A view toward the geometry of normalizing sequences in finitely semi-graded algebras

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May, 2020

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Thesis Work to Obtain the Degree of
Master of Science in Mathematics

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Bogotá, D.C.
May, 2020

Title in English

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Título en español

Una mirada hacia la geometría de las sucesiones normalizadoras en álgebras finitamente semigraduadas

Abstract: In this work, we see the behavior of normalizing sequence in semi-graded algebras. First, we define regular sequence and the Koszul complex in commutative case. Using the variety of maximal ideals a geometry is reached between the sub-varieties and regular sequences in graded algebras. Then, we turn to non-commutative case. The normalizing sequence is defined and we see that it appears related to the height of an ideal. Then we see that the normalizing sequence appears in graded skew Clifford algebras. Also, we define the right point modules and the zero locus. With these definitions, we consider the relation between the zero locus with the normalizing sequences in graded algebras. We present the context of finitely semi-graded algebras and geometry of normalizing sequence in this case. Thus, we define the concept of finitely semi-graded ring and finitely semi-graded algebra. We note that skew PBW extension are finitely semi-graded rings. Also, we see the appearance of normalizing sequence in a type of semi-graded algebra. For this, we consider the theory of the universal enveloping algebra of a Lie algebra. We define a completely solvable algebra, a ring of fractions and we arrive at some properties in which the normalizing sequence appears. Finally, we see how point modules can be parameterized with a certain scheme in the case of graded algebras. With the help of this, we get to the main purpose of this work, which is to see the geometry of the normalizing sequences in certain finitely semi-graded algebras. Here, we find a geometry of the normalizing sequences in graded skew PBW extensions (Theorem 2.3.33). Then, a way is left to continue investigating the geometry of normalizing sequences in more general semi-graded objects.

Resumen: En este trabajo vemos el comportamiento de la sucesión normalizadora en álgebras semi-graduadas. Primero, definimos la sucesión regular y el complejo de Koszul en el caso conmutativo. Usando la variedad de ideales máximos, llegamos a una geometría entre las subvariedades y las sucesiones regulares en álgebras graduadas. Luego, pasamos al caso no conmutativo. Definimos la sucesión normalizadora y vemos que está relacionada con la altura de un ideal. Luego, vemos que la sucesión normalizadora aparece en álgebras de Clifford torcidas graduadas. Además, definimos los módulos punto derechos y el *zero locus*. Con estas definiciones, vemos la relación entre el *zero locus* con las sucesiones normalizadoras en álgebras graduadas. Presentamos el contexto de álgebras finitamente semi-graduadas y la geometría de la sucesión normalizadora en este caso. Así, definimos el concepto de anillo finitamente semi-graduado y álgebra finitamente semi-graduada. Vemos que las extensiones PBW torcidas son anillos finitamente semi-graduados. También, vemos la aparición de la sucesión normalizadoras en un tipo de álgebra semi-graduada. Para esto, consideramos la teoría del álgebra envolvente universal de un álgebra de Lie. Definimos un álgebra completamente solucionable, un anillo de fracciones y llegamos a algunas propiedades en las que aparece la sucesión normalizadora. Finalmente, vemos

cómo los módulos de puntos pueden parametrizarse con un cierto esquema en el caso de álgebras graduadas. Con la ayuda de esto, llegamos al objetivo principal de este trabajo, que es ver la geometría de las sucesiones normalizadoras en ciertas álgebras finitamente semi-graduadas. Aquí, encontramos una geometría de las sucesiones normalizadoras en las extensiones PBW torcidas graduadas (Teorema 2.3.33). Luego, se deja una vía para continuar investigando las geometría de las sucesiones normalizadoras en objetos semi-graduados más generales.

Keywords: Normalizing sequence, semi-graded algebra, regular sequence, sheaf, scheme, point module.

Palabras clave: Sucesión normalizadora, álgebra semigraduada, sucesión regular, haz, esquema, módulo punto.

Dedication

Dedicated to my mom, Viviana, who has always made me strong.

Acknowledgments

This work has been very important for me, because it has made me enter into very interesting and wide-ranging topics. This has been a very satisfying trip for me. For this reason, I would like to give my special appreciation to my thesis director, Armando Reyes, who has continued to teach me a lot about the topics we are discussing in this work. Besides being my teacher, he is also a dear friend to me.

My grandmother Odilia, my grandfather Samuel, my mother Viviana and my sister Nicole have been absolutely fundamental in this work. They may not have helped me with issues related to this work, but they have helped me to strengthen myself, to move forward when I once wanted to give up, to be a person with correct ethical values.

My friends have also been a fundamental piece in my career. Thanks to Jorge, I have been with a student environment of the best, besides being like my brother. Thanks to Camila, I have been able to count on a childhood friend for all situations of everyday life, besides being like my sister. Thanks to Camilo and Ximena, I have been able to understand the value of perseverance and continuity. Thanks to Ricardo, I have been able to understand the value of responsibility and compliance.

After so many years, my girlfriend Daniela has taught me many things about life, love, friendship, among other things. With her I have learned not to give up, to learn what self-esteem and mutual effort means. Without her constant presence in my life, I doubt I could have reached where I am now.

Finally, there have been professors who have marked me in my career as a mathematician. Therefore, I would like to emphasize Professor Oswaldo Lezama, who thanks to his way of teaching algebra, managed to awaken in me (and I know that in many other people) the love for mathematics.

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Introduction

The main purpose of this work is to see the geometry behind normalizing sequences in finitely semi-graded algebras. In other words, we want to generalize the geometry found in graded case, as can be seen in [CV14]. In the commutative case, the normalizing sequences do not have any gain as in the non-commutative case. However, a concept closely related to normalizing sequences is the notion of regular sequence. In this context, there is a geometry behind these sequences in commutative case, as Futorny showed in [FO05].

Artin, Tate and Van den Bergh wanted to relate a certain type of graded algebra, namely the regular algebras of dimension 3, to automorphisms of elliptic curves. In [ATVdB07], p. 44, they denoted by $T = \mathbb{K}\langle x_0, \ldots, x_n\rangle$ the free associative $\mathbb{K}$-algebra on generators $x_0, \ldots, x_n$ of degree 1, and $V = T_1^*$ the dual space of T_1. A homogeneous element $f \in T_d = T_1^{\otimes d}$ of degree d defines a linear map $\widetilde{f} : V \otimes \cdots \otimes V = V^{\otimes d} \to k$. This form may be viewed as a section of the sheaf $\mathcal{O}(1, 1, \ldots, 1) = \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \mathrm{pr}_d^*\mathcal{O}_{\mathbb{P}}(1)$, on the product $\mathbb{P} \times \cdots \times \mathbb{P}$ of d copies of the projective space $\mathbb{P}$ of lines in V (or hyperplanes in T_1). This product is denoted by $(\mathbb{P})^d$. Since $\widetilde{f}$ is multilinear and hence multihomogeneous, its zeros define a ***locus*** in $(\mathbb{P})^d$. Let $A = T/I$ be a graded quotient ring of T and let $\widetilde{I}_d$ be the space of multilinear forms $\widetilde{f}$ such that $f \in I_d$. Γ_d denotes the scheme of zeros of $\hat{I}_d$ in the product $(\mathbb{P})^d$, where $\hat{I}_d$ is the space of multilinear forms $\hat{f}$. This scheme is precisely a subset of d copies of projective plane $\mathbb{P}$. Then, with the definition of ***truncated point module of length*** $d + 1$, we have that there is a one-to-one correspondence between points $(p_1, \ldots, p_d) \in \Gamma_d$ and truncated point modules of length $d + 1$. The main part of the parameterization is when the inverse limit of the sets Γ_d, called Γ, is taken. The points of Γ are in one-to-one correspondence with the module points.

Twelve years later, Shelton and Vancliff in [SV02] retook the parameterization made by Artin, Tate and Van den Bergh, but they go beyond that, because the idea is to parameterize modules called *d-**linear modules***. This is a generalization of Artin's parameterization, because in the case when $d = 0$, the 0-linear modules are the point modules. In [SV02], Section 1, it was defined a truncated d-linear module of length r. Then, it was considered a functor $F_{d,r}$ from $\mathbb{K}$-algebras to sets. With this, it was introduced the ***Grassmannian scheme*** of n-dimensional subspaces of a finite-dimensional $\mathbb{K}$-vector space W, denoted by $\mathbb{G}_n(W)$; also it was defined the scheme of subspaces of W of codimension n, denoted by $\mathbb{G}^n(W)$. Now, it was considered the following product of schemes: for each $r \in \mathbb{N}$ and for

each $d \in \{0, 1, \ldots, d\}$, let $\gamma_r(V, d) = \prod_{i=1}^{r} \mathbb{G}^{C(d+i,d)}((V^*)^{\otimes i})$, where $C(d+i, d)$ denotes the binomial coefficient $\binom{d+i}{d}$. Shelton and Vancliff built, using all previous constructions, a scheme called $\Omega_r(A, d)$ which represents the functor $F_{d,r+1}$ of isomorphism classes of truncated d-linear modules of length $r+1$. At the same way as Artin, they take a projective limit of sets $\Omega_r(A, d)$, denoted by $\Omega_\infty(A, d)$. [SV02], Corollary 1.5 tells us that the set of closed points of $\Omega_\infty(A, d)$ is in one-to-one correspondence with the set of isomorphism classes of d-linear A-modules.

Until now, it had worked with graded rings and graded algebras, both in Artin, Tate and Van den Bergh [ATVdB07] and Shelton and Vancliff [SV02]. However, Lezama and Latorre in [LL17] introduced the concept of point module for semi-graded rings. The difference between graded rings and semi-graded rings is that, in the first case, $B_nB_m \subseteq B_{m,n}$, while in the second case $B_nB_m \subseteq B_0 \oplus \cdots \oplus B_{m,n}$. As we can see, the concept of graded is more particular than that of semi-graded. The profit that is obtained is that we can locate algebras that are not graded in the context of semigraduation (examples of this fact are shown in Chapter 2). With this new definition, in [Lez20] it was introduced a concept of point module in finitely semi-graded rings and the collection of isomorphism classes of point modules for that algebra is defined, called $P(B)$, where B is a finitely semi-graded ring. Then, a scheme is defined ([Lez20], Section 2) that, later, parameterizes the class of isomorphisms of point modules. We focus on these details in Chapter 2.

This work is organized into two chapters and three appendices. Chapter 1 is divided into three sections. In Section 1.1, it is defined ***regular sequence*** and the ***Koszul complex*** in commutative case. Using the variety of maximal ideals, denoted by Specm, a geometry is reached between the sub-varieties of Specm and regular sequences in graded algebras ([FO05], Corollary 3.5). In Section 1.2, we turn to non-commutative case. The ***normalizing sequence*** is defined and we see that it appears related to the ***height*** of an ideal ([MRS01], Theorem 4.1.13). Then we see that the normalizing sequence appears in ***graded skew Clifford algebras***. Finally, in Section 1.3, we define the ***right point modules*** and the ***zero locus***. With these definitions, we see the relation between the zero locus (geometric object) with the normalizing sequences (algebraic object) in graded algebras ([CV14], Corollary 11).

Attempting to generalize the geometry of normalizing sequences in graded context of Chapter 1, in Chapter 2, we present the context of finitely semi-graded algebras and geometry of normalizing sequence in this case. Chapter 2 is divided into 3 sections. In Section 2.1, we define the concept of ***finitely semi-graded ring*** and ***finitely semi-graded algebra***. We see that ***skew PBW extensions*** are finitely semi-graded rings. Examples of semi-graded algebras that are not graded, of skew PBW extensions that are finitely semi-graded algebras, and of finitely graded algebras that are not skew PBW extensions, are exposed. In Section 2.2, we see the appearance of normalizing sequence in a type of semi-graded algebra. For this, we consider to the theory of the universal enveloping algebra of a Lie algebra. We define a ***completely solvable algebra***, a ***ring of fractions*** and we arrive at some properties in which the normalizing sequence appears. In Section

2.3, we see how point modules can be parameterized with a certain scheme in the case of graded algebras. Also, we see some properties about ***Gelfand-Kirillov dimension*** that help us to find a geometry in ***graded skew PBW extensions***. Finally, in Section 2.4 we find a ***scheme*** that allows us to parameterize point modules of certain semi-graded algebras. With the help of this, we get to the main purpose of this work, which is to see how can we find a geometry of the normalizing sequences in certain finitely semi-graded algebras. We find a geometry of the normalizing sequences in graded skew PBW extensions (Theorem 2.3.33). Then, a way is left to continue investigating the geometry of normalizing sequences in more general semi-graded objects.

During these two chapters the details of certain results and examples that are omitted in the literature are completed. This is said as this work progresses.

In carrying out this work, there were many important concepts that were used from homological algebra, category theory, and algebraic geometry. Likewise, many algebras were named as examples throughout the chapters. Therefore, there are three appendices that complement the knowledge of the topics mentioned above.

Appendix A is divided into three sections. In Section A.1, we see the most important notions of ideals and modules, such as being ***Noetherian***, being ***maximal***, being ***minimal***, among others. In Section A.2, we define all the most important homological algebra concepts during this work. First, ***complexes*** and ***resolutions*** are defined. Then, all the dimensions used are defined. And finally, we define some of the fundamental properties for the proof of certain propositions and theorems during this work. In Section A.3, we define the basic concepts of category theory and the properties that we were used.

Throughout this work, many types of algebras were named as examples of certain properties. In Appendix B, these algebras are explicitly defined.

In Appendix C, we remember notions of algebraic geometry. Concepts like the ***affine plane*** and the ***projective plane*** are defined in this part, along with their most important properties. The fundamental concepts of ***presheaf***, ***sheaf*** and ***scheme*** are also defined here along with important examples and notions about these geometric objects.

Finally, after the appendices, certain topics are left that it can be seen as future research works in relation to what has been worked on here.

Notation

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{K}$	An algebraically closed field with $\mathrm{char}(\mathbb{K}) = 0$
$\mathrm{Ker}(f)$	Kernel of the function f
$\mathrm{Im}(f)$	Image of the function f
A/C	The set $\{aC \mid a \in A\}$
$S^\times$	The subset of invertible elements of the ring S
$\mathbb{A}$	Affine space
$\mathbb{P}$	Projective space
C_n	The set of homogeneous elements of degree n in a graded algebra C
$\dim_{\mathbb{K}}(B)$	Dimension of B as $\mathbb{K}$-vectorial space
$\mathrm{M}_n(\mathbb{K})$	Square matrices of size n with entries in $\mathbb{K}$
$(M)_{ij}$	Entry of a matrix M in the position i, j
$\mathbb{K}[z_1, \ldots, z_n]$	Ring of polynomials in n variables on $\mathbb{K}$
V^*	Dual space of V, where V is a vectorial space
$\bigoplus_{i \in I} A_i$	Direct sum of all $A_i, i \in I$
$\bigotimes_{i \in I} A_i$	Tensor product of all $A_i, i \in I$
$T(V)$	Tensor algebra of a vector space V
Λ	An affine $\mathbb{K}$-algebra
$\mathrm{Specm}\Lambda$	The variety of all maximal ideals of Λ
$\mathcal{V}(I)$	The subset of $\mathrm{Specm}\Lambda$, which is all zeroes of an ideal I of Λ
$\mathcal{V}(f_1, \ldots, f_s)$	Denoted the same $\mathcal{V}(I)$, when $I = \langle f_1, \ldots, f_s \rangle \subseteq \Lambda$

CHAPTER 1

Normalizing and regular sequences

It is important to consider what happens in the commutative case to see the differences and similarities that it has in the non-commutative case. Although the central concept of this work is normalizing sequence, we are going to see another type of sequence that is closely linked to the normalizing sequence. Thus, in this chapter we see the definition of regular sequence in commutative algebra. Next, we see the normalizing sequence and regular sequence in non-commutative algebra, more specifically, in the context of Clifford algebras. Then we come to a section that talks about the geometry of these sequences.

1.1 Regular sequences in commutative algebra

In this section we see the behavior of regular sequences in commutative algebra. Moreover, we see relation between the variety of all maximal ideals of certain algebra with a regular sequence of the same algebra. Regular sequences are important since there is a connection between these and normalizing sequences in non-commutative algebra (Section 1.2).

Throughout this section, by dim we denote the dimension of vector spaces and algebraic varieties.

Definition 1.1.1 ([GP12], Definition 7.6.1) Let A be a ring and M an A-module. The set $\{a_1, \ldots, a_n\} \subseteq A$ is called a ***regular sequence*** with respect to M, or, in short, an M-***sequence*** if,

(i) a_i is not a zerodivisor for $M/\langle a_1, \ldots, a_{i-1}\rangle M$, for $i = 1, \ldots, n$.

(ii) $M \neq \langle a_1, \ldots, a_{i-1}\rangle M$.

The next example and remark are presented in [GP12], p. 411 without details. However, details are included below.

Example 1.1.2 ([GP12], p.411) Let $a_1 = x(y-1), a_2 = y, a_3 = z(y-1) \in \mathbb{K}[x,y,z]$. Then $\{a_1, a_2, a_3\}$ is a $\mathbb{K}[x,y,z]$-sequence. Indeed, $x(y-1)$ is regular in $\mathbb{K}[x,y,z]$, because $\mathbb{K}[x,y,z]$ is an integral domain. Now, consider $\mathbb{K}[x,y,z]/\langle x(y-1)\rangle$. If $\overline{yp} = \overline{0}$, for $p \in \mathbb{K}[x,y,z]$, then $yp = qx(y-1)$, for some $q \in \mathbb{K}[x,y,z]$. We have that $q = yq'$, for some $q' \in \mathbb{K}[x,y,z]$ and $p = q'x(y-1)$. So $\overline{p} = \overline{q'x(y-1)} = \overline{0}$. Then a_2 is regular in $\mathbb{K}[x,y,z]/\langle x(y-1)\rangle$. Finally, consider $\mathbb{K}[x,y,z]/\langle x(y-1), y\rangle$. We have $\overline{y} = \overline{0}$ and $\overline{x(y-1)} = \overline{xy-x} = \overline{-x} = \overline{0}$, so $\overline{x} = \overline{0}$. If $z(y-1)p = \overline{0}$, for $p \in \mathbb{K}[x,y,z]$, then $z(y-1)p = qx(y-1)+ry$ for some $q, r \in \mathbb{K}[x,y,z]$. We have that $q = zq'$ and $r = z(y-1)r'$ for some $q', r' \in \mathbb{K}[x,y,z]$. Then, $p = q'x + r'y$ and $\overline{p} = \overline{q'x + r'y} = \overline{q'x} + \overline{r'y} = \overline{0}$. So, $z(y-1)$ is regular in $\mathbb{K}[x,y,z]/\langle x(y-1), y\rangle$. With this, we can see that $\{a_1, a_2, a_3\}$ is a regular sequence with respect to $\mathbb{K}[x,y,z]$.

Remark 1.1.3 ([GP12], Remark 7.6.2) The permutation of a regular sequence need not be a regular sequence. If we consider Example 1.1.2, we saw that $\{a_1, a_2, a_3\}$ is a $\mathbb{K}[x,y,z]$-sequence, but the sequence $\{a_1, a_3, a_2\}$ is not regular. Indeed, $a_1 = x(y-1)$ is regular in $\mathbb{K}[x,y,z]$ for the same reason shown in the previous example. Now, consider $\mathbb{K}[x,y,z]/\langle x(y-1)\rangle$. In this case $\overline{x} \neq \overline{0}$ and we have that $\overline{z(y-1)x} = \overline{zx(y-1)} = \overline{z}\overline{x(y-1)} = \overline{0}$. So, $z(y-1)$ is not regular in $\mathbb{K}[x,y,z]/\langle x(y-1)\rangle$, and then $\{a_1, a_3, a_2\}$ is not a regular sequence with respect to $\mathbb{K}[x,y,z]$.

With the following definition, we can define a relation between ring ideals and regular sequences.

Definition 1.1.4 ([GP12], Definition 7.6.4) Let A be a ring, $I \subseteq A$ an ideal and M an A-module. If $M \neq IM$, then the maximal length n of an M-sequence $\{a_1, \ldots, a_n\} \in I$ is called the I-***depth*** of M and denoted by $\operatorname{depth}(I, M)$. If $M = IM$ then the I-depth of M is by convention ∞. If $(A, \mathfrak{m})$ is a local ring, then the $\mathfrak{m}$-depth of M is simply called the ***depth*** of M, that is, $\operatorname{depth}(M) := \operatorname{depth}(\mathfrak{m}, M)$.

Example 1.1.5 ([GP12], Example 7.6.5 (2)) Let A be a ring, $I \subseteq A$ an ideal and M an A-module. Then the I-depth of M is 0 if and only if every element of I is a zerodivisor for M. Hence, $\operatorname{depth}(I, M) = 0$ if and only if I is contained in some associated prime ideal of M.

In particular, for a local ring $(A, \mathfrak{m})$, we have $\operatorname{depth}(\mathfrak{m}, A/\mathfrak{m}) = 0$.

In the case that we take a finitely generated ideal, we can ask ourselves when the set of its generators is a regular sequence on a module. For this, we have the following proposition that gives us an answer to this question.

Proposition 1.1.6 ([GP12], Corollary 7.6.12) Let A be a Noetherian ring (Definition A.1.21), $I = \langle f_1, \ldots, f_n\rangle$ an ideal of A and M a finitely generated A-module. Assume that $M \neq IM$. The following holds:

(i) $\{f_1, \ldots, f_n\}$ is an M-sequence if and only if $\operatorname{depth}(I, M) = n$.

(ii) Let $J \subseteq A$ be an ideal and $\{f_1, \ldots, f_n\}$ an M-sequence in J, then

$$\operatorname{depth}(J, M/\langle f_1, \ldots, f_n\rangle M) = \operatorname{depth}(J, M) - n.$$

The details of the following example are not shown in [GP12]. Nevertheless , we include every detail explicitly.

Example 1.1.7 ([GP12], Example 7.6.13) Let A be $\mathbb{K}[x_1, \ldots, x_n]$. Consider $\{x_1, \ldots, x_n\}$. This sequence is an A-sequence. Indeed, x_1 is regular in A because A is an integral domain. Let x_i, $A/\langle x_1, \ldots, x_{i-1}\rangle$, for $2 \leq i \leq n$, and $p \in A/\langle x_1, \ldots, x_{i-1}\rangle$. If $\overline{x_i p} = \overline{0}$, we have that $x_i p = r_1 x_1 + \cdots + r_{i-1} x_{i-1}$, $r_j \in A$, for all $1 \leq j \leq i-1$. With this equality, we can see that $r_j = x_i r'_j$, $r'_j \in A$, for all $1 \leq j \leq i-1$. Then, $p = r'_1 x_1 + \cdots + r'_{i-1} x_{i-1}$. Thus, $\overline{p} = \overline{r'_1 x_1 + \cdots + r'_{i-1} x_{i-1}} = \overline{0}$. Thereby, x_i is regular in $A/\langle x_1, \ldots, x_{i-1}\rangle$, for all $2 \leq i \leq n$. With this we show that $\{x_1, \ldots, x_n\}$ is an A-sequence.

Now, by Proposition 1.1.6 (i), we have that $\operatorname{depth}(\{x_1, \ldots, x_n\}, A) = n = \dim_{\mathbb{K}}(A)$.

We remember that the goal is to see the geometry behind the regular sequences. For this, we need the following definition that allows us to define an important concept: ***complete intersection***.

Definition 1.1.8 ([FO05], p.188) Let U be an associative (not necessarily commutative) algebra, and let M be a U-$\mathbb{K}[x_1, \ldots, x_t]$ bimodule. The ***associated Koszul complex*** $\mathbf{K}_\bullet = \mathbf{K}_\bullet(M)$ of left U-modules is defined as follows. Let $e_1, \ldots, e_t$ be the standard basis of $\mathbb{K}^t$. Set

$$\mathbf{K}_i = \begin{cases} 0, & \text{for } i < 0; \\ M \otimes_{\mathbb{K}} \bigwedge_{\mathbb{K}}^{i} \mathbb{K}^t, & \text{for each } i \geq 0. \end{cases}$$

Define the differential $d_i : \mathbf{K}_i \to \mathbf{K}_{i-1}$ on the element $m \otimes (e_{j_1} \wedge \cdots \wedge e_{j_i})$ of $\mathbf{K}_i$, $1 \leq j_1 < \cdots < j_i \leq t, m \in M$, as follows:

$$d_i(m \otimes (e_{j_1} \wedge \cdots \wedge e_{j_i})) = \sum_{k=1}^{i} (-1)^{k-1} m \cdot x_{j_k} \otimes (e_{j_1} \wedge \cdots \hat{e_{j_k}} \cdots \wedge e_{j_i}),$$

where $\hat{e_{j_k}}$ indicates that e_{j_k} is omitted.

Now, let $\mathbf{g} = \{g_1, \ldots, g_t\} \subseteq U$ be a sequence of mutually commuting elements, let $M = U$ with the natural structure of a left U-module, and let $a \cdot x_i = a g_i$, $a \in U$, where

$1 \leq i \leq t$. We denote the corresponding Koszul complex by $\mathbf{K}_\bullet(\mathbf{g}, U)$. The sequence $\mathbf{g}$ is called a ***complete intersection for*** U (or simply a ***complete intersection***, if U is fixed). In the special case when we consider an affine algebra Λ, we have that a regular sequence $\mathbf{g}$ is a complete intersection for Λ.

An affine algebra Λ is called a ***complete intersection***, provided that the isomorphism $\Lambda \cong \mathbb{K}[x_1, \ldots, x_n]/I$, for some ideal I generated by a regular sequence $\mathbf{g} = \{g_1, \ldots, g_t\}$ in $\mathbb{K}[x_1, \ldots, x_n]$.

As we saw in Remark 1.1.3, in general we have that the permutation of a regular sequence is not a regular sequence. A natural question would be, when does the permutation of a regular sequence remain regular? We have the following proposition, which its proof in [GP12], Corollary 7.6.15 is presented without details. However, theses details are included below.

Proposition 1.1.9 ([GP12], Corollary 7.6.15) Let $(A, \mathfrak{m})$ be a local ring, M a finitely generated A-module and $x_1, \ldots, x_n$ an M-sequence. Then $x_{\pi(1)}, \ldots, x_{\pi(n)}$ is an M-sequence for all permutations π.

Proof. By [GP12], Corollary 7.6.8, we have that for all permutation π, $K(x, M)_\bullet \cong K(y, M)_\bullet$, where $x = (x_1, \ldots, x_n)$ and $y = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. Now, by Corollary 1.1.6 we have $\operatorname{depth}(M) = n$ and by [GP12], Corollary 7.6.11 $H_i(x, M) = 0$, for all $i \geq 0$, where $H_i(x, M)$ is called the i-***th homology*** of $K(x, M)_\bullet$, that is,

$$H_i(x, M) := \operatorname{Ker}(d_p \otimes 1_M)/\operatorname{Im}(d_{p+1} \otimes 1_M).$$

In particular $H_1(y, M) \cong H_1(x, M) = 0$. Finally, by [GP12], Theorem 7.6.14(2), we have that y is an M-sequence. □

When we have a regular sequence, we can build other different regular sequences from the first one as follows:

Corollary 1.1.10 ([GP12], Corollary 7.6.16) Let A be a ring, M an A-module, $I \subseteq A$ an ideal. If $\{x_1, \ldots, x_n\} \subseteq I$ is an M-sequence, then $\{x_1^m, \ldots, x_n^m\}$ is an M-sequence, for all $m \geq 1$.

Proof. We use induction on n. The case $n = 1$ is immediate. Assume that $n > 1$ and that $\{x_1^m, \ldots, x_{n-1}^m\}$ is an M-sequence. We first prove that x_n is not a zerodivisor of $M/\langle x_1^m, \ldots, x_{n-1}^m\rangle M$. It is enough to check this in the localization at any prime ideal P. If $P \not\supseteq \langle x_1, \ldots, x_{n-1}\rangle$, then either x_n is a unit in A_P, or we have $M_P = \langle x_1^m, \ldots, x_{n-1}^m\rangle M_P$, that is, in this case x_n is a non-zerodivisor of $M_P/\langle x_1^m, \ldots, x_{n-1}^m\rangle M_P$.

For the remaining cases, we may assume that $(A, \mathfrak{m})$ is already local and $x_1, \ldots, x_n \in \mathfrak{m}$.

Now, $x_1, \ldots, x_n$ being an M-sequence implies that $\{x_1, \ldots, x_{n-1}, x_n^m\}$ is an M-sequence. Using what was said in Proposition 1.1.9, we obtain that $\{x_n^m, x_1, \ldots, x_{n-1}\}$ is an M-sequence. Repeating this n times we obtain that $\{x_n^m, \ldots, x_1^m\}$ is an M-sequence and, again, Proposition 1.1.9 completes the proof. □

Example 1.1.11 If we consider Example 1.1.2, we have that $\{x^m(y-1)^m, y^m, z^m(y-1)^m\}$ is a $\mathbb{K}[x, y, z]$-sequence, for $m \geq 1$.

Now, we see a particular case of algebras where we can find a geometry behind of these. In this case, we see what happens with filtered algebras.

Remark 1.1.12 ([FO05], p.190) Let U be a filtered algebra, that is, an associative algebra over $\mathbb{K}$ endowed with an increasing filtration $\{U_i\}_{i\geq 0}$ such that $U_0 = \mathbb{K}$, $U_iU_j \subseteq U_{i+j}$ and $U = \bigcup_{i\geq 0} U_i$. If $u \in U_i \setminus U_{i-1}$, set $\deg(u) = i$ (by convention, $U_{-1} = \{0\}$). Let

$$\overline{U} = \mathrm{gr}U = \bigoplus_{i=0}^{\infty} U_i/U_{i-1},$$

be the associated graded algebra of U. If $u \in U$, let $\overline{u}$ be its image in $\overline{U}$, and for a subset $S \subseteq U$. Let $S_i = S \cap U_i$ and $\overline{S} = \{\overline{s} \mid s \in S\} \subseteq \overline{U}$.

Finally, with the following two results we can find a relation between geometry and regular sequences. In particular, we find a relation of Specm with regular sequence, when the algebra fulfills certain properties.

Proposition 1.1.13 ([FO05], Proposition 3.4 (iv)) Let U be a filtered algebra, and let $\mathbf{g} = \{g_1, \ldots, g_t\}$ be a sequence of mutually commuting elements from U such that $\{\overline{g_1}, \ldots, \overline{g_t}\}$ is a complete intersection for $\overline{U}$. Let $\Gamma = \mathbb{K}[g_1, \ldots, g_t]$. For any $\nu \in \mathrm{Specm}\Gamma$, there is a simple left U-module M, generated by $m \in M$, such that $\gamma \cdot m = \gamma(\nu)m$, for every $\gamma \in \Gamma$. That is, ν lifts to a simple U-module.

Proof. By [FO05], Proposition 3.4(iii), we have that there exists a simple left U-module M, generated by $m \in M$, such that $g_im = \mu_im$, $i = 1, \ldots, t$, where $\mu_i \in U$ such that $\deg(g_i) > \deg(\mu_i)$, for $i = 1, \ldots, t$. If we take $\mu_i = g_i(\nu)$, $i = 1, \ldots, t$, we finish the proof. □

The last important result below is taken from [FO05], Corollary 3.5, but its proof is presented without details there. However, these details are included below.

Corollary 1.1.14 ([FO05], Corollary 3.5) Let U be a filtered algebra such that the associated graded algebra $\overline{U}$ is Cohen-Macaulay (Definition A.2.30). If $g_1, \ldots, g_t$ are mutually commuting elements of U such that $\mathcal{V}(\overline{g_1}, \ldots, \overline{g_t}) \subseteq \mathrm{Specm}\overline{U}$ is equidimensional (that

is, all irreducible components have the same dimension), of codimension t, then every $\mu \in \operatorname{Specm}\mathbb{K}[g_1, \ldots, g_t]$ lifts to a simple U-module.

Proof. [FO05], Proposition 2.1(v) says that in a Cohen-Macaulay algebra A a sequence of elements of A $f_1, \ldots, f_t$ is a complete intersection if an only if the variety $\mathcal{V}(f_1, \ldots, f_t)$ is equidimensional, of dimension $n - t$. In this case, $\overline{U}$ is Cohen-Macaulay and $\mathcal{V}(\overline{g_1}, \ldots, \overline{g_t})$ is equidimensional, then $\{\overline{g_1}, \ldots, \overline{g_t}\}$ is a complete intersection. By Proposition 1.1.13, we finish the proof. □

1.2 Normalizing sequences in non-commutative algebra

In this section, we turn to the non-commutative case. We see that one of the occurrences of the normalizing sequences is with the prime ideals and with a concept called height of a prime ideal. Also, we see that it appears in the definition of the so-called graded skew Clifford algebras. Some examples of normalizing sequences are shown throughout this section.

To begin, we define the notion of a normal element in an algebra and, with this, the normalizing sequence.

Definition 1.2.1 ([CV10], Definition 1.9 (a)) Let B denote a $\mathbb{K}$-algebra. An element $b \in B$ is said to be a ***normal*** element, if $Bb = bB$. If $b_1, \ldots, b_m \in B$ and if the image of b_{k+1} in $B/\langle b_1, \ldots, b_k\rangle$ is normal, for all $k = 1, \ldots, m-1$, then $\{b_1, \ldots, b_m\}$ is called a ***normalizing sequence*** in B.

The next example is presented in [Van15b], Example 2.2 without details. However, details are included below.

Example 1.2.2 ([Van15b], Example 2.2) Let B be the ***coordinate ring of quantum*** 2×2 ***matrices***; that is, B is the quadratic $\mathbb{K}$-algebra on generators a, b, c, d with defining relations

$$\begin{aligned} ab &= qba, & bd &= qdb, \\ ac &= qca, & cd &= qdc, \\ bc &= cb, & ad - da &= (q - q^{-1})bc, \end{aligned}$$

where $q \in \mathbb{K}^\times$ and $q^2 \neq 1$. We have that $\{bc, b^2, c^2, ad\}$ is a normalizing sequence. Indeed, let $x \in B$. We can see $x = \sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} c^{\gamma_i} d^{\delta_i}$, where $k_i \in \mathbb{K}$, $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{Z}^+ \cup \{0\}$, for all $1 \leq i \leq n$, $n \geq 1$. Then

$$\begin{aligned} bcx &= bc\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} c^{\gamma_i} d^{\delta_i}\right) = \sum_{i=1}^{n} bck_i a^{\alpha_i} b^{\beta_i} c^{\gamma_i} d^{\delta_i} = \sum_{i=1}^{n} k_i bca^{\alpha_i} b^{\beta_i} c^{\gamma_i} d^{\delta_i} \\ &= \sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} bcb^{\beta_i} c^{\gamma_i} d^{\delta_i} = \sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} bcd^{\delta_i} \\ &= \sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} q^{2\delta_i} d^{\delta_i} bc \\ &= \left(\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} q^{2\delta_i} d^{\delta_i}\right) bc. \end{aligned}$$

so, $bcx \in B(bc)$. Analogously, we have that

$$\begin{aligned} xbc &= \left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} c^{\gamma_i} d^{\delta_i}\right) bc = \sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} c^{\gamma_i} d^{\delta_i} bc = \sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} c^{\gamma_i} bcq^{-2\delta_i} d^{\delta_i} \\ &= \sum_{i=1}^{n} k_i a^{\alpha_i} bcb^{\beta_i} c^{\gamma_i} q^{-2\delta_i} d^{\delta_i} = \sum_{i=1}^{n} k_i bcq^{2\alpha_i} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} q^{-2\delta_i} d^{\delta_i} \\ &= \sum_{i=1}^{n} bck_i q^{2\alpha_i} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} q^{-2\delta_i} d^{\delta_i} \\ &= bc\left(\sum_{i=1}^{n} k_i q^{2\alpha_i} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} q^{-2\delta_i} d^{\delta_i}\right). \end{aligned}$$

so, $xbc \in (bc)B$. Then $(bc)B = B(bc)$.

Now, let $\overline{x} \in B/\langle bc\rangle$. If $\overline{x} \neq \overline{0}$, we can see $\overline{x} = \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}}$, where $k_i, k_j \in \mathbb{K}$, $\alpha_i, \alpha_j, \beta_i, \gamma_j, \delta_i, \delta_j \in \mathbb{Z}^+ \cup \{0\}$, for all $1 \leq i \leq n$, $1 \leq j \leq m$, $n, m \geq 1$ (the elements that have bc are zero under the quotient). Then

$$\overline{b^2x} = \overline{b^2\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right)} = \overline{\sum_{i=1}^{n} b^2 k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} b^2 k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}}$$
$$= \overline{\sum_{i=1}^{n} k_i b^2 a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j b^2 a^{\alpha_j} c^{\gamma_j} d^{\delta_j}}$$
$$= \overline{\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} b^2 d^{\delta_i} + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} b^2 d^{\delta_j}}$$
$$= \overline{\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{2\delta_i} d^{\delta_i} b^2 + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{2\delta_j} d^{\delta_j} b^2}$$
$$= \overline{\left(\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{2\delta_j} d^{\delta_j}\right) b^2}.$$

so, $\overline{b^2x} \in (B/\langle bc\rangle)\,\overline{b^2}$. Analogously, we have that

$$\overline{xb^2} = \overline{\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right) b^2} = \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} b^2 + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j} b^2}$$
$$= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} b^2 q^{-2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} b^2 q^{-2\delta_j} d^{\delta_j}}$$
$$= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^2 b^{\beta_i} q^{-2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} b^2 c^{\gamma_j} q^{-2\delta_j} d^{\delta_j}}$$
$$= \overline{\sum_{i=1}^{n} k_i b^2 q^{2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{-2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j b^2 q^{2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{-2\delta_j} d^{\delta_j}}$$
$$= \overline{b^2\left(\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{2\delta_j} d^{\delta_j}\right)}.$$

so, $\overline{xb^2} \in \overline{b^2}\,(B/\langle bc\rangle)$. Then $\overline{b^2}\,(B/\langle bc\rangle) = (B/\langle bc\rangle)\,\overline{b^2}$.

Let $\overline{x} \in B/\langle bc, b^2\rangle$. If $\overline{x} \neq \overline{0}$, we can see $\overline{x} = \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}}$, where $k_i, k_j \in \mathbb{K}$, $\alpha_i, \alpha_j, \gamma_j, \delta_i, \delta_j \in \mathbb{Z}^+ \cup \{0\}$, $\beta_i = 0, 1$, for all $1 \leq i \leq n$, $1 \leq j \leq m$, $n, m \geq 1$ (the elements that have bc and b^2 are zero under the quotient). Then

$$\begin{aligned}
\overline{c^2x} &= \overline{c^2\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right)} = \overline{\sum_{i=1}^{n} c^2 k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} c^2 k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i c^2 a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j c^2 a^{\alpha_j} c^{\gamma_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} c^2 d^{\delta_i} + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} c^2 d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{2\delta_i} d^{\delta_i} c^2 + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{2\delta_j} d^{\delta_j} c^2} \\
&= \overline{\left(\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{2\delta_j} d^{\delta_j}\right) c^2}.
\end{aligned}$$

so, $\overline{c^2x} \in \left(B/\langle bc, b^2\rangle\right)\overline{c^2}$. Analogously, we have that

$$\begin{aligned}
\overline{xc^2} &= \overline{\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right) c^2} = \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} c^2 + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j} c^2} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} c^2 q^{-2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} c^2 q^{-2\delta_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} c^2 b^{\beta_i} q^{-2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^2 c^{\gamma_j} q^{-2\delta_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i c^2 q^{2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{-2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j c^2 q^{2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{-2\delta_j} d^{\delta_j}} \\
&= \overline{c^2\left(\sum_{i=1}^{n} k_i q^{-2\alpha_i} a^{\alpha_i} b^{\beta_i} q^{2\delta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j q^{-2\alpha_j} a^{\alpha_j} c^{\gamma_j} q^{2\delta_j} d^{\delta_j}\right)}.
\end{aligned}$$

so, $\overline{xc^2} \in \overline{c^2}\left(B/\langle bc, b^2\rangle\right)$. Then $\overline{c^2}\left(B/\langle bc, b^2\rangle\right) = \left(B/\langle bc, b^2\rangle\right)\overline{c^2}$.

Finally, let $\overline{x} \in B/\langle bc, b^2, c^2\rangle$. If $\overline{x} \neq \overline{0}$, we can see $\overline{x} = \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}}$, where $k_i, k_j \in \mathbb{K}$, $\alpha_i, \alpha_j, \delta_i, \delta_j \in \mathbb{Z}^+ \cup \{0\}$, $\beta_i, \gamma_j = 0, 1$, for all $1 \leq i \leq n$, $1 \leq j \leq m$, $n, m \geq 1$ (the elements that have bc, b^2 and c^2 are zero under the quotient). It should be noted that, under this quotient, we have $\overline{ad - da} = \overline{(q - q^{-1})bc} = \overline{0}$. Then

$$\begin{aligned}
\overline{adx} &= \overline{ad\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right)} = \overline{\sum_{i=1}^{n} adk_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} adk_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i ada^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j ada^{\alpha_j} c^{\gamma_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} adb^{\beta_i} cd^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} adc^{\gamma_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} add^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} q^{-2\gamma_j} c^{\gamma_j} add^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} ad + \sum_{j=1}^{m} k_j a^{\alpha_j} q^{-2\gamma_j} c^{\gamma_j} d^{\delta_j} ad} \\
&= \overline{\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} q^{-2\gamma_j} c^{\gamma_j} d^{\delta_j}\right) ad}.
\end{aligned}$$

so, $\overline{adx} \in \left(B/\langle bc, b^2, c^2\rangle\right)\overline{ad}$. Analogously, we have that

$$\begin{aligned}
\overline{xad} &= \overline{\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right) ad} = \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} ad + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j} ad} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} add^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} add^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i a^{\alpha_i} adb^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} adc^{\gamma_j} d^{\delta_j}} \\
&= \overline{\sum_{i=1}^{n} adk_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} adk_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}} \\
&= \overline{ad\left(\sum_{i=1}^{n} k_i a^{\alpha_i} b^{\beta_i} d^{\delta_i} + \sum_{j=1}^{m} k_j a^{\alpha_j} c^{\gamma_j} d^{\delta_j}\right)}.
\end{aligned}$$

so, $\overline{xad} \in \overline{ad}\left(B/\langle bc, b^2, c^2\rangle\right)$. Then $\overline{ad}\left(B/\langle bc, b^2, c^2\rangle\right) = \left(B/\langle bc, b^2, c^2\rangle\right)\overline{ad}$. Effectively, $\{bc, b^2, c^2, ad\}$ is a normalizing sequence.

Remark 1.2.3 With Definition 1.2.1, we can see why the normalizing sequences were not worked in the commutative case. This is because any sequence of elements in an algebra would be a normalizing sequence due to their commutativity.

Now, we start with some previous definitions in order to arrive at the relation that normalizing sequences have with prime ideals.

Lemma 1.2.4 ([MRS01], Lemma 4.1.10) Let R be a right Noetherian prime ring (Definition A.1.1) and $0 \neq a \in R$ a normal element. Then, given any right ideal I of R, there exists $n \geq 0$ such that $I \cap a^n R \subseteq IaR$.

Proof. We have the following chain

$$Ia^{-1}R \cap R \subseteq Ia^{-2}R \cap R \subseteq \cdots \subseteq Ia^{-n}R \cap R \subseteq \cdots .$$

Since R is a right Noetherian ring, $Ia^{-n+1}R \cap R = Ia^{-n}R \cap R$, for some n. We can multiply by a^n, from the right, and we have $IaR \cap a^n R = IR \cap a^n R$. Hence $I \cap a^n R \subseteq IaR$. □

Example 1.2.5 Consider $R = \mathbb{Z}$, $a = 8$ and $I = \langle 2 \rangle$. We have $IaR = \langle 2 \rangle 8\mathbb{Z} = \langle 2 \rangle \langle 3 \rangle = \langle 6 \rangle$. When $n = 2$, we have $\langle 2 \rangle \cap 8^2\mathbb{Z} = \langle 2 \rangle \cap \langle 16 \rangle = \langle 16 \rangle$.

The following result involves the ***height*** of prime P (denoted by $\mathrm{ht}(P)$), that is, the largest length of a chain of prime ideals contained in P, or ∞ if no bound exists.

Proposition 1.2.6 ([MRS01], Theorem 4.1.11) Let R be a right Noetherian ring, a any normal element which is not a unit, and P a prime ideal of R minimal over aR (Definition A.1.7), that is, it is minimal among all prime ideals containing aR. Then P has height at most 1.

Proof. Some details of this proof that are not included in [MRS01], but we include them here. Suppose it does not happen. Then $P \supset P' \supset P''$. Factoring out P'' means we can suppose that R is prime; then a is regular. Since P' is a nonzero ideal of R, it contains some regular element b.

By Lemma 1.2.4, we have $bR \cap a^n R \subseteq baR$, for some n. Then

$$\begin{aligned} bR \cap a^{n+1}R &= bR \cap a^n R \cap a^{n+1}R \subseteq baR \cap a^{n+1}R \quad (\text{since} \quad a^{n+1}R \subseteq a^n R) \\ &= (bR \cap a^n R)a \subseteq (baR)a = ba^2 R. \end{aligned}$$

Following the same way we have that $bR \cap a^{2n}R \subseteq ba^n R$.

Hence, since P is minimal among all primes ideals containing aR and $a^n R \subseteq aR$, then P is minimal over $a^n R$ and a^n is a normal non-unit (because a is a normal element and $(aR)^n = a^n R$). Thereby, replacing a by a^n, we can assume that $bR \cap a^2 R \subseteq baR$. Then $bR \cap (a^2 R + baR) = baR$. So

$$\frac{a^2R + bR}{a^2R + baR} = \frac{a^2R + baR + bR}{a^2R + baR}.$$

We consider

$$\begin{aligned} f : bR &\to \frac{bR + a^2R + baR}{a^2R + baR} \\ br &\mapsto f(br) := (br)(a^2R + baR). \end{aligned}$$

We can see that f is surjective and $\mathrm{Ker}(f) = bR \cap (a^2R + baR)$. By fundamental isomorphism theorem, we have that

$$\frac{a^2R + baR + bR}{a^2R + baR} \cong \frac{bR}{(a^2R + baR) \cap bR}.$$

Now, since $bR \cap (a^2R + baR) = baR$

$$\frac{bR}{(a^2R + baR) \cap bR} = \frac{bR}{baR}.$$

We can consider

$$\begin{aligned} g : R &\to \frac{bR}{baR} \\ r &\mapsto g(r) := r(baR) \end{aligned}$$

We can see that g is surjective and $\mathrm{Ker}(g) = bR$. By fundamental isomorphism theorem, we have that

$$\frac{bR}{baR} \cong \frac{R}{aR}.$$

With this, we have the following isomorphism

$$\frac{aR + bR}{a^2R + baR} \cong \frac{R}{aR}.$$

Before continuing, we need to define the concept of reduced rank, since it will be used heavily in this proof. If M is a right R-module, the ***reduced rank*** of M is defined to be $\rho_R(M) = \mathrm{udim}(M \otimes Q)$[1], where $Q = Q(R)$ is the field of fractions of R.

Now, if $I \lhd R$, we consider the following set

$$\mathcal{C}_R(I) = \{s \in R \mid (s + I) \text{ is regular in} R/I\}.$$

We can see that this set is multiplicatively closed.

Therefore $\rho((a^2R + bR)/(a^2R + baR)) = \rho(R/aR)$ with ρ being the reduced rank[2] as modules over $\overline{R} = R/aR$.

[1] Uniform dimension is defined in Definition A.2.16

[2] For more details of reduced rank applications, see [MRS01] section 4.1.

The aplication $r \mapsto a^{-1}ra$ gives an automorphism of R and of $\overline{R}$. It yields a lattice automorphism on the set of right ideals of $\overline{R}$ which preserves sums, intersections and the property of being torsion with respect to $\mathcal{C}(N)$, where N/aR is the prime radical of $\overline{R}$. accordingly it preserves the reduced rank on subfactors. So, if $a^{-1}ba = c$, then

$$\rho((aR + bR)/aR) = \rho((aR + cR)/aR).$$

Thus

$$\begin{aligned}\rho\left(\frac{aR+bR}{a^2R+baR}\right) &= \rho\left(\frac{aR+bR}{a^2R+acR}\right) = \rho\left(\frac{aR+bR}{aR}\right) + \rho\left(\frac{aR}{a^2R+acR}\right)\\ &= \rho\left(\frac{aR+cR}{aR}\right) + \rho\left(\frac{R}{aR+cR}\right) = \rho\left(\frac{R}{aR}\right)\\ &= \rho\left(\frac{a^2R+bR}{a^2R+baR}\right).\end{aligned}$$

We have that

$$\left(\frac{aR+bR}{a^2R+baR}\right) = 0$$

Therefore, there exists $d \in \mathcal{C}_R(N)$ such that $ad \in a^2R + bR$ and so $ad = a^2r + bs$ for some $r, s \in R$. Consequently $a(d - ar) = bs \in P'$. But nevertheless, $a \notin P'$ and, as it is a normal element, is regular modulo P'. Therefore $d - ar \in P' \subseteq P$ and then $d \in P$. However, P/N is a minimal prime of R/N, so $\mathcal{C}(N) \subseteq \mathcal{C}(P)$. This shows that $d \in \mathcal{C}(P)$. Since $P \cap \mathcal{C}(P) = \emptyset$, this brings us to a contradiction. □

Remark 1.2.7 ([MRS01], p. 105) a must comply with the imposed condition of being normal. For example, let $T = \mathbb{Z}[x]$, $R = M_2(T)$ and

$$a = \begin{bmatrix} 2 & 0 \\ 0 & x \end{bmatrix}.$$

Then the only prime P of R containing aR is $M_2(I)$, where $I = (2, x) \lhd T$, i.e., I is a two-sided ideal of T; so, height$(P) = 2$.

With the following corollary, we are going to involve a normal element with a quotient of ideals.

Corollary 1.2.8 ([MRS01], Corollary 4.1.12) Let R be a right Noetherian ring, a any normal element which is not a unit and P a prime ideal of R containing aR. Then

$$\text{height}(P) \leq \text{height}(P/aR) + 1.$$

Proof. Suppose that $P = P_0 \supset P_1 \supset \cdots \supset P_m$. If $a \notin P_{m-1}$, we can choose k such that $a \in P_k$, $a \notin P_{k+1}$. Applying Theorem 1.2.6 to R/P_{k+2} we have that P_k is not minimal over $aR + P_{k+2}$. So, there exists P'_{k+1} with $P_k \supset P'_{k+1} \supset P_{k+2}$ and $P'_{k+1} \supseteq aR$. We replace P_{k+1} by P'_{k+1} and then we can repeat this process. Eventually we will have that $a \in P_{m-1}$; but then $m \leq \text{height}(P/aR) + 1$ as required. □

Example 1.2.9 We consider again Example 1.2.5. We have $\text{ht}(\langle 2\rangle) = 1$, $\text{ht}(\langle 2\rangle/\langle 8\rangle) = \text{ht}(\mathbb{Z}) = 1$. So, the inequality of Corollary 1.2.8 holds.

The main idea is to generalize Theorem 1.2.6 with the notion of normalizing sequence. For this, we see another definition (equivalent to Definition 1.2.1) of normalizing sequence.

Definition 1.2.10 ([MRS01], Definition 4.1.13) A sequence $a_1, \ldots, a_n$ of elements of a ring R is called a ***normalizing sequence*** or, more specifically, a ***centralizing sequence***, if for each $j \in \{0, \ldots, n-1\}$, the image of a_{j+1} in $R/\sum_{i=1}^{j} a_i R$ is a normal (respectively central, i.e., $\overline{a_{j+1}} \cdot \overline{r} = \overline{r} \cdot \overline{a_{j+1}}$, for all $\overline{r} \in R/\sum_{i=1}^{j} a_i R$) element and also $\sum_{i=1}^{n} a_i R \neq R$. The ideal generated by such a sequence is sometimes called ***polynormal*** or ***polycentral***.

The next example is presented in [Van15b], Example 2.4 without details. However, details are included below.

Example 1.2.11 ([Van15b], Example 2.4) Let B be the $\mathbb{K}$-algebra on degree-1 generators x, y with defining relations $xy + yx = -x^2 - y^2$, $x^2y = yx^2$. x^2 and y^2 are central elements in B. The sequence $\{x^2, y^2\}$ is normalizing. Indeed, let $p \in B$. We can see that $p = \sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i}$, where $k_i \in \mathbb{K}$, $\alpha_i, \beta_i \in \mathbb{Z}^+ \cup \{0\}$, for all $1 \leq i \leq n$. Then

$$x^2 p = x^2 \left(\sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i} \right) = \sum_{i=1}^{n} x^2 k_i x^{\alpha_i} y^{\beta_i} = \sum_{i=1}^{n} k_i x^2 x^{\alpha_i} y^{\beta_i} = \sum_{i=1}^{n} k_i x^{\alpha_i} x^2 y^{\beta_i}$$
$$= \sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i} x^2 = \left(\sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i} \right) x^2.$$

So, $x^2 p \in Bx^2$. Analogously, we have that

$$px^2 = \left(\sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i} \right) x^2 = \sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i} x^2 = \sum_{i=1}^{n} k_i x^{\alpha_i} x^2 y^{\beta_i} = \sum_{i=1}^{n} k_i x^2 x^{\alpha_i} y^{\beta_i}$$
$$= \sum_{i=1}^{n} x^2 k_i x^{\alpha_i} y^{\beta_i} = x^2 \left(\sum_{i=1}^{n} k_i x^{\alpha_i} y^{\beta_i} \right).$$

So, $px^2 \in x^2 B$. Then $x^2 B = Bx^2$.

Now, let $\overline{p} \in B/\langle x^2 \rangle$. If $\overline{p} \neq 0$, we can see $\overline{p} = \sum_{i=1}^{n} k_i y^{\beta_i} + \sum_{j=1}^{m} k_j x y^{\beta_j}$ where $k_i, k_j \in \mathbb{K}$, $\beta_i, \beta_j \in \mathbb{Z}^+ \cup \{0\}$, for all $1 \leq i \leq n$, $1 \leq j \leq m$ (the elements that have x^2 are zero under the quotient). It should be noted that, under this quotient, we have $\overline{xy + yx} = \overline{-x^2 - y^2} = \overline{-y^2}$. Then

$$\begin{aligned}
\overline{y^2 p} &= \overline{y^2 \left(\sum_{i=1}^{n} k_i y^{\beta_i} + \sum_{j=1}^{m} k_j x y^{\beta_j} \right)} = \overline{\sum_{i=1}^{n} y^2 k_i y^{\beta_i} + \sum_{j=1}^{m} y^2 k_j x y^{\beta_j}} \\
&= \overline{\sum_{i=1}^{n} k_i y^2 y^{\beta_i} + \sum_{j=1}^{m} k_j y^2 x y^{\beta_j}} = \overline{\sum_{i=1}^{n} k_i y^{\beta_i} y^2 + \sum_{j=1}^{m} k_j x y^{\beta_j} y^2} \\
&= \overline{\left(\sum_{i=1}^{n} k_i y^{\beta_i} + \sum_{j=1}^{m} k_j x y^{\beta_j} \right) y^2}
\end{aligned}$$

So, $\overline{y^2 p} \in \left(B/\langle x^2 \rangle\right) y^2$. Analogously, we have that

$$\begin{aligned}
\overline{p y^2} &= \overline{\left(\sum_{i=1}^{n} k_i y^{\beta_i} + \sum_{j=1}^{m} k_j x y^{\beta_j} \right) y^2} = \overline{\sum_{i=1}^{n} k_i y^{\beta_i} y^2 + \sum_{j=1}^{m} k_j x y^{\beta_j} y^2} \\
&= \overline{\sum_{i=1}^{n} k_i y^2 y^{\beta_i} + \sum_{j=1}^{m} k_j x y^2 y^{\beta_j}} = \overline{\sum_{i=1}^{n} y^2 k_i y^{\beta_i} + \sum_{j=1}^{m} y^2 k_j x y^{\beta_j}} \\
&= \overline{y^2 \left(\sum_{i=1}^{n} k_i y^{\beta_i} + \sum_{j=1}^{m} k_j x y^{\beta_j} \right)}
\end{aligned}$$

So, $\overline{p y^2} \in y^2 \left(B/\langle x^2 \rangle\right)$. Then $y^2 \left(B/\langle x^2 \rangle\right) = \left(B/\langle x^2 \rangle\right) y^2$. Effectively, $\{x^2, y^2\}$ is a normalizing sequence.

Finally, we arrive at the principal result that connects the prime ideals with the normalizing sequences. The details of proof of this result are not explicitly in [MRS01]. However, the proof is written in detail here.

Proposition 1.2.12 ([MRS01], Theorem 4.1.13) Let R be a right Noetherian ring. Let $a_1, \ldots, a_n$ be a normalizing sequence and P a prime ideal containing $A = \sum_{i=1}^{n} a_i R$. Then

$$\text{height}(P) \leq \text{height}(P/A) + n$$

Proof. We use induction over n.

If $n = 1$, then we are in the case of Corollary 1.2.8. Suppose that the statement is true for $n-1$, i.e., we have

$$\begin{aligned} \text{height}(P) &\leq \text{height}\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) + n - 1 \\ \text{height}(P) + 1 - n &\leq \text{height}\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \end{aligned}$$

Since R is Noetherian, then $R/\sum_{i=1}^{n-1} a_i R$ is Noetherian. a_n is a normal element in $R/\sum_{i=1}^{n-1} a_i R$, because it belongs to $\{a_1, \ldots, a_n\}$ which is a normalizing sequence. In the same way, since P is an prime ideal in R, then $P/\sum_{i=1}^{n-1} a_i R$ is an prime ideal in $R/\sum_{i=1}^{n-1} a_i R$ that contains $a_n R(R/\sum_{i=1}^{n-1} a_i R)$. With these conditions, we can apply Corollary 1.2.8 to obtain

$$\text{height}\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \leq \text{height}\left(\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ \left(a_n R\left(R \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right)\right)\right) + 1$$

We have that $a_n R\left(R \Big/ \sum_{i=1}^{n-1} a_i R\right) \cong \left(a_n R + \sum_{i=1}^{n-1} a_i R\right) \Big/ \sum_{i=1}^{n-1} a_i R$. Indeed, we consider

$$\begin{aligned} f : a_n R + \sum_{i=1}^{n-1} a_i R &\to a_n R\left(R \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \\ a_n r_n + \cdots + a_1 r_1 &\mapsto \overline{a_n r_n + \cdots + a_1 r_1}. \end{aligned}$$

We can see that f is surjective and $\text{Ker}(f) = \sum_{i=1}^{n-1} a_i R$. By fundamental isomorphism theorem, we have that

$$a_n R\left(R \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \cong \left(a_n R + \sum_{i=1}^{n-1} a_i R\right) \Big/ \sum_{i=1}^{n-1} a_i R.$$

Then, we have the following isomorphism

$$\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ a_n R\left(R \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \cong \left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ \left(\left(a_n R + \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right). \quad (1.2.1)$$

Now, we have that $a_n R + \sum_{i=1}^{n-1} a_i R = \sum_{i=1}^{n} a_i R$. So, Equation 1.2.1 is equal to

$$\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ \left(\textstyle\sum_{i=1}^{n} a_i R \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right). \quad (1.2.2)$$

If we consider the following homomorphism

$$g : \sum_{i=1}^{n} a_i R \to a_n R$$
$$a_n r_n + \cdots + a_1 r_1 \mapsto a_n r_n,$$

we can see that g is surjective and $\mathrm{Ker}(g) = \sum_{i=1}^{n-1} a_i R$. By fundamental isomorphism theorem, we have that

$$\textstyle\sum_{i=1}^{n} a_i R \Big/ \sum_{i=1}^{n-1} a_i R \cong a_n R.$$

So, Equation 1.2.2 is isomorphic to

$$\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ a_n R. \tag{1.2.3}$$

Now, we consider h as follows:

$$h : P \to \left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ a_n R$$
$$p \mapsto \overline{p}(a_n R),$$

We can see that h is surjective and $\mathrm{Ker}(h) = \sum_{i=1}^{n} a_i R$. By fundamental isomorphism theorem, we have that

$$\left(P \Big/ \textstyle\sum_{i=1}^{n-1} a_i R\right) \Big/ a_n R \cong P \Big/ \textstyle\sum_{i=1}^{n} a_i R.$$

And we have that

$$P \Big/ \textstyle\sum_{i=1}^{n} a_i R = (P/A).$$

Finally,

$$\mathrm{height}\left(P/\sum_{i=1}^{n-1} a_i R\right) \leq \mathrm{height}\,(P/A) + 1.$$

Hence,

$$\mathrm{height}(P) + 1 - n \leq \mathrm{height}\left(P/\sum_{i=1}^{n-1} a_i R\right) \leq \mathrm{height}\,(P/A) + 1$$
$$\mathrm{height}(P) + 1 - n \leq \mathrm{height}\,(P/A) + 1$$
$$\mathrm{height}(P) \leq \mathrm{height}\,(P/A) + n.$$

□

Remark 1.2.13 Although we have reached the concept of normalizing sequence by two ways, Definition 1.2.1 and Definition 1.2.10, both definitions are the same. The difference is that in Definition 1.2.1 normalizing sequence is defined for $\mathbb{K}$-algebras, while in Definition 1.2.10 it is defined for rings.

Now, we move on to the second part of this section. In this part we see how the normalizing sequence appears to define graded skew Clifford algebras. We begin defining graded Clifford algebras and, with it, we generalize them to arrive at the algebras that interest us.

Definition 1.2.14 ([Vee17], Definition 2.1) Let $M_1, \ldots, M_n \in M_n(\mathbb{K})$ be symmetric matrices. A ***graded Clifford algebra*** is a $\mathbb{K}$-algebra C on degree-one generators $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with the following defining relations:

(i) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$, for all $i, j = 1, \ldots, n$;

(ii) y_k is central, for all $k = 1, ..., n$.

Next, we show a simple example of an algebra that meets both conditions to be graded Clifford algebra.

Example 1.2.15 ([Vee17], Example 1) Suppose $M_1 = \begin{bmatrix} 2 & \lambda \\ \lambda & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, where $\lambda \in \mathbb{K}$. Let C be the graded $\mathbb{K}$-algebra on degree-one generators x_1, x_2 and on degree-two generators y_1, y_2 with definitions given by Definition 1.2.14 (i). That is, for $i, j = 1, 2$, the defining relations of C are: $2x_1^2 = 2y_1$, $x_2^2 = y_2$, and $x_1x_2 + x_2x_1 = \lambda y_1 = \lambda x_1^2$.

In order to define graded skew Clifford algebras, we must first have the generalized concept of symmetric matrix.

Definition 1.2.16 ([Vee17], Definition 3.2) For $\{i, j\} \subset \{1, \ldots, n\}$, let $\mu_{ij} \in \mathbb{K}^\times$ such that $\mu_{ij}\mu_{ji} = 1$, for all i, j where $i \neq j$, and $\mu_{ii} = 1$, for all $i = 1, \ldots, n$. We write $\mu = (\mu_{ij}) \in M_n(\mathbb{K})$.

A matrix $M \in M(n, \mathbb{K})$ is called μ-***symmetric***, if $(M)_{ij} = \mu_{ij}(M)_{ji}$, for all $i, j = 1, \ldots, n$. We write $M^\mu(n, \mathbb{K})$ for the set of μ-symmetric matrices in $M(n, \mathbb{K})$.

Example 1.2.17 We have that $M = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 3 \\ \frac{1}{4} & 1 & 2 \end{bmatrix}$ is a μ-symmetric matrix with $\mu =$

$$\begin{bmatrix} 1 & 2 & 4 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{4} & \frac{1}{3} & 1 \end{bmatrix}.$$

Finally, we see the appearance of normalizing sequences in Clifford algebras.

Definition 1.2.18 ([Vee17], Definition 3.2) A ***graded skew Clifford algebra*** $A = A(\mu, M_1, \ldots, M_n)$ associated to μ and $M_1, \ldots, M_n \in M^{\mu}(n, \mathbb{K})$ is a $\mathbb{Z}$-graded $\mathbb{K}$-algebra on degree-one generator $x_1, \ldots, x_n$ and on degree-two generators $y_1, \ldots, y_n$ with defining relations given by:

(i) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$, for all $i, j = 1, \ldots, n$;

(ii) the existence of a normalizing sequence $\{y_1', \ldots, y_n'\}$ that spans $\mathbb{K}y_1 + \cdots + \mathbb{K}y_n$.

Example 1.2.19 ([Van15a], Examples 2.5) (Quantum affine plane) Let $n = 2$, and consider $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. The degree-2 relations of $A(\mu, M_1, M_2)$ have the form

$$2x_1^2 = 2y_1, \qquad 2x_2^2 = 2y_2, \qquad x_1x_2 + \mu_{12}x_2x_1 = 0,$$

so that $\mathbb{K}\langle x_1, x_2\rangle / \langle x_1x_2 + \mu_{12}x_2x_1\rangle$ is a graded skew Clifford algebra with isomorphism $M_i \mapsto 2x_i^2$, for $i = 1, 2$.

1.3 The geometry of normalizing sequences

With the definition of normalizing sequence, we proceed to see how it behaves with geometry from an algebraic point of view. In this section, we see notions such as point module, scheme and complete intersections.

First, we introduce an algebra whose role is important in algebraic geometry.

Definition 1.3.1 ([Van15a], Motivation 1A) Consider the $\mathbb{K}$-algebra S on generators $z_1, \ldots, z_n$ with defining relations

$$z_j z_i = \mu_{ij} z_i z_j, \quad \text{for all distinct} \quad i, j,$$

where $\mu_{ij} \in \mathbb{K}^{\times}$, for all i, j, and $\mu_{ij}\mu_{ji} = 1$, for all distinct i, j.

Remark 1.3.2 Let A be a graded skew Clifford algebra. We consider $V = A_1$ and we can view V^* as S_1 (S as Definition 1.3.1), so that U is the span in $V^* \otimes_{\mathbb{K}} V^*$ of the defining relations of S.

Definition 1.3.3 ([CV10], Definition 1.5) For any $q \in S_2$ we call the ***zero locus*** of q in $\mathcal{V}(U)$ the ***quadric associated to*** q, and denote it by $\mathcal{V}_U(q)$; in other words, $\mathcal{V}_U(q) = \mathcal{V}(\mathbb{K}\widehat{q} + U) = \mathcal{V}(\widehat{q}) \cap \mathcal{V}(U)$, where $\widehat{q}$ is any lift of q to $T(S_1)_2$. The span of elements $q_1, \ldots, q_m$ in S_2 will be called the ***quadric system*** associated to $q_1, \ldots, q_m$.

Definition 1.3.4 ([CV10], Definition 1.7) We call a point $(a, b) \in \mathcal{V}(U)$ a ***base point*** of the quadric system associated to $q_1, \ldots, q_m \in S_2$, if $(a, b) \in \mathcal{V}_U(q_k)$ for all $k = 1, \ldots, m$. We say such a quadric system is ***base point free*** if $\bigcap_{k=1}^{m} \mathcal{V}_U(q_k)$ is empty.

Now, we are going to define a geometry on modules, which are called point modules.

Definition 1.3.5 ([CV10], Definition 3.1) A ***right (respectively, left) point module*** over S is a cyclic (Definition A.1.15) graded right (respectively, left) S-module $M = \bigoplus_{i\geq 1} M_i$ such that $M = M_0 S$ or (respectively, SM_0) and $\dim_{\mathbb{K}}(M_i) = 1$, for all i.

Definition 1.3.6 ([Van15a], Definition 1.2) Let $A = \bigoplus_{i=0}^{\infty} A_i$ denote an $\mathbb{N}$-graded, connected $\mathbb{K}$-algebra generated by A_1 where $\dim_{\mathbb{K}}(A_1) = n < \infty$. We define a ***right base-point module*** over A to be a graded 1-critical with respect to GK-dimension (Definition A.2.32) right A-module M such that $M = \bigoplus_{i=0}^{\infty} M_i = M_0 A$ and M has Hilbert series $H_M(t) = c/(1-t)$, for some $c \in \mathbb{N}$.

In Definition 1.3.6, if $c = 1$ the module is a point module; if $c \geq 2$, then the module is called a ***fat point module***.

Remark 1.3.7 ([CV10], p.101) We write $\Gamma_2 = \mathcal{V}(U) \subset \mathbb{P}(V) \times \mathbb{P}(V)$, where $\mathbb{P}(V)$ is projective plane (Definition C.0.24), $\mathcal{V}(U)$ is the zero locus in $\mathbb{P}((S_1)^*) \times \mathbb{P}((S_1)^*)$ of U. and we use $\Gamma_1 \subset \mathbb{P}(V)$ to denote the image of the projection of Γ_2 onto the first copy of $\mathbb{P}(V)$. Then, for all $i \geq 3$, let Γ_i denote all elements $(p_1, \ldots, p_i)$ of $\mathbb{P}(V) \times \cdots \times \mathbb{P}(V)$ such that $(p_j, p_{j+1}) \in \Gamma_2$, for all $j = 1, \ldots, i-1$, and let

$$\Gamma = \{(p_1, p_2, \ldots) \in \mathbb{P}(V) \times \mathbb{P}(V) \times \cdots \mid (p_j, p_{j+1}) \in \Gamma_2, \text{ for all } j \geq 1\}.$$

By Proposition 2.3.7 and Corollary 2.3.8, Γ parametrizes the isomorphism classes of right (respectively, left) point modules over S (see Section 2.3). Let $M = \bigoplus_{i\geq 1} M_i$ be a right point module over S, and, for all $i \geq 0$, we have $M_i = \mathbb{K}u_i$. For all i, j, we can find that $u_i z_j = \alpha_{ij} u_{i+1}$, where $\alpha_{ij} \in \mathbb{K}$. Thereby, to M we can associate the point $p = (p_1, p_2, \ldots) \in \mathbb{P}(V) \times \mathbb{P}(V) \times \cdots$, where $p_{i+1} = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \in \mathbb{P}(V)$, for all $i \geq 0$. Even more, p is well defined, $p \in \Gamma$, and $\Gamma_i \cong \Gamma$, for all $i \geq 1$.

With the following lemma, we can find a characterization of the zero locus with point modules.

Lemma 1.3.8 ([CV10], Lemma 3.2) Suppose that $p = (p_1, p_2, \ldots) \in \Gamma$ and $M = M_0 S$ is the right point module over S corresponding to p. Let $f \in S_d$, where $d > 0$.

(i) We have $f(p_1, \ldots, p_d) = 0$ if and only if $M_0 f = 0$.

(ii) If f is normal in S, then $f(p_1, \ldots, p_d) = 0$ if and only if $Mf = 0$.

Proof. If we can prove (i), we have (ii). We write $M = \bigoplus_{i \geq 0} ku_i$, so that $u_i z_j = z_j(p_{i+1})u_{i+1}$, for all i, j. Then, $u_0 f = f(p_1, \ldots, p_d)u_d$, since $f \in S_d$, we have that (i) holds. □

Definition 1.3.9 ([CV10], Definition 3.3) If $I = \langle f_1, \ldots, f_m \rangle \subset S$, where $f_i \in S_{d_i}$ and $d_i > 0$, for all $i = 1, \ldots, m$, then we say that ***p belongs to*** $\mathcal{V}_U(I)$, if $p = (p_1, p_2, \ldots) \in \Gamma$ and if $f_i(p_1, \ldots, p_{d_i}) = 0$, for all i. In other words, by Lemma 1.3.8(i), it is clear that $p \in \mathcal{V}_U(I)$ if and only if $M_0 f_i = 0$, for all $i = 1, \ldots, m$, where $M = M_0 S$ is the right point module over S corresponding to p.

Example 1.3.10 Consider S in two indeterminates with relations $xy = 2yx$, $I = \langle x, y^2 \rangle$, $f_1 = x \in S_1$, $f_2 = y \in S_2$ and $d_1 = 1$, $d_2 = 2$. Let $p = (p_1, p_2, \ldots) \in \Gamma$, where $p_1 = p_2 = 0$. $p \in \mathcal{V}_U(I)$. Indeed, $f_1(p_1, \ldots, p_{d_i}) = f_1(0) = 0$; $f_2(p_1, \ldots, p_{d_2}) = f_1(0, 0) = 0$.

Remark 1.3.11 We write $\widehat{\mathbf{V}_U(I)}$ to denote the set of isomorphism classes of right base-point modules over S/I.

With this in mind, we have the following proposition that connects the normalizing sequences with the geometry described above.

Proposition 1.3.12 ([CV14], Proposition 10) Let $\mu = (\mu_{ij}) \in M_n(\mathbb{K}^\times)$ such that $\mu_i \mu_{ji} = 1$, for all $1 \leq i, j \leq n$, let S as Definition 1.3.1 and let $I = \langle f_1, \ldots, f_\nu \rangle$, where $f_i \in S \setminus \mathbb{K}^\times$ is homogeneous, for all $i = 1, \ldots, n$. If $\{f_1, \ldots, f_\nu\}$ is a normalizing sequence in S, then $\widehat{\mathcal{V}_U(I)}$ is empty if and only if $\dim_{\mathbb{K}}(S/I)$ is finite.

Proof. Let $M = \bigoplus_{i=0}^{\infty} M_i \in \widehat{\mathbf{V}_U(I)}$. Since $M = M_0(S/I)$ and $\dim_{\mathbb{K}}(M_0) < \infty$, it follows that $\dim_{\mathbb{K}}(S/I) = \infty$ since $\dim_{\mathbb{K}}(M) = \infty$.

Conversely, suppose $\dim_{\mathbb{K}}(S/I) = \infty$. If $n = 1$, then S is commutative and $I = 0$, so the result follows. Henceforth, we suppose inductively that $\widehat{\mathbf{V}_U(I)} \neq \emptyset$, for all $n \leq m - 1$. We will prove that $\widehat{\mathbf{V}_U(I)}$ is nonempty for $n = m$.

If there exists k such that $\dim_{\mathbb{K}}(S/(I + \langle z_k \rangle)) = \infty$, then we may apply the induction hypothesis to $S/\langle z_k \rangle$ to prove that there exists a base-point module M over $S/(I + \langle z_k \rangle)$. Since M is also a base-point module over S/I, it follows that $\widehat{\mathbf{V}_U(I)} \neq \emptyset$. Hence, we may

assume $\dim_{\mathbb{K}}(S/(I+\langle z_k\rangle)) < \infty$ for all k and that $f_i \neq 0$, for all i.

It follows that there exists $t \in \mathbb{N}$ such that $z_i^t \in I + \langle z_k\rangle$, for all i, k. Since I is homogeneous and z_k is normal, there exists $g_{ik} \in S_{t-1}$ such that $z_i^t - g_{ik}z_k \in I$, for all i, k.

On the other hand, since $\dim_{\mathbb{K}}(S/I) = \infty$, $S_d \not\subset I$ for all d, so there exists $z \in \{z_1, \ldots, z_n\}$ such that $z^d \notin I$ for all $d \in \mathbb{N}$. By relabelling, we may assume $z = z_n$. Since z is normal and regular in S, we may form $S' = (S[z^{-1}])_0$. Due to the relations of S, each generator of S', namely, $z_i' = z_i z^{-1}$ for all $i \neq n$, is normal in S'. In fact, S' has $\mathbb{K}$-basis

$$\{(z_1')^{i_1} \cdots (z_{n-1}')^{i_{n-1}} \mid i_k \in \mathbb{N} \cup \{0\} \text{ for all } k\},$$

and we henceforth view $\deg(z_i') = 1$, for all i.

Let $f_i' = f_i z_i^{-d_i}$, where $d_i = \deg(f_i)$, for all i, and let $I' = \sum_{i=1}^{\nu} f_i' S'$. If $I' = S'$, then $1 \in I'$, so $z^t \in I$, for some $t \in \mathbb{N}$, which is false; thus $I' \neq S'$. Taking $z_k = z$ in $z_i^t - g_{ik}z_k \in I$ implies that there exists $g_i' \in S'$, of degree at most $t-1$, such that $(z_i')^t - g_i' \in I'$, for all $i \neq n$. Since $\deg(g_i') \leq t-1$ for all $i \neq n$, and since z_i' is normal in S', for all i, $\{(z_1')^{i_1} \cdots (z_{n-1}')^{i_{n-1}} \mid i_k \in \mathbb{N} \cup \{0\} \text{ for all } k\}$ and $(z_i')^t - g_i' \in I'$ imply that S'/I' has a finite basis, whence $\dim_{\mathbb{K}}(S'/I') < \infty$.

By [CV14], Corollary 9, there exists a base-point module M over S and $w \in M_0^{\times}$ such that $wf_i = 0$, for all i. By [CV14], Lemma 5, $MI = 0$ since $\{f_1, \ldots, f_\nu\}$ is a normalizing sequence in S. Hence, $M \in \widehat{\mathbf{V}_U(I)}$, so $\widehat{\mathbf{V}_U(I)} \neq \emptyset$. □

Before stating an important corollary, we return to regular sequences from Definition 1.1.1 but this time in non-commutative case. Thus, we have the following definition.

Definition 1.3.13 ([CV10], Definition 1.9 (b)) Let B denote a $\mathbb{K}$-algebra. A normalizing sequence $\{b_1, \ldots, b_m\}$ in B is said to be a ***regular sequence***, if b_{k+1} is regular in $B/\langle b_1, \ldots, b_k\rangle$, for all $k = 1, \ldots, m-1$.

The following example is presented in [GP12], Remark 7.6.2 without details. Here, we include the details.

Example 1.3.14 ([GP12], Remark 7.6.2) Consider the ring $\mathbb{K}[x, y, z]$. The sequence $\{x, y(1-x), z(1-x)\}$ is regular. Indeed, if $xw = 0$ for $w \in \mathbb{K}[x, y, z]$, we have that $w = 0$. Now if $\overline{y(1-x)w} = \overline{0}$, we have that $\mathbb{K}[x, y, z]/\langle x\rangle \cong \mathbb{K}[y, z]$. $\mathbb{K}[y, z]$ is an integral domain, so $w = 0$. Finally, if $\overline{z(1-x)w} = \overline{0}$, we have that $\mathbb{K}[x, y, z]/\langle x, y(1-x)\rangle \cong \mathbb{K}[x, y, z]/\langle x\rangle \oplus \mathbb{K}[x, y, z]/\langle y(1-x)\rangle \cong \mathbb{K}[y, z] \oplus \mathbb{K}[x, y, z]/\langle y(1-x)\rangle$. Then $w = 0$.

Finally, we come to the last two results of this section that show us a series of equiva-

lences for normalizing sequences, in which the geometry of these sequences is involved.

Corollary 1.3.15 ([CV10], Corollary 2.6) Let $S = \mathbb{K}[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then $\{f_1, \ldots, f_n\}$ is a regular sequence in S if and only if

$$\dim_{\mathbb{K}} (S/\langle f_1, \ldots, f_n\rangle) < \infty.$$

In this case, for each $k = 1, \ldots, n$, we find that $\{f_1, \ldots, f_n\}$ is a regular sequence in S if and only if $\operatorname{GKdim}(S/\langle f_1, \ldots, f_k\rangle) = n - k$ ([CV10], Proposition 2.3).

(Gelfand-Kirillov dimension, GKdim, is defined in Definition A.2.19)

Proof. S is Noetherian and $\operatorname{GKdim}(S) = n$. Indeed, by Proposition 2.3.24, we have that $\operatorname{GKdim}(S) = \operatorname{GKdim}(\mathbb{K}) + n = 0 + n = n$. Since the elements $z_1, \ldots, z_n$ are normal in S and form a regular sequence, [CV10], Proposition 2.3(c) implies that S is Auslander-Gorenstein (Definition A.2.30 (ii)) of injective dimension (Definition A.2.14 (ii)) n and satisfies the Cohen-Macaulay property (Definition A.2.30 (iv)). Hence, [CV10], Theorem 2.5 applies to S, which completes the proof. □

Corollary 1.3.16 ([CV14], Corollary 11) As above, let $S = \mathbb{K}[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$, for $1 \leq i, j \leq n$. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then the following are equivalent:

(i) $\{f_1, \ldots, f_n\}$ is a regular sequence in S.

(ii) $\dim_{\mathbb{K}}(S/\langle f_1, \ldots, f_n\rangle) < \infty$.

(iii) For each $k = 1, \ldots, n$, we have $\operatorname{GKdim}(S/\langle f_1, \ldots, f_k\rangle) = n - k$.

(iv) $\widehat{\mathbf{V}_U(I)}$ is empty, where $I = \langle f_1, \ldots, f_n\rangle$.

Proof. The equivalence of (i) trough (iv) follows from combining Proposition 1.3.12 with Corollary 1.3.15. □

Example 1.3.17 ([Van15a], Examples 3.2 (a)) We revisit the quantum affine plane from Example 1.2.19, where $n = 2$. In this case, $M_i \mapsto q_i = 2z_i^2 \in S_2$, for $i = 1, 2$. The sequence $\{q_1, q_2\}$ is normalizing in S and $\dim_{\mathbb{K}}(S/\langle q_1, q_2\rangle) < \infty$. Indeed, any element $s \in S$ can be viewed as $s = \sum_{j=1}^{n} k_j z_1^{l_{1j}} z_2^{l_{2j}}$ with $k_j \in \mathbb{K}$, $l_{1j}, l_{2j} \in \mathbb{N} \cup \{0\}$, $1 \leq j \leq n$, $n \geq 1$. So

$$q_1 s = 2z_1^2 s = 2z_1^2 \left(\sum_{j=1}^{n} k_j z_1^{l_{1j}} z_2^{l_{2j}} \right) = \sum_{j=1}^{n} k_j z_1^{l_{1j}} 2z_1^2 z_2^{l_{2j}} = \left(\sum_{j=1}^{n} k_j (\mu_{21})^{2l_{2j}} z_1^{l_{1j}} z_2^{l_{2j}} \right) q_1.$$

Then $q_1S = Sq_1$. Now, we consider $S/\langle q_1\rangle$. If $\overline{0} = \overline{s} \in S/\langle q_1\rangle$, it is clear that $\overline{q_2 s} = \overline{sq_2}$. So, let $\overline{0} \neq \overline{s} \in S/\langle q_1\rangle$. We can see $\overline{s} = \overline{\sum_{j=1}^{n} k_j z_1^{l_{1j}} z_2^{l_{2j}}}$ with $k_j \in \mathbb{K}$, $l_{1j}, l_{2j} \in \mathbb{N} \cup \{0\}$, $k_j z_1^{l_{1j}} \neq 2z_1^2$, $1 \leq j \leq n$, $n \geq 1$. Then

$$\overline{q_2 s} = \overline{2z_2^2 s} = \overline{2z_2^2 \left(\sum_{j=1}^{n} k_j z_1^{l_{1j}} z_2^{l_{2j}}\right)} = \overline{\sum_{j=1}^{n} k_j 2z_2^2 z_1^{l_{1j}} z_2^{l_{2j}}} = \overline{\left(\sum_{j=1}^{n} k_j (\mu_{12})^{2l_{1j}} z_1^{l_{1j}} z_2^{l_{2j}}\right) q_2}.$$

We have that $\overline{q_2} S/\langle q_1\rangle = S/\langle q_1\rangle \overline{q_2}$. Then, $\{q_1, q_2\}$ is a normalizing sequence in quantum plane. So by Proposition 1.3.12, the corresponding quadric system is base-point free.

We have the definition of complete intersection in the commutative case in Definition 1.1.8. However, we can make a definition of this for the non-commutative case as follows:

Definition 1.3.18 ([CV14], Definition 13) As Corollary 1.3.16, let $S = \mathbb{K}[z_1, \ldots, z_n]$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$, for $1 \leq i, j \leq n$. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in S of homogeneous elements of positive degree, then we call $S/\langle f_1, \ldots, f_n\rangle$ a ***complete intersection***, if any of the statements established in Corollary 1.3.16 (i)-(iv) hold.

Example 1.3.19 The set $\{q_1, q_2\}$ from Example 1.3.17 is a complete intersection.

CHAPTER 2

Normalizing sequences in semi-graded algebras

We present the context of finitely semi-graded algebras and geometry of normalizing sequence in this case. For that, we define the concept of finitely semi-graded ring and finitely semi-graded algebra. We see that skew PBW extensions are finitely semi-graded rings. Examples of semi-graded algebras that are not graded, of skew PBW extensions that are finitely semi-graded algebras, and of finitely graded algebras that are not skew PBW extensions, are exposed. Then, we see the appearance of normalizing sequence in a type of semi-graded algebra. For this we consider the theory of the universal enveloping algebra of a Lie algebra. We define a completely solvable algebra, a ring of fractions and we arrive at some properties in which the normalizing sequence appears. Finally, we see how point modules can be parameterized with a certain scheme in the case of graded algebras. Guided by this, we define a scheme that allows us to parameterize point modules of certain semi-graded algebras. With the help of this, we get to the main purpose of this work, which is to see the geometry of the normalizing sequences in certain finitely semi-graded algebras. Finally, we come to Theorem 2.3.33, which tells us the behavior of the normalizing sequence geometry in the graded skew PBW extensions. With this, we leave a way to investigate what happens to this geometry in more general FSG algebras. It should be noted that these results will be included in a manuscript which will be submitted for publication.

2.1 Semi-graded algebras

Finitely graded algebras over fields cover many important classes of commutative and non-commutative rings and algebras coming from mathematical physics. Examples of these algebras are the multi-parameter quantum affine n-space, the Jordan plane, the Manin algebra $Mq(2)$, the multiplicative analogue of the Weyl algebra, among many others. There exist recent interest in developing the non-commutative projective algebraic geometry for finitely graded algebras (e.g. [CV14], [CV10], [NV15], [Van15a], [Vee17] and [Van15b]).

We are interested in investigating normalizing sequences behavior for algebras over fields not being necessarily $\mathbb{N}$-graded. For this purpose, we consider the finitely semi-graded

algebras. These type of algebras extend finitely graded algebras over fields generated in degree one and conform a particular subclass of finitely semi-graded rings.

Definition 2.1.1 ([LL17], Definition 2.1) Let B be a ring. We say that B is ***semi-graded (SG)***, if there exists a collection $\{B_n\}_{n\geq 0}$ of subgroups B_n of the additive group B^+ such that the following conditions hold:

(i) $B = \bigoplus_{n\geq 0} B_n$.

(ii) For every $m, n \geq 0$, $B_m B_n \subseteq B_0 \oplus \cdots \oplus B_{m+n}$.

(iii) $1 \in B_0$.

The collection $\{B_n\}_{n\geq 0}$ is called a ***semi-graduation*** of B and we say that the elements of B_n are ***homogeneous of degree*** n. Let B and C be semi-graded rings and let $f : B \to C$ be a ring homomorphism. We say that f is ***homogeneous***, if $f(B_n) \subseteq C_n$, for every $n \geq 0$.

Definition 2.1.2 ([LL17], Definition 2.2) Let B be a SG ring and let M be a B-module. We say that M is a $\mathbb{Z}$-***semi-graded***, or simply ***semi-graded***, if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups M_n of additive group M^+, such that the following conditions hold:

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) For every $m \geq 0$ and $n \in \mathbb{Z}$, $B_m M_n \subseteq \bigoplus_{k\leq m+n} M_k$.

We say that M is ***positively semi-graded***, also called $\mathbb{N}$-***semi-graded***, if $M_n = 0$, for every $n < 0$. Let $f : M \to N$ be an homomorphism of B-modules, where M and N are semi-graded B-modules. We say that f is ***homogeneous*** if $f(M_n) \subseteq N_n$ for every $n \in \mathbb{Z}$.

The collection $\{M_n\}_{n\in\mathbb{Z}}$ is called a ***semi-graduation*** of M and we say that the elements of M_n are ***homogeneous*** of degree n.

Definition 2.1.3 ([LL17], Definition 2.3) Let B be a SG ring and M be a semi-graded module over B. Let N be a submodule of M, we say that N is a ***semi-graded submodule*** of M, if $N = \bigoplus_{n\in\mathbb{Z}} N_n$.

Now, we introduce an important class of semi-graded rings.

Definition 2.1.4 ([LL17], Definition 2.4) Let B be a ring. We say that B is ***finitely semi-graded (FSG)***, if B satisfies the following conditions:

(i) B is semi-graded.

(ii) There exists finitely many elements $x_1, \ldots, x_n \in B$ such that the subring generated by B_0 and $x_1, \ldots, x_n$ coincides with B.

(iii) For every $n \geq 0$, B_n is a free B_0-module of finite dimension.

Moreover, if M is a B-module, we say that M is FGS if M is semi-graded, finitely generated, and for every $n \in \mathbb{Z}$, M_n is a free B_0-module of finite dimension.

We see some important properties regarding semi-graded rings.

Proposition 2.1.5 ([LL17], Proposition 2.6) Let $B = \bigoplus_{n\geq 0} B_n$ be a SG ring and I be a proper two-sided ideal of B semi-graded as left ideal. Then,

(i) B_0 is a subring of B. Moreover, for any $n \geq 0$, $B_0 \oplus \cdots \oplus B_n$ is a B_0-B_0-bimodule, as well as B.

(ii) B has a standard $\mathbb{N}$-filtration given by

$$F_n(B) := B_0 \oplus \cdots \oplus B_n.$$

(iii) The associated graded ring $\mathrm{Gr}(B)$ satisfies

$$\mathrm{Gr}(B)_n \cong B_n, \text{ for every } n \geq 0 \text{ (isomorphism of abelian groups).}$$

(iv) Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a semi-graded B-module and N a submodule of M. The following conditions are equivalent:

(I) N is semi-graded.

(II) For every $z \in N$, the homogeneous components of z are in N.

(III) M/N is semi-graded with semi-graduation given by

$$(M/N)_n := (M_n + N)/N, n \in \mathbb{Z}.$$

(v) B/I is SG.

(vi) If B is FSG and $I \cap B_n \subseteq IB_n$ for every n, then B/I is FSG.

Finitely graded algebras over fields are examples of FSG rings. In order to present many other examples of FSG rings not being necessarily graded algebras, we recall the notion of skew PBW extension defined firstly in [GL10]. Skew PBW extensions are important since they are a generalization of PBW extensions defined by Bell and Goodearl in [BG88]. In general, skew PBW extensions are not trivially graded rings (see Example 2.1.9). Therefore, the main motivation for the definition of semi-graded rings is to see that all skew PBW extensions are semi-graded rings (see Proposition 2.1.16).

Definition 2.1.6 ([GL10], Definition 1) Let R and A be rings. We say that A is a ***skew PBW extension*** of R (also called a σ-PBW extension of R), if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) There exist finitely many elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\text{Mon}(A) := \{x^\alpha := x^{\alpha_1} \cdots x^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \text{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For every $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that we have the expression $x_i x_j - c_{i,j} x_j x_i \in R + Rx_1 + \cdots + Rx_n$.

Under these conditions, we write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

Proposition 2.1.7 ([GL10], Proposition 3) Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$ for each $r \in R$.

Definition 2.1.8 ([Suá17], Definition 2.5) Let A be a skew PBW extension.

(a) A is ***quasi-commutative***, if the conditions (iii) and (iv) in Definition 2.1.6 are replaced by the following:

(iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For every $1 \leq j < i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) A is ***bijective***, if σ_i is bijective, for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(c) A is ***constant***, if the condition (iii) in Definition 2.1.6 is replaced by the following: for every $1 \leq i \leq n$ and $r \in R$, $x_i r = r x_i$

(d) A is ***pre-commutative***, if the condition (iv) in Definition 2.1.6 is replaced by the following: for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in Rx_1 + \cdots + Rx_n$.

(e) A is called ***semi-commutative***, if A is quasi-commutative and constant.

Example 2.1.9 ([LG19], Example 1.10) Many important algebras and rings coming from mathematical physics are examples of skew PBW extensions:

(i) ***The algebra of q-differential operators*** $D_{q,h}[x,y]$: let $q,h \in \mathbb{K}, q \neq 0$; consider the ring $\mathbb{K}[y][x;\sigma,\delta]$, $\sigma(y) := qy$, $\delta(y) := h$. Then $xy = \sigma(y)x + \delta(y) = qyx + h$ whence $xy - qyx = h$. Thus $D_{q,h}[x,y] \cong \sigma(\mathbb{K}[y])\langle x\rangle \cong \sigma(\mathbb{K})\langle x,y\rangle$.

(ii) ***Algebras of diffusion type***: a diffusion algebra $\mathcal{A}$ with parameters $a_{ij} \in \mathbb{C}^{\times}$, $1 \leq i,j \leq n$ is an algebra over $\mathbb{C}$ generated by variables $x_1,\ldots,x_n$ subject to relations

$$a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j,$$

whenever $i<j$, $b_{ij}, r_i \in \mathbb{C}$, for all $i<j$. From Definition 2.1.6, (iii) and (iv), it is clear that the family of skew PBW extensions are more general than diffusion algebras.

We denote $q_{ij} := \frac{b_{ij}}{a_{ij}}$. If all coefficients q_{ij} are nonzero, then the corresponding diffusion algebra have a PBW basis of standard monomials $x^{i_1}\cdots x^{i_n}$ and hence these algebras are skew PBW extensions. More precisely, $\mathcal{A} \cong \sigma(\mathbb{C})\langle x_1,\ldots,x_n\rangle$.

(iii) ***Quadratic algebras in 3 variables***: a quadratic algebra in 3 variables $\mathcal{A}$ is a $\mathbb{K}$-algebra generated by x,y,z subject to the relations

$$\begin{aligned} yx &= xy + a_1xz + a_2y^2 + a_3yz + \xi_1z^2,\\ zx &= xz + \xi_2y^2 + a_5yz + a_6z^2,\\ zy &= yz + a_4z^2. \end{aligned}$$

If we consider the case $a_1 = a_4 = 0$, we obtain the relations

$$\begin{aligned} yx &= xy + a_2y^2 + a_3yz + \xi_1z^2,\\ zx &= xz + \xi_2y^2 + a_5yz + a_6z^2,\\ zy &= yz. \end{aligned}$$

And we have that $\mathcal{A} \cong \sigma(\mathbb{K}[y,z])\langle x\rangle$. We can also consider the situation when $\xi_2 = 0$ and $a_5 = a_3 = 0$ which implies $a_2 = a_6 = 0$ and thus there is a family of algebras with relations

$$\begin{aligned} yx &= xy + a_1xz + \xi_1z^2,\\ zx &= xz,\\ zy &= yz + a_4z^2. \end{aligned}$$

These algebras are skew PBW extensions of the form $\sigma(\mathbb{K}[x,z])\langle y\rangle$.

(iv) ***Quantum algebra***: $\mathcal{U}'(\mathfrak{so}(3,\mathbb{K}))$ is the $\mathbb{K}$-algebra generated by the variables I_1,I_2,I_3 subject to relations

$$I_2I_1 - qI_1I_2 = -q^{1/2}I_3, I_3I_1 - q^{-1}I_1I_3 = q^{-1/2}I_2, I_3I_2 - qI_2I_3 = -q^{1/2}I_1,$$

where $q \in \mathbb{K}^{\times}$. We have $\mathcal{U}'(\mathfrak{so}(3,\mathbb{K})) \cong \sigma(\mathbb{K})\langle I_1,I_2,I_3\rangle$.

(v) ***Dispin algebra***: $\mathcal{U}(\mathrm{osp}(1,2))$ is the algebra generated by x, y, z over the commutative ring K satisfying the relations

$$yz - zy = z, zx + xz = y, xy - yx = x.$$

Thus, $\mathcal{U}(\mathrm{osp}(1,2)) \cong \sigma(\mathbb{K})\langle x, y, z\rangle$.

(vi) ***Hayashi algebra***: It is the quantized $\mathbb{K}$-algebra $W_q(J)$ defined as the algebra generated by the variables x_i, y_i, z_i, $1 \leq i \leq n$, $i \in J$, where $\mid J \mid = n$, and relations

$$\begin{aligned}
&x_jx_i = x_ix_j, \quad z_jz_i = z_iz_j, \quad y_jy_i = y_iy_j, 1 \leq i, j \leq n,\\
&z_jy_i = y_iz_j, \quad z_jx_i = x_iz_j, \quad y_ix_j = x_jy_i, i \neq j,\\
&z_iy_i = qy_iz_i, \quad (z_ix - i - qxiz_i)y_i = 1 = y_i(z_ix - i - qxiz_i), y_ix_i = qx_iy_i, 1 \leq i \leq n,
\end{aligned}$$

with $q \in \mathbb{K}^\times$. We remark that $W_q(J)$ is a skew PBW extension of the Laurent polynomial ring $\mathbb{K}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$. Indeed, if we consider

$$x_iy_j^{-1} = y_j^{-1}x_i, z_iy_j^{-1} = y_j^{-1}z_i, y_jy_j^{-1} = y_j^{-1}y_j = 1, z_ix_i = qx_iz_i + y_i^{-1},$$

for $1 \leq i, j \leq n$, then $W_q(J) \cong \sigma(\mathbb{K}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}])\langle x_1, \ldots, x_n; z_1, \ldots, z_n\rangle$.

For more details of theses examples, see [Rey13b]. Other algebras that are skew PBW extensions are the following: Classical polynomial rings (Example B.0.5), skew polynomial ring of derivation type (Example B.0.6), Weyl algebras (Example B.0.8), enveloping algebras of finite dimensional Lie algebras (Example B.0.9), Ore algebras of derivation type (Example B.0.7), additive and multiplicative analogues of the Weyl algebra (Examples B.0.11 and B.0.12), Woronowicz algebra (Example B.0.16), Manin algebra (Example B.0.20), algebra of linear partial shift operators (Example B.0.23), q-Heisenberg algebra (Example B.0.13), differential operators on a quantum space (Example B.0.17), multiparameter quantized Weyl algebra (Example B.0.18), quantum symplectic space (Example B.0.19), some 3-dimensional skew polynomial algebras (Example B.0.21), particular types of Sklyanin algebras (Example B.0.14), homogenized enveloping algebra (Example B.0.15), Sridharan enveloping algebra of 3-dimensional Lie algebra $\mathcal{G}$, among many others. Ring and module theoretical properties of these extensions have been investigated (e.g., [Art15], [LAR15] - [LFG$^+$], [Rey14], [RS17b], [NR17], [SR17b]). Relations between graded skew Clifford algebras and skew PBW extensions can be found in [RR].

Proposition 2.1.10 ([LFG$^+$], Theorem 1.3.1) There exists a skew PBW extension A satisfying the following conditions:

(i) $R \subseteq A$,

(ii) A is a left R-free module with basis $\mathrm{Mon}\{x_1, \ldots, x_n\}$,

(iii) $x_ir = \sigma_i(r)x_i + \delta_i(r)$, $1 \leq i \leq n$, $r \in R$,

(iv) $x_jx_i = c_{ij}x_ix_j + a_{ij}^{(1)}x_1 + \cdots + a_{ij}^{(n)}x_n + d_{ij}$, $1 \leq i < j \leq n$,

if and only if

1. for $1 \leq i \leq n$, σ_i is an injective ring homomorphism and δ_i is a σ_i-derivation,
2. $h(x_jx_ir) = h(p(x_jx_i)r)$, for $i < j$ and $r \in R$,
3. $h(x_kx_jx_i) = h(p(x_kx_j)x_i)$, for $i < j < k$.

Proposition 2.1.11 (Universal property) ([LFG+], Theorem 1.2.2) Consider $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension with parameters $R, n, \sigma_k, \delta_k, c_{ij}, d_{ij}, a_{ij}^{(k)}$, $1 \leq i < j \leq n$, $1 \leq k \leq n$. Let B be a ring with a homomorphism $\phi : R \to B$ and elements $y_1, \ldots, y_n \in B$ such that

(i) $y_k\phi(r) = \phi(\sigma_k(r))y_k + \phi(\delta_k(r))$, for every $r \in R$, $1 \leq k \leq n$.

(ii) $y_jy_i = \phi(c_{ij})y_iy_j + \phi(a_{ij}^{(1)})y_1 + \cdots + \phi(a_{ij}^{(n)})y_n + \phi(d_{ij})$, $1 \leq 1 < j \leq n$.

Then, there exists an unique ring homomorphism $\hat{\phi} : A \to B$ such that $\hat{\phi}\mathfrak{i} = \phi$ and $\hat{\phi}(x_i) = y_i$, where $\mathfrak{i}$ is the inclusion of R in A, $1 \leq i \leq n$.

Next, we see what the invariance in skew PBW extensions means and how this helps us to say if a quotient between a skew PBW extension by an ideal, is a skew PBW extension.

Definition 2.1.12 ([LAR15], Definition 2.1) Let R be a ring. $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ be a finite set of endomorphisms of R and $\Delta = \{\delta_i, \ldots, \delta_n\}$ be a finite set such that δ_i is a σ_i-derivation of R, $1 \leq i \leq n$. In this situation we say that (Σ, Δ) is a system of endomorphisms and Σ-derivations of R.

(i) If I is an ideal of R, I is called Σ-***invariant*** if $\sigma_i(I) \subseteq I$, for every $1 \leq i \leq n$. Δ-***invariant*** ideals are defined similarly. If I is both Σ and Δ-invariant, we say that I is (Σ, Δ)-***invariant***.

(ii) A proper Σ-invariant ideal I of R is Σ-***prime***, if whenever a product of two Σ-invariant ideals is contained in I, one of the ideals is contained in I. R is a Σ-***prime***, if the ideal 0 is Σ-prime. Δ-*prime* and (Σ, Δ)-***prime*** ideals and rings are defined similarly.

(iii) The system Σ is commutative if $\sigma_i\sigma_j = \sigma_j\sigma_i$, for every $1 \leq i, j \leq n$. The ***commutativity*** for Δ is defined similarly. The system (Σ, Δ) is ***commutative***, if both Σ and Δ are commutative.

Proposition 2.1.13 ([LAR15], Proposition 2.2) Let R be a ring, (Σ, Δ) a system of endomorphism and Σ-derivations of R, I a proper ideal if R and $\overline{R} := R/I$.

(i) If I is (Σ, Δ)-invariant, then over $\overline{R}$ it is induced a system $(\overline{\Sigma}, \overline{\Delta})$ of endomorphism and $\overline{\Sigma}$-derivations by $\overline{\sigma_i}(\overline{r}) := \overline{\sigma_i(r)}$ and $\overline{\sigma_i}(\overline{r}) := \overline{\delta_i(r)}$, $1 \leq i \leq n$. If σ_i is bijective and $\sigma_i(I) = I$, then $\overline{\sigma_i}$ is bijective.

(ii) If I is Σ-invariant and Σ is commutative, then $\overline{\Sigma}$ is commutative. This property holds for Δ and (Σ, Δ).

(iii) Let I be Σ-invariant. I is Σ-prime if and only if $\overline{R}$ is $\overline{\Sigma}$-prime. Similar properties hold for Δ and (Σ, Δ).

With properties of Σ and Δ below, we can introduce some special classes of skew PBW extensions.

Definition 2.1.14 ([LAR15], Definition 2.3) Let A be a skew PBW extensions of a ring R with a set of endomorphism $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and Σ-derivations $\Delta := \{\delta_1, \ldots, \delta_n\}$.

(i) If $\sigma_i = i_R$, for every $1 \leq i \leq n$, we say that A is a ***skew PBW extension of derivation type***.

(ii) If $\delta_i = 0$, for every $1 \leq i \leq n$, we say that A is a ***skew PBW extension of endomorphism type***. In addition, if every σ_i is bijective, A is a ***skew PBW extension of automorphism type***.

(iii) A is Σ-***commutative*** if the set Σ is commutative. Δ and (Σ, Δ)-***commutative*** of A are defined similarly.

The most important result for this work on skew PBW extensions is given in the following Proposition. This tells us when a quotient of a skew PBW extension is also a skew PBW extension.

Proposition 2.1.15 ([LAR15], Proposition 2.6) Let A be a skew PBW extension of a ring R and I a (Σ, Δ)-invariant ideal of R. Then

(i) IA is an ideal of A and $IA \cap R = I$. IA is proper if and only if I is proper. Moreover, if for every $1 \leq i \leq n$, σ_i is bijective and $\sigma_i(I) = I$, then $IA = AI$.

(ii) If I is proper and $\sigma_i(I) = I$, for every $1 \leq i \leq n$, then A/IA is a skew PBW extension of R/I. Moreover, if A is of automorphism type, then A/IA is of automorphism type. If A is bijective, then A/IA is bijective. In addition, if A is Σ-commutative, then A/IA is $\overline{\Sigma}$-commutative. Similar properties are true for the $\overline{\Delta}$ and $(\overline{\Sigma}, \overline{\Delta})$ commutativity.

(iii) Let A be of derivation type and I proper. Then, $IA = AI$ and A/IA is a skew PBW extension of derivation type of R/I.

(iv) Let R be left (right) Noetherian and σ_i bijective for every $1 \leq i \leq n$. Then, A/IA is a bijective skew PBW extension of R/I.

Now, going back to FSG rings, the following proposition tells us how skew PBW extensions are with respect to semi-graduation.

Proposition 2.1.16 ([LL17], Proposition 2.7) We have the following assertions:

(i) Any $\mathbb{N}$-graded ring is SG.

(ii) Any finitely graded $\mathbb{K}$-algebra is FSG ring.

(iii) Any skew PBW extension is a FSG ring.

Proof. (i) Let A be a $\mathbb{N}$-graded ring. $A = \bigoplus_{n\geq 0} A_n$. $1 \in A_0$; and $A_iA_j \subseteq A_{i+j} \subseteq A_0 \oplus \cdots \oplus A_{i+j}$, for all $i, j \in \mathbb{N}$. Then $\{A_n\}_{n\in\mathbb{N}}$ is a semi-graduation of A.

(ii) Let A be a finitely graded $\mathbb{K}$-algebra. From (i) we have that A is semi-graded. We have that $A_0 = \mathbb{K}$ and that A is finitely generated as $\mathbb{K}$-algebra, then (ii) from 2.1.4 holds. The last condition (iii) from 2.1.4 holds because $A_0 = \mathbb{K}$.

(iii) Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension. Then $A = \bigoplus_{k\geq 0} A_k$, where

$$A_k :=_R \langle x^\alpha \in \text{Mon}(A) \mid \deg(x^\alpha) := \alpha_1 + \cdots + \alpha_n = k\rangle.$$

Thus, A_k is a free left R-module with

$$\dim_R(A_k) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} = \begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix}$$

(Definition A.1.19).

□

With the above definitions, we now consider a particular class of these types of rings.

Definition 2.1.17 ([LFG+], Definition 18.1.1) Let R be a commutative ring and B be an R-algebra. We say that B is ***finitely semi-graded (FSG)***, if the following conditions hold:

(i) B is a FSG ring with semi-graduation $B = \bigoplus_{p\geq 0} B_p$.

(ii) For every $p, q \geq 1$, $B_pB_q \subseteq B_1 \oplus \cdots \oplus B_{p+q}$.

(iii) B is connected, i.e., $B_0 = R$.

(iv) B is generated in degree 1.

There are many examples of finitely semi-graded algebras. We recall some of them.

Example 2.1.18 ([LG19], Example 2.4) (Skew PBW extensions that are FSG algebras). Note that a skew PBW extension of a field is a FSG algebra if and only if it is constant and pre-commutative. Thus, we have:

(i) The following skew PBW extensions of the field $\mathbb{K}$ are FSG algebras: The classical polynomial algebra (Example B.0.5); the particular Sklyanin algebra (Example B.0.14); the universal enveloping algebra of a Lie algebra (Example B.0.9); the quantum algebra $\mathcal{U}'(\mathfrak{so}(3,\mathbb{K}))$ (Example 2.1.9 (iv)); the dispin algebra (Example 2.1.9 (v)); among others.

(ii) In the class of skew quantum polynomials, the multi-parameter quantum affine n-space is another example of skew PBW extension of the field $\mathbb{K}$ that is a FSG (actually finitely graded) algebra. In particular, this is the case for the quantum plane.

Example 2.1.19 ([LG19], Example 2.5) (FSG algebras that are not skew PBW extensions of $\mathbb{K}$). One example that FSG algebras that are not skew PBW extensions of field $\mathbb{K}$ is the Jordan plane. This algebra A is the free $\mathbb{K}$-algebra generated by x, y with relation $yx = xy + x^2$, so $A = \mathbb{K}\{x,y\}/\langle yx - xy - x^2\rangle$. A is not a skew PBW extension of $\mathbb{K}$, but of course, it is a FSG algebra over $\mathbb{K}$, actually, it is a finitely graded algebra over $\mathbb{K}$ (observe that A can be viewed as a skew PBW extension of $\mathbb{K}[x]$, i.e., $A = \sigma(\mathbb{K}[x])\langle y\rangle$) (see [BRS+16] Example 1.3.1).

Example 2.1.20 ([LG19], Example 2.6) (FSG algebras that are not skew PBW extensions). The following FSG algebras are not skew PBW extensions:

(i) Consider the ***Sklyanin algebra*** generated by x, y, z with the following relations

$$ayx + bxy + cz^2 = 0, azy + byz + cx^2 = 0, axz + bzx + cy^2 = 0, a, b, c \in \mathbb{K}.$$

If we consider $c \neq 0$, then S is not a skew PBW extension, but clearly it is a FSG algebra over $\mathbb{K}$.

(ii) The finitely graded $\mathbb{K}$-algebra

$$B = \mathbb{K}[x]/\langle x^2\rangle.$$

In fact, B is a finite dimensional $\mathbb{K}$-algebra, $B = \mathbb{K} \oplus \mathbb{K}x$, so B is Artinian, but none skew PBW extension is an Artinian ring. Indeed, Let $C = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension, then

$$x_1 C \supsetneq x_1^2 C \supsetneq x_1^3 C \supsetneq \cdots.$$

2.2 Universal enveloping algebra of a Lie algebra

Now, we see a first appearance of the normalizing sequence in a semi-graded algebra: the universal enveloping of a Lie algebra.

Definition 2.2.1 ([MRS01], Definition 1.7.1) A Lie algebra $\mathbf{g}$ over a field $\mathbb{K}$, or $\mathbb{K}$-Lie algebra for short, is a $\mathbb{K}$-vector space equipped with a Lie product, i.e. a $\mathbb{K}$-bilinear map $\mathbf{g} \times \mathbf{g} \to \mathbf{g}$, $(x, y) \mapsto [x, y]$, such that $[x, y] = -[y, x]$, $[x, x] = 0$, and satisfying the ***Jacobi identity***

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Note that if A is an associative $\mathbb{K}$-algebra, one can define a $\mathbb{K}$-Lie algebra structure on A by setting $[a, b] = ab - ba$, this product being called the ***bracket product***.

A ***representation*** of a $\mathbb{K}$-Lie algebra $\mathbf{g}$ is defined to be a Lie algebra homomorphism from $\mathbf{g}$ to any such A.

Example 2.2.2 ([MRS01], Example 1.7.3) If A is a $\mathbb{K}$-algebra three-dimensional such that $[x, y] = z$, $[y, z] = x$, $[z, x] = y$, with basis $\{x, y, z\}$, then A is a Lie algebra.

Example 2.2.3 ([Sam12], Example 2) Take for A the algebra of all operators (endomorphism) of a vector spaces V; the corresponding A_L (also called A as Lie algebra) is called the ***general Lie algebra of V***, $\mathfrak{gl}(V)$. Concretely, taking number space $\mathbb{R}^n$ as V, this is the ***general linear Lie algebra*** $\mathfrak{gl}(n, \mathbb{R})$ of all $n \times n$ real matrices, with $[X, Y] = XY - YX$.

Example 2.2.4 ([Sam12], Example 3) The ***special linear Lie algebra*** $\mathfrak{sl}(n, \mathbb{R})$ consists of all $n \times n$ real matrices with trace 0 (and has the same linear and bracket operations as $\mathfrak{gl}(n, \mathbb{R})$-it is a "sub Lie algebra"); similarly for $\mathbb{C}$. For any vector space V we have $\mathfrak{sl}(V)$, the ***special linear Lie algebra of V***, consisting of the operators on V of trace 0.

Now, with the concept of representation and Lie algebra, we see how the universal enveloping algebra is constructed.

Definition 2.2.5 ([MRS01], Definition 1.7.2) If $\mathbf{g}$ is a k-Lie algebra then the ***universal enveloping algebra*** of $\mathbf{g}$ is an (associative) $\mathbb{K}$-algebra $\mathcal{U} = \mathcal{U}(\mathbf{g})$ together with a representation $\theta : \mathbf{g} \to \mathcal{U}$ which is universal; i.e. given any (associative) $\mathbb{K}$-algebra A and representation $\varphi : \mathbf{g} \to A$, there exists a unique algebra homomorphism $\psi : \mathcal{U} \to A$ such that $\psi\theta = \varphi$. As usual with universal objects, $\mathcal{U}(\mathbf{g})$ is uniquely determined up to isomorphism.

Example 2.2.6 ([MRS01], Example 1.6.11) Let S be the $\mathbb{K}$-algebra $\mathbb{K}[x, y, z]$ with relations $xy - yx = z$, $yz - zy = x$, $zx - xz = y$. This is the universal enveloping algebra of the $\mathbb{K}$-algebra described in Example 2.2.2. Indeed, $[x, y] = z$ and with the relation $[x, y] = xy - yz$ of universal enveloping algebra, we have the previous relation. In the same way, we get the other two relations.

Example 2.2.7 ([LFG+], Example 1.1 e)) ***The universal enveloping algebra of the Lie algebra*** $\mathfrak{sl}(2,K)$, $\mathcal{U}(\mathfrak{sl}(2,K))$, is the $\mathbb{K}$-algebra generated by the variables x,y,z subject to the relations $xy-yx=z$, $yz-zy=2y$, $xz-zx=-2x$. Recall that $\mathfrak{sl}(2,K)$ is the Lie algebra of matrices over $\mathbb{K}$ of size 2×2 with null trace (see Example 2.2.4)

Example 2.2.8 ([LFG+], Example 1.1 (e)) $\mathcal{U}(\mathfrak{so}(3,K))$ is the $\mathbb{K}$-algebra generated by the variables x,y,z subject to the relations $xy-yx=z$, $yz-zy=x$, $xz-zx=-y$. $\mathfrak{so}(3,K)$ is the Lie algebra of matrices over $\mathbb{K}$ of size 3×3 that satisfy $F+F^T=0$.

Proposition 2.2.9 (Poincaré-Birkhoff-Witt theorem) ([LFG+], Theorem 2.1.1) The standard monomials $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\alpha_i\geq 0$, $1\leq i\leq n$, conform a $\mathbb{K}$-basis of $\mathcal{U}(\mathcal{G})$.

Proof. For $\mathcal{U}(\mathcal{G})$ we take:

$$X:=\{x_1,\ldots,x_n\},\sigma_i:=i_k,\delta_i:=0$$
$$c_{i,j}:=1,d_{ij}:=0,[x_i,x_j]=a_{ij}^{(1)}x_1+\cdots+a_{ij}^{(n)}x_n,1\leq i<j\leq n.$$

Instead of demonstrating directly, we will use Theorem 2.1.10 to see that conditions (1)-(3) hold. It is clear that condition (i) holds. Recalling that $h=qp$, we have: For (2), let $i<j$ and $r\in\mathbb{K}$, then

$$h(x_jx_ir)=h(x_jrx_i)=h(rx_jx_i)=h(rx_ix_j)+h(r[x_j,x_i])=rx_ix_j+r[x_j,x_i];$$
$$h(p(x_jx_i)r)=h(x_ix_jr)+h([x_j,x_i]r)=h(x_irx_j)+h(r[x_j,x_i])=rx_ix_j+r[x_j,x_i].$$

Condition (3) of Theorem 2.1.10 also holds. For $1<j<k$,

$$\begin{aligned}h(p(x_kx_j)x_i)&=h(x_jx_kx_i)+h([x_k,x_j]x_i)=h(x_jx_ix_k)+h(x_j[x_k,x_i])+h([x_k,x_j]x_i)\\&=x_ix_jx_k+h([x_j,x_i]x_k)+h(x_j[x_k,x_i])+h([x_k,x_j]x_i),\end{aligned}$$

but expanding the brackets, we have that

$$\begin{aligned}h([x_j,x_i]x_k)&=h(x_k[x_j,x_i])+h([[x_j,x_i],x_k]),\\h(x_j[x_k,x_i])&=h([x_k,x_i]x_j)+h([x_j,[x_k,x_i]]),\\h([x_k,x_j]x_i)&=h(x_i[x_k,x_j])+h([[x_k,x_j],x_i]);\end{aligned}$$

where Jacobi identity is used in the last equality. On the other hand,

$$\begin{aligned}h(x_kx_jx_i)&=h(x_kx_ix_j)+h(x_k[x_j,x_i])=h(x_ix_kx_j)+h([x_k,x_i]x_j)+h(x_k[x_j,x_i])\\&=h(x_ix_jx_k)+h(x_i[x_k,x_j])+h([x_k,x_i]x_j)+h(x_k[x_j,x_i])\\&=x_ix_jx_k+h(x_i[x_k,x_j])+h([x_k,x_i]x_j)+h(x_k[x_j,x_i]).\end{aligned}$$

From Theorem 2.1.10 we get that there exists a skew PBW extension $A=\sigma(k)\langle x_1,\ldots,x_n\rangle$. Then, the monomials $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\alpha_i\geq 0$, $1\leq i\leq n$, conform a $\mathbb{K}$-basis of A. Applying Theorem 2.1.11 of A and the universal property of the free algebra $\mathbb{K}\{x_1,\ldots,x_n\}$ we have $\mathcal{U}(\mathcal{G})\cong A$, whence $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\alpha_i\geq 0$, $1\leq i\leq n$, is a $\mathbb{K}$-basis of $\mathcal{U}(\mathcal{G})$. □

Other alternative proofs of Proposition 2.2.9 can be found in [Hig69], Section 3 or [Coh63].

Now, we focus on the universal enveloping of Lie algebras and we will see an important lemma which relates it to the normalizing sequences. However, before we must define some concepts concerning Lie algebras.

Definition 2.2.10 ([MRS01], Definition 14.1.7) Let $\mathbf{g}$ be a $\mathbb{K}$-Lie algebra. We say that $\mathbf{g}$ is ***solvable***, if it has a $\mathbb{K}$-basis $\{x_1, \ldots, x_n\}$ such that $\mathbb{K}x_1 + \cdots + \mathbb{K}x_i \lhd \mathbb{K}x_1 + \cdots \mathbb{K}x_{i+1}$ for each i. This is equivalent to requiring that $\mathbf{g}$ have a chain of Lie subalgebras

$$0 = \mathbf{g}_0 \lhd \mathbf{g}_1 \lhd \cdots \lhd \mathbf{g}_m = \mathbf{g},$$

with each an ideal in the next and each factor being Abelian.

Definition 2.2.11 ([Kir08], Definition 5.22) For a Lie algebra $\mathfrak{g}$, define the series of ideals $D^i\mathfrak{g}$ (called the ***derived series***) by $D^0\mathfrak{g} = \mathfrak{g}$ and $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$.

With Definition 2.2.11, we can arrive at a proposition that gives us some equivalences of solvable.

Proposition 2.2.12 ([Kir08], Proposition 5.23) Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent:

(i) $D^n\mathfrak{g} = 0$, for large enough n.

(ii) $\mathfrak{g}$ is solvable.

(iii) For large enough n, every commutador of the form $[\ldots, [[x_1, x_2], [x_3, x_4]]\ldots]$ (2^n terms, arranged in a binary tree of length n) is zero.

Proof. We begin with the equivalence of (i) and (iii). Suppose that $D^n\mathfrak{g} = 0$ for large enough n. Then

$$D^n\mathfrak{g} = [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}] = [[D^{n-2}\mathfrak{g}, D^{n-2}\mathfrak{g}], [D^{n-2}\mathfrak{g}, D^{n-2}\mathfrak{g}]] = [\ldots, [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]\ldots] = 0$$

This happens for all $\mathfrak{g}$. Then we have that every commutador of the form

$$[\ldots, [[x_1, x_2], [x_3, x_4]]\ldots], \quad x_i \in \mathfrak{g}, \quad 1 \leq i \leq 2n,$$

is zero. The other implication is demonstrated in the same way. Implication $(i) \Rightarrow (ii)$ is clear: we take $\mathfrak{a}^i = D^i\mathfrak{g}$. $(ii) \Rightarrow (i)$, if $\mathfrak{a}^i$ satisfies the conditions of the proposition, then by [Kir08], Lemma 5.20, we have $\mathfrak{a}^{i+1} \supset [\mathfrak{a}^i, \mathfrak{a}^i]$. Thus, using induction, we see that $\mathfrak{a}^i \supset D^i\mathfrak{g}$. □

Definition 2.2.13 ([Kir08], Definition 5.25) For a Lie algebra $\mathfrak{g}$, define the series of ideals $D_i\mathfrak{g}$ (called the ***lower central series***), by $D_0\mathfrak{g} = \mathfrak{g}$ and $D_{i+1}\mathfrak{g} = [\mathfrak{g}, D_i\mathfrak{g}]$.

With Definition 2.2.13, we can arrive at a proposition that gives us some equivalences of ***nilpotent***.

Proposition 2.2.14 ([Kir08], Proposition 5.26) Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent:

(i) $D_n\mathfrak{g} = 0$, for large enough n.

(ii) There exists a sequence of ideals $\mathfrak{a}_0 = \mathfrak{g} \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_k = \{0\}$ such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$.

(iii) For large enough n, every commutador of the form $[\ldots, [[x_1, x_2], x_3], x_4] \ldots x_n]$ (n terms) is zero.

If $\mathfrak{g}$ satisfies any of the equivalent conditions, $\mathfrak{g}$ is called ***nilpotent***.

Proof. We begin with the equivalence of (i) and (iii). Suppose that $D_n\mathfrak{g} = 0$ for large enough n. Then

$$D_n\mathfrak{g} = [\mathfrak{g}, D_{n-1}\mathfrak{g}] = [\mathfrak{g}, [\mathfrak{g}, D_{n-2}\mathfrak{g}]] = [\ldots, [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \mathfrak{g}] \ldots \mathfrak{g}] = 0$$

This happens for all $\mathfrak{g}$, then we have that every commutador of the form

$$[\ldots, [[x_1, x_2], x_3], x_4] \ldots x_n], \quad x_i \in \mathfrak{g}, \quad 1 \leq i \leq n,$$

is zero. The other implication is demonstrated in the same way. Implication $(i) \Rightarrow (ii)$ is clear: we take $\mathfrak{a}_i = D_i\mathfrak{g}$. $(ii) \Rightarrow (i)$, if $\mathfrak{g}_i$ satisfies the conditions of the proposition, then using induction, we can see that $\mathfrak{a}_i \supset D_i\mathfrak{g}$. □

Example 2.2.15 ([Kir08], Example 5.28) Let $\mathfrak{b} \subseteq \mathfrak{gl}(n, \mathbb{K})$ be the subalgebra of upper triangular matrices, and $\mathfrak{n}$ be the subalgebra of all strictly upper triangular matrices. Then $\mathfrak{b}$ is solvable, and $\mathfrak{n}$ is nilpotent.

Let us first generalize it. Namely, if $\mathcal{F}$ is a ***flag*** in a finite-dimensional vector space V, that is

$$\mathcal{F} = (\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V)$$

with $\dim_{\mathbb{K}}(V_i) < \dim_{\mathbb{K}}(V_{i+1})$ (we do not require that $\dim_{\mathbb{K}}(V_i) = i$), then define

$$\mathfrak{b}(\mathcal{F}) = \{\, x \in \mathfrak{gl}(V) \mid xV_i \subset V_i \text{ for all } i \,\},$$
$$\mathfrak{n}(\mathcal{F}) = \{\, x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-1} \text{ for all } i \,\}.$$

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for V. By taking $\mathcal{F}$ to be the standard flag in $\mathbb{K}^n$ (i.e. when $V_i = \langle b_1, \ldots, b_i \rangle$ and $\dim_{\mathbb{K}}(V_i) = i$) we recover the Lie algebras $\mathfrak{b}$, $\mathfrak{n}$ defined above.

We claim that $\mathfrak{n}(\mathcal{F})$ is nilpotent. Indeed, we define more general algebras

$$\mathfrak{a}_k(\mathcal{F}) = \{\, x \in \mathfrak{gl}(V) \mid xV_i \subset V_{i-k} \text{ for all } i \,\}.$$

so that $\mathfrak{b}(\mathcal{F}) = \mathfrak{a}_0$, $\mathfrak{n}(\mathcal{F}) = \mathfrak{a}_1$. Then it is obvious that for $x \in \mathfrak{a}_k$, $y \in \mathfrak{a}_l$, we have $xy \in \mathfrak{a}k + l$ (here xy is the usual product in $\mathrm{End}(V)$); this, $[\mathfrak{a}_k, \mathfrak{a}_l] \subset \mathfrak{a}_{k+l}$, so $D_i\mathfrak{n} \subset \mathfrak{a}_{i+1}$. This proves nilpotency of $\mathfrak{n}(\mathcal{F})$ (see [Kir08], Proposition 5.26).

We can show solvability of $\mathfrak{b}$ (for the standard flag $\mathcal{F}$), noting that even though for $x, y \in \mathfrak{b}$ we can only say that $xy \in \mathfrak{b}$, for the commutator we have a stronger condition: $[x, y] \in \mathfrak{n} = \mathfrak{a}_1$. Indeed, diagonal entries of xy and yx coincide. This, $D^1\mathfrak{b} \subset \mathfrak{n} = \mathfrak{a}_1$. From here it easily follows by induction that $D^{i+1}\mathfrak{b} \subset \mathfrak{a}_{2^i}$.

Definition 2.2.16 ([MRS01], Definition 14.1.8) We also recall that $\mathbf{g}$ is ***completely solvable*** if, in the notation of Definition 2.2.10, it can also be arranged that each $\mathbb{K}x_1 + \cdots + \mathbb{K}x_i$ is an ideal of $\mathbf{g}$ or, equivalently, that each $\mathbf{g}_i$ is an ideal of $\mathbf{g}$ and each $\mathbf{g}_{i+1}/\mathbf{g}_i$ is one-dimensional over $\mathbb{K}$.

Example 2.2.17 ([Kna13], Example 1.1.12) The Lie algebra of all matrices

$$\begin{bmatrix} t & 0 & x \\ 0 & t & y \\ 0 & 0 & 0 \end{bmatrix}$$

is an example of completely solvable Lie algebra. This is isomorphic with the Lie algebra of the group of translations and dilations of the plane.

With the above definitions, we now look at some properties and definitions of multiplicative closed subsets. This allow us to arrive at a proposition where we can find the normalizing sequence in certain universal enveloping algebras.

Definition 2.2.18 ([MRS01], Definition 2.1.6) A multiplicatively closed subset S of R is said to satisfy the ***right Ore condition*** if, for each $r \in R$ and $s \in S$, there exist $r' \in R$, $s' \in S$ such that $rs' = sr'$.

Example 2.2.19 Consider the quantum plane described in Example 1.2.19. If we consider $S = \mathbb{K}[y]$, then S satisfies the right Ore condition. Indeed, if we consider $r = ax^{\alpha_1}y^{\alpha_2}$, $s = by^{\alpha_3}$, where $\alpha_3 > \alpha_2$, i.e., there is $\gamma \in \mathbb{N}$ such that $\alpha_2 + \gamma = \alpha_3$. Then $r' = b^{-1}q^{\alpha_1\alpha_3}x^{\alpha_1}$ and $s' = a^{-1}y^{\alpha_3}$, and so Ore condition is fulfilled.

Definition 2.2.20 ([MRS01], Definition 2.1.1) If R is a commutative ring and S is a multiplicatively closed subset of R, the localization $S^{-1}R$ can be constructed (as in the commutative case. If S satisfies right Ore condition, we say that $R_S = S^{-1}R$ is a ***ring of fractions*** of R with respect to S.

Example 2.2.21 Some rings of fractions are the following:

(i) Let $R = \mathbb{Z}$ and $S = R\backslash\{0\}$. Then the ring of fractions is $R_S = \mathbb{Q}$.

(ii) If $R = \mathbb{K}[x]$ and $S = \{x^n \mid n \in \mathbb{Z}\}$, then $R_S = \mathbb{K}[x, x^{-1}]$, the ***Laurent polynomials*** over $\mathbb{K}$.

(iii) If R is an integral domain and $S = R\backslash\{0\}$, then R_S is a field, called the ***field of fractions***.

Definition 2.2.22 ([MRS01], Definition 2.1.2) An element $x \in R$ is ***right regular***, if $xr = 0$ implies $r = 0$ for $r \in R$. Similarly ***left regular*** is defined. ***Regular*** means both right and left regular (and hence not a zero divisor).

Example 2.2.23 Some regular and not regular elements are the following

(i) Let R be an integral domain. Each $r \in R\backslash\{0\}$ is regular.

(ii) If $R = \mathbb{Z}_6$, then $\overline{2}$ it is not regular, because $\overline{2} \cdot \overline{3} = \overline{3} \cdot \overline{2} = \overline{0}$.

Definition 2.2.24 ([MRS01], Definition 2.1.3) Let S be a (non-empty) multiplicatively closed subset of a ring R, and let

$$\operatorname{ass}(S) = \{\, r \in R \mid rs = 0 \text{ for some } s \in S \,\}.$$

Then a ***right quotient ring*** of R with respect to S is a ring Q together with a homomorphism $\theta : R \to Q$ such that:

(i) for all $s \in S$, $\theta(s)$ is a unit in Q;

(ii) for all $q \in Q$, $q = \theta(r)\theta(s)^{-1}$, for some $r \in R$, $s \in S$; and

(iii) $\operatorname{Ker}(\theta) = \operatorname{ass}(S)$.

If, further, $\operatorname{ass}(S) = 0$, one can identify R with its image under θ, and then each $q \in Q$ takes the form rs^{-1}.

Corollary 2.2.25 ([MRS01], Corollary 2.1.4) Let R be a ring and let S be a multiplicatively closed subset of R. Then

(i) If there exists a right quotient ring Q of R with respect to S, then it is unique up to isomorphism.

(ii) If R also has a left quotient ring Q' with respect to S then $Q \cong Q'$.

Remark 2.2.26 The latter statement shows that, when both Q and Q' exist, then $\mathrm{ass}(S)$ equals its left-hand version, namely $\{r \in R \mid sr = 0, \text{ for some } s \in S\}$. The uniqueness given by Corollary 2.2.25 makes razonable the notation R_S for Q. This ring is also called the (***right***) ***localization*** of R at S.

The examples below show that the right Ore condition is not always satisfied.

Example 2.2.27 ([MRS01], Example 2.1.7)

(i) Let $R = \mathbb{K}\langle x, y\rangle$ be the free associative algebra in two indeterminates over a field $\mathbb{K}$. Let $S = R \setminus 0$. It is immediate that the right Ore condition fails, choosing, say, $r = x$ and $s = y$. This also shows, by [MRS01], Proposition 2.1.6 (it says that if a right quotient ring R_S exists then S satisfies the right Ore condition), that R_S does not exist.

(ii) Let $R = \mathbb{K}[x, y]$ with $xy = y(x + 1)$. This is a Noetherian integral domain and $P = xR + yR$ is a prime ideal with $R/P \cong \mathbb{K}$. Let $S = R/P$. The right Ore condition again fails, choosing $r = y$ and $s = x - 1$.

Definition 2.2.28 ([MRS01], Definition 2.1.13) A multiplicatively closed subset S of a ring R which satisfies the right Ore condition is called a ***right Ore set***. A right Ore set such that the elements of $\overline{S}$ are regular in $\overline{R} = R/\mathrm{ass}S$ (and so R_S exists) will be called a ***right denominator set***.

Example 2.2.29 Again, we consider quantum plane and S like in Example 2.2.19. S is a right denominator set.

Definition 2.2.30 Let R be a ring, we define ***Z(R)*** as follows

$$Z(R) \coloneqq \{\, z \in R \mid zr = rz \text{ for all } r \in R \,\}.$$

Definition 2.2.31 ([MRS01], Definition 14.1.3) A ***right $\mathfrak{g}$-module*** is a vector space M together with a k-bilinear map $M \times \mathfrak{g} \to M$, $(m, x) \mapsto [m, x]$ such that

$$[m, [x, y]] = [[m, x], y] - [[m, y], x], \quad \text{for each} m \in M, x, y \in \mathfrak{g}.$$

For example, $\mathfrak{g}$ is itself a right $\mathfrak{g}$-module under its own product; and its ideals are the submodules.

Definition 2.2.32 ([MRS01], Definition 14.1.15) Let M be a nonzero $\mathfrak{g}$-module. The nonzero elements $m \in M$ such that $\mathbb{K}m \lhd M$ are called the ***eigenvectors*** (or ***semi-invariants***) of $\mathfrak{g}$ in M. The set of eigenvectors is denoted by $E(M;\mathfrak{g})$ or simply $E(M)$. Those eigenvectors which also have $m\mathfrak{g} = 0$ are the ***invariants*** of $\mathfrak{g}$, and $M^{\mathfrak{g}}$ denotes the set of these.

All that is described throughout this section, is to get to the next Lemma. This is important because the notion of normalizing sequence appears in the case of the universal enveloping algebra of a Lie algebra.

Lemma 2.2.33 ([MRS01], Lemma 14.3.4) Let $\mathfrak{g}$ be completely solvable, R be a factor ring of $U(\mathfrak{g})$, and S be a right denominator set in R. Then:

(i) if $0 \neq I \lhd R_S$ then $I \cap E(R) \neq \emptyset$;

(ii) if $a \in E(R_S)$ then $aR = Ra$;

(iii) $R^{\mathfrak{g}} = Z(R) \setminus \{0\}$;

(iv) each ideal of R has a normalizing sequence of generators; and if $\mathfrak{g}$ is nilpotent, a centralizing sequence.

Proof. (i) It is clear because we have $I \lhd R_S$ implies that $I \subseteq E(R_S)$. Then $I \cap E(R) \neq \emptyset$.

(ii) if $\lambda \in \mathfrak{g}^*$ is the corresponding eigenvalue then $[a, x] = \lambda(x)a$ for each x in $\mathfrak{g}$. Thus $a\bar{x} = \bar{x}a + \lambda(x)a$ and so $aR = Ra$.

(iii) If we take $\lambda = 0$ from the proof of ii), $a\bar{x} = \bar{x}a$. So, $R^{\mathfrak{g}} = Z(R) \setminus \{0\}$.

(iv) We will using induction over the generators of an ideal $I \subseteq R$. If $I = \langle a \rangle$, using i) and ii), we have that $aR = Ra$.

Now, suppose that $I = \langle a_1, \ldots, a_n \rangle$ and $\{a_1, \ldots, a_{n-1}\}$ is a normalizing sequence. We have that $\hat{I} \subset \hat{R}$ is an ideal, where $\hat{I} = I/\langle a_1, \ldots, a_{n-1} \rangle$ and $\hat{R} = R/\langle a_1, \ldots, a_{n-1} \rangle$. But $\hat{I} = \langle \overline{a_n} \rangle$. We are in the case of generation by a single element, then applying the case when $n = 1$, we have that $\overline{a_n}\hat{R} = \hat{R}\overline{a_n}$. So, $\{a_1, \ldots, a_n\}$ is a normalizing sequence.

□

2.3 Point modules and schemes in graded algebras

The main idea in this section is to see how schemes can parametrize point modules in graded algebras, and try to find an analogy for point modules in semi-graded algebras.

By achieving this, we can build a notion of locus zero on a finitely semi-graded algebras.

Definition 2.3.1 ([Van15a], p. 373) Let S be the $\mathbb{K}$-algebra defined in Section 1.2. If we take $\mu_{ij} = 1$, for all $1 \leq i \leq n$, then S is the commutative polynomial ring and its algebraic geometry is quite extensive, as shown in Section 1.3. In particular, we have that the point of $\mathbb{P}(S_1^*)$ are in one-to-one correspondence with certain ideals of S via

$$(\alpha_1, \ldots, \alpha_n) \leftrightarrow (\alpha_i z_1 - \alpha_1 z_i, \ldots, \alpha_i z_n - \alpha_n z_i), \quad \alpha_i \neq 0.$$

Another important fact is that we have an ideal I, the graded module S/I has Hilbert series $H(t) = 1/(1-t)$ and that S/I is a 1-critical graded cyclic module over S.

Now, in the case where $\mu_{ij} \neq 1$, we can see S "close" to commutative. Therefore, we can expect that there is a geometry behind. The geometrics objects are modeled on the module S/I.

We will see how we can have geometry in S. For that, next definition is very important.

Definition 2.3.2 ([Van15a], Definition 1.1) Let $A = \bigoplus_{i=0}^{\infty} A_i$ denote an $\mathbb{N}$-graded, connected $\mathbb{K}$-algebra generated by A_1 where $\dim_{\mathbb{K}}(A_1) = n < \infty$. A graded right A-module $M = \bigoplus_{i=0}^{\infty} M_i$ is called a ***line module***, if

(i) M is cyclic with $M = M_0 A$, and

(ii) $\dim_{\mathbb{K}}(M_i) = i + 1$, for all i.

Remark 2.3.3 ([Van15a], p. 374) If A is the polynomial ring S, then the module S/I is a point module. In general, we may associate some geometry to point and line modules. Indeed, first condition implies that A maps onto M via $a \mapsto ma$, for all $a \in A$, where $\{m\}$ is a $\mathbb{K}$-basis for M_0, and this map restricts via the grading to a linear map $\beta : A_1 \to M_1$. Let $K \subset A_1$ denote the kernel of β. The second condition implies that $\dim_{\mathbb{K}}(K) = n - 1$ ($n - 2$ if we take a line module), so that $K^{\perp} \subset A_1^*$ has dimension one (respectively, two). Thus, $\mathbb{P}(K^{\perp})$ is a point (respectively, a line) in the geometric space $\mathbb{P}(A_1^*)$.

For point modules we have that its Hilbert series is $1/(1-t)$. On the other hand, the Hilbert series for line modules is $1/(1-t)^2$. With this, we can define a plane module, changing the condition (ii) of Definition 2.3.2 by the requirement that the module has a Hilbert series $1/(1-t)^3$. In the same way, we may define d-linear modules, where the definition is modeled on Definition 2.3.2, but the module has Hilbert series $1/(1-t)^{d+1}$.

Now, we focus on how point modules can be parameterized by schemes. Categorically speaking, *there is a scheme that represents the functor of point modules.*

Although we are working on non-commutative algebraic geometry, it is important to see an example of scheme in the commutative version. Therefore, the following example is presented.

Example 2.3.4 ([LFG+], Example A.4.2) Consider $\mathbb{K}$ an algebraically closed field and let $\mathbb{A}^n$ be the affine space (Definition C.0.1). We consider the Zariski topology for $\mathbb{A}^n$ (Definition C.0.3). Let V be any nonempty variety of $\mathbb{A}^n$. We associate a scheme $(\widetilde{V}, \mathcal{F})$ to V. Now, we have that $\widetilde{V}$ is the set of nonempty varieties contained in V; we consider $U \subseteq V$ an open set and let $\widetilde{U}$ be the set of nonempty varieties contained in U. The collection $\{\widetilde{U}\}_{u \subseteq V \text{open}}$ define a topology for $\widetilde{V}$. Then, we define

$$\mathcal{F}(\widetilde{U}) := \left\{ \widetilde{z} = \frac{\overline{f}}{\overline{g}} \in \mathcal{K}(V) \mid \widetilde{z} \text{ is defined in every point } P \in U \right\}.$$

$$\mathcal{F}^{\widetilde{U}}_{\widetilde{U'}} : \mathcal{F}(\widetilde{U}) \to \mathcal{F}(\widetilde{U'}), \widetilde{z} \mapsto \widetilde{z}, \text{ for } U' \subseteq U,$$

where $\mathcal{K}(V) := Q(\mathcal{A}(V))$ is the field of fractions of $\mathcal{A}(V) := K[x_1, \ldots, x_n]/I(V)$, the coordinate ring of V. Recall that $\mathcal{F}(\widetilde{U}) := \{\widetilde{z} : U \to K \mid \widetilde{z} \text{ is regular } \}$.

Having the notion of a scheme, we can now focus on how a scheme can parameterize the point modules.

Definition 2.3.5 ([ATVdB07], Definition 3.8) A ***truncated point module of length*** $d+1$ is a module M which satisfies (i) and (ii) from Definition 2.3.2, and whose Hilbert function is

$$\dim_{\mathbb{K}}(M_i) = \begin{cases} 1, & \text{if } 0 \geq i \geq d; \\ 0, & \text{otherwise .} \end{cases}$$

Let $T = \mathbb{K}\langle x_0, \ldots, x_n \rangle$ denote the free associative k-algebra on generators $x_0, \ldots, x_n$ of degree 1, and let $V = T_1^*$ denote the dual space of $T_1 =_{\mathbb{K}} \langle x_0, \ldots, x_n \rangle$. A homogeneous element $f \in T_d = T_1^{\otimes d}$ of degree d, i.e., $f = f_1 \otimes \cdots \otimes f_d$, for $f_i \in T_1, 1 \leq i \leq d$, defines a linear map

$$\widetilde{f} : V \otimes \cdots \otimes V = V^{\otimes d} \to \mathbb{K},$$

or equivalently, a multilinear form $\widetilde{f} : V \times \cdots \times V = V^d \to \mathbb{K}$. Indeed, let $g_1, \ldots, g_d \in V$

$$\widetilde{f}(g_1, \ldots, g_d) = g_1(f_1) \cdots g_d(f_d).$$

Now, let $r, s \in \mathbb{K}$ and $g_1, \ldots, g_i, g_i', \ldots, g_d \in V$; then:

$$\begin{aligned}\widetilde{f}(g_1,\dots,r\cdot g_i+s\cdot g_i',\dots,g_d) &= g_1(f_1)\cdots(r\cdot g_i+s\cdot g_i')(f_i)\cdots g_d(f_d)\\ &= g_1(f_1)\cdots(r\cdot g_i(f_i)+s\cdot g_i'(f_i))\cdots g_d(f_d)\\ &= g_1(f_1)\cdots r\cdot g_i(f_i)\cdots g_d(f_d)+g_1(f_1)\cdots s\cdot g_i'(f_i)\cdots g_d(f_d)\\ &= r\cdot g_1(f_1)\cdots g_i(f_i)\cdots g_d(f_d)+s\cdot g_1(f_1)\cdots g_i'(f_i)\cdots g_d(f_d)\\ &= r\cdot\widetilde{f}(g_1,\dots,g_i,\dots,g_d)+s\cdot\widetilde{f}(g_1,\dots,g_i',\dots,g_d),\end{aligned}$$

for all $1\le i\le d$.

As in [ATVdB07], we consider $\mathbb{P}\approx\mathbb{P}^n$. We want define a sheaf on the product $\mathbb{P}\times\cdots\times\mathbb{P}$ of d copies of the projective space $\mathbb{P}$ of lines in V. We denote this product by $(\mathbb{P})^d$. We consider twisting sheaf $\mathcal{O}_{\mathbb{P}}(1)$ (Definition C.0.32) on $\mathbb{P}$. Now we have functions $\mathrm{pr}_i:(\mathbb{P})^d\to\mathbb{P}$ such that for $(p_1,\dots,p_i,\dots,p_d)\mapsto p_i$, for all $1\le i\le d$. Since we have a sheaf on $\mathbb{P}$, $\mathcal{O}_{\mathbb{P}}(1)$, we can take pullback of function pr_i, called pr_i^* (Definition C.0.29). With this, we have a presheaf $pr_i^*(\mathcal{O}_{\mathbb{P}}(1))$ on $(\mathbb{P})^d$, for all $1\le i\le d$. Applying tensor product between theses presheaves (Definition C.0.29), we obtain a presheaf $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}}(1)\otimes\cdots\otimes\mathrm{pr}_d^*\mathcal{O}_{\mathbb{P}}(1)$. In general, tensor product between sheaves is not a sheaf; for this, we have to sheaving [1] for see a presheaf $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}}(1)\otimes\cdots\otimes\mathrm{pr}_d^*\mathcal{O}_{\mathbb{P}}(1)$ as a sheaf. We denoted this sheaf as $\mathcal{O}(1,1,\dots,1)$.

Each $g\in V$ is a regular function. In particular, $\widetilde{f}$ is a regular function. Then, $\widetilde{f}\in(\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}}(1)\otimes\cdots\otimes\mathrm{pr}_d^*\mathcal{O}_{\mathbb{P}}(1))((\mathbb{P})^d)$. It is equivalent to say that $\widetilde{f}$ is a section of sheaf $\mathcal{O}(1,1,\dots,1)$.

Since $\widetilde{f}$ is multilinear and hence multihomogeneous, its zeros define a locus in $(\mathbb{P})^d$.

Definition 2.3.6 [ATVdB07] Let $A=T/I$ be a graded quotient ring of T, let $\widetilde{I}_d$ be the space of multilinear forms $\widetilde{f}$ such that $f\in I_d$. We will denote by Γ_d the scheme of zeros if $\widetilde{I}_d$ in the product $(\mathbb{P})^d$:

$$\Gamma_d=Z(\widetilde{I}_d)\subseteq(\mathbb{P})^d.$$

With this, we can get to the most important proposition of this section.

Proposition 2.3.7 ([ATVdB07], Proposition 3.9) With the above notation, there is a one-to-one correspondence between points $(p_0,\dots,p_d)\in\Gamma_d$ and truncated point modules of length $d+1$.

Proof. Let M be a family of truncated point modules. Since point modules have been rigidified by the requirement that $M_0=R$, they are compatible with descent. So we can localize R, to reduce to the case that each graded piece M_j, $1\le j\le d$, is a free R-module of rank 1. We choose a basis b_j for each M_j. For $1\le j\le d$ and $0\le i\le n$, we write out the products of the bases b_j by the generators x_i of A:

[1] For more details of turning a presheaf into a sheaf see [DFI+04], Chapter VII.

$$b_{j-1}x_i = b_j a_{ij},$$

for some $a_{ij} \in R$. In this way, we obtain a set of d points $a_j = (a_{0j}, \ldots, a_{nj}) \in \mathbb{P}$ with coordinates in R, i.e., an R-valued point of $\mathbb{P}^d$. This point has the property that if $f(x)$ is any polynomial in $R \otimes T$ of degree d, then the formula $b_0 f(x) = b_d \widetilde{f}(a_1, \ldots, a_d)$ holds, where M is given the structure of T-module via the map $T \to A$. □

Corollary 2.3.8 ([ATVdB07], Corollary 3.13) Let Γ denote the inverse limit of the sets Γ_d (Definition A.3.10). Then the points of Γ are in one-to-one correspondence with point modules.

The collection of isomorphic classes of point modules for an algebra A is denoted by $P(A)$. The following is an example of correspondence of point modules and it is stated in [LFG+], but the details are not there. However, such details are included here.

Example 2.3.9 ([LFG+], Example A.3.4 (ii)) Point modules of Jordan plane are parameterized by $\mathbb{P}^1$, that is, there is a bijective correspondence between the projective space $\mathbb{P}^1$ and the isomorphism class collection of the Jordan plane point modules $\mathcal{J} := \mathbb{K}\{x, y\}/\langle yx - xy - x^2\rangle$:

$$\mathbb{P}^1 \longleftrightarrow P(\mathcal{J}).$$

We define $\mathbb{P}^1 \to^{\varphi} P(\mathcal{J}), P := (a_0 : a_1) \mapsto \mathcal{J}/I(P)$, where $I(P) := \mathcal{J}f$, with $f := a_0 y - a_1 x$. Also, $I(P)$ is a $\mathcal{J}$-module $\mathbb{N}$-graded and therefore, $\mathcal{J}/I(P)$ is a cyclic $\mathbb{N}$-graded $\mathcal{J}$-module generated by $1 + I(P) \in (\mathcal{J}/I(P))_0$; besides, for all $p \geq 0$, $\dim_{\mathbb{K}}((\mathcal{J}/I(P))_p) = \dim_{\mathbb{K}}([\mathcal{J}_p + I(P)]/I(P)) = 1$.

In fact, $[\mathcal{J}_p + I(P)]/I(P) \cong \mathcal{J}_p/[\mathcal{J}_p \cap I(P)]$; note that $a_0 \neq 0$ or $a_1 \neq 0$. Just consider the first case, since the second is similar. Note that $x^p \in \mathcal{J}_p$ and $x^p \notin I(P)$. Moreover, suppose that $x^p = h(a_0 y - a_1 x)$ for some $h \in \mathcal{J} - \{0\}$, let $h = h_0(x) + h_1(x)y + \cdots h_t(x)y^t$, for some $t \geq 0$ and $h_t(x) \neq 0$, then $a_0 h_t(x) y^{t+1} = 0$, which is a contradiction.

The idea is to show that $\mathcal{J}_p/[\mathcal{J}_p \cap I(P)] = \mathbb{K}\overline{x^p}$. It is clear that $\mathbb{K}\overline{x^p} \subseteq \mathcal{J}_p/[\mathcal{J}_p \cap I(P)]$; let $\overline{g} \in \mathcal{J}_p/[\mathcal{J}_p \cap I(P)]$, with $g \in \mathcal{J}_p$, then g is a $\mathbb{K}$-linear combination of monomials of the form $x^{\alpha_0} y^{\alpha_1}$, with $\alpha_0 + \alpha_1 = p$; let $x^{\alpha_0} y^{\alpha_1}$ any of these polynomials, if we prove that

$$\overline{x^{\alpha_0} y^{\alpha_1}} = \lambda_{\alpha_0,\alpha_1} \overline{x^p}, \text{ for some } \lambda_{\alpha_0,\alpha_1} \in \mathbb{K},$$

then there exists $\lambda \in \mathbb{K}$ such that $\overline{g} = \lambda \overline{x^p}$. If $\alpha_1 = 0$, the statement is trivially true; let $\alpha_1 \geq 1$, we divide

$$x^{\alpha_0}y^{\alpha_1} = y^{\alpha_1}x^{\alpha_0} - \alpha_0\alpha_1 y^{\alpha_1-1}x^{\alpha_0+1} + (-1)^{\alpha_1-1}\prod_{j=0}^{\alpha_1-2}(\alpha_0+j)(\alpha_1-1)yx^{\alpha_0+\alpha_1-1}$$
$$+ (-1)^{\alpha_1}\prod_{j=0}^{\alpha_1-1}(\alpha_0+j)x^{\alpha_0+\alpha_1} + \sum_{p=2}^{\alpha_1-2}\left[(-1)^p\prod_{j=0}^{p-1}(\alpha_0+j)[(\alpha_1-3)(\alpha_1-2)+2]\right],$$

by $f = a_0y - a_1x$. Doing the division, we get that $x^{\alpha_0}y^{\alpha_1} - \lambda_{\alpha_0,\alpha_1}x^p \in \mathcal{J}f$.

Now, consider the function

$$P(\mathcal{J}) \to^{\phi} \mathbb{P}^1$$
$$M \mapsto P,$$

where P is defined as follows: let $M = \mathcal{J}m_0$, we know that $M \cong \mathcal{J}/J$, with $J := \text{Ann}(m_0) \neq 0$; M is graded, then J is homogeneous, that is, $\mathbb{N}$-graded like $\mathcal{J}$-module. For the homogeneous component of J of degree 1, we have that $J_1 = \mathcal{J}_1 \cap J$, so J_1 is a subspace of dimension 1 of $\mathcal{J}_1 = \mathbb{K}x + \mathbb{K}y$: Indeed, $M_1 \cong (\mathcal{J}/J)_1 = (\mathcal{J}_1 + J)/J \cong \mathcal{J}_1/\mathcal{J}_1 \cap J$ has dimension 1, but $1 = \dim_{\mathbb{K}}(\mathcal{J}_1/\mathcal{J}_1 \cap J) = \dim_{\mathbb{K}}(\mathcal{J}_1) - \dim_{\mathbb{K}}(\mathcal{J}_1 \cap J) = 2 - \dim_{\mathbb{K}}(\mathcal{J}_1 \cap J)$, so $\dim_{\mathbb{K}}(\mathcal{J}_1 \cap J) = 1$. Thus, $J_1 =_{\mathbb{K}} \langle f \rangle$, with $f = a_0y - a_1x$, for some $(a_0, a_1) \neq (0,0)$; besides, the $\mathcal{J}$-module generated by J_1 is $\mathcal{J}f$. Note that J coincides with the $\mathcal{J}$-module generated by J_1, that is, $J = \mathcal{J}f$: Indeed, $\dim_{\mathbb{K}}(M_2) = 1 = \dim_{\mathbb{K}}(\mathcal{J}_2) - \dim_{\mathbb{K}}(J_2)$, so $\dim_{\mathbb{K}}(J_2) = 2$, but $\{xf, yf\}$ is $\mathbb{K}$-linearly independent, so, $J_2 \subset \mathcal{J}f$; similarly, $\dim_K(J_3) = 3$ and $\{x^2f, xyf, y^2f\}$ is a $\mathbb{K}$-base of J_3, where $J_3 \subset \mathcal{J}f$; continuing on this path we have that $\dim_{\mathbb{K}}(J_p) = p$ and $\{x^{p-1}f, x^{p-2}yf, \ldots, xy^{p-2}f, y^{p-1}f\}$ is a $\mathbb{K}$-base of J_p for all $p \geq 2$, that is, $J_p \subset \mathcal{J}f$, so $J = \mathcal{J}f$.

We define $\phi(M) := P$ with $P := (a_0, a_1)$. ϕ is well defined since if M' is a point module with $M \cong^{\beta} M'$, then $J = \text{Ann}(m_0) = \text{Ann}(\beta(m_0))$ and $M' = \mathcal{J}\beta(m_0)$; besides, if we change the generator of M, that is $M = \mathcal{J}z_0$, then for $J' := \text{Ann}(z_0)$ and $J' = \mathcal{J}g$, with $g = b_0y - b_1x$ for some $(b_0, b_1) \neq (0,0)$, we have $g = \lambda f$ for some $\lambda \in \mathbb{K}^{\times}$: indeed, $gz_0 = 0$, but $z_0 = \lambda' m_0$, with $\lambda' \in \mathcal{J} - \{0\}$, but since $\text{gr}(z_0) = 0 = \text{gr}(m_0)$, then $\lambda' \in \mathbb{K}^{\times}$; $g\lambda' m_0 = 0$, so $g\lambda' \in J$, also $g\lambda' = \lambda'' f$, with $\lambda'' \in \mathcal{J} - \{0\}$, then comparing the degrees we come to that $\lambda'' \in \mathbb{K}^{\times}$, so $g = \lambda f$, with $\lambda = \lambda'^{-1}\lambda''$.

Moreover, $(b_0, b_1) = \lambda(a_0, a_1)$, and P is unique.

$$\varphi\phi = i_{P(\mathcal{J})} : \varphi\phi(M) = \varphi(P) = \mathcal{J}/I(P) = \mathcal{J}/\mathcal{J}f = \mathcal{J}/\text{Ann}(m_0) \cong M.$$
$$\phi\varphi = i_{\mathbb{P}^1} : \phi\varphi(P) = \phi(\mathcal{J}/I(P)) = \phi(\mathcal{J}/\mathcal{J}f) = P \quad \text{as} \quad \text{Ann}(\overline{1}) = \mathcal{J}f.$$

Now, we see another parameterization of point modules. The following example is an adaptation of Example 2.3.9. However, the different parts in the proof are shown in detail in this work.

Example 2.3.10 Consider the $\mathbb{K}$-algebra $A = \mathbb{K}\{x,y\}/\langle y - x^2 + yx\rangle$. The bijective correspondence between the projective space $\mathbb{P}^1$ and the isomorphism class collection of A is exactly the same that in Jordan plane but in division of polynomials. In this case, we divide

$$x^{\alpha_0}y^{\alpha_1} = (-1)^{\alpha_0\alpha_1}y^{\alpha_1}x^{\alpha_0} + \left(\frac{1+(-1)^{\alpha_1}}{2}\right)\left(\frac{1+(-1)^{\alpha_0}}{2}\right)y^{\alpha_1-1}x^{\alpha_0+1},$$

by $f = a_0y - a_1x$. We have three cases. If α_1 is even, doing the division we get

$$\begin{aligned} x^{\alpha_0}y^{\alpha_1} &= (a_0^{-1}y^{\alpha_1-1}x^{\alpha_0} - a_0^{-2}a_1y^{\alpha_1-2}x^{\alpha_0-1} + \cdots \\ &\quad - (a_0^{-\binom{\frac{\alpha_1+2}{2}}{}}a_1^{\binom{\frac{\alpha_1}{2}}{}} - \binom{\frac{\alpha_1-2}{2}}{1}a_0^{-\binom{\frac{\alpha_1+2}{2}+1}{}}a_1^{\binom{\frac{\alpha_1}{2}+1}{}} + \cdots \\ &\quad + (-1)^{\omega}\binom{\frac{\alpha_1-2}{2}}{\frac{\alpha_1-2}{2}-1}a_0^{-\alpha_1+1}a_1^{\alpha_1-2} + (-1)^{\omega+1}a_0^{-\alpha_1}a_1^{\alpha_1-1})x^{\alpha_0+\alpha_1-1})f + \lambda_{\alpha_0,\alpha_1}x^p, \end{aligned}$$

where $\omega = 0$ if $4 \mid (\alpha_1)$ and $\omega = 1$, otherwise; and

$$\begin{aligned} \lambda_{\alpha_0,\alpha_1} &= a_0^{-\binom{\frac{\alpha_1}{2}}{}}a_1^{\binom{\frac{\alpha_1}{2}}{}} - \binom{\frac{\alpha_1}{2}}{1}a_0^{-\binom{\frac{\alpha_1}{2}+1}{}}a_1^{\binom{\frac{\alpha_1}{2}+1}{}} + \cdots \\ &\quad + (-1)^{\omega-1}\binom{\frac{\alpha_1}{2}}{\frac{\alpha_1}{2}-1}a_0^{-\alpha_1+1}a_1^{\alpha_1-1} + (-1)^{\omega}a_0^{-\alpha_1}a_1^{\alpha_1} \end{aligned}$$

If α_1 is odd and α_0 is even, doing the division we get

$$\begin{aligned} x^{\alpha_0}y^{\alpha_1} &= (a_0^{-1}y^{\alpha_1-1}x^{\alpha_0} + a_0^{-2}a_1y^{\alpha_1-2}x^{\alpha_0-1} + \cdots \\ &\quad - (a_0^{-\binom{\frac{\alpha_1-3}{2}}{}}a_1^{\binom{\frac{\alpha_1+1}{2}}{}} + \binom{\frac{\alpha_1-3}{2}}{1}a_0^{-\binom{\frac{\alpha_1+3}{2}+1}{}}a_1^{\binom{\frac{\alpha_1+1}{2}+1}{}} + \cdots \\ &\quad + (-1)^{\omega-1}\binom{\frac{\alpha_1-3}{2}}{\frac{\alpha_1-3}{2}-1}a_0^{-\alpha_1+1}a_1^{\alpha_1-2} + (-1)^{\omega}a_0^{-\alpha_1}a_1^{\alpha_1-1})x^{\alpha_0+\alpha_1-1})f + \lambda_{\alpha_0,\alpha_1}x^p, \end{aligned}$$

where $\omega = 0$ if $4 \mid (\alpha_1+1)$ and $\omega = 1$, otherwise; and

$$\begin{aligned} \lambda_{\alpha_0,\alpha_1} &= a_0^{-\binom{\frac{\alpha_1+1}{2}}{}}a_1^{\binom{\frac{\alpha_1+1}{2}}{}} + \binom{\frac{\alpha_1-1}{2}}{1}a_0^{-\binom{\frac{\alpha_1+1}{2}+1}{}}a_1^{\binom{\frac{\alpha_1+1}{2}+1}{}} + \cdots \\ &\quad + (-1)^{\omega}\binom{\frac{\alpha_1-1}{2}}{\frac{\alpha_1-1}{2}-1}a_0^{-\alpha_1+1}a_1^{\alpha_1-1} + (-1)^{\omega+1}a_0^{-\alpha_1}a_1^{\alpha_1} \end{aligned}$$

Finally, if α_0, α_1 are odd, doing the division we get

$$\begin{aligned}x^{\alpha_0}y^{\alpha_1} &= (-a_0^{-1}y^{\alpha_1-1}x^{\alpha_0} + (a_0^{-1} - a_0^{-2}a_1)y^{\alpha_1-2}x^{\alpha_0-1} + \cdots \\ &- (a_0^{-\binom{\frac{\alpha_1+1}{2}}{}}a_1^{\binom{\frac{\alpha_1-1}{2}}{}} + \binom{\frac{\alpha_1-1}{2}}{1}a_0^{-\binom{\frac{\alpha_1+1}{2}+1}{}}a_1^{\binom{\frac{\alpha_1-1}{2}+1}{}} + \cdots \\ &+ (-1)^{\omega}\binom{\frac{\alpha_1-1}{2}}{\frac{\alpha_1-1}{2}-1}a_0^{-\alpha_1+1}a_1^{\alpha_1-2} + (-1)^{\omega+1}a_0^{-\alpha_1}a_1^{\alpha_1-1})x^{\alpha_0+\alpha_1-1})f + \lambda_{\alpha_0,\alpha_1}x^p,\end{aligned}$$

where $\omega = 0$ if $4 \mid (\alpha_1 + 1)$ and $\omega = 1$, otherwise; and

$$\begin{aligned}\lambda_{\alpha_0,\alpha_1} &= a_0^{-\binom{\frac{\alpha_1-1}{2}}{}}a_1^{\binom{\frac{\alpha_1-1}{2}}{}} + \binom{\frac{\alpha_1+1}{2}}{1}a_0^{-\binom{\frac{\alpha_1-1}{2}+1}{}}a_1^{\binom{\frac{\alpha_1-1}{2}+1}{}} + \cdots \\ &+ (-1)^{\omega-1}\binom{\frac{\alpha_1+1}{2}}{\frac{\alpha_1+1}{2}-1}a_0^{-\alpha_1+1}a_1^{\alpha_1-1} + (-1)^{\omega}a_0^{-\alpha_1}a_1^{\alpha_1}\end{aligned}$$

Then, in the three cases, we have that $x^{\alpha_0}y^{\alpha_1} - \lambda_{\alpha_0,\alpha_1}x^p \in \mathcal{J}f$.

The rest of the proof is the same as in Example 2.3.9.

In [VVR97], the scheme Γ was called the ***point scheme***. The idea now is to generalize what Artin demonstrated and see that d-linear modules are parameterized by a scheme.

Definition 2.3.11 ([SV02], Definition 0.1) We define ***Proj B*** to be the following triple $((\mathrm{gr}-B)/\mathscr{J}, \mathcal{O}, \sigma)$, where $\mathscr{J}$ denotes the subcategory of gr-B of torsion modules, $\mathcal{O}$ denotes an object of $(\mathrm{gr}-B)/\mathscr{J}$ that is represented by the right modules B, and σ is the operation $\mathscr{M} \to \mathscr{M}[1]$ on $(\mathrm{gr}-B)/\mathscr{J}$ induced by the shift of degree on a B-module.

Remark 2.3.12 We consider A from Definition 2.3.6. Now for any commutative $\mathbb{K}$-algebra R, we denote the graded algebras $A \otimes R$ by $A(R)$.

Definition 2.3.13 ([SV02], Definition 1.1) Let d be a non-negative integer and let R be a commutative $\mathbb{K}$-algebra.

(i) Let $r \in \mathbb{N}$. We say that a graded $A(R)$-module $M = \bigoplus_{i=0}^{\infty} M_i$ is a ***truncated d-linear module*** of length r, if M is cyclic, generated by M_0, and has Hilbert series

$$H_M(t) = \sum_{i=0}^{r-1}\binom{d+i}{d}t^i.$$

(ii) We say that a graded $A(R)$-module $M = \bigoplus_{i=0}^{\infty} M_i$ is a d-**linear module** if M is cyclic, generated by M_0, and has Hilbert series

$$H_M(t) = \frac{1}{(1-t)^{d+1}}.$$

Any d-linear module can be "truncated" to a truncated d-linear modules of any required length. We refer to 0-linear modules as point modules, 1-linear modules as line modules, and 2-linear modules as plane modules, with similar terminology for their truncated analogues.

Definition 2.3.14 ([SV02], Definition 1.2) For $d \in \{0, 1, \ldots, v\}$ and $r \in \mathbb{N}$, let $F_{d,r}$ denote the functor from $\Bbbk$-algebras to sets, which assigns to each $\Bbbk$-algebra R the set of all isomorphism classes of truncated d-linear $A(R)$-modules of length r, and to each $\Bbbk$-algebra homomorphism $f : R \to R'$ the induction functor $__ \otimes_R R'$. Similarly, we write F_d for the functor of d-linear modules. As they are functors on schemes over $\Bbbk$, we will refer to these functors as the functors of flat families of d-linear, or truncated d-linear, modules.

Now, let us fix r, d and recall $v = \dim_{\mathbb{K}}(V)$. Also, we introduce the following notation.

Let W denote any finite-dimensional $\mathbb{K}$-vector space, and let U denote any subspace of W. We write $\mathbb{G}_n(W)$ for the Grassmannian scheme[2] of n-dimensional subspaces of W, and $\mathbb{G}^n(W)$ for the scheme of subspaces of W of codimension n. There is a natural embedding of the scheme $\mathbb{G}^n(W/U)$ into $\mathbb{G}^n(W)$ as the subscheme of codimension n subspaces that contain U. we denote this closed subscheme by $\mathbb{G}^n_U(W)$.

For convenience, $C(a, b)$ denotes the binomial coefficient $\binom{a}{b}$.

For each $r \in \mathbb{N}$ and for each $d \in \{0, 1, \ldots, v\}$, let

$$\Upsilon_r(V, d) = \prod_{i=1}^{r} \mathbb{G}^{C(d+i,d)}((V^*)^{\otimes i}).$$

With the definition of $\Upsilon_r(V, d)$, we have that, there is a closed subscheme related to the ideal I defining the algebra A,

$$\Upsilon_r(V, I, d) = \prod_{i=1}^{r} \mathbb{G}_{I_i}^{C(d+i,d)}((V^*)^{\otimes i}),$$

where I_i denotes the homogeneous elements of degree i in I.

Now it is defined a flag variety as follows: given a decreasing sequence of integers $e_1 \geq e_2 \geq \cdots \geq e_r \geq 0$, we denote by $\mathbb{F}(e_1, \ldots, e_r, W)$ the ***flag variety*** of flags

[2]For more details about Grassmannian scheme, see [KT+03]

$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r \subseteq W$, where the codimension of the vector subspace U_i in W is e_i.

Now, we define a decreasing sequence of integers k_i by $k_i = C(d+i,d)v^{r-i}$, $1 \leq i \leq r$. There is a natural scheme morphism $\omega_r : \Upsilon_r(V,d) \to \prod_i \mathbb{G}^{k_i}((V^*)^{\oplus r})$. This morphism is given by

$$\omega_r(Q) = (Q_1 \otimes (V^*)^{\otimes(r-1)}, Q_2 \otimes (V^*)^{\otimes(r-2)}, \ldots, Q_i \otimes (V^*)^{\otimes(r-i)}, \ldots, Q_r),$$

where $Q = (Q_1, \ldots, Q_r)$ is a closed point of $\Upsilon_r(V,d)$.

Now, let $\Phi_r(V,d) = \omega_r^{-1}(\mathbb{F}(k_1, \ldots, k_r, (V^*)^{\otimes r}))$, which is a closed subscheme of $\Upsilon_r(V,d)$. And we define $\Omega_r(A,d)$ as the scheme-theoretic intersection

$$\Omega_r(A,d) = \Phi_r(V,d) \cap \Upsilon(V,I,d).$$

To compute the set of closed points of $\Omega_r(A,d)$, we consider a closed point $(Q_1, \ldots, Q_r)$ of $\Upsilon_r(A,d)$, that is, each Q_i is a subspace of $(V^*)^{\otimes i}$ of codimension $C(d+i,d)$. It follows that the set of closed points of $\Omega_r(A,d)$ is

$$\{(Q_1, \ldots, Q_r) \mid Q_i \otimes V^* + I_{i+1} \subseteq Q_{i+1}, 1 \leq i \leq r-1\}.$$

For each $r \geq 2$, let $\pi_{r-1} : \Upsilon_r(V,d) \to \Upsilon_{r-1}(V,d)$ be the projection onto the first $r-1$ factors. Since $\pi_{r-1}(\Omega_r(A,d)) \subseteq \Omega_{r-1}(A,d)$ for all r, we can define

$$\Omega_\infty(A,d) = \varprojlim \Omega_r(A,d),$$

which will play a role analogous to that played by the scheme Γ that appeared in Corollary 2.3.8.

With that, the following proposition is a generalisation, from 0-linear to d-linear, of Proposition 2.3.7, concerning a scheme, Γ_r, that represents the truncated point modules of length r.

Proposition 2.3.15 ([SV02], Theorem 1.4) The scheme $\Omega_r(A,d)$ represents the functor $F_{d,r+1}$ of isomorphism classes of truncated d-linear modules of length $r+1$.

Proof. We fix d and r and let $F = F_{d,r+1}$. Let G be the functor from the category of commutative $\mathbb{K}$-algebras to the category of sets that is represented by $\Omega_r(A,d)$. The $\mathbb{K}$-valued points of G have been described above. It suffices to give an isomorphism from the

functor F to the functor G.

Let R denote a commutative $\mathbb{K}$-algebra and let M denote a d-linear $A(R)$-module of length $r+1$; that is, M represents an element of $F(R)$. Consider M as a $T(R)$-module that is annihilated by $I(R) = I \otimes R$. We have that $M_0 = R$. There exists $u \in M_0$ such that $M = uA(R)$. Let Q denote the annihilator of u in $T(R)$. Since u is unique up to multiplication by a unit of R, and since R is central in $A(R)$, it follows that Q is independent to the choice of u. Moreover, Q has the form $\bigoplus_{i=1}^{\infty} Q_i$, where Q_i is the kernel of the epimorphism $T(R) : i \to M_i$ given $t \mapsto ut$. Since each M_i, where $0 \leq i \leq r$, is a projective direct summand of the free R-module $T(R)_i$ of rank $C(d+i, d)$, Q_i is projective with well defined rank. For $i > r$, we have $Q_i = T(R)_i$, and, for $1 \leq i \leq r$, Q_i has corank $C(d+i, d)$ in $T(R)_i$. Since M is an $A(R)$-module, we have $Q_i \supseteq I_i(R)$. It follows that the point $(Q_1, \ldots, Q_r)$ is an element of $G(R)$. We have defined a transformation, $M \mapsto (Q_1, \ldots, Q_r)$, from F to G.

Conversely, let $(Q_1, \ldots, Q_r)$ be an R-valued point of $\Omega_r(A, d)$; that is, $(Q_1, \ldots, Q_r) \in G(R)$. Let $Q = \bigoplus_{i=1}^{\infty} Q_i$, where, for $i > r$, we take $Q_i = T(R)_i$. The structure of $\Omega_r(A, d)$ implies that Q is a right ideal of $T(R)$ containing $I(R)$. Let M denote the right $T(R)$-module, also $A(R)$-module, $T(R)/Q$. By definition, M is a cyclic $A(R)$-module generated by M_0, and, for $i > 0$, we have $M_i = T(R)_i/Q_i$. By the definition of G, for $1 \leq i \leq r$, Q_i is a direct summand of $T(R)_i$ of well defined corank $C(d+i, d)$. Thus every M_i is projective of the appropriate well defined rank, so M represents a point in $F(R)$. □

Finally, we have an important result of parameterization of d-linear A-modules.

Corollary 2.3.16 ([SV02], Corollary 1.5) The set of closed points of $\Omega_\infty(A, d)$ is in one-to-one correspondence with the set of isomorphism classes of d-linear A-modules, and the scheme $\Omega_\infty(A, d)$ represents the functor of flat families of d-linear A-modules.

The scheme $\Omega_\infty(A, 0)$, that represents the functor of point modules of A is called in [VVR97] **the point scheme of** A.

With the results previously exposed, we saw how the bijective correspondence exists between a scheme and the class of isomorphism of point modules in the context of graded algebras. Next, we see some important results of this bijection for the case of semi-graded algebras.

In [Lez20] we have an example about correspondence between point modules for a free algebra and the points of infinite product $\mathbb{P}^n \times \mathbb{P}^n \times \cdots = \prod_{i=0}^{\infty} \mathbb{P}^n$; this example is shown below.

Example 2.3.17 ([Lez20], Example 1.3) The isomorphism classes of point modules for the free algebra $A := \mathbb{K}\{x_0, \ldots, x_n\}$ are in bijective correspondence with $\mathbb{N}$-indexed sequences of points in $\mathbb{P}^n$, $\{(\lambda_{0,i} : \cdots : \lambda_{n,i}) \in \mathbb{P}^n \mid i \geq 0\}$. In other words, with the points of the

infinite product $\mathbb{P}^n \times \mathbb{P}^n \times \cdots = \prod_{i=0}^{\infty} \mathbb{P}^n$. Thus, there exists a bijective correspondence between $\prod_{i=0}^{\infty} \mathbb{P}^n$ and $P(\mathbb{K}\{x_0, \ldots, x_n\})$:

$$\prod_{i=0}^{\infty} \mathbb{P}^n \longleftrightarrow P(A).$$

Also it is said in [Lez20] that, thanks to the previous example, it is possible to compute the collection of point modules for any finitely presented algebra and then, we could calculate them for previous algebras. In Example 2.3.9, we saw the correspondence for Jordan plane. For homogenized enveloping algebra the correspondence should be with $\mathbb{P}^n$ and for Manin algebra the correspondence should be with $\mathbb{P}^3$.

Now, we see another parameterization of point modules from a generalization of Jordan algebra. The following example is an adaptation of Example 2.3.9. However, the different parts in the proof are shown in detail in this work.

Example 2.3.18 Let $\mathrm{s}\mathcal{J}$ an $\mathbb{K}$-algebra defined as follows: $\mathrm{s}\mathcal{J}$ is the free $\mathbb{K}$-algebra generated by x, y with relation $yx = qxy + x^2$, $q \in \mathbb{K}^{\times}$, so $\mathrm{s}\mathcal{J} = \mathbb{K}\{x, y\}/\langle yx - qxy - x^2 \rangle$. $\mathrm{s}\mathcal{J}$ is called ***skew Jordan plane***. The correspondence of point modules of skew Jordan plane is given by $\mathbb{P}^1$. The proof of this fact is exactly the same as on Jordan plane (Example 2.3.9). However, the only thing different is the division of a monomial by f. In this case, we divide

$$\begin{aligned}
x^{\alpha_0} y^{\alpha_1} = q^{-\alpha_1\alpha_0} y^{\alpha_1} x^{\alpha_0} &- \left(\sum_{i=1}^{\alpha_0} q^{-1}\right)\left(\sum_{k=0}^{\alpha_1-1} q^{-(\alpha_1-1)\alpha_0-k}\right) y^{\alpha_1-1} x^{\alpha_0+1} \\
&+ (-1)^{\alpha_1-1} \prod_{j=0}^{\alpha_1-2} \left(\sum_{i=0}^{\alpha_0+j} q^{-i}\right)\left(\sum_{k=0}^{\alpha_1-1} q^{-\alpha_0-k}\right) y x^{\alpha_0+\alpha_1-1} \\
&+ (-1)^{\alpha_1} \prod_{j=0}^{\alpha_1-1} \left(\sum_{i=0}^{\alpha_0+j} q^{-i}\right) x^{\alpha_0+\alpha_1} \\
&+ \sum_{p=2}^{\alpha_1-2} ((-1)^p \prod_{j=0}^{p-1} \left(\sum_{i=0}^{\alpha_0+j} q^{-i}\right) (q^{-(\alpha_1-p)\alpha_0} + q^{-(\alpha_1-p)\alpha_0-1} + \\
&q^{-(\alpha_1-p)\alpha_0-(\alpha_1-1)} + q^{-(\alpha_1-p)\alpha_0-\alpha_1} + \sum_{m=2}^{\alpha_1-3} 2q^{(\alpha_1-p)\alpha_0-m}))
\end{aligned}$$

by $f = a_0 y - a_1 x$. Doing the division, we get that $x^{\alpha_0} y^{\alpha_1} - \lambda_{\alpha_0,\alpha_1} x^p \in \mathcal{J}f$.

The parameterization of the point modules is important, since we want to know how many point modules has an algebra, in order to try to adapt Corollary 1.3.16 to more general algebras. In particular, we start by extending Corollary 1.3.16 for graded skew

PBW extensions.

For this, we see a slightly more general version of Corollary 1.3.16. To do this, we see what happens with graded skew Clifford algebras.

Proposition 2.3.19 ([Van15b], Proposition 2.3) Let A denote a graded skew Clifford algebra and let F denote the normalizing sequence of length n given in Definition 1.2.18(ii). Then

(i) The algebra A is Noetherian.

(ii) If F is regular in A, then conditions (i)-(iv) from Corollary 1.3.16 hold for A and F.

The previous proposition allows us to generalize these conditions for graded skew PBW extensions. These extensions are defined as follows:

Definition 2.3.20 ([Suá17], Definition 2.6) Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension (Definition 2.1.8) of an $\mathbb{N}$-graded $\mathbb{K}$-algebra R. We said that A is a ***graded skew PBW extension***, if the following conditions hold:

i) $x_1, \ldots, x_n$ have degree 1 in A.

ii) σ_i is a graded ring homomorphism and $\sigma_i : R(-1) \to R$ is a graded σ_i-derivation, for all $1 \leq i \leq n$.

iii) $x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, $c_{i,j} \in R_0$.

Example 2.3.21 ([Suá17], Example 2.8) Quasi-commutative skew PBW extensions with the trivial graduation of R are graded skew PBW extensions: let $r \in R = R_0$. Then $\sigma_i(r) = c_{i,r} \in R_0$, $\delta_i = 0$ and $x_j x_i - c_{i,j} x_i x_j = 0 \in R_2 + R_1 x_1 + \cdots + R_1 x_n$; if we assume that R has a different graduation to the trivial graduation, then A is graded skew PBW extension provided that σ_i is graded and $c_{i,j} \in R_0$, $1 \leq, i, j \leq n$.

In [Suá17], Example 2.9, we have many examples of graded skew PBW extensions of the classical polynomial ring R with coefficients in $\mathbb{K}$, which are not quasi-commutative and where R has the usual graduation. Some of these algebras are the Jordan plane, the homogenized enveloping algebra, Manin algebra, and others. Definitions of these algebras can be founded in Appendix B [B].

We need some previous results to arrive to adaptation of Corollary 1.3.16 for graded skew PBW extensions.

In the case of skew PBW extensions, there are important results on GK-dimension.

Proposition 2.3.22 ([Rey13a], Proposition 4.3.3) Let B be a $\mathbb{K}$-algebra with finite filtration $\{B_i\}_{i\in\mathbb{Z}}$ such that $G(B)$ is finitely generated, where $G(B)$ is the associated graded ring of B. Then

$$\text{GKdim}(G(B)_{G(B)}) = \text{GKdim}(B_B)$$

Lemma 2.3.23 ([Rey13a], Lemma 4.3.4) Let B be a $\mathbb{K}$-algebra with a finite dimensional generating subspace V, σ a $\mathbb{K}$-automorphism of B and δ a σ-derivation. If $\sigma(V) \subseteq V$, then

$$\text{GKdim}(B[x;\sigma,\delta]) = \text{GKdim}(B) + 1$$

Now, we consider the automorphisms σ_i, $1 \leq i \leq n$ of R in Proposition 2.1.7.

Proposition 2.3.24 ([Rey13a], Theorem 4.3.5) Let R be a $\mathbb{K}$-algebra with a finite dimensional generating subspace V and let A be a bijective skew PBW extension of R given by $A = \sigma(R)\langle x_1,\dots,x_n\rangle$. If $\sigma_i(V) \subseteq V$, for $1 \leq i \leq n$, then

$$\text{GKdim}(A) = \text{GKdim}(B) + n.$$

Remark 2.3.25 In the previous proposition, an error that is in [Rey13a], Theorem 4.3.5 was corrected, because in that Theorem it says that only $\sigma_n(V) \subseteq V$ is considered, when in fact all $\sigma_i(V) \subseteq V$, $1 \leq i \leq n$, have to be considered.

Definition 2.3.26 ([Rey13a], Definition 4.3.7) For a $\mathbb{K}$-algebra B, an automorphism σ of B is said to be ***locally algebraic***, if for any $b \in B$ the set $\{\sigma^m(b) \mid m \in \mathbb{N}\}$ is contained in a finite dimensional subspace of B.

With Definition 2.3.26, we can formulate another result about the GK-dimension for certain types of skew PBW extensions. Again, consider the automorphisms σ_i, $1 \leq i \leq n$, of R in Proposition 2.1.7.

Proposition 2.3.27 ([Rey13a], Theorem 4.3.9) Let R be a $\mathbb{K}$-algebra with a finite dimensional generating subspace V and let A be a bijective skew PBW extension of R given by $A = \sigma(R)\langle x_1,\dots,x_n\rangle$. If σ_i is locally algebraic, for $1 \leq i \leq n$, then

$$\text{GKdim}(A) = \text{GKdim}(B) + n.$$

Again, as Remark 2.3.25, in previous Proposition we have to consider all σ_i, $1 \leq i \leq n$ to be locally algebraic.

Now, with the following proposition we can find a relation between the Gelfand-Kirillov dimension and the injective dimension of a ring. The details of the proof are not presented in [ASZ98]. However, we include them with details here.

Proposition 2.3.28 ([ASZ98], Corollary 1.4) Let A be an Auslander-Gorenstein and Cohen-Macaulay ring. For every non-zero Noetherian A-module M,

$$\mathcal{K}\dim(M) \leq \operatorname{GKdim}(M) - (\operatorname{GKdim}(A) - \operatorname{id}(A))$$

where $\operatorname{GKdim}(A) - \operatorname{id}(A) \geq 0$. ($\mathcal{K}$dim is the Krull dimension defined in Definition A.1.9 and id is the injective dimension defined in Definition A.2.14).

Proof. Let $d = \operatorname{id}(A)$. By [ASZ98], Corollary 1.3, we have the following inequality

$$\mathcal{K}\dim(M) \leq d - j(M), \tag{2.3.1}$$

with M a finitely generated A-module, where $j(M)$ is the graded number of M (Definition A.2.9). Now, A is a Cohen-Macaulay ring, that is

$$\begin{aligned} j(M) + \operatorname{GKdim}(M) &= \operatorname{GKdim}(A) \\ j(M) &= \operatorname{GKdim}(A) - \operatorname{GKdim}(M), \end{aligned} \tag{2.3.2}$$

for every non-zero Noetherian A-module M. Equation 2.3.1 is true for every non-zero Noetherian A-module M, since M is finitely generated (any Noetherian module is finitely generated). With Equation 2.3.1 with Equation 2.3.2, we have

$$\begin{aligned} \mathcal{K}\dim(M) &\leq d - (\operatorname{GKdim}(A) - \operatorname{GKdim}(M)) \\ &= d - \operatorname{GKdim}(A) + \operatorname{GKdim}(M) \\ &= \operatorname{GKdim}(M) - (\operatorname{GKdim}(A) - d) \\ &= \operatorname{GKdim}(M) - (\operatorname{GKdim}(A) - \operatorname{id}(A)), \end{aligned} \tag{2.3.3}$$

for every non-zero Noetherian A-module M. Now, we see that $\operatorname{GKdim}(A) - \operatorname{id}(A) \geq 0$. Indeed, by Equation 2.3.2, we have $\operatorname{GKdim}(A) - j(M) = \operatorname{GKdim}(M) \geq 0$, for every non-zero Noetherian A-module M. Thus, as A is Auslander-Gorenstein, we have that

$$\operatorname{id}(A) = \max\{j(M) \mid 0 \neq M \text{ finitely generated right } A\text{-module}\}. \tag{2.3.4}$$

Hence, A is a two-sided Noetherian ring, since A is Auslander-Gorenstein. This implies that every finitely generated A-module is Noetherian. Equation 2.3.2 holds for every non-zero Noetherian A-module M, then it holds for Equation 2.3.4. Let N be the A-module such that $j(N) = \operatorname{id}(A)$. Then

$$\begin{aligned} j(N) + \operatorname{GKdim}(N) &= \operatorname{GKdim}(A) \\ \operatorname{id}(A) + \operatorname{GKdim}(N) &= \operatorname{GKdim}(A) \\ \operatorname{GKdim}(N) &= \operatorname{GKdim}(A) - \operatorname{id}(A) \geq 0. \end{aligned}$$

□

With Proposition 2.3.28 we can ensure that when a ring has the Cohen-Macaulay property, it is enough to show that the Gelfand-Kirillov dimension is finite to know that its injective dimension is also finite.

Proposition 2.3.29 ([CV10], Proposition 2.3)

(i) Suppose that B is a Noetherian $\mathbb{K}$-algebra on which GK-dimension is exact (Definition A.2.31). If B is homogeneous with respect to GK-dimension (Definition A.2.32), then the set X of regular elements in B satisfies

$$\begin{aligned} X &= \{b \in B \mid \operatorname{GKdim}(B/(Bb)) < \operatorname{GKdim}(B)\} \\ &= \{b \in B \mid \operatorname{GKdim}(B/(bB)) < \operatorname{GKdim}(B)\}. \end{aligned}$$

(ii) If B is a Noetherian, Auslander-Gorenstein $\mathbb{K}$-algebra of finite injective dimension and finite integral GK-dimension (i.e. $\operatorname{GKdim}(B) = \omega \in \mathbb{N}$) that satisfies the Cohen-Macaulay property, then B is homogeneous with respect to GK-dimension. Additionally, if $f \in B$ is normal and regular, then $B/\langle f \rangle$ is Auslander-Gorenstein of finite injective dimension, satisfies the Cohen-Macaulay property and is homogeneous with respect to GK-dimension.

(iii) Suppose that B is finitely generated $\mathbb{N}$-graded $\mathbb{K}$-algebra with $\dim_{\mathbb{K}}(B_0) < \infty$ such that B has polynomial growth (Definition A.2.33). Let $f \in B$ be a homogeneous normal element of positive degree.

(i)' If f is not a zero divisor on a finitely graded B-module M, then

$$\operatorname{GKdim}(M/(fM)) = \operatorname{GKdim}(M) - 1.$$

(ii)' If f is regular in B, and if $B/\langle f \rangle$ is Auslander-Gorenstein of dimension ν, satisfies the Cohen-Macaulay property and has polynomial growth, then B is Auslander-Gorenstein of dimension $\nu + 1$, satisfies the Cohen-Macaulay property and has polynomial growth.

Proof. (i) By [MRS01], Theorem 6.8.15, we have that

$$X = \{b \in B \mid \operatorname{GKdim}(B/(bB)) < \operatorname{GKdim}(B)\}.$$

By [MRS01], Corollary 8.3.16, we can conclude that

$\{b \in B \mid \mathrm{GKdim}(B/(bB)) < \mathrm{GKdim}(B)\} = \{b \in B \mid \mathrm{GKdim}(B/(Bb)) < \mathrm{GKdim}(B)\}$.

(ii) By [Lev92], Remark 5.8(3), we have that $\gamma(M) = \mathrm{id}(B) - j(M)$ defines a dimension function (Definition A.2.31) with M been finitely generated B-module and also we have that $\gamma(M) + (\mathrm{GKdim}(B) - \mathrm{id}(B)) = \mathrm{GKdim}(M)$. By [Lev92], Proposition 4.5, we have γ is an exact dimension function. Now, by [Lev92], p.297 (4.6.1), we have that B is homogeneous with respect to γ. With this, we can conclude that B is homogeneous with respect to GKdim.

Now, [Lev92], Remark 3.4(3) and Proposition 3.5(a) we have $B/\langle f\rangle$ is Noetherian, Auslander-Gorenstein of finite injective dimension and satisfies the Cohen-Macaulay property. Hence, $B/\langle f\rangle$ is homogeneous with respect to GKdim.

(iii) By [Lev92], Lemma 5.7, we have (i)'. Now, by [Lev92], Theorem 3.6(2) and Proposition 5.6, we have that B is Auslander-Gorenstein of dimension $\nu + 1$ and B has polynomial growth. For Cohen-Macaulay property, [Lev92], Lemma 5.8, says that it is enough showing Cohen-Macaulay property for non-zero graded finitely generated B-modules (It is called ***graded Cohen-Macaulay property***). Then, we have to show that $\mathrm{GKdim}(M) + j(M) = \mathrm{GKdim}(B/\langle f\rangle) + 1$ for all M been non-zero graded finitely generated B-module. Let $\overline{M} = M/(fM)$, $M' = M/\mathrm{Ker}_M(f)$, where $\mathrm{Ker}_M(f^p) = \{x \in M \mid f^p x = 0\}$, $n(M)$ is the smallest integer q such that $\mathrm{Ker}_M(f^q) = \mathrm{Ker}_M(f^{q+1})$. We noticed that $n(M') \leq n(M) - 1$ and we induct on $n(M)$. if $M = \mathrm{Ker}_M(f)$, i.e. $M' = 0$ then M is a graded finitely generated $B/\langle f\rangle$-module. Hence $\mathrm{GKdim}_{B/\langle f\rangle}(M) + j_{B/\langle f\rangle}(M) = \mathrm{GKdim}(B/\langle f\rangle)$, $\mathrm{GKdim}_{B/\langle f\rangle}(M) = \mathrm{GKdim}_B(M)$, $j_{B/\langle f\rangle}(M) = j_B(M) - 1$ (by [Lev92], Proposition 3.4(b)). Thus $\mathrm{GKdim}_A(M) + j_A(M) = \mathrm{GKdim}(B/\langle f\rangle) + 1$. If $n(M) = 0$, f is not a zero divisor in M. By [Lev92], Lemma 5.7, $\mathrm{GKdim}_B(M/(fM)) = \mathrm{GKdim}_B(M) - 1$, $j_B(M/(fM)) = j_{B/\langle f\rangle}(M/(fM)) + 1$ and $j_B(M/(fM)) = j_B(M) + 1$ by [Lev92], Corollary 4.4. Since $\mathrm{GKdim}_B(M/(fM)) = \mathrm{GKdim}_{B/\langle f\rangle}(M/(fM)) = \mathrm{GKdim}(B/\langle f\rangle) - j_{(B/\langle f\rangle)}(M/(fM))$ we obtain $\mathrm{GKdim}_B(M) + j_B(M) = \mathrm{Gkdim}(B/\langle f\rangle) + 1$. If $n(M) \geq 1$, we may assume $M' \neq 0$. Consider the exact sequence $0 \to \mathrm{Ker}_M(f) \to M \to M' \to 0$. By [Lev92], Sublemma 5.10, induction and the results above, we obtain $\mathrm{GKdim}_B(M) + j_B(M) = \mathrm{Gkdim}(B/\langle f\rangle) + 1$.

□

Remark 2.3.30 ([CV10], Remark 2.4) In the setting of Proposition 2.3.29(i(i)'), where f is not a zero divisor on M, since $\mathrm{GKdim}(M/(fM)) = \mathrm{Gkdim}(M) - 1$, it follows that if g is a zero divisor on M, then $\mathrm{GKdim}(M/(gM)) \geq \mathrm{GKdim}(M) - 1$.

Now, the main idea is to be able to use Pproposition 2.3.29 to proof an analog of Corollary 1.3.16 for graded skew PBW extensions. For this, we see some important results.

Proposition 2.3.31 ([Lez17], Theorem 2.9) Let A be a bijective skew PBW extension of

a ring R such that R is Auslander-Gorenstein, respectively Auslander-regular, then so too is A

Proposition 2.3.32 ([Lez17], Theorem 3.9) Let A be a bijective skew PBW extension of a ring R such that R is Auslander-Gorenstein, Cohen-Macaulay, and $R = \sum_{i\geq 0} R_i$ is a connected graded $\mathbb{K}$-algebra such that $\sigma_j(R_i) \subseteq R_i$, for each $i \geq 0$ and $1 \leq j \leq n$, then A is Cohen Macaulay.

For algebraic and dimensional properties of graded skew PBW extensions, see [SLR17].

With this, we come to the main result of this section and the main theorem of this work that will be included in a manuscript which will be submitted for publication.

Theorem 2.3.33 Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective graded skew PBW extension of a finitely generated $\mathbb{K}$-algebra R such that R is Auslander-Gorenstein, Cohen-Macaulay, finite dimensional generating subspace V, and $R = \sum_{i\geq 0} R_i$ is a connected graded $\mathbb{K}$-algebra such that either $\sigma_j(V) \subseteq V$ or σ_j is locally algebraic and $\sigma_j(R_i) \subseteq R_i$, for each $i \geq 0$ and $1 \leq j \leq n$. If $\{f_1, \ldots, f_n\}$ is a normalizing sequence in A of homogeneous elements of positive degree, then the following are equivalent:

(i) $\{f_1, \ldots, f_n\}$ is a regular sequence in A.

(ii) $\dim_{\mathbb{K}}(A/\langle f_1, \ldots, f_n\rangle) < \infty$.

(iii) For each $k = 1, \ldots, n$, we have $\mathrm{GKdim}(A/\langle f_1, \ldots, f_k\rangle) = n - k$.

(iv) If A is quadratic quasi-commutative skew PBW extension, with finite global dimension and R is a field, or if $A/\langle f_1, \ldots, f_n\rangle \cong S/\langle G\rangle$, where S is a regular skew polynomial ring on m generators and G is a normalizing sequence in $S \setminus \mathbb{K}^{\times}$ of m homogeneous elements, then there are no right base-point modules over $A/\langle f_1, \ldots, f_n\rangle$ (if $A/\langle f_1, \ldots, f_n\rangle$ is connected and generated by its degree-1 elements).

Proof. By Proposition 2.3.31 and Proposition 2.3.32, we can see that A is Auslander-Gorenstein and Cohen-Macaulay. Since R is finite dimensional, then $\mathrm{GKdim}(R) = 0$. By Proposition 2.3.24 and Proposition 2.3.27, $\mathrm{GKdim}(A) = n$, since $\mathrm{GKdim}(R) = 0$ and, hence, by Proposition 2.3.28 we have that A has finite injective dimension.

For equivalence between (i), (ii) and (iii). Suppose that $\{f_1, \ldots, f_n\}$ is a regular sequence. By Proposition 2.3.29(iii), we have $\mathrm{GKdim}(A/\langle f_1, \ldots, f_k\rangle) = n - k$ for all k, and hence $\dim_{\mathbb{K}}(A/\langle f_1, \ldots, f_n\rangle) < \infty$.

Now, to prove the converse, suppose that $\{f_1, \ldots, f_k\}$ is a regular sequence for some $k = 1, \ldots, n$, but that f_{k+1} is not regular in $A/\langle f_1, \ldots, f_k\rangle$. By Proposition 2.3.29(ii), we have that $A/\langle f_1, \ldots, f_k\rangle$ is homogeneous, and hence, by Proposition 2.3.29(i) and (iii), we have that $\mathrm{GKdim}(A/\langle f_1, \ldots, f_{k+1}\rangle) = \mathrm{GKdim}(A/\langle f_1, \ldots, f_k\rangle) = n - k$. Now, it follows,

from Proposition 2.3.29(iii) and Remark 2.3.30, that, for any $N = k+2, \ldots, n$, we have $\mathrm{GKdim}(A/\langle f_1, \ldots, f_N\rangle) \geq (n-k) - (N-k-1) = n - N + 1$. Hence, we have that $\mathrm{GKdim}(A/\langle f_1, \ldots, f_n\rangle) \geq 1$, so that $\dim_{\mathbb{K}}(A/\langle f_1, \ldots, f_k\rangle) = \infty$.

If A is quadratic quasi-commutative skew PBW extension, with finite global dimension and R is a field, then A is a graded skew Clifford algebra ([RR])[3]. By Proposition 2.3.19, (iv) is equivalent to (i), (ii) and (iii).

Now, if $A/\langle f_1, \ldots, f_n\rangle \cong S/\langle G\rangle$, where S is a regular skew polynomial ring on m generators and G is a normalizing sequence in $S \setminus \mathbb{K}^{\times}$ of m homogeneous elements, we have that condition (iv) holds if and only if $S/\langle G\rangle$ has no right base-point modules. By Corollary 1.3.16, this situation happens if and only if $\dim_{\mathbb{K}}(S/\langle G\rangle) < \infty$. It follows that condition (iv) is equivalent to condition (ii). □

Next, we will see an example of a skew PBW extension that meets the conditions of Theorem 2.3.33.

Example 2.3.34 We consider the Ore extensions defined in [ALF16] as follows: R is a commutative ring and A denote the Ore extension $A := R[x_1, \ldots, x_n; \sigma]$ for which σ is an automorphism of R, $x_i x_j = x_j x_i$ and $x_i r = \sigma(r) x_i$, for every $1 \leq i, j \leq n$.

1. If $R = \mathbb{K}$, we have the conditions of Theorem 2.3.33 with $\sigma(V) \subseteq V$, since σ is an automorphism of $\mathbb{K}$.
2. If we consider $R = \mathbb{K}$ and $n = 1$, by Proposition 2.3.23, $\mathrm{GKdim}(\mathbb{K}[x; \sigma]) = 1$. By [Zha97], Theorem 1.2, we have that σ is locally algebraic.

2.4 Point modules in semi-graded algebras

Finally, we are going to look at the theory of point modules in the semi-graded context. If we realize, in the previous sections the concepts of point module, right base-point module have been defined and the parameterization has been seen through a scheme in the graded case. All of these definitions were given based on the graduation of an algebra. In the last part of Section 2.3, we were able to see the geometry of the normalizing sequences of the graded skew PBW extensions. In this section, we define the geometric and algebraic concepts in the semi-graded context so that we can see the geometry of the normalizing sequences in finitely semi-graded algebras.

Definition 2.4.1 ([Lez20], Definition 2.1) Let $B = \bigoplus_{n\geq 0} B_n$ be a FSG ring that is generated in degree 1.

[3]This statement is the result of the work we do to obtain the degree of mathematician in 2018. Currently, we have a preprint with the aim of submitting to publication.

(i) A ***point module*** for B is a finitely $\mathbb{N}$-semi-graded B-module $M = \bigoplus_{n\in\mathbb{N}} M_n$ such that M is cyclic, generated in degree 0, i.e., there exists an element $m_0 \in M_0$ such that $M = Bm_0$, and $\dim_{B_0}(M_n) = 1$ for all $n \geq 0$.

(ii) Two points modules M and M' for B are ***isomorphic*** if there exists a homogeneous B-isomorphism between them.

(iii) $P(B)$ is the collection of isomorphism classes of point modules for B.

With definition of point modules over a FSG ring, we want to see how is its geometry. First, we have to construct a generalization of right base-point module from Definition 1.3.6, since this definition is given by consider graded algebras. For this new definition, we use the generalized Gelfand-Kirillov dimension (Definition A.2.26) and the generalized Hilbert series (Definition A.2.35).

Definition 2.4.2 Let B a FSG $\mathbb{K}$-algebra. We define a *right* (respectively, ***left***) ***base-point module*** over B to be any 1-critical with respect to GGK-dimension semi-graded right (respectively, left) B-module that is generated by its homogeneous degree-zero elements and which has generalized Hilbert series $\mathrm{Gh}(t) = c/(1-t)$ for some $c \in \mathbb{N}$.

Remark 2.4.3 Although we have that, in general, the classical Gelfand-Kirillov dimension is different from the generalized Gelfand-Kirillov dimension, in this particular case where we consider B as FSG $\mathbb{K}$-algebra, $\mathrm{GGKdim}(B) = \mathrm{GKdim}(B)$. Also, this happens for generalized Hilbert series and classical Hilbert series.

Now, we have an important result of a scheme that parameterizes the point modules in SG rings. This helps us to see the geometry of the normalizing sequences that we see later.

Proposition 2.4.4 ([LG19], Theorem 11) Let $B = \bigoplus_{n\geq 0} B_n$ be a FSG ring generated in degree 1 such that $B_0 = \mathbb{K}$ and B is a $\mathbb{K}$-algebra. Let X be a $\mathbb{K}$-scheme that parametrizes $P(B)$. Then, there exists a bijective correspondence between the closed points of X and $P(B)$.[4]

Remark 2.4.5 As in the graded case, we write $\widehat{\mathcal{V}_B(I)}$ to denote the set of isomorphism classes of (right) base-point modules over B/I, where $B = \bigoplus_{i=0}^{\infty}$ is a FSG $\mathbb{K}$-algebra and I is any homogeneous ideal of B.

Remark 2.4.6 After Theorem 2.3.33, the natural way to generalize would be to see what happens to skew PBW extensions, not necessarily graded. Since we are in the case of finitely semi-graded algebras, we must consider the constant and pre-commutative skew PBW extensions, as stated in [LG19], Example 2.

[4] For all details of this parameterization, see [LG19]

Now, if we want to get an analogous result, we should study the behavior of right base-point modules in the semi-graded context. This definition has not been found in the literature so far. Therefore, a possible notion was made in Definition 2.4.2 based on the graded case.

If we work on the case where the FSG algebra is a $\mathbb{K}$-algebra, Remark 2.4.3 tells us that the generalized concepts of Gelfand-Kirillov dimension and Hilbert series coincide with the classic ones. Therefore, the differences in the graded case to the semi-graded case of the notion of right base-point module should be studied.

Proposition 2.3.29(iii) is valid for graded case and its proof is strongly based on the graduation of an algebra, since with it important results are obtained that involve well filtration, the concept of separation, among others. Therefore, for this part it is necessary to see what analogous properties of finitely graded algebras exist for finitely semi-graded algebras.

If it is possible to do the aforementioned, we arrive at a possible way for the geometry of the normalizing sequences in FSG $\mathbb{K}$-algebras.

Finally, we would have the case of FSG R-algebras, where R is not necessarily a field. For this case, we must study the properties used of Gelfand-Kirillov dimension and Hilbert series in the right base-point modules of the graded context and try to show analogous properties for generalized Gelfand-Kirillov dimension and generalized Hilbert series.

APPENDIX A

Homological properties and category theory

In this appendix all homological properties that have been used (directly or indirectly) in the realization of the normalizing sequences in the semi-graded algebras are presented.

A.1 Some properties of ideals and modules

In this section, we see some properties of rings and modules that are mentioned throughout the work.

Definition A.1.1 ([Hun74], Definition 3.2.14) An ideal P in a ring R is said to be ***prime***, if $P \neq R$ and for any ideals A, B in R, if $AB \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$.

We say that a ring R is ***prime*** if and only if the zero ideal $\{0\}$ is a prime ideal.

Example A.1.2 Let $R = \mathbb{Z}[x]$ be the ring of polynomials over $\mathbb{Z}$ and the ideal $I = \langle x \rangle$. We have that I is prime.

Proposition A.1.3 ([Hun74], Theorem 3.2.15) If P is an ideal in a ring R such that $P \neq R$ and for all $a, b \in \mathbb{R}$, $ab \in P \Rightarrow a \in P$ or $b \in P$, then P is prime.

Proof. If A and B are ideals such that $AB \subseteq P$ and $A \nsubseteq P$, then there exists an element $a \in A - P$. For every $b \in B$, $ab \in AB \subseteq P$, whence $a \in P$ or $b \in P$. Since $a \notin P$, we must have $b \in P$, for all $b \in B$; that is, $B \subseteq P$. Therefore, P is prime. □

Example A.1.4 ([Hun74], Examples 3.2.15) The zero ideal in any integral domain is prime since $ab = 0$ if and only if $a = 0$ or $b = 0$. If p is a prime integer, then the principal ideal $\langle p \rangle$ in $\mathbb{Z}$ is prime since $ab \in \langle p \rangle \Rightarrow p \mid ab \Rightarrow p \mid a$ or $p \mid b \Rightarrow a \in \langle p \rangle$ or $b \in \langle p \rangle$.

Definition A.1.5 ([Hun74], Definition 3.2.17) A right ideal (respectively left ideal) M in a ring R is said to be ***maximal***, if $M \neq R$ and for every right ideal (respectively left ideal) N such that $M \subseteq N \subseteq R$, either $N = M$ or $N = R$.

Analogously, a right ideal (respectively left ideal) M in a ring R is said to be ***minimal***, if $M \neq \{0\}$ and for every right ideal (respectively left ideal) N such that $\{0\} \subseteq N \subseteq M$, either $N = M$ or $N = \{0\}$.

Example A.1.6 ([Hun74], p.127) The ideal $\langle 3 \rangle$ is maximal in $\mathbb{Z}$; but the ideal $\langle 4 \rangle$ is not since $\langle 4 \rangle \subsetneq \langle 2 \rangle \subsetneq \mathbb{Z}$.

One of the most important ideals in this work are the prime ideals, since these are related to the normalizing sequences. For this, we have the following definition.

Definition A.1.7 ([Hun74], Exercise 8.2.6) Let R be a ring with identity. A prime ideal P in R is called a ***minimal prime*** over I, if $I \subseteq P$ and there is no prime ideal P' such that $I \subseteq P' \subsetneq P$.

Example A.1.8 Consider $\mathbb{C}[x, y]/\langle xy \rangle$. The ideals $\langle x \rangle$, $\langle y \rangle$ are the minimal prime ideals in $\mathbb{C}[x, y]/\langle xy \rangle$, since they are the extension of prime ideals for the morphism $\mathbb{C}[x, y] \to \mathbb{C}[x, y]/\langle xy \rangle$, contain the zero ideal and are not contained in any other prime ideal.

In this point, it is important to define a dimension that is tied to prime ideals.

Definition A.1.9 ([GP12], Definition 3.3.1) Let A be a ring and let $\mathcal{C}(A)$ be the set of all chains of prime ideals in A, that is,

$$\mathcal{C}(A) := \{\mathfrak{p} = (P_0 \subsetneq \cdots \subsetneq P_m \subsetneq A) \mid P_i \text{ prime ideal } \}.$$

(i) If $\mathfrak{p} = (P_0 \subsetneq \cdots \subsetneq P_m \subsetneq A) \in \mathcal{C}(A)$ then ***length***$(\mathfrak{p}) := m$.

(ii) The ***Krull dimension*** of A is defined as $\mathcal{K}\dim(A) = \sup\{\text{length}(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{C}(A)\}$.

(iii) For an arbitrary ideal $I \subseteq A$ the ***Krull dimension*** of I is defined by $\mathcal{K}\dim(I) := \mathcal{K}\dim(A/I)$.

Example A.1.10 If we consider the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, we have that the Krull dimension $\mathcal{K}\dim(\mathbb{K}[x_1, \ldots, x_n]) = n$[1]

One of the most important constructions in algebra is the localization. In it, prime ideals are used to define a multiplicative system. This is defined as follows:

[1]This claim is not easy to proof. For all details, see [Nat18].

Definition A.1.11 ([Hun74], p. 147) Let R be a commutative ring with identity and P a prime ideal of R. Then $S = R \setminus P$ is a multiplicative subset. The ring of quotients $S^{-1}R$ is called the ***localization of*** R ***at*** P and is denoted R_P. If I is an ideal in R, then the ideal $S^{-1}I$ in R_P is denoted I_P.

Example A.1.12 ([Mor98], Example 10) Let R_1 and R_2 be commutative rings, and let $R = R_1 \oplus R_2$. Let P_1 be a prime ideal of R_1, and set $P = P_1 \oplus R_2$. Then P is a prime ideal of R. Moreover, we claim that $R_P \cong (R_1)_{P_1}$. To prove this, let σ and σ_1 be the canonical maps $\sigma : R \to R_P$ and $\sigma_1 : R_1 \to (R_1)_{P_1}$. Let $\pi : R \to R_1$ be the standard projection. Then $\pi(P) = P_1$, and so $\pi(S - P) \subseteq (R_1)^{\times}_{P_1}$. So, by the universal property, there is a ring homomorphism $\pi' : R_P \to (R_1)_{P_1}$, and this map satisfies $\pi'((r_1, r_2)/(s_1, t_2)) = r_1/s_1$. Conversely, there is a ring homomorphism $\tau : R_1 \to R_P$ given by $\tau(r) = (r, 0)/(1, 1)$. Note that since $(1, 0)/(1, 1) = (1, 1)/(1, 1)$ in R_P, the map τ does send 1 to 1, so τ is a ring homomorphism. The map τ send $R_1 - P_1$ into the units of R_P, so the universal mapping property gives us a map $\tau' : (R_1)_{P_1} \to R_P$. This map satisfies $\tau'(r/s) = (r, 0)/(s, 0)$. From definition of localization, $(r, r_2)/(s, t_2) = (r, 0)/(s, 0)$ for any $r \in R_1$, $s \in R_1 - P_1$ and $r_2, t_2 \in R_2$. This proves that $\pi' \circ \sigma' = \mathrm{id}$, and also equality $\sigma' \circ \tau' = \mathrm{id}$. So, $R_P \cong (R_1)_{P_1}$.

Definition A.1.13 ([Hun74], Definition 3.4.12) A ***local ring*** is a commutative ring with identity which has a unique maximal ideal.

Example A.1.14 ([Hun74], Example 3.4.12) If p is a prime and $n \geq 1$, then $\mathbb{Z}_{p^n}$ is a local ring with unique maximal $\langle p \rangle$.

Now, we define modules that are fundamental in the development of this work.

Definition A.1.15 ([Hun74], Definition 4.1.4) If X is a subset of a module A over a ring R, then the intersection of all submodules of A containing X is called the ***submodule generated by*** X (or ***spanned by*** X).

If X is finite, and X generates the module B, B is said to be ***finitely generated***. If X consists of a single element, $X = \{a\}$, then the submodule generated by X is called the ***cyclic (sub)module*** generated by a.

Example A.1.16 Let $A_n[x]$, $n \geq 0$, be the set of all polynomials of degree less than or equal to n with coefficients in A. This set is an A-submodule of $A[x]$ and it is finitely generated: $\langle 1, x, \ldots, x^n \rangle = A_n[x]$. At the same time, the set $\{x^k\}_{k=0}^{\infty}$ is a generator system for $A[x]$.

Proposition A.1.17 ([Hun74], Theorem 4.1.15) Let R be a ring and $\{A_i \mid i \in I\}$ a family of submodules of an R-module A such that

(i) A is the sum of the family $\{A_i \mid i \in I\}$.

(ii) For each $k \in I$, $A_k \cap A_k^* = 0$, where A_k^* is the sum of the family $\{A_i \mid i \neq k\}$.

Then there is an isomorphism $A \cong \sum_{i \in I} A_i$.

A module A is said to be the (***internal***) ***direct sum*** of a family of submodules $\{A_i \mid i \in I\}$ provided that A and $\{A_i\}$ satisfy the hypotheses of Proposition A.1.17.

Proposition A.1.18 ([Hun74], Theorem 4.2.1) Let R be a ring with identity. The following conditions on a unitary R-module F are equivalent:

(i) F has a nonempty basis.

(ii) F is the internal direct sum of a family of cyclic R-modules, each of which is isomorphic as a left R-module of R.

(iii) F is R-module isomorphic to a direct sum of copies of the left R-module R.

A unitary module F over ring R with identity, which satisfies the equivalent conditions of Proposition A.1.18, is called a ***free R-module*** on the set X.

Since, in general, the modules do not have a finite base, it is important to highlight those that do have a finite base and, even more, those whose bases have the same cardinality. For this, we have the following definition.

Definition A.1.19 ([Hun74], Definition 4.2.8) Let R be a ring with identity such that for every free R-module F, any two bases of F have the same cardinality. Then R is said to have the ***invariant dimension property*** and the cardinal number of any basis of F is called the ***dimension*** (or ***rank***) of F over R, denoted by $\dim_R(F)$.

Example A.1.20 ([Rot08], Proposition 2.37) If R is division ring, then R has invariant dimension property. Indeed, it is enough to show that if F is a free R-module with two finite bases $B = \{b_1, \ldots, b_n\}$ and $B' = \{b'_1, \ldots, b'_m\}$ then $n = m$. We use induction with respect to n.

Assume that $n = 1$, and so $B = \{b_1\}$. If $B' = \{b'_1, \ldots, b'_m\}$ for some $m > 1$, we have $b'_1 = r_1 b_1$ and $b'_2 = r_2 b_1$, for some $r_1, r_2 \in R$, $r_2 \neq 0$. Therefore $r_1^{-1} b'_1 - r_2^{-1} b'_2 = 0$, which contradicts the assumption that B' is a linearly independent set. So, $m = 1$.

Now, we assume that for $n \geq 1$ if a free R-modules has a basis consisting of n elements then every other basis of that module also has n elements.

Let F be a free R-module with a basis $B = b_1, \ldots, b_{n+1}$ consisting of $n+1$ elements and let $B' = \{b'_1, \ldots, b'_m\}$ be another basis of F. Since $\langle B' \rangle = F$ we have

$$b_{n+1} = r_1 b'_1 + \cdots + r_m b'_m,$$

for some $r_1, \ldots, r_m \in R$. Since $b_{n+1} \neq 0$, we have $r_i \neq 0$ for some $1 \leq i \leq m$. We can assume that $r_m \neq 0$. Let $B'' = \{b'_1, \ldots, b'_{m-1}, b_{n+1}\}$. We have that B'' is another basis of F.

Now, we have the canonical epimorphism

$$\pi : F \to F/\langle b_{n+1} \rangle.$$

Since F is a free module with basis $B = \{b_1, \ldots, b_n, b_{n+1}\}$, thus $F/\langle b_{n+1} \rangle$ is a free module with basis $\{\pi(b_1), \ldots, \pi(b_n)\}$. On the other hand, since F has a basis $B'' = \{b'_1, \ldots, b'_{m-1}, b_{n+1}\}$ there $\{\pi(b'_1), \ldots, \pi(b'_{m-1})\}$ is a basis of $F/\langle b_{n+1} \rangle$. By inductive assumption we obtain that $n = m - 1$, and so $n + 1 = m$.

The dimension on the modules defined in Definition A.1.19 is a very important concept since, throughout this work, this dimension is used for higher calculations and for larger results.

Next, we show the most important property of modules and rings, since this property is fundamental to be able to apply enough theorems and propositions that are cited during this work.

Definition A.1.21 ([Hun74], Definition 8.1.1) A module A is said to satisfy the ***ascending chain condition*** (***ACC***) ***on submodules*** (or to be ***Noetherian***), if for every chain $A_1, \subseteq A_2 \subseteq \cdots$ of submodules of A, there is an integer n such that $A_i = A_n$, for all $i \geq n$.

A module B is said to satisfy the ***descending chain condition*** (***DCC***) ***on submodules*** (or to be ***Artinian***), if for every chain $B_1, \supseteq B_2 \supseteq \cdots$ of submodules of B, there is an integer m such that $B_i = B_m$, for all $i \geq m$.

Example A.1.22 ([Hun74], Example 8.1.1) The $\mathbb{Z}$-module (abelian group) $\mathbb{Z}$ satisfies the ascending but not the descending chain condition on submodules. Indeed, since $\mathbb{Z}$ is a principal ideal domain, any ideal is generated by some single element, and containment is the same as divisibility. Hence, an ascending chain ideals $\langle n_1 \rangle \subseteq \langle n_2 \rangle \subseteq \langle n_3 \rangle \subseteq \cdots$ give us a chain of nonnegative integers, each a factor of the one before $\cdots \mid n_3 \mid n_2 \mid n_1$. That means that $n_1 \geq n_2 \geq n_3 \geq \cdots \geq 0$, so certainly the chain stabilises, and hence $\mathbb{Z}$ is a Noetherian $\mathbb{Z}$-module.

Now, consider the following descending chain of ideals $\langle 1 \rangle \supseteq \langle 2 \rangle \supseteq \langle 4 \rangle \supseteq \langle 8 \rangle \supseteq \cdots$.

This is not stabilise. So, $\mathbb{Z}$ is not an Artinian $\mathbb{Z}$-module.

Definition A.1.23 ([Hun74], Definition 8.1.2) A ring R is ***left*** (respectively ***right***) ***Noetherian***, if R satisfies the ascending chain condition on left (respectively right) ideals. R is said to be ***Noetherian***, if R is both left and right Noetherian.

A ring R is ***left*** (respectively ***right***) ***Artinian***, if R satisfies the descending chain condition on left (respectively right) ideals. R is said to be ***Artinian***, if R is both left and right Artinian.

A.2 General homological elements

A.2.1 Complex and resolutions

Before arriving at the most advanced homological elements, it is necessary to define some basic elements that allow to facilitate understanding of those that will be defined later on.

Definition A.2.1 ([Rot08], p. 28) A sequence or chain of A-modules homomorphisms of the form

$$\mathcal{C} : \cdots \longrightarrow M_{-1} \xrightarrow{f_{-1}} M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots,$$

is called a ***complex***, if $\mathrm{Im}(f_i) \subseteq \mathrm{Ker}(f_{i+1})$, for each $i \in \mathbb{Z}$. The quotient module

$$\mathrm{Ker}(f_{i+1})/\mathrm{Im}(f_i),$$

is called the ***i-th module of homology*** of the complex $\mathcal{C}$. The sequence $\mathcal{C}$ is called ***exact*** if $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$, for each $i \in \mathbb{Z}$.

Definition A.2.2 ([Rot08], p. 47) A finite exact sequence of the form

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0,$$

is said to be a ***short exact sequence***. Two short exact sequences

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0,$$

and

$$0 \longrightarrow N_1 \xrightarrow{a} N_2 \xrightarrow{b} N_3 \longrightarrow 0,$$

are said to be ***equivalents***, if there are A-homomorphisms h, k, l such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{a} & N_2 & \xrightarrow{b} & N_3 & \longrightarrow & 0
\end{array}$$

With these definitions, we can get to a very important type of modules

Definition A.2.3 ([Rot08], p. 106) Let M be an A-module. It is said that M is ***finitely presented***, if there exists an exact sequence of the form

$$A^n \xrightarrow{f} A^m \xrightarrow{g} M \longrightarrow 0,$$

where A^n and A^m are free A-modules of finite bases with $n \geq 1$ and $m \geq 1$ elements, respectively.

An important proposition about Noetherian modules is the following:

Proposition A.2.4 ([Rot08], Corollary 3.19) Let A_A Noetherian. Then, each A-module finitely generated M is finitely presented.

Proof. If we consider the null module $M = 0$ the statement is trivial. So, let $0 \neq M = \{x_1, \ldots, x_m\rangle$ be an A-module. Now, we define

$$\begin{aligned} g : A^m &\to M \\ g(e_i) &\mapsto x_i, \end{aligned}$$

with $1 \leq i \leq m$, where $\{e_i\}_{i=1}^m$ is the canonical basis of A^m. Since A^m is a noetherian module, $K := \mathrm{Ker}(g)$ is finitely generated and then we have the following exact sequence

$$0 \longrightarrow K \xrightarrow{\imath} A^m \xrightarrow{g} M \longrightarrow 0,$$

where $\imath$ is the inclusion. □

Remark A.2.5 ([Lez], Definition 3.2.1) A functor (Definition A.3.4) $F : \mathscr{B} \to \mathscr{C}$ is said exact to the right, if an exact sequence $A \to B \to C \to 0$ in $\mathscr{B}$ implies that sequence $F(A) \to F(B) \to F(C) \to 0$ in $\mathscr{C}$ is exact. F is said exact to the left, if an exact sequence $0 \to A \to B \to C$ in $\mathscr{B}$ implies that sequence $0 \to F(A) \to F(B) \to F(C)$ in $\mathscr{C}$ is exact. If F is a contravariant functor (Definition A.3.6), F is exact to the right if an exact sequence $0 \to A \to B \to C$ in $\mathscr{B}$ implies that sequence $F(C) \to F(B) \to F(A) \to 0$ in $\mathscr{C}$ is exact and F is exact to the left if an exact sequence $A \to B \to C \to 0$ in $\mathscr{B}$ implies that sequence $0 \to F(C) \to F(B) \to F(A)$ in $\mathscr{C}$ is exact.

Definition A.2.6 ([Rot08], p. 100) Consider an A-module P. P is said to be ***projective***, if $\mathrm{Hom}_A(P, _)$ is an exact functor.

Definition A.2.7 ([Wei95], Definition 2.2.4) Let M be an A-module. A ***left resolution*** of M is a complex $\mathcal{P}$, with $P_i = 0$, for $i < 0$, together with a map $\epsilon : P_0 \to M$ such that the augmented complex

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0\ ,$$

is exact. It is a ***projective resolution***, if each P_i is projective.

With Definition A.2.7, we can define a very important element in homological algebra and that is used a lot, implicitly, in this work.

Definition A.2.8 ([Wei95], Definition 2.52) Let M and N be two A-modules. Consider a projective resolution of M

$$\cdots \longrightarrow P_{i+1} \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0.$$

Now, we can take $\mathrm{Hom}_A(_, N)$ to resolution. We obtain the complex

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \xrightarrow{\epsilon^*} \mathrm{Hom}_A(P_0,N) \xrightarrow{f_1^*} \mathrm{Hom}_A(P_1,N) \longrightarrow \cdots\ ,$$

with $f_i^* : \mathrm{Hom}_A(P_{i-1},N) \to \mathrm{Hom}_A(P_i,N)$, $h \mapsto hf_i$, for all $i \geq 0$.

With this complex, it is defined the following abelian groups

$$\begin{aligned} \mathrm{Ext}^0_A(M,N) &:= \mathrm{Hom}_A(M,N), \\ \mathrm{Ext}^i_A(M,N) &:= \mathrm{Ker}(f^*_{i+1})/\mathrm{Im}(f^*_i), \quad i \geq 1. \end{aligned}$$

Definition A.2.9 ([Lev92], Definition 1.2) Let M be an A-module. The ***grade number*** of M is defined by

$$j_A(M) = \inf\{i \mid \mathrm{Ext}^i_A(M,A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

Definition A.2.10 ([Rot08], p. 116) Consider an A-module N. N is said to be ***injective***, if $\mathrm{Hom}_A(_, N)$ is an exact functor.

Definition A.2.11 ([Wei95], Definition 2.3.5) Let M be an A-module. A ***right resolution*** of M is a complex $\mathcal{L}$, with $L_i = 0$, for $i < 0$, together with a map $f : M \to L_0$ such that the augmented complex

$$0 \longrightarrow M \xrightarrow{f} L_0 \xrightarrow{f_0} L_1 \xrightarrow{f_1} L_2 \longrightarrow \cdots\ ,$$

is exact. It is a ***injective resolution*** if each L_i is projective.

Definition A.2.12 ([GWJ04], p. 89) Let $M \neq 0$ be an A-module and let $0 \neq N \leq M$ be a submodule such that for all submodule $X \leq M$ with $X \neq 0$, we have that $N \cap X \neq 0$. It is said that N is an ***essential (or large) submodule*** of M, and we also say that M is an ***essential extension*** of N. This relation is denoted by $N \leq_e M$.

Definition A.2.13 ([GWJ04], p. 95) A module U is ***uniform***, if $U \neq 0$ and every nonzero submodule of U is essential in U.

A.2.2 Dimensions

Definition A.2.14 ([Wei95], Definition 4.1.1) Let M be a right R-module.

(i) The ***projective dimension*** of M, denoted by $\mathrm{pd}_R(M)$, is the minimum integer n (if it exists) such that there is a projective resolution of M

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

(ii) The ***injective dimension*** of M, denoted by $\mathrm{id}_R(M)$, is the minimum integer n (if it exists) such that there is an injective resolution of M

$$0 \longrightarrow M \longrightarrow L_0 \longrightarrow L_1 \longrightarrow \cdots \longrightarrow L_n \longrightarrow 0.$$

If no finite resolutions exists, we set $\mathrm{pd}(M)$ or $\mathrm{id}(M)$ equal to ∞. pd measures how far a module is from being projective, while id measures how far a module is from being injective.

Definition A.2.15 ([Wei95], Theorem 4.1.2) Let A be a ring. The ***right projective global dimension*** of A, denoted by $\mathrm{rpgld}(A)$, is defined by

$$\mathrm{rpgld}(A) := \sup\{\, \mathrm{pd}(M) \mid M \text{ is a right } A\text{-module} \,\}.$$

The ***right injective global dimension*** of A, denoted by $\mathrm{rigld}(A)$, is defined by

$$\mathrm{rigld}(A) := \sup\{\, \mathrm{id}(M) \mid M \text{ is a right } A\text{-module} \,\}$$

We have that $\mathrm{rpgld}(A) = \mathrm{rigld}(A)$. Then, the ***right global dimension*** of A is defined by $\mathrm{rgld}(A) := \mathrm{rpgld}(A) = \mathrm{rigld}(A)$.

Analogously, we can define the ***left global dimension*** of A by $\mathrm{lgld}(A) := \mathrm{lpgld}(A) = \mathrm{ligld}(A)$, where

$$\operatorname{lpgld}(A) \coloneqq \sup\{\operatorname{pd}(M) \mid M \text{ is a left } A\text{-module}\},$$
$$\operatorname{ligld}(A) \coloneqq \sup\{\operatorname{id}(M) \mid M \text{ is a left } A\text{-module}\}.$$

We say that A has ***finite global dimension*** (respectively ***finite injective dimension***), if the left and right global dimensions of A are finite and equal (respectively the modules A_A and ${}_AA$ have finite injective dimensions which are equal). In that case we denote these numbers by $\operatorname{gld}(A)$ (respectively $\operatorname{id}(A)$).

Definition A.2.16 ([GWJ04], p. 98) A submodule has ***finite uniform dimension***, if it does not have infinite direct sums of nonzero submodules.

Proposition A.2.17 ([GWJ04], Proposition 5.20) Let $M \neq 0$ be an A-module and n a nonnegative integer. Then the following conditions are equivalent:

(i) M has finite uniform dimension.

(ii) M has an essential submodule which is a direct sum of n uniform submodules.

(iii) M contains a direct sum of n nonzero submodules but no direct sum of $n+1$ nonzero submodules.

Remark A.2.18 Let $M \neq 0$ be an A-module with finite uniform dimension. The integer n from Proposition A.2.16 is called the ***uniform dimension*** of M, also denominated ***Goldie dimension***, and it is denoted by $\operatorname{udim}(M) = n$. If M does not have finite uniform dimension we write $\operatorname{udim}(M) = \infty$. Uniform dimension to the right of a ring A is defined by $\operatorname{rudim}(A) := \operatorname{udim}(A_A)$. The udim measures how far a module is from being uniform.

Next dimension measures some growth of algebra, in such a way that it establishes how far it is from being of finite dimension as vector space.

Definition A.2.19 ([Lez], Definition 4.9.1) Let $\mathbb{K}$ be a field and let A be a $\mathbb{K}$-algebra. The ***Gelfand-Kirillov dimension*** of A is defining by

$$\operatorname{GKdim}(A) := \overline{\lim_{n\to\infty}} \log_n (\dim_{\mathbb{K}} V^n),$$

where V goes through all frames of A and $V^n :=_{\mathbb{K}} \langle v_1 \cdots v_n | v_i \in V\rangle$; a frame of A is a $\mathbb{K}$-subspace of A with finite dimension such that $1 \in V$. It is said that a frame V is generator if A is generated by V as $\mathbb{K}$-algebra.

But we do not just have GK-dimension for algebras. We can also calculate GK-dimension for both rings and modules. Next, this dimension will be defined for these algebraic objects.

Definition A.2.20 ([LV20], Definition 1.1) Let R be a commutative domain and Q be the field of fractions of R. Let B be a R-algebra, then $Q \otimes B$ is a Q-algebra and its classical Gelfand-Kirillov dimension is denote by $\mathrm{GKdim}(Q \otimes B)$. If M is a finitely generated R-module, then the ***rank*** of M is defined by

$$\mathrm{rank}M := \dim_Q(Q \otimes_R M) < \infty.$$

The ***Gelfand-Kirillov dimension*** of B is defined to be

$$\mathrm{GKdim}(B) := \sup_V \overline{\lim_{n\to\infty}} \log_n \mathrm{rank}V^n,$$

where V varies over all frames of B and $V^n :=_R \langle v_1 \cdots v_n | v_i \in V, 1 \leq i \leq n\rangle$; a frame of B is a finitely generated R-submodule of B containing 1. A frame V generates B if B is generated by V as R-algebra.

Proposition A.2.21 ([LV20], Proposition 1.3) Let B be a R-algebra and V be a frame that generates B. Then,

$$\mathrm{GKdim}(B) = \overline{\lim_{n\to\infty}} \log_n(\mathrm{rank}V^n).$$

Moreover, this equality is independent of the generator frame.

Next, we see some important properties of GK-dimension for certain algebraic constructions.

Proposition A.2.22 ([LV20], Theorem 2.1) Let B a R-algebra.

(i) If B is finitely generated, then

$$\mathrm{GKdim}(B) = 0 \text{ if and only if } \mathrm{rank}B < \infty.$$

Moreover, if B is a domain with $\mathrm{GKdim}(B) < \infty$, then B is an Ore domain.

(ii) $\mathrm{GKdim}(B[x_1, \ldots, x_m]) = \mathrm{GKdim}(B) + m$.

(iii) For $m \geq 2$, $\mathrm{GKdim}(R\{x_1, \ldots, x_m\}) = \infty$.

(iv) $\mathrm{GKdim}(M_n(B)) = \mathrm{GKdim}(B)$.

(v) If I is a proper two-sided ideal of B, then $\mathrm{GKdim}(B/I) \leq \mathrm{GKdim}(B)$.

(vi) Let C be a subalgebra of B. Then, $\mathrm{GKdim}(C) \leq \mathrm{GKdim}(B)$. Moreover, if $C \subseteq Z(B)$ (the center of B) and B is finitely generated as C-module, then $\mathrm{GKdim}(C) = \mathrm{GKdim}(B)$.

(vii) Let C be a R-algebra. Then,

$$\mathrm{GKdim}(B \times C) = \max\{\mathrm{GKdim}(B), \mathrm{GKdim}(C)\}.$$

(viii) Let C be a R-algebra. Then,

$$\max\{\mathrm{GKdim}(B), \mathrm{GKdim}(C)\} \leq \mathrm{GKdim}(B \otimes C) \leq \mathrm{GKdim}(B) + \mathrm{GKdim}(C).$$

In addition, suppose that C contains a finitely generated subalgebra C_0 such that $\mathrm{GKdim}(C_0) = \mathrm{GKdim}(C)$, then

$$\mathrm{GKdim}(B \otimes C) = \mathrm{GKdim}(B) + \mathrm{GKdim}(C).$$

(ix) Let S be a multiplicative system of B consisting of central regular elements. Then,

$$\mathrm{GKdim}(BS^{-1}) = \mathrm{GKdim}(B).$$

Now, we can extend Definition A.2.20 to modules.

Definition A.2.23 ([LV20], Definition 3.1) Let B a R-algebra and M be a right B-module. The ***Gelfand Kirillov*** of M is defined by

$$\mathrm{GKdim}(M) := \sup_{V,F} \overline{\lim_{n\to\infty}} \log_n \mathrm{rank} FV^n,$$

where V varies over all frames of B and F over all finitely generated R-submodules of M. In addition, $\mathrm{GKdim}(0) := -\infty$.

Proposition A.2.24 ([LV20], Proposition 3.2) Let B be a R-algebra and M be a right B-module. Then,

$$\mathrm{GKdim}(M) = \mathrm{GKdim}(Q \otimes_R M).$$

Now, we see some important properties of GK dimension.

Proposition A.2.25 ([LV20], Theorem 3.3) Let B a R-algebra and M be a right B-module. Then,

(i) $\mathrm{GKdim}(B_B) = \mathrm{GKdim}(B)$.

(ii) $\mathrm{GKdim}(M) \leq \mathrm{GKdim}(B)$.

(iii) Let $0 \to K \to M \to L \to 0$ be an exact sequence of right B-modules, then

$$\mathrm{GKdim}(M) \geq \max\{\mathrm{GKdim}(K), \mathrm{GKdim}(L)\}.$$

(iv) Let I be a two-sided ideal of B and $MI = 0$, then

$$\mathrm{GKdim}(M_B) = \mathrm{GKdim}(M_{B/I}).$$

(v) $\mathrm{GKdim}\left(\sum_{i=1}^{n} M_i\right) = \max\{\mathrm{GKdim}(M_i)\}_{i=1}^{n} = \mathrm{GKdim}\left(\bigoplus_{i=1}^{n} M_i\right)$.

Now, we see how the Gelfand-Kirillov dimension can be generalized with the following definition.

Definition A.2.26 ([LL17], Definition 4.1) Let B be a FSG ring such that B_0 is a left Noetherian domain. The ***generalized Gelfand-Kirillov dimension*** of B is defined by

$$\mathrm{GGKdim}(B) := \sup_V \overline{\lim_{k\to\infty}} \log_k \mathrm{udim} V^k,$$

where V ranges over all frames of B and $V^k :=_{B_0} \langle v_1 \cdots v_k | v_i \in V \rangle$ (a frame of B is a finite dimensional B_0-free submodule of B such that $1 \in V$).

The word "generalize" comes from the fact that if V is a free R-module of finite dimension, $\dim_R(V) = k$, then

$$\mathrm{udim}(V) = \mathrm{udim}({}_R R \oplus \cdots \oplus_R R) = \mathrm{udim}({}_R R) + \cdots + \mathrm{udim}({}_R R) = k.$$

that is, GGK-dimension coincides with GK-dimension.

Remark A.2.27 ([LL17], Remark 4.2)

(i) Note that B has at least one frame: B_0 is a frame of dimension 1. We say that V is a ***generating frame*** of B, if the subring of B generating by V and B_0 is B.

(ii) The notion of generalized Gelfand-Kirillov dimension of a FSG ring B depends on the semi-graduation, in particular, depends on B_0.

(iii) As above, if B is a finitely graded $\mathbb{K}$-algebra, then the classical Gelfand-Kirillov dimension of B coincides with the generalized Gelfand-Kirillov dimension.

Now, we see an analogous of Proposition A.2.21 for GGKdim.

Proposition A.2.28 ([LL17], Proposition 4.3) Let B be a FSG ring such that B_0 is a left Noetherian domain. Let V be a generating frame of B, then

$$\mathrm{GGKdim}(B) = \overline{\lim_{k\to\infty}} \log_k(\mathrm{udim} V^k).$$

Moreover, this equality does not depend on the generating frame V.

Now, we see the generalized Gelfand-Kirillov for some skew PBW extensions.

Proposition A.2.29 ([LL17], Theorem 4.4) Let R be a left Noetherian domain and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of R. Then,

$$\mathrm{GGKdim}(A) = \overline{\lim_{k\to\infty}} \log_k \left(\sum_{i=0}^{k} \dim_R A_i \right) = n.$$

A.2.3 Some important properties of homological algebra

We want to conclude this section of homological algebra, defining some of the most important properties that were used for certain proofs throughout this work.

Definition A.2.30 ([Lev92], Definition 2.1) Let B be a Noetherian ring.

(i) A B-module M satisfies the ***Auslander-condition***, if $\forall p \geq 0$, $j_B(N) \geq p$, for all B-submodule N of $\mathrm{Ext}^p_B(M, B)$.

(ii) The ring B is said to be ***Auslander-Gorenstein*** of dimension q, if $\mathrm{id}(B) = q < \infty$, and every left or right finitely generated B-module satisfies the Auslander-condition.

(iii) The ring B is said to be ***Auslander-regular*** of dimension q, if $\mathrm{gld}(B) = q < \infty$ and every left or right finitely generated B-module satisfies the Auslander-condition.

(iv) Let B be an algebra. If $\mathrm{GKdim}(B) = j_B(M) + \mathrm{GKdim}(M)$ for every non-zero Noetherian B-module M, then B is called ***Cohen-Macaulay***.

Now, we have some important properties for the dimensions that we have considered. However, we study theses properties for a general dimension. For that, we start with definition of dimension function.

Definition A.2.31 ([MRS01], Definition 6.8.4) A ***dimension function*** δ for a right Noetherian ring R assigns a value $\delta(M)$ to each finitely generated module M and has the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is exact, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$, with equality if the sequence is split.

(iii) If $MP = 0$ for some prime P and M is a torsion module over R/P, then $\delta(M)+1 \leq \delta(R/P)$.

If, in condition (ii), equality is always true, then δ is called an ***exact dimension function***.

We are going to define some properties for a dimension function

Definition A.2.32 ([ASZ98], Definition 0.2) Given a dimension function δ, exact or not, we say that a module M is

- s-***pure*** (or s-***homogeneous***) ***with respect to*** δ, if $\delta(N) = s$, for all non-zero Noetherian submodules $N \subseteq M$;
- ***Essentially*** s-***pure*** (or ***essentially*** s-***homogeneous***) ***with respect to*** δ, if it contains an essential submodule which is s-pure with respect to δ;
- s-***critical with respect to*** δ, if it is s-pure and $\delta(M/N) < s$ for all non-zero submodules $n \subseteq M$.

Definition A.2.33 ([Lev92], p. 288, 5.1) Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a finitely generated graded $\mathbb{K}$-algebra. A has ***polynomial growth***, if there exists positive real numbers c and ϵ such that $\dim_{\mathbb{K}}(A_i) \leq ci^{\epsilon}$, for all $i \geq 0$.

Definition A.2.34 ([Lev92], p. 289) Suppose $B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$ is Noetherian. If $M = \bigoplus_n M_n$ is a B-module, the ***Hilbert series*** is defined by

$$h_M(t) := \sum_{n\in\mathbb{Z}} (\dim_{\mathbb{K}} M_n)\, t^n.$$

In particular,

$$h_B(t) := \sum_{n=0}^{\infty} (\dim_{\mathbb{K}} B_n)\, t^n.$$

As in the case of the Gelfand-Kirillov dimension, we can generalize the Hilbert series for semi-graded modules.

Definition A.2.35 ([LL17], Definition 3.1) Let $B = \sum_{n\geq 0} \oplus B_n$ be a FSG ring and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a FSG B-module. The ***generalized Hilbert series*** of M is defined by

$$\mathrm{Gh}_M(t) := \sum_{n\in\mathbb{Z}} (\dim_{B_0} M_n)\, t^n.$$

In particular,

$$\mathrm{Gh}_B(t) := \sum_{n=0}^{\infty} (\dim_{B_0} B_n)\, t^n.$$

Remark A.2.36 ([LL17], Remark 3.2)

(i) Note that if B is a finitely graded $\mathbb{K}$-algebra, then the generalized Hilbert series coincides with the usual Hilbert series.

(ii) Observe that if a semi-graded ring B has another semi-graduation, then its generalized Hilbert series can change, in particular, generalized Hilbert series depends on B_0.

For skew PBW extensions, we have the following result:

Proposition A.2.37 ([LL17], Theorem 3.3) Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be an arbitrary skew PBW extension. Then,

$$\mathrm{Gh}_A(t) = \frac{1}{(1-t)^n}$$

A.3 General elements of category theory

Definition A.3.1 ([Lei14], Definition 1.1.1) A ***category*** $\mathscr{A}$ consists of:

(i) a collection $\mathrm{ob}(\mathscr{A})$ of ***objects***;

(ii) for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathrm{Mor}_{\mathscr{A}}(A, B)$ of ***morphisms*** from A to B;

(iii) for each $A, B, C \in \mathrm{ob}(\mathscr{A})$, a function

$$\begin{array}{ccccc} \mathrm{Mor}_{\mathscr{A}}(B,C) \times \mathrm{Mor}_{\mathscr{A}}(A,B) & \to & \mathrm{Mor}_{\mathscr{A}}(A,C) \\ (g, f) & \mapsto & g \circ f \end{array}$$

called ***composition***.

(iv) for each $A \in \mathrm{ob}(\mathscr{A})$, an element i_A of $\mathrm{Mor}_{\mathscr{A}}(A, A)$, called the ***identity*** on A, satisfying the following axioms:

- ***associativity***: for each $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$, $g \in \mathrm{Mor}_{\mathscr{A}}(B, C)$ and $h \in \mathrm{Mor}_{\mathscr{A}}(C, D)$, we have
$$(h \circ g) \circ f = h \circ (g \circ f);$$

- ***identity law***: for each $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ we have

$$f \circ i_A = f = i_B \circ f.$$

Example A.3.2 ([Lei14], Examples 1.1.3) We can see the sets with their functions, as a category. Indeed, we consider a category ***Set*** as follows.

(i) $\mathrm{ob}(\boldsymbol{Set})$ are sets.

(ii) Given A, B sets, a morphism from A to B is exactly a function between A and B.

(iii) Composition of morphism is the usual composition of functions.

(iv) Identity of $\mathrm{Mor}_{\boldsymbol{Set}}(A, A)$ is the identity function.

Remark A.3.3 Given a category $\mathscr{A}$, we can construct an ***opposite*** or ***dual*** category $\mathscr{A}^{op}$, as follows

(i) $\mathrm{ob}(\mathscr{A}^{op}) = \mathrm{ob}(\mathscr{A})$

(ii) for each $A, B \in \mathrm{ob}(\mathscr{A}^{op})$, $\mathrm{Mor}_{\mathscr{A}^{op}}(B, A) = \mathrm{Mor}_{\mathscr{A}}(A, B)$

(iii) for each $f \in \mathrm{Mor}_{\mathscr{A}^{op}}(A, B)$ and $g \in \mathrm{Mor}_{\mathscr{A}^{op}}(B, C)$

$$g \circ f = (f^{op} \circ g^{op})^{op}$$

(iv) identity is the same as category $\mathscr{A}$.

Now, we have to see how categories interact with each other. Therefore, we come to notion of ***functor***.

Definition A.3.4 ([Lei14], Definition 1.2.1)Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A ***functor*** $F : \mathscr{A} \to \mathscr{B}$ consists of:

(i) a function

$$\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}, (\mathscr{B})$$

written as $A \mapsto F(A)$;

(ii) for each $A, A' \in \mathrm{ob}(\mathscr{A})$, a function

$$\mathrm{Mor}_{\mathscr{A}}(A, A') \to \mathrm{Mor}_{\mathscr{B}}(F(A), F(A')),$$

written as $f \mapsto F(f)$, satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A'$, $f' : A' \to A''$ in $\mathscr{A}$;
- $F(i_A) = i_{F(A)}$ whenever $A \in \mathrm{ob}(\mathscr{A})$.

Example A.3.5 ([Lei14], Examples 1.2.3) There is a functor between category **Grp** (category of groups) and category **Set**. Indeed, we consider U : **Grp** $\to$ **Set** defined as follows:

(i) for $G \in \mathrm{ob}(\mathbf{Grp})$, $U(G)$ is the underlying set of G;

(ii) if $f : G \to H$ is a group homomorphism then $U(f)$ is the function itself.

This type of functors are the so-called ***forgetful functors***.

Definition A.3.4 describes a functor that is usually called a ***covariant functor***. Now we will see another type of functor which is important in theory of categories.

Definition A.3.6 ([Lei14], Definition 1.2.10)Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A ***contravariant functor*** from $\mathscr{A}$ to $\mathscr{B}$ is a functor $\mathscr{A}^{op} \to \mathscr{B}$.

Example A.3.7 ([Lei14], Examples 1.2.11) Given a topological space X, let $C(X)$ be the ring of continuous real-valued functions on X. The ring operations are defined *pointwise*: for instance, if $p_1, p_2 : X \to \mathbb{R}$ are continuous maps then the map $p_1 + p_2 : X \to \mathbb{R}$ is defined by

$$(p_1 + p_2)(x) = p_1(x) + p_2(x),$$

for $x \in X$. A continuous map $f : X \to Y$ induces a ring homomorphism $C(f)$: $C(Y) \to C(X)$, defined at $q \in C(Y)$ by taking $(C(f))(q)$ to be the composite map

$$X \xrightarrow{f} Y \xrightarrow{q} \mathbb{R}.$$

$C(f)$ goes in the opposite direction from f. Then, C is a contravariant functor from **Top** to **Ring**.

Definition A.3.8 ([Lei14], Definition 1.2.16) A functor $F : \mathscr{A} \to \mathscr{B}$ is ***faithful*** (respectively, ***full***) if for each $A, A' \in \mathscr{A}$, the function

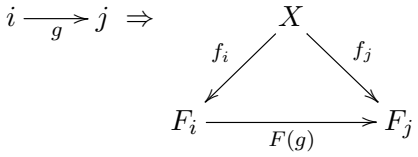
$$\begin{aligned} \mathscr{A}(A, A') &\to \mathscr{B}(F(A), F(A')) \\ f &\mapsto F(f), \end{aligned}$$

is injective (respectively, surjective).

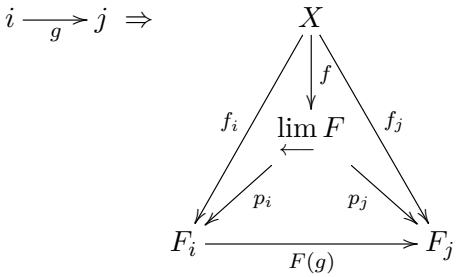
Definition A.3.9 ([Lei14], Proposition 3.2.4) A category $\mathscr{A}$ is ***small*** if $\mathrm{Ob}(\mathscr{A})$ is a set.

Definition A.3.10 ([Lez], Definition 3.3.1) Let $F : \mathscr{L} \to \mathscr{C}$ be a functor, with $\mathscr{L}$ a small category and $\mathscr{C}$ any category. For $X \in \mathrm{Ob}(\mathscr{C})$ and each $i \in I := \mathrm{Ob}(\mathscr{L})$, let $F_i := F(i)$

and $f_i : X \to F_i$ be a morphism. The family $(f_i)_{i\in I}$ is ***compatible*** if for each morphism $g : i \to j$ in $\mathscr{L}$ we have that $f_j = F(g) \circ f_i$:



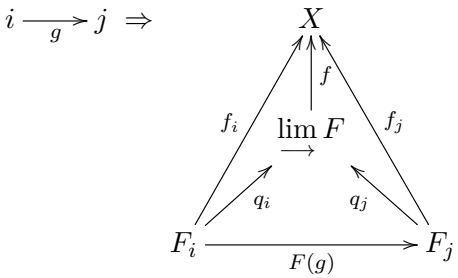
A ***limit*** (or ***inverse limit***) of F is an object $\varprojlim F$ of $\mathscr{C}$ along with a compatible family of morphisms $p_i : \varprojlim F \to F_i$, such that for any other compatible family $f_i : X \to F_i$ there exists an unique morphism $f : X \to \varprojlim F$ for which $p_i \circ f = f_i$, that is



It is said that $\mathscr{C}$ is a ***complete category*** if for each small category $\mathscr{L}$ and each functor $F : \mathscr{L} \to \mathscr{C}$ there exists the inverse limit.

Definition A.3.11 ([Lez], Definition 3.3.2) Let $F : \mathscr{L} \to \mathscr{C}$ be a functor, with $\mathscr{L}$ a small category and $\mathscr{C}$ any category. For $X \in \mathrm{Ob}(\mathscr{C})$ and each $i \in I := \mathrm{Ob}(\mathscr{L})$, let $F_i := F(i)$ and $f_i : X \to F_i$ be a morphism.

A ***colimit*** (or ***direct limit***) of F is an object $\varinjlim F$ of $\mathscr{C}$ along with a compatible family of morphisms $q_i : F_i \to \varinjlim F$, such that for any other compatible family $f_i : F_i \to X$ there exists an unique morphism $f : \varinjlim F \to X$ for which $f \circ q_i = f_i$, that is



It is said that $\mathscr{C}$ is a ***cocomplete category*** if $\varinjlim F$ exists for each small category $\mathscr{L}$ and for each functor $F : \mathscr{L} \to \mathscr{C}$.

APPENDIX B

Some important algebras

In this section we consider several algebras that are important throughout this work. We have to define certain types of algebras and certain properties.

Throughout this work, examples of skew PBW extensions are mentioned. It is important to take into account their origins and the motivation that was had at the time of defining them. Ore in [Ore33] defined the ***Ore extensions***, which are important since they were generalized to define PBW extensions and then skew PBW extensions.

Definition B.0.1 ([GWJ04], p. 34) Let R be a ring, σ a ring endomorphism of R, and δ and σ-derivation on R, that is

(i) δ is a endomorphism of R as an additive group, and

(ii) $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in R$.

We shall write $S = R[x; \sigma, \delta]$ provided

(i) S is a ring, containing R as a subring.

(ii) x is an element of S.

(iii) S is a free left R-module with basis $\{1, x, x^2, \ldots\}$.

(iv) $xr = \sigma(r)x + \delta(r)$, for all $r \in R$.

Such a ring S is called an ***Ore extension of*** R.

This definition is important since it will be used later in the propositions and definitions on PBW extensions.

A important subset of Ore extension is Ore algebras. These are defined as follows.

Definition B.0.2 ([RS16], Definition 4.2) An ***Ore algebra*** is an Ore extension in which the ring of coefficients is $A := \mathbb{K}[t_1, \ldots, t_m], m \geq 0$ and for each $1 \leq i \leq n$, σ_i, δ_i are $\mathbb{K}$-linear. Note that this is equivalent to $\sigma_i(k) = k$, $\delta_i(k) = 0$, for each $k \in \mathbb{K}$; moreover, A is indeed a $\mathbb{K}$-algebra. Thus, an Ore algebra is an extension with the form

$$\mathbb{K}[t_1, \ldots, t_m][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n], m \geq 0.$$

Remark B.0.3 In the case where $\sigma_i = i_R$, it is said that Ore extension is derivation type.

Example B.0.4 ([Lez], Example 1.4.2) Let R be a ring and A an Ore extension of R, $A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$. We say that A is an Ore extension of derivation type if $\sigma_i = i_R$, for each $1 \leq i \leq n$. Note that these are skew PBW extensions. Indeed, in this case $c_{i,r} = r$, $c_{i,j} = 1$, $x_i r - r x_i = \delta_i(r)$, $x_i x_j - x_j x_i = 0$.

In particular, any Ore algebra of derivation type is a skew PBW extension. With this, it is enough to see that some algebra are Ore algebras of derivation type to say that they are skew PBW extensions.

Now, we see definition of algebras mentioned in this work and their properties.

Example B.0.5 ([GL10], Example 1.(a)) (***Classical polynomial ring***) Let $A = R[t_1, \ldots, t_n]$ be the classical polynomial ring, so $t_i r - r t_i = 0$ and $t_i t_j - t_j t_i = 0$, for any $1 \leq i, j \leq n$. The R-free basis is $\mathrm{Mon}(A)$.

Example B.0.6 ([GL10], Example 1.(b)) (***Skew polynomial ring of derivation type***) Any skew polynomial ring $A = R[x; \sigma, \delta]$ of derivation type is a skew PBW extension. In this case $xr - rx = \delta(r)$ and $xx - xx = 0$; the R-free basis is $\{x^l \mid l \geq 0\}$.

Example B.0.7 ([GL10], Example 1.(c)) (***Ore algebra of derivation type***) Let $R = \mathbb{K}[t_1, \ldots, t_m]$. If $A = R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is an Ore algebra of derivation type, then A is a skew PBW extension of R. In this case $x_i r - r x_i = \delta(r)$, $x_i x_j - x_j x_i = 0$ and $\delta_i(r) \in R$, for any $1 \leq i, j \leq n$, $r \in R$.

Example B.0.8 ([GL10], Example 1.(d)) (***Weyl algebra***) $A_n(\mathbb{K}) = \mathbb{K}[t_1, \ldots, t_n][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is another important example of skew PBW extension. In fact, the Weyl algebra $A_n(\mathbb{K})$ is an Ore algebra with $\sigma_i = i_{\mathbb{K}[t_1, \ldots, t_n]}$ and $\delta_i = \partial/\partial t_i$, for $1 \leq i \leq n$. In this situation, $x_i p - p x_i = \partial p/\partial t_i$, $x_i x_j - x_j x_i = 0$, for any $p \in \mathbb{K}[t_1, \ldots, t_n]$ and $1 \leq i, j \leq n$.

Example B.0.9 ([GL10], Example 1.(e)) (***Enveloping algebras of finite dimensional Lie algebras***) Let $\mathbf{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$ with basis $\{x_1, \ldots, x_n\}$; the universal enveloping algebra of $\mathbf{g}$, $\mathcal{U}(\mathbf{g})$, is a skew PBW extension of $\mathbb{K}$.

In this case, $x_ik - kx_i = 0$ and $x_ix_j - x_jx_i = [x_i, x_j] \in \mathbf{g} = \mathbb{K} + \mathbb{K}x_1 + \cdots + \mathbb{K}x_n$, for any $k \in \mathbb{K}$ and $1 \leq i, j \leq n$.

Now, we see an important example to show that certain algebras are skew PBW extensions.

Example B.0.10 ([GL10], Example 5.(iii)) Any iterated skew polynomial ring $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is a skew PBW extension, if it satisfies the following conditions:

- For $1 \leq i \leq n$, σ_i is injective.
- For every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$.
- For $i < j$, $\sigma_j(x_i) = cx_i + d$, with $c, d \in R$, and c has a left inverse.
- For $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_i$.

Under these conditions we have

$$R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] = \sigma(R)\langle x_1, \ldots, x_n\rangle.$$

In particular, any Ore algebra $\mathbb{K}[t_1, \ldots, t_m][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is a skew PBW extension if it satisfies the following condition:

- For $1 \leq i \leq n$, σ_i is injective.

In fact, in Ore algebra for every $r \in \mathbb{K}[t_1, \ldots, t_m]$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in \mathbb{K}[t_1, \ldots, t_m]$, for $i < j$, $\sigma_i(x_i) = x_i$ and $\delta_i(x_i) = 0$. Under these conditions we get

$$\mathbb{K}[t_1, \ldots, t_m][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] = \sigma(\mathbb{K}[t_1, \ldots, t_m])\langle x_1, \ldots, x_n\rangle.$$

Example B.0.11 ([GL10], Example 5.(iv)) (***Additive analogue of the Weyl algebra***) The $\mathbb{K}$-algebra $A_n(q_1, \ldots, q_n)$ is generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and subject to the relations

$$\begin{aligned} x_jx_i = x_ix_j, y_jy_i = y_iy_j, 1 \leq i, j \leq n, \\ y_ix_j = x_jy_i, i \neq j, \\ y_ix_i = q_ix_iy_i + 1, 1 \leq i \leq n, \end{aligned}$$

where $q_i \in \mathbb{K}^\times$. We observe that $A_n(q_1, \ldots, q_n)$ is isomorphic to the iterated skew polynomial ring $\mathbb{K}[x_1, \ldots, x_n][y_1; \sigma_1, \delta_1] \cdots [y_n; \sigma_n, \delta_n]$ over the commutative polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$

$$\begin{aligned} \sigma_j(y_i) := y_i, \delta_j(y_i) := 0, 1 \leq i < j \leq n, \\ \sigma_i(x_j) := x_j, \delta_i(x_j) := 0, i \neq j, \\ \sigma_i(x_i) := q_ix_i, \delta_i(x_i) := 1, 1 \leq i \leq n. \end{aligned}$$

Thus, $A_n(q_1,\dots,q_n)$ satisfies the conditions of Example B.0.10 and is bijective. We have

$$A_n(q_1,\dots,q_n)=\sigma(\mathbb{K}[x_1,\dots,x_n])\langle y_1,\dots,y_n\rangle.$$

Example B.0.12 ([GL10], Example 5.(v)) (***Multiplicative analogue of the Weyl algebra***) The $\mathbb{K}$-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by $x_1,\dots,x_n$ and subject to the relations

$$x_jx_i=\lambda_{ji}x_ix_j, 1\leq i<j\leq n,$$

where $\lambda_{ji}\in\mathbb{K}^\times$. We note that $\mathcal{O}_n(\lambda_{ji})$ is isomorphic to the iterated skew polynomial ring $\mathbb{K}[x_1][x_2:\sigma_2]\cdots[x_n;\sigma_n]$

$$\sigma_j(x_i):=\lambda_{ji}x_i, 1\leq i<j\leq n. \tag{B.0.1}$$

Thus, $\mathcal{O}_n(\lambda_{ji})$ satisfies the conditions of Example B.0.10, and hence

$$\mathcal{O}_n(\lambda_{ji})=\sigma(\mathbb{K}[x_1])\langle x_2,\dots,x_n\rangle.$$

Example B.0.13 ([GL10], Example 5.(vi)) (***q-Heisenberg algebra***) The $\mathbb{K}$-algebra $h_n(q)$ is generated by $x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n$ and subject to the relations

$$\begin{aligned}
&x_jx_i=x_ix_j, \quad z_jz_i=z_iz_j, \quad y_jy_i=y_iy_j, 1\leq i,j\leq n,\\
&z_jy_i=y_iz_j, \quad z_jx_i=x_iz_j, \quad y_ix_j=x_jy_i, i\neq j,\\
&z_iy_i=qy_iz_i, \quad z_ix_i=q^{-1}x_iz_i+y_i, \quad y_ix_i=qx_iy_i, 1\leq i\leq n,
\end{aligned}$$

with $q\in\mathbb{K}^\times$. Note that $h_n(q)$ is isomorphic to the iterated skew polynomial ring $\mathbb{K}[x_1,\dots,x_n][y_1;\sigma_1]\cdots[y_n;\sigma_n][z_1;\theta_1,\delta_1]\cdots[z_n;\theta_n,\delta_n]$ on the commutative polynomial ring $\mathbb{K}[x_1,\dots,x_n]$:

$$\begin{aligned}
\theta_j(z_i):=z_i,&\delta_j(z_i):=0,\sigma_j(y_i):=y_i, 1\leq i<j\leq n,\\
\theta_j(y_i):=y_i,&\delta_j(y_i):=0,\theta_j(x_i):=x_i,\delta_j(x_i):=0,\sigma_j(x_i):=x_i, i\neq j,\\
\theta_i(y_i):=qy_i,&\delta_i(y_i):=0,\theta_i(x_i):=q^{-1}x_i,\delta_i(x_i):=y_i,\sigma_i(x_i):=qx_i, 1\leq i\leq n.
\end{aligned}$$

Since $\delta_i(x_i)=y_i\notin\mathbb{K}[x_1,\dots,x_n]$, then $h_n(q)$ is not a skew PBW extension of $\mathbb{K}[x_1,\dots,x_n]$. However $h_n(q)$ is bijective skew PBW extension of $\mathbb{K}$:

$$h_n(q)=\sigma(\mathbb{K})\langle x_1,\dots,x_n;y_1,\dots,y_n;z_1,\dots,z_n\rangle.$$

Example B.0.14 ([SR17a], Example 2.6.3) (***Particular Sklyanin algebra***) The $\mathbb{K}$-algebra $S=\mathbb{K}\langle x,y,z\rangle/\langle ayx+bxy+cz^2,axz+bzx+cy^2,azy+byz+cx^2\rangle$, where $a,b,c\in\mathbb{K}$. If $c\neq 0$ then S is not a skew PBW extension. If $c=0$ and $a,b\neq 0$ then in S:

$$yx=-\frac{b}{a}xy, zx=-\frac{a}{b}xz, zy=-\frac{b}{a}yz$$

In this case, we have $\sigma_1 = \mathrm{id}_{\mathbb{K}}$, $\sigma_2(x) = -\frac{b}{a}x$ and $\sigma_3(x) = -\frac{a}{b}x$, $\sigma_3(y) = -\frac{b}{a}y$. σ_1 corresponds to x, σ_2 corresponds to y and σ_3 corresponds to z. Therefore, $S = \sigma(\mathbb{K})\langle x, y, z\rangle$ is a skew PBW extension of $\mathbb{K}$.

Example B.0.15 ([SR17a], Example 2.6.5) (***Homogenized enveloping algebra***) Let $\mathbf{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$ with basis $\{x_1, \ldots, x_n\}$ and $\mathcal{U}(\mathbf{g})$ its enveloping algebra. The homogenized enveloping algebra of $\mathbf{g}$ is $\mathcal{A}(\mathbf{g}) := T(\mathbf{g} \oplus \mathbb{K}z)/\langle R\rangle$, where $T(\mathbf{g} \oplus \mathbb{K}z)$ is the tensor algebra, z is a new variable and R is spanned by $\{z \otimes x - x \otimes z \mid x \in \mathbf{g}\} \cup \{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathbf{g}\}$. From Proposition 2.2.9 for $\mathbf{g} \otimes \mathbb{K}(z)$, considered as a Lie algebra over $\mathbb{K}(z)$, we get that $\mathcal{A}(\mathbf{g})$ is a skew PBW extension of $\mathbb{K}[z]$.

Example B.0.16 ([Rey13b], p. 26) (***Woronowicz algebra***) It is the algebra denoted by $\mathcal{W}_\nu(\mathfrak{sl}(2, \mathbb{K}))$, generated by x, y, z subject to the relations

$$xz - \nu^4 zx = (1 + \nu^2)x, \quad xy - \nu^2 yx = \nu z, \quad zy - \nu^4 yz = (1 + \nu^2)y,$$

where $\nu \in \mathbb{K}^\times$ is not a root of unity. If $1 + \nu^2 = \nu$, we have the isomorphism $\mathcal{W}_\nu(\mathfrak{sl}(2, \mathbb{K})) \cong \sigma(\mathbb{K})\langle x, y, z\rangle$ (for more details see [Rey13b], Example 3.2.11).

Example B.0.17 ([Rey13b], p. 28) (***The algebra of differential operators on a quantum space***) Let K be a commutative ring and let $\mathbf{q} = [q_{ij}]$ be a matrix with entries in $K^\times$ such that $q_{ii} = 1 = q_{ij}q_{ji}$, for all $1 \leq i, j \leq n$. The K-algebra $S_{\mathbf{q}}$ is generated by x_i, $1 \leq i \leq n$, subject to relations

$$x_i x_j = q_{ij} x_j x_i.$$

The algebra $S_{\mathbf{q}}$ is regarded as the ***algebra of functions on a quantum space***. The algebra $D_{\mathbf{q}}(S_{\mathbf{q}})$ of $\mathbf{q}$-differential operators on $S_{\mathbf{q}}$ is defined by

$$\partial_i x_j - q_{ij} x_j \partial_i = \delta_{ij}, \text{ for all } i, j.$$

The relations between ∂_i, are given by

$$\partial_i \partial_j = q_{ij} \partial_j \partial_i, \text{ for all } i, j.$$

Therefore, $D_{\mathbf{q}}(S_{\mathbf{q}}) \cong \sigma(\sigma(K)\langle x_1, \ldots, x_n\rangle)\langle \partial_1, \ldots, \partial_n\rangle$.

Example B.0.18 ([Rey13b], p. 30) (***Multiparameter quantized Weyl algebra***) Let $Q := [q_1, \ldots, q_n]$ be a vector in $\mathbb{K}^n$ with no zero components, and let $\Gamma = [\gamma_{ij}]$ be a multiplicatively antisymmetric $n \times n$ matrix over $\mathbb{K}$. The multiparameter Weyl algebra $A_n^{Q,\Gamma}(\mathbb{K})$ is defined to be the algebra generated by $\mathbb{K}$ and the indeterminates $y_1, \ldots, y_n, x_1, \ldots, x_n$ subject to the relations

$$\begin{aligned}
y_iy_l &= \gamma_{ij}y_jy_i, && 1 \leq i, j \leq n,\\
x_ix_j &= q_i\gamma_{ij}x_jx_i, && 1 \leq i < j \leq n,\\
x_iy_j &= \gamma_{ji}y_jx_i, && 1 \leq i < j \leq n,\\
x_iy_j &= q_j\gamma_{ji}y_jx_i, && 1 \leq j < i \leq n,\\
x_jy_j &= q_jy_jx_j + 1 + \sum_{l<j}(q_l - 1)y_lx_l, && 1 \leq j \leq n.
\end{aligned}$$

From relations above we have that $A_n^{Q,\Gamma}(\mathbb{K})$ is isomorphic to a skew PBW extension, that is

$$A_n^{Q,\Gamma}(\mathbb{K}) \cong \sigma(\sigma(\sigma(\sigma(\cdots\sigma(\sigma(\mathbb{K})\langle x_1\rangle)\langle y_1\rangle)\cdots)\langle x_{n-1}\rangle)\langle y_{n-1}\rangle)\langle x_n\rangle)\langle y_n\rangle.$$

Example B.0.19 ([Rey13b], p. 31) (***Quantum symplectic space***) For every nonzero element q in $\mathbb{K}$, one defines this quantum algebra $\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n}))$ to be the algebra generated by $\mathbb{K}$ and the variables $y_1, \ldots, y_n, x_1, \ldots, x_n$ subject to the relations

$$\begin{aligned}
y_jx_i &= q^{-1}x_iy_j, \qquad y_jy_i = qy_iy_j, && 1 \leq i < j \leq n,\\
x_jx_i &= q^{-1}x_ix_j, \qquad x_jy_i = qy_ix_j, && 1 \leq i < j \leq n,\\
x_iy_i - q^2y_ix_i &= (q^2 - 1)\sum_{l=1}^{i-1} q^{i-l}y_lx_l, && 1 \leq i \leq n.
\end{aligned}$$

From relations above we have that $\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n}))$ is isomorphic to a skew PBW extension, that is

$$\mathcal{O}_q(\mathfrak{sp}(\mathbb{K}^{2n})) \cong \sigma(\sigma(\sigma(\sigma(\cdots\sigma(\sigma(\mathbb{K})\langle x_1\rangle)\langle y_1\rangle)\cdots)\langle x_{n-1}\rangle)\langle y_{n-1}\rangle)\langle x_n\rangle)\langle y_n\rangle.$$

Example B.0.20 ([Rey13b], p. 26) (***Manin algebra***) By definition, $\mathcal{O}_q(M_2(\mathbb{K}))$, also denoted $\mathcal{O}(M_q(2))$, is the coordinate algebra of the quantum matrix space $M_2(\mathbb{K})$. It is the $\mathbb{K}$-algebra generated by the variables x, y, u, v satisfying the relations

$$ux = qxu, \quad yu = quy, \quad vu = uv,$$

and

$$vx = qxv, \quad yv = qvy, \quad yx - xy = (q^{-1} - q)uv,$$

where $q \in \mathbb{K}^\times$. Thus, $\mathcal{O}_q(M_2(\mathbb{K})) \cong \sigma(\sigma(\sigma(\sigma(\mathbb{K}[x]))\langle u\rangle)\langle v\rangle)\langle y\rangle$.

Now we see the general definition of 3-dimensional skew polynomial algebras. It should be noted that all are skew PBW extensions.

Example B.0.21 ([Ros13], C4.3) (***3-dimensional skew polynomial algebras***) A 3-dimensional skew polynomial algebra $\mathcal{A}$ is the $\mathbb{K}$-algebra generated by the variables x, y, z restricted to relations

$$yz - \alpha zy = \lambda, \quad zx - \beta xz = \mu, \quad xy - \gamma yx = \nu,$$

such that

(i) $\lambda, \mu, \nu \in \mathbb{K} + \mathbb{K}x + \mathbb{K}y + \mathbb{K}z$, and $\alpha, \beta, \gamma \in \mathbb{K}^{\times}$;

(ii) Standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\mathbb{K}$-basis of the algebra.

Proposition B.0.22 ([Ros13], Theorem C4.3.1) If $\mathcal{A}$ is a 3-dimensional skew polynomial algebra, then $\mathcal{A}$ is one of the following algebras:

(a) if $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$. In this case $\mathcal{A} \cong \mathbb{K}[x; \sigma_1][y; \sigma_2][z; \sigma_3]$ with $\sigma_1 = \mathrm{id}_{\mathbb{K}}$, $\sigma_2(x) = \alpha^{-1}x$ and $\sigma_3(x) = \beta^{-1}x$, $\sigma_3(y) = \gamma^{-1}y$.

(b) if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then $\mathcal{A}$ is one of the following algebras:

(i) $yz - zy = z, \quad zx - \beta xz = y, \quad xy - yx = x;$
(ii) $yz - zy = z, \quad zx - \beta xz = b, \quad xy - yx = x;$
(iii) $yz - zy = 0, \quad zx - \beta xz = y, \quad xy - yx = 0;$
(iv) $yz - zy = 0, \quad zx - \beta xz = b, \quad xy - yx = 0;$
(v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x;$
(vi) $yz - zy = z, \quad zx - \beta xz = 0, \quad xy - yx = 0,$

where a, b are any elements of $\mathbb{K}$. All nonzero values of b give isomorphic algebras.

(c) If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then $\mathcal{A}$ is one of the following algebras:

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$
(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

In this case, b is an arbitrary element of $\mathbb{K}$. Again, any nonzero values of b give isomorphic algebras.

(d) If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is the algebra defined by the relations $yz - \alpha zy = a_1x + b_1$, $zx - \alpha xz = a_2y + b_2$, $xy - \alpha yx = a_3z + b_3$. If $a_i = 0$ $(i = 1, 2, 3)$, then all nonzero values of b_i give isomorphic algebras. If $a_i \neq 0$ $(i = 1, 2, 3)$, then $\mathcal{A}$ is not an Ore extension.

(e) If $\alpha = \beta = \gamma = 1$, then $\mathcal{A}$ is isomorphic to one of the following algebras:

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$

(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$

(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$

(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

Parameters $a, b \in \mathbb{K}$ are arbitrary, and all nonzero values of b generate isomorphic algebras.

Example B.0.23 ([Rey13b], p. 21) (***Algebra of linear partial shift operators***) The $\mathbb{K}$-algebra of linear partial shift operators with polynomial coefficients is the algebra $\mathbb{K}[t_1, \ldots, t_n][E_1, \ldots, E_m]$ subject to the relations

$$\begin{aligned} t_j t_i &= t_i t_j, && 1 \leq i < j \leq n, \\ E_i t_i &= (t_i + 1)E_i = t_i E_i + E_i, && 1 \leq i \leq m, \\ E_j t_i &= t_i E_j, && i \neq j, \\ E_j E_i &= E_i E_j, && 1 \leq i < j \leq n. \end{aligned}$$

These relations imply that this algebra can be expressed as the iterated skew PBW extension $\sigma(\sigma(\cdots\sigma(\sigma(\mathbb{K}[E_1, \ldots, E_m])\langle t_1\rangle)\langle t_2\rangle)\cdots)\langle t_{n-1}\rangle)\langle t_n\rangle$.

Example B.0.24 ([LFG+], Example 18.2.1) (***Sridharan enveloping algebra of 3-dimensional Lie algebra*** $\mathcal{G}$) Let $\mathcal{G}$ be a finite dimensional Lie algebra, and let $f \in Z^2(\mathcal{G}, \mathbb{K})$ be an arbitrary 2-***cocycle***, that is, $f : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$ such that $f(x, x) = 0$ and

$$f(x, [y, z]) + f(z, [x, y]) + f(y, [z, x]) = 0$$

for all $x, y, z \in \mathcal{G}$. The Sridharan enveloping algebra of $\mathcal{G}$ is defined to be the associative algebra $\mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I$, where $T(\mathcal{G})$ is the tensor algebra of $\mathcal{G}$ and I is the two-sided ideal of $T(\mathcal{G})$ generated by the elements

$$(x \otimes y) - (y \otimes x) - [x, y] - f(x, y), \text{ for all } x, y \in \mathcal{G}.$$

If $\mathcal{G}$ is a Lie $\mathbb{K}$-algebra of dimension three, then the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ for $f \in Z^2(\mathcal{G}, \mathbb{K})$ is isomorphic to one of ten following associative $\mathbb{K}$-algebras, defined by three generators x, y, z and the following commutation relations (see [Nus91], Theorem 1.3)

Type	$[x,y]$	$[y,z]$	$[z,x]$
1	0	0	0
2	0	x	0
3	x	0	0
4	0	αy	$-x$
5	0	y	$-(x+y)$
6	z	$-2y$	$-2x$
7	1	0	0
8	1	x	0
9	x	1	0
10	1	y	x

where $\alpha \in \mathbb{K}^{\times}$. Therefore the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is a skew PBW extension of $\mathbb{K}$, $\mathcal{U}_f(\mathcal{G}) \cong \sigma(\mathbb{K})\langle x,y,z\rangle$.

The examples shown in this appendix are skew PBW extensions. This means that the algebras shown have PBW basis. An algorithm to determine a basis of skew PBW extensions can be found in [RS17a].

APPENDIX C

Algebraic geometry

In this appendix, we see the most important notions of algebraic geometry that it were used throughout this work.

First, we define the notion of affine space, which is the motivation to then arrive at the projective plane.

Definition C.0.1 ([Har13], p. 1) Let $\mathbb{K}$ be a fixed algebraically closed field. We defined ***affine n-space*** over $\mathbb{K}$, denoted by $\mathbb{A}^n_{\mathbb{K}}$, or simply $\mathbb{A}^n$, to be the set of all n-tuples of elements of $\mathbb{K}$. An element $P \in \mathbb{A}^n$ will be called a ***point***, and if $P = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{K}$, then the a_i will be called the ***coordinates*** of P.

Let $A = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. We will interpret the elements of A as functions from the affine space to $\mathbb{K}$, by defining $f(P) = f(a_1, \ldots, a_n)$, where $f \in A$ and $P \in \mathbb{A}^n$. Thus if $f \in A$ is a polynomial, we can talk about the set of ***zeros*** of f, namely $Z(f) = \{P \in \mathbb{A}^n \mid f(P) = 0\}$. More generally, if T is any subset of A, we define the ***zero set*** of T to be the common zeros of all elements of T, namely

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0, \text{ for all } f \in T\}.$$

Definition C.0.2 ([Har13], p. 2) A subset Y of $\mathbb{A}^n$ is an ***algebraic set***, if there exists a set $T \subset A$ such that $Y = Z(T)$.

Definition C.0.3 ([Har13], p. 2) We define the ***Zariski topology*** on $\mathbb{A}^n$ by taking the open subsets to be the complements of the algebraic sets.

Now, we see the analogy that we have with the previous concepts when defining the projective space.

Definition C.0.4 ([Har13], p. 8) Let $\mathbb{K}$ be an algebraically closed field. We defined ***projective n-space*** over $\mathbb{K}$, denoted by $\mathbb{P}^n_{\mathbb{K}}$, or simply $\mathbb{P}^n$, to be the set of equivalence classes of $(n+1)$-tuples $(a_0, \ldots, a_n)$ of elements of $\mathbb{K}$, not all zero, under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in \mathbb{K}$, $\lambda \neq 0$.

An element of $\mathbb{P}^n$ is called a ***point***. If P is a point, then any $(n+1)$-tuple $(a_0, \ldots, a_n)$ in the equivalence class P is called a ***set of homogeneous coordinate for*** P.

Let S be the polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$. We can talk about the ***zeros*** of a homogeneous polynomial, namely $Z(f) = \{P \in \mathbb{P}^n \mid f(P) = 0\}$. If T is any set of homogeneous elements of S, we define the ***zero set*** of T to be

$$Z(T) = \{P \in \mathbb{P}^n \mid f(P) = 0, \text{ for all } f \in T\}.$$

Definition C.0.5 ([Har13], p. 9) A subset Y of $\mathbb{P}^n$ is an ***algebraic set***, if there exists a set T of homogeneous elements of S such that $Y = Z(T)$.

Definition C.0.6 ([Har13], p. 10) We define the ***Zariski topology*** on $\mathbb{P}^n$ by taking the open sets to be the complements of algebraic sets.

Definition C.0.7 ([Har13], p. 10) A ***projective algebraic variety*** (or simply ***projective variety***) is an irreducible algebraic set in $\mathbb{P}^n$, with the induced topology. An open subset of a projective variety is a ***quasi-projective variety***.

If Y is any subset of $\mathbb{P}^n$, we define the ***homogeneous ideal*** of Y in S, denoted $I(Y)$, to be the ideal generated by $\{f \in S \mid f \text{ is homogeneous and } f(P) = 0, \text{ for all } P \in Y\}$. If Y is an algebraic set, we define the ***homogeneous coordinate ring*** of Y to be $S(Y) = S/I(Y)$.

Definition C.0.8 ([Har13], p. 15) Let us consider a quasi-projective variety $Y \subseteq \mathbb{P}^n$. A function $f : Y \to \mathbb{K}$ is ***regular at a point*** $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and homogeneous polynomials $g, h \in S = \mathbb{K}[x_0, \ldots, x_n]$, of the same degree, such that h is nowhere zero on U, and $f = g/h$ on U. We say that f is ***regular on*** Y, if it is regular at every point.

Now, we define an essential concept that was used throughout this work, which is associated with some given variety.

Definition C.0.9 ([Har13], p. 16) Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the ***local ring of*** P ***on*** Y, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the ring of germs of regular functions on Y near P. In other words, an element of $\mathcal{O}_P$ is a pair $\langle U, f\rangle$ where U is an open subset of Y containing P, and f is

a regular function on U, and where we identify two such pairs $\langle U, f\rangle$ and $\langle V, g\rangle$, if $f = g$ on $U \cap V$.

Notation C.0.10 If S is a graded ring, and $\mathfrak{p}$ a homogeneous prime ideal in S, then we denote by $S_{(\mathfrak{p})}$ the subring of elements of degree 0 in the localization of S with respect to the multiplicative subset T consisting of the homogeneous elements of S not in $\mathfrak{p}$. In particular, if S is a domain, then for $\mathfrak{p} = (0)$ we obtain a field $S_{((0))}$.

Now, we define another topology as follows: Let A be a local ring with maximal ideal $\mathfrak{m}$. The powers of $\mathfrak{m}$ define a topology on A, called the ***$\mathfrak{m}$-adic topology***. By completing with respect to this topology, one defines the ***completion*** of A, denoted $\hat{A}$. Alternatively, one can define $\hat{A}$ as the inverse limit $\varprojlim A\mathfrak{m}^n$ (Definition A.3.10).

Now we define an important concept about module theory. Let S be a graded ring. A ***graded S-modules*** is an S-module M, together with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$, such that $S_d \cdot M_e \subseteq M_{d+e}$. For any graded S-module M, and for any $l \in \mathbb{Z}$, we define the ***twisted*** module $M(l)$ by shifting l places to the left, i.e., $M(l)_d = M_{d+l}$.

With the definitions mentioned above, we come to the notion of ***presheaf***. Although there is a general notion of presheaf, we will define it over a topological space.

Definition C.0.11 ([LFG+], Definition A.4.0.1)Let X be a topological space. A ***presheaf*** $\mathcal{F}$ over X is defined with the following conditions:

(i) To every open set $U \subseteq X$ is associated a set $\mathcal{F}(U)$. $\mathcal{F}(\emptyset)$ consists only of one element.

(ii) To any open sets $U \subseteq V$ of X is associated a function $\mathcal{F}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ such that the following conditions hold:

a) $\mathcal{F}_U^U = i_{\mathcal{F}(U)}$, for any open set U of X.

b) for any open sets $U \subseteq V \subseteq W$ of X we have $\mathcal{F}_U^W = \mathcal{F}_U^V \circ \mathcal{F}_V^W$.

The sets $\mathcal{F}(U)$ are called the ***sections*** of $\mathcal{F}$.

Example C.0.12 Let R be an integral domain and let $X = \mathrm{Spec}(R)$ be the set of its primes ideals with the Zariski topology. We consider $K = Q(R)$ and $U \subseteq X$ open set. Then $\mathcal{R}$ is a presheaf defined by:

- $\mathcal{R}(U) = \{0\}$, if $U = \emptyset$,
- $\mathcal{R}(U) = \{z \in K \mid z$ has at least one representation of the form $z = \frac{q}{r}, r \notin P,$ for all $P \in U\}$, if $U \neq \emptyset$
- $U \subseteq V \subseteq X$ open sets, then

$$\mathcal{R}_U^V = \begin{cases} \mathcal{R}(U) \to \mathcal{R}(V), & z \mapsto 0, \quad \text{if } U = \emptyset \\ \\ \mathcal{R}(U) \to \mathcal{R}(V), & z \mapsto z, \quad \text{if } U \neq \emptyset \end{cases}$$

Definition C.0.13 ([LFG+], Definition A.4.0.2) A presheaf $\mathcal{F}$ on a topological space X is a ***sheaf***, if for any open set $U \subseteq X$ and any open cover $U = \bigcup_{i \in \mathcal{C}} U_i$ of U the following two conditions hold:

(i) If $z, z' \in \mathcal{F}(U)$ and $\mathcal{F}_{U_i}^U(z) = \mathcal{F}_{U_i}^U(z')$, for all $i \in \mathcal{C}$ then $z = z'$.

(ii) Let $\{z_i \mid z_i \in \mathcal{F}(U_i)\}_{i \in \mathcal{C}}$ be a collection of elements such that for any $i, j \in \mathcal{C}$ we have $\mathcal{F}_{U_i \cap U_j}^{U_i}(z_i) = \mathcal{F}_{U_i \cap U_j}^{U_j}(z_j)$. Then there exists $z \in \mathcal{F}(U)$ such that $\mathcal{F}_{U_i}^U(z) = z_i$, for any $i \in \mathcal{C}$.

Example C.0.14 If R is a commutative ring, then $\mathcal{R}$ from Examples C.0.12 is a sheaf for $X = \mathrm{Spec}(R)$.

Definition C.0.15 ([LFG+], Definition A.4.0.3) A ***ringed space*** is a pair $(X, \mathcal{F})$ consisting of a topological space X and a sheaf of rings $\mathcal{F}$ over X.

It is clear that $(X, \mathcal{R})$ from Example C.0.14 is a ringed space.

Now we will see how two ringed spaces can be related.

Definition C.0.16 ([LFG+], Definition A.4.0.4) A ***morphism of ringed spaces*** $\varphi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is defined by:

(i) A continous function $\varphi : X \to Y$.

(ii) Given an open set U of Y, we have a ring homomorphism $\varphi_U : \mathcal{G}(U) \to \mathcal{F}(\varphi^{-1}(U))$.

(iii) For any open subsets $U \subseteq V$ of Y the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(\varphi^{-1}(V)) & \xrightarrow{\mathcal{F}_{\varphi^{-1}(U)}^{\varphi^{-1}(V)}} & \mathcal{F}(\varphi^{-1}(U)) \\ \uparrow{\scriptstyle \psi_V} & & \uparrow{\scriptstyle \psi_U} \\ \mathcal{G}(V) & \xrightarrow{\mathcal{G}_U^V} & \mathcal{G}(U) \end{array}$$

A morphism of ringed spaces $\varphi : (X, \mathcal{F}) \to (Y_{\mathcal{G}})$ is an ***isomorphism***, if there exists a morphism of ringed spaces $\phi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ such that $\phi\varphi = i_{(x,\mathcal{F})}$ and $\varphi\phi = i_{(Y,\mathcal{G})}$.

With all the above definitions, we have what is necessary to define the central element of this chapter.

Definition C.0.17 ([LFG+], Definition A.4.0.6) A ***scheme*** is a ringed space $(X, \mathcal{F})$ for which every point $x \in X$ has a neighborhood U_x such that the induced ringed space $(U_x, \mathcal{F}|_{U_x})$ is isomorphic to $(\mathrm{Spec}(R), \mathcal{R})$, where R is some commutative ring depending of x.

Example C.0.18 ([LFG+], Example A.4.2) Let R be a commutative ring and let $\mathrm{Spec}(R)$ be the set of its primes ideals with the Zariski topology. Given U an open set of $\mathrm{Spec}(R)$ we define

$$\mathcal{R}(U) := \varprojlim_{U_f \subseteq U} R_f,$$

where U_f ranges over all basic open sets contained in U and R_f is the localization of R with respect to $f \in R$ (if f is nilpotent, we take $U_f := \emptyset$ and $R_f := \{0\}$). We have that

$$\mathcal{R}(U) = \left\{ (z_f) \in \prod_{U_f \subseteq U} R_f \mid z_h = \mathcal{R}_{U_h}^{U_g}(z_g), \text{ for } U_h \subseteq U_g \subseteq U \right\},$$

where

$$\begin{aligned} \mathcal{R}_{U_h}^{U_g} : & R_g \to R_h \\ & \frac{a}{g^l} \mapsto \frac{au^l}{h^{kl}}, \end{aligned}$$

with $h^k = gu$, for some $k \geq 1$ and $u \in R$. $\mathcal{R}_{X_f}^{X_g}$ is a rign homomorphism, and hence, $\mathcal{R}(U)$ is a subring of $\prod_{X_f \in U} R_f$. Now, let $U \subseteq V$ be open sets of $\mathrm{Spec}(R)$, we define

$$\begin{aligned} \mathcal{R}_U^V : \mathcal{R}(V) &\to \mathcal{R}(U) \\ (z_f) &\mapsto (z_h), \end{aligned}$$

where (z_h) is the subsequence of (z_f) such that $U_h \subseteq U$.

With this construction, we have $(\mathrm{Spec}(R), \mathcal{R})$ is a scheme, named the ***affine scheme defined by*** R.

Definition C.0.19 ([Har13], p. 62) If $\mathcal{F}$ is a presheaf on X, and if P is a point of X, we define the ***stalk*** $\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P, via the restriction maps $\mathcal{F}_U$.

Definition C.0.20 ([Har13], p. 65) Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the ***direct image*** sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subseteq Y$. For any sheaf $\mathcal{G}$ on Y, we define the ***inverse image*** sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \to \lim_{U \supseteq f(U)} \mathcal{G}(U)$, where U is any open set in X, and the limit is taken over all open sets V of Y containing $f(U)$.

An important example of sheaf is the sheaf of regular functions; it is defined as follows

Example C.0.21 ([Har13], Example 1.0.1) Let X be a variety over the field $\mathbb{K}$. For each open set $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions from U to $\mathbb{K}$, and for each $V \subseteq U$, let $\mathcal{F}_V^U : \mathcal{O}(U) \to \mathcal{O}(V)$ be the restriction map. Then $\mathcal{O}$ is a sheaf of rings on X. We call $\mathcal{O}$ the ***sheaf of regular functions*** on X.

Now, we construct the space Spec$A$ associated to a ring A. As a set, we define Spec$A$ to be the set of all prime ideals of A. If $\mathfrak{a}$ is any ideal of A, we define the subset $V(\mathfrak{a}) \subseteq$ Spec$A$ to be the set of all prime ideals which contain $\mathfrak{a}$.

We can define a topology on Spec$A$ by taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets.

As in Example C.0.21, we define a sheaf of ring $\mathcal{O}$ on Spec(A). For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localization of A at $\mathfrak{p}$. For an open set $U \subseteq$ Spec(A), we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} A_\mathfrak{p}$, such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p}$, and such that s is locally a quotient of elements of A.

Definition C.0.22 ([Har13], p. 70) Let A be a ring. The ***spectrum*** of A is the pair consisting of the topological space Spec$A$ together with the sheaf of ring $\mathcal{O}$ defined above.

Above definition is an important example of ringed space (Definition C.0.15).

Definition C.0.23 ([Har13], p. 72) A ***morphism*** of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on Y.

Now, we return to definition of scheme (Definition C.0.17). We define an important class of schemes constructed using graded rings.

Let S be a graded ring. We denoted by S_+ the ideal $\bigoplus_{d>0} S_d$. We define the ***set*** Proj$S$ to be the set of all homogeneous prime ideals $\mathfrak{p}$, which do not contain all of S_+. If $\mathfrak{a}$ is a homogeneous ideal of S, we define the subset $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj}S \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

Example C.0.24 ([Har13], Example 2.5.1) If A is a ring, we define ***projective n-space*** over A to be the scheme $\mathbb{P}^n_A = \text{Proj}A[x_0, \ldots, x_n]$.

Definition C.0.25 ([Har13], p. 78) Let S be a fixed scheme. A ***scheme over*** A is a scheme X, together with a morphism $X \to S$. If X and Y are schemes over S, a morphism of X to Y as schemes over S, is a morphism $f : X \to Y$ which is compatible with the given morphism to S. We denote by $\mathfrak{Sch}(S)$ the category of schemes over S.

Definition C.0.26 ([Har13], p. 86) The ***dimension*** of a scheme X, denoted dim$X$, is

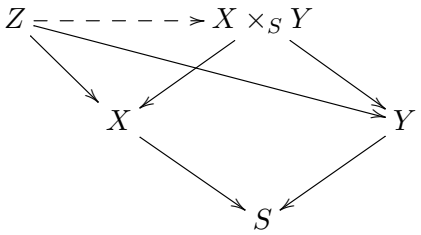
its dimension as a topological space. If Z is an irreducible closed subset of X, then the ***codimension*** of Z in X, denoted $\operatorname{codim}(Z, X)$ is the supremum of integers n such that there exists a chain

$$Z = Z_0 < Z_1 < \cdots < Z_n$$

of distinct closed irreducible subsets of X, beginning with Z.

Now, we define an important product between schemes.

Definition C.0.27 ([Har13], p. 87) Let S be a scheme, and let X, Y be schemes over S. We define the ***fibred product*** of X and Y over S, denoted $X \times_S Y$, to be a scheme, together with morphism $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, such that given any scheme Z over S, and given morphism $f : Z \to X$ and $g : Z \to Y$ which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$, and $g = p_2 \circ \theta$. The morphisms p_1 and p_2 are called the ***projection morphisms*** of the fibred product onto its factors



If X and Y are schemes given without reference to any base scheme S, we take $S = \operatorname{Spec}\mathbb{Z}$ and define the ***product*** of X and Y, denoted $X \times Y$, to be $X \times_{\operatorname{Spec}\mathbb{Z}} Y$.

With Definition C.0.27, we can extend concept of projective n-space.

Definition C.0.28 ([Har13], p. 103) If Y is any scheme, we define ***projective n-space*** over Y, denoted $\mathbb{P}^n_Y$, to be $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$. A morphism $f : X \to Y$ of schemes is ***projective*** if it factors into closed immersion $i : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$.

We are interesting in sheaves of modules. So, we begin by defining sheaves of modules on a ringed space.

Definition C.0.29 ([Har13], p. 109) Consider $(X, \mathcal{O}_X)$ be a ringed space. A ***sheaf of $\mathcal{O}_X$-modules*** (or simply an ***$\mathcal{O}_X$-module***) is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$,

the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A ***morphism*** $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

We define the ***tensor product*** $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$-modules to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Since we have the morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings, this gives $f_*\mathcal{F}$ a natural structure of $\mathcal{O}_Y$-module. We call it the ***direct image*** of $\mathcal{F}$ by the morphism f.

Now let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1}, we have a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X. We define $f^*\mathcal{G}$ to be the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module. We call it the ***inverse image*** of $\mathcal{G}$ by the morphism f.

Now we see the definition of associated sheaf.

Definition C.0.30 ([Har13], p. 110) Let A be a ring and let M be an A-module. We define the ***sheaf associated*** to M on Spec$A$, denoted by $\widetilde{M}$, as follows. For each prime ideal $\mathfrak{p} \subseteq A$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq \text{Spec}A$, we define the group $\widetilde{M}(U)$ to be the set of functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$.

Taking into account Definition C.0.30, we will now see what happens with associated sheaf when we consider a graded ring and a graded module.

Definition C.0.31 ([Har13], p. 116) Let S be a graded ring and let M be a graded S-module. We define the ***sheaf associated*** to M on Proj$S$, denoted by $\widetilde{M}$, as follows. For each $\mathfrak{p} \in \text{Proj}S$, let $M_{(\mathfrak{p})}$ be the group of elements of degree 0 in the localization $T^{-1}M$, where T is the multiplicative system of homogeneous elements of S not in $\mathfrak{p}$. For any open subset $U \subseteq \text{Proj}S$ we define $\widetilde{M}(U)$ to be the set of functions s from U to $\bigsqcup_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$ which are locally fractions. This means that for every $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements $m \in M$ and $f \in S$ of the same degree, such that for every $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_{(\mathfrak{q})}$.

The following definition is quite important, because Artin uses it in [ATVdB07].

Definition C.0.32 ([Har13], p. 117) Let S be a graded ring, and let $X = \text{Proj}S$. For any $n \in \mathbb{Z}$, we define the sheaf $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$. We call $\mathcal{O}_X(1)$ the ***twisting sheaf*** of Serre. For any sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$, we denote by $\mathcal{F}(n)$ the ***twisted sheaf*** $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Definition C.0.33 ([Har13], p. 120) For any scheme Y, we define the ***twisting sheaf*** $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ to be $g^*(\mathcal{O}(1))$, where $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ is the natural map.

Definition C.0.34 ([Har13], p. 143) For any ringed space X, we define the ***Picard group*** of X, Pic$X$, to be the group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$.

Conclusions

Throughout this work, we saw the geometry of two very important sequences: normalizing sequence and regular sequence. We also saw that these sequences are not far from each other, since both concepts appear precisely in Theorem 2.3.33, our main result. Also, in both chapters the details of propositions, examples and remarks that are not in the original literatures were completed.

Mainly, we sought to find the geometry of the normalizing sequences in finitely semi-graded algebras. To do this, we wanted to try a generalization of Corollary 1.3.16.

Therefore, we studied graded skew PBW extensions looking for a way to generalize to skew PBW extensions.

Citing what was said in Remark 2.4.6, if we want to generalize Theorem 2.3.33 to the case of FSG R-algebras, where R is not necessarily a field, we have to study the properties of Gelfand-Kirillov dimension and Hilbert series in the right base-point modules of the graded context and try to show analogous properties for generalized Gelfand-Kirillov dimension and generalized Hilbert series.

Future work

Since the concept of semi-graded algebra is relatively new, there are questions about whether certain properties that graded algebras meet also apply to semi-graded algebra. Some of these questions are asked in [Lez19]. However, we want to leave some research options related to algebraic geometry, as it was done in this work.

Although we work with normalizing sequences and arrive at their geometry in the semi-graded context, we still do not have a characterization of normalizing sequences. It would be interesting to study about a reciprocal of Corollary 1.3.16.

In this work, the concept of right base-point module for semi-graded algebra was defined, based on the generalized Gelfand-Kirillov dimension and the generalized Hilbert series. However, since we worked on FSG $\mathbb{K}$-algebras, these notions agreed with the classical ones. An interesting job would be to see what happens when we do not work on $\mathbb{K}$, but on a Noetherian domain R.

Finally, an important and interesting work that could be carried out would be to see the relation between filtration and semi-graduation of a SG algebra. It is well known that some properties of the associated graded algebra of an algebra are inherited from the algebra itself. It would be interesting to see what that is like in the semi-graded context.

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