreted as the quantification of the system's resilience in catastrophic situations. Analytical calculations of this expected value is highly demanding, for this reason it is preferable to use the simulation procedure to estimate this survival probability.

4. 1998 Banking crisis in Colombia [1]

In the early 1990s Colombia started a financial liberalization process with measures such as the elimination of foreign investment limit which was in 49 % (Law 9 of 1991), an extension in the operations that financial intermediaries could perform (Law 35 of 1993) and relaxation of legal reserve requirement. Besides, the credit system had several types of intermediaries: Savings and Housing Corporations (SHC), Commercial Financing Companies(CFC), Financial Corporations (FC), Cooperative Banks and traditional banks; of which some had not enough assets nor administrative capacity to face an external shock, neither they had an appropriate regulation that considered the high market and liquidity risks associated with their debtors nor enough oversight from the Superintendencia Bancaria[2]. Due to all of these, the combination of the forces of internal financial liberalization along with strong fluctuations of external financing, was the most accentuated financial cycle experimented by the Colombian economy [Ocampo, 2015].

What is more, by this time, an international capital flow boom started towards emergent economies which, together with domestic demand, led to an internal credit boom and the rise in public spending. In Figure 4-1 we can see how in the early 1990s there is a rapid credit growth in banks and SHCs (getting to grow up to 60 %) followed by a sudden decrease in 1998. The effect of this credit boom was more noticeable in the growth of mortgages than in other lines of credit and because of this, SHCs where highly affected by the crisis since their main goal was to promote private savings and lead them to construction industry towards mortgage loans (it is evident in Figure 4-2 the rapid growth this line of credit had during this period and in Figure 4-3 how significant it was in their portfolios). Furthermore, the participation of mortgages in banks portfolios increased importantly and got to be over 20 % while at the beginning of the 1990s was significantly under 5 %.

According to public financial statements of 1997 (tables 4-1 and 4-2), the banks' exposure, just at the beginning of the crisis, was over 50 % except for Bank of America Colombia, and over 60 % for SHCs which means that, in general, credits represented more than half of the banks' assets and around three-quarters of the SHCs' assets. On the other hand, the Cen-

[1]The graphs and tables presented in this chapter were personally created with information of financial statements extracted from the Superintendencia Financiera de Colombia web page: `https://www.superfinanciera.gov.co`, except when specified otherwise.

[2]According to [Arango, 2006]

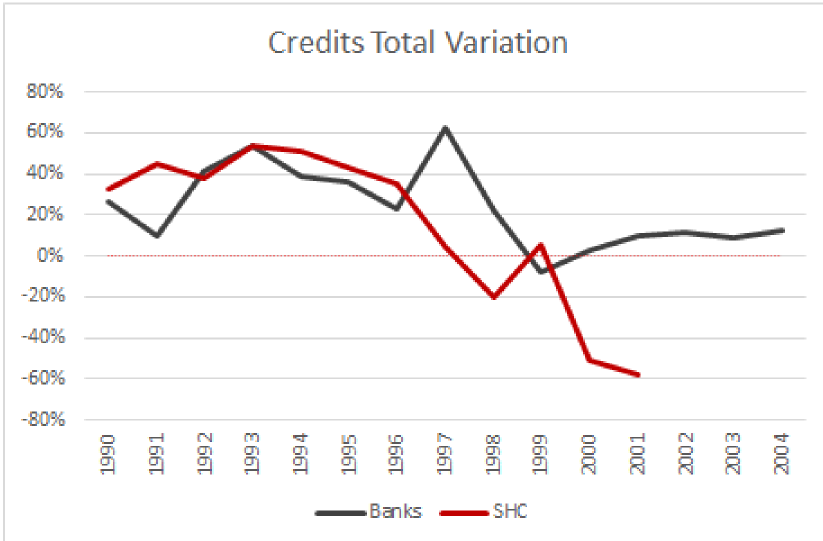

Figure 4-1.: Colombian Banks and Savings and Housing Corporations Credits Variation.

tral Hipotecario Bank was the one with the biggest mortgage-based portfolio and because of this, it was at higher risk with respect to other banks. Other banks with representative mortgage-based portfolio were Popular Bank and BBVA Ganadero Bank. With regard to the SHCs, it is easy to see that mortgage loans were their primary product where this type of credits got up until 85 % of the credit portfolio.

BANK	**(Credits/ Assets) %**	**% Past-due credits**	**(Mortgages/ Credits) %**
BANKBOSTON S.A.	80,79 %	0,00 %	0,00 %
STANDARD CHARTERED COLOMBIA	76,06 %	1,29 %	0,00 %
BANCO COLPATRIA	74,34 %	1,58 %	0,00 %
BANCO CAJA SOCIAL S.A.	73,81 %	2,59 %	2,98 %
MERCANTIL DE COLOMBIA S.A.	72,93 %	1,44 %	0,00 %
BCH (Banco Central Hipotecario)	72,64 %	13,34 %	71,67 %
INTERBANCO S.A.	71,31 %	2,04 %	3,01 %
CITIBANK	70,99 %	2,10 %	0,00 %
BANCOOP	70,63 %	4,21 %	0,30 %
BANCO POPULAR S.A.	70,48 %	2,66 %	9,02 %
LLOYDS TSB BANK S.A.	70,00 %	2,32 %	0,11 %

BANK	(Credits/ Assets) %	% Past-due credits	(Mortgages/ Credits) %
PACIFICO S.A	69,30 %	2,64 %	0,25 %
UNION COLOMBIANO	68,95 %	2,21 %	0,00 %
COOPDESARROLLO	67,07 %	2,31 %	0,90 %
BBV BANCO GANADERO S.A.	66,61 %	2,81 %	7,25 %
BANCO UCN	66,27 %	6,24 %	3,68 %
TEQUENDAMA S.A.	64,23 %	0,00 %	0,00 %
CAJA DE CREDITO AGRARIO	63,46 %	9,25 %	4,60 %
BANCAFE	61,64 %	4,19 %	0,00 %
SUDAMERIS COLOMBIA	60,58 %	2,14 %	0,94 %
BANCO SANTANDER COLOMBIA S.A.	60,56 %	2,22 %	0,00 %
BANESTADO	60,50 %	0,81 %	0,00 %
NACIONAL DEL COMERCIO	59,08 %	2,72 %	0,00 %
CREDITO	58,27 %	0,12 %	0,00 %
COLOMBIA	57,35 %	2,36 %	0,15 %
BANCO DE BOGOTA	57,14 %	2,50 %	0,00 %
BANSUPERIOR	56,64 %	8,25 %	0,00 %
ANDINO	56,26 %	2,00 %	3,28 %
ABN AMRO BANK COLOMBIA S.A.	52,89 %	0,00 %	0,00 %
OCCIDENTE	51,09 %	1,46 %	0,00 %
BANCOLOMBIA S.A.	50,74 %	0,98 %	0,00 %
SELFIN	50,04 %	1,45 %	0,00 %
BANK OF AMERICA COLOMBIA	21,31 %	0,00 %	0,00 %
Banks's Total	**63,47 %**	**4,01 %**	**14,18 %**

Table 4-1.: Exposure of Banks according to its credit portfolio in 1997.

Along with this, the financial liberalization made the existence of small entities filling specific niches possible which resulted in a substantial raise in the number of credit institutions [3],

[3]Savings and Housing Corporations (SHC), Commercial Financing Companies(CFC), Financial Corporations (FC), Cooperative Banks and traditional banks

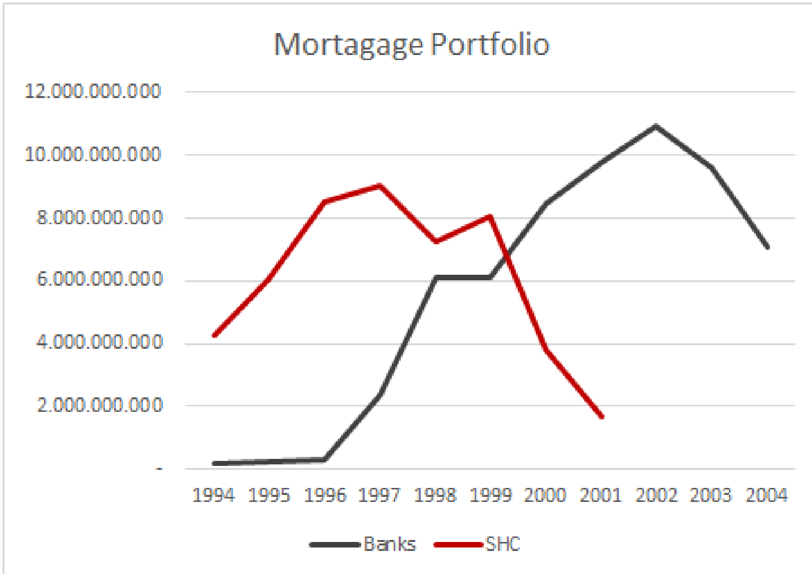

Figure 4-2.: Credits corresponding to mortgage loans.

from 87 in 1985 to 147 in 1996 [Arango, 2006]. We can see in Figure 4-4 how the number of banks raised in the first half of the decade, nonetheless, the number of SHCs was constant in this period.

SCH	**(Credits/ Assets) %**	**% Past-due credits**	**(Mortgages/ Credits) %**
COLMENA S.A.	84,05 %	7,56 %	80,78 %
GRANAHORRAR	82,73 %	6,32 %	85,39 %
CONAVI	82,57 %	4,50 %	79,90 %
COLPATRIA	82,47 %	6,30 %	80,62 %
CONCASA	81,51 %	10,31 %	86,73 %
AHORRAMAS	77,15 %	9,12 %	64,70 %
DAVIVIENDA S.A.	75,40 %	5,39 %	81,64 %
AV VILLAS	73,60 %	7,48 %	79,36 %
SCHs' Total	**80,99 %**	**7,04 %**	**80,52 %**

Table 4-2.: Exposure of SHCs according to its credit portfolio in 1997.

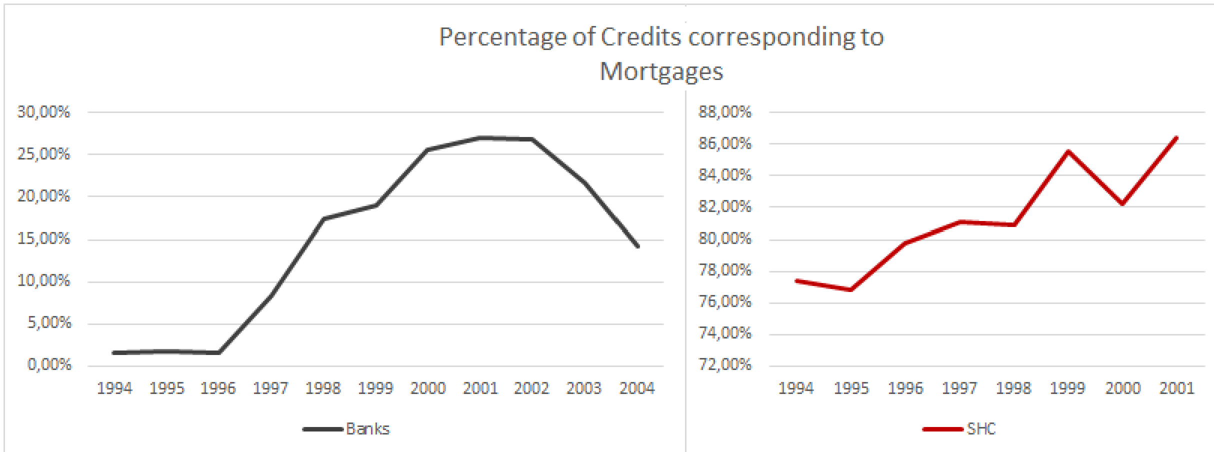

Figure 4-3.: Percentage of credits corresponding to mortgage loans.

Followed by this credit boom and thanks, mainly, to Asiatic and Russian crisis and its extension to emergent economies, Colombia had the deepest and most expensive banking crisis of its recent history largely caused by a sudden suspension of external capital flow. By that time there was a rise in the perception of credit risk which made the composition of portfolios change towards government bonds on the part of the financial intermediaries [ElTiempo, 2002].

In this context, as shown in Figure 4-5, there was an excessive rise in interest rates that affected the UPAC (constant purchasing power unit) which measured the price-level of money in long-term mortgage loans and changed with the financial rate DTF (rate for fixed-term deposits). Interest rates are a reflection of what was happening since it is visible how both, deposit and placement rates, decreased in the early 1990s, which stimulated the supply of credits (low-interest loans) and became savings demotivating for financial consumers (low-interest Term Deposit Certificates). This effect was absolutely clear in mortgage loans. Subsequently, in the second half of the decade, volatility of Colombian interest rates increased and the crisis became manifest, thus the risk of default raised and the uncertainty around the Colombian financial system was wide.

This rise made the UPAC (Figure 4-6) became higher than inflation from 1995 to 1999 and made the credits impossible to be paid for debtors and about three million of families lost or had the threat of losing their houses, which caused a serious reduction in banks' income flows and a estate devaluation up to 50 % [Clavijo et al., 2011]. This was followed by high unemployment rates, closure of companies, shrinkage of construction activity among others. In Figure 4-7 is evident how in 1998 the debts with due payment raised dramatically in both banks and SHCs and Figure 4-8 shows how in mortgage loans the situation was even more worrying, specially in banks, although in SHCs the rise seen in 1999 represented the double

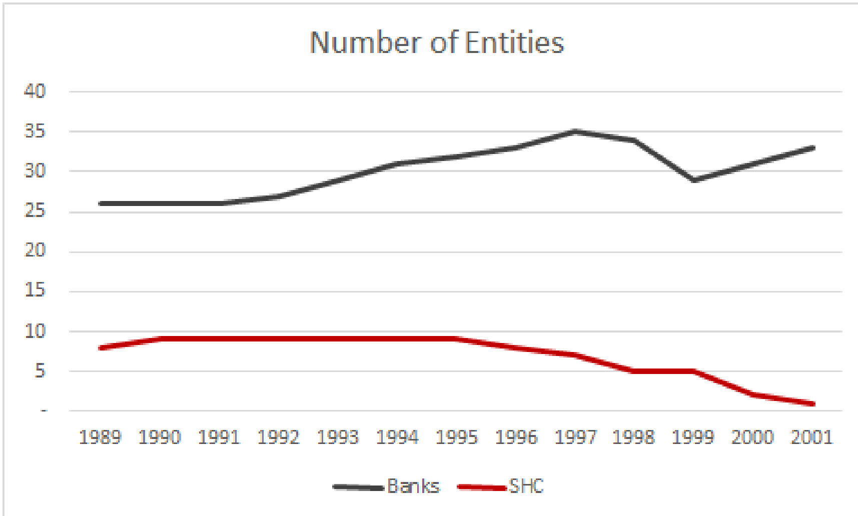

Figure 4-4.: Total of Banks and Saving Housing Corporations.

of past-due loans with regard to the immediately preceding year.

In order to understand the crisis impact it is sufficient to remember some numbers that illustrate the difficult moment that the country faced. First of all, between 1997 and 2001, the then called Superintendencia Bancaria intervened 24 credit institutions with liquidation ends (from a total of 104 that operated in 1998) which contrasts with the 20 intervened institutions during the 30 years before the crisis; from this outlook is clear the rapid deterioration of the financial system [Aparicio, 2009]. For example, in 1997 was needed to liquidate the capital leasing companies Cauca and Arfín, in addition to the commercial financing company La Fortaleza; also Granahorrar, one of the most important SHCs of the country, was intervened in October of 1998 and in the following months institutions such as the Caja Agraria and the Banco Central Hipotecario (mortgage central bank), Banco del Estado, among others, disappeared [ElTiempo, 2002]. It is notorious the case of the Central Hipotecario Bank that financed houses as a part of social housing schemes of the government in power, approved financing of mega-projects (clinics, malls, luxurious buildings, etc.) and purchased loans assuming the administrative costs and giving benefits to debtors which made the credit portfolio go from $1,6$ trillion of pesos in 1996 to $2,4$ trillion in 1997; it was in this precise moment when interest rates grew up to 20 points and people could not meet their financial obligations which made them give up their homes (the bank got to have 40 thousand properties and losses of 140 billion pesos). It is also remarkable how the cooperative sector was affected and the decision

[4]Personally created with information from the Banco de la República de Colombia web page: https://www.banrep.gov.co/en/statistics-0.

[5]Personally created with information from the Banco de la República de Colombia web page: https://www.banrep.gov.co/es/estadisticas/unidad-poder-adquisitivo-constante-upac

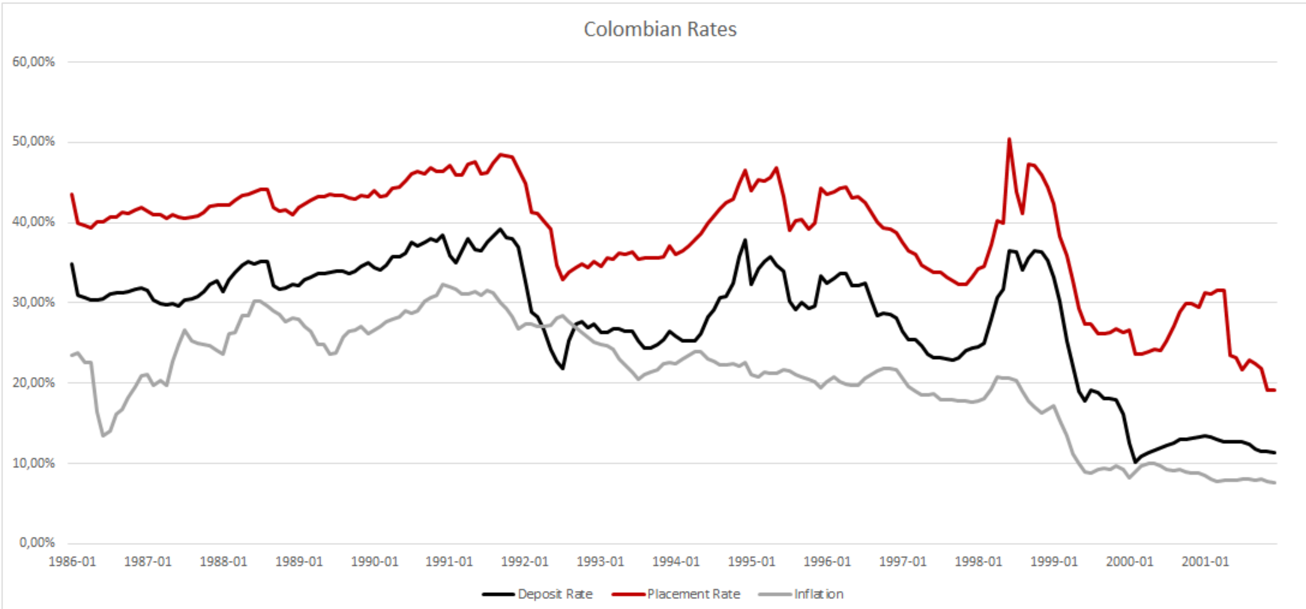

Figure 4-5.: Deposit, Placement and Inflation Rates in Colombia before and during the crisis.[4]

of merging Cupocrédito, Coopsibaté, Bancoop and Coopdesarrollo (the biggest and most representative entities of the sector) in order to avoid bankruptcy; Megabanco was born from this fusion with four different legal persons.

Secondly, there was a generalized mistrust in the financial institutions, together with the unfavorable expectations around the Colombian economy. The general feeling came along with corruption acts such as the Andino Bank that took 111 billion pesos out of the country and in which the prosecutors found out about destruction of documents and e-mails that were proof of illegal transactions, or Estado Bank where managers were accused of creating bank accounts with false documents and authorizing huge overdrafts to their supposed holders.

Thirdly, the crisis had a significant impact from the fiscal point of view. The national government was forced to allocate about $26,4$ trillion pesos to solve the crisis (which is equivalent to $15,4$ per cent of the GDP of 1999), resources that in normal circumstances would have been intended to other kind of public investment [Aparicio, 2009]. These resources were destined to cover measures taken in response to this huge crisis for which the government declared the 'economic emergency' state in November 1998. Among the actions taken, it was decided to subsidize debtors not in default of more than three months, to issue employment insurance to debtors of mortgage loans and to introduce a banking tax (first it was two pesos per thousand pesos of each financial transaction, then it was raised to four and it is called four per thousand or *cuatro por mil* in Spanish).

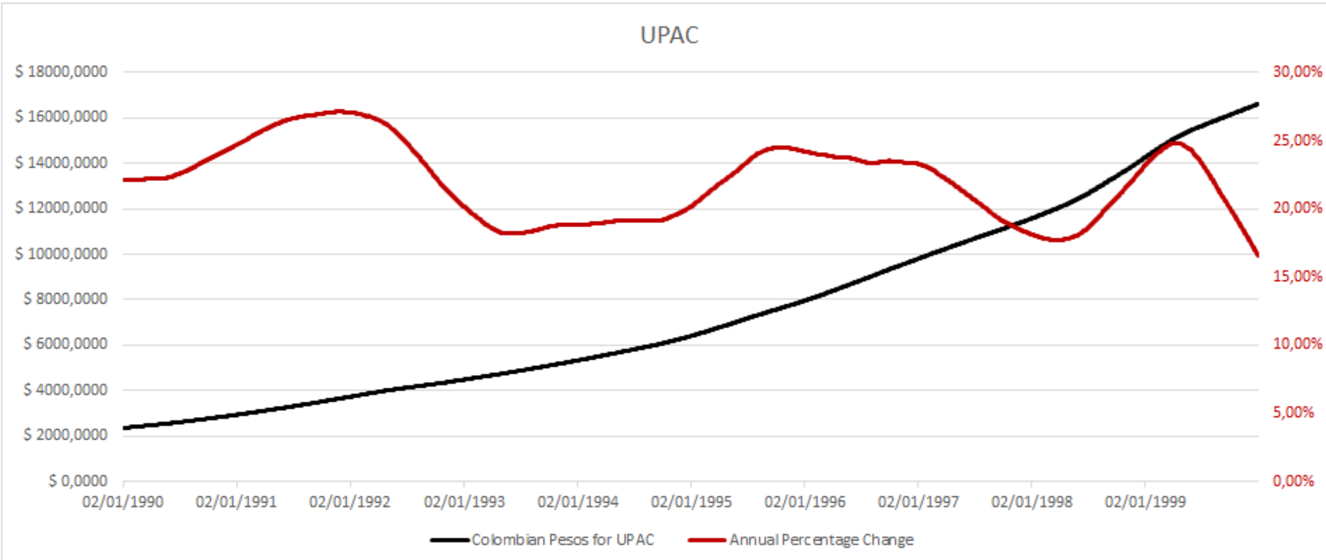

Figure 4-6.: Constant Purchasing Power Unit (UPAC) before and during the crisis.[5]

For construction industry was difficult to recover from the crisis. Even though, sector's entrepreneurs started to make provisions in 1997 to face which seemed a normal downturn, the fall was longer and deeper than in previous periods and with consequences such as hundreds of companies closed, thousands of laborers fired and an important loss in the real estate value. It was not until 2001 that this industry shows a slight growth thanks to the approbation of a new housing law that created incentives to buyers, gave tax benefits to constructors and modified the housing financing scheme [ElTiempo, 2002]. Among the changes considered was the shifting of SHCs to banks where some of them merge with other SHCs like Ahorramas and Las Villas from Grupo Aval organization, and others like Granahorrar purchased loans of extinct banks, in this case the Central Hipotecario Bank.

In our context of study, it is necessary to determine what we are going to considered as a defaulted bank or SHC. For this purpose, we are going to take into account that not just bankruptcy is a sign of default but the need to be intervened by the Government for capitalization is also an indication of affectation in an economic crisis. Additionally, financial companies that did not disappear completely but had to be merged with others to receive financial support. In Table **4-3** is exposed the number of banks and SHCs that were merged, liquidated and capitalized in a period of four years according to newspaper news and the information in financial states from the web page of the Superintendencia Financiera de Colombia. Also, we can see the quantity of surviving banks and SHCs, with the clarity that the SHCs that survived continued functioning as banks by Government's decree. What is

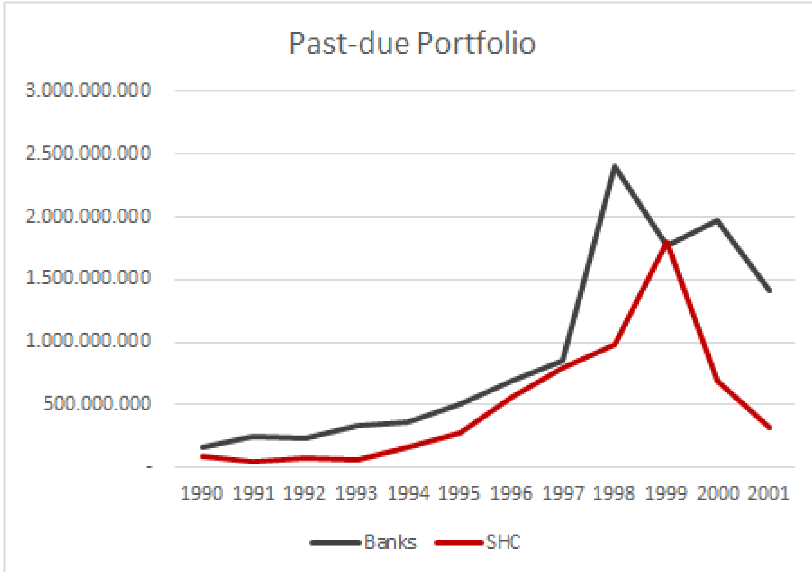

Figure 4-7.: Past-due portfolio of Banks and Saving Housing Corporations.

	Number of Banks	Number of SHCs
Merged	5	2
Liquidated	7	0
Capitalized	3	1
Total in Default	**15**	**3**
Surviving	17	5
Total During the Crisis	**32**	**8**

Table 4-3.: Banks and SHCs in default during the crisis

evident is that nearly half of the banks and SHCs fell in default, even when some of them recovered after the crisis and exist up to now.

It is important to emphasize here that 'contagion' in this context represents the clustering phenomena in the default of credit and financial entities in Colombia, which is evident from the number of companies merged, liquidated and capitalized. On the basis of what we saw before, the period of study will be from September 1997 to September 2001 and, graphically, we can see the clustering defaults in Figure 4-9 where the dates when these defaults happened are exposed. Each peak represents that, according to newspaper news, around that date a Bank or a SHC defaulted. Now, we clarify that the information shown in the present work only covers banks and SHCs, since we consider that they are the principal entities involved in the crisis because the biggest compromised portfolio was the mortgage-related, nevertheless, there were other financial institutions affected (as we said before, the Superin-

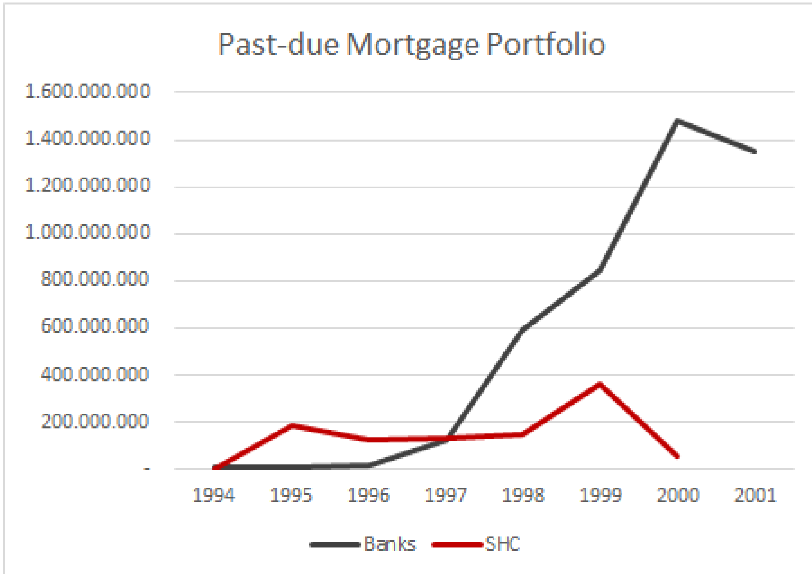

Figure 4-8.: Past-due Mortgage portfolio of Banks and Saving Housing Corporations.

tendencia Bancaria intervened 24 institutions). The details of defaulted banks and SHCs are in Table 4-4, according to newspaper news and public information related to the liquidation or interventions carried out to these entities.

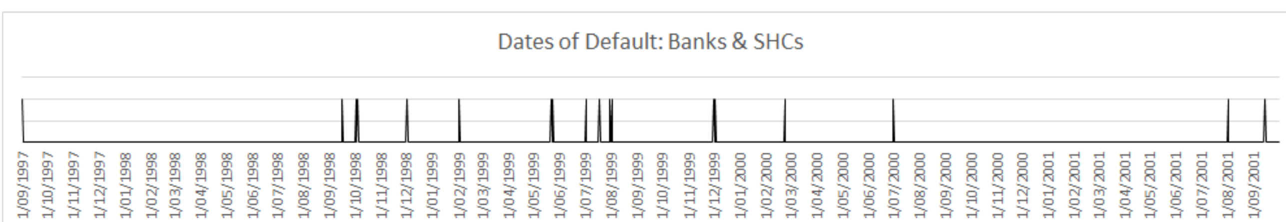

Figure 4-9.: Default dates of Banks and Saving Housing Corporations.

According to public financial statements from 1990 to 2004, the crisis not only affected the entities referred to in Table 4-4, but all the market suffered. This can be seen from their assets variation, where most of them had negative variations among the years 1998 and 2004. However, the strongest entities survived the crisis and most of them took advantage by merging with the weaker ones and huge economic empires were born such as Grupo Aval and Grupo Bancolombia.

	Entity	Default Date	Details
Banks	BANCOOP	sep-98	Merged with Coopdesarrollo
	CAJA DE CREDITO AGRARIO	oct-98	Liquidated (known as Caja Agraria)
	MERCANTIL DE COLOMBIA S.A.	dic-98	Its size decreased and it closed 70 % of its offices
	BANCO UCN	feb-99	Liquidated
	ANDINO	may-99	Liquidated
	PACIFICO S.A	may-99	Liquidated
	SUPERBANCO	jul-99	Capitalized by the Government
	SELFIN	jul-99	Liquidated
	UNION COLOMBIANO	jul-99	Capitalized by the Government
	RED MULTIBANCA COLPATRIA S.A.	jul-99	Capitalized by the Government
	COOPDESARROLLO	nov-99	Merged with other cooperatives, it becomes Megabanco
	BCH (Banco Central Hipotecario)	feb-00	Stoped functioning as bank
	BANESTADO	jun-00	Acquired by Bancafé
	INTERBANCO S.A.	ago-01	Merged with Aliadas
	BANK OF AMERICA COLOMBIA	sep-01	Assets and liabilities transfered to Banco de Bogotá
SCH	CONCASA	sep-97	Acquired by Bancafé
	GRANAHORRAR	oct-98	Intervened
	AHORRAMAS	dic-99	Merged with SCH Las Villas, it became Av Villas Bank

Table 4-4.: Details of Banks and SHCs in default during the crisis

	Entity	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
	BANCOOP				97 %	75 %	53 %	55 %	14 %	-57 %	-100 %	-48 %	-95 %	354 %	-100 %	
	CAJA DE CREDITO AGRARIO	27 %	10 %	14 %	26 %	21 %	6 %	16 %	33 %	-7 %	-100 %					
	MERCANTIL DE COLOMBIA S.A.	6 %	2 %	166 %	58 %	43 %	50 %	44 %	-7 %	-21 %	-33 %	-48 %	-100 %			
	BANCO UCN				122 %	101 %	28 %	36 %	63 %	84 %	-100 %	5 %	-100 %			
	ANDINO	2 %	-20 %	115 %	57 %	27 %	35 %	21 %	53 %	-25 %	-100 %					
	PACIFICO S.A				267 %	79 %	77 %	36 %	32 %	17 %	-100 %					
	SUPERBANCO					80 %	-9 %	19 %	33 %	3 %	-10 %	13 %	9 %	23 %	20 %	20 %
	SELFIN							175 %	25 %	-37 %	-100 %					
	UNION COLOMBIANO	29 %	11 %	103 %	20 %	37 %	36 %	8 %	8 %	16 %	-14 %	0 %	31 %	10 %	23 %	22 %
	COOPDESARROLLO						64 %	66 %	28 %	73 %	-100 %					
	BCH (Banco Central Hipotecario)		20 %	11 %	26 %	-100 %	98 %	248984 %	65 %	-11 %	10 %	-86 %	-13 %	-25 %	-100 %	
	BANESTADO	13 %	18 %	49 %	45 %	22 %	16 %	2 %	33 %	-24 %	78 %	-69 %	-38 %	44 %	7 %	1 %
	INTERBANCO S.A.					198 %	123 %	139 %	56 %	68 %	-8 %	-33 %	-100 %			
	BANK OF AMERICA COLOMBIA	49 %	1 %	-2 %	5 %	10 %	67 %	86 %	79 %	27 %	0 %	-14 %	-100 %			
	BANCO DE BOGOTA	23 %	21 %	89 %	25 %	27 %	30 %	15 %	43 %	13 %	21 %	14 %	13 %	17 %	13 %	31 %
	BANCO POPULAR S.A.	24 %	18 %	14 %	49 %	25 %	7 %	16 %	32 %	23 %	8 %	12 %	9 %	4 %	13 %	24 %
	BANCAFE	25 %	16 %	19 %	27 %	27 %	31 %	17 %	18 %	62 %	-1 %	13 %	-9 %	10 %	4 %	15 %
	BANCO SANTANDER COLOMBIA S.A.	31 %	29 %	63 %	63 %	24 %	24 %	21 %	46 %	24 %	1 %	2 %	-2 %	-8 %	-1 %	21 %
	BANCOLOMBIA S.A.	13 %	26 %	47 %	49 %	25 %	30 %	24 %	35 %	1 %	-4 %	8 %	15 %	19 %	15 %	26 %
	ABN AMRO BANK COLOMBIA S.A.	13 %	30 %	23 %	46 %	-2 %	26 %	23 %	118 %	80 %	11 %	32 %	21 %	-27 %	6 %	-10 %
	CITIBANK	86 %	21 %	35 %	40 %	38 %	36 %	28 %	52 %	61 %	29 %	10 %	8 %	-12 %	-4 %	23 %
	LLOYDS TSB BANK S.A.	33 %	30 %	52 %	45 %	19 %	24 %	21 %	38 %	22 %	-14 %	21 %	20 %	20 %	9 %	5 %
Banks	SUDAMERIS COLOMBIA	54 %	2 %	51 %	32 %	28 %	42 %	37 %	40 %	22 %	-8 %	-3 %	24 %	6 %	-14 %	37 %
	BBV BANCO GANADERO S.A.	34 %	25 %	51 %	46 %	39 %	22 %	27 %	26 %	17 %	-5 %	1 %	9 %	-9 %	26 %	15 %
	CREDITO	60 %	25 %	66 %	54 %	45 %	25 %	14 %	54 %	12 %	2 %	10 %	13 %	1 %	27 %	21 %
	NACIONAL DEL COMERCIO	42 %	60 %	30 %	26 %	11 %	21 %	73 %	-17 %	-100 %						
	OCCIDENTE	34 %	30 %	41 %	38 %	27 %	26 %	26 %	29 %	25 %	17 %	13 %	14 %	20 %	17 %	18 %
	STANDARD CHARTERED COLOMBIA	41 %	2 %	47 %	59 %	39 %	21 %	8 %	42 %	-4 %	1 %	8 %	10 %	10 %	-32 %	-34 %
	TEQUENDAMA S.A.	9 %	43 %	92 %	77 %	37 %	29 %	27 %	28 %	23 %	-3 %	28 %	29 %	-15 %	11 %	15 %
	BANCO CAJA SOCIAL S.A.	37 %	33 %	49 %	41 %	38 %	45 %	22 %	32 %	9 %	3 %	2 %	14 %	6 %	16 %	9 %
	BANKBOSTON S.A.						98 %	77 %	64 %	75 %	63 %	33 %	5 %	-13 %	-2 %	-54 %
	BANCO AGRARIO DE COLOMBIA S.A.											5 %	18 %	17 %	14 %	19 %
	MEGABANCO S.A.											-4 %	13 %	10 %	19 %	10 %
	BANCO ALIADAS S.A.													15 %	22 %	9 %
	COMERCIO	41 %	21 %	-100 %												
	BANCO SANTANDER	14 %	11 %	-100 %												
	CONCASA	25 %	50 %	35 %	38 %	59 %	43 %	31 %	15 %	-100 %						
	GRANAHORRAR	16 %	55 %	29 %	60 %	60 %	49 %	37 %	21 %	-21 %	21 %	86 %	-7 %	2 %	3 %	-7 %
	COLPATRIA	30 %	49 %	44 %	44 %	52 %	47 %	55 %	29 %	101 %	-2 %	-37 %	6 %	7 %	19 %	-2 %
	DAVIVIENDA	24 %	50 %	30 %	51 %	59 %	41 %	111 %	208 %	-25 %	8 %	2 %	3 %	8 %	3 %	25 %
	AHORRAMAS	25 %	62 %	39 %	43 %	49 %	36 %	35 %	-100 %							
SCHs	AV VILLAS	26 %	71 %	51 %	42 %	35 %	57 %	69 %	39 %	13 %	5 %	55 %	2 %	-4 %	5 %	5 %
	COLMENA S.A.	27 %	59 %	53 %	42 %	44 %	35 %	36 %	33 %	26 %	4 %	-2 %	3 %	1 %	3 %	2 %
	CONAVI	31 %	64 %	45 %	42 %	46 %	35 %	36 %	26 %	20 %	8 %	2 %	3 %	7 %	6 %	11 %
	CORPAVI	27 %	44 %	21 %	29 %	30 %	32 %	-100 %								

Table 4-5.: Assets Variation of Banks and SHCs.

5. Dynamic Contagion in Colombian Banking Crisis

Finally, we have to relate the model to the data. Recall that the intensity of the Dynamic Contagion Process defined in Definition 17, has two parts: The externally excited jumps which are called immigrants and in our context would be the risk of default caused by international economy and their repercussions on Colombian economy. The second part is the self-excited jumps that represent the risk of default caused by a bankruptcy in a Colombian bank or SHC. We want to emphasize that this kind of model is able to capture stress conditions and instability in the system, caused by external or internal factors, because the probability of default is affected by these events. Moreover, in order to take advantage of the properties where the jumps size are exponential, we are going to assume these distributions for both, external and self-excited jumps.

At this point it is important to remark that these shocks are latent, which means they are not directly observable and we have to inferred them from other observable variables. Now, since we could see the process N_t as in Figure **5-1**, where it is evident that the process has its defaults mainly between 1998 and 1999, our first approach will be to inferred the internal impact by this data and the external shocks with a relatable financial series that represents the effect of international crisis on Latin American economy, besides, since this is a crisis from over 20 years ago, we are going to fix the reversion level to the actual proportion of banks that have defaulted after the crisis in 1998. The series chosen to represent the external shocks should be one that has rare positive jumps during this time and that could somehow explain when the crisis arrived and how bad it was. This approach allows us to decrease the number of parameters to estimate and simplifies our problem.

The second approach would be to estimate all the parameters from the default data. This procedure assumes that both types of jumps are not observable and all the information that we have is the defaults arrival. For this kind of processes, this is the standard procedure and we want to be able to compare the two methods and to determine if both are plausible.

[1] Personally created with information collected from newspaper articles and the financial statements extracted from the Superintendencia Financiera de Colombia web page: https://www.superfinanciera.gov.co.

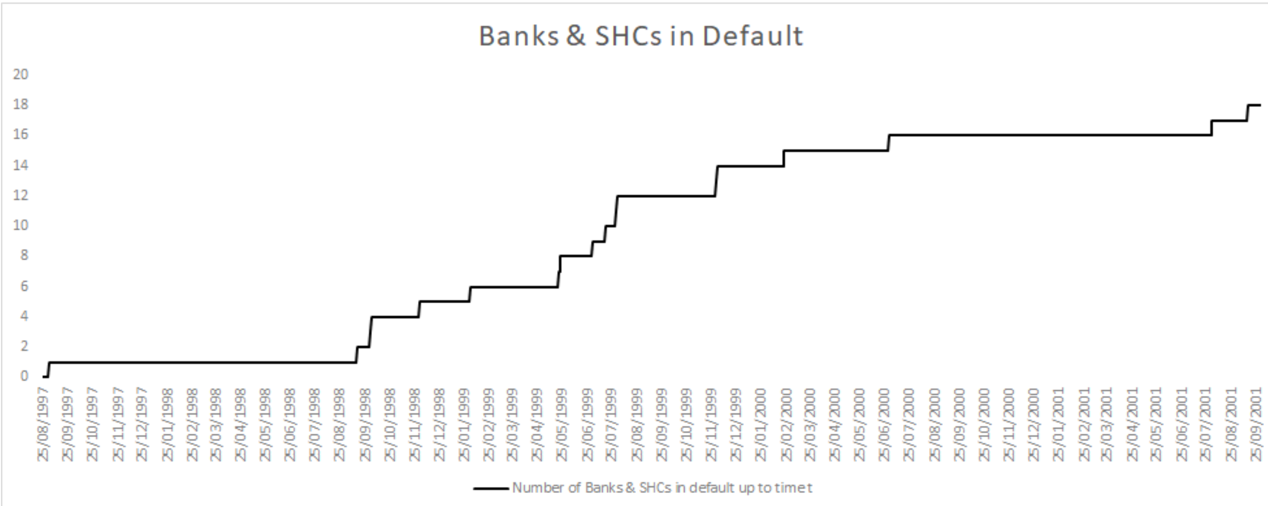

Figure 5-1.: Process N_t of number of defaults in banks and SHCs up to time t [1]

In consequence, the estimation methodology and simulation study would by the same except for the number of parameters to be estimated. Because of this, we review the general method of estimation and simulation and later we will give the results for each approach.

5.1. Estimation methodology

Thanks to the results of the dynamic contagion process moments given in [Dassios and Zhao, 2011], we have the possibility to compute moments of the process N_t explicitly and then, for estimating the parameters, we are going to use the fast calibration method applied by [Da Fonseca and Zaatour, 2014] to Hawkes processes.

This procedure is an application of the generalized method of moments (GMM) for Hawkes-like processes. This is a brief review of the method.

Definition 18. *([Hall, 2005], page 14) Let θ_0 be a vector of unknown parameters which are to be estimated, v_t be a vector of random variables and $f(\cdot)$ a vector of functions, then a **population moment condition** takes the form*

$$E[f(v_t, \theta_0)] = 0, \tag{5-1}$$

for all t.

If we have that the number, K, of unknown parameters in θ_0 is greater than the number, R, of functions in f, we obtain the **Generalized Method of Moments** (GMM); this characteristic is called **Over-identification**. In the case where $K < R$, we have **Under-identification** in which case the parameters can not be consistently estimated and it is necessary to find instruments that give the sufficient amount of moment conditions. Also, in the case were $K = R$ we have **Just-identification** from where we derive the classical method of moments.

GMM estimators are obtained minimizing a quadratic of the sample moment $T^{-1}\sum_{t=1}^{T} f(v_t, \theta)$ weighting by a matrix W_T [Hall, 2005].

Definition 19. *([Hall, 2005], page 14) The* ***Generalized Method of Moments estimator*** *based on (5-1) is the value of theta which minimizes:*

$$Q_T(\theta) = T^{-1}\sum_{t=1}^{T} f(v_t, \theta)' W_T T^{-1} \sum_{t=1}^{T} f(v_t, \theta),$$

where W_T is a positive semi-definite matrix which may depend on the data but converges in a probability to a positive definite matrix of constants.

According to [Hall, 2005], so as to have in $Q_T(\theta)$ a meaningful measure of distance, the weighting matrix has to have some restrictions. Also, [Hall, 2005] specified that the whole analysis is based on asymptotic theory when $T \to \infty$. In this manner, our inference problem is written as:

$$\hat{\theta} = argmin \left\{ (M - g(\theta))' W (M - g(\theta)) \right\}$$

where M is the vector of empirical moments (or restrictions over the parameters), $g(\theta)$ is the vector of corresponding theoretical moments and W is a symmetric positive definite weighting matrix. This minimization process has the restrictions derived from the dynamic contagion process, that is to say, all parameters are positive and a can be zero, also $\delta\beta > 1$.

The moment restrictions recommended by [Da Fonseca and Zaatour, 2014] in situations of clustering phenomena are the mean, variance and autocorrelation function. Also, so as to be able to use the data to develop the estimation procedure, it is necessary to use the restrictions but over an interval of size τ, which means that we need the mean, variance and autocorrelation function for the process $N_{t+\tau} - N_t$, for a given τ. In the following result, developed for the purposes of this work, we calculate these quantities [2].

Proposition 1. *Let N_t a dynamic contagion process as in Definition 17, and define the number of events in an interval τ as $N_{t+\tau} - N_t$. Then, if $\gamma = \delta - \mu_{1_G}$ we have*

1. $E[N_{t+\tau} - N_t] \xrightarrow[t\to\infty]{} \Lambda\tau,$

[2] We are grateful for the support of Johanna Garzón, professor of the mathematics department of the Universidad Nacional de Colombia, in the development of Proposition 1 results

2. $Var[N_{t+\tau} - N_t] \xrightarrow[t\to\infty]{} \Lambda\tau + \kappa\left(\tau + \frac{e^{-\gamma\tau}-1}{\gamma}\right)$

3. $Cov[N_{t+\tau} - N_t, N_{t+2\tau+\epsilon} - N_{t+\tau+\epsilon}] \xrightarrow[t\to\infty]{} e^{-\gamma\epsilon}\left(\frac{1-e^{-\gamma\tau}}{\gamma}\right)\left[2\bar{k}(1 - e^{-\gamma\tau} - \Lambda^2\tau)\right]$

4. $Acf(\tau, \epsilon) \xrightarrow[t\to\infty]{} \frac{e^{-\gamma\epsilon}(1-e^{-\gamma\tau})[2\bar{k}(1-e^{-\gamma\tau})-\Lambda^2\tau]}{\gamma\Lambda\tau+\kappa(\tau\gamma+e^{-\gamma\tau}-1)}$

where

- $\Lambda = \frac{\mu_{1_G}\rho + a\delta}{\gamma}$
- $\kappa = \frac{\mu_{2_G}\Lambda + \mu_{2_H}\rho}{\gamma^2} + \frac{2\mu_{1_G}\Lambda}{\gamma}$
- $\bar{k} = \frac{\mu_{1_G}\Lambda}{\gamma}\left(1 + \frac{1}{2\gamma}\right) + \frac{\mu_{2_H}\rho}{2\gamma^2}$

Proof: Due to the length of calculations required in the proof of this proposition, it is necessary to leave the proof in the Appendix A.

We need to remark from the previous theorem that $Acf(\tau, \epsilon)$ is the autocorrelation function between two intervals of size τ separated by a distance of ϵ and since we must have overidentification, we chose as many lags of the autocorrelation function as needed to obtain one more functions than parameters. Also, it is necessary to point out that in the previous theorem we take the limit when t tends to infinity because we are going to use the stationary distribution of the process and as we do that, these formulas do not depend on λ_0.

Let us recall the comment made by [Arango, 2006] that implies that before the 1998 Colombian banking crisis, 20 credit entities had been liquidated throughout the previous 30 years. We consider that this information gives us a good approximation of the initial intensity $\lambda_0 = 0{,}056$.

The generalized method of moments (GMM) algorithm that we used in the present work was developed by [Cao, 2020] for MATLAB.

5.2. Simulation study

As suggested by [Da Fonseca and Zaatour, 2014], we performed a Monte Carlo experiment to determine the robustness of the estimation procedure. But before addressing this, we want to make a descriptive and graphical representation of the process we are dealing with.

We are going to see two dynamic contagion processes with different parameters that are chosen to illustrate the distinctions when the parameters change. In Figure **5-2** there is a path

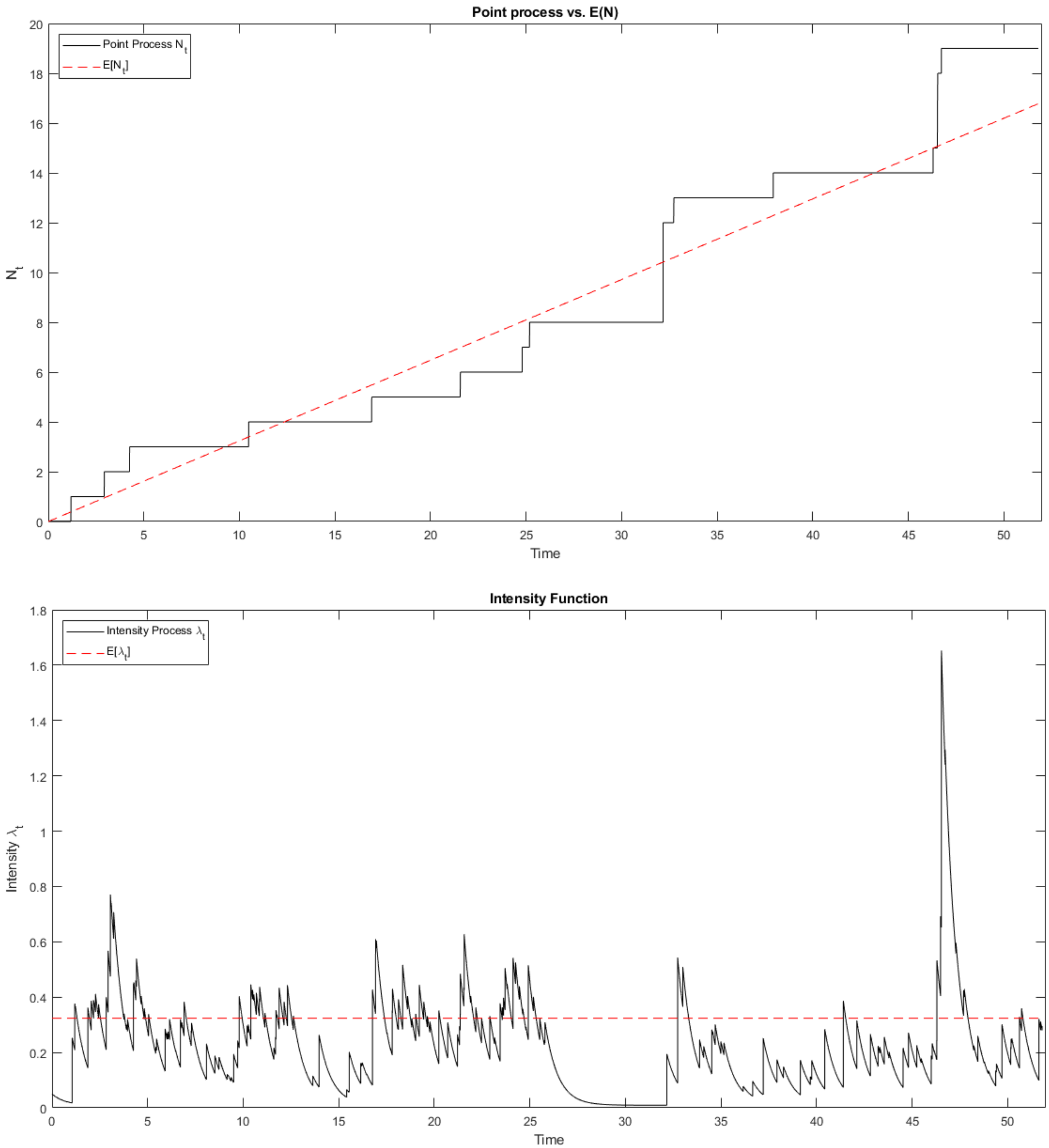

Figure 5-2.: Simulated path of a dynamic contagion process with parameters $(a, \rho, \delta, \alpha, \beta, \lambda_0) = (0.009, 6.7, 1.5, 17.1, 4, 0.05)$.

of a dynamic contagion process with parameters $(a, \rho, \delta, \alpha, \beta, \lambda_0) = (0.01{,}6.7{,}1.5{,}17.1{,}4{,}0.05)$. For this process is expected that, in $T = 50$, there are around 17 arrivals and for this particular path, the arrivals up to this time were 19. Now, in process N_t there are periods of more current arrivals, for example near time 46 where N_t goes from 14 to 19 in a very short period of time, but this process is mainly represented by long periods of time with no arrivals.

This phenomenon can be explained by the magnitude of the jump sizes, α and β: Since we are assuming that the jump sizes are exponentially distributed, the mean jump size for the externally excited jumps is $\frac{1}{\alpha} = 0{,}058$ and for the self-excited jumps is $\frac{1}{\beta} = 0{,}25$. In this case, the impact of shocks within the system or external to it are not exaggeratedly strong, and so there exist clusters of events but they are small, nevertheless, they are considered clusters because usually they come after a period of quietness in the system.

It is specially interesting to analyze how the intensity process behaves. Even when there are periods without defaults in the process N_t, there are peaks in the intensity process λ_t and that happens because ρ is relatively large, which means, the arrival of external shocks are really frequent but, as mentioned before, most of them have not large impact in the probability of default. The effect of that most of external impacts are not causing defaults in the immediate time, but all of them together stress gradually the system until the default happens.

A second example can be seen in Figure **5-3** where a path of a dynamic contagion process with parameters $(a, \rho, \delta, \alpha, \beta, \lambda_0) = (0.7{,}0.6{,}3{,}2{,}2.5{,}0.5)$ is shown. To start with, the first evident difference is that at time $T = 50$, the expected value of defaults or events is 45 while the arrivals in the process were 49 and in general, the process N_t has more arrivals and hence, shorter periods between arrivals. On the other hand, having ρ smaller than 1 makes the intensity have less peaks but since β and α are smaller than in the first example, the impact of each external or internal shock is higher which increases the probability of defaults arrivals.

Parameter	**Lower bound**	**Upper bound**
δ	0	10
β	$\frac{1}{\delta}$	$\frac{1}{\delta} + 20$
α	0	20
ρ	$\frac{\alpha}{3}$	$\frac{\alpha}{3} + 5$
a	0	5
λ_0	a	$a + 2$

Table 5-1.: Intervals where each parameter is randomly selected in the simulation study.

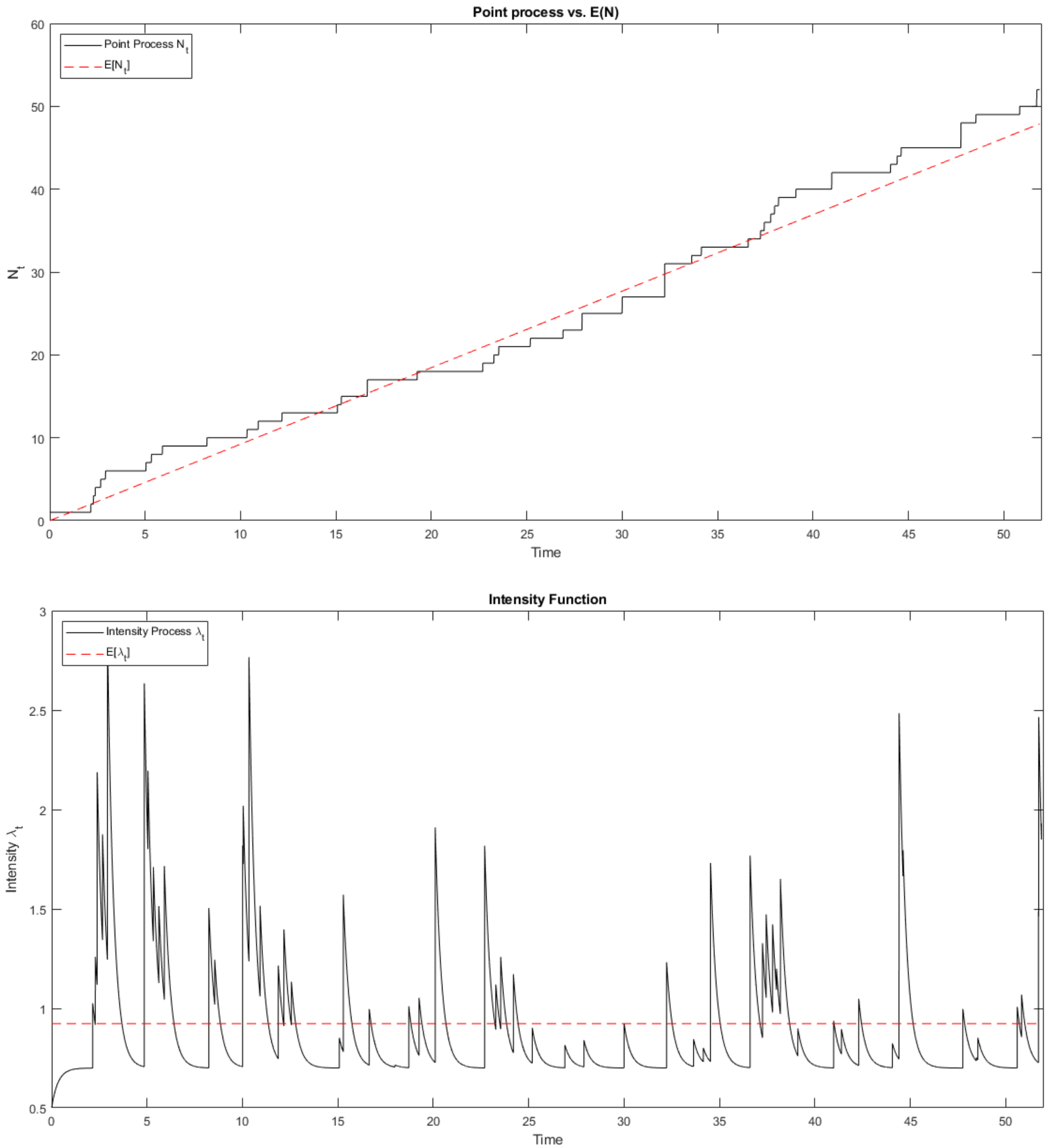

Figure 5-3.: Simulated intensity of a dynamic contagion process with parameters $(a, \rho, \delta, \alpha, \beta, \lambda_0) = (0.7,0.6,3,2,2.5,0.5)$.

With the objective of assessing if the estimation procedure is appropriate, first we randomly generate the parameters. As we have seen so far, our data resembles to the first kind of dynamic contagion process, that is the reason why, empirically we conclude that our parameters should be in the intervals shown in Table **5-1** and is from those intervals that we are going to randomly generate the simulation parameters.

It is of the highest importance to emphasize that this choosing of intervals is purely empirical, mainly because in a first approximation there were indications of this parameter order for the process of Colombian banking defaults, and it is our main goal to estimate adequate parameters for this process. In particular, β does have a mandatory lower bound because, according to the results about the dynamic contagion process, it is necessary that $\beta\delta > 1$.

Moreover, after several repetitions of the simulation algorithm, it was evident that for the estimation procedure to work properly, it is necessary that the external shock of the process will have a visible impact but that this impact will not be too large. This means that α and ρ can not simultaneously be large or small, because when they are both large the process explodes and N_t becomes uncontrollably huge in very little time, which is intuitive clear because if the intensity ρ of arrival of external shocks is large and also the jump size is large, the intensity process will not have enough time to decay. Now, if they are both small, the estimation procedure is not capable of capturing their little effect and the estimation of these parameters is not accurate. Also, similar to the result obtained by [Da Fonseca and Zaatour, 2014], for this process, the identity matrix as W generates less estimation error and therefore this is what we use in the whole procedure.

Finally, [Dassios and Zhao, 2011] set the initial conditions of the simulation procedure as $t_0 = 0$ and $\lambda_0 > a$, so in order to select a random λ_0 that fits this restriction, we take the interval presented in Table **5-1**. After randomly select the parameters, we simulate a path of the corresponding dynamic contagion process and estimate the parameters with the estimation procedure explained in Section 5.1. We simulate 25000 times and we will show the mean relative error and the standard deviation of the error calculated as the root mean squared relative error.

5.3. Estimation with external data and fixed reversion level

In the first case, we take into account that, since it has been almost 20 years since the crisis, we could assume the reversion level at which the intensity tended after the crisis. It was found that after the crisis, only 2 banks had to be liquidated due to instability, so the monthly reversion level intensity of our process is estimated as $a = 0{,}009$. Also, we are going to assume that the impact of international crisis on Latin American economy can be seen in Brazilian interest rate jumps. All of these assumptions help us to reduce the complexity of the minimization problem because at this point we are left with three parameters to estimate.

5.3.1. Externally excited jumps

According to historical bibliography and experts' recommendations [3], one Latin American country that reflected clearly the crisis after the, so-called, Asian Tigers' decline was Brazil. In the early 1990's, the Brazilian government implemented the Plan Real (Real Plan in English) that aimed at reducing inflation and included measures such as external financial and trade openness which caused plentiful entrance of direct foreign funding mainly in short-term financial inflows through banking indebtedness, also placement of securities abroad or active policies of privatization [Calcagno and Sáinz, 1999].

In addition, there were a huge expansion of imports and high interest rates which increased the country's vulnerability to international economy context, vulnerability that was dangerously underestimated. In 1996, symptoms of difficulties began to arise but it was in the last trimester of 1997 when the impact of the Asian crisis is felt [Calcagno and Sáinz, 1999] and it is evident the external and fiscal imbalance in addition to economic stagnation, situation that was even more complicated when the Russian crisis arose.

During the crisis, external deficit grew even more along with the rise in public debt. This situation made Brazil increasingly dependent on foreign funding and in order to attract external capital, higher interest rates were offered and so high returns to investments, this is remarkably evident in Figure **5-4** where after having a decreasing interest rate from April 1995, in the last months of 1997 there is a significant rise and it is maintained unstable until mid 1999. In this manner, there was evident interrelation between currency appreciation and interest rates [Calcagno and Sáinz, 1999].

Now it is necessary to define what we are going to call a jump, this is mainly because the intuitive idea of a jump is a change from the previous state and it turns out that sometimes from one month to the next, there is no much change in the interest rate. Additionally, in our context, a jump must have a negative impact on the probability of a new default arriving,

[3] We would like to recognize the time dedicated in conversations with us by Alexander Guarín, professor of the Master in Actuary and Finance of the Universidad Nacional de Colombia, Bogotá venue and Chief Scientific Advisor of the Banco de la República Model Development Section. Also, we are grateful to Nicolás Verano, statistician and magister in Actuary and Finance from the Universidad Nacional de Colombia.

[4] Personally created with information of Brazilian interest rates extracted from the Banco Central do Brasil webpage: `https://www.bcb.gov.br/controleinflacao/historicotaxasjuros`.

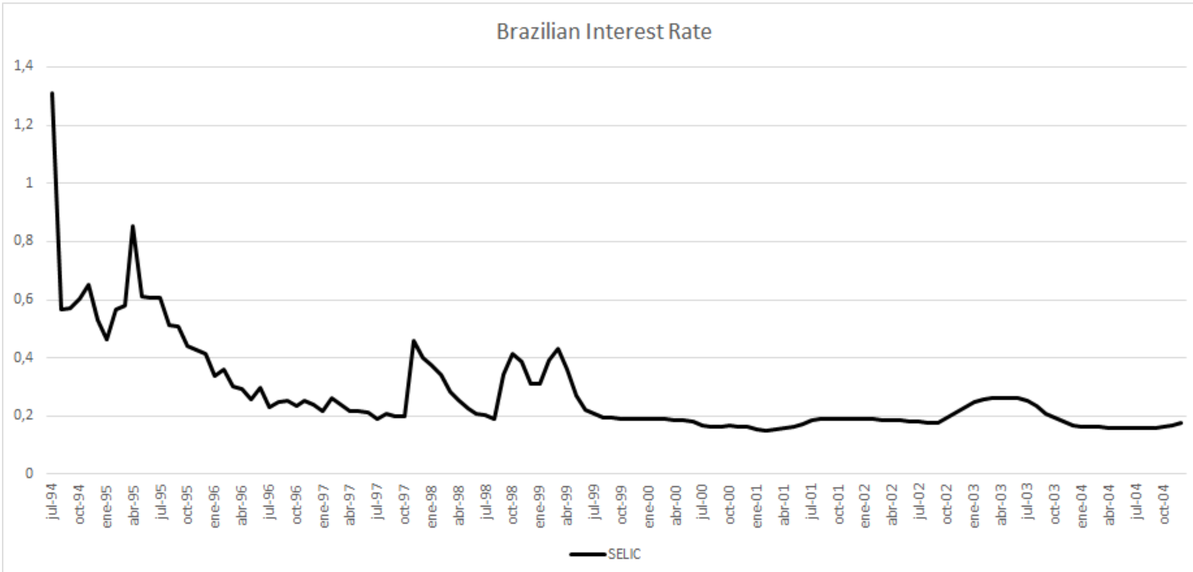

Figure 5-4.: Interest Rate of Brazil, typically known as the Special System of Clearance and Custody (SELIC) [4]

so we will only work with positive changes.

As remarked in Section 3.2, the case where jump sizes have exponential distribution has very interesting properties and so forth, it has an intuitive meaning: Big jumps are rare and therefore they have small probability. Following this idea, we are going to consider positive and significant monthly changes in the interest rate. To find what positive changes are significant, we are going to remove from the series several portions of the smallest changes until we have a good fit with an exponential distribution.

To do so, in Figure **5-5** we can see that, when removing the 60 % of the smallest changes in the Brazilian interest rates, the data fits an exponential distribution with a p-value of 35,85 %, which means that with a level of confidence of 10 % the null hypothesis of exponential distribution is accepted. It is important to clarify that even when 60 % of the changes removed sounds like too much to eliminate, the changes left out are not greater than 0,85 % and since we are looking for changes that could indicate that there is a negative external impact over Colombian economy, this kind of elimination is acceptable for our problem.

In conclusion, the changes in SELIC greater than 0,85 % are going to be considered external jumps for our dynamic contagion process and they fit an exponencial distribution of parameter $\alpha = 17{,}14$, as it can be seen in Figure **5-6**.

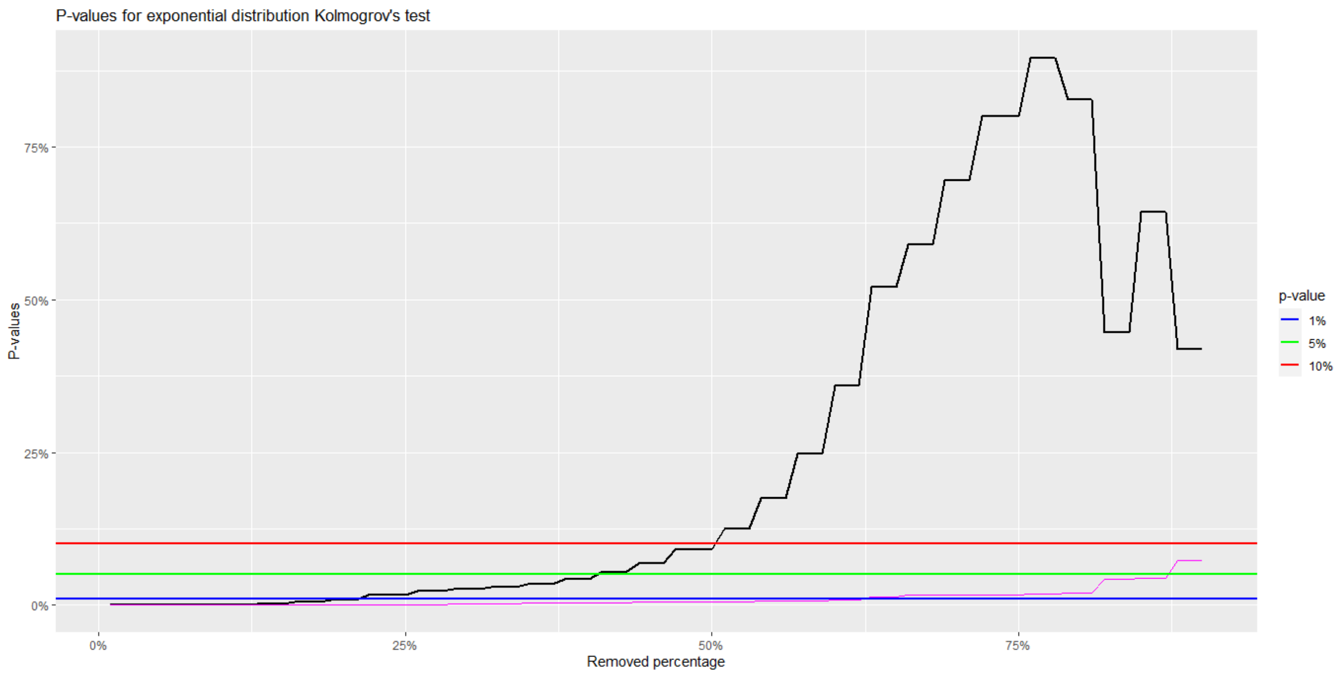

Figure 5-5.: Kolmogrov test for positive changes in Brazilian interest rate after removing several percentages of the smallest changes

5.3.2. Simulation study results

Now, we are going to carry out the simulation procedure when having reversion level fixed and β given by Brazilian interest rate jumps. To begin with, while performing the simulation, the need of having upper bounds of the parameters arose and empirically we chose to let the program to search for a solution under 1.5 times the real parameter of the simulation, except for ρ that allowed an upper bound 5 times greater than its real value without increasing significantly the error. This means that when we are going to estimate parameters is essential to understand how the dynamic contagion process works and according to our data, what parameters are most likely to represent our problem for being able to choose adequate upper bounds for our parameters. Moreover, proper knowledge of the financial problem that we are dealing with and its economic context becomes of high importance.

We obtained the mean relative error and deviation presented in Table 5-2. It is remarkable how, even when ρ has a window of more than three times bigger than the other parameters, its mean relative error is close to the ones from other parameters, but its deviation is affected by the upper bound used. From here, we can conclude that the estimation procedure presents acceptable errors and standard deviations and so, we can apply it to our data.

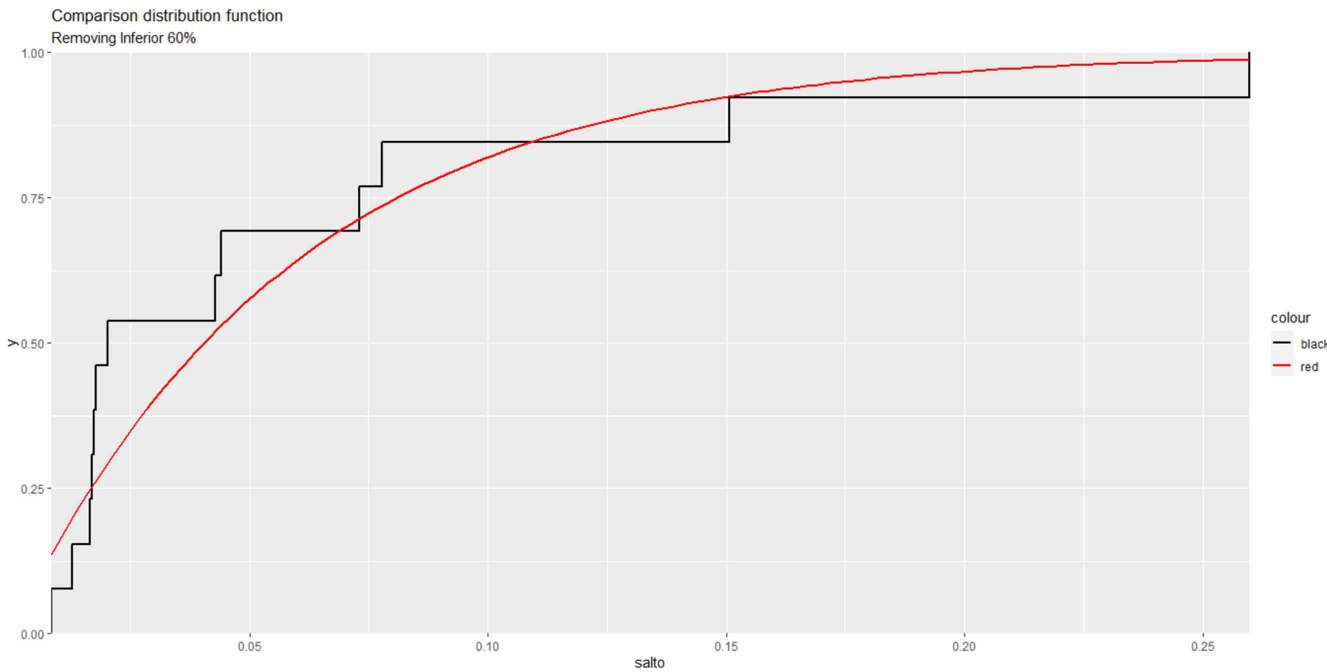

Figure 5-6.: Comparison between exponential distribution and sample distribution when removing the 60 % smallest changes in Brazilian interest rate.

Parameter	Mean Relative Error	Error Standard Deviation
δ	-0.036	0.359
β	-0.098	0.268
ρ	0.117	0.648

Table 5-2.: Mean relative error and standard deviation of estimated parameters vs simulation parameters.

5.3.3. Results for Colombian banking crisis

Finally, we implement the estimation procedure to Colombian data and obtain the results in Table **5-3**. As we can see, the estimated mean internal impact is $\frac{1}{\beta}$ =0.242, this agrees with the simulated representation of the process done in Section 5.2. Now, we also estimate that the arrival's rate of external shocks is 6.671 but this usual arrival is compensated by the fact that the mean impact (with α =17.14) is $\frac{1}{\alpha}$ =0.058.

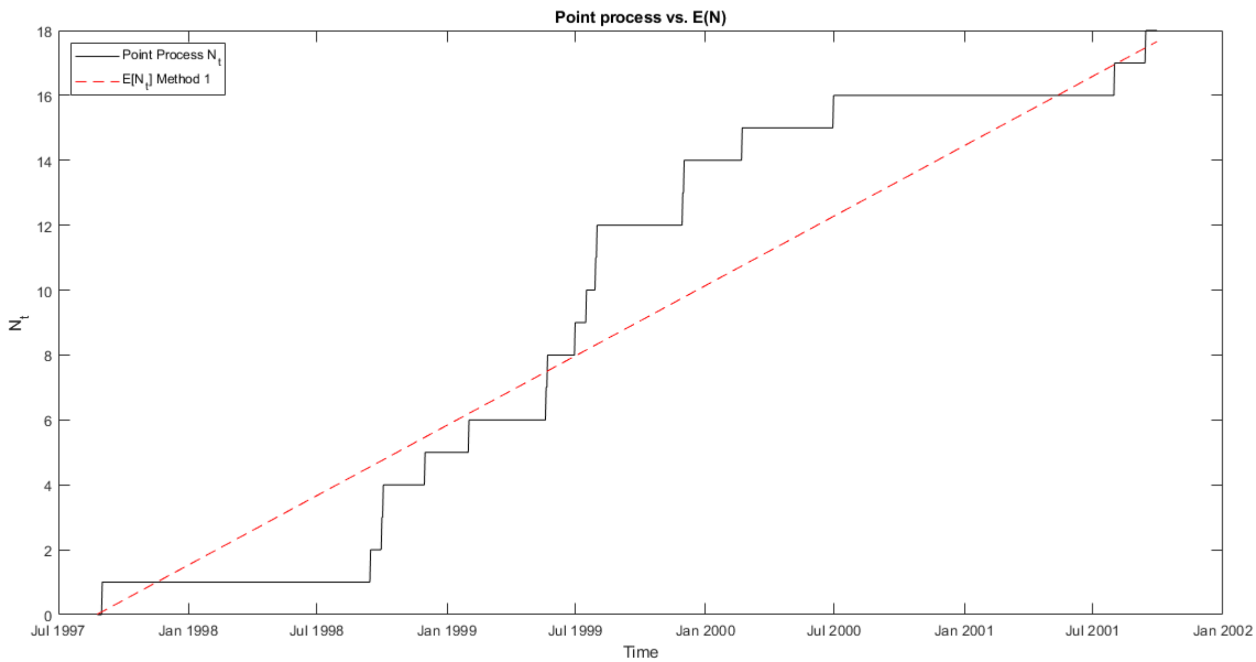

Figure 5-7.: Colombian banking defaults during the crisis of 1998 and the expected value of N_t estimated from the model.

Parameter	**Estimated parameters**	**Standard Deviation**	**Confidence Interval with two standard deviations**	
δ	1.427	0.476	0.475	2.380
β	4.132	0.029	4.075	4.190
ρ	6.671	0.084	6.503	6.839

Table 5-3.: Fit results for Colombian banking defaults.

The GMM algorithm designed by [Cao, 2020] also performs an over-identifying test where the null hypothesis is that moment conditions are sufficient to find the GMM estimator. This is crucial, because if we remember from Section 5.1, what allows us to use this method of estimation is over-identification of the parameters on the functions used to estimate them. In the case of our problem, the p-value of this test is 0,4618 then we accept the null hypothesis with a significance level of 10 %. Also, we can measure the mean squared error on the restrictions evaluated in the estimated parameters and we have that the MSE of the fitted model is $1{,}9275 \times 10^{-04}$.

Lastly, in Figure **5-7** it is possible to observe the estimated expected defaults versus the actual banking defaults in Colombia during this crisis and we can conclude that this model is able to adequately represent their tendency of growth.

5.3.4. Survival probability

If we consider that our underlying system is the Colombian economy and the events that might arrive are bank defaults or bankruptcies, we might be interested in estimating the probability that the economy survives the arrivals of defaults. In this manner, we could define an interesting Colombian bank credit risk measure just as the default probability, that is to say, the probability of Colombian economy non survival to arrivals of bank defaults. For this purpose we need to estimate the probability that a bank default causes the country to become economically insolvent. A simplification of this problem would be to consider that a default can result in insolvency with constant probability d and it is going to depend on the resistance that Colombia has to overcome banks bankruptcies. In this case, the survival probability conditional on the initial intensity λ_0 at time T is $P_s(T) = E[(1-d)^{N_T}|\lambda_0]$ and it can be computed from Theorem 6.

Table **5-4** shows the survival probability of the country to the crisis if it lasts 1 to 6 years, depending on supposed probabilities d; for instance, if $d = 20\,\%$ the survival probability after two years of crisis is of 45 %.

Time (Years)	**1**	**2**	**3**	**4**	**5**	**6**
$d = 2\,\%$	99.65 %	92.57 %	92.52 %	92.35 %	91.85 %	91.15 %
$d = 10\,\%$	98.30 %	66.63 %	66.08 %	66.12 %	65.74 %	64.97 %
$d = 20\,\%$	96.66 %	45.24 %	43.94 %	44.00 %	43.37 %	42.84 %
$d = 40\,\%$	93.55 %	21.97 %	20.82 %	19.90 %	18.24 %	20.88 %
$d = 100\,\%$	85.34 %	3.66 %	2.52 %	3.59 %	3.55 %	2.23 %

Table 5-4.: Survival probability under the model estimated in Subsection 5.3.3.

With this in mind, we are going to use the Moody's long term risk ratings along the period of interest to estimate the probability d, ratings which scale, as far as we know, have not changed since 1998. But first, in Table **5-5**, the global long-term rating scale used by Moody's is explained.

Table **5-5** comes along with the note below explaining the subdivision of each global rating scale:

[5]Information taken from Moody's web page:'https://www.moodys.com/sites/products/AboutMoodysRatingsAttachments/MoodysRatingSymbolsandDefinitions.pdf'

Moody's Global Rating Scales	
Global Long-Term Rating Scale	
Aaa	Obligations rated Aaa are judged to be of the highest quality, subject to the lowest level of credit risk.
Aa	Obligations rated Aa are judged to be of high quality and are subject to very low credit risk.
A	Obligations rated A are judged to be upper-medium grade and are subject to low credit risk.
Baa	Obligations rated Baa are judged to be medium-grade and subject to moderate credit risk and as such may possess certain speculative characteristics.
Ba	Obligations rated Ba are judged to be speculative and are subject to substantial credit risk.
B	Obligations rated B are considered speculative and are subject to high credit risk.
Caa	Obligations rated Caa are judged to be speculative of poor standing and are subject to very high credit risk
Ca	Obligations rated Ca are highly speculative and are likely in, or very near, default, with some prospect of recovery of principal and interest.
C	Obligations rated C are the lowest rated and are typically in default, with little prospect for recovery of principal or interest.

Table 5-5.: Moody's rating scales explanation [5].

Rating	Lineal Probability d Associated
Aaa1	1.0 %
Aa1	5.9 %
Aa2	10.8 %
Aa3	15.7 %
A1	20.6 %
A2	25.5 %
A3	30.4 %
Baa1	35.3 %
Baa2	40.2 %
Baa3	45.1 %
Ba1	50.0 %
Ba2	54.9 %
Ba3	59.8 %
B1	64.7 %
B2	69.6 %
B3	74.5 %
Caa1	79.4 %
Caa2	84.3 %
Caa3	89.2 %
Ca	94.1 %
C	99.0 %

Table 5-6.: Linear assignation of probability of insolvency for each Moody's rating.

Note[6]: "Moody's appends numerical modifiers 1, 2, and 3 to each generic rating classification from Aa through Caa. The modifier 1 indicates that the obligation ranks in the higher end of its generic rating category; the modifier 2 indicates a mid-range ranking; and the modifier 3 indicates a ranking in the lower end of that generic rating category. Additionally, a "(hyb)" indicator is appended to all ratings of hybrid securities issued by banks, insurers, finance companies, and securities firms."

According to these ratings, we decide to lineally assign a probability to each category (Table 5-6), being 1 % the lowest and 99 % the greatest, in order to be able to have an estimation of Colombian probability of insolvency when a bank goes bankrupt.

Now, if we analyze the Colombian debt rating from Table 5-7, in local or foreign currency,

[6]Information taken from Moody's web page: ‘`https://www.moodys.com/sites/products/AboutMoodysRatingsAttachments/MoodysRatingSymbolsandDefinitions.pdf`

Local or Foreign Currency	
Date	Rating for Colombia
23/11/2004	Ba2
11/08/1999	Ba2
19/06/1998	Baa2
19/09/1995	Baa3
04/08/1993	Ba1

Table 5-7.: Worst Moody's long term risk rating for Colombia in local and foreign currency.

during the banking crisis (that is to say, between 1998 and 1999), we can conclude that Colombian debt was considered speculative and substantially risky by that time, and even when for a short period the rating tried to improve, the after crisis period made the rating to go down again. By our assignment of probabilities, we estimate d for Colombia between 40 % and 55 %, and then, our model estimates a survival probability of 93,55 % for the first year and it drops substantially for the second year to 21,97 %.

We are aware that this is an oversimplification of a very complex subject, but the objective of this example was to show how this kind of models can give us useful information and allow us to take action in crisis situations, besides being an interesting theoretical contribution to the study of modeling in economical high stress situations, a field that has hardly been explored. For instance, the model estimated here urged us to take measures before the second year passes so as to avoid serious consequences in the country's economy.

5.4. Estimation without restricted parameters

Finally, we are going to considered the model as it was originally designed. In this section we will deal with the jump sizes of internal and external shocks as latent variables, and in this manner we are going to estimate the model without fixed parameters. As explained before, this changes the estimation procedure in the number of parameters and, as over-identification is mandatory to get to GMM estimation, it is also necessary to augment the number of functions that work as restrictions. In this case, we are going to take more lags of the autocorrelation function, then we will have as restrictions: the mean, the variance and four lags of autocorrelation what makes six restrictions to estimate five parameters.

5.4.1. Simulation study results

The simulation study, in this case, is going to determine how accurate is our estimation procedure for getting the five real values of a dynamic contagion process. As in the previous case, we were forced to determine upper bounds to the parameters in order to obtain better results, so one more time, it is necessary to have indications of where the appropriate parameters should be. In this section we will again use upper bounds 1.5 times greater than the real value of the simulation, except for ρ because it allowed a greater upper bound without its error growing importantly.

In Table 5-8 the mean relative error and standard deviation of estimation in simulated processes are presented. As we can see, the errors of the parameters that this model shares with the previous one are of similar order but the standard deviations were reduced in half. The standard deviation of ρ is a higher than that from the other parameters, but this is due to the amplitude of the interval of estimation. Despite this, we conclude that this method improves de standard deviation of estimation and is acceptable, so we can proceed to fit to our data.

Parameter	**Mean Relative Error**	**Error Standard Deviation**
δ	-0.067	0.168
β	-0.051	0.133
ρ	-0.130	0.389
α	0.052	0.130
a	0.041	0.155

Table 5-8.: Mean relative error and standard deviation of estimated parameters vs simulation parameters.

5.4.2. Results for Colombian banking crisis

After implementing our estimation procedure to our data we obtained the estimated parameters shown in Table 5-9. The first big difference among these results and the model of the previous section is that δ is three times higher here which means that this model stabilizes faster than the other. The rate of arrival of external shocks is smaller than previous one but it remains in the same order, and the parameters α are very similar and that means that the impact of external shocks is the same in both models. In this model, the estimated mean external impact is $\frac{1}{\alpha} = 0{,}059$ versus the previous 0,058.

The internal impact when default happens is also estimated similar for both models. In this model, the estimated mean internal impact is $\frac{1}{\beta} = 0{,}250$ versus the previous 0,242. Finally, the reversion level is greater than the one we formerly assumed which could mean that we did not take into account enough factors to estimate it. The big surprise is the standard deviations of this model because they are much more smaller than before which makes us more confident in the precision of estimation.

Regarding the over-identifying test, the p-value is 0,4356 which means that once again, we accept the null hypothesis and conclude that the chosen function for estimating are appropriate. Lastly, in Figure 5-8 we can see the expected number of defaults in the estimated process versus the defaults seen during the crisis, this graph is identical to the one of the previous model and also represents the tendency of defaults growth.

Parameter	**Estimated parameters**	**Standard Deviation**	**Confidence Interval with two standard deviations**	
δ	4.002	2.60×10^{-03}	3.998	4.008
β	3.996	7.00×10^{-04}	3.995	3.998
ρ	3.993	1.90×10^{-03}	3.990	3.997
α	17.001	4.00×10^{-04}	17.000	17.002
a	0.260	1.29×10^{-01}	0.002	0.517

Table 5-9.: Fit results for Colombian banking defaults.

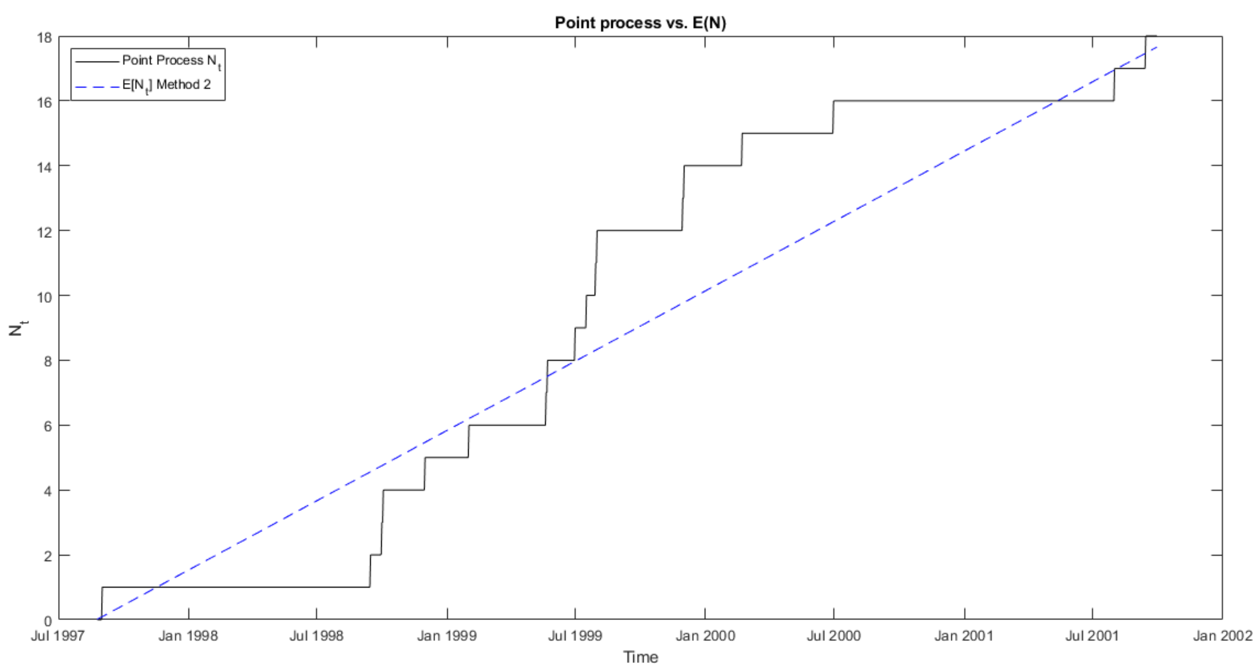

Figure 5-8.: Colombian banking defaults during the crisis of 1998 and the expected value of N_t estimated from the model.

Time (Years)	**1**	**2**	**3**	**4**	**5**	**6**
$d = 2\,\%$	99.47 %	97.70 %	97.77 %	97.63 %	97.33 %	97.77 %
$d = 10\,\%$	97.39 %	86.36 %	88.16 %	87.91 %	88.09 %	88.15 %
$d = 20\,\%$	94.88 %	75.95 %	75.52 %	77.09 %	76.93 %	74.29 %
$d = 40\,\%$	90.10 %	58.87 %	57.41 %	58.77 %	58.84 %	53.27 %
$d = 100\,\%$	77.62 %	27.25 %	27.00 %	24.71 %	21.46 %	27.21 %

Table 5-10.: Survival probability under the model estimated in Subsection 5.4.2.

5.4.3. Survival probability

Similarly to what we did in the past section, with our model we estimate the probability of surviving the crisis while years pass, depending on the country's level of resistance. The probabilities estimated by this model are in Table **5-10**, and when we compare them to the ones from the first model, we notice that these probabilities are, in general, greater than the past ones except when the resistance level of the system is null.

We could say that this model is more realistic since when d is less than 50 % is not alarmist but when the resistance level is low (meaning a bid probability of default d), it reacts and shows huge fall in the survival probability. This model's probabilities also have inflection points in the second year, what indicates that it is necessary to intervene the crisis before the second year to avoid irreparable damage and gain survival probability.

Again, in Colombian case, according to Table **5-7**, the probability d is around 40 % and 55 % then the mean survival probability was 62 % with tendency to 53 %.

6. Conclusions

The innovation of this work was to apply to the Colombian banking crisis, a stochastic process with intensity modeling including the stylized economical fact of reversion level, impact of external arrival of events and impact of self-occurrence of events in the studied system. Additionally, those impacts have exponential decay, meaning that their effect is not maintained but, as time passes, the impact on the underlying system is decreasing which is consistent with what happens in crisis periods.

In order to approach our problem, we applied tho methods. The first method worked as indication of the magnitude of the parameters and the second method allowed us to decrease the standard deviation of the estimated parameters. In this particular case, the second method also showed us that choosing the Brazilian series seemed accurate and it gets to represent the external impact over Colombian economy during that period of time.

Regarding Colombian fits, we saw that, in general, both models represent what happened during the crisis. Nevertheless, it was evident when fitting the second method that the reversion level was underestimated in the first one and that there exist other factors needed to determine its magnitude, apart from the number of banks that defaulted after the crisis. Moreover, underestimating this parameter can lead to overestimate the others since the optimization method tries to compensate the effect of one parameter with the others.

On the other hand, in problems or situations like the one addressed in the present work, where the number of events or defaults are not substantial, the parameters α and β are greater than one because there is an inverse relation between the exponential parameter of the jumps size and the effective number of events.

In addition, both methods reflect that the choosing of functions for the estimation procedure was accurate since in both of them the over-identifying hypothesis was accepted, and that means that there are no problem of identifiability in the estimation.

We want to emphasize that this is a first step for addressing this kind of problem with processes of this type, problem involving crisis situations, high stress in the system and with dependence on occurrence of events. Additionally, in the process of developing this work we got the Proposition 1 that allowed us to implement the generalized method of moments to estimate the model properly.

Furthermore, this work had a great effort in historic revision because this is the only crisis of this magnitude that has happened in Colombia and it was over 20 years ago. This review involved mostly search for specialized literature and press articles that exposed the scene of the crisis and how each bank that defaulted was affected by it.

Finally, we would like to leave the question open about the choice of upper bounds, necessity that even when it was not a theoretical requirement, we saw it was needed in the computational simulation. We feel that it is a crucial matter in the estimation procedure that can change completely the results and should be studied profoundly in order to give higher credibility to this contagion modeling procedure.

A. Appendix: Proof of Proposition 1

Let us recall that

- $\gamma = \delta - \mu_{1_G}$,
- $\Lambda = \frac{\mu_{1_H}\rho + a\delta}{\gamma}$,
- $\kappa = \frac{\mu_{2_G}\Lambda + \mu_{2_H}\rho}{\gamma^2} + \frac{2\mu_{1_G}\Lambda}{\gamma}$,
- $\bar{k} = \frac{\mu_{1_G}\Lambda}{\gamma}\left(1 + \frac{1}{2\gamma}\right) + \frac{\mu_{2_H}\rho}{2\gamma^2}$.

Now, we are going to proof each part of Preposition 1:

1. $E[N_{t+\tau} - N_t] \xrightarrow[t\to\infty]{} \Lambda\tau$:

   From Theorem 4, we can see that

$$E[N_{t+\tau} - N_t] = \Lambda\tau, \tag{A-1}$$

   which does not depend on t. And so, we have the result.

2. $Var[N_{t+\tau} - N_t] \xrightarrow[t\to\infty]{} \Lambda\tau + \kappa\left(\tau + \frac{e^{-\gamma\tau} - 1}{\gamma}\right)$:

   We begin by calculating the second moment of the process $N_{t+\tau} - N_t$. Let $t_1 < t_2$, conditioning by $\mathcal{F}_{t_1}$ we have

$$\begin{aligned} I &= E\left[(N_{t_2} - N_{t_1})^2\right], \\ &= E\left[E_{t_1}(N_{t_2}^2) - 2N_{t_1}E_{t_1}(N_{t_2}) + N_{t_1}^2\right]. \end{aligned} \tag{A-2}$$

   From the development made by [Dassios and Zhao, 2011] in the proof of Theorem 5 with the infinitesimal generator of the process given by (3-2) and function $f(\lambda, n, t) = n^2$, we have that

$$E_{t_1}[N_{t_2}^2] = N_{t_1}^2 + 2\int_{t_1}^{t_2} E_{t_1}[\lambda_u N_u]du + \int_{t_1}^{t_2} E[\lambda_u]du. \tag{A-3}$$

Replacing (A-3) in (A-2),

$$
\begin{aligned}
I &= E\left[2N_{t_1}^2 + 2\int_{t_1}^{t_2} E_{t_1}(\lambda_u N_u)du + \int_{t_1}^{t_2} E_{t_1}(\lambda_u)du - 2N_{t_1}E_{t_1}(N_{t_2})\right] \\
&= 2E[N_{t_1}^2] + 2\int_{t_1}^{t_2} E[\lambda_u N_u]du + \int_{t_1}^{t_2} E[\lambda_u]du - 2E\left[N_{t_1}(N_{t_1} + \int_{t_1}^{t_2} E_{t_1}(\lambda_u)du)\right] \\
&= 2\int_{t_1}^{t_2} E[\lambda_u N_u]du + \int_{t_1}^{t_2} E[\lambda_u]du - 2E\left[N_{t_1}\int_{t_1}^{t_2} E_{t_1}(\lambda_u)du\right],
\end{aligned}
\tag{A-4}
$$

where in the second line we used that $E_{t_1}(N_{t_2}) = N_{t_1} + \int_{t_1}^{t_2} E_{t_1}(\lambda_u)du$, again, from the martingale property of the infinitesimal generator, now with function $f(\lambda, n, t) = n$. Later, for $f(\lambda, n, t) = \lambda n$ and $f(\lambda, n, t) = \lambda$, respectively, we obtain

$$
\frac{dE[\lambda_t N_t]}{dt} = -\gamma E[\lambda_t N_t] + \gamma\Lambda E[N_t] + E[\lambda_t^2] + \mu_{1_G}E[\lambda_t],
$$

$$
\frac{dE_0[\lambda_t]}{dt} = -\gamma E_0[\lambda_t] + \gamma\Lambda,
$$

and using that for an EDO of the form

$$
g'(t) = ag(t) + b(t) \qquad \text{with } g(0) = g_0,
$$

the solution would be

$$
g(t) = g_0 e^{at} + \int_0^t b(s)e^{a(t-s)}ds,
$$

then, the first integral in (A-4) can be computed as

$$
\begin{aligned}
I_1 &= \int_{t_1}^{t_2} E\left[\lambda_u N_u\right] du \\
&= \int_{t_1}^{t_2} e^{-\gamma(u-t_1)} E\left[\lambda_{t_1} N_{t_1}\right] du + \int_{t_1}^{t_2}\int_{t_1}^{u} e^{-\gamma(u-s)}\left\{\gamma\Lambda E[N_s] + E[\lambda_s^2] + \mu_{1_G}E[\lambda_s]\right\} ds\, du,
\end{aligned}
$$

and the third integral in (A-4) is computed as

$$\begin{aligned} I_3 &= E\left[N_{t_1}\int_{t_1}^{t_2} E_{t_1}(\lambda_u)du\right] \\ &= E\left[N_{t_1}\left(\int_{t_1}^{t_2} e^{-\gamma(u-t_1)}\lambda_{t_1}du + \int_{t_1}^{t_2}\int_{t_1}^{u}\gamma\Lambda e^{-\gamma(u-r)}dr\ du\right)\right] \\ &= E[N_{t_1}\lambda_{t_1}]\int_{t_1}^{t_2} e^{-\gamma(u-t_1)}du + \gamma\Lambda E[N_{t_1}]\int_{t_1}^{t_2}\int_{t_1}^{u} e^{-\gamma(u-r)}dr\ du. \end{aligned}$$

We have that $E[N_s] = E[N_{t_1}] + \int_{t_1}^{s} E[\lambda_r]dr$, so if we substitute this in I_1, join I_1 and I_3 into I from (A-4) and simplify, we obtain:

$$I = \int_{t_1}^{t_2} E[\lambda_u]du + 2\int_{t_1}^{t_2}\int_{t_1}^{u} e^{-\gamma(u-s)}\left\{\gamma\Lambda\int_{t_1}^{s} E[\lambda_r]dr + E[\lambda_s^2] + \mu_{1_G}E[\lambda_s]\right\}ds\ du.$$

Then,

$$E\left[(N_{t+\tau}-N_t)^2\right] = \int_t^{t+\tau} E[\lambda_u]du + 2\int_t^{t+\tau}\int_t^{u} e^{-\gamma(u-s)}\left\{\gamma\Lambda\int_t^{s} E[\lambda_r]dr + E[\lambda_s^2] + \mu_{1_G}E[\lambda_s]\right\}ds\ du.$$

We also know, from Corollary 1, that $E[\lambda_t] = \Lambda$ and $E[\lambda_t^2] =: \Lambda_2$. Therefore, assuming that $\mu_{1_G} < \delta$, under the stationary distribution of the process (taking the limit when $t \to \infty$) we get

$$\begin{aligned} \lim_{t\to\infty} E\left[(N_{t+\tau}-N_t)^2\right] = \lim_{t\to\infty}\Bigg\{&\Lambda\tau + 2\int_t^{t+\tau}\int_t^{u} e^{-\gamma(u-s)}\gamma\Lambda^2(s-t)ds\ du \\ &+2\int_t^{t+\tau}\int_t^{u} e^{-\gamma(u-s)}(\Lambda_2 + \mu_{1_G}\Lambda)ds\ du\Bigg\}, \end{aligned} \quad \text{(A-5)}$$

Now, each integral (without the constants) solves to:

$$\begin{aligned} \int_t^{t+\tau}\int_t^{u} e^{-\gamma(u-s)}(s-t)ds\ du &= \frac{\tau^2}{2\gamma} - \frac{\tau}{\gamma^2} - \frac{e^{-\gamma\tau}-1}{\gamma^3}, \\ \int_t^{t+\tau}\int_t^{u} e^{-\gamma(u-s)}ds\ du &= \frac{\tau}{\gamma} + \frac{e^{-\gamma\tau}-1}{\gamma^2}. \end{aligned}$$

Hence, replacing them in (A-5)

$$\lim_{t\to\infty} E\left[(N_{t+\tau}-N_t)^2\right] = \Lambda\tau + 2\gamma\Lambda^2\left\{\frac{\tau^2}{2\gamma} - \frac{\tau}{\gamma^2} - \frac{e^{-\gamma\tau}-1}{\gamma^3}\right\} + 2(\Lambda_2 + \mu_{1_G}\Lambda)\left\{\frac{\tau}{\gamma} + \frac{e^{-\gamma\tau}-1}{\gamma^2}\right\}. \quad \text{(A-6)}$$

For computing

$$\lim_{t\to\infty} Var[N_{t+\tau}-N_t] = \lim_{t\to\infty}\left\{E\left[(N_{t+\tau}-N_t)^2\right]-E[N_{t+\tau}N_t]^2\right\},$$

we replace here: (A-1), (A-6) and $\Lambda_2=\Lambda^2+\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{2\gamma}$,

$$\begin{aligned}
\lim_{t\to\infty} Var[N_{t+\tau}-N_t] =&\Lambda\tau+2\gamma\Lambda^2\left(\frac{\tau^2}{2\gamma}-\frac{\tau}{\gamma^2}-\frac{e^{-\gamma\tau}-1}{\gamma^3}\right)\\
&+2\left(\Lambda^2+\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{2\gamma}+\mu_{1_G}\Lambda\right)\left(\frac{\tau}{\gamma}+\frac{e^{-\gamma\tau}-1}{\gamma^2}\right)\\
&-\Lambda^2\tau^2\\
=&\Lambda\tau+2\gamma\Lambda^2\left(-\frac{\tau}{\gamma^2}-\frac{e^{-\gamma\tau}-1}{\gamma^3}\right)\\
&+\left(2\Lambda^2+\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)\left(\frac{\tau}{\gamma}+\frac{e^{-\gamma\tau}-1}{\gamma^2}\right)\\
=&\Lambda\tau-\frac{2\Lambda^2\tau}{\gamma}-\frac{2\Lambda^2(e^{-\gamma\tau}-1)}{\gamma^2}+\frac{2\Lambda^2\tau}{\gamma}+\frac{\tau}{\gamma}\left(\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)\\
&+\frac{e^{-\gamma\tau}-1}{\gamma^2}\left(2\Lambda^2+\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)\\
=&\Lambda\tau+\frac{\tau}{\gamma}\left(\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)+\frac{e^{-\gamma\tau}-1}{\gamma^2}\left(\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)\\
=&\Lambda\tau+\frac{1}{\gamma}\left(\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)\left(\tau+\frac{e^{-\gamma\tau}-1}{\gamma}\right)
\end{aligned}$$

Then, defining $\kappa=\frac{1}{\gamma}\left(\frac{\mu_{2_G}\Lambda+\mu_{2_H}\rho}{\gamma}+2\mu_{1_G}\Lambda\right)$ we obtain the result.

3. $Cov[N_{t+\tau}-N_t:N_{t+2\tau+\epsilon}-N_{t+\tau+\epsilon}]\xrightarrow[t\to\infty]{} e^{-\gamma\epsilon}\left(\frac{1-e^{-\gamma\tau}}{\gamma}\right)\left[2\bar{k}(1-e^{-\gamma\tau}-\Lambda^2\tau)\right]$:

Let $t<t_1<t_2<t_3$, by properly conditioning on $\mathcal{F}_{t_2}$, $\mathcal{F}_{t_1}$ and $\mathcal{F}_t$ we get

$$E[(N_{t_1}-N_t)(N_{t_3}-N_{t_2})]=E\left\{E_t\left[E_{t_1}\left(E_{t_2}[(N_{t_1}-N_t)(N_{t_3}-N_{t_2})]\right)\right]\right\}.$$

We are going to compute inside out this expected values. First of all, using the martingale property of the infinitesimal generator with $f(\lambda,n,t)=n$ we have,

$$\begin{aligned}
E_{t_2}((N_{t_1}-N_t)(N_{t_3}-N_{t_2}))&=(N_{t_1}-N_t)E_{t_2}(N_{t_3}-N_{t_2})\\
&=(N_{t_1}-N_t)\int_{t_2}^{t_3}E_{t_2}(\lambda_r)dr.
\end{aligned}$$

Applying Theorem 3, we know that

$$\begin{aligned}\int_{t_2}^{t_3} E_{t_2}(\lambda_r)dr &= \int_{t_2}^{t_3} \left[\Lambda + (\lambda_{t_2} - \Lambda)e^{-\gamma(r-t_2)}\right] dr \\ &= \Lambda\Delta_2 + (\lambda_{t_2} - \Lambda)\left(\frac{1-e^{-\gamma\Delta_2}}{\gamma}\right),\end{aligned}$$

where $\Delta_2 = t_3 - t_2$. Let us define $A = \Lambda\Delta_2 - \Lambda\left(\frac{1-e^{-\gamma\Delta_2}}{\gamma}\right)$ and $B = \frac{1-e^{-\gamma\Delta_2}}{\gamma}$, then the next expected value will be

$$E_{t_1}[(N_{t_1} - N_t)(A + \lambda_{t_2}B)] = (N_{t_1} - N_t)\{A + BE_{t_1}(\lambda_{t_2})\}.$$

Again, applying Theorem 3 we have that

$$\begin{aligned}E_{t_1}(\lambda_{t_2}) &= \Lambda + (\lambda_{t_1} - \Lambda)e^{-\gamma\Delta_1} \\ &= \Lambda(1 - e^{-\gamma\Delta_1}) + \lambda_{t_1}e^{-\gamma\Delta_1},\end{aligned}$$

where $\Delta_1 = t_2 - t_1$. If we define $C = \Lambda(1 - e^{-\gamma\Delta_1})$, the next expected value to compute is

$$\begin{aligned}D &= E_t[(N_{t_1} - N_t)\{A + B(C + \lambda_{t_1}e^{-\gamma\Delta_1})\}] \\ &= E_t[N_{t_1}](A + BC) - N_t(A + BC) + Be^{-\gamma\Delta_1}E_t[N_{t_1}\lambda_{t_1}] - Be^{-\gamma\Delta_1}N_tE_t[\lambda_{t_1}].\end{aligned} \tag{A-7}$$

Applying one last time Theorem 3, we obtain

$$E_t[\lambda_{t_1}] = \Lambda + (\lambda_t - \Lambda)e^{-\gamma\Delta_0}, \tag{A-8}$$

where $\Delta_0 = t_1 - t$. Also, thanks to the martingale property of the infinitesimal generator with $f(\lambda, n, t) = n$, we know that

$$\begin{aligned}E_t[N_{t_1} - N_t] &= \int_t^{t_1} E[\lambda_s]ds, \\ E_t[N_t] &= N_t + \Lambda\Delta_0.\end{aligned} \tag{A-9}$$

Moreover, we need Lemma 1 for computing the following

$$E_t[N_{t_1}\lambda_{t_1}] = \lambda_tN_t + \bar{k}(1 - e^{-\gamma\Delta_0}) + \Lambda^2\Delta_0. \tag{A-10}$$

Replacing (A-8), (A-9) and (A-10) in (A-7) we get the last expected value

$$\begin{aligned}E[D] = E\,\big[&(N_t + \Lambda\Delta_0)(A + BC) - N_t(A + BC) + Be^{-\gamma\Delta_1}\{\lambda_tN_t + \bar{k}(1 - e^{-\gamma\Delta_0}) + \Lambda^2\Delta_0\} \\ &- Be^{-\gamma\Delta_1}N_t\{\Lambda + e^{-\gamma\Delta_0}(\lambda_t - \Lambda)\}\big] \\ = \Lambda\Delta_0&(A + BC) + Be^{-\gamma\Delta_1}(1 - e^{-\gamma\Delta_0})\{E[\lambda_tN_t] + \bar{k} - \Lambda E[N_t]\} + Be^{-\gamma\Delta_1}\Lambda^2\Delta_0.\end{aligned}$$

$$\tag{A-11}$$

For Theorem 4 and Lemma 1, we know that

$$\begin{aligned} E[\lambda_t N_t] + \bar{k} - \Lambda E[N_t] &= \bar{k}(1 - e^{-\gamma t}) + \Lambda^2 t + \bar{k} - \Lambda^2 t \\ &= \bar{k}(2 - e^{-\gamma t}). \end{aligned} \tag{A-12}$$

On the other hand,

$$\begin{aligned} A + BC &= \Lambda \Delta_2 - \Lambda \left(\frac{1 - e^{-\gamma \Delta_2}}{\gamma} \right) + \left(\frac{1 - e^{-\gamma \Delta_2}}{\gamma} \right) \Lambda (1 - e^{-\gamma \Delta_1}) \\ &= \Lambda \left[\Delta_2 - e^{-\gamma \Delta_1} \left(\frac{1 - e^{-\gamma \Delta_2}}{\gamma} \right) \right]. \end{aligned} \tag{A-13}$$

Substituting (A-12) and (A-13) in (A-11), besides associating common factors we get

$$E[D] = \Lambda^2 \Delta_0 \Delta_2 + e^{-\gamma \Delta_1} \left(\frac{1 - e^{-\gamma \Delta_2}}{\gamma} \right) [\bar{k}(1 - e^{-\gamma \Delta_0})(2 - e^{-\gamma t}) - \Lambda^2 \Delta_0].$$

Now, if we consider that $t_1 = t + \tau$, $t_2 = t + \tau + \epsilon$ and $t_2 = t + 2\tau + \epsilon$ then $\Delta_0 = t_1 - t = \tau$, $\Delta_1 = t_2 - t_1 = \epsilon$, $\Delta_2 = t_3 - t_2 = \tau$ and

$$\begin{aligned} E[(N_{t+\tau} - N_t)(N_{t+2\tau+\epsilon} - N_{t+\tau+\epsilon})] &= \Lambda^2 \tau^2 + e^{-\gamma \epsilon} \left(\frac{1 - e^{-\gamma \tau}}{\gamma} \right) [\bar{k}(1 - e^{-\gamma \tau})(2 - e^{-\gamma t}) - \Lambda^2 \tau] \\ &\xrightarrow[t \to \infty]{} \Lambda^2 \tau^2 + e^{-\gamma \epsilon} \left(\frac{1 - e^{-\gamma \tau}}{\gamma} \right) [2\bar{k}(1 - e^{-\gamma \tau}) - \Lambda^2 \tau]. \end{aligned}$$

Finally, since $E[N_{t+\tau} - N_t] = E[N_{t+2\tau+\epsilon} - N_{t+\tau+\epsilon}] = \Lambda \tau$, then

$$\lim_{t \to \infty} Cov(N_{t+\tau} - N_t, N_{t+2\tau+\epsilon} - N_{t+\tau+\epsilon}) = e^{-\gamma \epsilon} \left(\frac{1 - e^{-\gamma \tau}}{\gamma} \right) [2\bar{k}(1 - e^{-\gamma \tau}) - \Lambda^2 \tau].$$

4. $Acf(\tau, \epsilon) \xrightarrow[t \to \infty]{} \frac{e^{-\gamma \epsilon}(1 - e^{-\gamma \tau})[2\bar{k}(1 - e^{-\gamma \tau}) - \Lambda^2 \tau]}{\gamma \Lambda \tau + \kappa(\tau \gamma + e^{-\gamma \tau} - 1)}$:

It is given that

$$\begin{aligned} \lim_{t \to \infty} Acf(\tau, \epsilon) &= \lim_{t \to \infty} \frac{Cov[(N_{t+\tau} - N_t)(N_{t+2\tau+\epsilon} - N_{t+\tau+\epsilon})]}{\sqrt{Var(N_{t+\tau} - N_t)} \sqrt{Var(N_{t+2\tau+\epsilon} - N_{t+\tau+\epsilon})}} \\ &= \frac{e^{-\gamma \epsilon}(1 - e^{-\gamma \tau})[2\bar{k}(1 - e^{-\gamma \tau}) - \Lambda^2 \tau]}{\gamma \Lambda \tau + \kappa(\tau \gamma + e^{-\gamma \tau} - 1)}. \end{aligned}$$

B. Appendix: Newspaper articles links and their descriptions

In this Appendix we are going to list the newspaper articles used to determine when the banks or SHCs were merged, liquidated or capitalized and if when they did, they were in crisis. This last part is important because we are aware that there are entities that merge just as part of business development but not because they were in crisis, and we are interested in the ones that disappeared because they were badly and irretrievably affected by the crisis.

B.1. Savings and Housing Corporations

1. - "In the 1999's, the Grupo AVAL carried out the merging of the SHCs Las Villas and Ahorramas": https://www.dinero.com/caratula/edicion-impresa/articulo/grupo-aval/12167.

2. "Merging of Concasa and Bancafé": https://www.eltiempo.com/archivo/documento/MAM-645646.

3. "Granahorrar was the first financial entity that the Government had to intervened during the crisis of the lately 1990's. A lot more followed, but its case was emblematic since it survived": https://www.dinero.com/edicion-impresa/caratula/articulo/la-crisis-financiera-90/182419.

B.2. Banks

1. "The Banco del Pacífico de Colombia, of the Grupo Pacífico del Ecuador property one of the biggest of that country and the Andino, also from acuatorian capital, were intervened yesterday and immediately the Superintendencia Bancaria ordered their liquidation": https://www.eltiempo.com/archivo/documento/MAM-875288.

2. "In six months Bancoop's end":https://www.eltiempo.com/archivo/documento/MAM-820673.

3. "Government decided slow death for the bank Uconal":https://www.eltiempo.com/archivo/documento/MAM-862645.

4. “The Superintendencia Bancaria approved the transfer of part of Bank of America Colombia’s assets and liabilities to Banco de Bogotá”: https://www.eltiempo.com/archivo/documento/MAM-650318.

5. “The Banco Central Hipotecario, that during its 68 years of life was the leading entity of housing credits in Colombia, will suffer this week its coup de grace”:https://www.eltiempo.com/archivo/documento/MAM-1218822.

6. “Caja Agraria: farewell to another nation’s insignia”: https://www.portafolio.co/economia/finanzas/caja-agraria-adios-insignia-nacional-230350.

7. “Banco Nacional del Comercio (BNC) and Banco Ganadero’s merging has near 1.700 merchants suffering, they bought these entities’ shares three years ago and don’t know who is gong to answer for them”: https://www.eltiempo.com/archivo/documento/MAM-818258.

8. “Government capitalized the banks Colpatria and Unión Colombiano: In Colpatria’s case, it is the biggest capitalization carried in the financial system”: https://www.eltiempo.com/archivo/documento/MAM-894794.

9. “End of Banestado”: https://www.eltiempo.com/archivo/documento/MAM-1272833.

10. “The Mercantil bank decided to close offices in the whole country and liquidate 70 per cent of their plant personnel”: https://www.eltiempo.com/archivo/documento/MAM-813908.

11. “Selfin bank is ordered to be liquidated”: https://www.eltiempo.com/archivo/documento/MAM-913967

12. “Interbanco and Aliadas bank integration phase was completed”: https://www.eltiempo.com/archivo/documento/MAM-452963

13. “Government liquidated Cafetero bank and created Granbanco-Bancafé”: https://www.eltiempo.com/archivo/documento/MAM-1630361

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