UNIVERSIDAD NACIONAL DE COLOMBIA

On-shell methods in QCD and for WBF Higgs

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Abstract

We study recursive techniques for efficient computation of perturbative scattering amplitudes in gauge theory, in particular tree and one-loop processes in QCD theory.
By using the spinor-helicity formalism, we discuss BCFW recursion to get amplitudes at tree-level and the unitarity of the S-matrix to get the cut-constructible and rational parts at one-loop.
We propose a new formalism to compute one loop scattering amplitudes in dimensional regularized quantum chromodynamics (QCD). Our proposal combines the generalized D-dimensional unitarity together with an extension of the Dirac equation with mass $m + i\mu\gamma_5$. We prove that, by this procedure, is possible to reconstruct the full scattering amplitudes by performing only four dimensional unitarity cuts in a formalism based on helicity spinors. The calculation of tree level scattering amplitude, in this framework, allows for an automatized computation of cut-constructible and rational parts of one loop scattering amplitudes. The method is checked by computing the QCD one loop correction to the scattering amplitude of two gluons production by quark anti-quark annihilation and the processes at two-loop of three gluons fusion in a Higgs.

Keywords: QCD, Color decomposition, Spinor-Helicity formalism, On Shell, BCFW, Rational contribution, Cut-constuctible amplitude, Unitarity, Generalized Unitarity.

Resumen

Se estudian técnicas recursivas para calcular de forma eficiente amplitudes de scattering a nivel perturbativo en teorías gauge, en particular procesos a tree-level y one-loop en la teoría QCD.
Usando el formalismo de los espinores de helicidad, se discute la fórmula de recursión BCFW para obtener amplitudes de tree-level y la unitariedad de la matriz-S para obtener las partes cut-constructible y racional a one-loop.
Se propone un nuevo formalismo para calcular amplitudes a one-loop en regularización dimensional en cromodinamica cuántica (QCD). Nuestro propósito combina la unitariedad generalizada junto con una extensión de la ecuación de Dirac con masa $m + i\mu\gamma_5$. Se prueba que por este método, es posible reconstruir la amplitud de scattering con el uso de cortes unitarios en un formalismo basado en los espinores de helicidad. El cálculo de las amplitudes a tree-level en este escenario permite la automatización del cálculo de la partes cut-constructible y racional de una amplitud a one-loop. Este formalismo es verificado al calcular la corrección a one-loop en QCD a la amplitud de scattering de la producción de dos gluones por la aniquilación de quark anti-quark y el proceso a two-loop de la fusión de tres gluones en un Higgs.

Palabras Claves: QCD, Descomposión del color, formalismo de los espinores de helicidad, On Shell, BCFW, contribución Rational, Amplitud Cut-constuctible, Unitariedad, Unitariedad Generalizada.

Contents

1. Introduction

The experimental program at CERN Large Hadron Collider demands that we refine our understanding of events originating in known physics. High precision predictions in such background processes are necessary in order to find and understand new physics at the TeV scale. An important class of such computations is the ones in Quantum Chromodynamics (QCD), the quantum theory that describes the strong interactions. QCD is asymptotically free, so the strong coupling constant g becomes weak at large momentum transfers[1, 2], justifying a perturbative expansion. However, perturbative QCD amplitudes are notoriously difficult to calculate even at tree level, because of the proliferation of Feynman diagrams as the number of external legs or the order of perturbation grow.

In the present thesis, we study recursive techniques for efficient computation of perturbative scattering amplitudes in Yang-Mills theory, the general non-Abelian gauge theory that includes QCD. In particular, we study tree and one-loop scattering amplitudes involving gauge bosons (gluons) but also matter states (quarks). The main goal of the thesis is to discuss the modern methods for computation of multi-partons scattering amplitudes in QCD, the so called on-shell methods. The well known analytic properties of the one-loop amplitudes [3] lie at the heart of these techniques. Scattering amplitudes, in fact, can be constructed in terms of their singularities. For tree amplitudes, these are complex poles. In loop amplitudes, there are branch cuts, as well as other singularities associated with generalized cuts. All of these singularities probe factorization limits of the amplitude: they select kinematics where some propagators are put on shell. Thus, the calculation can be packaged in terms of lower-order amplitudes instead of the complete sum of Feynman diagrams[4, 5, 6].

This framework define the so called unitarity methods, nowadays developed in a consistent way just to perform one-loop calculations. Instead of the explicit set of loop Feynman diagrams, the basic reference point is the linear expansion of the amplitude function in a basis of master integrals, multiplied by coefficients that are rational functions of the kinematic variables, already known as Passarino Veltman reduction theorem[7]. The point is that the most difficult part of the calculation, namely integration over the loop momentum, can be done once and for all, with explicit evaluations of the master integrals. The master integrals contain all the logarithmic functions. It then remains to find their coefficients[8].

If an amplitude is uniquely determined by its branch cuts, it is said to be cut-constructible. All one-loop amplitudes are cut-constructible in dimensional regularization, provided that the full dimensional dependence is kept in evaluating the branch cut. Each master integral has a distinct branch cut, uniquely identified by its logarithmic and di-logarithm's arguments. Therefore, the decomposition in master integrals can be used to solve for their coefficients separately using analytic properties[6]. Also, D- dimensional unitarity cuts of higher-loop amplitudes involve lower-order amplitudes which still contain loops and yet have D-dimensional momenta on some external legs (the cut lines). Analytic calculations are simplest in massless theories, where formulas can be written compactly in the spinor-helicity formalism[9, 10]. Spinor variables are helpful inside the loop as well, when propagators associated to massless field are placed on shell in an unitarity cut. For this reason as well, we work mostly in four-dimensional Minkowski space and its D-dimensional analytic continuations.

The purpose of this thesis is to reach two main goals, the first consisting in a full review of the framework developed over the last two decades for studying perturbative amplitudes efficiently, the second is to perform an orginal calculation where such methods are applied to the physics of the gluon fusion Higgs production. In doing so in this master thesis will be reached an unified four-dimensional formalism, still not present in literature [11], where in the framework of the dimensional regularization and by an appropriate extension of the spinor helicity formalism, the calculation of the four-dimensional cut-constructible part of a scattering amplitude as well the reconstruction of the so-called rational part of the amplitude will be provided at once. The rational part of the scattering amplitude are all those contributions to the scattering amplitude not related to the position and features of the branch cuts in the complex plane.

For the one-loop analysis we will start by reviewing the master integrals and the Passarino-Veltman reduction[12].We will use the unitarity methods by evaluating the cuts of master integrals and therefore general cut amplitudes by using the list of formulas for the coefficients of master integrals, given a general one-loop integrand. We will discuss generalized unitarity cuts for one-loop amplitudes, from quadruple, triple and double cut, this is because only the massless gauge theories will be considered. We will address D-dimensional unitarity methods, which are a very efficient way for solving the problem of the calculation of the rational terms, which, by definition, are not affected by branch-cut and therefore in this framework could be ambigously defined. The extension to a massive theory, is in principle easy in formalism proposed in this dissertation, however we will not fully develop the massive cases in the presented examples. Our study will be a journey in the huge literature devoted in the last 20 years to the formulation of the computational framework of the on-shell methods. We benefited a lot of the recent and very comphensive reviews on the subject [14], [15].

The thesis is organized as follows.

The first chapter introduces the subject. In the second chapter the non Abelian gauge theories are discussed and an alternative expansion to the standard Feynman diagram is introduced, namely the reduction to the color ordered amplitudes and their gauge invariant color stripped factors called primitive amplitudes[8, 20, 21]. Colour information is lacking in primitive, which will be decomposed into helicity partial amplitudes. Studying these objects for different helicity configurations of the external gluons, is more convenient because certain helicity configurations vanish, while very compact formulas are reached, namely the so called Maximally-helicity-violating amplitudes (MHV). The simplicity and efficiency of such amplitudes will be strongly enhanced by the use of the helicity spinor formalism. The MHV amplitude will be the building blocks for building more complicated amplitudes in a recursive fashion. Finally, we discuss factorization properties which are powerful tools for checking the correctness of the results and constructing recursion relations. Important property of tree-level amplitudes is that the singularities they possess are always poles and never branch cuts[9]. Based on this, we will introduce the BCFW (Britto, Cachazo, Feng and Witten) recursion relations, a powerful method in gauge field theory that here we will exploit for what concerns recursive calculations of multi-gluon tree-level scattering amplitudes. The main ingredient of the proof of the recursion relations is the factorisability of amplitudes on multi-particle poles[8, 9, 20]. The recursion is performed on-shell in the number of legs and it gives very compact formulas. Any tree-level amplitude of gluons is expressed as a sum over terms constructed from the product of two analytically continued subamplitudes with fewer gluons times a Feynman propagator. The two subamplitudes in each term have momenta shifted in such a way that all the particles are on-shell and the momentum conservation is preserved. Applying the recursion relations one or more times, we can calculate any amplitude by just using the information contained in the MHV amplitudes[21, 22].

In the third chapter we will manage with the unitarity framework for calculations of one-loop gluon scattering amplitudes. Similarly to tree-level amplitudes, we perform a colour decomposition to obtain purely kinematic objects. Unitarity of the S-matrix and factorization properties put constraints on the amplitudes and, in some cases, they fully determine them. Dimensionally regularised amplitudes are expressed in a basis of integrals with unknown coefficients that are fixed by means of unitarity. After revising the Cutkosky method[24] in which just two legs are cutted, we will introduce the generalized unitarity involving quadrupole, triple and double cut associated to log-like branch cut [4, 5, 6] and provide a well defined prescription to fix uniquely the coefficient of the cut-constructible contribution to the Passarino Veltman decomposition. In general, one is also interested in massive theories such as QCD with heavy quarks. The amplitudes of such theories contain logarithms that depend only on masses; such functions do not have cuts in any kinematic variable. This might seem to imply that one cannot obtain all of the terms in massive amplitudes via unitarity, however we will review the methods, that using the generalized unitarity in D dimensions and knowing the ultraviolet and infrared behaviour of renormalizable gauge theories, allows to determine uniquely the rational part of scattering amplitudes involving massive particles[6].

In the fourth chapter we will study the formalism that allows for the calculation of the cut-constructible and rational parts at once, providing an explicit prescription for the unitarity cuts in $D = 4-2\epsilon$. By that prescription any full one-loop amplitude can be obtained from tree amplitudes in four dimensions, where the particles across the cuts are treated as massive, such a mass (μ) [17] encodes the extra-dimensional dpendence. To achieve this objective we will review regularization schemes and in the rest of this thesis the FDH(Four Dimensional Helicity) scheme will be used. Because the FDH scheme is used where To treat the internal particles a first useful tool to be studied will be the Quigley-Rozali brackets [23], later a generalization of the Dirac spinors obeying to a generalized form of the Dirac equation with a mass term $m + i\mu\gamma_5$ (m is the physical mass) will allow to treat properly the internal fermionic legs in the unitarity cuts. The internal gluons will just require a generalization of the polarization vectors to include the mass μ from the extra dimensions, we found that it is enough to replace the two massless $4D$ polarization vectors with the three massive $4D$ polarization vectors. This formalism is checked against the previous results obtained by OPP method [26], where we compute rational contributions to the amplitude of QED processes like γe^+e^- and $\gamma\gamma \to \gamma\gamma$ at one loop.

In the fifth chapter we will do applications of how our formalism works by computing the QCD one loop correction to the scattering amplitude of two gluons production by quark anti-quark annihilation. The reached result is checked with the one obtained by Feynman diagrams by Kunszt et all. [18]. The studied primitive amplitudes are $A_4^{1-loop}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$ and $A_4^{1-loop}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right)$, in which we will find the box, triangle and bubble coefficients for the cut-constructible part, likewise these contributions to the rational part.

In the sixth chapter we derive the effective vertex of two gluons fusion in a Higgs, to calculate it we take into account certain approaches like the top mass goes to infinity ($m_t \rightarrow \infty$) and the fermionic loop momentum is much greater than Higgs momentum. With this effective vertex we do the process at two-loop of three gluons fusion in a Higgs where the reached result is checked with previous results of Schmidt [19].

In the last chapter we will comment on our results and draw our conclusions.

2. Tree Level Amplitudes

2.1. Color-ordered Amplitudes

In perturbative QCD the calculation of multi-gluon scattering amplitudes, even at tree level, is very challenging. The number of diagrams describing a given process grows very quickly, and the redundancy due to the gauge invariance leads to a rapid proliferation of terms. One way to simplify these calculations is to divide all of the diagrams contributing to a given matrix element into subsets of diagrams which are independently gauge invariant, meaning invariant under redefinition of the polarization:

$$\varepsilon_i^\mu (p_i) \rightarrow \varepsilon_i^\mu (p_i) + \alpha_i (p_i) p_i^\mu, \tag{2-1}$$

with the $\alpha_i (p_i)\,'$s being arbitrary functions. It might then be possible to choose different gauges for these different subsets in such a way as to simplify the calculation as much as possible[22]. It is remarkable as this point of view is based on the S- matrix scattering amplitudes more than the Lagrangian approach. In fact having written the terms contributing to the S matrix as the sum of gauge invariant pieces, we may choose the appropriate functions α_i without changing the relative phase between the different gauge invariant terms. The solution of the issue of dividing in gauge invariant pieces a general scattering amplitude will be done in this chapter, without loosing any generality, for a $SU(N_c)$ gauge field theory. Here N_c denotes the number of colors.

A general scattering amplitude in a non-Abelian gauge theory can be decomposed in an orthogonal basis in the color space, which brings to gauge invariant pieces because of the orthogonal character of the decomposition. Here we identify such orthogonal linear indipendent color structures by the traces of the Lie goup generators and we define the color-ordered amplitudes as the terms emerging from such a "trace-based" color decomposition.

The external asymptotic states fill two $SU(N_c)$ representations: the adjoint representation for the gluon, where the adjoint color indices are denoted by $a, b, c, a_i, \ldots \in \left\{1, 2, \ldots, N_c^2 - 1\right\}$, and the fundamental representation N_c with its conjugate representation $\overline{N}_c$ for quarks and antiquarks respectively. Fundamental color indices are denoted by $i_1, i_2, \ldots \in \{1, 2, \ldots, N_c\}$, and anti-fundamental $\overline{N}_c$ indices by $\bar{j}_1, \bar{j}_2, \ldots \in \{1, 2, \ldots, N_c\}$ (see appendix A).

We represent the generators of $SU(N_c)$ by the Hermitean traceless matrices $(T^a)_i^{\bar{j}}$, with the convention about the normalization [9, 27],

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}, \tag{2-2}$$

which differs by the usually used in the textbooks, see for instance [1], by a factor $\frac{1}{2}$. The group theory factors of QCD Feynman rules, relevant in this analysis, are $(T^a)_i^{\bar{j}}$ for the gluon-quark-antiquark vertex but a trilinear gluon vertex is proportional to the $SU(N_c)$ structure constants

$$f^{abc} = \mathrm{Tr}\left(\left[T^a, T^b\right] T^c\right) \tag{2-3}$$

In the present convention the structure constants differ by a factor of $i\sqrt{2}$ factor with respect to those of the textbooks[1], such a convention avoids a proliferation of $\sqrt{2}$ factors in the adopted calculation procedures. Another useful property is the color Fierz identity

$$(T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1} - \frac{1}{N_c} \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1}, \tag{2-4}$$

which will be put at work in the next section and proved in the appendix (A), together with other results of the color algebra.

2.1.1. Trace-based color decomposition

Having introduced the color ordered amplitudes, we are going to provide a prescription about how to extract them in a given tree level amplitude of a $SU(N_c)$ gauge theory. Consider a tree level n-gluon scattering amplitude, let's prove that it can be decomposed into tree graphs color factors represented by the sum of traces of generators T^a in the fundamental representation of $SU(N_c)$:

$$\mathrm{Tr}(T^{a_1} T^{a_2} \ldots . T^{a_n}) + \text{all non cyclic permutations.}$$

By the explicit form of (2-3),

$$f^{abc} = (T^a)_{i_1}^{\bar{j}_1} \delta_{\bar{j}_1}^{i_2} \left(T^b\right)_{i_2}^{\bar{j}_2} \delta_{\bar{j}_2}^{i_3} (T^c)_{i_3}^{\bar{j}_3} \delta_{\bar{j}_3}^{i_1} - \left(T^b\right)_{i_2}^{\bar{j}_2} \delta_{\bar{j}_2}^{i_1} (T^a)_{i_1}^{\bar{j}_1} \delta_{\bar{j}_1}^{i_3} (T^c)_{i_3}^{\bar{j}_3} \delta_{\bar{j}_3}^{i_2} \tag{2-5}$$

the color factors of the vertex of three gluons and of the gluon propagator, using Fierz identity (2-4), amounts diagrammatically to

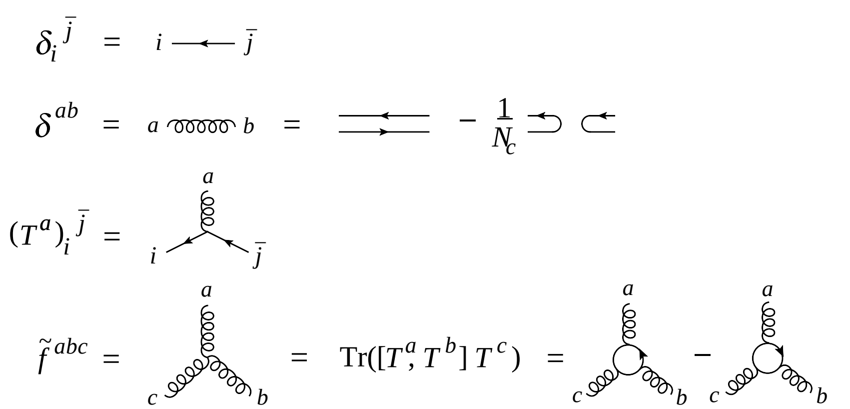


Figure **2-1**.: Diagrammatic equations for simplifying $SU(N_c)$ color algebra.

Adopting the rules of the fig. **6-1** and the eq. (2-3) in the Feynman diagrams, we get the decomposition into single trace terms for planar diagrams, with its own cyclic ordering, the color ordering. Non-planar diagrams will allow also multitrace factors. Moreover, if the amplitude has external quarks legs, there will be also the strings of T^a's terminated by fundamental indices of the form $(T^{a_1} \ldots T^{a_m})_{i_2}^{\bar{\imath}_1}$, one for each external quark-antiquark pair.

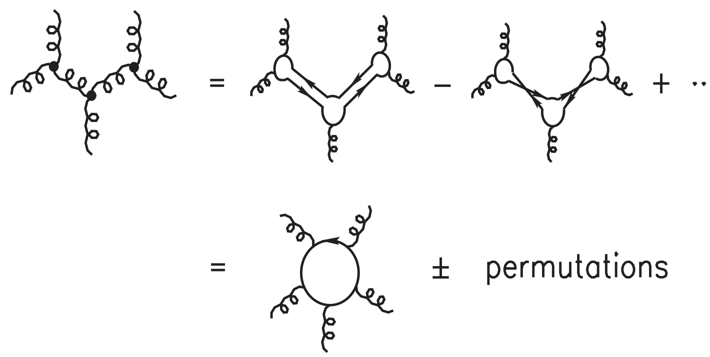


Figure **2-2.**: Reduction of color factors for n-gluons tree amplitudes to a singlet trace of T^a generators[27]

Therefore an n-gluon tree amplitude can be reduced by a trace-based color decomposition to sum of color-ordered amplitudes[27, 28],

$$\mathcal{A}_n^{\text{tree}}\left(\{k_i, h_i, a_i\}\right) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(,n)}}\right) A_n^{\text{tree}}\left(\sigma\left(1^h\right), \ldots, \sigma\left(n^{h_n}\right)\right) \tag{2-6}$$

Here $\mathcal{A}_n^{\text{tree}}$ is the full amplitude, with dependence on the external gluon momentum $k_i, i = 1, 2, \ldots, n$, helicities $h_i = \pm 1$, and adjoint indices a_i. A_n^{tree} are the primitive amplitudes stripped by the color factors but with all the kinematic informations. Cyclic permutation of the arguments of a primitive amplitude, denoted by Z_n, leave this invariant, because the associated trace is invariant under these operations. All $(n-1)!$ non-cyclic permutations, or orderings of the primitive amplitude appear in eq. (2-6). These permutations are denoted by $\sigma \in S_n/Z_n = S_{n-1}$.

Similarly, tree amplitudes with two external quarks and $(n-2)$ gluons can be reduced to single strings of T^a matrices,

$$\mathcal{A}_n^{\text{tree}}\left(q_1, g_2, \ldots, g_{n-1}, \bar{q}_n\right) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \left(T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(,n-1)}}\right)_{i_1}^{\bar{j}_n} A_n^{\text{tree}}\left(1_q, \sigma\left(2\right), \ldots, \sigma\left(n-1\right), n_{\bar{q}}\right) \tag{2-7}$$

in (2-7), we have omitted the helicity labels, and numbers without subscripts in the argument of A_n^{tree} refer to gluons. There are $(n-2)!$ terms corresponding to all possible gluon orderings between quarks.

The primitive amplitudes, denoted here generically by $A\left(1, 2, \ldots, n\right)$, are by construction color independent and satisfy a number of important properties and relationships[22, 28]:

1. $A\left(1, 2, \ldots, n\right)$ is gauge invariant.

   The proof follows the same lines of the QED Ward Identity, since after stripping the color factor, primitive amplitudes behave like in an Abelian theory. For a given color ordering a gluon field couples to a gauge invariant conserved current, because of the linear independence of the basis in the decomposition of (2-6) and (2-7) which does not allow any mixing of the given traces. Therefore the amplitude with at least one gluon has the form,

$$A\left(k\right) = A'_{\mu}\left(k\right) \varepsilon^{\mu}\left(k\right) \tag{2-8}$$

where gluons are created by the interaction term,

$$\int d^4x \, J^{a\mu} \, A^a_\mu \tag{2-9}$$

here $J^{a\mu}$ is the conserved color vector current [1],[2] which becomes gauge invariant by imposing the limitation of calculation of color ordered amplitudes. $A'_\mu(k)$ amounts to the matrix element of the Heisenberg field J^μ:

$$A'_\mu(k) = \int d^4x \, e^{ik\cdot x} \left\langle f \left| j_\mu(x) \right| i \right\rangle \tag{2-10}$$

where the initial (i) and final (f) states include all particles except the given gluon. Dotting by k_μ into eq. (2-10),

$$k^\mu A'_\mu(k) = \int d^4x \, e^{ik\cdot x} k^\mu \left\langle f \left| j_\mu(x) \right| i \right\rangle = i \int d^4x \, e^{ik\cdot x} \left\langle f \left| \partial^\mu j_\mu(x) \right| i \right\rangle = 0 \tag{2-11}$$

we show the requested gauge invariance.

2. $A(1, 2, \ldots, n)$ is invariant under cyclic permutations of $1, 2, \ldots, n$.

   Since the traces of generators are invariant of cyclic permutation we obtain the same physical result if we do a cyclic permutation in the primitive amplitude.

3. $A(n, n-1, \ldots, 2, 1) = (-1)^n A(1, 2, \ldots, n)$

4. The dual Ward identity,

   $$A(1, 2, 3, \ldots, n) + A(2, 1, 3, \ldots, n) + A(2, 3, 1, \ldots, n) + \ldots + A(2, 3, \ldots, 1, n) = 0$$

   The properties 3 and 4 will be verified once in the section 2.3, a specific form of the primitive amplitudes will be provided.

5. Factorization of $A(1, 2, 3, \ldots, n)$ on multi-gluon poles.

   This property can be seen by studying for instance the amplitude of five gluons (ver fig.**2-2**). In the corresponding Feynman diagrams, which are planar we have three consecutive propagators, bringing to the factorization of multi-gluons poles of the form: $s_{1,2}, s_{2,3}, s_{3,4}, s_{4,5}$ and $s_{5,1}$.

With the previous prescriptions we can write (see fig.**2-3**) the color-ordered Feynman rules for QCD, they are readily obtained by the usual Feynman rules just by imposing a given ordering. Even the four-gluon color ordered vertex is a trivial consequence of the quadrilinear non Abelian gluon interaction, after the ordering $(1, 2, 3, 4)$ which extracts the color factor $\frac{ig^2}{2}\text{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right]$ time the kinematical factor written in the figure **2-3**.

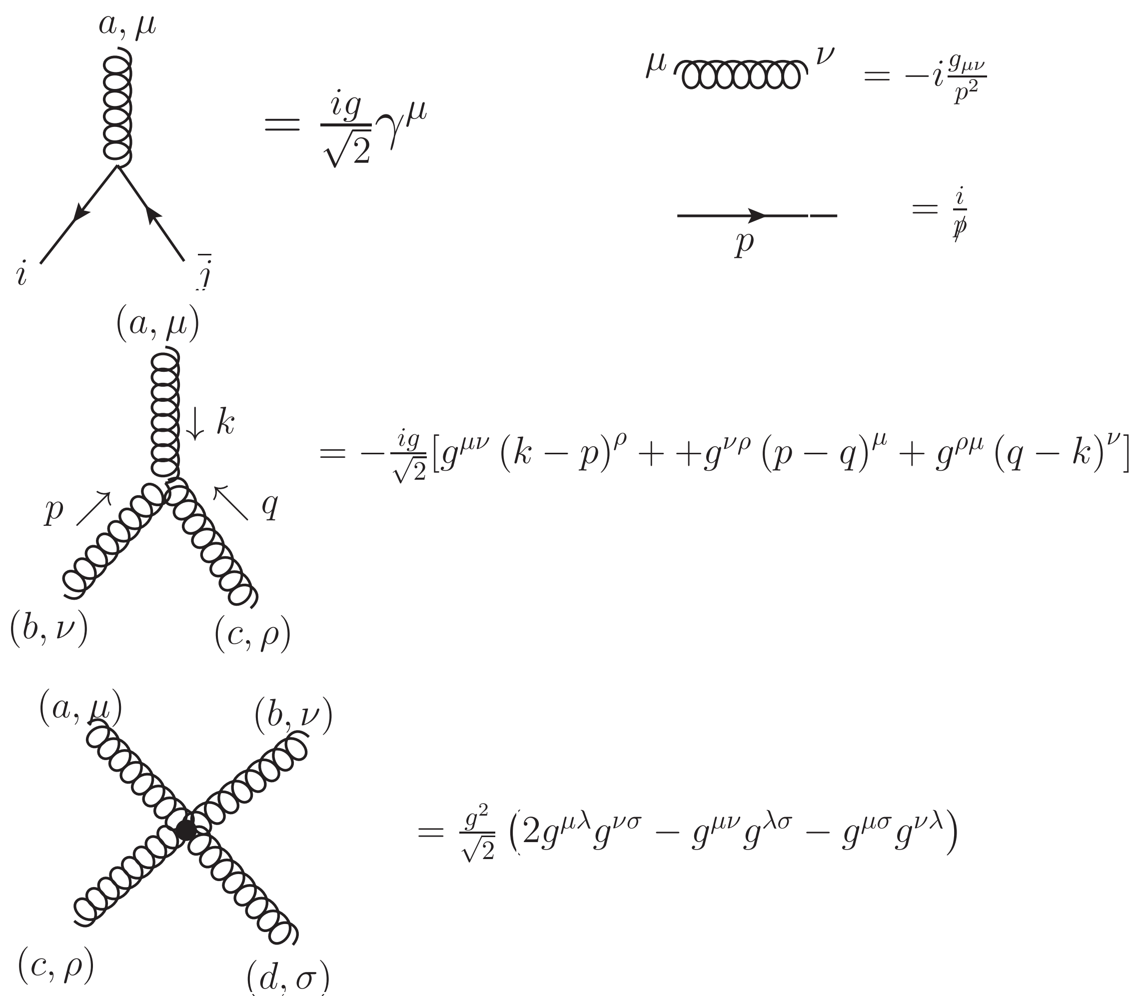


Figure **2-3**.: Color-ordered Feynman rules in 't Hooft-Feynman gauge. All momenta are taken outgoing.

Color decomposition for the process of four gluons

To get a better understanding about the color decomposition, the color factor of the treel level four-gluon amplitude is computed. The process depicted in the figure **2-4** at tree level in terms of the usual Feynman diagrams is given by,

$$0 \to g\,(\mu, a)\;\; g\,(\nu, b)\;\; g\,(\sigma, c)\;\; g\,(\tau, d) \tag{2-12}$$

here a, b, c and d are the color indices and the convention of all outgoing momenta has been made.

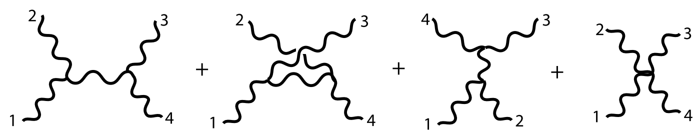


Figure **2-4**.: Feynman diagrams for $gg \to gg$.

By studying each diagram

$$A_1 = C_1 f^{abe} f^{ecd}, \tag{2-13}$$
$$A_2 = C_2 f^{ace} f^{ebd}, \tag{2-14}$$
$$A_3 = C_3 f^{ade} f^{ebc}, \tag{2-15}$$
$$A_4 = C_{4;1} f^{abe} f^{ecd} + C_{4;2} f^{ace} f^{ebd} + C_{4;3} f^{ade} f^{ebc} \tag{2-16}$$

Here $C_i, i = 1,2,3,4$ contains all kinematic information from Feynman diagrams. However, we are interested in the color factor, then by replacing $f^{abc} = \mathrm{Tr}\left\{\left[T^a, T^b\right] T^c\right\}$, we obtain the explicit form of $f^{abe} f^{ecd}$

$$i f^{abe} f^{ecd} = i\mathrm{Tr}\left\{\left[T^a, T^b\right] T^e\right\} f^{ecd} = \mathrm{Tr}\left\{\left[T^a, T^b\right] f^{cde} T^e\right\} = -i\mathrm{Tr}\left\{\left[T^a, T^b\right]\left[T^c, T^d\right]\right\} \tag{2-17}$$
$$= \mathrm{Tr}\left[T^a T^b T^c T^d - T^a T^b T^d T^c - T^b T^a T^c T^d + T^b T^a T^d T^c\right] \tag{2-18}$$
$$= \mathrm{Tr}\left[T^a T^b T^c T^d\right] - \mathrm{Tr}\left[T^a T^b T^d T^c\right] - \mathrm{Tr}\left[T^b T^a T^c T^d\right] + \mathrm{Tr}\left[T^b T^a T^d T^c\right] \tag{2-19}$$

Doing the same procedure for all A_i contributions and summing these results, we find

$$\mathcal{A} = A_1 + A_2 + A_3 + A_4 \tag{2-20}$$
$$\begin{aligned} &= (C_1 - C_3 + C_{4;1} - C_{4;3})\mathrm{Tr}\left(T^a T^b T^c T^d\right) - (C_1 + C_2 + C_{4;1} + C_{4;2})\mathrm{Tr}\left(T^a T^b T^d T^c\right) \\ &\quad - (C_1 + C_2 + C_{4;1} + C_{4;2})\mathrm{Tr}\left(T^a T^c T^d T^b\right) + (C_2 + C_3 + C_{4;2} + C_{4;3})\mathrm{Tr}\left(T^a T^c T^b T^d\right) \\ &\quad + (C_1 - C_3 + C_{4;1} - C_{4;3})\mathrm{Tr}\left(T^a T^d T^c T^b\right) + (C_2 + C_3 + C_{4;2} + C_{4;3})\mathrm{Tr}\left(T^a T^d T^b T^c\right) \end{aligned} \tag{2-21}$$
$$\begin{aligned} &= A(1,2,3,4)\,\mathrm{Tr}\left(T^a T^b T^c T^d\right) + A(1,2,3,4)\,\mathrm{Tr}\left(T^a T^b T^d T^c\right) \\ &\quad + A(1,3,4,2)\,\mathrm{Tr}\left(T^a T^c T^d T^b\right) + A(1,3,2,4)\,\mathrm{Tr}\left(T^a T^c T^b T^d\right) \\ &\quad + A(1,4,3,2)\,\mathrm{Tr}\left(T^a T^d T^c T^b\right) + A(1,4,2,3)\,\mathrm{Tr}\left(T^a T^d T^b T^c\right) \end{aligned} \tag{2-22}$$

confirming the general statement (2-6). Only the primitive amplitude $A(1,2,3,4)$ is needed, the other amplitudes can be found by non-cyclic permutations of the external legs.

2.1.2. Color and strings

The color ordered decomposition introduced in the previous paragraph is very natural in the calculation of string theoretical amplitudes and actually it was previously derived from string theory[28].

In a basic course of String Theory, like the one taken by the author at Universidad Nacional de Colombia, it is possible to see that an open string state is described by

$$|\text{oscillator state}, k; I, J\rangle \,,$$

where, in particular, $I, J \in \{1, 2, ...N\}$ denote the Chan-Paton internal degrees of freedom of the extrema of the open string[29]. In an interaction process of n string states each string is then characterized by the Chan-Paton wave functions λ_{IJ}. In order for an interaction to happen the right endpoint of each string must be in the same state as the left endpoint of the next one.

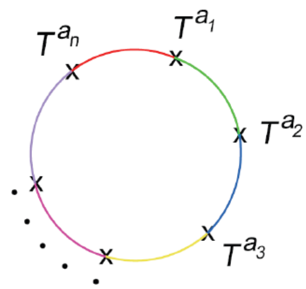


Figure **2-5**.: Color decomposition derived from string theory.

For instance for the case of a tree level open string three tachyons amplitude the Chan-Paton factor would be

$$\lambda^1_{IJ}\lambda^2_{JK}\lambda^3_{KI} + \lambda^1_{IJ}\lambda^3_{JK}\lambda^2_{KI} = \text{Tr}(\lambda^1\lambda^2\lambda^3 + \lambda^1\lambda^3\lambda^2) \tag{2-23}$$

if the cyclic order is 123 and so on. By introducing a complete basis of Hermitean matrices T^a the Chan-Paton factor can be expressed as

$$\text{Tr}(T^{a_1}T^{a_2}T^{a_3}).$$

The Chan-Paton factors have been therefore expressed in terms of the generators of the $U(N)$ algebra and all open string states transform in the adjoint representation of such a non-simple Lie Algebra: $U(N) \cong SU(N) \otimes U(1)$ which for gauge interactions always reduces to $SU(N)$ because of the photon decoupling from the amplitudes. In open string theory a four-tachyon amplitude is written as

$$\begin{aligned} S_{D_2}\left(k_1, a_1; k_2, a_2; k_3, a_3; k_4, a_4\right) = \frac{ig_0^2}{\alpha'}\left(2\pi\right)^{26}\delta^{26}\left(\sum_i k_i\right) \\ \times \left[\text{Tr}\left(T^{a_1}T^{a_2}T^{a_4}T^{a_3} + T^{a_1}T^{a_3}T^{a_4}T^{a_2}\right)B\left(-\alpha_0\left(s\right), -\alpha_0\left(t\right)\right)\right. \\ + \text{Tr}\left(T^{a_1}T^{a_3}T^{a_2}T^{a_4} + T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right)B\left(-\alpha_0\left(s\right), -\alpha_0\left(u\right)\right) \\ \left. + \text{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4} + T^{a_1}T^{a_4}T^{a_3}T^{a_2}\right)B\left(-\alpha_0\left(t\right), -\alpha_0\left(u\right)\right)\right] \end{aligned} \tag{2-24}$$

with the same color structure as in (2-22), B is the Euler Beta-function characteristic of the Veneziano amplitude and $\alpha_0(x)$ represents the linear Regge trajectories proper of the flat space string theory. From the previous arguments it is easy to extrapolate that an n-point tree level string amplitude will have the general structure of $2-6$, however we do not reach the structure (2-7), even if matter is included by superstring theory because there are no states transforming in the fundamental representation of $U(N)$. Moreover the color stripped primitive amplitudes still satisfy the properties the gauge invariance, the cyclic, the reverse permutation, the dual Ward identities and of course they have pole singularites in single adiacent channels.

To conclude this brief connection to the string theoretical color ordered amplitudes it is worth to remark that with respect to the original paper of Chan-Paton in 1969[30], nowadays the $U(N)$ gauge symmetry in string theory is related to a stack of N D-branes and the generators T^a are seen as transition factors for gauge vectors which have ends on different branes.

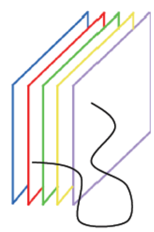


Figure **2-6**.: Color decomposition derived from string theory.

2.2. Spinor-Helicity Formalism

The spinor-helicity formalism [9, 10, 32, 33, 34] for scattering amplitudes has proven an invaluable tool in perturbative computation since its development in the eighties, being responsible for the discovery of compact representations of tree and loop amplitudes. Instead of Lorentz inner products of momenta, it relies on the more fundamental spinor products. These neatly capture the analytic properties of on-shell scattering amplitudes, like the factorization behavior on multi-particle-channels. The recent boost in the progress of evaluating on-shell scattering amplitudes is due to turning qualitative information on their analytic properties into quantitative tools for computing them.

2.2.1. Fermion Wave Functions for Real Massless Momenta

Consider a massless fermion of momentum p, the helicity spinor for this fermion satisfy the Dirac equation [9, 35]

$$\not{p}u\left(p\right)=0 \tag{2-25}$$

We can construct general helicity spinors of momentum p, $u_+\left(p\right)$ and $u_-\left(p\right)$ if we choose a simple set of momenta k_0^μ, k_1^μ that are fixed and satisy $k_0^2=0, k_1^2=-1, k_0\cdot k_1=0$ and,

$$u_\lambda\left(k_0\right)\bar{u}_\lambda\left(k_0\right)=\left(1+\lambda\gamma^5\right)\not{k}_0 \tag{2-26}$$

with $\lambda=\pm1$. From these spinors we define basic spinors in the follow way: let $u_-\left(k_0\right)$ be the left-handed spinor for a fermion with a momentum k_0 and $u_0\left(k_0\right)=\not{k}_1 u_-\left(k_0\right)$. Then, for any p

such that p is lightlike ($p^2 = 0$), define[10, 32]:

$$u_-(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_+(k_0), \qquad u_+(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_-(k_0) \tag{2-27}$$

with $u_+(k_0) = \not{k}_1 u_-(k_0)$. This set of conventions defines the phases of spinors unambiguously, except when p is parallel to k_0. Writing explicitly the 4-momentum for k_0^μ, k_1^μ as,

$$k_0^\mu = (E, 0, 0, -E), \qquad k_1^\mu = (0, 1, 0, 0) \tag{2-28}$$

we can construct $u_-(k_0), u_+(k_0), u_-(p)$, and $u_+(p)$ explicitly.
From Dirac equation and chirality, we get

$$\not{k}_0 u_-(k_0) = 0, \qquad u_-(k_0) = \frac{1-\gamma_5}{2} u_-(k_0) \tag{2-29}$$

writing $u_-(k_0)$ as

$$u_-(k_0) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{2-30}$$

With the normalization of $a = \sqrt{2E}$, we obtain

$$u_-(k_0) = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u_+(k_0) = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2-31}$$

$$u_-(p) = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} -p^1 + ip^2 \\ p^0 + p^3 \\ 0 \\ 0 \end{pmatrix}, \qquad u_+(p) = \frac{1}{\sqrt{p^0 + p^3}} \begin{pmatrix} 0 \\ 0 \\ p^0 + p^3 \\ p^1 + ip^2. \end{pmatrix} \tag{2-32}$$

The components of the momentum p can be expressed in terms of the $p_\pm$ and $p_\perp$

$$p_\pm = p^0 \pm p^3, \tag{2-33}$$

$$p_\perp = p^1 + ip^2 = |p_\perp| \, \mathrm{e}^{i\varphi_p} = \sqrt{p_+ p_-} \mathrm{e}^{i\varphi_p}, \tag{2-34}$$

where

$$\mathrm{e}^{\pm i\varphi_p} = \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p_+ p_-}}. \tag{2-35}$$

By this choice

$$u_-(p) = \begin{pmatrix} -\sqrt{p_+} \\ \sqrt{p_-} \mathrm{e}^{-i\varphi_p} \\ 0 \\ 0 \end{pmatrix}, \qquad u_+(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p_-} \mathrm{e}^{-i\varphi_p} \\ \sqrt{p_+}. \end{pmatrix} \tag{2-36}$$

For the four-components spinors $u_{\pm}(p)$ it is possible to deduce the two-components spinors u_R and u_L, related each other by

$$u_R(p) = i\sigma^2 u_L^*(p), \tag{2-37}$$

as it will be proven in the appendix B.1.

A bra and ket notation spinors is introduced corresponding to the massless momentum p_i and labelled by the index i, with the phase convention for physical particles and antiparticles given as

$$v_+(p_i) = \bar{u}_-(p_i) = \langle i|, \qquad v_-(p_i) = \bar{u}_+(p_i) = [i|, \tag{2-38}$$
$$\bar{v}_+(p_i) = u_-(p_i) = |i], \qquad \bar{v}_-(p_i) = u_+(p_i) = |i\rangle. \tag{2-39}$$

Lorentz-invariant spinor products can the be constructed as,

$$\bar{u}_-(p_i)\, u_+(p_j) = \langle ij\rangle, \qquad \bar{u}_+(p_i)\, u_-(p_j) = [ij] \tag{2-40}$$

with the explicit form of left and right -handed spinors (see eq. 2-36), these spinor products become,

$$\langle ij\rangle = \sqrt{p_{i-}p_{j+}}\mathrm{e}^{i\varphi_{p_i}} - \sqrt{p_{i+}p_{j-}}\mathrm{e}^{-i\varphi_{p_j}} = \sqrt{|s_{ij}|}\mathrm{e}^{i\phi_{ij}} \tag{2-41}$$
$$[ij] = \sqrt{p_{i+}p_{j-}}\mathrm{e}^{-i\varphi_{p_j}} - \sqrt{p_{i-}p_{j+}}\mathrm{e}^{-i\varphi_{p_i}} = -\sqrt{|s_{ij}|}\mathrm{e}^{-i\phi_{ij}} \tag{2-42}$$

where $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$ and

$$\cos\phi_{ij} = \frac{p_i^1 p_j^+ - p_j^1 p_i^+}{\sqrt{|s_{ij}|\, p_{i+}p_{j+}}}, \qquad \sin\phi_{ij} = \frac{p_i^2 p_j^+ - p_j^2 p_i^+}{\sqrt{|s_{ij}|\, p_{i+}p_{j+}}}. \tag{2-43}$$

These products appear to be antisymmetric explicitly and are related to their 4-vectors by the identities,

$$|p\rangle\,[p| = u_R(p)\,\bar{u}_R(p) = \left(\frac{1+\gamma^5}{2}\right)\not{p}, \tag{2-44}$$
$$|p]\,\langle p| = u_L(p)\,\bar{u}_L(p) = \left(\frac{1-\gamma^5}{2}\right)\not{p} \tag{2-45}$$

eqs. (2-41) and (2-42) show,

$$\langle ij\rangle = -\langle ji\rangle, \qquad [ij] = -[ji], \qquad \langle ij\rangle = [ji]^* \tag{2-46}$$

so that,

$$|\langle pq\rangle|^2 = |[pq]|^2 = \langle ij\rangle\,[ji] = s_{pq} = 2p\cdot q \tag{2-47}$$

The phase convention

$$|i\rangle^c = |i], \qquad {}^c\langle i| = [i|, \tag{2-48}$$

where c indicates charge conjugation, is adopted in the following. For vector currents built by spinors, the following identities, to be proven in the appendix B.1, are very useful

$$u_L^\dagger(p)\,\bar{\sigma}^\mu u_L(q) = u_R^\dagger(q)\,\sigma^\mu u_R(p) \tag{2-49}$$

$$\langle p|\gamma^\mu|q] = [q|\gamma^\mu|p\rangle \tag{2-50}$$

The Fierz identity, the identity of sigma matrices (see appendix B.1)

$$(\bar{\sigma}^\mu)_{ab}(\bar{\sigma}_\mu)_{cd} = 2\left(i\sigma^2\right)_{ac}\left(i\sigma^2\right)_{bd} \tag{2-51}$$

allows the simplification of contractions of spinor expressions, for instance

$$\langle p|\gamma^\mu|q]\,\langle k|\gamma_\mu|l] = 2\,\langle pk\rangle\,[lq]\,, \qquad \langle p|\gamma^\mu|q]\,[k|\gamma_\mu|l\rangle = 2\,\langle pl\rangle\,[kq] \tag{2-52}$$

Finally the spinor products obey the Schouten identity (see appendix B.1)

$$\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0 \tag{2-53}$$

$$[ij]\,[kl] + [ik]\,[lj] + [il]\,[jk] = 0. \tag{2-54}$$

2.2.2. Massless Vector Boson Wave-functions for Real Momenta

We construct the massless polarization vectors by considering k to be the momentum of a photon (gluon), and p be another lightlike vector, chosen so that $p\cdot k \neq 0$. $u_-(p), u_-(p)$ are the spinors of definite helicity for fermions with the light-like momentum p, defined according to the conventions of eq. (2-27). The helicity one photon polarization vectors are

$$\varepsilon_+^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_+(k)\gamma^\mu u_+(p), \qquad \varepsilon_-^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_-(k)\gamma^\mu u_-(p) \tag{2-55}$$

In the shorthand notation,

$$\varepsilon_+^\mu(k;q) = -\frac{\langle k|\gamma^\mu|q]}{\sqrt{2}\,[qk]}, \qquad \varepsilon_-^\mu(k;q) = \frac{[k|\gamma^\mu|q\rangle}{\sqrt{2}\,\langle qk\rangle} \tag{2-56}$$

$$\varepsilon_+^{*\mu}(k;q) = \frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle}, \qquad \varepsilon_-^{*\mu}(k;q) = -\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[qk]} \tag{2-57}$$

these polarization vectors are defined in terms of both the momentum vector k and a reference vector q. The gauge invariance of the scattering amplitudes of the spin-1 field manifests itself in the arbitrariness of the reference momentum q.

Now, we consider an azimuthal rotation about p_i axis, spinors left and right -handed transform as,

$$|i\rangle \to |i'\rangle = e^{i\phi/2}\,|i\rangle \tag{2-58}$$

$$|i] \to |i'] = e^{-i\phi/2}\,|i] \tag{2-59}$$

and the polarization vectors with helicity $\pm$,

$$\varepsilon_+^\mu(i) \to \frac{\langle i'|\gamma^\mu|q]}{\sqrt{2}\,[qi']} = e^{i\phi}\varepsilon_+^\mu(i) \tag{2-60}$$

$$\varepsilon_-^\mu(i) \to \frac{[i'|\gamma^\mu|q\rangle}{\sqrt{2}\,[qi']} = e^{-i\phi}\varepsilon_-^\mu(i) \tag{2-61}$$

eqs. (2-60) and (2-61) as required for helicity +1 and −1 respectively[2]

The polarization vectors have the usual properties

$$\left(\varepsilon^{\pm}\right)^{*} = \varepsilon^{\mp}, \tag{2-62}$$

$$\varepsilon^{\pm}\cdot\varepsilon^{\pm} = 0, \tag{2-63}$$

$$\varepsilon^{\pm}\cdot\varepsilon^{\mp} = -1, \tag{2-64}$$

$$\varepsilon_{+}^{\mu}\varepsilon_{+}^{*\nu} + \varepsilon_{-}^{\mu}\varepsilon_{-}^{*\nu} = -g^{\mu\nu} + \frac{k^{\mu}q^{\nu} + q^{\mu}k^{\nu}}{q\cdot k} \tag{2-65}$$

these properties can be obtained easily by using the Fierz identity and for the last one see appendix B.1. The arbitrariness of the choice of q can be seen by examining the difference between two choices of q:

$$\begin{aligned}
\varepsilon_{+}^{*\mu}(k;r) - \varepsilon_{+}^{*\mu}(k;s) &= \frac{1}{\sqrt{2}}\left(\frac{\langle r\left|\gamma^{\mu}\right|k]}{\langle rk\rangle} - \frac{\langle s\left|\gamma^{\mu}\right|k]}{\langle sk\rangle}\right)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\left(-\langle r\left|\gamma^{\mu}\right|k]\langle ks\rangle + \langle s\left|\gamma^{\mu}\right|k]\langle kr\rangle\right)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\left(-\langle r\left|\gamma^{\mu}k\right|s\rangle + \langle s\left|\gamma^{\mu}k\right|r\rangle\right)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\left(\langle s\left|k\gamma^{\mu}\right|r\rangle + \langle s\left|\gamma^{\mu}k\right|r\rangle\right)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle rk\rangle\langle sk\rangle}\langle s\left|k\gamma^{\mu} + \gamma^{\mu}k\right|r\rangle\\
&= \sqrt{2}\frac{\langle sr\rangle}{\langle rk\rangle\langle sk\rangle}k^{\mu}
\end{aligned} \tag{2-66}$$

the last line follows from the anticommutator of Dirac matrices. The final result of this calculation is that

$$\varepsilon_{+}^{*\mu}(k;r) - \varepsilon_{+}^{*\mu}(k;s) = f(r,s)\,k^{\mu} \tag{2-67}$$

where $f(r,s)$ is a function of the reference vectors. This expression will not give any contribution to the amplitudes because of the Ward identity at work (see figure **2-7**). Thus, the difference between the polarization vectors generated by two choices of q is proportional to k^{μ} and it is therefore a pure gauge term[9, 35].

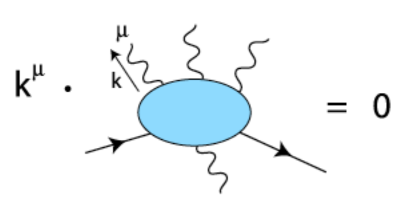


Figure **2-7**.: Ward identity obeyed by a gauge-invariant sum of diagrams with all external particles on shell[9].

2.2.3. Parity and Charge Conjugation

We might worry that the color and helicity decompositions will lead to a huge proliferation in the number of primitive amplitudes that have to be computed. This does not happen, thanks to the group theory relations and the discrete symmetries of parity and charge conjugation. Parity simultaneously reverses all helicities in an amplitude; for example eqs. (2-56) and (2-57) show that it is implemented by the exchange $\langle qk\rangle \longleftrightarrow [kq]$. Charge conjugation is related to the anti-symmetry of the color-ordered rules; for pure gluon primitive amplitudes it takes the form of a reflected identity[36],

$$A_n^{\text{tree}}\left(1,2,\ldots,n\right) = \left(-1\right)^n A_n^{\text{tree}}\left(n,\ldots,2,1\right)$$

For amplitudes with external quarks, it allows us to exchange a quark and anti-quark.

As an example, with the use of parity and charge conjugation symmetry, we can reduce the five-gluon amplitude at tree level to a combination of just four independent partial amplitudes:

$$\begin{aligned} &A_5^{\text{tree}}\left(1^+,2^+,3^+,4^+,5^+\right), \qquad && A_5^{\text{tree}}\left(1^-,2^+,3^+,4^+,5^+\right), \\ &A_5^{\text{tree}}\left(1^-,2^-,3^+,4^+,5^+\right). && A_5^{\text{tree}}\left(1^-,2^+,3^-,4^+,5^+\right). \end{aligned} \tag{2-68}$$

Furthemore, as it will be seen later in section 2.3 the first two primitive tree-level amplitudes vanish and there is a group theory ($U\left(1\right)$ decoupling) relation between the last two, so there is only one independent non-vanishing object to calculate. In the next chapter it will be discovered that at one-loop of the four previuosly listed independent primitive amplitudes only the last two contribute to the NLO cross-section, due to the tree level vanishings.

2.2.4. Examples

$\boldsymbol{e^+e^- \to qg\bar{q}}$

As a warming up exercise consider the gluonsstralhung process

Figure **2-8**.: Feynman diagram for the process $e^+e^- \to qg\bar{q}$.

The amplitude can be written by using Feynman diagrams as,

$$\mathcal{A}_5 = -\frac{ige^2}{\sqrt{2}}\left\{\frac{1}{s_{12}s_{34}}\bar{u}_+\left(p_1\right)\gamma^\mu u_+\left(p_2\right)\bar{u}_+\left(p_3\right)\not{\epsilon}\left(p_4\right)\left(\not{k}_3+\not{k}_4\right)\gamma_\mu u_+\left(p_5\right)\right.$$
$$\left.-\frac{1}{s_{12}s_{45}}\bar{u}_+\left(p_1\right)\gamma_\nu u_+\left(p_2\right)\bar{u}_+\left(p_3\right)\gamma^\nu\left(\not{k}_4+\not{k}_5\right)\not{\epsilon}\left(p_4\right)u_+\left(p_5\right)\right\}T^a \tag{2-69}$$

The color indices of the quark and anti-quark are implicitly included in the generator T^a.
In the following calculations we will write the polarization vectors $\not{\varepsilon}_+(4) = \varepsilon_+^\mu(4)\gamma_\mu = \varepsilon_+(4)$, then the amplitude with the shorthand notation previously studied takes the form,

$$\mathcal{A}_5 = -\frac{ige^2}{\sqrt{2}}\left\{\frac{1}{s_{12}s_{34}}[1|\gamma^\mu|2\rangle[3|\varepsilon_+(4)(\not{k}_3+\not{k}_4)\gamma_\mu|5\rangle - \frac{1}{s_{12}s_{45}}[1|\gamma^\mu|2\rangle[3|\gamma_\mu(\not{k}_4+\not{k}_5)\varepsilon_+(4)|5\rangle\right\}T^a \tag{2-70}$$

Using Fierz identity and writting explicitly $\varepsilon_+(4)$,

$$\mathcal{A}_5 = \frac{2ige^2}{\sqrt{2}}\left\{\frac{1}{s_{12}s_{34}}\langle 25\rangle[1|(\not{k}_3+\not{k}_4)\varepsilon_+(4)|3] + \frac{1}{s_{12}s_{45}}[13]\langle 2|(\not{k}_4+\not{k}_5)\varepsilon_+(4)|5\rangle\right\}T^a \tag{2-71}$$

$$= 2ige^2\left\{\frac{1}{s_{12}s_{34}}\frac{\langle 25\rangle}{\langle 4q\rangle}[1|(\not{k}_3+\not{k}_4)|q\rangle[43] + \frac{1}{s_{12}s_{45}}\frac{[13]}{\langle 4q\rangle}\langle 2|(\not{k}_4+\not{k}_5)|4]\langle q5\rangle\right\}T^a \tag{2-72}$$

From the Ward identity our result is independent of the reference vector, for simplicity we choose $q = 5$ to remove the second diagram,

$$\mathcal{A}_5 = \frac{2ige^2}{s_{12}s_{34}}\frac{\langle 25\rangle}{\langle 45\rangle}[1|(\not{k}_3+\not{k}_4)|5\rangle[43]T^a \tag{2-73}$$

using momentum conservation i.e. $\not{k}_3+\not{k}_4 = -\not{k}_1-\not{k}_2$,

$$\mathcal{A}_5 = -\frac{2ige^2}{s_{12}s_{34}}\frac{\langle 25\rangle[1|2|5\rangle[43]}{\langle 45\rangle}T^a \tag{2-74}$$

for compactness we write $[1|\not{k}_2|5\rangle = [1|2|5\rangle$

$$\mathcal{A}_5 = 2ige^2\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}T^a \tag{2-75}$$

separating the kinematics and the color factors, we get the primitive amplitude,

$$A_5 = \frac{\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle} \tag{2-76}$$

$q\bar{q} \to gg$

Now, we consider a process in QCD. The Feynman diagrams are given by

The amplitude for this process is,

$$\begin{aligned}\mathcal{A} = &-\frac{ig^2}{2}\bar{u}(p_1)\varepsilon^\mu(p_2)\varepsilon^\nu(p_3)\left\{\gamma_\mu\frac{\not{p}_1+\not{p}_2}{s_{12}}\gamma_\nu T^aT^b + \gamma_\nu\frac{\not{p}_1+\not{p}_3}{s_{13}}\gamma_\mu T^bT^a\right\}\bar{v}(p_4)\\ &-\frac{ig^2 f^{abc}T^c}{2s_{14}}\bar{u}(p_1)\gamma^\rho\bar{v}(p_4)\{\varepsilon^\rho(p_2)(-2p_2-p_3)\cdot\varepsilon(p_3)+\\ &+\varepsilon^\rho(p_3)(2p_3+p_2)\cdot\varepsilon(p_2)+(p_2-p_3)^\rho\varepsilon(p_2)\cdot\varepsilon(p_3)\}\end{aligned} \tag{2-77}$$

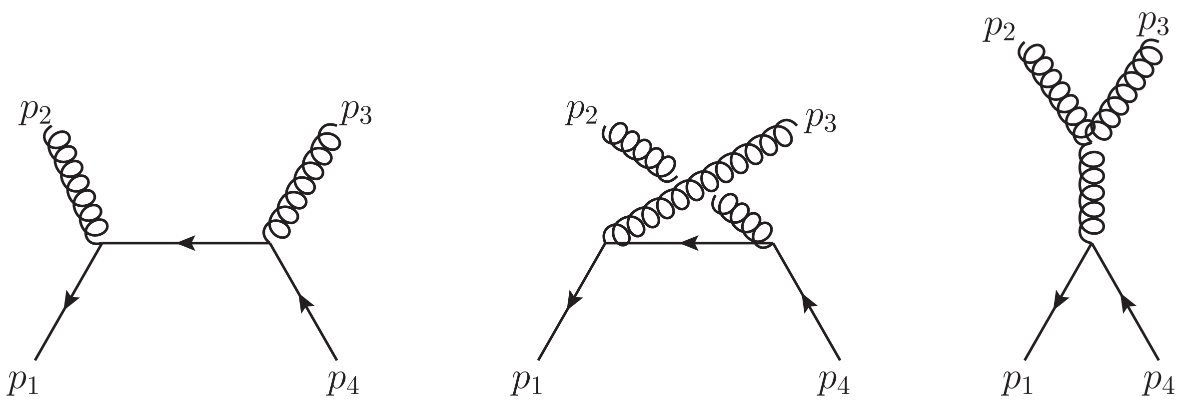


Figure **2-9**.: Feynman diagrams for the process $q\bar{q} \to gg$

First we study the channel $q_L \bar{q}_R \to g_L g_L$, the (2-77) takes the form

$$\begin{aligned}\mathcal{A} = &-\frac{ig^2}{2} \langle 1| \varepsilon_-^{\mu}(2)\, \varepsilon_-^{\nu}(3) \left\{ \gamma_\mu \frac{\not{p}_1 + \not{p}_2}{s_{12}} \gamma_\nu T^a T^b + \gamma_\nu \frac{\not{p}_1 + \not{p}_3}{s_{13}} \gamma_\mu T^b T^a \right\} |4] \\ &- \frac{ig^2 f^{abc} T^c}{2 s_{14}} \langle 1| \gamma_\rho |4] \left\{ \varepsilon^{\rho}(2) (-2p_2 - p_3) \cdot \varepsilon(3) \right. \\ &\left. + \varepsilon^{\rho}(3) (2p_3 + p_2) \cdot \varepsilon(2) + (p_2 - p_3)^{\rho} \varepsilon(2) \cdot \varepsilon(3) \right\} \end{aligned} \tag{2-78}$$

studying only the contribution to the amplitude that comes from the interaction vertex for the quarks with gluons,

$$\begin{aligned} &-\frac{ig^2}{2} \langle 1| \varepsilon_-^{\mu}(p_2)\, \varepsilon_-^{\nu}(p_3) \left\{ \gamma_\mu \frac{\not{p}_1 + \not{p}_2}{s_{12}} \gamma_\nu T^a T^b + \gamma_\nu \frac{\not{p}_1 + \not{p}_3}{s_{13}} \gamma_\mu T^b T^a \right\} |4] = \\ &= -\frac{ig^2}{2} \frac{[q_2 |\gamma^\mu| 2\rangle}{\sqrt{2}\,[q_2 2]} \frac{[q_3 |\gamma^\nu| 3\rangle}{\sqrt{2}\,[q_2 3]} \langle 1| \left\{ \gamma_\mu \frac{\not{p}_1 + \not{p}_2}{s_{12}} \gamma_\nu T^a T^b + \gamma_\nu \frac{\not{p}_1 + \not{p}_3}{s_{13}} \gamma_\mu T^b T^a \right\} |4] \\ &= -\frac{ig^2}{4\,[q_2 2]\,[q_2 3]} [q_2 |\gamma^\mu| 2\rangle\, [q_3 |\gamma^\nu| 3\rangle \langle 1| \left\{ \gamma_\mu \frac{\not{p}_1 + \not{p}_2}{s_{12}} \gamma_\nu T^a T^b + \gamma_\nu \frac{\not{p}_1 + \not{p}_3}{s_{13}} \gamma_\mu T^b T^a \right\} |4] \\ &= -\frac{ig^2}{[q_2 2]\,[q_2 3]} \left\{ \frac{\langle 21\rangle \langle 3 |1+2| q_2]\,[4 q_3]}{s_{12}} T^a T^b + \frac{\langle 31\rangle \langle 1 |1+3| q_3]\,[4 q_2]}{s_{13}} T^b T^a \right\} \end{aligned} \tag{2-79}$$

if we take the reference vectors $q_2 = q_3 = 4$ this contribution vanishes.

Given the choice of the reference vectors the last diagram amounts to

$$\begin{aligned} &-\frac{ig^2 f^{abc} T^c}{2 s_{14}} \langle 1| \gamma_\rho |4] \times \\ &\times \left\{ \varepsilon_-^{\rho}(p_2) (-2p_2 - p_3) \cdot \varepsilon_-(p_3) + \varepsilon_-^{\rho}(p_3) (2p_3 + p_2) \cdot \varepsilon_-(p_2) + (p_2 - p_3)^{\rho} \varepsilon_-(p_2) \cdot \varepsilon_-(p_3) \right\} \end{aligned} \tag{2-80}$$

where due to

$$\langle 1|\,\gamma_\rho\,|4]\,\varepsilon_-^\rho\,(p_2) = \langle 1|\,\gamma_\rho\,|4]\,\frac{[q_2\,|\gamma^\rho|\,2\rangle}{\sqrt{2}\,[q_2 2]} = \sqrt{2}\frac{\langle 12\rangle\,[q_2 4]}{[q_2 2]} = 0, \tag{2-81}$$

$$\langle 1|\,\gamma_\rho\,|4]\,\varepsilon_-^\rho\,(p_3) = \langle 1|\,\gamma_\rho\,|4]\,\frac{[q_3\,|\gamma^\rho|\,3\rangle}{\sqrt{2}\,[q_3 3]} = \sqrt{2}\frac{\langle 13\rangle\,[q_3 4]}{[q_3 3]} = 0, \tag{2-82}$$

$$\varepsilon_-\,(p_2)\cdot\varepsilon_-\,(p_3) = \frac{[q_2\,|\gamma_\rho|\,2\rangle}{\sqrt{2}\,[q_2 2]}\frac{[q_3\,|\gamma^\rho|\,3\rangle}{\sqrt{2}\,[q_3 3]} = \frac{\langle 23\rangle\,[q_3 q_2]}{[q_2 2]\,[q_3 3]} = 0, \tag{2-83}$$

this contribution also vanishes.
In conclusion for this channel

$$\mathcal{A}\,(q_L\,(1)\,\bar{q}_R\,(4) \to g_L\,(2)\,g_L\,(3)) = 0 \tag{2-84}$$

and using parity and charge conjugation,

$$\mathcal{A}\,(q_L\,(1)\,\bar{q}_R\,(4) \to g_L\,(2)\,g_L\,(3)) = 0 \tag{2-85}$$
$$\mathcal{A}\,(q_R\,(1)\,\bar{q}_L\,(4) \to g_L\,(2)\,g_L\,(3)) = 0 \tag{2-86}$$
$$\mathcal{A}\,(q_R\,(1)\,\bar{q}_L\,(4) \to g_R\,(2)\,g_R\,(3)) = 0 \tag{2-87}$$
$$\mathcal{A}\,(q_L\,(1)\,\bar{q}_R\,(4) \to g_R\,(2)\,g_R\,(3)) = 0. \tag{2-88}$$

Now We compute this process in another channel $q_L\bar{q}_R \to g_R g_L$.
The Feynman diagrams which have only the interaction vertex of quarks and gluons give

$$\begin{aligned}
&-\frac{ig^2}{2}\,\langle 1|\,\varepsilon_+^\mu\,(2)\,\varepsilon_-^\nu\,(3)\left\{\gamma_\mu\frac{\not{p}_1+\not{p}_2}{s_{12}}\gamma_\nu T^a T^b + \gamma_\nu\frac{\not{p}_1+\not{p}_3}{s_{13}}\gamma_\mu T^b T^a\right\}|4] \\
&\qquad = -\frac{ig^2}{2}\frac{\langle q_2\,|\gamma^\mu|\,2]}{\sqrt{2}\,\langle q_2 2\rangle}\frac{[q_3\,|\gamma^\nu|\,3\rangle}{\sqrt{2}\,[q_3 3]}\,\langle 1|\left\{\gamma_\mu\frac{\not{p}_1+\not{p}_2}{s_{12}}\gamma_\nu T^a T^b + \gamma_\nu\frac{\not{p}_1+\not{p}_3}{s_{13}}\gamma_\mu T^b T^a\right\}|4] \\
&\qquad = -ig^2\left\{\frac{\langle q_2 1\rangle\,\langle 3\,|1+2|\,2]\,[4q_3]}{[q_3 3]\,\langle q_2 2\rangle\,s_{12}}T^a T^b + \frac{\langle 31\rangle\,\langle q_2\,|1+3|\,q_3]\,[42]}{\langle q_2 2\rangle\,[q_3 3]\,s_{13}}T^b T^a\right\} \\
&\qquad\qquad = -ig^2\left\{\frac{\langle 31\rangle\,\langle 3\,|1|\,2]\,[42]}{[23]\,\langle 32\rangle\,s_{12}}T^a T^b + \frac{\langle 31\rangle\,\langle 3\,|1|\,2]\,[42]}{\langle 32\rangle\,[23]\,s_{13}}T^b T^a\right\}
\end{aligned} \tag{2-89}$$

putting the reference vectors as $q_2 = 3$ and $q_3 = 2$

$$\mathcal{A}\,(q_L\,(1)\,\bar{q}_R\,(4) \to g_R\,(2)\,g_L\,(3)) = -ig^2\left\{\frac{\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}T^a T^b + \frac{\langle 12\rangle^3\langle 42\rangle}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}T^b T^a\right\} \tag{2-90}$$

Here, the self interactions of gluons also vanishes. However, eq.(2-90) can be studied as

$$\mathcal{A}\,(q_L\,(1)\,\bar{q}_R\,(4) \to g_R\,(2)\,g_L\,(3)) = A\,(1234)\,T^a T^b + A\,(1324)\,T^b T^a \tag{2-91}$$

$$A\,(1234) = -ig^2\frac{\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{2-92}$$

the second term in (2-91), is given by the same expression with $(2, \varepsilon\,(2))$ exchanged with $(3, \varepsilon\,(3))$. Here A is the color-ordered primitive amplitude and $\mathcal{A}\,(q_L\,(1)\,\bar{q}_R\,(4) \to g_R\,(2)\,g_L\,(3))$ is the full

amplitude.
Using parity and charge conjugation

$$\mathcal{A}\left(q_L\left(1\right)\bar{q}_R\left(4\right)\to g_R\left(2\right)g_L\left(3\right)\right)=-ig^2\left\{\frac{\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}T^aT^b+\frac{\langle 12\rangle^3\langle 42\rangle}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}T^bT^a\right\}, \tag{2-93}$$

$$\mathcal{A}\left(q_R\left(1\right)\bar{q}_L\left(4\right)\to g_L\left(2\right)g_R\left(3\right)\right)=-ig^2\left\{\frac{[13]^3\,[43]}{[12]\,[23]\,[34]\,[41]}T^aT^b+\frac{[12]^3\,[42]}{[13]\,[32]\,[24]\,[41]}T^bT^a\right\}, \tag{2-94}$$

$$\mathcal{A}\left(q_R\left(1\right)\bar{q}_L\left(4\right)\to g_R\left(2\right)g_L\left(3\right)\right)=-ig^2\left\{\frac{\langle 43\rangle^3\langle 13\rangle}{\langle 42\rangle\langle 23\rangle\langle 31\rangle\langle 14\rangle}T^aT^b+\frac{\langle 42\rangle^3\langle 12\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}T^bT^a\right\}, \tag{2-95}$$

$$\mathcal{A}\left(q_L\left(1\right)\bar{q}_R\left(4\right)\to g_L\left(2\right)g_R\left(3\right)\right)=-ig^2\left\{\frac{[43]^3\,[13]}{[42]\,[23]\,[31]\,[14]}T^aT^b+\frac{[42]^3\,[12]}{[12]\,[23]\,[34]\,[41]}T^bT^a\right\}. \tag{2-96}$$

2.3. MHV amplitudes

If the color-ordered primitive amplitude for gluons $1,\ldots,n$, of momenta $p_1,\ldots,p_n$ and helicities $h_1,\ldots,h_n$, is $A_n\left(1^{h_1},\ldots,n^{h_n}\right)$, where the momenta and helicities are labeled for all outgoing particles, then the three primitive amplitudes of interest are [9, 21, 22, 37]

$$A_n\left(1^+,\ldots,i^-,\ldots,j^-\ldots,n^+\right)=i\frac{\langle ij\rangle^4}{\langle 12\rangle\,\langle 23\rangle\cdots\langle (n-1)\,n\rangle\,\langle n1\rangle} \tag{2-97}$$

$$A_n\left(1^-,\ldots,i^+,\ldots,j^+\ldots,n^-\right)=\left(-1\right)^n i\frac{[ij]^4}{[12]\,[23]\cdots[(n-1)\,n]\,[n1]} \tag{2-98}$$

$$A_n\left(1^\pm,\ldots,i^+,\ldots,j^+\ldots,n^+\right)=0 \tag{2-99}$$

The “maximally helicity violating” or MHV amplitudes are those with two negative and the rest positive helicity, the other non-zero amplitude is usually called anti-MHV. The origin of these names is due to the fact that at tree level the violation of the helicity conservation to the maximal possible extent, of course no Lorentz symmetry violation is involved. They are also known as Parke-Taylor amplitudes.

A proof of some of the Parke-Taylor amplitudes will be given in the subsection (2.4.2) in the context of the so called BCFW recursive relations.

We can derive the MHV amplitudes for processes with a pair of massless quark-antiquark

$$A_n\left(1_q^-,2^+,\ldots,i^-,\ldots,(n-1)^+,n_{\bar{q}}^+\right)=i\frac{\langle 1i\rangle^3\,\langle ni\rangle}{\langle 12\rangle\,\langle 23\rangle\cdots\langle (n-1)\,n\rangle\,\langle n1\rangle} \tag{2-100}$$

$$A_n\left(1_q^-,2^-,\ldots,i^+,\ldots,(n-1)^-,n_{\bar{q}}^+\right)=\left(-1\right)^{n-1} i\frac{[1i]^3\,[ni]}{[12]\,[23]\cdots[(n-1)\,n]\,[n1]} \tag{2-101}$$

Consider the first member of the sequence of MHV amplitudes for n gluons,

$$A_3^{\text{tree}}\left(1^-,2^-,3^+\right)=i\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle}., \tag{2-102}$$

For real momenta the momentum conservation has the following implication

$$p_1 + p_2 + p_3 = 0 \to s_{12} = s_{23} = s_{13} = 0. \tag{2-103}$$

Since the three point gluon amplitude is a very useful tool it is possible to extend to complex momenta in order that the all three channels could make sense. The procedure is the following: choose all three left-handed spinors to be proportional, $|1] = c_1 |3], |2] = c_2 |3]$, while the right-handed spinors are not proportional, but obey the relation, $c_1 |1\rangle + c_2 |2\rangle + |3\rangle = 0$, which follows from momentum conservation $p_1 + p_2 + p_3 = 0$ and from the momentum representation (2-44), (2-45). Then

$$[12] = [23] = [31] = 0 \tag{2-104}$$

while $\langle 12\rangle, \langle 23\rangle$ and $\langle 31\rangle$ are all novanishing[4]. For such a kinematic choice, the tree-level primitive amplitude for two negative helicities and one positive helicity, $A_3^{\rm tree}$, is no-nul, even though all momentum invariants $s_{jl}, j, l = 1, 2, 3$ vanish according to eq.(2-103). For three gluons, $A_3^{\rm tree}$ can be evaluated using the three-gluon vertex obtaining eq. (2-102). There is a class of complex momenta conjugate to eq. (2-104), for which

$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0 \tag{2-105}$$

while $[12], [23]$ and $[31]$ are all novanishing. By this kinematics the no-vanishing amplitude is the parity-conjugate three-point amplitude,

$$A_3^{\rm tree}\left(1^+, 2^+, 3^-\right) = -i\frac{[12]^4}{[12]\,[23]\,[31]}. \tag{2-106}$$

When the amplitude $A_3^{\rm tree}\left(1^-, 2^-, 3^+\right)$ appears in the 'wrong' kinematics (2-105), it must be set to zero, because more vanishing spinor products appear in the numerator than in the numerator.

In chapter 2.1 we studied the properties of primitive amplitudes for gluons and we did not prove the properties 3 and 4, namely the invariance under reverse permutations and the dual Ward Identity. Now, using the explicit form for these amplitudes, we show those statements.

- The property 3 can be seen by taking the MHV amplitude with n-external gluons. The Mathematica allows a very efficient automatization of that procedure. The following box is full self-explanatory and the check has been performed for 3, 4 and 6 external gluons and the Schouten's identity has been used many times.

```
n=3

In[570]:= gMHV[{1, 2, 3}, {1, 2}] + gMHV[{2, 1, 3}, {1, 2}]
Out[570]= 0

n=4

In[569]:= Schouten[gMHV[{1, 2, 3, 4}, {1, 2}] + gMHV[{2, 1, 3, 4}, {1, 2}] + gMHV[{2, 3, 1, 4}, {1, 2}] //
            Simplify, 1, 4, 2, 3]
Out[569]= 0

n=6:

In[589]:= Schouten[
           Schouten[
             Schouten[gMHV[{1, 2, 3, 4, 5, 6}, {1, 2}] + gMHV[{2, 1, 3, 4, 5, 6}, {1, 2}] +
                gMHV[{2, 3, 1, 4, 5, 6}, {1, 2}] + gMHV[{2, 3, 4, 1, 5, 6}, {1, 2}] +
                gMHV[{2, 3, 4, 5, 1, 6}, {1, 2}] // Simplify, 1, 5, 3, 4], 1, 6, 2, 3] // Simplify,
           1, 6, 3, 5]
Out[589]= 0
```

- For the property 2 consider any of the MHV amplitudes of section (2.3). The numerator is always even under the momenta labels exchange, because of the factors $\langle ij\rangle^4$. On the contrary in the denominator we have the product of n Lorentz invariant products, by their antisymmetry$\langle ab\rangle = -\langle ba\rangle$, the desired property is recovered.

2.4. BCFW recursion relation

The BCFW recursion relation uses the main ideas of the analyticity of S-matrix to reconstruct the full scattering amplitudes, this is performed by the previously made extension to the complex momenta. The extension of the scattering amplitudes to the complex plane allows in fact for reusing also nul amplitudes, which vanish for real momenta as well as to exploit the analytic properties of the corresponding functions of complex variables. BCFW introduce an algorithm to calculate efficiently, and in a recursive way, all tree-level scattering amplitudes for various theories under certain conditions. Since at tree level the singularities required by unitarity of the theory are simple poles in the two-particle and multi-particle kinematic invariants, precise recursions can be extracted starting from the smallest building blocks, namely three-point amplitudes, exactly those nul for real momenta[25].

2.4.1. Derivation of the recursion

Consider a color-ordered primitive amplitude $A\left(p_1,\ldots,p_n\right)$, and select two legs for special treatment; we define the $[j,l\rangle$ shift to be [39]

$$|j] \rightarrow |j] - z\,|l] \tag{2-107}$$

$$|l\rangle \rightarrow |l\rangle + z\,|j\rangle \tag{2-108}$$

where z is a complex paramenter. The shift leaves untouched $|j\rangle, |l]$, and the spinors for all the other particles in the process. Under that shift the corresponding momentum is

$$k_j^\mu \to k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j|\gamma^\mu|l]$$
$$k_l^\mu \to k_l^\mu(z) = k_l^\mu + \frac{z}{2}\langle j|\gamma^\mu|l]$$

Now, without loss of generality we apply the shift $[n,1\rangle$. We can shift non-adjacent particles but this would lead to recursion relations involving more terms. One shifts the two momenta as

$$\hat{p}_1(z) = p_1 + z\eta, \qquad \hat{p}_n(z) = p_n - z\eta. \tag{2-109}$$

These shifts are chosen in a particular form in order not to alter the momentum conservation condition. Furthermore, we would like to preserve the on-shell condition for particles 1 and n, which is possible if $p_1\eta = p_n\eta = 0$. In real Minkowsky space there are no solutions to these constraints but in complex Minkowsky space there are two solutions, $\eta = |1\rangle[n| + |n]\langle 1|$ and $\eta = |n\rangle[1| + |1]\langle n|$, where $p_i = |i\rangle[i| + |i]\langle i|, i = 1, n$, as usual.

By this prescription we define the complex function,

$$A(z) := A(\hat{p}_1, p_2, p_3, \ldots, \hat{p}_n) \tag{2-110}$$

where the external momenta are on shell but complex. In fact $\hat{p}_1^2(z) = \hat{p}_n^2(z) = 0$ for all values of z. Being the continuation of a tree level amplitude $A(z)$ is a rational function of z with only simple poles in these variables. By the polology theorem [2] the poles correspond to the exchanged virtual particles and the corresponding residues to the the coupling of such particles to all the spectrum of the theory, the physical amplitude is given by $A(0)$.

Let $\hat{P}_{ij} = p_i + \cdots + p_j$ the momentum flowing in a given propagator. There are three possibilities: either leg one or two belong to $\hat{P}_{ij}$ or both legs, or none, belong to $\hat{P}_{ij}$. It is only in the first case that $\hat{P}_{ij}$ depends of z (see fig. **2-10**), since in the other two cases, such dependence is either not present or cancels since $\hat{p}_1 + \hat{p}_n = p_1 + p_n$. Focussing on the first case and assuming for definiteness that the particle 1 belongs to $\hat{P}_{ij}$, we can write $\hat{P}_{ij}^\mu$ as

$$\hat{P}_{ij}^\mu = P_{ij}^\mu - \frac{z}{2}\langle 1|\gamma^\mu|n] \tag{2-111}$$

and the propagator

$$\frac{i}{\hat{P}_{ij}^2} = -\frac{z_{ij}}{P_{ij}^2}\frac{i}{z - z_{ij}}, \tag{2-112}$$

$$z_{ij} = \frac{P_{ij}^2}{\langle 1|P_{ij}|n]} \tag{2-113}$$

where z_{ij} is the solution of $\hat{P}_{ij}^2 = 0$.

The on-shell complex continued scattering amplitude $A(z)$ can be computed, for instance, by the usual Feynman rules. Momentum conservation suggests that both the momenta of external particles and the spinors of massless particles are linear functions of z. Consider the contour integral

$$\oint_C \frac{dz}{2\pi i}\frac{A(z)}{z},$$

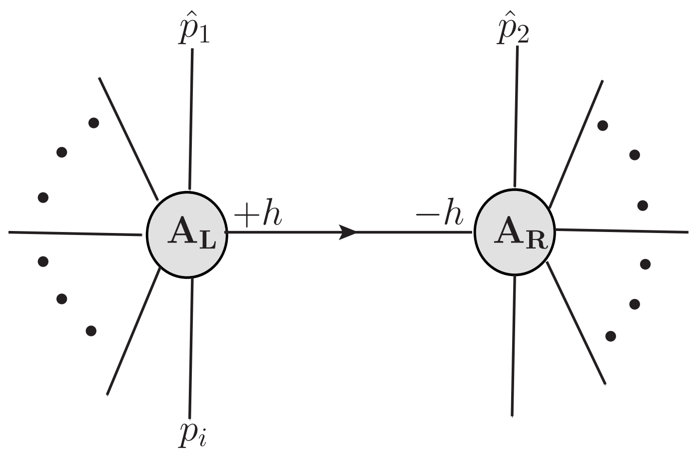


Figure **2-10**.: One of the recursive diagrams contributing to the BCFW recursion relation for a colour-ordered amplitude $A(1,\ldots,n)$. The particles with shifted momenta are adjacent - namely 1 and n

where the contour is taken around the circle at infinity. If $A(z) \to 0$ as $z \to \infty$, the contour integral vanishes and we obtain a relationship between the physical amplitude, at $z = 0$, and a sum over residues for the poles of $A(z)$, located at z_α[40],

$$\lim_{C\to\infty} \oint_C \frac{dz}{2\pi i}\frac{A(z)}{z} = \mathrm{Res}_{z\to 0}\left[\frac{A(z)}{z}\right] + \sum_{\text{poles } \alpha} \mathrm{Res}_{z\to z_\alpha}\frac{A(z)}{z} = 0,$$

$$A(0) = -\sum_{\text{poles } \alpha} \mathrm{Res}_{z\to z_\alpha}\frac{A(z)}{z} \tag{2-114}$$

To determine the residues at each pole, we use the general factorization properties that any amplitude must satisfy as an intermediate momentum K^μ goes on-shell, $K^2 \to 0$. In general, the residue is given by a product of lower-point on-shell amplitudes. To get the precise form of the contribution, using eq. (2-114), we need to evaluate the residue,

$$-\mathrm{Res}_{z\to z_\alpha}\left(\frac{1}{z}\frac{i}{\hat{P}_{ij}^2}\right) = \frac{i}{P_{ij}^2},$$

the final form of the tree level recursion relation is[20, 39]

$$A_n(1,2,\ldots,n) = \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}^{\text{Left}}\left(\widehat{1},2,\ldots,k,-\widehat{P}_{1,k}^{-h}\right)\frac{i}{P_{1,k}^2} A_{n-k+1}^{\text{Right}}\left(\widehat{P}_{1,k}^{h},k+1,\ldots,\widehat{n}\right) \tag{2-115}$$

Generally we have a recursive sum over diagrams, with legs 1 and n always appearing on opposite side of the pole (see fig. **2-11**). There is also a sum over the helicity h of the intermediate state. The squared momentum P_{ij}^2, is evaluated in the unshifted kinematics. The on-shell blocks tree amplitudes A_{Left} and A_{Right} are evaluated in kinematics that have been shifted by eq. (2-109),

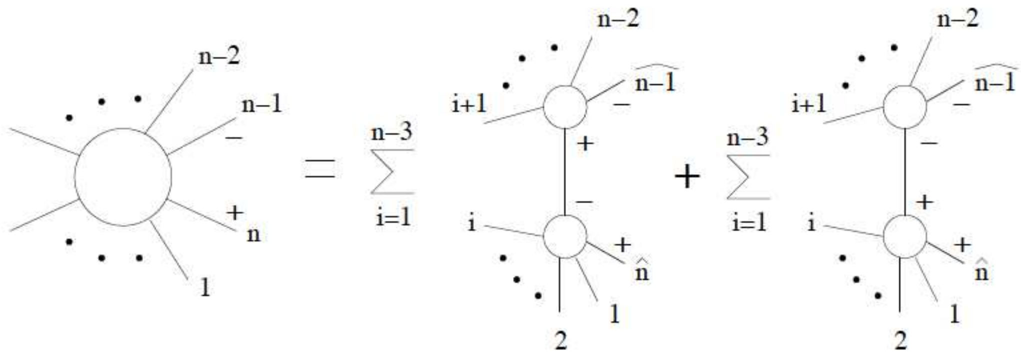


Figure **2-11**.: Pictorial representation of the recursion relation. Note that the difference between the terms in the two sums is just the helicity assignment of the internal line[20]

with $z = z_{ij}$, by definitions of residues and in agreement with the polology theorem. The shifted momenta for such kinematics are indicated by hats.

Recursive diagrams containing three-point amplitudes often vanish because the 'wrong' kinematics, in the sense explained in the section (2.3). In general, if a $[j, l\rangle$ shift is used, meaning that the momenta p_j and p_l are shifted, and the recursive diagram contains a three-subamplitude with two positive helicities, one of which is j, the the diagram vanishes. The reason is that the spinor $|j\rangle$ is unaffected by the shift, so its product with the spinor for the other external leg a in the three-point amplitude, $\langle ja\rangle$, remains non-vanishing. Therefore $[ja]$, and all of the left-handed spinor products, must vanish, and so the three-vertex with two helicities vanishes. Similarly, three-vertices with two negative helicities can also dropped, when one of the three legs is l.

There is one subtlety that should be clarified to evaluate the right-hand side in eq. (2-115). These amplitudes involve angle brackets and squared brackets of the complex momentum $\hat{P}_{ij}$. In our calculations, we will evaluate these brackets by assembling them into complete factors of the momentum $\hat{P}_{ij}$. To do this, we will need to relate the brackets $\left|-\hat{P}_{ij}\right\rangle$ and $\left|-\hat{P}_{ij}\right]$ in the amplitude on the left to $\left|\hat{P}_{ij}\right\rangle$ and $\left|\hat{P}_{ij}\right]$. It is consistent always to take[9]

$$\left|-\hat{P}_{ij}\right\rangle = i\left|\hat{P}_{ij}\right\rangle, \qquad \left|-\hat{P}_{ij}\right] = i\left|\hat{P}_{ij}\right],$$

one special circumstance should be noted. If the line on which the amplitude factorizes is a fermion propagator, the value of this propagator is

$$i\frac{|P_{ij}]\,\langle P_{ij}|}{P_{ij}^2}, \qquad i\frac{|P_{ij}\rangle\,[P_{ij}|}{P_{ij}^2}.$$

The one of the brackets in the left-hand amplitude is $|P_{ij}]$, not a $|-P_{ij}]$. To compensate for this, we need to add a factor $(-i)$ for a cut through a fermion propagator.

We use the following identities to compute any spinor product involving $\hat{P}_{1,k}$,

$$\left\langle \bullet \hat{P}_{1,k} \right\rangle = \frac{\langle \bullet \left| P_{1,k} \right| n]}{\left[\hat{P}_{1,k} n \right]} \tag{2-116}$$

$$\left[\hat{P}_{1,k} \bullet \right] = \frac{\langle 1 \left| P_{1,k} \right| \bullet]}{\left\langle 1 \hat{P}_{1,k} \right\rangle} \tag{2-117}$$

2.4.2. Proof of the MHV formula

As an application of the BCFW recursive relation we give here a proof of the MHV formula for n-gluon amplitudes. The proof will proceed by induction. The MHV amplitudes have been verified for $n = 3, 4$ MHV gluons amplitudes. Let us assume that the MHV formula is correct of the case $n = N - 1$ and use that hypothesis to evaluate the $n = N$ gluon MHV amplitude.
Without loss of generally, we choose the BCFW shift over particles 1 and n,

$$|n\rangle \to |n\rangle - z\,|1\rangle$$
$$|1] \to |1] + z\,|n]$$

With the BCFW shift, the color-ordered primitive amplitude takes the form,

$$\begin{aligned}
A_n\left(1^-, 2^+, \ldots, i^-, \ldots, n^+\right) &= \sum_{h=\pm} \sum_{k=2}^{n-2} A_{k+1}\left(\widehat{1}, 2, \ldots, k, -\widehat{K}_{1,k}^{-h}\right) \frac{i}{K_{1,k}^2} A_{n-k+1}\left(\widehat{K}_{1,k}^{h}, k+1, \ldots, \widehat{n}\right) \\
&= \sum_{h=\pm} \left\{ A_3\left(\widehat{1}^-, 2^+, -\widehat{K}_{1,2}^{-h}\right) \frac{i}{K_{1,2}^2} A_{n-2}\left(\widehat{K}_{1,2}^{h}, 3^+, \ldots,, i^-, \ldots, \widehat{n}^+\right) + \right. \\
&\quad + A_4\left(\widehat{1}^-, 2^+, 3^+, -\widehat{K}_{1,3}^{-h}\right) \frac{i}{K_{1,3}^2} A_{n-3}\left(\widehat{K}_{1,2}^{h}, 4^+, \ldots,, i^-, \ldots, \widehat{n}^+\right) + \\
&\quad \left. + \ldots + A_{n-1}\left(\widehat{1}^-, \ldots, i^-, \ldots, (n-2)^+, -\widehat{K}_{1,(n-2)}^{-h}\right) \frac{i}{K_{1,(n-2)}^2} A_4\left(\widehat{K}_{1,(n-2)}^{h}, (n-1)^+, \widehat{n}^+\right) \right\} \\
&= \left\{ A_3\left(\widehat{1}^-, 2^+, -\widehat{K}_{1,2}^{+}\right) \frac{i}{K_{1,2}^2} A_{n-2}\left(\widehat{K}_{1,2}^{-}, 3^+, \ldots,, i^-, \ldots, \widehat{n}^+\right) + \right. \\
&\quad \left. + A_{n-1}\left(\widehat{1}^-, \ldots, i^-, \ldots, (n-2)^+, -\widehat{K}_{1,(n-2)}^{+}\right) \frac{i}{K_{1,(n-2)}^2} A_4\left(\widehat{K}_{1,(n-2)}^{-}, (n-1)^+, \widehat{n}^+\right) \right\}
\end{aligned} \tag{2-118}$$

We choose the kinematics

$$s_{12} = 0 \to [21] = 0, \langle 12 \rangle \neq 0$$

In this kinematic the amplitude $A_3\left(1^-, 2^+, \hat{K}_{12}^{+}\right) = 0$ and the remaining amplitudes have the form $A\left(+ - - - - \ldots\right)$ or $A\left(- + + + + \ldots\right)$. Then this amplitude has only one BCFW diagram that contributes to the primitive amplitude,

$$\begin{aligned}
&A_n\left(1^-,2^+,\ldots,i^-,\ldots,n^+\right) = \\
&\quad = A_{n-1}\left(\widehat{1}^-,\ldots,i^-,\ldots,(n-2)^+,-\widehat{K}^+_{1,(n-2)}\right)\frac{i}{K^2_{1,(n-2)}}A_4\left(\widehat{K}^-_{1,(n-2)},(n-1)^+,\widehat{n}^+\right) \\
&\qquad = \left(-i\frac{\left\langle\widehat{1}i\right\rangle^4}{\left\langle\widehat{1}2\right\rangle\langle 23\rangle\cdots\left\langle(n-2),\widehat{K}_{1,(n-2)}\right\rangle\left\langle\widehat{K}_{1,(n-2)}\widehat{1}\right\rangle}\right)\frac{i}{K^2_{1,(n-2)}} \\
&\qquad\quad \times\left(-i\frac{\left[(n-1)\,\widehat{n}\right]^4}{\left[\widehat{K}_{1,(n-2)}\,(n-1)\right]\left[(n-1)\,\widehat{n}\right]\left[\widehat{n}\widehat{K}_{1,(n-2)}\right]}\right) \\
&\quad = -i\frac{1}{K^2_{1,(n-2)}}\frac{\left\langle\widehat{1}i\right\rangle^4\left[(n-1)\,\widehat{n}\right]^4}{\left\langle\widehat{1}2\right\rangle\langle 23\rangle\cdots\left\langle(n-2)\left|\widehat{K}_{1,(n-2)}\right|\widehat{n}\right]\left[(n-1)\,\widehat{n}\right]\left\langle\widehat{1}\left|\widehat{K}_{1,(n-2)}\right|(n-1)\right]} \\
&\quad = -i\frac{1}{\left[(n-1)\,n\right]\langle n\,(n-1)\rangle}\frac{\left\langle\widehat{1}i\right\rangle^4\left[(n-1)\,\widehat{n}\right]^4}{\left\langle\widehat{1}2\right\rangle\langle 23\rangle\cdots\langle(n-2)\left|(n-1)\right|\widehat{n}]\left[(n-1)\,\widehat{n}\right]\left\langle\widehat{1}\left|\widehat{n}\right|(n-1)\right]} \\
&\qquad\qquad = i\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle(n-2)\,(n-1)\rangle\langle(n-1)\,n\rangle\langle n1\rangle}
\end{aligned} \tag{2-119}$$

which is exactly the Parke-Taylor amplitude for the case of n legs. by induction, this formula applies for all n.

2.4.3. Examples

$A_6\left(1^-,2^-,3^-,4^+,5^+,6^+\right)$

We compute the amplitude $A_6\left(1^-,2^-,3^-,4^+,5^+,6^+\right)$.
We do the $[1,6\rangle$ shift

$$\begin{aligned}
|6\rangle &\to |6\rangle + z\,|1\rangle \\
|1] &\to |1] - z\,|6]
\end{aligned} \tag{2-120}$$

the BCFW recursive relation is given by

$$A_n\left(1,2,\ldots,n\right) = \sum_{h=\pm}\sum_{k=2}^{n-2}A_{k+1}\left(\widehat{1},2,\ldots,k,-\widehat{K}^{-h}_{1,k}\right)\frac{i}{K^2_{1,k}}A_{n-k+1}\left(\widehat{K}^h_{1,k},k+1,\ldots,\widehat{n}\right).$$

specifically to our case,

$$
\begin{aligned}
A_6\left(1^-,2^-,3^-,4^+,5^+,6^+\right) &= \sum_{h=\pm}\sum_{k=2}^{4} A_{k+1}\left(\hat{1}^-,\ldots,k,-\hat{K}_{1,k}^{-h}\right)\frac{i}{K_{1,k}^2}A_{n-k+1}\left(\hat{K}_{1,k}^{h},k+1,\ldots,\hat{6}\right)\\
&= \sum_{h=\pm} A_3\left(\hat{1}^-,2^-,-\hat{K}_{12}^{-h}\right)\frac{i}{K_{12}^2}A_5\left(\hat{K}_{1,k}^{h},3^-,4^+,5^+,\hat{6}^+\right)\\
&\qquad + A_4\left(\hat{1}^-,2^-,3^-,-\hat{K}_{123}^{-h}\right)\frac{i}{K_{123}^2}A_4\left(\hat{K}_{123}^{h},4^+,5^+,\hat{6}^+\right)\\
&\qquad + A_5\left(\hat{1}^-,2^-,3^-,4^+,-\hat{K}_{1234}^{-h}\right)\frac{i}{K_{1234}^2}A_3\left(\hat{K}_{1,234}^{h},5^+,\hat{6}^+\right)\\
&= A_3\left(\hat{1}^-,2^-,-\hat{K}_{12}^{+}\right)\frac{i}{K_{12}^2}A_5\left(\hat{K}_{1,k}^{-},3^-,4^+,5^+,\hat{6}^+\right)\\
&\quad + A_4\left(\hat{1}^-,2^-,3^-,-\hat{K}_{123}^{+}\right)\frac{i}{K_{123}^2}A_4\left(\hat{K}_{123}^{-},4^+,5^+,\hat{6}^+\right)\\
&\quad + A_5\left(\hat{1}^-,2^-,3^-,4^+,-\hat{K}_{1234}^{+}\right)\frac{i}{K_{1234}^2}A_3\left(\hat{K}_{1,234}^{-},5^+,\hat{6}^+\right),
\end{aligned}
$$

which corresponds to three BCFW diagrams (see figure **2-12**).

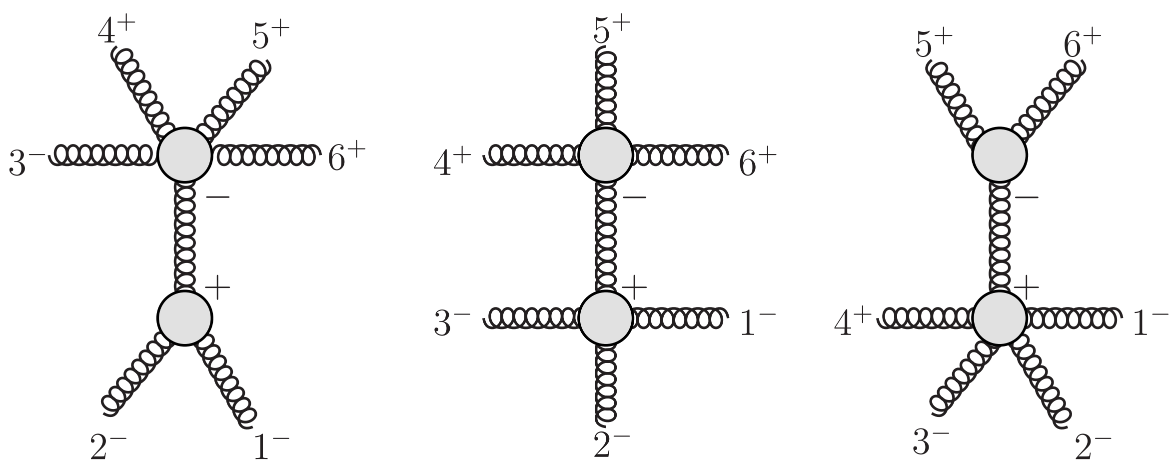


Figure **2-12**.: Configurations contributing to the six-gluon amplitude $A_6\left(1^-,2^-,3^-,4^+,5^+,6^+\right)$. The first and third BCFW diagram are related by symmetry and the second BCFW diagrams vanish for either helicity configuration of the internal line.

Consider in the first BCFW diagram the product of the tree-level amplitudes

$$A_3\left(\hat{1}^-,2^-,-\hat{K}_{12}^{+}\right)\frac{i}{K_{12}^2}A_5\left(\hat{K}_{1,k}^{-},3^-,4^+,5^+,\hat{6}^+\right) = -\frac{i\langle\hat{1}|2\rangle^3\langle\hat{K}|3\rangle^3}{s_{12}\langle 3|4\rangle\langle 4|5\rangle\langle 5|\hat{6}\rangle\langle\hat{K}|1\rangle\langle\hat{K}|2\rangle\langle\hat{K}|\hat{6}\rangle} \quad (2\text{-}121)$$

and writing $\langle\bullet K\rangle$ as

$$\left\langle\bullet\hat{K}\right\rangle = -\frac{\langle\bullet\,|3+4+5|\,6]}{\left[\hat{K}6\right]} \quad (2\text{-}122)$$

taking the shift of eq. (2-120) and using momentum conservation

$$
\begin{aligned}
&A_3\left(\hat{1}^-,2^-,-\hat{K}_{12}^+\right)\frac{i}{K_{12}^2}A_5\left(\hat{K}_{1,k}^-,3^-,4^+,5^+,\hat{6}^+\right)\\
&\qquad=\frac{i(\langle 3|4|6]+\langle 3|5|6])^3}{(s_{12}+s_{16}+s_{26})\,[2|1][6|1](\langle 5|3|2]+\langle 5|4|2])\langle 3|4\rangle\langle 4|5\rangle}\\
&\qquad\qquad=-\frac{i}{\langle 5|3+4|2]}\frac{\langle 3|4+5|6]^3}{[6|1][1|2]\langle 3|4\rangle\langle 4|5\rangle s_{612}}
\end{aligned}
\tag{2-123}
$$

here $s_{612}=s_{12}+s_{16}+s_{26}$.

Using the parity operation, as explained in the section (2.2.3), in eq. (2-123)

$$
A_3\left(\hat{1}^+,2^+,-\hat{K}_{12}^-\right)\frac{i}{K_{12}^2}A_5\left(\hat{K}_{12}^+,3^+,4^-,5^-,\hat{6}^-\right)=\frac{i}{[5|3+4|2\rangle}\frac{[3|4+5|6\rangle^3}{\langle 6|1\rangle\,\langle 1|2\rangle\,[3|4]\,[4|5]\,s_{612}}. \tag{2-124}
$$

Moreover

$$
\begin{aligned}
&A_5\left(\hat{1}^-,2^-,3^-,4^+,-\hat{K}_{1234}^+\right)\frac{i}{K_{1234}^2}A_3\left(\hat{K}_{1,234}^-,5^+,\hat{6}^+\right)=\\
&\qquad=A_5\left(\hat{1}^-,2^-,3^-,4^+,\hat{K}_{56}^+\right)\frac{i}{K_{56}^2}A_3\left(-\hat{K}_{56}^-,5^+,\hat{6}^+\right)
\end{aligned}
\tag{2-125}
$$

and we may reuse the previous computed amplitudes by flipping as follows

$$
\begin{aligned}
6&\to 1\\
5&\to 2\\
4&\to 3\\
3&\to 4\\
2&\to 5\\
1&\to 6
\end{aligned}
$$

obtaining

$$
\begin{aligned}
A_5\left(\hat{1}^-,2^-,3^-,4^+,-\hat{K}_{1234}^+\right)\frac{i}{K_{1234}^2}A_3\left(\hat{K}_{1,234}^-,5^+,\hat{6}^+\right)&=\frac{i}{[2|4+3|5\rangle}\frac{[4|3+2|1\rangle^3}{\langle 6|1\rangle\,\langle 6|5\rangle\,[4|3]\,[3|2]\,s_{561}}\\
&=-\frac{i}{\langle 5|3+4|2]}\frac{\langle 1|2+3|4]^3}{[2|3]\,[3|4]\,\langle 5|6\rangle\,\langle 6|1\rangle\,s_{561}}.
\end{aligned}
\tag{2-126}
$$

Adding both contributions (2-123) and (2-126) we get

$$
A_6\left(1^-,2^-,3^-,4^+,5^+,6^+\right)=-\frac{i}{\langle 5|3+4|2]}\left(\frac{\langle 1|2+3|4]^3}{[3|4]\,[2|3]\,\langle 5|6\rangle\,\langle 6|1\rangle\,s_{561}}+\frac{\langle 3|4+5|6]^3}{[6|1][1|2]\langle 3|4\rangle\langle 4|5\rangle s_{612}}\right)
\tag{2-127}
$$

Another six point amplitude $A_6\left(1^+,2^-,3^+,4^-,5^+,6^-\right)$

$$A_6\left(1^+,2^-,3^+,4^-,5^+,6^-\right) = A_3\left(\hat{1}^+,2^-,-\hat{K}^+_{12}\right)\frac{i}{K^2_{12}}A_5\left(\hat{K}^-_{1,2},3^+,4^-,5^+,\hat{6}^-\right)$$
$$+A_4\left(\hat{1}^+,2^-,3^+,-\hat{K}^+_{123}\right)\frac{i}{K^2_{123}}A_4\left(\hat{K}^-_{123},4^-,5^+,\hat{6}^-\right)$$
$$+A_5\left(\hat{1}^+,2^-,3^+,4^-,-\hat{K}^+_{1234}\right)\frac{i}{K^2_{1234}}A_3\left(\hat{K}^-_{1,234},5^+,\hat{6}^-\right)$$

From the first BCFW diagram we obtain,

$$A_3\left(\hat{1}^+,2^-,-\hat{K}^+_{12}\right)\frac{i}{K^2_{12}}A_5\left(\hat{K}^-_{1,2},3^+,4^-,5^+,\hat{6}^-\right) = -\frac{i\langle 2|6\rangle^4[3|5]^4}{\left(s_{12}+s_{16}+s_{26}\right)\langle 6|1\rangle\langle 1|2\rangle[3|4][4|5]\langle 2|3+4|5]\langle 6|4+5|3]} \tag{2-128}$$

We can obtain the third contribution by using the parity operation in eq. (2-128)

$$A_3\left(\hat{1}^-,2^+,-\hat{K}^-_{12}\right)\frac{i}{K^2_{12}}A_5\left(\hat{K}^+_{1,2},3^-,4^+,5^-,\hat{6}^+\right)$$
$$= \frac{i\,[2|6]^4\,\langle 3|5\rangle^4}{\left(s_{12}+s_{16}+s_{26}\right)[6|1]\,[1|2]\,\langle 3|4\rangle\,\langle 4|5\rangle\,[2|3+4|5\rangle\,[6|4+5|3\rangle} \tag{2-129}$$

by the flip

$$6\to 1$$
$$5\to 2$$
$$4\to 3$$
$$3\to 4$$
$$2\to 5$$
$$1\to 6$$

we find

$$A_5\left(\hat{1}^+,2^-,3^+,4^-,-\hat{K}^+_{1234}\right)\frac{i}{K^2_{1234}}A_3\left(\hat{K}^-_{1,234},5^+,\hat{6}^-\right) =$$
$$= -\frac{i\,[1|5]^4\,\langle 2|4\rangle^4}{\left(s_{65}+s_{16}+s_{15}\right)\langle 2|3\rangle\,\langle 3|4\rangle\,[5|6]\,[6|1]\,[5|4+3|2\rangle\,[1|3+2|4\rangle}.$$

Finally, the second contribution,

$$A_4\left(\hat{1}^+,2^-,3^+,-\hat{K}^-_{123}\right)\frac{i}{K^2_{123}}A_4\left(\hat{K}^+_{123},4^-,5^+,\hat{6}^-\right) =$$
$$= -\frac{i[1|3]^4\langle 4|6\rangle^4}{\left(s_{12}+s_{13}+s_{23}\right)[1|2][2|3]\langle 4|5\rangle\langle 5|6\rangle\langle 6|1+2|3]\langle 4|2+3|1]}$$

The total color-ordered primitive amplitudes is given by

$$\begin{aligned} A_6\left(1^+,2^-,3^+,4^-,5^+,6^-\right) = &-\frac{i\langle 2|6\rangle^4[3|5]^4}{\left(s_{12}+s_{16}+s_{26}\right)\langle 6|1\rangle\langle 1|2\rangle[3|4][4|5]\langle 2|3+4|5]\langle 6|4+5|3]} \\ &-\frac{i[1|3]^4\langle 4|6\rangle^4}{\left(s_{12}+s_{13}+s_{23}\right)[1|2][2|3]\langle 4|5\rangle\langle 5|6\rangle\langle 6|1+2|3]\langle 4|2+3|1]} \\ &-\frac{i\,[1|5]^4\,\langle 2|4\rangle^4}{\left(s_{65}+s_{16}+s_{15}\right)\langle 2|3\rangle\,\langle 3|4\rangle\,[5|6]\,[6|1]\,[5|4+3|2\rangle\,[1|3+2|4\rangle} \end{aligned} \quad (2\text{-}130)$$

These two expressions are in agreement with [41] and even much simpler compare to them. Actually the relative simplicity is even more striking for higher number of external legs.

However, these results can be obtained in a simple way by using mathematica with the packager `s@m` (see appendix C). These computations are showed in details in appendix D.

It is also available a calculation of seven partons amplitudes based on BCFW on-shell recursive relations in ([43]), the calculation proceeds essentially on the same lines described in this paragraph.

3. One-loop amplitudes

The tree-level amplitudes studied before do not give relevant information when we compare theory with experimentation, therefore is necessary to go to higher orders. In this chapter we are going to study how to compute one-loop amplitudes using analytic methods. As before, is important to establish a relation among kinematic and color information, for this, we consider the color decomposition to one-loop. To obtain kinematic informations we review many ways to compute one-loop primite amplitudes as passarino-Veltman decomposition, optical theorem and unitarity of the S-matrix. We focus in the unitarity of the S-matrix by studing the contributions that coming from box, triangle and bubble configurations, the tadpole configuration does not give any contributions because we only consider internal massless loop.

3.1. Color-Ordered amplitudes at one-loop level

In the chapter 2 we studied the color-ordered amplitudes at tree level, following the same procedure for the case of amplitudes at one-loop, we obtain [27, 28, 31],

$$\begin{aligned}
\mathcal{A}_n^{1-loop}\left(\{k_{i,}, h_i, a_i\}\right) = \\
&= g^n \left[\sum_{\sigma \in S_n/Z_n} N_c \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right) A_{n;1}\left(\sigma\left(1^{\lambda_1}\right), \ldots, \sigma\left(n^{\lambda_n}\right)\right) \right. \\
&+ \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}\right) \mathrm{Tr}\left(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}\right) A_{n;c}\left(\sigma\left(1^{\lambda_1}\right), \ldots, \sigma\left(n^{\lambda_n}\right)\right)
\end{aligned} \tag{3-1}$$

where $A_{n;c}$ are the partial amplitudes that can be obtained from the primitive amplitudes $A_{n;1}$ by summing over all its permutations, Z_n and $S_{n;c}$ (previosly defined) that leave the corresponding single and double trace structures invariant, and $\lfloor m \rfloor$ is the greatest integer less than or equal to m.

The primitive amplitudes $A_{n;1}$ can be computed using the color-ordered Feyman rules of section 2.1.1.

3.1.1. Color factors for $\mathcal{A}_4^{1-loop}(1_g, 2_g, 3_g, 4_g)$

To get a better understanding about the color factors we consider the process of five gluons in pure Yang-Mills. We are going to extract naively the color factors using properties of the generators and then we compare our result with eq. (3-1).
Consider the color-ordered Feynman diagram in fig. **3-5**,

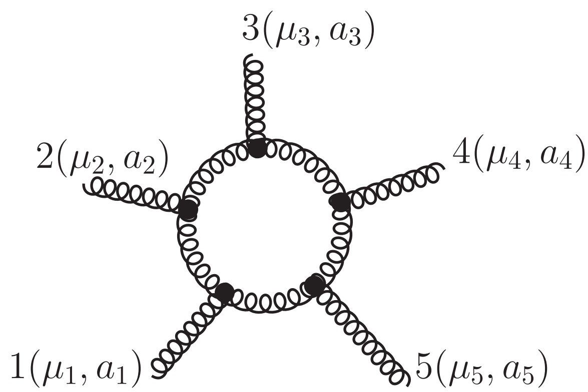


Figure **3-1**.: Color-ordered Feynman diagram for a process of five gluons in pure YM.

We write down the amplitude for this diagram and separate color factors and kinematic information,

$$\mathcal{A} = f^{a_1 c_1 b_2} f^{a_2 b_2 c_3} f^{a_3 c_3 b_4} f^{a_4 b_4 c_5} f^{a_5 c_5 c_1} A'$$

we are interested in how the color factor works. Then, using the following identities,

$$\mathrm{Tr}\left\{T^{a_1} T^{a_2} T^{a_I}\right\} \mathrm{Tr}\left\{T^{a_I} T^{a_3} T^{a_4}\right\} = \mathrm{Tr}\left\{T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5}\right\} \tag{3-2}$$

$$\mathrm{Tr}\left\{T^{a_1} \dots T^{a_m} T^{a_I} T^{a_2} \dots T^{a_3} T^{a_I} T^{a_{m+1}} \dots T^{a_n}\right\} = \mathrm{Tr}\left\{T^{a_1} \dots T^{a_m} T^{a_{m+1}} \dots T^{a_n}\right\} \mathrm{Tr}\left\{T^{a_2} \dots T^{a_3}\right\} \tag{3-3}$$

$$\mathrm{Tr}\left\{T^{a_1} \dots T^{a_m} T^{a_I} T^{a_I} T^{a_{m+1}} \dots T^{a_n}\right\} = N_c \mathrm{Tr}\left\{T^{a_1} \dots T^{a_n}\right\} \tag{3-4}$$

and the product of constant structures,

$$\begin{aligned} & f^{a_1 c_1 b_2} f^{a_2 b_2 c_3} f^{a_3 c_3 b_4} f^{a_4 b_4 c_5} f^{a_5 c_5 c_1} = f^{a_1 b_2 c_1} f^{a_2 b_2 c_3} f^{a_3 b_4 c_3} f^{a_4 b_4 c_5} f^{a_5 c_5 c_1} \\ & = -i \mathrm{Tr}\left\{\left[T^{a_1}, T^{b_2}\right] T^{c_1}\right\} \mathrm{Tr}\left\{\left[T^{a_2}, T^{b_2}\right] T^{c_3}\right\} \mathrm{Tr}\left\{\left[T^{a_3}, T^{b_4}\right] T^{c_3}\right\} \mathrm{Tr}\left\{\left[T^{a_4}, T^{b_4}\right] T^{c_5}\right\} \mathrm{Tr}\left\{\left[T^{a_5}, T^{c_5}\right] T^{c_1}\right\} \end{aligned} \tag{3-5}$$

Expanding the commutator and using identities (3-2), (3-3) and (3-4),

$$
\begin{aligned}
&-if^{a_1c_1b_2}f^{a_2b_2c_3}f^{a_3c_3b_4}f^{a_4b_4c_5}f^{a_5c_5c_1}= \\
&= N_c\left[\mathrm{Tr}\left\{T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_4}T^{a_3}T^{a_2}T^{a_1}\right\}\right] \\
&+\mathrm{Tr}\left\{T^{a_1}T^{a_2}\right\}\left[\mathrm{Tr}\left\{T^{a_3}T^{a_4}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_4}T^{a_3}\right\}\right]+\mathrm{Tr}\left\{T^{a_1}T^{a_3}\right\}\left[\mathrm{Tr}\left\{T^{a_2}T^{a_4}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_4}T^{a_2}\right\}\right] \\
&+\mathrm{Tr}\left\{T^{a_1}T^{a_4}\right\}\left[\mathrm{Tr}\left\{T^{a_2}T^{a_3}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_3}T^{a_2}\right\}\right]+\mathrm{Tr}\left\{T^{a_1}T^{a_5}\right\}\left[\mathrm{Tr}\left\{T^{a_2}T^{a_3}T^{a_4}\right\}-\mathrm{Tr}\left\{T^{a_4}T^{a_3}T^{a_2}\right\}\right] \\
&+\mathrm{Tr}\left\{T^{a_2}T^{a_3}\right\}\left[\mathrm{Tr}\left\{T^{a_1}T^{a_4}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_4}T^{a_1}\right\}\right]+\mathrm{Tr}\left\{T^{a_2}T^{a_4}\right\}\left[\mathrm{Tr}\left\{T^{a_1}T^{a_3}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_3}T^{a_1}\right\}\right] \\
&+\mathrm{Tr}\left\{T^{a_2}T^{a_5}\right\}\left[\mathrm{Tr}\left\{T^{a_1}T^{a_3}T^{a_4}\right\}-\mathrm{Tr}\left\{T^{a_4}T^{a_3}T^{a_1}\right\}\right]+\mathrm{Tr}\left\{T^{a_3}T^{a_4}\right\}\left[\mathrm{Tr}\left\{T^{a_1}T^{a_2}T^{a_5}\right\}-\mathrm{Tr}\left\{T^{a_5}T^{a_2}T^{a_1}\right\}\right] \\
&+\mathrm{Tr}\left\{T^{a_3}T^{a_5}\right\}\left[\mathrm{Tr}\left\{T^{a_1}T^{a_2}T^{a_4}\right\}-\mathrm{Tr}\left\{T^{a_4}T^{a_2}T^{a_1}\right\}\right]+\mathrm{Tr}\left\{T^{a_4}T^{a_5}\right\}\left[\mathrm{Tr}\left\{T^{a_1}T^{a_2}T^{a_3}\right\}-\mathrm{Tr}\left\{T^{a_3}T^{a_2}T^{a_1}\right\}\right] \\
&+\mathrm{Tr}\left\{T^{a_1}\right\}\left(\mathrm{Tr}\left\{T^{a_4}T^{a_5}T^{a_2}T^{a_3}\right\}-\mathrm{Tr}\left\{T^{a_3}T^{a_2}T^{a_5}T^{a_4}\right\}\right) \\
&+\mathrm{Tr}\left\{T^{a_2}\right\}\left(\mathrm{Tr}\left\{T^{a_1}T^{a_5}T^{a_4}T^{a_3}\right\}-\mathrm{Tr}\left\{T^{a_3}T^{a_4}T^{a_5}T^{a_1}\right\}\right) \\
&+\mathrm{Tr}\left\{T^{a_3}\right\}\left(\mathrm{Tr}\left\{T^{a_2}T^{a_1}T^{a_5}T^{a_4}\right\}-\mathrm{Tr}\left\{T^{a_4}T^{a_5}T^{a_1}T^{a_2}\right\}\right) \\
&+\mathrm{Tr}\left\{T^{a_4}\right\}\left(\mathrm{Tr}\left\{T^{a_2}T^{a_1}T^{a_5}T^{a_3}\right\}-\mathrm{Tr}\left\{T^{a_3}T^{a_5}T^{a_1}T^{a_2}\right\}\right) \\
&+\mathrm{Tr}\left\{T^{a_5}\right\}\left(\mathrm{Tr}\left\{T^{a_2}T^{a_1}T^{a_4}T^{a_3}\right\}-\mathrm{Tr}\left\{T^{a_3}T^{a_4}T^{a_1}T^{a_2}\right\}\right)
\end{aligned}
\tag{3-6}
$$

Finally, the amplitude can be written in compact form as,

$$
\begin{aligned}
\mathcal{A}_4 = &\sum_{\sigma\in S_5/Z_5} N_c\mathrm{Tr}\left(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}\right)A_{5;1}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right) \\
&+\sum_{\sigma\in S_5/Z_4}\mathrm{Tr}\left(T^{a_{\sigma(1)}}\right)\mathrm{Tr}\left(T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}\right)A_{5;2}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right) \\
&+\sum_{\sigma\in S_5/Z_4}\mathrm{Tr}\left(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\right)\mathrm{Tr}\left(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}\right)A_{5;3}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right)
\end{aligned}
\tag{3-7}
$$

where the amplitudes $A_{5;2}$ and $A_{5;3}$ are obtained from $A_{5;1}$ as we mentioned above.
This result is in agreement with eq. (3-1)

3.2. Passarino-Veltman reduction

When we do processes to one-loop, integrals appear as the following

$$
I_n\left[f(l)\right] = -i\left(4\pi\right)^{D/2}\int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{\left(l^2-m_0^2\right)\left((l+q_1)^2-m_1^2\right)\cdots\left((l+q_{n-1})^2-m_{n-1}^2\right)}
\tag{3-8}
$$

where

$$
q_i = p_1+p_2+\ldots+p_i,
\tag{3-9}
$$

p_i being the external momenta (in $D=4$ dimensions). $D=4-2\epsilon$ is the number of dimensions in which we perform the loop integral in order to regularize either ultraviolet or infrared divergencies. $f(l)$ contains all information from the loop momentum i.e. powers of loop momentum.

If we consider $f(l) = 1$ we obtain *the scalar master integrals* (see appendix G).

$$I_n[1] = -i(4\pi)^{D/2} \int \frac{d^D l}{(2\pi)^D} \frac{1}{\left(l^2 - m_0^2\right)\left((l+q_1)^2 - m_1^2\right)\cdots\left((l+q_{n-1})^2 - m_{n-1}^2\right)} \tag{3-10}$$

Integral reduction [7, 8, 47] is a clearly defined procedure for expressing any one-loop Feynman integral as a linear combination of scalar boxes, scalar triangles, scalar bubbles, and scalar tadpoles, with rational coefficients:

$$A^{\text{1-loop}} = \sum_n \sum_{K_r} c_n(K) I_n(K) \tag{3-11}$$

In four dimensions, n ranges from 1 to 4.
Additionally, in Passarino-Veltman reduction [7, 46], we work in $D = 4 - 2\epsilon$ dimensions and the coefficients of the loop integral functions depend on the dimensional regulator ϵ. Rational terms develop when ϵ-dependent pieces of the coefficients multiply poles in ϵ from the loop integral. The tadpole contributions with $n = 1$ arise only with internal masses. If we keep higher order contribution in ϵ, we find that the pentagons ($n = 5$) are independent as well.

If we consider $f(l) = l^\mu$, one power of loop momentum in numerator

$$I_n[l^\mu] = -i(4\pi)^{D/2} \int \frac{d^D l}{(2\pi)^D} \frac{l^\mu}{\left(l^2 - m_0^2\right)\left((l+q_1)^2 - m_1^2\right)\cdots\left((l+q_{n-1})^2 - m_{n-1}^2\right)} \tag{3-12}$$

the result for this integral must be constructed from the vectors $p_1, \ldots, p_{n-1}$, (by momentum conservation $p_1 + p_2 + \cdots + p_{n-1} = -p_n$).

$$I_n[l^\mu] = \sum_{i=1}^{n-1} C_{n;i} p_i^\mu \tag{3-13}$$

Contracting both sides with p_j^μ,

$$I_n[l \cdot p_j] = -i(4\pi)^{D/2} \int \frac{d^D l}{(2\pi)^D} \frac{l \cdot p_j}{\left(l^2 - m_0^2\right)\left((l+q_1)^2 - m_1^2\right)\cdots\left((l+q_{n-1})^2 - m_{n-1}^2\right)} = \sum_{i=1}^{n-1} C_{n;i} \triangle^{ij} \tag{3-14}$$

with $\triangle^{ij} = p_i \cdot p_j$ is the “Gram” matrix.
Since $p_j = q_j - q_{j-1}$ (with $q_0 = 0$) we can write the numerator of the integral as,

$$l \cdot p_j = \frac{1}{2}\left(\left((l+q_j)^2 - m_j^2\right) - \left((l+q_{j-1})^2 - m_{j-1}^2\right) + m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2\right) \tag{3-15}$$

this is the Passarino-Veltman reduction formula[7].
Here the terms $\left((l+q_j)^2 - m_j^2\right)$ and $\left((l+q_{j-1})^2 - m_{j-1}^2\right)$ in the numerator can be used to cancel the j^{th} and $(j-1)^{th}$ propagators respectively and so we end with a set of $n-1$ linear equations

for the coefficients $C_{n;i}$.

$$\sum_{i=1}^{n-1} C_{n;i}\triangle^{ij} = \frac{1}{2}\left(I_{n-1}^{(j)}[1] - I_{n-1}^{(j-1)}[1] + \left(m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2\right) I_n[1]\right) \tag{3-16}$$

$$C_{n;i} = \frac{1}{2}\sum_j \triangle_{ij}^{-1}\left(I_{n-1}^{(j)}[1] - I_{n-1}^{(j-1)}[1] + \left(m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2\right) I_n[1]\right) \tag{3-17}$$

eq. (3-17) represents the set of linear equations.

Now we consider $f(l) = l^\mu l^v$, two powers of loop momentum in numerator[12]. The integral is a rank two tensor which can be formed out of the outer products of external momenta $p_i^\mu p_j^v$ and the metric $g^{\mu\nu}$,

$$\begin{aligned} I_n[l^\mu l^\nu] &= -i(4\pi)^{D/2}\int \frac{d^D l}{(2\pi)^D}\frac{l^\mu l^\nu}{(l^2 - m_0^2)\left((l+q_1)^2 - m_1^2\right)\cdots\left((l+q_{n-1})^2 - m_{n-1}^2\right)} \\ &= C_{n:00}g^{\mu\nu} + \sum_{i=1}^{n-1} C_{n;i}p_i^\mu p_j^\nu \end{aligned} \tag{3-18}$$

the first equation can be derived by contracting both sides with $g^{\mu\nu}$,

$$\begin{aligned} I_n\left[l^2\right] &= -i(4\pi)^{D/2}\int \frac{d^D l}{(2\pi)^D}\frac{l^2}{(l^2 - m_0^2)\left((l+q_1)^2 - m_1^2\right)\cdots\left((l+q_{n-1})^2 - m_{n-1}^2\right)} \\ &= C_{n:00}D + \sum_{i=1}^{n-1} C_{n;i}\Delta^{ij} \end{aligned} \tag{3-19}$$

the other equations are obtained by contracting both sides with p_i, p_j and using eq. (3-17).

For $f(l) = l^\mu l^v l^\rho$ and $f(l) = l^\mu l^v l^\rho l^\sigma$, more power of l we follow the same procedure,

$$I_n[l^\mu l^v l^\rho] = \sum_{i=4}^{4} C_{n;00i}g^{\{\mu\nu}p_i^{\rho\}} + \sum_{i,j,k=4}^{4} C_{n;ijk}p_i^{\{\mu}p_j^\nu p_k^{\rho\}} \tag{3-20}$$

to obtain a set of linear equations for the coefficients $C_{n;00i}$ or $C_{n;ijk}$ we need to contract with $g^{\mu\nu}p^\rho$ or with $p_r^\mu p_s^\nu p_t^\rho$.

And, for four powers of loop momentum we have,

$$I_n[l^\mu l^v l^\rho l^\sigma] = C_{n;0000}g^{\{\mu\nu}p_i^{\rho\sigma\}} + \sum_{i,j=1}^{4} C_{n;00ij}g^{\{\mu\nu}p_i^\rho p_j^{\rho\}} + \sum_{i,j,k,h=1}^{4} C_{n;ijk,h}p_i^{\{\mu}p_j^\nu p_k^\rho p_h^{\sigma\}} \tag{3-21}$$

Here we need to contract with $g^{\mu\nu}g^{\rho\sigma}, g^{\mu\nu}p_r^\rho p_s^\sigma$ and $p_r^\mu p_s^\nu p_t^\rho p_u^\sigma$ in order to project out the coefficients $C_{n;0000}, C_{n;00ij}$ and $C_{n;ijkh}$

We list the necessary master integrals in appendix G

3.3. Unitarity method

The "unitarity method" started as a framework for one-loop calculations. Instead of the explicit set of loop Feynman diagrams, the basic reference point is the linear expansion of the amplitude in a basis of "master integrals", multiplied by coefficients that are rational contributions of the kinematic variables. The point is that the most difficult part of the calculation, namely integration over the loop momentum, can be done once and for all, with explicit calculation of the master integrals. The master integrals contain all the logarithmic functions. It then remains to find their coefficients[8].

If an amplitude is uniquely determined by its branch cuts, it is said to be "cut-constructible". All one-loop amplitudes are cut-constructible in dimensional regularization, provided that the full dimensional dependence is kept in evaluating the branch cut. Each master integral has different branch cut, uniquely indentified by its logarithmic arguments. Therefore, the decomposition in master integrals can be used to solve for their coefficients separately using analytic properties. It is not necessary to reconstruct the amplitude from the cut in a traditional way from a dispersion integral. Rather, we overlay information from various cuts separately[8].

Unitarity cuts can be "generalized" in the sense of putting a different number of propagators on-shell. This operation selects different kinds of singularities of the amplitude; they are not physical momentum channels like ordinary cuts and do not have an interpretation relating to the unitarity of the S-matrix.

3.3.1. Optical Theorem

The optical theorem is a straightforward consequence of the unitarity of the S-matrix: $S^\dagger S = 1$. Inserting $S = 1 + iT$, where T is the interaction matrix[1, 3, 8],

$$-i\left(T - T^\dagger\right) = T^\dagger T \tag{3-22}$$

Let us take the matrix element of this equation between two particles states $|\boldsymbol{p}_1\boldsymbol{p}_2\rangle$ and $|\boldsymbol{k}_1\boldsymbol{k}_2\rangle$. To evaluate the right-hand side, insert a complete set of intermediate states

$$\left\langle \boldsymbol{p}_1\boldsymbol{p}_2 \left| T^\dagger T \right| \boldsymbol{k}_1\boldsymbol{k}_2 \right\rangle = \sum_n \prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3}\frac{1}{2E_i}\left\langle \boldsymbol{p}_1\boldsymbol{p}_2 \left| T^\dagger \right| \{\boldsymbol{q}_i\}\right\rangle \langle\{\boldsymbol{q}_i\}\left|T\right|\boldsymbol{k}_1\boldsymbol{k}_2\rangle \tag{3-23}$$

Now express the T-matrix elements as invariant matrix elements $\mathcal{A}$ times for 4-momentum-convervation delta functions. Identity (3-22) then becomes

$$\begin{aligned} & -i\left[\mathcal{A}\left(k_1k_2 \to p_1p_2\right) - \mathcal{A}^*\left(p_1p_2 \to k_1k_2\right)\right] = \\ & = \sum_n \prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3}\frac{1}{2E_i}\mathcal{A}^*\left(p_1p_2 \to \{q_i\}\right)\mathcal{A}\left(k_1k_2 \to \{q_i\}\right)(2\pi)^4\delta^4\left(k_1 + k_2 - \sum_i q_i\right) \end{aligned} \tag{3-24}$$

times an overall delta function $(2\pi)^4\delta^4\left(k_1 + k_2 - p_1 - p_2\right)$. Let us abbreviate $k_i, p_i, i = 1.2$ as,

$$|\boldsymbol{p}_1\boldsymbol{p}_2\rangle = |b\rangle \tag{3-25}$$

$$|\boldsymbol{k}_1\boldsymbol{k}_2\rangle = |a\rangle \tag{3-26}$$

$$|\{\boldsymbol{q}_i\}\rangle = |f\rangle \tag{3-27}$$

eq. (3-24) takes the form,

$$-i\left[\mathcal{A}\left(a\to b\right)-\mathcal{A}^{*}\left(b\to a\right)\right]=\sum_{f}\int d\Pi_{f}\,\mathcal{A}^{*}\left(b\to f\right)\mathcal{A}\left(a\to f\right)\left(2\pi\right)^{4}\delta^{4}\left(k_{1}+k_{2}-\sum_{i}q_{i}\right) \tag{3-28}$$

where $d\Pi_f = \prod_{i=1}^{n}\int\frac{d^3q_i}{(2\pi)^3}\frac{1}{2E_i}$ and the sum is over all posible sets f of final-states particles. Although we have so far assumed that a and b are two-particle states, they could equally well be one-particle or multiparticle asymptotic states.

For the important special case of forward scattering, we can set $p_i = k_i$ to obtain a simpler identity, shown pictorially in fig. **3-2**. Finally, the standard form of the optical theorem

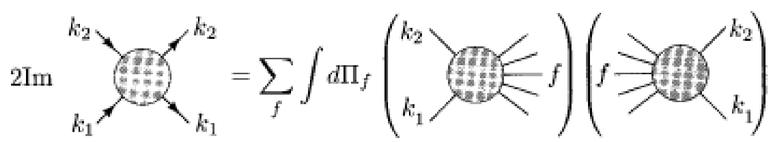


Figure **3-2**.: The optical theorem: the imaginary part of a forward scattering amplitudes arises from a sum of contributions from all posible intermediate state particles[1]

$$2\mathrm{Im}\mathcal{A}^{1-loop}\left(k_1k_2\to k_1k_2\right)=\sum_f\int d\Pi_f\,\mathcal{A}^{tree*}\left(k_1k_2\to f\right)\mathcal{A}^{tree}\left(k_1k_2\to f\right) \tag{3-29}$$

where we see that the imaginarity part of the one-loop amplitude is related to a product of two tree amplitudes. Effectively, two propagators within the loop are put on-shell. The imaginary part should be viewed more generally as a discontinuity across a branch cut singularity of the amplitude.

Taking into account the expression of the cross section for a process $2\to 2$, we can write the optical theorem as[1],

$$\mathrm{Im}\mathcal{A}^{1-loop}\left(k_1k_2\to k_1k_2\right)=2E_{cm}p_{cm}\sigma\left(k_1k_2\to \text{anything}\right) \tag{3-30}$$

Here E_{cm} is the total center of mass energy and p_{cm} is the momentum of either particle in the center of mass frame.

We study the behavior of the $\mathcal{A}^{1-loop}$. To compute this amplitude we use the perturbation theory which allows us to consider $\mathcal{A}^{1-loop}(s)$ as analytic function of the complex variable $s=E_{cm}^2$.
we consider $s_0$to be the threshold energy for production of the lightest multiparticle state. For real $s<s_0$ the intermediate state cannot go on-shell, so $\mathcal{A}^{1-loop}(s)$ is real and we have the identity

$$\mathcal{A}^{1-loop}\left(s\right)=\left[\mathcal{A}^{1-loop}\left(s^{*}\right)\right]^{*} \tag{3-31}$$

each side of this equation is an analytic function of s, so it can be analytically coontinued to the entire complex s plane. In particular, near the real axis for $s>s_0$, eq. (3-31) implies

$$\mathrm{Re}\;\mathcal{A}^{1-loop}\left(s+i\epsilon\right)=\mathrm{Re}\;\mathcal{A}^{1-loop}\left(s-i\epsilon\right) \tag{3-32}$$

$$\mathrm{Im}\;\mathcal{A}^{1-loop}\left(s+i\epsilon\right)=-\mathrm{Im}\;\mathcal{A}^{1-loop}\left(s-i\epsilon\right) \tag{3-33}$$

there is a branch cut across the real axis, starting at the threshold energy s_0; the discontinuity across the cut is

$$\text{Disc}\ \mathcal{A}^{1-loop}(s) = 2i\text{Im}\ \mathcal{A}^{1-loop}(s - i\epsilon)$$

The $i\epsilon$ prescription indicates that physical scattering amplitudes should be evaluated above the cut, at $s + i\epsilon$.

If we want to calculate the discontinuity in the s-channel, we most consider the sum of all Feynman diagrams and then the optical theorem dictates the we have to cut the diagram in two tree diagrams (see figure **3-3**) ,

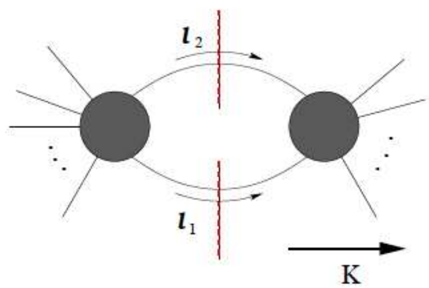


Figure **3-3**.: Unitarity cut of a one-loop amplitude in the s-channel ($s = K^2$). The two propagators are constrained to their respective mass shells. The disks represent the sum of all Feynman diagrams linking the fixed external lines and the two cut propagators.

The Cutkosky rules for computing the physical discontinuity of a specified diagram are given by the following algorithm[24]:

1. We cut the diagram so that the two propagators can simultaneosly be put on-shell
2. For each cut propagator, we replace

$$\frac{i}{P^2 - m^2 + i\epsilon} \to -2\pi i\delta^{(+)}\left(P^2 - m^2\right) \tag{3-34}$$

   here, the superscript $(+)$ on the delta functions for the cut propagators denotes the choice of a positive-energy solution.

3. Then, perfom the loop integrals
4. And finally, sum the contributions of all cuts

Using these rules "cutting rules", it is posible to prove the optical theorem to all orders in pertubation theory.

The Cutkosky rules are expressed in the cut integral[3, 5, 8, 24, 48]

$$\triangle A^{\text{1-loop}} \equiv \int d\mu\, A^{\text{tree}}_{\text{Left}} \times A^{\text{tree}}_{\text{Right}} \tag{3-35}$$

where $A^{\text{1-loop}}$ is the color-ordered primitive amplitude and $d\mu$ the Lorentz-invariant phase space (LIPS) measure is defined by

$$d\mu = d^4 l_1\, d^4 l_2\, \delta^4\left(l_1 + l_2 - K\right) \delta^{(+)}\left(l_1^2\right) \delta^{(+)}\left(l_2^2\right) \tag{3-36}$$

To compute the amplitude, we apply the cut $\triangle$ in various momentum channels where we get information about the coefficients of master integrals.
If we apply a unitarity cut to the expansion (3-11) of an amplitude in master integrals. Since the coefficients are rational functions, the branch cuts are located only in the master integrals. Thus we find that

$$\triangle A^{\text{1-loop}} = \sum_n \sum_{K_r} c_n\left(K_r\right) \triangle I_n\left(K_r\right) \tag{3-37}$$

Eq. (3-37) is the key to the unitarity method. It has two important features. First, we see from (3-35) that it is a relation involving tree-level quantities. Second, many of the terms on the right-hand side vanish, because only a subset of master integrals have a cut involving the given momentum K[8].

The problem is to obtain the individual coefficients c_i. With generalized unitarity these coefficients are obtained easily.

3.4. Generalized unitarity

In this section we discuss a consequence of using internal lines in $(4-2\epsilon)$-dimensions [35, 49, 53] One consequence was obtaining of an effective mass μ^2 en 4 dimensions.

The one-loop color-ordered amplitude for n massless particles in D-dimensions, can be written as

$$A_n^{1-loop} = \int \frac{d^D\ell}{(4\pi)^{D/2}} \frac{\mathcal{N}\left(\{p_i\}, \ell\right)}{\left(\ell^2 - m_1^2\right)\left(\left(\ell - K_1\right)^2 - m_2^2\right)\cdots\left(\left(\ell + K_n\right)^2 - m_n^2\right)} \tag{3-38}$$

The numerator $\mathcal{N}$ contains all information from external momenta and polarization states and tensor structure from the loop momenta. We restrict external momenta to be in four dimensions while internal momenta to be in D-dimensions.
Using D-dimensional Passarino-Veltman reduction techniques on (3-38), the one-loop amplitude can be written as

$$A_n^{1-loop} = \sum_{K_5} \tilde{C}_{5;K_5}(D)\, I_{5;K_5}^D + \sum_{K_4} \tilde{C}_{4;K_4}(D)\, I_{4;K_4}^D + \sum_{K_3} C_{3;K_3}(D)\, I_{3;K_3}^D + \sum_{K_2} C_{2;K_2}(D)\, I_{2;K_2}^D + C_1(D)\, I_1^D \tag{3-39}$$

This expansion shows that any one-loop amplitude can be expanded in a linear combination of master integrals (I_n^D), where the coefficients of each master integral will be found by using generalized unitarity. The rational contributions to the amplitude arise with $D = 4-2\epsilon$. Here K_r refers to the set of all ordered partitions of the external momenta into r distinct groups.

We are interested on internal loop momenta in $D = 4 - 2\epsilon$, then is useful to decompose the loop momenta as

$$\ell^{\nu} = \bar{\ell}^{\nu} + \tilde{\ell}^{\nu}_{[-2\epsilon]}, \tag{3-40}$$

$$\ell^2 = \bar{\ell}^2 - \mu^2 = 0, \tag{3-41}$$

where $\bar{\ell}$ contains the four-dimensional components and $\tilde{\ell}_{[-2\epsilon]}$ the remaining (-2ϵ)-dimensional components. We see then that any dimensional dependence of the numerators arises only through dependence on (μ^2)[1]. In QCD, the maximum number of power of loop momentum appearing in the numerator of an n-point tensor integral is n, so the boxes can have at most a μ^4 while the triangles and bubbles can have up to a μ^2 . The pentagon integral is an independent function in D dimensions since we can find poles in the $D-4$ dimensional sub-space, then the coefficient of this function in $D = 4 - 2\epsilon$, residue around the extra dimensional poles, can have no dependence on ϵ[49].
With this prescription, the master integrals in $D = 4 - 2\epsilon$-dimensions would take the form,

$$(4\pi)^{2-\epsilon} \int \frac{d^4 p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\left(\mu^2\right)^r}{\mathcal{D}_n} \tag{3-42}$$

Using the following identity (See appendix H),

$$I_n\left[\left(\mu^2\right)^r\right] = (4\pi)^{2-\epsilon} \int \frac{d^4 p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\left(\mu^2\right)^r}{\mathcal{D}_n} = \frac{1}{2^r} I_n^{D+2r} \prod_{k=0}^{r-1} (D-4+2k) \tag{3-43}$$

We can remove the μ^2 dependence in the numerator, this dependence is removed just by taking into account $p^2 = \mu^2$. However, this procedure changes the dimension of the integral and that D -dependence appears in the coefficients of the master integrals.
Writting A_n^{1-loop} in D-dimension in terms of $\left(\mu^2\right)^k, k = 0, 1, 2,$

$$\begin{aligned} A_n^{1-loop} = & \sum_{K_5} \tilde{C}_{5;K_5} I^D_{5;K_5} + \sum_{K_4} C^{[0]}_{4;K_4} I^D_{4;K_4} + \sum_{K_4} C^{[2]}_{4;K_4} I^D_{4;K_4}\left[\mu^2\right] + \sum_{K_4} C^{[4]}_{4;K_4} I^D_{4;K_4}\left[\mu^4\right] + \\ & + \sum_{K_3} C^{[0]}_{3;K_3} I^D_{3;K_3} + \sum_{K_3} C^{[2]}_{3;K_3} I^D_{3;K_3}\left[\mu^2\right] + \sum_{K_2} C^{[0]}_{2;K_2} I^D_{2;K_2} + \sum_{K_2} C^{[2]}_{2;K_2} I^D_{2;K_2}\left[\mu^2\right] + \\ & + C_1 I_1^D \end{aligned} \tag{3-44}$$

However, the pentagon integral can be decomposed in[54]

$$I_5^D = \frac{D-4}{2} I_5^{D+2} \left(\sum_{i,j} S^{-1}_{ij} \right) + \frac{1}{2} \sum_{i=1}^{5} \sum_j S^{-1}_{ij} I^D_{4;K_5^i} \tag{3-45}$$

$$S_{ij} = \frac{1}{2}\left(m_i^2 + m_j^2 - p_{ij}^2\right) \tag{3-46}$$

The master integral I_5^{D+2} in $D = 4 - 2\epsilon$ is independent of ϵ, then by taking $\epsilon \to 0$, the pentagon integral is written as linear combinations of all posible boxes contributions,

$$I_5^{D=4-2\epsilon} = \frac{1}{2} \sum_{i=1}^{5} \sum_j S^{-1}_{ij} I^{D=4-2\epsilon}_{4;K_5^i} \tag{3-47}$$

[1] If we would have a dependence of a odd power of μ ($\mu^{2k+1}, k = 0, 1, 2, \dots$) the integral (3-38) vashishes.

From identity (3-43) we obtain:

$$I^D_{4;K_4}\left[\mu^2\right] = \frac{D-4}{2} I^{D+2}_{4;k_4}\left[1\right] \tag{3-48}$$

$$I^D_{4;K_4}\left[\mu^4\right] = \frac{(D-4)(D-2)}{4} I^{D+4}_{4;k_4}\left[1\right] \tag{3-49}$$

$$I^D_{3;K_3}\left[\mu^2\right] = \frac{D-4}{2} I^{D+2}_{3;k_3}\left[1\right] \tag{3-50}$$

$$I^D_{2;K_2}\left[\mu^2\right] = \frac{D-4}{2} I^{D+2}_{2;k_2}\left[1\right] \tag{3-51}$$

And the full amplitude:

$$\begin{aligned} A^{(1),D}_n = & \sum_{K_5} \tilde{C}_{5;K_5}\left[\frac{D-4}{2} I^{D+2}_{5;K_5}\left(\sum_{i,j} S^{-1}_{ij}\right) + \frac{1}{2}\sum_{i=1}^{5}\sum_j S^{-1}_{ij} I_{4;K^i_5}\right] + \\ & + \sum_{K_4} C^{[0]}_{4;K_4} I^D_{4;K_4} + \frac{D-4}{2}\sum_{K_4} C^{[2]}_{4;K_4} I^{D+2}_{4;k_4} + \frac{(D-4)(D-2)}{4}\sum_{K_4} C^{[4]}_{4;K_4} I^{D+4}_{4;k_4} \\ & + \sum_{K_3} C^{[0]}_{3;K_3} I^D_{3;K_3} + \frac{D-4}{2}\sum_{K_3} C^{[2]}_{3;K_3} I^{D+2}_{3;k_3} + \sum_{K_2} C^{[0]}_{2;K_2} I^D_{2;K_2} + \frac{D-4}{2}\sum_{K_2} C^{[2]}_{2;K_2} I^{D+2}_{2;k_2} + \\ & + C_1 I^D_1 \end{aligned} \tag{3-52}$$

Using (3-47) we obtain the total box coefficient then renaming $\tilde{C}_{5;K_5}$ and $C^{[0]}_{4;K_4}$ in this way:

$$C_{4;K_4} = C^{[0]}_{4;K_4} + \frac{1}{2}\tilde{C}_{5;K_5}\sum_{i=1}^{5}\sum_j S^{-1}_{ij} \tag{3-53}$$

Now, we take the 4 -dimensional limit $D = 4 - 2\epsilon$ around $\epsilon \to 0$:

$$A^{1-loop}_n = \text{Cut-Constructible} + \text{Rational Terms} \tag{3-54}$$

The cut-contructible amplitude can be obtained just by studying our amplitude in $D = 4$ -dimensions and is given by,

$$\text{Cut-Constructible} = \sum_{K_4} C^{[0]}_{4;K_4} I^{4-2\epsilon}_{4;K_4} + \sum_{K_3} C^{[0]}_{3;K_3} I^{4-2\epsilon}_{3;K_3} + \sum_{K_2} C^{[0]}_{2;K_2} I^{4-2\epsilon}_{2;K_2} + C_1 I^{4-2\epsilon}_1 \tag{3-55}$$

Rational terms, R_n, arise in $D = 4 - 2\epsilon$ dimensions,

$$\begin{aligned} R_n = & \frac{D-4}{2}\sum_{K_5} C_{5;K_5} I^{D+2}_{5;K_5} + \frac{D-4}{2}\sum_{K_4} C^{[2]}_{4;K_4} I^{D+2}_{4;k_4} + \frac{(D-4)(D-2)}{4}\sum_{K_4} C^{[4]}_{4;K_4} I^{D+4}_{4;k_4} \\ & + \frac{D-4}{2}\sum_{K_3} C^{[2]}_{3;K_3} I^{D+2}_{3;k_3} + \frac{D-4}{2}\sum_{K_2} C^{[2]}_{2;K_2} I^{D+2}_{2;k_2} \end{aligned} \tag{3-56}$$

$$= -\epsilon\sum_{K_5} C_{5;K_5} I^{6-2\epsilon}_{5;K_5} - \epsilon\sum_{K_4} C^{[2]}_{4;K_4} I^{6-2\epsilon}_{4;k_4} - \epsilon(1-\epsilon)\sum_{K_4} C^{[4]}_{4;K_4} I^{8-2\epsilon}_{4;k_4} - \epsilon\sum_{K_3} C^{[2]}_{3;K_3} I^{6-2\epsilon}_{3;k_3} - \epsilon\sum_{K_2} C^{[2]}_{2;K_2} I^{6-2\epsilon}_{2;k_2} \tag{3-57}$$

The first two terms in eq. (3-57) go to zero because $I^{6-2\epsilon}_{5;K_5}$ and $I^{6-2\epsilon}_{4}$ do not depend of ϵ.
The other contributions can be computed by using recursive formulas[6] and the scalar bubble integral (See appendix H)

$$I_2^{D=4-2\epsilon} = r_\Gamma \left(\frac{1}{\varepsilon} - \ln\left(-K^2\right) + 2\right) + \mathcal{O}\left(\varepsilon\right) \tag{3-58}$$

Finally, The rational term contributions are:

$$R_n = -\frac{1}{6}\sum_{K_4} C^{[4]}_{4;K_4} - \frac{1}{2}\sum_{K_3} C^{[2]}_{3;K_3} - \frac{1}{6}\sum_{K_2}\left(K_2^2 - 3\left(m_1^2 + m_2^2\right)\right) C^{[2]}_{2;K_2} \tag{3-59}$$

In agreement with [49, 55]. It is worth here to mention the existence [50] of a semi-analytic method for the integrand reduction of one-loop amplitudes, based on the systematic application of the Laurent expansions to the integrand decomposition. With the aim of performing fully analitical computation the approach of [50] will be considered for future studies.

3.5. Extracting the integral coefficients using massive propagators

In this section, we follow Kilgore and Badger[35, 49] directly for coefficients of integrals contributing to the rational piece of the amplitudes, while for the cut constructible parts we simply take $\mu^2 \to 0$. Then to extract the integral coefficient using generalized unitarity we need to solve the constraints which put the various propagators on-shell. Moreover in $D = 4 - 2\epsilon$ we need to extract the μ dependence of the coefficients.
By studying internal lines in $D = 4 - 2\epsilon$ we obtain an effective mass term, therefore it is posible to construct the full amplitude from tree amplitudes where the internal lines have an uniform mass,

$$l_i^2 = \bar{l}_i^2 - \mu^2 = 0 \Rightarrow \bar{l}_i^2 = \mu^2 \tag{3-60}$$

as we saw in the previous section, $\bar{l}_i$ is in 4 dimensions. This method has been used successfully within the standard unitarity cut technique by Bern and Morgan[6].

3.5.1. Box coefficient

We begin by choosing the four-momentum, $\bar{l}_i$, to be parametrized by[35, 49, 53],

$$\bar{l}_1^\mu = aK_4^{\flat\mu} + bK_1^{\flat\mu} + \frac{c}{2}\left\langle K_4^\flat\left|\gamma^\mu\right|K_1^\flat\right] + \frac{d}{2}\left\langle K_1^\flat\left|\gamma^\mu\right|K_4^\flat\right] \tag{3-61}$$

Here $\bar{l}_1$ is in terms of the spinor representation of two massless real momenta ($K_1^\flat$ and $K_4^\flat$) which can be constructed from any two adjacent external momenta on the box. We choose two adjacent external momenta, K_1 and K_4 and project them onto one another to form two massless momenta $K_1^\flat$ and $K_4^\flat$.
where we define a massless basis in terms of two of the external momenta:

$$K_4^{\flat\mu} = \frac{\gamma_{14}\left(\gamma_{14}K_4^\mu - S_4 K_1^\mu\right)}{\gamma_{14}^2 - S_1 S_4}, \qquad K_1^{\flat\mu} = \frac{\gamma_{14}\left(\gamma_{14}K_1^\mu - S_1 K_4^\mu\right)}{\gamma_{14}^2 - S_1 S_4}, \tag{3-62}$$

$$K_4^\mu = K_4^{\flat\mu} + \frac{S_1}{\gamma_{14}}K_1^{\flat\mu}, \qquad K_1^\mu = K_1^{\flat\mu} + \frac{S_4}{\gamma_{14}}K_4^{\flat\mu}, \tag{3-63}$$

$$S_i = K_i^2, \qquad \gamma_{14} = K_1\cdot K_4 \pm \sqrt{\left(K_1\cdot K_4\right)^2 - S_1 S_4} \tag{3-64}$$

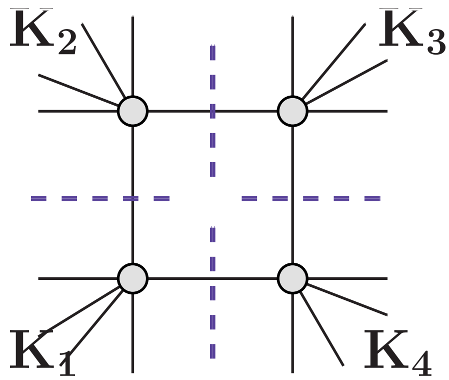


Figure **3-4**.: A general quadrupole cut. Loop momenta flow clockwise.

For the quadrupole cut (Fig. **3-4**) in $4-2\epsilon$-dimensions, the on-shell cut conditions as

$$l_1^2 = l_2^2 = l_3^2 = l_4^2 = 0, \tag{3-65}$$

or equivalently with $\bar{l}_i, i = 1, 2, 3, 4$, in 4-dimensions

$$\bar{l}_1^2 = \bar{l}_2^2 = \bar{l}_3^2 = \bar{l}_4^2 = \mu^2, \tag{3-66}$$

where μ^2 represents an effective mas term that comes from the (-2ϵ)-dimensional components
Using momentum conservation and writting all loop momenta in terms of $\bar{l}_1$,

$$\bar{l}_1^2 = \mu^2 \tag{3-67}$$
$$\bar{l}_2^2 = \left(\bar{l}_1 - K_2\right)^2 = \mu^2 \tag{3-68}$$
$$\bar{l}_3^2 = \left(\bar{l}_1 - K_2 - K_3\right)^2 = \mu^2 \tag{3-69}$$
$$\bar{l}_4^2 = \left(\bar{l}_1 + K_4\right)^2 = \mu^2 \tag{3-70}$$

from eq. (3-67) d takes the form

$$d = \frac{\gamma_{14}ab - \mu^2}{\gamma_{14}c} \tag{3-71}$$

then eq. (3-61) can be written in terms of two massless momenta,

$$\bar{l}_1^\mu = aK_4^{\flat\mu} + bK_1^{\flat\mu} + \frac{c}{2}\left\langle K_4^\flat\right|\gamma^\mu\left|K_1^\flat\right] + \frac{\gamma_{14}ab - \mu^2}{2\gamma_{14}c}\left\langle K_1^\flat\right|\gamma^\mu\left|K_4^\flat\right], \tag{3-72}$$

$$\bar{l}_1^\mu = \bar{l}_1^{\flat\mu} - \frac{\mu^2}{2\gamma_{14}c}\left\langle K_1^\flat\right|\gamma^\mu\left|K_4^\flat\right], \tag{3-73}$$

Here $\left(\bar{l}_1^{\flat\mu}\right)^2 = \left\langle K_1^\flat\right|\gamma^\mu\left|K_4^\flat\right]^2 = 0$
solving the other on-shell conditions, we find

$$a = \frac{S_1\left(S_4 + \gamma_{14}\right)}{\gamma_{14}^2 - S_1 S_4}, \qquad b = -\frac{S_4\left(S_1 + \gamma_{14}\right)}{\gamma_{14}^2 - S_1 S_4} \qquad c_\pm = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0c_2}}{2c_2}$$

$$
\begin{aligned}
c_1 &= a\left\langle K_4^\flat \left|K_2\right| K_4^\flat\right] + b\left\langle K_1^\flat \left|K_2\right| K_1^\flat\right] - S_2 \\
c_2 &= \left\langle K_4^\flat \left|K_2\right| K_1^\flat\right] \\
c_0 &= \left(ab - \frac{\mu^2}{\gamma_{14}}\right)\left\langle K_1^\flat \left|K_2\right| K_4^\flat\right]
\end{aligned}
\tag{3-74}
$$

there are two solutions for $c_\pm$ and it might appear that, combined with the two solutions for γ_{14}, there are four solutions for $\bar{l}_1$. However, it works out that[35]

$$
\begin{aligned}
\bar{l}_1\left(\gamma_{14}^+, c_+\right) &= \bar{l}_1\left(\gamma_{14}^-, c_-\right), \\
\bar{l}_1\left(\gamma_{14}^+, c_-\right) &= \bar{l}_1\left(\gamma_{14}^-, c_+\right),
\end{aligned}
\tag{3-75}
$$

If we have $S_1 = 0$ and $S_4 = 0$ there is only one solution for γ_{14} (but still two solutions to the on-shell conditions).
To determine the full box coefficient, we must average over these solutions

The coefficients associated with our integral basis choice are (see appendix I),

$$
C_4^{[0]} = \frac{i}{2}\sum_\sigma A_1A_2A_3A_4\left(\bar{l}_1^\sigma\right) \tag{3-76}
$$

$$
C_4^{[4]} = \frac{i}{2}\sum_\sigma \mathrm{Inf}_{\mu^2}\left[A_1A_2A_3A_4\left(\bar{l}_1^\sigma\right)\right]_{\mu^4} \tag{3-77}
$$

the sum is over the two solutions of the quadrupole cut; the product $A_1A_2A_3A_4$ must be computed for each.

- To find $C_4^{[0]}$, our cut-constructible piece, we put $\mu^2 = 0$.
- The Inf term in $C_4^{[4]}$ contains the information from the boundary of the μ contour integral.

$$
\lim_{\mu^2\to\infty}\left\{\mathrm{Inf}_{\mu^2}\left[A_1A_2A_3A_4\left(\mu^2\right) - A_1\left(\mu\right)A_2\left(\mu\right)A_3\left(\mu\right)A_4\left(\mu\right)\right]\right\} = 0 \tag{3-78}
$$

here we are interested in how the product of tree amplitudes behaves at infinite. To study this behavior we need to expand the product of tree amplitudes,

$$
\mathrm{Inf}_{\mu^2}\left[A_1A_2A_3A_4\left(\mu^2\right)\right] = \sum_{i=0}^{2} C_i\mu^{2i} \tag{3-79}
$$

then restrict $C_4^{[4]}$ to be the coefficient of the μ^4 term.

Quadrupole cut coefficient for one-loop five gluons amplitude in pure YM

Consider the process of five gluons

$$
0 \to g\left(1^-\right)g\left(2^-\right)g\left(3^+\right)g\left(4^+\right)g\left(5^+\right) \tag{3-80}
$$

In this example we are going to show how to compute the quadrupole box coefficient in the s_{12} channel (see fig. **3-5**)

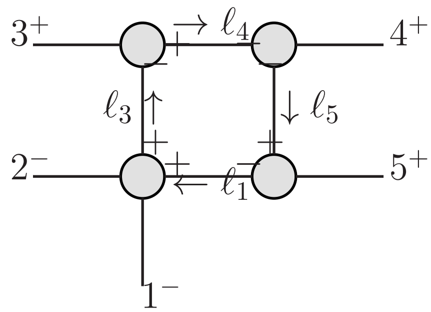


Figure **3-5**.: Box configuration for the process of five gluons. All external and internal legs are gluons.

For this process, we have the following conditions:

$$\ell_1^2 = 0, \qquad \ell_3^2 = 0, \qquad \ell_4^2 = 0, \qquad \ell_5^2 = 0. \tag{3-81}$$

If we write ℓ_1, ℓ_3 and ℓ_5 in terms of ℓ_4 and all external momenta, we obtain:

$$\ell_1^2 = (\ell_4 - 4 - 5)^2 = 0 \tag{3-82}$$
$$\ell_3^2 = (\ell_4 + 3)^2 = 0 \tag{3-83}$$
$$\ell_4^2 = 0 \tag{3-84}$$
$$\ell_5^2 = (\ell_4 - 4)^2 \tag{3-85}$$

From eq. (3-83) and (3-85):

$$2\,(\ell_4 \cdot 3) = \langle \ell_4 3\rangle\,[3\ell_4] = 0 \quad \Rightarrow |\ell_4\rangle \propto |3\rangle \text{ or } |\ell_4] \propto |3]$$
$$2\,(\ell_4 \cdot 4) = \langle \ell_4 4\rangle\,[4\ell_4] = 0 \quad \Rightarrow |\ell_4\rangle \propto |4\rangle \text{ or } |\ell_4] \propto |4]$$

From eq. (3-61) we assume,

$$\ell_4^\mu = \frac{1}{2}\xi\,\langle 3\,|\gamma^\mu|\,4] \tag{3-86}$$

Because $2\,(\ell_4 \cdot 3) = \xi\,\langle 3\,|3|\,4] = 0$ and $2\,(\ell_4 \cdot 4) = \xi\,\langle 3\,|4|\,4] = 0$.
The parameter ξ is determinated with the condition (3-82):

$$2\,(\ell_4 \cdot 5) = 2\,(4 \cdot 5) = \langle 45\rangle\,[54] \tag{3-87}$$
$$2\,(\ell_4 \cdot 5) = \xi\,\langle 3\,|\gamma^\mu|\,4]\,5_\mu = \xi\,\langle 3\,|5|\,4] = \xi\,\langle 35\rangle\,[54] \tag{3-88}$$
$$\xi\,\langle 35\rangle\,[54] = \langle 45\rangle\,[54] \quad \Rightarrow \quad \xi = \frac{\langle 45\rangle}{\langle 35\rangle} \tag{3-89}$$

Now we do the quadrupole cut, by sewing tree amplitudes [2]

$$C_{1345}^{[0]} = \frac{i}{2} A_4^{tree}\left(1^-, 2^-, \ell_3^+, -\ell_1^+\right) A_3^{tree}\left(3^+, \ell_4^+, -\ell_3^-\right) A_3^{tree}\left(\ell_5^-, -\ell_4^-, 4^+\right) A_3^{tree}\left(-\ell_5^-, 5^-, \ell_1^-\right) \tag{3-90}$$
$$= \frac{i}{2}\frac{\langle 12\rangle^3}{\langle 2\ell_3\rangle\,\langle \ell_3\ell_1\rangle\,\langle \ell_1 1\rangle}\frac{[3\ell_4]^3}{[\ell_4\ell_3]\,[\ell_3 3]}\frac{\langle \ell_5\ell_4\rangle^3}{\langle \ell_4 4\rangle\,\langle 4\ell_5\rangle}\frac{[\ell_5 5]^3}{[5\ell_1]\,[\ell_1\ell_5]} \tag{3-91}$$

[2]For these tree amplitudes we have taken into account the MHV amplitudes and the sequence MHV $-$ $\overline{\text{MHV}}$

Using spinor identities and momentum conservation (see equations (3-82)(3-83)(3-84)(3-85)),

$$C_{1345}^{[0]} = -\frac{i}{2}\frac{\langle 12\rangle^3 \langle 4\,|\ell_4|\,3]^2\,[45]^3}{\langle 1\,|\ell_1|\,5]\,\langle 4\,|\ell_4|\,5]\,\langle 2\,|\ell_4|\,3]\,\langle 35\rangle} \tag{3-92}$$

and writting the explicit solution for ℓ_4 (eq. (3-86)),

$$C_{1345}^{[0]} = -\frac{i}{2}\frac{\langle 12\rangle^3\langle 43\rangle\,[43]\,[45]}{\langle 23\rangle\,\langle 34\rangle\,\langle 15\rangle}$$

$$C_{1345}^{[0]} = \frac{1}{2}s_{34}s_{45}A_5^{tree}\left(1^-,2^-,3^+,4^+,5^+\right) \tag{3-93}$$

This result can be obtained by utilizing of mathematica (see appendix J)

3.5.2. Triangle Coefficient

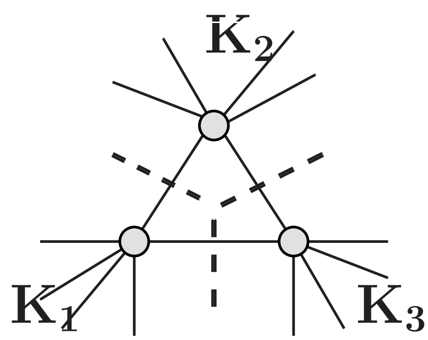


Figure **3-6**.: A general triple cut. Loop momenta flow clockwise.

Defining $K_{1,3}^\flat$ analogously to eq. (3-62) The three delta function constraints imposed by the cuts[35]:

$$\delta\left(l^2-\mu^2\right) \qquad \delta\left((l-K_1)^2-\mu^2\right) \qquad \delta\left((l+K_2)^2-\mu^2\right) \tag{3-94}$$

Using the same parametrization of the box and the on-shell conditons, we find

$$\bar{l}_1^\mu = aK_1^{\flat\mu} + bK_3^{\flat\mu} + \frac{t}{2}\left\langle K_1^\flat\,|\gamma^\mu|\,K_3^\flat\right] + \frac{\gamma_{13}ab-\mu^2}{2\gamma_{13}t}\left\langle K_3^\flat\,|\gamma^\mu|\,K_1^\flat\right] \tag{3-95}$$

$$a = \frac{S_1\left(S_3+\gamma_{13}\right)}{\gamma_{13}^2-S_1S_3}, \tag{3-96}$$

$$b = -\frac{S_3\left(S_1+\gamma_{13}\right)}{\gamma_{13}^2-S_1S_3} \tag{3-97}$$

$$\gamma_{13} = K_1\cdot K_3 \pm \sqrt{\left(K_1\cdot K_3\right)^2 - S_1S_3} \tag{3-98}$$

If $S_1=0$ or $S_3=0$, γ_{13} has only one solution and we do not need to average our result. However, for a fixed value of γ_{13}, we must also average over the coefficients given by the conjugate solution,

$$\bar{l}_1^{*\mu} = aK_1^{\flat\mu} + bK_3^{\flat\mu} + \frac{t}{2}\left\langle K_3^\flat\,|\gamma^\mu|\,K_1^\flat\right] + \frac{\gamma_{13}ab-\mu^2}{2\gamma_{13}t}\left\langle K_1^\flat\,|\gamma^\mu|\,K_3^\flat\right] \tag{3-99}$$

In both solutions the complex parameter t is free.
Box integrals also contain triple cuts, so we must extract the triangle coefficients using the limiting behavior of the integrand. The coefficients therefore contain an Inf term that is a polynomial expansion in t, but ontly the order t^0 is retained (see appendix I).

$$C_3^{[0]} = -\frac{1}{2n_\gamma}\sum_\sigma \left[\mathrm{Inf}_t A_1 A_2 A_3\left(\bar{l}_1^\sigma\right)\right](t)\Big|_{t^0} \tag{3-100}$$

$$C_3^{[2]} = -\frac{1}{2n_\gamma}\sum_\sigma \mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t A_1 A_2 A_3\left(\bar{l}_1^\sigma\right)\right](t)\Big|_{\mu^2,t^0} \tag{3-101}$$

The sum is over the solutions, including the conjugate momentum solution. There may be either two or four solutions depending on the number of solutions n_γ for γ_{13}. In $C_3^{[0]}$, μ^2 is set to zero, while the expansion in $C_3^{[2]}$ is restricted to the coefficients of the μ^2 term.

If we consider a massless loop momenta ($\mu^2 = 0$) we can decompose $\bar{l}_1$ as spinors components[56]

$$\langle\bar{l}_1| = t\left\langle K_1^\flat\right| + \alpha_{11}\left\langle K_3^\flat\right|, \tag{3-102}$$

$$[\bar{l}_1| = \frac{\alpha_{12}}{t}\left[K_1^\flat\right| + \left[K_3^\flat\right|, \tag{3-103}$$

where

$$\alpha_{11} = \frac{S_1\left(\gamma_{13}-S_3\right)}{\left(\gamma^2 - S_1 S_3\right)}, \qquad \alpha_{12} = \frac{S_3\left(\gamma_{13}-S_1\right)}{\left(\gamma^2 - S_1 S_3\right)}. \tag{3-104}$$

we can also use momentum conservation to write component forms for the other two cut momenta l_i with $i = 2, 3$

$$\langle\bar{l}_2| = t\left\langle K_1^\flat\right| + \alpha_{21}\left\langle K_3^\flat\right| \qquad \langle\bar{l}_3| = t\left\langle K_1^\flat\right| + \alpha_{31}\left\langle K_3^\flat\right| \tag{3-105}$$

$$[\bar{l}_2| = \frac{\alpha_{22}}{t}\left[K_1^\flat\right| + \left[K_3^\flat\right|, \qquad [\bar{l}_3| = \frac{\alpha_{32}}{t}\left[K_1^\flat\right| + \left[K_3^\flat\right|, \tag{3-106}$$

the constants α's are given by,

$$\alpha_{21} = -\frac{S_1 S_3\left(1 - S_1/\gamma_{13}\right)}{\gamma_{13}^2 - S_1 S_3}, \qquad \alpha_{12} = \frac{\gamma_{13}\left(S_3-\gamma\right)}{\gamma^2 - S_1 S_3}, \tag{3-107}$$

$$\alpha_{12} = \frac{\gamma_{13}\left(S_1-\gamma\right)}{\gamma^2 - S_1 S_3}, \qquad \alpha_{21} = -\frac{S_1 S_3\left(1 - S_3/\gamma_{13}\right)}{\gamma_{13}^2 - S_1 S_3} \tag{3-108}$$

Triple cut coefficient for gluon production by quark anti-quark annihilation

Consider the process,

$$0 \to g\left(1^-\right) g\left(2^+\right) \bar{q}\left(3^-\right) q\left(4^+\right) \tag{3-109}$$

We are going to compute the coefficient that comes from the triangle in the channel s_{34}.

For this process, we have the following conditions:

$$l_1^2 = 0 \qquad l_2^2 = 0 \qquad l_4^2 = 0 \tag{3-110}$$

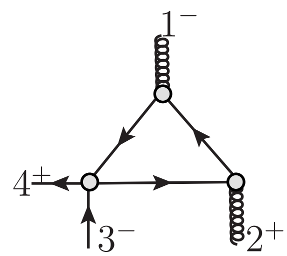


Figure **3-7**.: Triangle configuration for gluon production by quark anti-quark annihilation.

writing l_2 y l_4 in terms of l_1, on-shell conditions take the form,

$$l_2^2 = (l_1 + 2)^2 = 0 \tag{3-111}$$

$$l_4^2 = (l_1 - 1)^2 = 0 \tag{3-112}$$

$$l_1^2 = 0 \tag{3-113}$$

With these conditions, l_1, l_2 and l_4 can be written as,

$$l_1^\mu \gamma_\mu = \frac{t}{2} \langle 1 | \gamma^\mu | 2] \gamma_\mu = t (|1\rangle [2| + |2] \langle 1|) \tag{3-114}$$

$$l_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 | \gamma^\mu | 2] - K_1^\mu \right) \gamma_\mu = |1\rangle (t [2| - [1|) + (t |2] - |1]) \langle 1| \tag{3-115}$$

$$l_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 | \gamma^\mu | 2] + K_2^\mu \right) \gamma_\mu = (t |1\rangle + |2\rangle) [2| + |2] (t \langle 1| + \langle 2|) \tag{3-116}$$

and the conjugate solution,

$$\bar{l}_1^\mu \gamma_\mu = \frac{t}{2} \langle 2 | \gamma^\mu | 1] \gamma_\mu = t (|2\rangle [1| + |1] \langle 2|) \tag{3-117}$$

$$\bar{l}_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 | \gamma^\mu | 1] - K_1^\mu \right) \gamma_\mu = (t |2\rangle - |1\rangle) [1| + |1] (t \langle 2| - \langle 1|) \tag{3-118}$$

$$\bar{l}_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 | \gamma^\mu | 1] + K_2^\mu \right) \gamma_\mu = |2\rangle (t [1| + [2|) + (t |1] + |2]) \langle 2| \tag{3-119}$$

By sewing the tree level amplitudes,

$$C_{421} = A_4^{tree} \left(-l_4^-, 4_q^+, 3_{\bar{q}}^-, l_2^+ \right) A_3^{tree} \left(-l_2^-, 2^+, l_1^+ \right) A_3^{tree} \left(-l_1^-, 1^-, l_4^+ \right) = -\frac{i[2|l_1][4|l_2]^2 \langle 1|l_1|2]}{[4|1][3|l_2][l_2|l_1]} \tag{3-120}$$

putting the explicit solution for l_1 and l_2 and taking $\mathrm{Inf_t}$,

$$C_{421} = -\frac{i[2|1] \left(t[4|1] + [4|2] \right)^2 s_{12} t}{[4|1] \left(t[3|1] + [3|2] \right) [2|1]} \tag{3-121}$$

$$\inf_{t^0} \left[-\frac{i[2|1] \left(t[4|1] + [4|2] \right)^2 s_{12} t}{[4|1] \left(t[3|1] + [3|2] \right) [2|1]} \right] = \frac{i \langle 13 \rangle^3 \langle 14 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{s_{12}^3}{s_{13}^3} s_{12} \tag{3-122}$$

conjugate solutions for l_1 and l_2 do not give any contribution.
Finally, the triple cut coefficient is written as,

$$C^{[0]}_{3;34}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = \frac{1}{2}\frac{s^4_{12}}{s^3_{13}} A^{tree}_4\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) \tag{3-123}$$

Using mathematica we recover this result (see appendix J)

3.5.3. Bubble coefficients

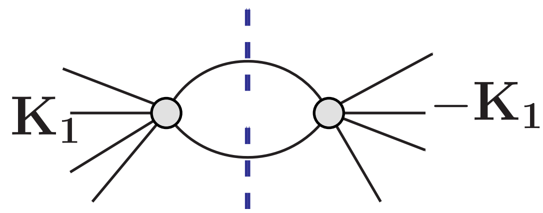


Figure **3-8**.: A double triple cut. Loop momenta flow clockwise.

To extract the coefficients of bubble integrals, we impose the cuts that define the bubble topology:

$$\delta\left(\bar{l}^2_1 - \mu^2\right) \qquad\qquad \delta\left(\left(\bar{l}_1 - K_1\right)^2 - \mu^2\right) \tag{3-124}$$

Only one bubble configuration will satisfy these cuts, but multiple triangle and box configurations will do so.
Since we only have one external momentum, K_1, in a bubble configuration, we can choose an arbitrary massless momentum χ^μ to define our parametrization[56]:

$$K_1 = K^\flat_1 + \frac{S_1}{\gamma_{1\chi}}\chi, \qquad\qquad \gamma_{1\chi} = 2K_1 \cdot \chi = 2K^\flat_1 \cdot \chi \tag{3-125}$$

using the on-shell conditions, we have only one solution for the bubble cut contribution,

$$\bar{l}^\mu_1 = yK^{\flat\mu}_1 + \frac{S_1}{\gamma_{1\chi}}(1-y)\chi^\mu + \frac{t}{2}\left\langle K^\flat_1\right|\gamma^\mu\left|\chi\right] + \frac{1}{2\gamma_{1\chi}t}\left(y(1-y)S_1 - \mu^2\right)\left\langle\chi\right|\gamma^\mu\left|K^\flat_1\right] \tag{3-126}$$

where l^μ_1 has two free complex parameters y and t.
Moreover, the two particle cut is contaminated by both boxes and triangles, so we have to take them into account:

- As before, the box contribution only gives the scalar box coefficients and therefore not of interest to extracting bubble coefficients
- Furthermore, triple cuts that share two of their cuts with the double cut contribute to tensor triangle integrals that reduce to scalar bubbles, so we must take them into account for the full bubble coefficient.

By studying the triangle contribution to the bubble coefficient, we fix the parameter y by using another on-shell condition,

$$\delta\left(\left(\bar{l}_1+K_3\right)^2-\mu^2\right) \tag{3-127}$$

the solutions for y (see appendix I)

$$y_\pm = \frac{B_1 \pm \sqrt{B_1^2+4B_0B_2}}{2C_2} \tag{3-128}$$

$$B_2 = S_1\left\langle \chi\left|K_3\right|K_1^\flat\right], \tag{3-129}$$

$$B_1 = \bar{\gamma}t\left\langle K_1^\flat\left|K_3\right|K_1^\flat\right] - S_1 t\left\langle \chi\left|K_3\right|\chi\right] + S_1\left\langle \chi\left|K_3\right|K_1^\flat\right], \tag{3-130}$$

$$B_0 = \gamma t^2\left\langle K_1^\flat\left|K_3\right|\chi\right] + \gamma t S_3 + tS_1\left\langle \chi\left|K_3\right|\chi\right] \tag{3-131}$$

We then calculate the triple cut integrand $A_1A_2A_3$ for all triple cuts that share two cuts with the original double cut. The bubble coefficients are given by

$$C_2^{[0]} = -i\ \mathrm{Inf}_t\left(\mathrm{Inf}_y\left[A_1A_2\left(l_1\left(y,t\right)\right)\right]\right)\big|_{t^0,y^i\to Y_i} - \frac{1}{2}\sum_{C_{tri}}\sum_{\sigma_y}\mathrm{Inf}_t\left[A_1A_2A_3\left(l_1\left(t\right)\right)\right]\big|_{t^i\to T_j} \tag{3-132}$$

$$C_2^{[2]} = -i\ \mathrm{Inf}_{\mu^2}\left(\mathrm{Inf}_t\left(\mathrm{Inf}_y\left[A_1A_2\left(l_1\left(y,t\right)\right)\right]\right)\right)\big|_{\mu^2,t^0,y^i\to Y_i} - \frac{1}{2}\sum_{C_{tri}}\sum_{\sigma_y}\mathrm{Inf}_{\mu^2}\left(\mathrm{Inf}_t\left[A_1A_2A_3\left(l_1\left(t\right)\right)\right]\right)\big|_{\mu^2,t^i\to T_j} \tag{3-133}$$

the functions T_i and Y_i have been computed in appendix I for arbitrary kinematics. Explicitly with an uniform mass we have,

$$Y_0=1,\quad Y_1=\frac{1}{2},\quad Y_2=\frac{1}{3}\left(1-\frac{\mu^2}{S_1}\right),\quad Y_3=\frac{1}{4}\left(1-2\frac{\mu^2}{S_1}\right),\quad Y_4=\frac{1}{5}\left(1-3\frac{\mu^2}{S_1}+\frac{\mu^4}{S_1^2}\right). \tag{3-134}$$

$$T_0 = 0 \tag{3-135}$$

$$T_1 = -\frac{S_1\langle\chi|K_3|K_1]}{2\gamma\Delta} \tag{3-136}$$

$$T_2 = -\frac{3S_1\langle\chi|K_3|K_1]^2}{8\gamma^2\Delta^2}\left(S_1S_3+K_1\cdot K_3S_1\right) \tag{3-137}$$

$$T_3 = -\frac{\langle\chi|K_3|K_1]^3}{48\gamma^3\Delta^3}\left(15S_1^3S_3^3+30\left(K_1\cdot K_3\right)S_1^3S_3+11\left(K_1\cdot K_3\right)^2S_1^3+4S_1^4S_3+16\mu^2S_1^2\Delta\right) \tag{3-138}$$

where $\Delta=\left(K_1\cdot K_3\right)^2-S_1S_3$. An alternative procedure to find the bubble coefficient in ingeniously done by using the Stokes theorem in the complex plane [51]. The application of the Cauchy-Pompeiu formula allows to perform the cut-integration.

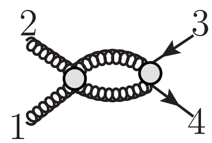


Figure **3-9**.: Bubble configuration for gluon production by quark anti-quark annihilation.

Double cut coefficient for gluon production by quark anti-quark annihilation

Consider the process,

$$0 \to g\left(1^{+}\right) g\left(2^{-}\right) \bar{q}\left(3^{-}\right) q\left(4^{+}\right) \tag{3-139}$$

We are going to compute the coefficient that comes from the bubble in the channel s_{12}.

For this process we have two on-shell conditions,

$$l_1^2 = 0, \qquad l_3^2 = 0. \tag{3-140}$$

taking into account this conditions, l_1 and l_3 can be expressed as,

$$\langle l_1| = t\left\langle K_1^{\flat}\right| + \frac{S_1}{\gamma}(1-y)\langle\chi|, \qquad \langle l_3| = \left\langle K_1^{\flat}\right| - \frac{S_1}{\gamma}\frac{y}{t}\langle\chi|, \tag{3-141}$$

$$[l_1| = \frac{y}{t}\left[K_1^{\flat}\right| + [\chi|, \qquad [l_3| = (y-1)\left[K_1^{\flat}\right| + t\,[\chi|. \tag{3-142}$$

where $K_1^{\flat}$ and χ are given by

$$K_1^{\flat} = 2, \qquad \chi = 1 \tag{3-143}$$

$$\langle l_1| = t\langle 2| + (1-y)\langle 1|, \qquad \langle l_3| = \langle 2| - \frac{y}{t}\langle 1|, \tag{3-144}$$

$$[l_1| = \frac{y}{t}[2| + [1|, \qquad [l_3| = (y-1)[2| + t\,[1|. \tag{3-145}$$

Now, we do the double cut,

$$\begin{aligned} C_{13} &= A_4^{tree}\left(-l_1^{-}, 1_g^{+}, 2,^{-} l_3^{+}\right) A_4^{tree}\left(-l_3^{-}, 3_{\bar{q}}^{-}, 4_q^{+}, l_1^{+}\right) + A_4^{tree}\left(-l_1^{+}, 1_g^{+}, 2,^{-} l_3^{-}\right) A_4^{tree}\left(-l_3^{+}, 3_{\bar{q}}^{-}, 4_q^{+}, l_1^{-}\right) \\ &= \frac{i}{\langle 4|l_1|4]}\left\{\frac{i[4|l_1][1|l_3]^2\langle 3l_3\rangle^2[l_3 1]^2}{[2|1][1|l_1][2|l_3]\langle 34\rangle[4l_3]} + \frac{i\langle l_3|2\rangle\langle l_1 4\rangle\langle 2|3|4]^2}{\langle 1|2\rangle\langle l_1|1\rangle[34]\langle 4l_3\rangle}\right\} \end{aligned} \tag{3-146}$$

Using momentum conservation and putting the solution for l_1 and l_3,

$$C_{13} = \left\{\frac{(y-1)^4[1|2]\left(t\langle 32\rangle - y\langle 31\rangle\right)^2}{t^2 y\langle 34\rangle\left((y-1)[42] + t[41]\right)\left(t\langle 42\rangle + (1-y)\langle 41\rangle\right)} + \frac{y\langle 23\rangle[14]}{\left(t\langle 42\rangle - y\langle 41\rangle\right)\left(y[24] + t[14]\right)}\right\} \tag{3-147}$$

taking Inf functions,

$$\inf_{t^0}\left[\inf_y [C_{13}]\right] = -2\frac{\langle 2|3\rangle^3 \langle 24\rangle}{\langle 12\rangle \langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}\frac{s_{12}}{s_{14}} - \frac{1}{2}\frac{\langle 24\rangle \langle 2|3\rangle^3}{\langle 12\rangle \langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}\frac{s_{12}^2}{s_{14}^2} + 0$$
$$= A_4^{tree}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)\left(\frac{3}{2}\frac{s_{12}}{s_{14}} - \frac{1}{2}\frac{s_{12}s_{13}}{s_{14}^2}\right) \tag{3-148}$$

Now we study the triangle contributions,
For the contributions that come from the triangle we have another on shell condition,

$$l_4^2 = 0 \qquad \text{or} \qquad l_2^2 = 0 \tag{3-149}$$

we are interested in the contribution of $l_2^2 = 0$ because the another one does not contribute to the coefficient. From this condition y is determined,

$$y_\pm = \alpha_{1,\pm} t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} \tag{3-150}$$

$$\alpha_{1,\pm} = \frac{s_{24} - s_{14} \pm (s_{24} + s_{14})}{2\langle 1|4|2]} = \begin{cases}\alpha_{1,+} = \frac{s_{24}}{\langle 1|4|2]}\\ \alpha_{1,-} = -\frac{s_{14}}{\langle 1|4|2]}\end{cases} \tag{3-151}$$

$$\alpha_{2,\pm} = \frac{1}{2}(1 \pm 1) = \begin{cases}\alpha_{2,+} = 1\\ \alpha_{2,-} = 0\end{cases} \tag{3-152}$$

$$\alpha_{3,\pm} = 0 \tag{3-153}$$

We compute the triangle contributions to the bubble coefficient,

$$C_{134} = \frac{i}{l_4^2} A_4^{tree}\left(-l_1^-, 1_g^+, 2,^- l_3^+\right) A_4^{tree}\left(-l_3^-, 3_{\bar{q}}^-, 4_q^+, l_1^+\right) + A_4^{tree}\left(-l_1^+, 1_g^+, 2,^- l_3^-\right) A_4^{tree}\left(-l_3^+, 3_{\bar{q}}^-, 4_q^+, l_1^-\right)$$
$$= \frac{i}{\langle 4|l_1|4]}\left\{\frac{i[4|l_1][1|l_3]^2\langle 3l_3\rangle^2 [l_3 1]^2}{[2|1][1|l_1][2|l_3]\langle 34\rangle [4l_3]} + \frac{i\langle l_3|2\rangle\langle l_1 4\rangle\langle 2|3|4]^2}{\langle 1|2\rangle\langle l_1|1\rangle [34]\langle 4l_3\rangle}\right\} \tag{3-154}$$

putting the solution for l_1 and l_3,

$$C_{134} = \left\{\frac{i\left(y[4|2] + t[4|1]\right)(y-1)^4 [1|2]\left(t\langle 32\rangle - y\langle 31\rangle\right)^2}{t^3 y\langle 34\rangle\left((y-1)[42] + t[41]\right)} + \frac{iy\left(t\langle 24\rangle + (1-y)\langle 14\rangle\right)\langle 23\rangle [14]}{t\left(t\langle 42\rangle - y\langle 41\rangle\right)}\right\}$$

taking $\mathrm{Inf}_{t^m\to T_m}$

$$\inf_{t^m\to T_m}[C_{134}] = A_4^{tree}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)\left(3\frac{s_{13}}{s_{14}} + \frac{s_{13}s_{12}}{s_{!4}}\right) \tag{3-155}$$

And the full bubble coefficient is given by,

$$C_{2;12}^{[0]} = -i\inf_{t^0}\left[\inf_y [C_{13}]\right] - \frac{1}{2}\inf_{t^m\to T_m}[C_{134}]$$
$$= -A_4^{tree}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)\left(\frac{3}{2}\frac{s_{12}}{s_{14}} - \frac{1}{2}\frac{s_{12}s_{13}}{s_{14}^2}\right) - A_4^{tree}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)\left(\frac{3}{2}\frac{s_{13}}{s_{14}} + \frac{1}{2}\frac{s_{13}s_{12}}{s_{14}}\right)$$
$$C_{2;12}^{[0]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = \frac{3}{2}A_4^{tree}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) \tag{3-156}$$

This example has been implemented in mathematica (see appendix J)

4. New Formalism

As we saw in the previous chapter, is important to establish a mechanism to find the rational contributions to the amplitude that come from box, triangle and bubble. For this purpose we study a new formalism, which provides us immediately the coefficients of the cut-constructible amplitude and the rational contribution. However, to reach this objective we have to consider internal particles in $D = 4 - 2\epsilon$ (gluons and fermions), studying these particles we have to generalize polarization vectors and spinors from 4 to $D = 4 - 2\epsilon$ to get right results.
In this chapter we will review the regularization schemes that are used to treat internal and external particles, the new formalism will be introduced and compared in front of OPP method by studying simple examples.

4.1. Regularization Schemes

By studying dimensional regularization we continue from 4 to D dimensions, because we want to avoid infrared and ultraviolet singularities. Generally, we choose $D = 4 - 2\epsilon$ with ϵ an arbitrary complex number. Infrared singularities are studied if we put $\mathfrak{Re}\,(\epsilon) < 0$ and ultraviolet singularities with $\mathfrak{Re}\,(\epsilon) > 0$
In the regularization schemes it is important to distinguish two class of particles: observed and unobserved ones. Unobserved particles are those virtual ones which circulate in internal loops as well as those which are external but soft or collinear with other external particles. All the rest are observed particles[18].
In order to formulatae those schemes, we need to study three spaces where each one is characterized by the metric tensor[58],

- the original 4-dimensional space (4S). The metric tensor is denoted by $\bar{g}^{\mu\nu}$,

$$\bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 4 \tag{4-1}$$

- the formally D-dimensional space for momenta and momentum integrals. This space is actually an infinite-dimensional vector space with certain D-dimensional properties, and is sometimes called "quasi-Ddimensional space" (QDS). The space 4S is therefore a subspace of QDS. The metric tensor is denoted by $\hat{g}^{\mu\nu}$,

$$\hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = D - 2\epsilon \tag{4-2}$$

- the formally 4-dimensional space for e.g. gluons in dimensional reduction. This space has to be a sub-space of QDS in order for the dimensionally reduced theory to be gauge invariant.

Hence it cannot be identified with the original 4S - it can only be constructed as a "quasi-4-dimensional space" (Q4S) with certain 4-dimensional properties. The metric tensor is denoted by $g^{\mu\nu}$,

$$g^{\mu\nu} g_{\mu\nu} = 4 \tag{4-3}$$

The particles both observed and unobserved should be treated in the following way (see table **4-1**),

- Unobserved or internal particles need to be regularized because they appear inside a divergent loop or for real correction diagrams.
- and for the observed or external particles (external gluons) the regularization is optional.

Now, since external gluons do not have to be treated in the same way as internal ones, it is in fact possible to distinguish two variants of each regularization.
The two variants of dimensional regularization are:

- CDR ("conventional dimensional regularization"): Here internal and external gluons (and other vector fields) are all treated as D-dimensional.
- HV (" 't Hooft Veltman scheme"): Internal gluons are treated as D-dimensional but external ones are treated as strictly 4-dimensional.

Note that the above definition of internal gluons in phase space integrals is necessary for unitarity but leads to complications in the treatment of phase space integrals in schemes where internal and external gluons are treated differently.
The two analogous variants of dimensional reduction are:

- DRED ("original/old dimensional reduction"): Internal and external gluons are all treated as quasi-4-dimensional.
- FDH ("four-dimensional helicity scheme"): Internal gluons are treated as quasi-4-dimensional but external ones are treated as strictly 4-dimensional.

	CDR	HV	FDH	DRED
Internal Gluon	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$g^{\mu\nu}$
External Gluon	$\hat{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$g^{\mu\nu}$

Table **4-1**.: Treatment of internal and external gluons in the four different regularization schemes, i.e. prescription for which metric tensor is to be used in propagator numerators and polarization sums[58].

In the following studies we are going to use the FDH scheme, where the external particles are kept in four dimensions and internal (or virtual) particles are put in $D = 4 - 2\epsilon$ dimensions.
By studying the virtual particles, is important to understand the behavior of these particles in $D = 4 - 2\epsilon$, therefore, we need a new formalism for gluons and quarks in $D = 4 - 2\epsilon$, which will be studied in the next sections for internal gluons and quarks. First we introduce our new formalism that will be studied and tested with previous results of [18], then we compared it with the OPP method [26]

4.2. Quigley & Rozali brackets

The Spinor-Helicity formalism methods have been used to compute tree level amplitudes in D-dimensions. In the FDH regularization scheme, the momentum is continued to $D = 4 - 2\epsilon$ dimensions, where L was decomposed as[6, 23],

$$L^\alpha = l^\alpha + \mu^\alpha \tag{4-4}$$

$$\not{L} = \not{l} + \not{\mu} \tag{4-5}$$

here l is the four dimensional component, and μ is a component in a formal (-2ϵ)-dimensional orthogonal space.

$$L^2 = l^2 - \mu^2 \tag{4-6}$$

The usual conventions for the Dirac algebra[1],

$$\left\{\gamma^\alpha, \gamma^\beta\right\} = 2g^{\alpha\beta} \tag{4-7}$$

In this equation, α and β are $(4 - 2\epsilon)$-dimensional Lorentz indices and the metric is $g^{\alpha\beta} = \mathrm{diag}\,(+, -, -, -, -, \ldots)$. It follows that,

$$2L \cdot L = \left\{\not{L}, \not{L}\right\} = \left\{\not{l}, \not{l}\right\} + \left\{\not{\mu}, \not{\mu}\right\} + 2\left\{\not{l}, \not{\mu}\right\} = 2l \cdot l + 2\mu \cdot \mu + 4l \cdot \mu = 2\left(l^2 - \mu^2\right) \tag{4-8}$$

here, $4l{\cdot}\mu$ vanishes because we have chosen μ^α to be in a sub-space othogonal to the four-dimensional space containing l^α and the minus sign in $\mu \cdot \mu = -\mu^2$ comes from the metric.
So on-shell massless momentum $\left(L^2 = 0\right)$ is equivalent to four-dimensional massive momentum $l^2 = \mu^2$. Therefore for scalars, working away from 4 dimensions is equivalent to adding a mass to the scalar field (see appendix E).

For fermionic lines we always have the sum over the intermediate spinor wavefunctions, so choice of basis is not necessary. We will use the Quigley-Rozali notation (QR brackets) $|L\}\,, \{L|$ to refer collectively to these wavefunctions, the sum over wavefunctions is perfomed using the identity[23],

$$|L\}\,\{L| = \not{L} = \not{l} + \not{\mu} \tag{4-9}$$

in eq. (4-9), $\not{L}$ has one component $\left(\not{l}\right)$ that preseves helicity and one that flips it $\left(\not{\mu}\right)$. This notation glosses over the distinction of spinors and antispinors and can be understood by summing overall internal states.
Similarly, in $4-2\epsilon$ dimensions the components $\not{l}$ and $\not{\mu}$ behave differently with respect to chirality, $\left\{\not{l}, \gamma^5\right\} = 0$ whereas $\left[\not{\mu}, \gamma^5\right] = 0$.

We want to use helicity-like states for external fermions, even when they are in $4-2\epsilon$-dimensions. As we keep γ^5 four dimensional, we can still use chiral basis by,

$$|L\rangle = \omega_+ \,|L\}\,, \qquad |L] = \omega_- \,|L\}\,, \tag{4-10}$$

$$\langle L| = \{L|\,\omega_+, \qquad [L| = \{L|\,\omega_-. \tag{4-11}$$

where $\omega_\pm = \frac{1\pm\gamma^5}{2}$. Moreover, the basis vectors $|L\rangle$ and $|L]$ do not individually satisfy the massless Dirac equation in D-dimensions. That equation is written in terms of the Weyl fermions,

$$\not{l}\,|l\rangle + \not{\mu}\,|l] = 0, \qquad \not{l}\,|l] + \not{\mu}\,|l\rangle = 0. \tag{4-12}$$

which is consistent with the mass-shell condition

$$l^2 = \mu^2 \tag{4-13}$$

Nevertheless, we can assemble the physical amplitudes with external wave functions

$$|L\} = |L\rangle + |L] \tag{4-14}$$

whenever we encounters an intermediate $4-2\epsilon$-dimensional fermion and the denominator of the propagator can be written as

$$|L\}\,\{L| = \not{L} = (|L\rangle + |L])\,(\langle L| + [L|) \tag{4-15}$$

here the propagator has both helicity preserving and helicity flipping parts.
The helicity preserving parts are the usual propagators,

$$|L\rangle\,[L| = \omega_+\not{l}, \qquad |L]\,\langle L| = \omega_-\not{l}, \tag{4-16}$$

whereas the helicity flipping parts are new,

$$|L\rangle\,\langle L| = \omega_+\not{\mu}, \qquad |L]\,[L| = \omega_-\not{\mu} \tag{4-17}$$

4.3. Generalized Dirac equation

We are concerned with extensions of the Dirac equation. The generalized, matrix-valued mass term M enters the Dirac equation in the form,

$$(i\gamma^\mu\partial_\mu - M)\,\psi\,(x) = 0 \tag{4-18}$$

It is quite surprising that a systematic presentation of the solutions of the generalized Dirac equations, in the helicity basis has not been recorded in the literature to the best of our knowledge[44].

Extensions of the Dirac equation with pseudoscalar mass term that contains the fifth current have been introduced in [45]. It shows that for a mass term of the form $M = m + i\mu\gamma^5$, the generalized Dirac equation is written as,

$$\left(i\gamma^\mu\partial_\mu - m - i\mu\gamma^5\right)\psi\,(x) = 0, \tag{4-19}$$

with the dispersion relation $E = \sqrt{\vec{l}^2 + m^2 + \mu^2}$.
We may indicate a further motivation for our study: the unitarity of the S-matrix implies the existence of uselful relations for even powers $(\mu)^{2n}$ obtained upon expanding at one-loop amplitude, formulated with a mass term $m + i\mu\gamma^5$, in powers of μ. This implies that a better understanding of the Dirac equation with two mass terms can be of much more general interests.
When we calculate the one-loop amplitudes from generalized unitarity (see section 3.4) the even powers of μ will be taken into account because these powers give us information about the contributions coming from the box, triangle and bubble configurations.

4.3.1. Generalized Massive Spinors

We need to find the generalized spinors that satisfy,

$$\left(\not{l} - m - i\mu\gamma_5\right) u\left(l,\sigma\right) = 0, \tag{4-20}$$

$$\bar{u}\left(-l,\sigma\right)\left(\not{l} + m + i\mu\gamma_5\right) = 0 \tag{4-21}$$

These spinors are

$$u_+\left(l\right) = \left|l^\flat\right\rangle + \frac{\left(m+i\mu\right)}{\left[l^\flat \bar{l}\right]}\left|\bar{l}\right], \tag{4-22}$$

$$u_-\left(l\right) = \left|l^\flat\right] + \frac{\left(m-i\mu\right)}{\left\langle l^\flat \bar{l}\right\rangle}\left|\bar{l}\right\rangle \tag{4-23}$$

where l is a massive 4-vector, such that

$$l^2 = m^2 + \mu^2 \tag{4-24}$$

and $\bar{l}$,$l^\flat$ are massless 4-vectors, such that,

$$\bar{l}^2 = \left(l^\flat\right)^2 = 0 \tag{4-25}$$

Here $\not{l}$ has been decomposed in

$$\not{l} = \not{l}^\flat + \frac{l^2}{2l\cdot\bar{l}}\bar{\not{l}} \tag{4-26}$$

and the completeness relation (see apendix M)

$$u_-\left(l\right)\bar{u}_-\left(k\right) + u_+\left(l\right)\bar{u}_+\left(l\right) = \not{l} + m - i\mu\gamma_5 \tag{4-27}$$

$$\sum_{\lambda=\pm} u_\lambda\left(l\right)\bar{u}_\lambda\left(l\right) = \not{l} + m - i\mu\gamma_5 \tag{4-28}$$

in eq. (4-27) the conjugated spinors are given by,

$$\bar{u}_-\left(l\right) = \left\langle l^\flat\right| + \frac{\left(m+i\mu\right)}{\left[\bar{l} l^\flat\right]}\left[\bar{l}\right|, \tag{4-29}$$

$$\bar{u}_+\left(l\right) = \left[l^\flat\right| + \frac{\left(m-i\mu\right)}{\left\langle \bar{l} l^\flat\right\rangle}\left\langle\bar{l}\right|, \tag{4-30}$$

For the antiparticles sector

$$v_-\left(l\right) = \left|l^\flat\right\rangle - \frac{\left(m+i\mu\right)}{\left[l^\flat \bar{l}\right]}\left|\bar{l}\right], \tag{4-31}$$

$$v_+\left(l\right) = \left|l^\flat\right] - \frac{\left(m-i\mu\right)}{\left\langle l^\flat \bar{l}\right\rangle}\left|\bar{l}\right\rangle, \tag{4-32}$$

and the conjugated

$$\bar{v}_+(l) = \left\langle l^\flat \right| - \frac{(m+i\mu)}{\left[\bar{l}l^\flat\right]} \left[\bar{l}\right|, \tag{4-33}$$

$$\bar{v}_-(l) = \left[l^\flat \right| - \frac{(m-i\mu)}{\left\langle\bar{l}l^\flat\right\rangle} \left\langle\bar{l}\right|, \tag{4-34}$$

by repeating the steps in eq. (4-27) we obtain

$$v_-(l)\,\bar{v}_-(l) + v_+(l)\,\bar{v}_+(l) = \not{l} - m + i\mu\gamma_5 \tag{4-35}$$

$$\sum_{\lambda=\pm} v_\lambda(l)\,\bar{v}_\lambda(l) = \not{l} - m + i\mu\gamma_5 \tag{4-36}$$

the sum rule (4-35) satisfies the condition

$$\sum_{\lambda=\pm} u_\lambda(-l)\,\bar{u}_\lambda(-l) = -\sum_{\lambda=\pm} v_\lambda(l)\,\bar{v}_\lambda(l) \tag{4-37}$$

By studying the generalized Dirac equation we can understand how the QR brackets work, due to

$$|l\} = U(l,\sigma) = u_-(l) + u_+(l) \tag{4-38}$$

$$\{l| = \bar{U}(l,\sigma) = \bar{u}_-(l) + \bar{u}_+(l) \tag{4-39}$$

Then, the QR brackets are written as,

$$|l\} = |l\rangle + |l] = u_+(l) + u_-(l) \tag{4-40}$$

The QR brackets provide us an useful and simply tool to compute tree-level amplitudes in $D = 4-2\epsilon$, since the calculation with fermionic lines becomes easier.
With these QR brackets or generalized spinors, we get tree-level amplitudes with lines in $D = 4-2\epsilon$ (see appendix F)

4.4. Generalized Polarization Vectors

As we saw in previous sections, in $D = 4-2\epsilon$-dimensions we obtain an effective mass in 4 dimensions (eq. (4-13)), then we have to consider three physical helicity states.
First, we write a massive momentum l^μ of the polarization vector and make the following decomposition

$$l^\mu = l^{\flat\mu} + \bar{l}^\mu, \qquad \left(l^\flat\right)^2 = \left(\bar{l}\right)^2 = 0. \tag{4-41}$$

Due to this decomposition, l^μ is written in terms of two masssless momenta.
Taking into account eq. (3-61), l^μ becomes,

$$l^\mu = \frac{t}{2}\left\langle K_1^\flat\right|\gamma^\mu\left|K_2^\flat\right] - \frac{\mu^2}{2\gamma t}\left\langle K_2^\flat\right|\gamma^\mu\left|K_1^\flat\right] \tag{4-42}$$

$$l^{\flat\mu} = \frac{t}{2}\left\langle K_1^\flat\right|\gamma^\mu\left|K_2^\flat\right], \tag{4-43}$$

$$\bar{l}^\mu = -\frac{\mu^2}{2\gamma t}\left\langle K_2^\flat\right|\gamma^\mu\left|K_1^\flat\right] \tag{4-44}$$

A posible completeness relation,

$$l^\flat = \left|l^\flat\right] \left\langle l^\flat\right| + \left|l^\flat\right\rangle \left[l^\flat\right| = t\left(\left|K_2^\flat\right] \left\langle K_1^\flat\right| + \left|K_1^\flat\right\rangle \left[K_2^\flat\right|\right) \tag{4-45}$$

$$\bar{l} = |\bar{l}] \langle\bar{l}| + |\bar{l}\rangle [\bar{l}| = -\frac{\mu^2}{\gamma t}\left(\left|K_1^\flat\right] \left\langle K_2^\flat\right| + \left|K_2^\flat\right\rangle \left[K_1^\flat\right|\right) \tag{4-46}$$

$$\left[l^\flat \bar{l}\right] = \left\langle \bar{l} l^\flat\right\rangle = \mu \tag{4-47}$$

The generalized polarization vectors

$$\varepsilon_+^\mu(l) = -\frac{\left[l^\flat\left|\gamma^\mu\right|\bar{l}\right\rangle}{\sqrt{2}\mu}, \qquad \varepsilon_-^\mu(l) = -\frac{\left\langle l^\flat\left|\gamma^\mu\right|\bar{l}\right]}{\sqrt{2}\mu}, \qquad \varepsilon_0^\mu(l) = \frac{l^{\flat\mu} - \bar{l}^\mu}{\mu} \tag{4-48}$$

These polarization vectors are orthonormal and display all of the usual properties expected of polarization vectors:

$$\begin{array}{rrr} \varepsilon_+ \cdot \varepsilon_+ = 0 & \varepsilon_+ \cdot \varepsilon_- = -1 & \varepsilon_+ \cdot \varepsilon_0 = 0 \\ \varepsilon_- \cdot \varepsilon_+ = -1 & \varepsilon_- \cdot \varepsilon_- = 0 & \varepsilon_- \cdot \varepsilon_0 = 0 \\ \varepsilon_0 \cdot \varepsilon_+ = 0 & \varepsilon_0 \cdot \varepsilon_- = 0 & \varepsilon_0 \cdot \varepsilon_0 = -1 \end{array} \tag{4-49}$$

and it is easy to prove that all characteristics requirements of massive spin polarization vectors are satisfied in particular the completeness relation

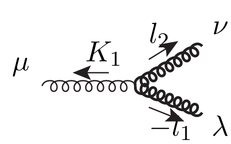
$$\sum_{\lambda=\pm,0} \varepsilon_\lambda^\mu(l)\varepsilon_\lambda^{*\nu}(l) = -g^{\mu\nu} + \frac{l^\mu l^\nu}{\mu^2} \tag{4-50}$$

It is worth to observe that

$$\varepsilon_+^\mu(l) = -\frac{\left[l^\flat\left|\gamma^\mu\right|\bar{l}\right\rangle}{\sqrt{2}\left[l^\flat \bar{l}\right]}, \qquad \varepsilon_-^\mu(l) = \frac{\left\langle l^\flat\left|\gamma^\mu\right|\bar{l}\right]}{\sqrt{2}\left\langle l^\flat \bar{l}\right\rangle}, \tag{4-51}$$

Moreover (4-50) just generalize the massless gluon polarization vectors with momentum $l^\flat$ and reference momentum $\bar{l}$.

4.4.1. Three point amplitudes



$$= \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}\left(K_1 - l_2\right)^\lambda + g^{\nu\lambda}\left(l_2 + l_1\right)^\mu + g^{\lambda\mu}\left(-l_1 - K_1\right)^\nu\right]$$

Figure **4-1**.: Three point vertex; the particle with momentym K_1 represents a gluon in 4 dimensions and the other two particles represent gluons in $4 - 2\epsilon$ dimensions.

Now consider the three point amplitude,

$$A_3\left(1,l_2,-l_1\right)=\frac{ig}{\sqrt{2}}\left[g^{\mu\nu}\left(K_1-l_2\right)^{\lambda}+g^{\nu\lambda}\left(l_2+l_1\right)^{\mu}+g^{\lambda\mu}\left(-l_1-K_1\right)^{\nu}\right]\varepsilon_{\mu}\left(1\right)\varepsilon_{\nu}\left(l_2\right)\varepsilon_{\lambda}\left(-l_1\right) \tag{4-52}$$

studying all posible helicity configurations, we obtain (for explicit calculation see appendix N),

$$A_3\left(1^+,l_2^-,l_1^+\right)=-ig\frac{\left[1l_1^\flat\right]^4}{\left[1l_2^\flat\right]\left[l_2^\flat l_1^\flat\right]\left[l_1^\flat 1\right]} \tag{4-53}$$

$$A_3\left(1^+,l_2^-,l_1^-\right)=ig\frac{\left\langle l_2^\flat l_1^\flat\right\rangle^4}{\left\langle 1l_2^\flat\right\rangle\left\langle l_2^\flat l_1^\flat\right\rangle\left\langle l_1^\flat 1\right\rangle} \tag{4-54}$$

$$A_3\left(1^+,l_2^+,l_1^0\right)=0 \tag{4-55}$$

$$A_3\left(1^+,l_2^-,-l_1^0\right)=0 \tag{4-56}$$

$$A_3\left(1^+,l_2^0,-l_1^0\right)=ig\frac{\langle q\left|l_1\right|1]}{\langle q1\rangle} \tag{4-57}$$

As we mentioned in eq. (4-51), the $\bar{l}_i$'s are the reference vectors of a gluon with momentum $l_i^\flat$. For our calculation we choose $\bar{l}_1=\bar{l}_2=\bar{l}$, with this the momentum conservation can be written as,

$$-l_1+p_1+l_2=0 \tag{4-58}$$

$$-l_1^\flat+p_1+l_2^\flat=0 \tag{4-59}$$

where l_1 and l_2 are

$$l_1=\bar{l}_1+l_1^\flat, \qquad l_2=\bar{l}_2+l_2^\flat. \tag{4-60}$$

Due to this formalism is possible to go from momentum conservation where two of the particles are massive to momentum conservation where all particles are massless now, which eases our calculation with the spinor-helicity formalism.

It is also important to see how eqs. (4-53) and (4-54) have the same structure of MHV amplitudes studied in section 2.3. On the other hand, eq. (4-57) has the same form of the three-point amplitude building block in scalar QCD (see appendix E). Finally, eqs. (4-55) and (4-56) show how is not possible to get a three-point amplitude where two gluons in $4-2\epsilon$ dimensions, one with zero polarization and the other one polarization $\pm$.

With these prescriptions we can obtain the cut-constructible and rational part by using generalized unitarity

- The Cut-constructible part is obtained by taking into account only the one-loop contributions that coming from the $\pm$ polarizations, because the $l_i^\flat$'s are massless momenta and we never obtain a term of the form $\mu^{2n}, n=1,2$. So the coefficients of the master integrals eq. (3-10) are obtained

- On the other hand, the rational part is obtained only if we take into account the zero polarization.
  As we saw in eq. (4-57) l_1 is a massive momentum, this suggests that we will find terms of the form $\mu^{2n}, n=1,2$ for each different cut.

In our calculations we are going to represent diagrammatically the gluons in $4-2\epsilon$ with zero polarization as dash lines and the gluons with $\pm$ polarizations as curl lines. Using this notation, let us draw processes to one-loop where external particles are gluons.
Consider the process $A_4^{1-loop}(1^+,2^+,3^+,4^+)$,

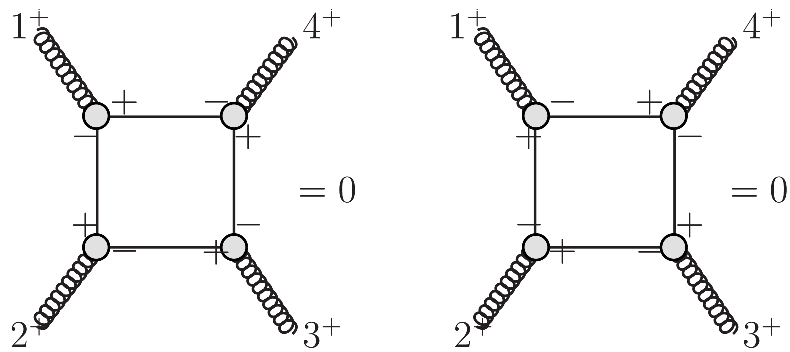
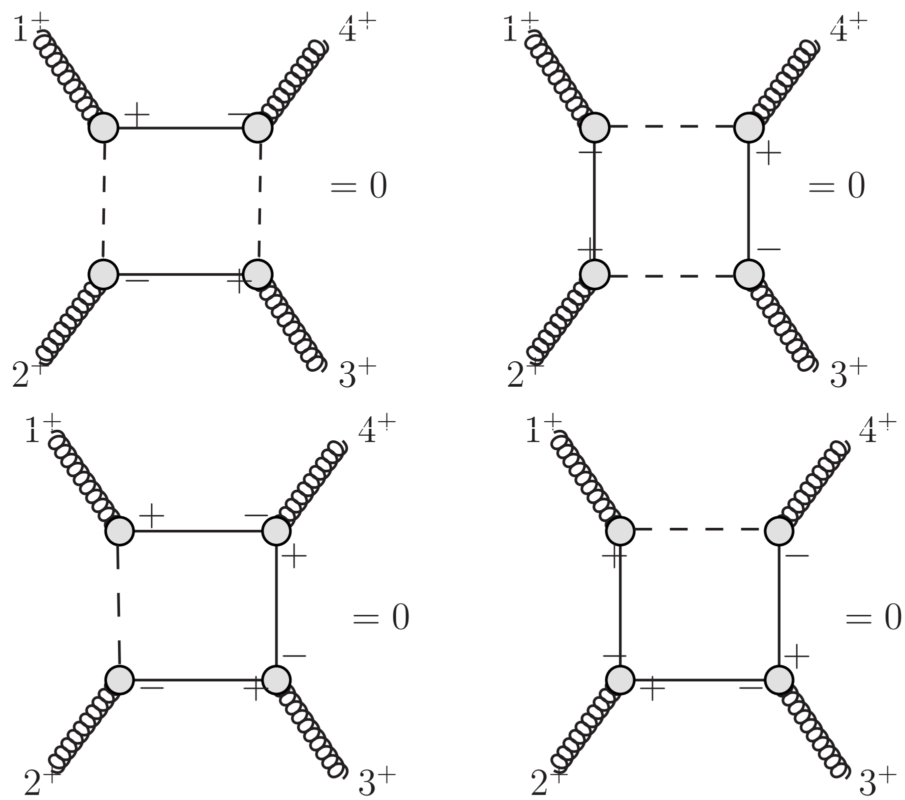


Figure **4-2**.: Cut constructible contributions to the process $A_4^{1-loop}(1^+,2^+,3^+,4^+)$



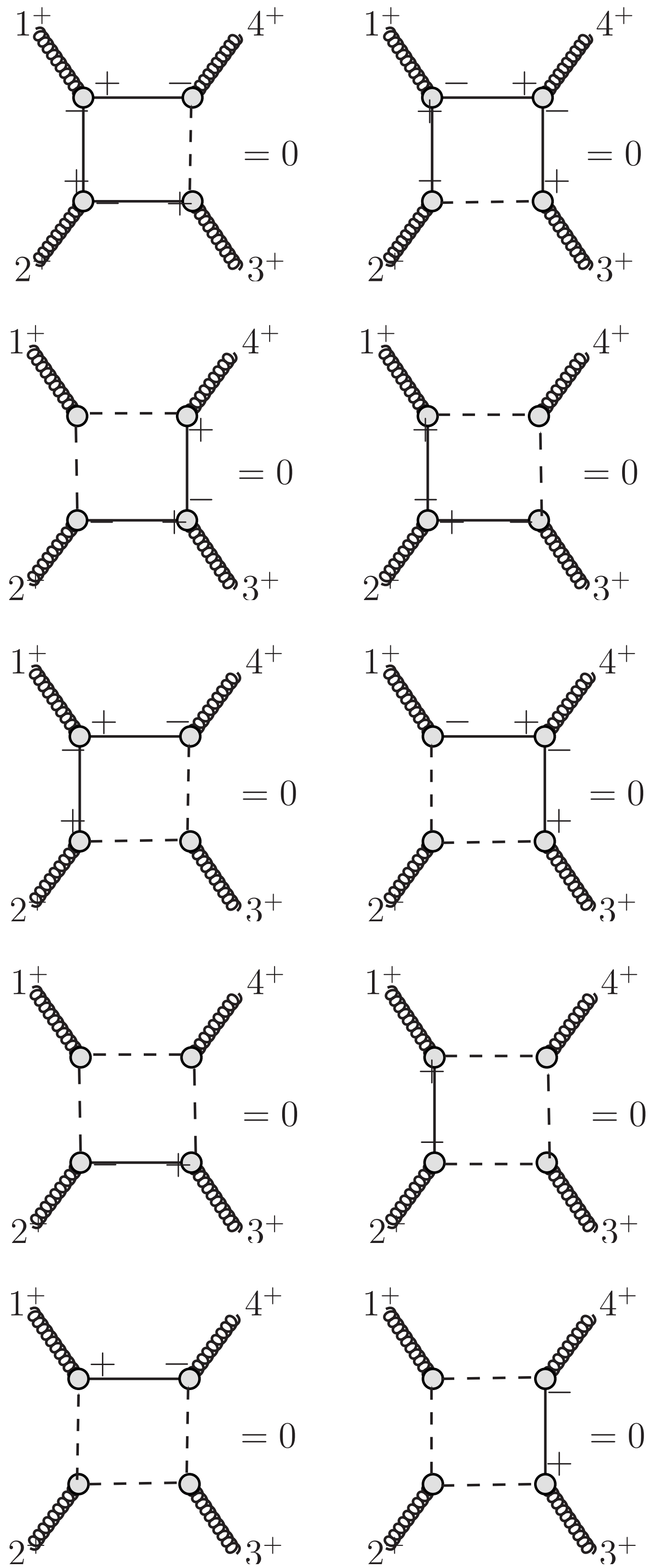


Figure **4-3**.: Box configurations where internal lines have polarizations ± and 0

The only remaing non-null diagram is

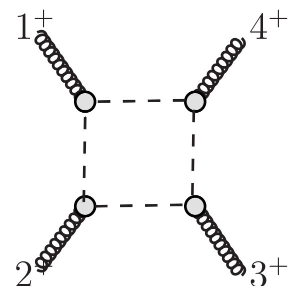


Figure **4-4**.: Rational contributions to the process $A_4^{1-loop}(1^+,2^+,3^+,4^+)$ from the box configuration.

4.4.2. Simple examples

In recent papers, Pittau [26, 59] has computed the rational part for the processes to one-loop of $e^+e^- \to \gamma$ and $\gamma\gamma \to \gamma\gamma$, so we can recover these results using our formalism.
Consider the QED process $e^+e^- \to \gamma$ to one-loop,

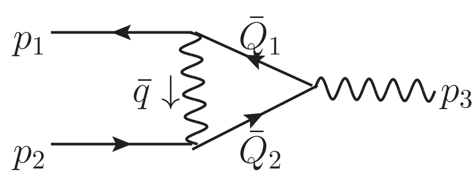


Figure **4-5**.: QED γe^+e^- diagram in $D = 4 - 2\epsilon$ dimensions.

the triple cut suggests us to cut the propagators and put them on-shel. Sewing the three three-point amplitudes,

$$C_{123} = (-ie)^3 \{L_1 |\gamma^\mu \varepsilon_\mu (k_3)| L_3\} \bar{u}(p_1) \varepsilon_0 (l_2) |L_1\} \{L_3| \varepsilon_0 (-l_2) u (p_2) \tag{4-61}$$

For the rational contribution we use QR brackets and zero polatization for unobserved (internal) particles.
Writing explicitly the polarization vectors and summing over internal states,

$$\begin{aligned} C_{123} &= -\frac{(-ie)^3}{\mu^2} \bar{u}(p_1) \left(l_2^\flat + \bar{l}_2\right) (L_1 + \mu) \gamma^\mu \varepsilon_\mu (k_3) (L_3 + \mu) \left(l_2^\flat + \bar{l}_2\right) u (p_2) \\ &= -\frac{(-ie)^3}{\mu^2} \bar{u}(p_1) \left[2 l_2^\flat \bar{l}_2 + \mu^2\right] \gamma^\mu \varepsilon_\mu (k_3) \left[2 \bar{l}_2 l_2^\flat + \mu^2\right] u (p_2) \\ &= -\mu^2 (-ie)^3 \bar{u}(p_1) \gamma^\mu \varepsilon_\mu (k_3) u (p_2) \end{aligned} \tag{4-62}$$

eq. (4-62) has been computed only for one Feynman diagram but we need consider the diagram where the fermions are exchanged, with this,

$$C_{123} = -2\mu^2 (-ie)^3 \bar{u}(p_1) \gamma^\mu \varepsilon_\mu (k_3) u (p_2) \tag{4-63}$$

we are interested in the rational contribution for this process, then taking $\mathrm{Inf_t}$ and Inf_{μ^2} and cutting external legs,

$$C_3^{[2]}\left(e^+e^- \to \gamma\right) = -2ie^3\bar{u}\left(p_1\right)\gamma^\mu\varepsilon_\mu\left(k_3\right)u\left(p_2\right) \tag{4-64}$$

with this result we can define an effective vertex,

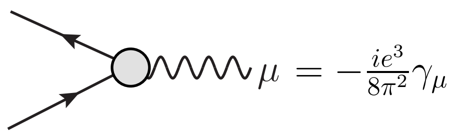


Figure **4-6**.: QED γe^+e^- effective vertex contributing to $\mathrm{R_2}$.

Following these procedures, we study the process in QED $\gamma\gamma \to \gamma\gamma$,

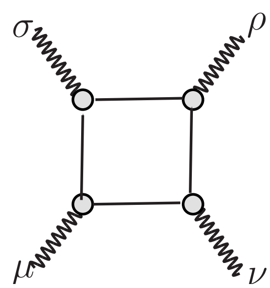


Figure **4-7**.: QED γe^+e^- diagram in $D = 4 - 2\epsilon$ dimensions.

Studing the box contribution to the rational part,

$$C_{1234} = (-ie)^4\left\{l_4\left|\varepsilon\left(1\right)\right|l_1\right\}\left\{l_1\left|\varepsilon\left(2\right)\right|l_2\right\}\left\{l_2\left|\varepsilon\left(3\right)\right|l_3\right\}\left\{l_3\left|\varepsilon\left(4\right)\right|l_4\right\} \tag{4-65}$$

summing over internal states,

$$C_{1234} = (-ie)^4\,\mathrm{Tr}\left[\left(l_4+\mu\right)\varepsilon\left(1\right)\left(l_1+\mu\right)\varepsilon\left(2\right)\left(l_2+\mu\right)\varepsilon\left(3\right)\left(l_3+\mu\right)\varepsilon\left(4\right)\right] \tag{4-66}$$

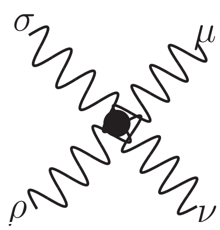
to compute this trace we remember that $\left[\gamma^5, \not{\mu}\right] = 0$.
Using momentum conservation, taking Inf_{μ^4} and cutting external legs,

$$C_{1234} = 4\left(-ie\right)^4\left(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\nu}g^{\rho\sigma}\right) \tag{4-67}$$

Moreover, this contribution is only for one specific box configuration, so we have to consider the other five contributions for this process. These contributions come from the exchange of photons. Using symmetry in eq. (4-67) we recover the result of [26],

$$C^{[4]}\left(\gamma\gamma \to \gamma\gamma\right) = \frac{2}{3}ie^4\left(g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\nu}g^{\rho\sigma}\right) \tag{4-68}$$

finally, we get an effective vertex,



$$= \frac{ie^4}{12\pi^2}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)$$

Figure **4-8**.: QED $\gamma\gamma \to \gamma\gamma$ effective vertex contributing to the rational part.

4.5. The OPP Method

In this section we give a briefly introduction of how Ossola-Papadopoulos-Pittau (OPP) method works. OPP method studies the internal particles following the FDH scheme[26].
The general expression for the integrand of a generic m-point color-ordered one-loop,

$$\bar{A}\left(\bar{q}\right) = \frac{\bar{N}\left(\bar{q}\right)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}}, \qquad \bar{D}_i = \left(\bar{q}_i + p_i\right)^2 - m_i^2, \quad p_0 \neq 0 \tag{4-69}$$

Following OPP notation, the bar denotes the objects living in $n = 4 - \epsilon$ dimensions. Furthemore $\bar{q}^2 = q^2 + \tilde{q}^2$, where $\tilde{q}^2$ is ϵ-dimensional and $q \cdot \tilde{q} = 0$. The numerator $\bar{N}\left(\bar{q}\right)$ can be also split into a 4-dimensional plus a ϵ-dimensional part,

$$\bar{N}\left(\bar{q}\right) = N\left(q\right) + \tilde{N}\left(\bar{q}^2, q, \epsilon\right) \tag{4-70}$$

$N\left(q\right)$ is 4-dimensional while $\tilde{N}\left(\bar{q}^2, q, \epsilon\right)$ gives rise to the Rational Terms of kind R_2, defined as

$$\mathrm{R}_2 = \frac{1}{\left(2\pi\right)^4}\int d^n\bar{q}\frac{\tilde{N}\left(\bar{q}\right)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} \tag{4-71}$$

$\tilde{N}\left(\bar{q}^2, q, \epsilon\right)$ is polynomial in μ^2 and linear in ϵ
The separation in eq. (4-70) implies,

$$\bar{q} = q + \tilde{q}, \tag{4-72}$$

$$\bar{\gamma}_{\bar{\mu}} = \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}, \tag{4-73}$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}} \tag{4-74}$$

To obtain R_2 we need to compute the whole contribution of $\bar{N}\left(\bar{q}\right)$ and then with eqs. (4-72), (4-73) and (4-74) we find $\tilde{N}\left(\bar{q}^2, q, \epsilon\right)$.
For clarify, we show the process to one-loop γe^+e^- (see fig. **4-5**) studied before, but now, we are going to use OPP method to compute Rational Terms of kind R_2.
The numerator $\bar{N}\left(\bar{q}\right)$ can be written as,

$$\begin{aligned}\bar{N}(\bar{q}) &\equiv e^3\left\{\bar{\gamma}_{\bar{\beta}}\ (\not{\bar{Q}}_1 + m_e)\ \gamma_\mu\ (\not{\bar{Q}}_2 + m_e)\ \bar{\gamma}^{\bar{\beta}}\right\} \\ &= e^3\left\{\gamma_\beta(\not{Q}_1 + m_e)\gamma_\mu(\not{Q}_2 + m_e)\gamma^\beta \right. \\ &\left. - \epsilon\left(\not{Q}_1 - m_e\right)\gamma_\mu(\not{Q}_2 - m_e) + \epsilon\tilde{q}^2\,\gamma_\mu - \tilde{q}^2\,\gamma_\beta\gamma_\mu\gamma^\beta\right\}\end{aligned} \tag{4-75}$$

The fisrt term in the l.h.s of eq. (4-75) is $N(q)$, while the sum of the remaining three define $\tilde{N}\left(\bar{q}^2, q, \epsilon\right)$. Inserting $\tilde{N}\left(\bar{q}^2, q, \epsilon\right)$ in eq. (4-71) R_2 gives

$$\mathrm{R}_2 = -\frac{ie^3}{8\pi^2}\gamma_\mu + \mathcal{O}(\epsilon) \tag{4-76}$$

this result agrees with ours.

Taking into account this effective vertex, the problem of computing R_2 is reduced to a tree level calculation and we consider it fully solved. The R_1 part is, instead, deeply connected to the structure of the one-loop amplitude[26, 59]. It is worth to mention that only the full $\mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2$ constitutes a physical gauge-invariant quantity in dimensional regularization.

5. Left and Right Turning contribution to the amplitude $A_4\left(1_g, 2_g, 3_{\bar{q}}, 4_q\right)$

In this section we show in details how the amplitudes $A_4^{1-loop}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$ and $A_4^{1-loop}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right)$ are obtained using the formalism descripted in chapter 4 where internal lines (like gluons or fermions) are in $4-2\epsilon$. These results for each amplitudes have been checked with [18].

In tree level amplitudes that include fermions and/or gluons we write down internal fermionic lines as capital letter (L) and gluonic lines as lowercase letter (l), external particles are represented by the number of each particle.

5.1. Cut Constructible part

5.1.1. Tree-Level Amplitudes

In this calculation we are going to use the same notation of ref. [18] where the tree-level amplitudes, $A_4\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$ and $A_4\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right)$ are given by,

$$c_{4;0}\left(+,-;+-\right) = -i\frac{\langle 24\rangle\langle 23\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{5-1}$$

$$c_{4;0}\left(-,+;+-\right) = -i\frac{\langle 14\rangle\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{5-2}$$

the other ones necessary amplitudes are given in appendices E, F.

5.1.2. Quadrupole cut coefficient

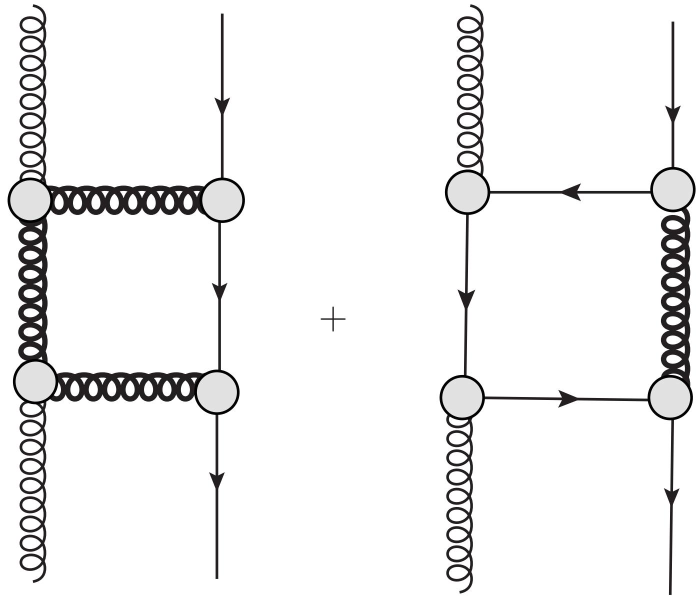


Figure **5-1**.: Left and Right turning configurations for the box contributions

For the quadrupole cut coefficient we have the following prescription

$$C_4^{[0]} = \frac{i}{2}\sum_{\sigma=\pm} A_1 A_2 A_3 A_4 \left(\bar{l}_1^{\sigma}\right) \tag{5-3}$$

$A_4^{box}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$

Left turning For the box contribution we get two configurations, in which the MHV-$\overline{\text{MHV}}$ sequence have been considered. Then the products of tree level amplitudes are given by,

$$\begin{aligned}
C_{1234}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) &= A_3^{tree}\left(-l_1^+, 1_g^+, l_2^-\right) A_3^{tree}\left(-l_2^+, 2_g^-, l_3^-\right) A_3^{tree}\left(-l_3^+, 3_{\bar{q}}^-, L_4^+\right) A_3^{tree}\left(-L_4^-, 4_q^+, l_1^-\right) \\
&+ A_3^{tree}\left(-l_1^-, 1_g^+, l_2^-\right) A_3^{tree}\left(-l_2^+, 2_g^-, l_3^+\right) A_3^{tree}\left(-l_3^-, 3_{\bar{q}}^-, L_4^+\right) A_3^{tree}\left(-L_4^-, 4_q^+, l_1^+\right) \\
&= \left(\frac{\langle 2|l_3|4] \langle l_3 2\rangle [l_1 1]}{[l_1 3] \langle l_3 1\rangle} + \frac{\langle 3|l_3|1]^2 \langle 21\rangle}{\langle 41\rangle \langle 3|l_1|2]}\right) s_{14} = \frac{4\langle 2|3\rangle^3 \langle 2|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle} \frac{s_{12}^2 s_{14}}{s_{13}} - \frac{\left(-s_{14} - 2s_{13}\right)\langle 23\rangle^2 \langle 2|4\rangle}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \frac{s_{12}s_{14}}{s_{13}}
\end{aligned} \tag{5-4}$$

Here l_1 is given by,

$$l_1^{\mu} = \frac{\xi}{2}\langle 1|\gamma^{\mu}|4], \qquad \xi = -\frac{[12]}{[24]} \tag{5-5}$$

Finally by averaging over two solutions,

$$\boxed{C_4^{[0]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = c_{4;0}\left(+,-;+-\right)\left(s_{12}s_{14} + \frac{1}{2}\frac{s_{12}^2 s_{14}}{s_{13}}\right)} \tag{5-6}$$

Right Turning Here we get only one configurations

$$C_{4321}\left(1_g^+,2_g^-,3_{\bar q}^-,4_q^+\right) = A_3^{tree}\left(-L_1^+,1_g^+,L_2^-\right)A_3^{tree}\left(-L_2^+,2_g^-,L_3^-\right)A_3^{tree}\left(-L_3^+,3_{\bar q}^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,L_1^-\right)$$
$$= \frac{\langle 2|l_3 l_4 l_1|1]^2}{\langle 4|l_1 l_2 l_3|3]} = \frac{\langle 2|34 l_1|1]^2}{\langle 4|l_1 12|3]} = \frac{\langle 24\rangle\langle 23\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{12}^2 s_{14}}{s_{13}} \tag{5-7}$$

Here we have written l_3 and l_2 in terms of l_1 and put the explicit solution for l_1,

$$l_1^\mu = \frac{\xi}{2}\langle 1|\gamma^\mu|2], \qquad \xi = \frac{[14]}{[24]} \tag{5-8}$$

Finally by averaging over two solutions,

$$\boxed{C_4^{box}\left(1_g^+,2_g^-,3_{\bar q}^-,4_q^+\right) = \frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}^2 s_{14}}{s_{13}}} \tag{5-9}$$

$A_4^{box}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right)$

Left Turning The possible two configurations taking into account the MHV-$\overline{\text{MHV}}$ sequence are given by,

$$C_{1234} = A_3^{tree}\left(-l_1^+,1_g^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2_g^+,l_3^-\right)A_3^{tree}\left(-l_3^+,3_{\bar q}^-,L_4^+\right)A_3^{tree}\left(-L_4^-,4_q^+,l_1^-\right)+$$
$$+A_3^{tree}\left(-l_1^-,1_g^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2_g^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3_{\bar q}^-,L_4^+\right)A_3^{tree}\left(-L_4^-,4_q^+,l_1^+\right)$$
$$= -\frac{\langle 1|l_3|4]\langle 1|l_3|4|l_1|1|l_3|4]}{\langle 2|l_1|3]\langle 2|l_3|4]} + \frac{\langle 1|l_1|4]\langle 3|l_3|2]^3}{[2|1]\langle 3|l_1 4|3\rangle} \tag{5-10}$$

Using momentum conservation and the explicit solution for l_1 (eq. (5-5)),

$$C_{1234} = -\frac{\langle 1|l_3|4]\langle 1|l_3|4|l_1|1|l_3|4]}{\langle 2|l_1|3]\langle 2|l_3|4]} + \frac{\langle 1|l_1|4]\langle 3|l_3|2]^3}{[2|1]\langle 3|l_1 4|3\rangle} = -\frac{[4|1][4|3]\langle 1|3\rangle^3\langle 1|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle} + \frac{(s_{12}+s_{24})^3\langle 1|4\rangle\langle 3|4\rangle}{[2|1]\langle 2|4\rangle^3} \tag{5-11}$$

$$= \frac{\langle 1|3\rangle^3\langle 1|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 34\rangle\langle 41\rangle}\left(1-\frac{s_{14}^3}{s_{13}^3}\right)s_{12}s_{14} \tag{5-12}$$

using eq. (5-3)

$$C_4^{[0]}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right) = \frac{i}{2}\frac{\langle 1|3\rangle^3\langle 1|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 34\rangle\langle 41\rangle}\left(1-\frac{s_{14}^3}{s_{13}^3}\right)s_{12}s_{14} = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1-\frac{s_{14}^3}{s_{13}^3}\right)s_{12}s_{14} \tag{5-13}$$

$$\boxed{C_4^{[0]}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right) = \frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1-\frac{s_{14}^3}{s_{13}^3}\right)s_{12}s_{14}} \tag{5-14}$$

Right Turning Only one possible appears,

$$C_{4321}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=A_3^{tree}\left(-L_1^+,1_g^-,L_2^-\right)A_3^{tree}\left(-L_2^+,2_g^+,L_3^-\right)A_3^{tree}\left(-L_3^+,3_{\bar{q}}^-,l_4^-\right)A_3^{tree}\left(-l_4^+,4_q^+,L_1^-\right)$$
$$=\frac{\langle 1|l_1|2]\langle 3|l_1|2]\langle 3|l_1|4]}{\langle 3|l_3|1]}=-\frac{(s_{23}+s_{24})^2 s_{34}\langle 1|4\rangle}{[3|1]\langle 2|4\rangle^2}=\frac{\langle 13\rangle^3\langle 1|4\rangle}{\langle 12\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 41\rangle}\frac{s_{12}^3}{s_{13}^3}s_{12}s_{14} \quad (5\text{-}15)$$

For this result we have used the the explicit solution of l_1 given in eq. (5-8).
Finally by averaging over two solutions,

$$\boxed{C_4^{[0]}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}^3}{s_{13}^3}s_{12}s_{14}} \quad (5\text{-}16)$$

5.1.3. Triple cut coefficients

The triangle contribution can be obtained from,

$$C_3^{[0]}=\frac{1}{2}\sum_{\sigma=\pm}Inf_t\left[A_1A_2A_3\left(\bar{l}_1^\sigma\right)\right]_{t^0} \quad (5\text{-}17)$$

Moreover, we can also write a product of three tree-level amplitudes as the product of two tree-level amplitudes multiplied by a propagator, i.e.

$$A_1A_2A_3=-il_1^2A_1'A_2' \quad (5\text{-}18)$$

Triangle contributions are obtained from four differents configurations, see figs. **5-2** and **5-3** for the left and right -turning respectively.

Left Turning solutions

Solutions for the channel s_{12}

$$l_4^\mu\gamma_\mu=\frac{t}{2}\langle 4|\gamma^\mu|3]\gamma_\mu=t\left(|4\rangle[3|+|3]\langle 4|\right) \quad (5\text{-}19)$$
$$l_3^\mu\gamma_\mu=\left(\frac{t}{2}\langle 4|\gamma^\mu|3]+K_3^\mu\right)\gamma_\mu=\left(t|4\rangle+|3\rangle\right)[3|+|3]\left(t\langle 4|+\langle 3|\right) \quad (5\text{-}20)$$
$$l_1^\mu\gamma_\mu=\left(\frac{t}{2}\langle 4|\gamma^\mu|3]-K_4^\mu\right)\gamma_\mu=|4\rangle\left(t[3|-[4|\right)+\left(t|3]-|4]\right)\langle 4| \quad (5\text{-}21)$$

and the conjugate solution,

$$\bar{l}_4^\mu\gamma_\mu=\frac{t}{2}\langle 3|\gamma^\mu|4]\gamma_\mu=t\left(|3\rangle[4|+|4]\langle 3|\right) \quad (5\text{-}22)$$
$$\bar{l}_3^\mu\gamma_\mu=\left(\frac{t}{2}\langle 3|\gamma^\mu|4]+K_3^\mu\right)\gamma_\mu=|3\rangle\left(t[4|+[3|\right)+\left(t|4]+|3]\right)\langle 3| \quad (5\text{-}23)$$
$$\bar{l}_1^\mu\gamma_\mu=\left(\frac{t}{2}\langle 3|\gamma^\mu|4]-K_4^\mu\right)\gamma_\mu=\left(t|3\rangle-|4\rangle\right)[4|+|4]\left(t\langle 3|-\langle 4|\right) \quad (5\text{-}24)$$

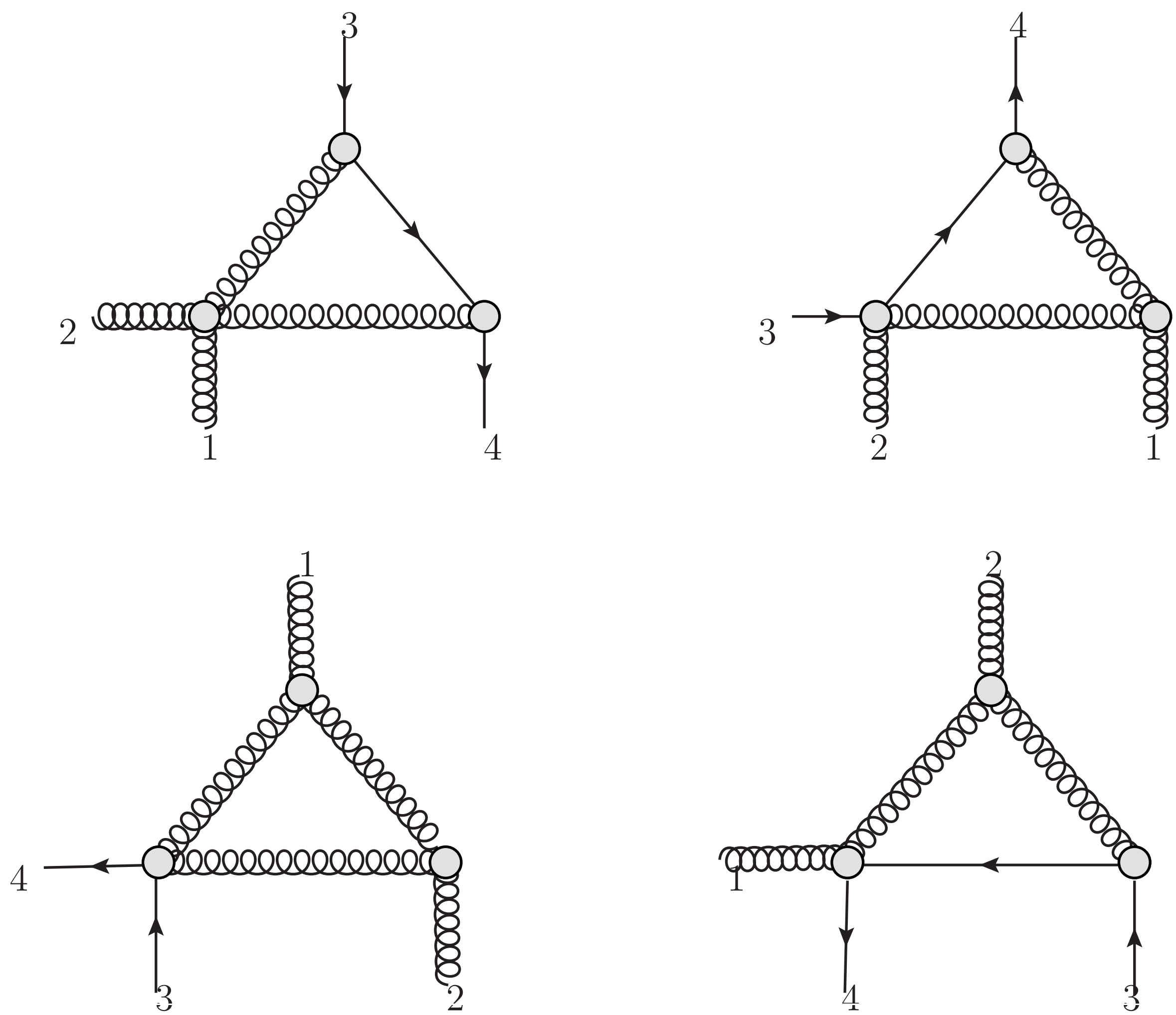


Figure **5-2**.: Left turning configurations for the triangle contributions

Solutions for the channel s_{23}

$$l_1^{\mu}\gamma_{\mu} = \frac{t}{2}\langle 1\,|\gamma^{\mu}|\,4]\,\gamma_{\mu} = t\left(|1\rangle\,[4| + |4]\,\langle 1|\right) \tag{5-25}$$

$$l_2^{\mu}\gamma_{\mu} = \left(\frac{t}{2}\langle 1\,|\gamma^{\mu}|\,4] - K_1^{\mu}\right)\gamma_{\mu} = |1\rangle\left(t\,[4| - [1|\right) + \left(t\,|4] - |1]\right)\langle 1| \tag{5-26}$$

$$l_4^{\mu}\gamma_{\mu} = \left(\frac{t}{2}\langle 1\,|\gamma^{\mu}|\,4] + K_4^{\mu}\right)\gamma_{\mu} = \left(t\,|1\rangle + |4\rangle\right)[4| + |4]\left(t\,\langle 1| + \langle 4|\right) \tag{5-27}$$

and the conjugate solution,

$$\bar{l}_1^\mu \gamma_\mu = \frac{t}{2} \langle 4 |\gamma^\mu| 1] \gamma_\mu = t \left(|4\rangle [1| + |1] \langle 4|\right) \tag{5-28}$$

$$\bar{l}_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 4 |\gamma^\mu| 1] - K_1^\mu\right) \gamma_\mu = \left(t |4\rangle - |1\rangle\right) [1| + |1] \left(t \langle 4| - \langle 1|\right) \tag{5-29}$$

$$\bar{l}_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 4 |\gamma^\mu| 1] + K_4^\mu\right) \gamma_\mu = |4\rangle \left(t [1| + [4|\right) + \left(t |1] + |4]\right) \langle 4| \tag{5-30}$$

Solutions for the channel s_{34}

$$l_2^\mu \gamma_\mu = \frac{t}{2} \langle 1 |\gamma^\mu| 2] \gamma_\mu = t \left(|1\rangle [2| + |2] \langle 1|\right) \tag{5-31}$$

$$l_1^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 |\gamma^\mu| 2] + K_1^\mu\right) \gamma_\mu = |1\rangle \left(t [2| + [1|\right) + \left(t |2] + |1]\right) \langle 1| \tag{5-32}$$

$$l_3^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 |\gamma^\mu| 2] - K_2^\mu\right) \gamma_\mu = \left(t |1\rangle - |2\rangle\right) [2| + |2] \left(t \langle 1| - \langle 2|\right) \tag{5-33}$$

and the conjugate solution,

$$\bar{l}_2^\mu \gamma_\mu = \frac{t}{2} \langle 2 |\gamma^\mu| 1] \gamma_\mu = t \left(|2\rangle [1| + |1] \langle 2|\right) \tag{5-34}$$

$$\bar{l}_1^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 |\gamma^\mu| 1] + K_1^\mu\right) \gamma_\mu = \left(t |2\rangle + |1\rangle\right) [1| + |1] \left(t \langle 2| + \langle 1|\right) \tag{5-35}$$

$$\bar{l}_3^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 |\gamma^\mu| 1] - K_2^\mu\right) \gamma_\mu = |2\rangle \left(t [1| - [2|\right) + \left(t |1] - |2]\right) \langle 2| \tag{5-36}$$

Solutions for the channel s_{41}

$$l_3^\mu \gamma_\mu = \frac{t}{2} \langle 2 |\gamma^\mu| 3] \gamma_\mu = t \left(|2\rangle [3| + |3] \langle 2|\right) \tag{5-37}$$

$$l_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 |\gamma^\mu| 3] - K_3^\mu\right) \gamma_\mu = \left(t |2\rangle - |3\rangle\right) [3| + |3] \left(t \langle 2| - \langle 3|\right) \tag{5-38}$$

$$l_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 |\gamma^\mu| 3] + K_2^\mu\right) \gamma_\mu = |2\rangle \left(t [3| + [2|\right) + \left(t |3] + |2]\right) \langle 2| \tag{5-39}$$

and the conjugate solution,

$$\bar{l}_3^\mu \gamma_\mu = \frac{t}{2} \langle 3 |\gamma^\mu| 2] \gamma_\mu = t \left(|3\rangle [2| + |2] \langle 3|\right) \tag{5-40}$$

$$\bar{l}_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 3 |\gamma^\mu| 2] - K_3^\mu\right) \gamma_\mu = |3\rangle \left(t [2| - [3|\right) + \left(t |2] - |3]\right) \langle 3| \tag{5-41}$$

$$\bar{l}_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 3 |\gamma^\mu| 2] + K_2^\mu\right) \gamma_\mu = \left(t |3\rangle + |2\rangle\right) [2| + |2] \left(t \langle 3| + \langle 2|\right) \tag{5-42}$$

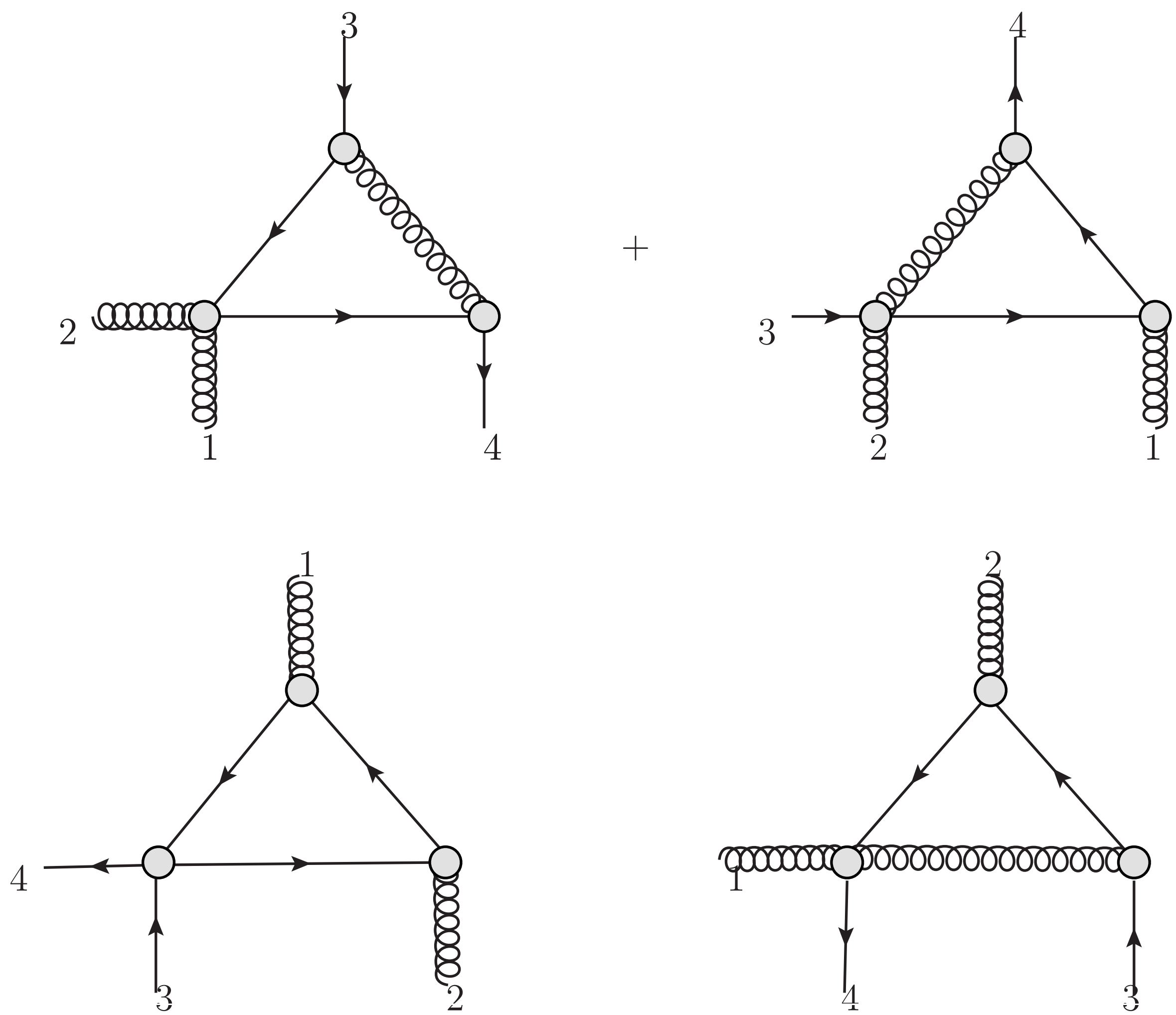


Figure **5-3**.: Right turning configurations for the triangle contributions

Right Turning solutions

Solutions for the channel s_{12}

$$l_3^\mu \gamma_\mu = \frac{t}{2} \langle 4 | \gamma^\mu | 3] \gamma_\mu = t \left(|4\rangle [3| + |3] \langle 4| \right) \tag{5-43}$$

$$l_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 4 | \gamma^\mu | 3] - K_3^\mu \right) \gamma_\mu = \left(t\, |4\rangle - |3\rangle \right) [3| + |3] \left(t \langle 4| - \langle 3| \right) \tag{5-44}$$

$$l_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 4 | \gamma^\mu | 3] + K_4^\mu \right) \gamma_\mu = |4\rangle \left(t\, [3| + [4| \right) + \left(t\, |3] + |4] \right) \langle 4| \tag{5-45}$$

and the conjugate solution,

$$\bar{l}_3^\mu \gamma_\mu = \frac{t}{2} \langle 3 | \gamma^\mu | 4] \gamma_\mu = t \left(|3\rangle [4| + |4] \langle 3| \right) \tag{5-46}$$

$$\bar{l}_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 3 | \gamma^\mu | 4] - K_3^\mu \right) \gamma_\mu = |3\rangle \left(t [4| - [3| \right) + \left(t |4] - |3] \right) \langle 3| \tag{5-47}$$

$$\bar{l}_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 3 | \gamma^\mu | 4] + K_4^\mu \right) \gamma_\mu = \left(t |3\rangle + |4\rangle \right) [4| + |4] \left(t \langle 3| + \langle 4| \right) \tag{5-48}$$

Solutions for the channel s_{23}

$$l_4^\mu \gamma_\mu = \frac{t}{2} \langle 1 | \gamma^\mu | 4] \gamma_\mu = t \left(|1\rangle [4| + |4] \langle 1| \right) \tag{5-49}$$

$$l_1^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 | \gamma^\mu | 4] + K_1^\mu \right) \gamma_\mu = |1\rangle \left(t [4| + [1| \right) + \left(t |4] + |1] \right) \langle 1| \tag{5-50}$$

$$l_3^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 | \gamma^\mu | 4] - K_4^\mu \right) \gamma_\mu = \left(t |1\rangle - |4\rangle \right) [4| + |4] \left(t \langle 1| - \langle 4| \right) \tag{5-51}$$

and the conjugate solution,

$$\bar{l}_4^\mu \gamma_\mu = \frac{t}{2} \langle 4 | \gamma^\mu | 1] \gamma_\mu = t \left(|4\rangle [1| + |1] \langle 4| \right) \tag{5-52}$$

$$\bar{l}_1^\mu \gamma_\mu = \left(\frac{t}{2} \langle 4 | \gamma^\mu | 1] + K_1^\mu \right) \gamma_\mu = \left(t |4\rangle + |1\rangle \right) [1| + |1] \left(t \langle 4| + \langle 1| \right) \tag{5-53}$$

$$\bar{l}_3^\mu \gamma_\mu = \left(\frac{t}{2} \langle 4 | \gamma^\mu | 1] - K_4^\mu \right) \gamma_\mu = |4\rangle \left(t [1| - [4| \right) + \left(t |1] - |4] \right) \langle 4| \tag{5-54}$$

Solutions for the channel s_{34}

$$l_1^\mu \gamma_\mu = \frac{t}{2} \langle 1 | \gamma^\mu | 2] \gamma_\mu = t \left(|1\rangle [2| + |2] \langle 1| \right) \tag{5-55}$$

$$l_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 | \gamma^\mu | 2] - K_1^\mu \right) \gamma_\mu = |1\rangle \left(t [2| - [1| \right) + \left(t |2] - |1] \right) \langle 1| \tag{5-56}$$

$$l_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 1 | \gamma^\mu | 2] + K_2^\mu \right) \gamma_\mu = \left(t |1\rangle + |2\rangle \right) [2| + |2] \left(t \langle 1| + \langle 2| \right) \tag{5-57}$$

and the conjugate solution,

$$\bar{l}_1^\mu \gamma_\mu = \frac{t}{2} \langle 2 | \gamma^\mu | 1] \gamma_\mu = t \left(|2\rangle [1| + |1] \langle 2| \right) \tag{5-58}$$

$$\bar{l}_4^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 | \gamma^\mu | 1] - K_1^\mu \right) \gamma_\mu = \left(t |2\rangle - |1\rangle \right) [1| + |1] \left(t \langle 2| - \langle 1| \right) \tag{5-59}$$

$$\bar{l}_2^\mu \gamma_\mu = \left(\frac{t}{2} \langle 2 | \gamma^\mu | 1] + K_2^\mu \right) \gamma_\mu = |2\rangle \left(t [1| + [2| \right) + \left(t |1] + |2] \right) \langle 2| \tag{5-60}$$

Solutions for the channel s_{41}

$$l_2^\mu\gamma_\mu = \frac{t}{2}\left\langle 2\left|\gamma^\mu\right|3\right]\gamma_\mu = t\left(|2\rangle\,[3| + |3]\,\langle 2|\right) \tag{5-61}$$

$$l_3^\mu\gamma_\mu = \left(\frac{t}{2}\left\langle 2\left|\gamma^\mu\right|3\right] + K_3^\mu\right)\gamma_\mu = \left(t\,|2\rangle + |3\rangle\right)[3| + |3]\left(t\,\langle 2| + \langle 3|\right) \tag{5-62}$$

$$l_1^\mu\gamma_\mu = \left(\frac{t}{2}\left\langle 2\left|\gamma^\mu\right|3\right] - K_2^\mu\right)\gamma_\mu = |2\rangle\left(t\,[3| - [2|\right) + \left(t\,|3] - |2]\right)\langle 2| \tag{5-63}$$

and the conjugate solution,

$$\bar{l}_2^\mu\gamma_\mu = \frac{t}{2}\left\langle 3\left|\gamma^\mu\right|2\right]\gamma_\mu = t\left(|3\rangle\,[2| + |2]\,\langle 3|\right) \tag{5-64}$$

$$\bar{l}_3^\mu\gamma_\mu = \left(\frac{t}{2}\left\langle 3\left|\gamma^\mu\right|2\right] + K_3^\mu\right)\gamma_\mu = |3\rangle\left(t\,[2| + [3|\right) + \left(t\,|2] + |3]\right)\langle 3| \tag{5-65}$$

$$\bar{l}_1^\mu\gamma_\mu = \left(\frac{t}{2}\left\langle 3\left|\gamma^\mu\right|2\right] - K_2^\mu\right)\gamma_\mu = \left(t\,|3\rangle - |2\rangle\right)[2| + |2]\left(t\,\langle 3| - \langle 2|\right) \tag{5-66}$$

$A_4^{Triangle}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$

Left turning The product of tree level amplitudes with the prescription given in eq. (5-18)

$$\begin{aligned} C_{134}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = &\left\{A_4^{tree}\left(-l_1^-, 1_g^+, 2,^-\, l_3^+\right)A_4^{tree}\left(-l_3^-, 3_{\bar{q}}^-, 4_q^+, l_1^+\right)\right. \\ &\left.+A_4^{tree}\left(-l_1^+, 1_g^+, 2,^-\, l_3^-\right)A_4^{tree}\left(-l_3^+, 3_{\bar{q}}^-, 4_q^+, l_1^-\right)\right\} il_4^2 \\ &= \frac{i[4|l_1]^2[1|l_3]^2\langle 3|l_3|1]^2}{[2|1][1|l_1][2|l_3]\langle 3|l_4|4][l_3|l_1]} + \frac{i\langle l_3|2\rangle\langle l_1|l_4 l_3|2\rangle^2}{\langle 1|2\rangle\langle 4|l_4|3]\langle l_1|1\rangle\langle l_1|l_3\rangle} \end{aligned} \tag{5-67}$$

Using momentum conservation, the explicit solution for l_4 and taking the Inf_t over (5-67),

$$\begin{aligned} C_{134}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) &= \frac{i[4|l_1]^2[1|l_3]^2\langle 3|l_4|1]^2}{[2|1][1|l_1][2|l_3]\langle 3|l_4|4][l_3|l_1]} + \frac{i\langle l_3|2\rangle\langle l_1|43|2\rangle^2}{\langle 1|2\rangle\langle 4|l_4|3]\langle l_1|1\rangle\langle l_1|l_3\rangle} \\ \Longrightarrow \inf_{t^0}\left[\frac{it^4[4|3]^2[1|3]^2\left\langle 34\right\rangle^2[31]^2}{[2|1]\left(t[1|3] - [1|4]\right)[2|3]ts_{34}[3|4]} - \frac{i\langle 3|2\rangle t^2\langle 3|43|2\rangle^2}{\langle 1|2\rangle ts_{34}\left(t\langle 3|1\rangle - \langle 4|1\rangle\right)\langle 4|3\rangle}\right] &= \frac{i\langle 2|3\rangle^3\langle 2|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}\frac{s_{14}s_{12}}{s_{13}} \end{aligned} \tag{5-68}$$

Averaging over two solutions,

$$\boxed{C_{3;12}^{[0]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}s_{14}}{s_{13}}} \tag{5-69}$$

We obtain the same result for the channel s_{34}

$$\boxed{C_{3;34}^{[0]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}s_{14}}{s_{13}}} \tag{5-70}$$

Now consider the channel s_{23}, the product of tree amplitudes is given by,

$$C_{124}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=A_4^{tree}\left(-l_2^+,2,^-3_{\bar{q}}^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_1^+\right)A_3^{tree}\left(-l_1^-,1^+,l_2^-\right)$$
$$=-\frac{i\langle 2|3\rangle^2[4|l_1]\langle l_1|l_2\rangle^2}{\langle l_1 1\rangle\langle l_2 1\rangle\langle l_2 3\rangle}=-\frac{i\langle 2|3\rangle^2[41]\langle 41\rangle^2}{\langle 41\rangle\langle 41\rangle\left(t\left\langle 43\right\rangle-\left\langle 13\right\rangle\right)}$$
$$=-\frac{i\langle 23\rangle^3\left\langle 24\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle}\frac{s_{12}s_{14}}{s_{13}}=-c_{4;0}\left(+,-;+-\right)\frac{s_{12}s_{14}}{s_{13}} \tag{5-71}$$

$$\boxed{C_{3;23}^{[0]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}s_{14}}{s_{13}}} \tag{5-72}$$

We obtain the same result for the channel s_{14}

$$\boxed{C_{3;41}^{[0]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}s_{14}}{s_{13}}} \tag{5-73}$$

The contributions of channels s_{23} and s_{14} have been obtained from only one triangle configuration, this is because we have followed the MHV-$\overline{\text{MHV}}$ sequence and had taken a three-point amplitude is zero if both fermions (quark - antiquark) have the same helicity.

Right turning The product of two three-point tree level amplitudes with another four-point tree level amplitude,

$$C_{432}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=A_4^{tree}\left(-l_2^-,2_g^-,1_g^+,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3_{\bar{q}}^-,l_2^+\right)$$
$$=\frac{i\langle 3|l_3|4]^2\langle l_2 2\rangle^2}{\langle 1|l_4|4]\langle l_2|1\rangle\langle l_2 3\rangle}=\frac{i\langle 3|l_3|4]^2\langle l_2 2\rangle^2}{\langle 1|l_3|4]\langle l_2 1\rangle\langle l_2 3\rangle}=\frac{is_{34}\left(t\left\langle 42\right\rangle-\left\langle 32\right\rangle\right)^2}{\left\langle 14\right\rangle\left(t\left\langle 41\right\rangle-\left\langle 31\right\rangle\right)}$$
$$\Longrightarrow\inf_{t^0}\left[\frac{is_{34}\left(t\left\langle 42\right\rangle-\left\langle 32\right\rangle\right)^2}{\left\langle 14\right\rangle\left(t\left\langle 41\right\rangle-\left\langle 31\right\rangle\right)}\right]=-\frac{is_{34}\left\langle 32\right\rangle^2}{\left\langle 14\right\rangle\left\langle 31\right\rangle}=\frac{i\left\langle 23\right\rangle^3\left\langle 24\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle}\frac{s_{12}s_{23}}{s_{13}} \tag{5-74}$$

The coefficients for the channel s_{12} and s_{34} take the form,

$$\boxed{C_{3;12}^{[0]}=-\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}s_{23}}{s_{13}}} \tag{5-75}$$

$$\boxed{C_{3;34}^{[0]}=-\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}s_{23}}{s_{13}}} \tag{5-76}$$

Finally we compute the coefficients for the channel s_{23} and s_{14},

$$C_{431}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=A_4^{tree}\left(-l_3^+,3_{\bar{q}}^-,2^-,l_1^+\right)A_3^{tree}\left(-l_1^-,1^+,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_3^-\right)$$
$$=\frac{i\langle 23\rangle^2\langle l_3|l_4|1]^2}{\langle l_3 3\rangle\langle l_3|l_1 l_4|4\rangle}=\frac{i\langle 23\rangle^2\langle l_3|4|1]^2}{\langle l_3 3\rangle\left\langle l_3 4\right\rangle s_{l_1,l_3}}=-\frac{i\langle 23\rangle^2\langle l_3|4|1]^2}{\langle l_3 3\rangle\left\langle l_3 4\right\rangle s_{23}}=-\frac{i\langle 23\rangle^2 t^2\langle 1|4|1]^2}{\left(t\left\langle 13\right\rangle-\left\langle 43\right\rangle\right)t\left\langle 14\right\rangle s_{23}}$$
$$\Longrightarrow\inf_{t^0}\left[-\frac{i\langle 23\rangle^2 t^2\langle 1|4|1]^2}{\left(t\left\langle 13\right\rangle-\left\langle 43\right\rangle\right)t\left\langle 14\right\rangle s_{23}}\right]=-\frac{i\langle 23\rangle^3\left\langle 24\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle}\frac{st}{u}=-c_{4;0}\left(+,-;+-\right)\frac{st}{u} \tag{5-77}$$

Here the coefficients are given by,

$$\boxed{C_{3;23}^{[0]} = \frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}s_{14}}{s_{13}}} \tag{5-78}$$

$$\boxed{C_{3;41}^{[0]} = \frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}s_{14}}{s_{13}}} \tag{5-79}$$

$A_4^{Triangle}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)$

Left Turning

$$\begin{aligned} C_{134}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) &= A_4^{tree}\left(-l_1^-,1^-,2^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3_{\bar{q}}^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_1^+\right) \\ &\quad + A_4^{tree}\left(-l_1^+,1^-,2^+,l_3^-\right)A_3^{tree}\left(-l_3^+,3_{\bar{q}}^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_1^-\right) \\ &= \frac{i\langle l3|1\rangle^2\langle l1|l4|l3|1\rangle^2}{\langle 1|2\rangle\langle 4|l4|3]\langle l1|1\rangle\langle l3|2\rangle\langle l1|l3\rangle} + \frac{i[4|l1]^2[2|l3]\langle 3|l3|2]^2}{[2|1][1|l1]\langle 3|l4|4][l3|l1]} \end{aligned} \tag{5-80}$$

taking the Inf_t and putting the explicit loop momentum solutions

$$\begin{aligned} &\Longrightarrow \inf_{t^0}\left[\frac{i\langle l3|1\rangle^2\langle l1|l4|l3|1\rangle^2}{\langle 1|2\rangle\langle 4|l4|3]\langle l1|1\rangle\langle l3|2\rangle\langle l1|l3\rangle} + \frac{i[4|l1]^2[2|l3]\langle 3|l3|2]^2}{[2|1][1|l1]\langle 3|l4|4][l3|l1]}\right] = -\frac{i[4|3]\langle 1|3\rangle^3}{\langle 1|2\rangle\langle 2|3\rangle} - \frac{i[3|2]^3[4|1]^2\langle 3|4\rangle}{[2|1][3|1]^3} \\ &= \frac{is_{12}\langle 1|3\rangle^3\langle 1|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle} + \frac{i\left\langle 13\right\rangle^3\left\langle 1|4\right\rangle}{\langle 1|2\rangle\langle 3|4\rangle\langle 2|3\rangle\langle 4|1\rangle}\frac{s_{14}^3}{s_{13}^3}s_{12} = -c_{4;0}\left(-,+;+,-\right)\left(1+\frac{s_{14}^3}{s_{13}^3}\right)s_{12} \end{aligned} \tag{5-81}$$

writting down the coefficient,

$$\boxed{C_{3;12}^{[0]}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1+\frac{s_{14}^3}{s_{13}^3}\right)s_{12}} \tag{5-82}$$

We study the channel s_{34},

$$\begin{aligned} C_{123}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) &= A_4^{tree}\left(-l_3^-,3_{\bar{q}}^-,4_q^+,l_1^+\right)A_3^{tree}\left(-l_1^-,1^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2^+,l_3^+\right) \\ &\quad + A_4^{tree}\left(-l_3^+,3_{\bar{q}}^-,4_q^+,l_1^-\right)A_3^{tree}\left(-l_1^+,1^-,l_2^-\right)A_3^{tree}\left(-l_2^+,2^+,l_3^-\right) \\ &= \frac{i[4|l3]^3\langle 1|l2|2]^3}{[4|3][2|l3]\langle 1|l1|4][l3|l2|l1|l3]} + \frac{i[2|l3]^3\langle 1|l1|3]\langle 1|l1|4]^2}{[4|3][3|l3]\langle 1|l2|l3][2|l2|l1|l3]} \end{aligned} \tag{5-83}$$

taking the Inf_t and putting the explicit loop momentum solutions

$$\begin{aligned} &\Longrightarrow \mathrm{inf}_{t^0}\left[\frac{i[4|l3]^3\langle 1|l2|2]^3}{[4|3][2|l3]\langle 1|l1|4][l3|l2|l1|l3]} + \frac{i[2|l3]^3\langle 1|l1|3]\langle 1|l1|4]^2}{[4|3][3|l3]\langle 1|l2|l3][2|l2|l1|l3]}\right] = \\ &= -\frac{i[4|2]^3\langle 1|2\rangle}{[4|1][4|3]} + \frac{i[3|2]^3[4|1]^2\langle 1|2\rangle}{[3|1]^3[4|3]} = \left[-\frac{i\langle 1|3\rangle^3\left\langle 14\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle} - \frac{i\left\langle 13\right\rangle^3\left\langle 14\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle}\frac{s_{14}^3}{s_{13}^3}\right]s_{12} \end{aligned} \tag{5-84}$$

Simple algebra reduces this expression to

$$\boxed{C^{[0]}_{3;34}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1-\frac{s_{14}^3}{s_{13}^3}\right)s_{12}} \tag{5-85}$$

The product of tree amplitudes in the channel s_{23},

$$\begin{aligned}C_{124}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)&=A_4^{tree}\left(-l_2^-,2^+,3_{\bar{q}}^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_1^-\right)A_3^{tree}\left(-l_1^+,1^-,l_2^+\right)\\&+A_4^{tree}\left(-l_2^-,2^+,3_{\bar{q}}^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_1^+\right)A_3^{tree}\left(-l_1^-,1^-,l_2^+\right)\\&=\frac{i[2|l4][l2|l1][2|l4|l1|l2]^2}{[1|l1][1|l2][2|l2][3|l4]\langle 4|l4|l2]}-\frac{i\langle 1|l1|4]^2\langle l1|1\rangle\langle l2|3\rangle^3}{\langle 2|3\rangle\langle 3|l4|4]\langle l2|1\rangle\langle l2|2\rangle\langle l1|l2\rangle}\end{aligned} \tag{5-86}$$

taking the Inf_t and putting the explicit loop momentum solutions

$$\begin{aligned}&\Longrightarrow\inf_{t^0}\left[\frac{i[2|l4][l2|l1][2|l4|l1|l2]^2}{[1|l1][1|l2][2|l2][3|l4]\langle 4|l4|l2]}-\frac{i\langle 1|l1|4]^2\langle l1|1\rangle\langle l2|3\rangle^3}{\langle 2|3\rangle\langle 3|l4|4]\langle l2|1\rangle\langle l2|2\rangle\langle l1|l2\rangle}\right]=\\&=0+\frac{is_{14}\left(\langle 1|4\rangle^2\langle 2|3\rangle^2+\langle 1|3\rangle\langle 1|4\rangle\langle 2|4\rangle\langle 2|3\rangle+\langle 1|3\rangle^2\langle 2|4\rangle^2\right)}{\langle 2|3\rangle\langle 2|4\rangle^3}=-\frac{i\langle 1|3\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(1+\frac{s_{14}^3}{s_{13}^3}\right)s_{14}\end{aligned} \tag{5-87}$$

writting down the coefficient,

$$\boxed{C^{[0]}_{3;23}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=-\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1+\frac{s_{14}^3}{s_{13}^3}\right)s_{14}} \tag{5-88}$$

Finally we study the channel s_{14},

$$\begin{aligned}C_{234}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)&=A_3^{tree}\left(-l_4^-,4^+,1^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2^+,l_3^-\right)A_3^{tree}\left(-l_3^+,3_{\bar{q}}^-,l_4^+\right)\\&+A_3^{tree}\left(-l_4^-,4^+,1^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3_{\bar{q}}^-,l_4^+\right)\\&=-\frac{i[2|l3]\langle 3|l3|2]^2\langle l4|1\rangle^3}{\langle 1|l2|2]\langle l4|3\rangle\langle l4|4\rangle\langle l4|l2|l3]}-\frac{i\langle l3|l2|4]\,[4|l2|l3|l4]^2}{[4|1][3|l4][4|l4]\langle 2|l2|1]\langle l3|2\rangle}\end{aligned} \tag{5-89}$$

taking the Inf_t and putting the explicit loop momentum solutions

$$\begin{aligned}&\Longrightarrow\inf_{t^0}\left[-\frac{i[2|l3]\langle 3|l3|2]^2\langle l4|1\rangle^3}{\langle 1|l2|2]\langle l4|3\rangle\langle l4|4\rangle\langle l4|l2|l3]}-\frac{i\langle l3|l2|4]\,[4|l2|l3|l4]^2}{[4|1][3|l4][4|l4]\langle 2|l2|1]\langle l3|2\rangle}\right]=\\&=\frac{i[3|2]\left(\langle 1|4\rangle^2\langle 2|3\rangle^2+\langle 1|3\rangle\langle 1|4\rangle\langle 2|4\rangle\langle 2|3\rangle+\langle 1|3\rangle^2\langle 2|4\rangle^2\right)}{\langle 2|4\rangle^3}+0\\&=-\frac{i\langle 1|3\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(1+\frac{s_{14}^3}{s_{13}^3}\right)s_{14}\end{aligned} \tag{5-90}$$

the coefficient takes the form,

$$\boxed{C^{[0]}_{3;41}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=-\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1+\frac{s_{14}^3}{s_{13}^3}\right)s_{14}} \tag{5-91}$$

Right Turning

Studying the channel s_{12},

$$\begin{aligned} C_{432}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) &= A_4^{tree}\left(-l_2^-,2^+,1^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3_{\bar{q}}^-,l_2^+\right) \\ &+ A_4^{tree}\left(-l_2^-,2^+,1^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_3^-\right)A_3^{tree}\left(-l_3^+,3_{\bar{q}}^-,l_2^+\right) \\ &= \frac{i[2|l4]^3\langle 3|l3|4]^2}{[2|1][1|l4][4|l4]\langle 3|l2|l4]} + \frac{i\langle l2|1\rangle\langle l4|l3|l2|1\rangle^2}{\langle 1|2\rangle\langle l2|2\rangle\langle l4|4\rangle\langle l4|l2|3]} \end{aligned} \quad (5\text{-}92)$$

taking the Inf_t and putting the explicit loop momentum solutions

$$\begin{aligned} &\Longrightarrow \inf_{t^0}\left[\frac{i[2|l4]^3\langle 3|l3|4]^2}{[2|1][1|l4][4|l4]\langle 3|l2|l4]} + \frac{i\langle l2|1\rangle\langle l4|l3|l2|1\rangle^2}{\langle 1|2\rangle\langle l2|2\rangle\langle l4|4\rangle\langle l4|l2|3]}\right] = \\ &= -\frac{i[3|2]\left([3|1]^2[4|2]^2+[2|1][3|1][4|3][4|2]+[2|1]^2[4|3]^2\right)\langle 3|4\rangle}{[2|1][3|1]^3} - \frac{i[4|3]\langle 1|3\rangle^3}{\langle 1|2\rangle\langle 2|3\rangle} = \\ &= \frac{i\langle 13\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(1+\frac{s_{12}^3}{s_{13}^3}\right)s_{12} + \frac{i\langle 1|3\rangle^3\langle 1|4\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}s_{12} \end{aligned} \quad (5\text{-}93)$$

Simple algebra reduces to

$$\boxed{C_{3;12}^{[0]}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) = \frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(2+\frac{s_{12}^3}{s_{13}^3}\right)s_{12}} \quad (5\text{-}94)$$

The channel s_{23},

$$\begin{aligned} C_{431} &= A_4^{tree}\left(-l_3^-,3_{\bar{q}}^-,2^+,l_1^+\right)A_3^{tree}\left(-l_1^-,1^-,l_4^+\right)A_3^{tree}\left(-l_4^-,4_q^+,l_3^+\right) \\ &= -\frac{i[2|l_1][4|l_3]^2\langle 1|l_1|2]}{[4|1][3|l_1][3|l_3]} \end{aligned} \quad (5\text{-}95)$$

taking the Inf_t and putting the explicit loop momentum solutions

$$\Longrightarrow \inf_{t^0}\left[-\frac{i[2|1]t^2[4|1]^2 t\langle 14\rangle[12]}{[4|1][3|1]\left(t[3|1]-[3|4]\right)}\right] = \frac{i\langle 1|2\rangle^2\langle 3|4\rangle^2}{\langle 2|3\rangle\langle 2|4\rangle^3}s_{14} = \frac{i\langle 1|3\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 2|3\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{12}^3}{s_{13}^3}s_{14} \quad (5\text{-}96)$$

writting down the coefficient,

$$\boxed{C_{3;23}^{[0]}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) = \frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}^3}{s_{13}^3}s_{14}} \quad (5\text{-}97)$$

Consider the channel s_{14},

$$\begin{aligned} C_{321} &= A_3^{tree}\left(-l_1^-,1^-,4^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3_{\bar{q}}^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2^+,l_1^+\right) \\ &= -\frac{i[2|l_1]\langle 3|l_2|4]^2}{[4|1][1|l_1]\langle 32\rangle} \end{aligned} \quad (5\text{-}98)$$

Using momentum conservation and the explicit solution for l_1,

$$\Longrightarrow \inf_{t^0}\left[-\frac{it[2|3]t^2\left\langle 32\right\rangle^2\left[34\right]^2}{[4|1]\left(t[1|3]-[1|2]\right)\left\langle 32\right\rangle}\right]=\frac{i[3|2]\langle 1|2\rangle^2\langle 3|4\rangle^2}{\langle 2|4\rangle^3}$$
$$=\frac{i\langle 1|3\rangle^3\left\langle 14\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle}\frac{s_{12}^3}{s_{13}^3}s_{14} \tag{5-99}$$

The coefficient is then,

$$\boxed{C_{3;41}^{[0]}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}^3}{s_{13}^3}s_{14}} \tag{5-100}$$

Finally, consider the channel s_{34}

$$C_{421}=A_4^{tree}\left(-l_4^-,4_q^+,3_{\bar{q}}^-,l_2^+\right)A_3^{tree}\left(-l_2^-,2^+,l_1^+\right)A_3^{tree}\left(-l_1^-,1^-,l_4^+\right)$$
$$=-\frac{i[2|l_1][4|l_2]^2\langle 1|l_1|2]}{[4|1][3|l_2][l_2|l_1]} \tag{5-101}$$

using momentum conservation and the explicit solution for l_2,

$$\Longrightarrow \inf_{t^0}\left[-\frac{i[2|1]\left(t[4|1]+[4|2]\right)^2s_{12}t}{[4|1]\left(t[3|1]+[3|2]\right)[2|1]}\right]=\frac{i\left\langle 13\right\rangle^3\left\langle 14\right\rangle}{\left\langle 12\right\rangle\left\langle 23\right\rangle\left\langle 34\right\rangle\left\langle 41\right\rangle}\frac{s_{12}^3}{s_{13}^3}s_{12} \tag{5-102}$$

The coefficient takes the form,

$$\boxed{C_{3;34}^{[0]}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}^3}{s_{13}^3}s_{12}} \tag{5-103}$$

5.1.4. Double cut coefficient

Left turning Configurations

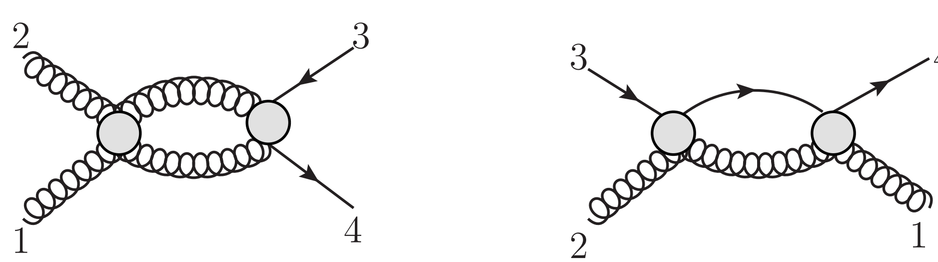


Figure **5-4**.: Left turning configurations for the Bubble contributions

Solution for the channel s_{12} l_1 and l_3 can be expressed as,

$$\langle l_1| = t\left\langle K_1^\flat\right| + \frac{S_1}{\gamma}(1-y)\langle\chi|, \qquad \langle l_3| = \left\langle K_1^\flat\right| - \frac{S_1}{\gamma}\frac{y}{t}\langle\chi|, \tag{5-104}$$

$$[l_1| = \frac{y}{t}\left[K_1^\flat\right| + [\chi|, \qquad [l_3| = (y-1)\left[K_1^\flat\right| + t[\chi|. \tag{5-105}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat = 2, \qquad \chi = 1 \tag{5-106}$$

$$\langle l_1| = t\langle 2| + (1-y)\langle 1|, \qquad \langle l_3| = \langle 2| - \frac{y}{t}\langle 1|, \tag{5-107}$$

$$[l_1| = \frac{y}{t}[2| + [1|, \qquad [l_3| = (y-1)[2| + t[1|. \tag{5-108}$$

To obtain the triangle contributions to the bubble coefficient we have to take another on-shell constrain given by $(l-K_3)^2 = 0$, in this case, $K_3 = 4$ (first triangle) and $K_3 = 2$ (second triangle). y is written in terms of t as,

$$y_\pm = \alpha_{1,\pm}t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} \tag{5-109}$$

$$\alpha_{1,\pm} = \frac{s_{24} - s_{14} \pm (s_{24}+s_{14})}{2\langle 1|4|2]} = \begin{cases}\alpha_{1,+} = \frac{s_{24}}{\langle 1|4|2]}\\ \alpha_{1,-} = -\frac{s_{14}}{\langle 1|4|2]}\end{cases} \tag{5-110}$$

$$\alpha_{2,\pm} = \frac{1}{2}(1\pm 1) = \begin{cases}\alpha_{2,+} = 1\\ \alpha_{2,-} = 0\end{cases} \tag{5-111}$$

$$\alpha_{3,\pm} = 0 \tag{5-112}$$

In the series expansion $[\inf_t A_1A_2](t) = \sum_{m=0}^{k} f_m t^m$, we make the replacements $t^m \to T_m$

$$T(0) = 0 \qquad T(1) = 2\frac{\langle 1|4|2]}{s_{12}^2}, \qquad T(2) = -3\frac{\langle 1|4|2]^2}{s_{12}^3}, \qquad T(3) = \frac{11}{3}\frac{\langle 1|4|2]^3}{s_{12}^4} \tag{5-113}$$

Solutions for the channel s_{23} l_2 and l_4 can be expressed as,

$$\langle l_2| = t\left\langle K_1^\flat\right| + \frac{S_1}{\gamma}(1-y)\langle\chi|, \qquad \langle l_4| = \left\langle K_1^\flat\right| - \frac{S_1}{\gamma}\frac{y}{t}\langle\chi|, \tag{5-114}$$

$$[l_2| = \frac{y}{t}\left[K_1^\flat\right| + [\chi|, \qquad [l_4| = (y-1)\left[K_1^\flat\right| + t[\chi|. \tag{5-115}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat = 3, \qquad \chi = 2 \tag{5-116}$$

$$\langle l_2| = t\langle 3| + (1-y)\langle 2|, \qquad \langle l_4| = \langle 3| - \frac{y}{t}\langle 2|, \tag{5-117}$$

$$[l_2| = \frac{y}{t}[3| + [2|, \qquad [l_4| = (y-1)[3| + t[2|. \tag{5-118}$$

To obtain the triangle contributions to the bubble coefficient we have to take another on-shell constrain given by $(l-K_3)^2=0$, in this case, $K_3=1$ (first triangle) and $K_3=3$ (second triangle). y is written in terms of t as,

$$y_{\pm}=\alpha_{1,\pm}t+\alpha_{2,\pm}+\frac{1}{t}\alpha_{3,\pm} \tag{5-119}$$

$$\alpha_{1,\pm}=\begin{cases}\alpha_{1,+}=\frac{s_{13}}{\langle 2|1|3]}\\ \alpha_{1,-}=-\frac{s_{12}}{\langle 2|1|3]}\end{cases} \tag{5-120}$$

$$\alpha_{2,\pm}=\frac{1}{2}(1\pm 1)=\begin{cases}\alpha_{2,+}=1\\ \alpha_{2,-}=0\end{cases} \tag{5-121}$$

$$\alpha_{3,\pm}=0 \tag{5-122}$$

In the series expansion $[\inf_t A_1A_2](t)=\sum_{m=0}^{k}f_mt^m$, we make the replacements $t^m\rightarrow T_m$

$$T(0)=0 \qquad T(1)=2\frac{\langle 2|1|3]}{s_{23}^2}, \qquad T(2)=-3\frac{\langle 2|1|3]^2}{s_{23}^3}, \qquad T(3)=\frac{11}{3}\frac{\langle 2|1|3]^3}{s_{23}^4} \tag{5-123}$$

Right turning Configurations

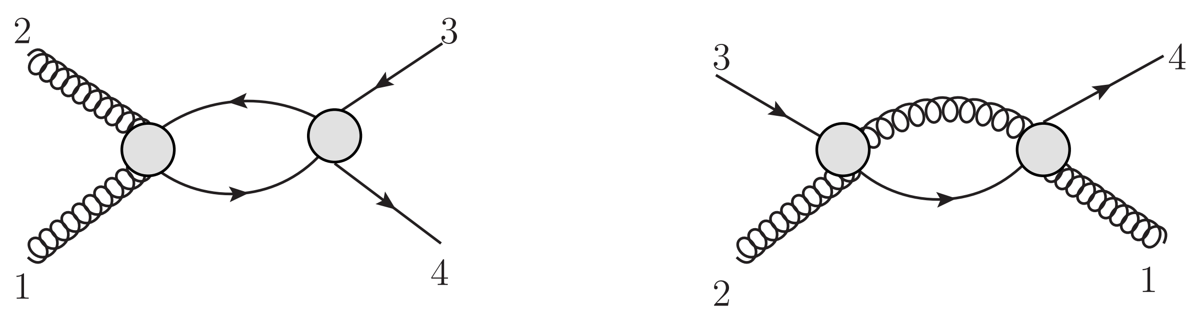


Figure **5-5**.: Left turning configurations for the Bubble contributions

Solution for the channel s_{12} l_2 and l_4 can be expressed as,

$$\langle l_4|=t\left\langle K_1^\flat\right|+\frac{S_1}{\gamma}(1-y)\langle\chi|, \qquad \langle l_2|=\left\langle K_1^\flat\right|-\frac{S_1}{\gamma}\frac{y}{t}\langle\chi|, \tag{5-124}$$

$$[l_4|=\frac{y}{t}\left[K_1^\flat\right|+[\chi|, \qquad [l_2|=(y-1)\left[K_1^\flat\right|+t[\chi|. \tag{5-125}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat=2, \qquad \chi=1 \tag{5-126}$$

$$\langle l_4|=t\langle 2|+(1-y)\langle 1|, \qquad \langle l_2|=\langle 2|-\frac{y}{t}\langle 1|, \tag{5-127}$$

$$[l_4|=\frac{y}{t}[2|+[1|, \qquad [l_2|=(y-1)[2|+t[1|. \tag{5-128}$$

To obtain the triangle contributions to the bubble coefficient we have to take another on-shell constrain given by $(l-K_3)^2=0$, in this case, $K_3=4$ (first triangle) and $K_3=2$ (second triangle). y is written in terms of t as,

$$y_{\pm}=\alpha_{1,\pm}t+\alpha_{2,\pm}+\frac{1}{t}\alpha_{3,\pm} \tag{5-129}$$

$$\alpha_{1,\pm}=\frac{s_{24}-s_{14}\pm(s_{24}+s_{14})}{2\langle 1|4|2]}=\begin{cases}\alpha_{1,+}=\frac{s_{24}}{\langle 1|4|2]}\\ \alpha_{1,-}=-\frac{s_{14}}{\langle 1|4|2]}\end{cases} \tag{5-130}$$

$$\alpha_{2,\pm}=\frac{1}{2}(1\pm 1)=\begin{cases}\alpha_{2,+}=1\\ \alpha_{2,-}=0\end{cases} \tag{5-131}$$

$$\alpha_{3,\pm}=0 \tag{5-132}$$

In the series expansion $[\inf_t A_1A_2](t)=\sum_{m=0}^{k}f_mt^m$, we make the replacements $t^m\to T_m$

$$T(0)=0 \qquad T(1)=2\frac{\langle 1|4|2]}{s_{12}^2}, \qquad T(2)=-3\frac{\langle 1|4|2]^2}{s_{12}^3}, \qquad T(3)=\frac{11}{3}\frac{\langle 1|4|2]^3}{s_{12}^4} \tag{5-133}$$

Solutions for the channel s_{23} l_1 and l_3 can be expressed as,

$$\langle l_1|=t\left\langle K_1^\flat\right|+\frac{S_1}{\gamma}(1-y)\langle\chi|, \qquad \langle l_3|=\left\langle K_1^\flat\right|-\frac{S_1}{\gamma}\frac{y}{t}\langle\chi|, \tag{5-134}$$

$$[l_1|=\frac{y}{t}\left[K_1^\flat\right|+[\chi|, \qquad [l_3|=(y-1)\left[K_1^\flat\right|+t[\chi|. \tag{5-135}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat=3, \qquad \chi=2 \tag{5-136}$$

$$\langle l_1|=t\langle 3|+(1-y)\langle 2|, \qquad \langle l_3|=\langle 3|-\frac{y}{t}\langle 2|, \tag{5-137}$$

$$[l_1|=\frac{y}{t}[3|+[2|, \qquad [l_3|=(y-1)[3|+t[2|. \tag{5-138}$$

To obtain the triangle contributions to the bubble coefficient we have to take another on-shell constrain given by $(l-K_3)^2=0$, in this case, $K_3=1$ (first triangle) and $K_3=3$ (second triangle). y is written in terms of t as,

$$y_{\pm}=\alpha_{1,\pm}t+\alpha_{2,\pm}+\frac{1}{t}\alpha_{3,\pm} \tag{5-139}$$

$$\alpha_{1,\pm}=\begin{cases}\alpha_{1,+}=\frac{s_{13}}{\langle 2|1|3]}\\ \alpha_{1,-}=-\frac{s_{12}}{\langle 2|1|3]}\end{cases} \tag{5-140}$$

$$\alpha_{2,\pm}=\frac{1}{2}(1\pm 1)=\begin{cases}\alpha_{2,+}=1\\ \alpha_{2,-}=0\end{cases} \tag{5-141}$$

$$\alpha_{3,\pm}=0 \tag{5-142}$$

In the series expansion $[\inf_t A_1 A_2](t) = \sum_{m=0}^{k} f_m t^m$, we make the replacements $t^m \to T_m$

$$T(0) = 0 \qquad T(1) = 2\frac{\langle 2|1|3]}{s_{23}^2}, \qquad T(2) = -3\frac{\langle 2|1|3]^2}{s_{23}^3}, \qquad T(3) = \frac{11}{3}\frac{\langle 2|1|3]^3}{s_{23}^4} \tag{5-143}$$

$A_4^{box}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$

Left Turning In the channel s_{12} the product of tree level amplitudes is given by,

$$\begin{aligned} C_{13} &= A_4^{tree}\left(-l_1^-, 1_g^+, 2,^- l_3^+\right) A_4^{tree}\left(-l_3^-, 3_{\bar{q}}^-, 4_q^+, l_1^+\right) + A_4^{tree}\left(-l_1^+, 1_g^+, 2,^- l_3^-\right) A_4^{tree}\left(-l_3^+, 3_{\bar{q}}^-, 4_q^+, l_1^-\right) \\ &= \frac{i}{l_4^2}\left\{\frac{i[4|l_1]^2[1|l_3]^2\langle 3|l_4|1]^2}{[2|1][1|l_1][2|l_3]\langle 3|l_4|4][l_3|l_1]} + \frac{i\langle l_3|2\rangle\langle l_1|43|2\rangle^2}{\langle 1|2\rangle\langle 4|l_4|3]\langle l_1|1\rangle\langle l_1|l_3\rangle}\right\} \end{aligned} \tag{5-144}$$

Writting the explicit solutions for l_1 and l_3, we find

$$\begin{aligned} C_{13} &= \frac{(y-1)^4 [1|2] (t\langle 32\rangle - y\langle 31\rangle)^2}{t^2 y\langle 34\rangle ((y-1)[42] + t[41])(t\langle 42\rangle + (1-y)\langle 41\rangle)} + \frac{y\langle 23\rangle [14]}{(t\langle 42\rangle - y\langle 41\rangle)(y[24] + t[14])} \\ &\Longrightarrow \inf_{t^0}\left[\inf_y [C_{13}]\right] = -2\frac{\langle 2|3\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}\frac{s_{12}}{s_{14}} - \frac{1}{2}\frac{\langle 24\rangle\langle 2|3\rangle^3}{\langle 12\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}\frac{s_{12}^2}{s_{14}^2} + 0 \end{aligned} \tag{5-145}$$

The pure bubble coefficient is,

$$C_{2;12}^{bubble[0]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = c_{4;0}(+,-;+-)\left(\frac{3}{2}\frac{s_{12}}{s_{14}} - \frac{1}{2}\frac{s_{12}s_{13}}{s_{14}^2}\right)$$

Now consider the triangle contributions,
Studying the triangle configuration for the channel s_{12},

$$C_{134} = \left\{\frac{i\left(y[4|2] + t[4|1]\right)(y-1)^4 [1|2] (t\langle 32\rangle - y\langle 31\rangle)^2}{t^3 y\langle 34\rangle ((y-1)[42] + t[41])} + \frac{iy(t\langle 24\rangle + (1-y)\langle 14\rangle)\langle 23\rangle [14]}{t(t\langle 42\rangle - y\langle 41\rangle)}\right\} \tag{5-146}$$

$$\begin{aligned} \inf_{t^m \to T_m}[C_{134}] &= \frac{2is_{24}\langle 2|4\rangle(\langle 1|4\rangle\langle 2|3\rangle - \langle 1|3\rangle\langle 2|4\rangle)(s_{14}\langle 1|4\rangle\langle 2|3\rangle + s_{14}\langle 1|3\rangle\langle 2|4\rangle + 2s_{24}\langle 1|3\rangle\langle 2|4\rangle)}{s_{12}(s_{14} + s_{24})\langle 1|2\rangle\langle 1|4\rangle^3\langle 3|4\rangle} + \\ &\quad + \frac{3is_{24}{}^2\langle 2|4\rangle(\langle 1|4\rangle\langle 2|3\rangle - \langle 1|3\rangle\langle 2|4\rangle)^2}{s_{12}{}^2\langle 1|2\rangle\langle 1|4\rangle^3\langle 3|4\rangle} + 0 \end{aligned} \tag{5-147}$$

$$= \frac{2i\langle 2|4\rangle\langle 2|3\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 4|1\rangle}(2s_{13} - s_{14})\frac{s_{13}}{s_{14}^2} - \frac{3i\langle 23\rangle^3\langle 2|4\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{24}^2}{s_{14}^2} \tag{5-148}$$

$$= -c_{4;0}(+,-;+-)\left[2\left(-\frac{s_{13}}{s_{14}} + 2\frac{s_{13}^2}{s_{14}^2}\right) - 3\frac{s_{13}^2}{s_{14}^2}\right] = c_{4;0}(+,-;+-)\left(3\frac{s_{13}}{s_{14}} + \frac{s_{13}s_{12}}{s_{!4}}\right) \tag{5-149}$$

The contribution from the triangle to the bubble is,

$$C_{2;12}^{triangle[0]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = -c_{4;0}(+,-;+-)\left(\frac{3}{2}\frac{s_{13}}{s_{14}} + \frac{1}{2}\frac{s_{13}s_{12}}{s_{!4}}\right) \tag{5-150}$$

The full bubble coefficient is given by,

$$C^{[0]}_{2;12} = C^{bubble[0]}_{2;12}\left(1^+_g, 2^-_g, 3^-_{\bar{q}}, 4^+_q\right) + C^{triangle[0]}_{2;12}\left(1^+_g, 2^-_g, 3^-_{\bar{q}}, 4^+_q\right)$$
$$\boxed{C^{[0]}_{2;12}\left(1^+_g, 2^-_g, 3^-_{\bar{q}}, 4^+_q\right) = \frac{3}{2}c_{4;0}\left(+,-;+-\right)} \tag{5-151}$$

In the channel s_{23} the product of tree level amplitudes is given by,

$$\begin{aligned} C_{24} &= \frac{i}{l_1^2}\left(-\frac{i\langle 2|3\rangle^2\left\langle l_2\left|1\right|4\right]^2}{\left\langle 1\left|l_2\right|4\right]\langle l_2 1\rangle\langle l_2 3\rangle}\right) = \frac{i}{\left\langle 1\left|l_2\right|1\right]}\left(\frac{i\langle 2|3\rangle^2\left[14\right]^2}{\left[l_2 4\right]\langle l_2 3\rangle}\right) \\ &= \frac{i}{\left(t\left\langle 13\right\rangle + (1-y)\left\langle 12\right\rangle\right)\left(\frac{y}{t}\left[31\right]+\left[21\right]\right)}\left(\frac{i\langle 2|3\rangle^2\left[14\right]^2}{\left(\frac{y}{t}\left[34\right]+\left[24\right]\right)(1-y)\left\langle 23\right\rangle}\right) \end{aligned} \tag{5-152}$$

this contribution vanishes,

$$\inf_y\left[C_{24}\right] = 0 \tag{5-153}$$

and the triangle contribution to the bubble coefficient also vanishes,

$$C_{124} = \frac{i\langle 2|3\rangle\left[14\right]^2}{\left(\frac{y}{t}\left[34\right]+\left[24\right]\right)(1-y)} \tag{5-154}$$

$$\inf_{t^m\to T_m}\left[C_{124}\right] = 0 \tag{5-155}$$

Finally,

$$\boxed{C^{[0]}_{2;23}\left(1^+_g, 2^-_g, 3^-_{\bar{q}}, 4^+_q\right) = 0} \tag{5-156}$$

$A^{box}_4\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right)$

Left Turning Studying the channel s_{12},

$$C_{13} = A^{tree}_4\left(-l^-_1, 1^-, 2^+, l^+_3\right)A^{tree}_4\left(-l^-_3, 3^-_{\bar{q}}, 4^+_q, l^+_1\right) + A^{tree}_4\left(-l^+_1, 1^-, 2^+, l^-_3\right)A^{tree}_4\left(-l^+_3, 3^-_{\bar{q}}, 4^+_q, l^-_1\right) \tag{5-157}$$

$$= \frac{[2|l3]^3\langle l3|3\rangle^2}{[2|1][1|l1]\langle l1|3\rangle\langle l1|4\rangle[l3|l1]} + \frac{[4|l3]^2\langle l3|1\rangle^4}{\langle 1|2\rangle[3|l1][4|l1]\langle l1|1\rangle\langle l3|2\rangle\langle l1|l3\rangle} \tag{5-158}$$

Putting the explicit solution for l_1 and l_3 and taking Inf_y,

$$\inf_y\left[\frac{t^2(t\langle 2|3\rangle - y\langle 1|3\rangle)^2}{y(t\langle 2|3\rangle - y\langle 1|3\rangle + \langle 1|3\rangle)(t\langle 2|4\rangle - y\langle 1|4\rangle + \langle 1|4\rangle)} + \frac{t^2(t[4|1] + y[4|2] - [4|2])^2}{y(t[3|1] + y[3|2])(t[4|1] + y[4|2])}\right] = 0$$

Triangle contributions,

$$\begin{aligned} C_{134} = {} & A^{tree}_4\left(-l^-_1, 1^-, 2^+, l^+_3\right)A^{tree}_3\left(-l^-_3, 3^-_{\bar{q}}, l^+_4\right)A^{tree}_3\left(-l^-_4, 4^+_q, l^+_1\right) \\ & + A^{tree}_4\left(-l^+_1, 1^-, 2^+, l^-_3\right)A^{tree}_3\left(-l^+_3, 3^-_{\bar{q}}, l^+_4\right)A^{tree}_3\left(-l^-_4, 4^+_q, l^-_1\right) \end{aligned} \tag{5-159}$$

$$= -\frac{i[4|l1]^2[2|l3]^3\langle l3|3\rangle^2}{[2|1][1|l1][4|l4]\langle l4|3\rangle[l3|l1]} - \frac{i\langle l3|1\rangle^4[l4|l3]^2\langle l1|l4\rangle^2}{\langle 1|2\rangle[3|l4]\langle l1|1\rangle\langle l3|2\rangle\langle l4|4\rangle\langle l1|l3\rangle} \tag{5-160}$$

with the explicit solutions for l_1 and l_3,

$$C_{134} = -\frac{it(t[4|1] + y[4|2])(t\langle 2|3\rangle - y\langle 1|3\rangle)^2}{y(t\langle 2|3\rangle - y\langle 1|3\rangle + \langle 1|3\rangle)} + \frac{it(t[4|1] + y[4|2] - [4|2])^2(-t\langle 2|4\rangle + y\langle 1|4\rangle - \langle 1|4\rangle)}{y(t[3|1] + y[3|2])} \tag{5-161}$$

$$\inf_{t^m \to T_m} [C_{134}] = \left(-\frac{2is_{14}\left(s_{14} - 2s_{13}\right)\langle 1|3\rangle^3}{{s_{13}}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} - \frac{3is_{14}\langle 1|3\rangle^3}{s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle}\right) + 0 \tag{5-162}$$

Finally,

$$\boxed{C^{[0]}_{2;12}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = c_{4;0}\left(-,+;+-\right)\left[\frac{s_{14}^2}{s_{13}^2} - \frac{1}{2}\frac{s_{14}}{s_{13}}\right]} \tag{5-163}$$

Studying the channel s_{23}

$$C_{24} = A^{tree}_4\left(-l^-_2, 2^+, 3^-_{\bar{q}}, l^+_4\right) A^{tree}_4\left(-l^-_4, 4^+_q, 1^-, l^+_2\right) \tag{5-164}$$

$$= \frac{[2|l4]^3\langle l4|1\rangle^3}{[2|l2][3|l4]\langle l2|1\rangle\langle l4|4\rangle[l4|l2]\langle l2|l4\rangle} \tag{5-165}$$

$$= \frac{(y-1)^3(y\langle 1|2\rangle - t\langle 1|3\rangle)^3}{t^2y\langle 2|3\rangle(-t\langle 1|3\rangle + y\langle 1|2\rangle - \langle 1|2\rangle)(y\langle 2|4\rangle - t\langle 3|4\rangle)} \tag{5-166}$$

$$\inf_y [C_{24}] = \frac{s_{12}s_{14}\left(s_{12} + 3s_{14}\right)\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} - \frac{1}{2}\frac{s_{12}{s_{14}}^2\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-167}$$

Triangle contributions,

$$C_{124} = A^{tree}_4\left(-l^-_2, 2^+, 3^-_{\bar{q}}, l^+_4\right) A^{tree}_3\left(-l^-_4, 4^+_q, l^+_1\right) A^{tree}_3\left(-l^-_1, 1^-, l^+_2\right) + A^{tree}_4\left(-l^-_2, 2^+, 3^-_{\bar{q}}, l^+_4\right) A^{tree}_3\left(-l^-_4, 4^+_q, l^-_1\right) A^{tree}_3\left(-l^+_1, 1^-, l^+_2\right)$$
$$= -\frac{i[2|l4]^3[l2|l1]^3\langle l1|l4\rangle^2}{[1|l1][1|l2][2|l2][3|l4]\langle l4|4\rangle[l4|l2]} + \frac{i[4|l1]^2\langle l1|1\rangle^3\langle l2|3\rangle^3}{\langle 2|3\rangle[4|l4]\langle l2|1\rangle\langle l2|2\rangle\langle l4|3\rangle\langle l1|l2\rangle} \tag{5-168}$$

with the explicit solutions for l_1, l_2 and l_3

$$C_{124} = \left(\frac{i(y-1)^3(t[4|2] + y[4|3])^3(y\langle 2|4\rangle - t\langle 3|4\rangle)}{t^3y[4|1](t[2|1] + y[3|1])}\right) + \left(\frac{i(y-1)^3(t[4|2] + y[4|3])^3(-t\langle 1|3\rangle + y\langle 1|2\rangle - \langle 1|2\rangle)}{t^3y[4|1](t[4|2] + y[4|3] - [4|3])}\right)$$
$$= 0 + \left(-\frac{6i{s_{12}}^2s_{14}\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} - \frac{3i{s_{12}}^3\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle}\right) \tag{5-169}$$

Using eqs. (5-167) and (5-169)

$$\boxed{C^{[0]}_{2;23}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = c_{4;0}\left(-,+;+-\right)\left[\frac{3}{2} - \frac{{s_{14}}^2}{{s_{13}}^2} + \frac{s_{14}}{2s_{13}}\right]} \tag{5-170}$$

Right Turning Studying the channel s_{12},

$$C_{13} = \frac{[4|l2]^2\langle l2|1\rangle^3}{[4|3]\langle 1|2\rangle[3|l2]\langle l2|2\rangle\langle l2|3\rangle} = \frac{t^2\langle 1|2\rangle(t[4|1]+y[4|2]-[4|2])^2}{y[4|3](t[3|1]+y[3|2]-[3|2])(t\langle 2|3\rangle - y\langle 1|3\rangle)} \tag{5-171}$$

$$\inf_y [C_{13}] = 0 \tag{5-172}$$

And the triangle contributions,

$$\begin{aligned} C_{234} &= -\frac{i[3|l2]\langle l2|1\rangle^3\langle l4|3\rangle^2}{\langle 1|2\rangle\langle l2|2\rangle\langle l4|4\rangle\langle l2|l4\rangle} + \frac{i[4|l2]^2[2|l4]^3\langle l2|3\rangle}{[2|1][1|l4][4|l4][l4|l2]} \\ &= \frac{it(t[3|1]+y[3|2]-[3|2])(t\langle 2|3\rangle - y\langle 1|3\rangle + \langle 1|3\rangle)^2}{y(t\langle 2|4\rangle - y\langle 1|4\rangle + \langle 1|4\rangle)} - \frac{it(t[4|1]+y[4|2]-[4|2])^2(t\langle 2|3\rangle - y\langle 1|3\rangle)}{y(t[4|1]+y[4|2])} \\ \inf_{t^m\to T_m}[C_{234}] &= 0 - \frac{i{s_{12}}^2t^3(t\langle 1|2\rangle\langle 3|4\rangle + \langle 1|3\rangle\langle 1|4\rangle)}{\langle 1|4\rangle(t\langle 2|4\rangle + \langle 1|4\rangle)(s_{12}t - [4|2]\langle 1|4\rangle)} \\ &= \frac{2is_{12}s_{14}\langle 1|3\rangle^3}{{s_{13}}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} - \frac{3is_{14}\langle 1|3\rangle^3}{s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \end{aligned}$$

finally,

$$C^{[0]}_{2;12}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = c_{4;0}\left(-,+;+-\right)\left[\frac{s_{12}s_{14}}{s_{13}^2} - \frac{3}{2}\frac{s_{14}}{s_{13}}\right] \tag{5-173}$$

$$\boxed{C^{[0]}_{2;12}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = c_{4;0}\left(-,+;+-\right)\left[\frac{s_{12}}{s_{13}}\left(\frac{s_{14}}{s_{13}} + \frac{3}{2}\right) + \frac{3}{2}\right]} \tag{5-174}$$

Consider the channel s_{23}

$$C_{24} = \frac{[4|l3]^2\langle l3|3\rangle^2}{[4|1]\langle 2|3\rangle[1|l1]\langle l1|2\rangle} = \frac{y^2(t[4|2]+y[4|3]-[4|3])^2}{t^2[4|1](t[2|1]+y[3|1])} \tag{5-175}$$

$$\inf_y [C_{24}] = \frac{s_{12}s_{14}\left(4s_{12}+s_{14}\right)\langle 1|3\rangle^3}{2{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} = \tag{5-176}$$

and the triangle contribution,

$$C_{124} = \frac{i[4|l3]\langle l1|1\rangle^2\langle l3|3\rangle^2}{\langle 2|3\rangle\langle l1|2\rangle\langle l1|l3\rangle} \tag{5-177}$$

$$= -\frac{iy^2(t[4|2]+y[4|3]-[4|3])(t\langle 1|3\rangle - y\langle 1|2\rangle + \langle 1|2\rangle)^2}{t^3\langle 2|3\rangle} \tag{5-178}$$

$$= -\frac{i[2|1]^2(t[3|2][4|1]-[3|1][4|3])\left(s_{14}t - [3|1]\langle 1|2\rangle\right)^2}{t[3|1]^5\langle 2|3\rangle} \tag{5-179}$$

$$\inf_{t^m\to T_m}[C_{234}] = -\frac{6i{s_{12}}^3\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} + \frac{3i{s_{12}}^3\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} = -\frac{3i{s_{12}}^3\langle 1|3\rangle^3}{{s_{13}}^3\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-180}$$

the total bubble coefficient,

$$C^{[0]}_{2;23}\left(1^-_g,2^+_g,3^-_{\bar{q}},4^+_q\right)=-\frac{is_{12}s_{14}\langle 1|3\rangle^3}{{s_{13}}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle}-\frac{3is_{12}\langle 1|3\rangle^3}{2s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-181}$$

$$\boxed{C^{[0]}_{2;23}\left(1^-_g,2^+_g,3^-_{\bar{q}},4^+_q\right)=-c_{4;0}\left(-,+;+-\right)\frac{s_{12}}{s_{13}}\left[\frac{s_{14}}{s_{13}}+\frac{3}{2}\right]} \tag{5-182}$$

5.2. Rational Parts

5.2.1. Box contributions

$A^{Box}_4\left(1^+_g,2^-_g,3^-_{\bar{q}},4^+_q\right)$

Left turning We define the product of four tree-level amplitudes as,

$$\begin{aligned}C_{1234}&=A^{tree}_3\left(-l^0_1,1^+_g,l^0_2\right)A^{tree}_3\left(-l^0_2,2^-,l^0_3\right)A^{tree}_3\left(-l^0_3,3^-_{\bar{q}},L_4\right)A^{tree}_3\left(-L_4,,4^+_q,l^0_1\right)\\&=\frac{\langle 2\left|l_1\right|1]^2\langle 3\left|l_1\right|4]}{s_{12}}\end{aligned} \tag{5-183}$$

The explicit solution for l_1 es given by,

$$l^\mu_1=\frac{c}{2}\langle 1\left|\gamma^\mu\right|2]-\frac{\mu^2}{2s_{12}c}\langle 2\left|\gamma^\mu\right|1] \tag{5-184}$$

$$c=\pm\mu\sqrt{\frac{\langle 2\left|3\right|1]}{\langle 1\left|3\right|2]}} \tag{5-185}$$

putting l_1 in C_{1234},

$$C_{1234}=c^2s_{12}\left(c\langle 31\rangle[24]-\frac{\mu^2}{c}\langle 32\rangle[14]\right)\propto\mu^3 \tag{5-186}$$

Taking the infinite function the box contribution vanishes,

$$\mathrm{Inf}_{\mu^4}\left[C_{1234}\right]=0 \tag{5-187}$$

$$\boxed{C^{[4]}_{4;1234}\left(1^+_g,2^-_g,3^-_{\bar{q}},4^+_q\right)=0} \tag{5-188}$$

Right turning We define the product of four tree-level amplitudes as,

$$\begin{aligned}C_{4321}&=A^{tree}_3\left(-L_4,4^+_q,l^0_3\right)A^{tree}_3\left(-l^0_3,3^-_{\bar{q}},L_2\right)A^{tree}_4\left(-L_2,2^-,L_1\right)A^{tree}_4\left(-L_1,1^+,L_4\right)\\&=-\frac{[1\left|L_2\right|2\rangle}{s_{12}\mu^2}\left(2\left\langle 3\left|l^\flat_3\bar{l}\right|2\right\rangle-\mu^2\langle 32\rangle\right)\left(2\left[1\left|\bar{l}l^\flat_3\right|4\right]-\mu^2[14]\right)\end{aligned} \tag{5-189}$$

l_3 can be written as,

$$l^\mu_3=\frac{c}{2}\langle 3\left|\gamma^\mu\right|4]-\frac{\mu^2}{2s_{34}c}\langle 4\left|\gamma^\mu\right|3] \tag{5-190}$$

$$c=\pm\mu\sqrt{\frac{\langle 4\left|1\right|3]}{\langle 3\left|1\right|4]}} \tag{5-191}$$

with $l_3^\flat$ and $\bar{l}$,

$$l_3^{\flat\mu} = \frac{c}{2}\left\langle 3\left|\gamma^\mu\right|4\right] \tag{5-192}$$

$$\bar{l}^\mu = -\frac{\mu^2}{2s_{12}c}\left\langle 4\left|\gamma^\mu\right|3\right] \tag{5-193}$$

Then C_{4321} takes the form,

$$\begin{aligned}
C_{4321} &= -\frac{\mu^2}{s_{12}}\left[1\left|L_2\right|2\right\rangle\langle 32\rangle[14] = -\frac{\mu^2}{s_{12}}\left[1\left|l_3-3\right|2\right\rangle\langle 32\rangle[14]\\
C_{4321} &= -\frac{\mu^2}{s_{12}}\left(c\langle 13\rangle[42] - \frac{\mu^2}{s_{12}c}\langle 14\rangle[32] - \left[1\left|3\right|2\right\rangle\right)\langle 32\rangle[14]\\
C_{4321} &\propto A\mu^3 + B\mu^2
\end{aligned} \tag{5-194}$$

In the last result we took into account $c \propto \mu$. Then the box contribution becomes,

$$\mathrm{Inf}_{\mu^4}\left[C_{4321}\right] = 0 \tag{5-195}$$

$$\boxed{C_{4;1234}^{[4]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-196}$$

$A_4^{Box}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right)$

Left turning We define C_{1234} as the product of tree level amplitudes,

$$\begin{aligned}
C_{1234} &= A_3^{tree}\left(-l_1^0, 1_g^-, l_2^0\right)A_3^{tree}\left(-l_2^0, 2_g^+, l_3^0\right)A_3^{tree}\left(-l_3^0, 3_{\bar{q}}^-, L_4\right)A_3^{tree}\left(-L_4, 4_q^+, l_1^0\right)\\
&= \frac{\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}}
\end{aligned} \tag{5-197}$$

The explicit solution for l_1 es given by,

$$l_1^\mu = \frac{c}{2}\left\langle 2\left|\gamma^\mu\right|1\right] - \frac{\mu^2}{2s_{12}c}\left\langle 1\left|\gamma^\mu\right|2\right] \tag{5-198}$$

$$c = \pm\mu\sqrt{\frac{\left\langle 1\left|3\right|2\right]}{\left\langle 2\left|3\right|1\right]}} \tag{5-199}$$

putting l_1 in C_{1234},

$$C_{1234} = c^2 s_{12}\left(c\langle 32\rangle[14] - \frac{\mu^2}{s_{12}c}\langle 31\rangle[24]\right) \propto \mu^3 \tag{5-200}$$

Taking the infinite function the box contribution vanishes,

$$\mathrm{Inf}_{\mu^4}\left[C_{1234}\right] = 0 \tag{5-201}$$

$$\boxed{C_{4;1234}^{[4]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-202}$$

Right turning We define the product of four tree-level amplitudes as,

$$C_{4321} = A_3^{tree}\left(-L_4, 4_q^+, l_3^0\right) A_3^{tree}\left(-l_3^0, 3_{\bar{q}}^-, L_2\right) A_4^{tree}\left(-L_2, 2^+, L_1\right) A_4^{tree}\left(-L_1, 1^-, L_4\right)$$
$$= \frac{1}{s_{12}\mu^2}\langle 1|\, L_1\,|2] \left(2\left[4\left|l_3^\flat \bar{l}\right|2\right] - \mu^2\,[42]\right)\left(2\left\langle 1\left|\bar{l} l_3^\flat\right|3\right\rangle - \mu^2\,\langle 13\rangle\right) \tag{5-203}$$

l_3 can be written as,

$$l_3^\mu = \frac{c}{2}\langle 3\,|\gamma^\mu|\,4] - \frac{\mu^2}{2s_{34}c}\langle 4\,|\gamma^\mu|\,3] \tag{5-204}$$

$$c = \pm\mu\sqrt{\frac{\langle 4\,|1|\,3]}{\langle 3\,|1|\,4]}} \tag{5-205}$$

with $l_3^\flat$ and $\bar{l}$,

$$l_3^{\flat\mu} = \frac{c}{2}\langle 3\,|\gamma^\mu|\,4] \tag{5-206}$$

$$\bar{l}^\mu = -\frac{\mu^2}{2s_{34}c}\langle 4\,|\gamma^\mu|\,3] \tag{5-207}$$

Then C_{4321} takes the form,

$$C_{4321} = \frac{\mu^2}{s_{12}}\langle 1|\,L_1\,|2]\,[42]\,\langle 13\rangle = \frac{\mu^2}{s_{12}}\langle 1|\,L_3 - 3\,|2]\,[42]\,\langle 13\rangle$$
$$C_{4321} = \frac{\mu^2}{s_{12}}\left(c\,\langle 13\rangle\,[42] - \frac{\mu^2}{s_{34}c}\langle 14\rangle\,[32] - \langle 1\,|3|\,2]\right)[42]\,\langle 13\rangle$$
$$C_{4321} \propto A\mu^3 + B\mu^2 \tag{5-208}$$

In the last result we took into account $c \propto \mu$. Then the box contribution becomes,

$$\mathrm{Inf}_{\mu^4}\left[C_{4321}\right] = 0 \tag{5-209}$$

$$\boxed{C_{4;1234}^{[4]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-210}$$

5.2.2. Triangle Contributions

The rational contribution that comes of the triangle is given by

$$C_3^{[2]} = \frac{1}{2}\sum_{\sigma=\pm}\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_1A_2A_3\left(\bar{l}_1^\sigma\right)\right]_{t^0}\right]_{\mu^2} \tag{5-211}$$

Left Turning solutions for loop momenta

Solutions for the channel s_{12}

$$l_4^\mu\gamma_\mu = t\left(|4\rangle\,[3| + |3]\,\langle 4|\right) - \frac{\mu^2}{2s_{34}t}\left(|3\rangle\,[4| + |4]\,\langle 3|\right) \tag{5-212}$$

$$\bar{l}_4^\mu\gamma_\mu = t\left(|3\rangle\,[4| + |4]\,\langle 3|\right) - \frac{\mu^2}{2s_{34}t}\left(|4\rangle\,[3| + |3]\,\langle 4|\right) \tag{5-213}$$

Solutions for the channel s_{23}

$$l_1^\mu \gamma_\mu = t\left(|4\rangle\,[1| + |1]\,\langle 4|\right) - \frac{\mu^2}{2s_{14}t}\left(|1\rangle\,[4| + |4]\,\langle 1|\right) \tag{5-214}$$

$$\bar{l}_1^\mu \gamma_\mu = t\left(|1\rangle\,[4| + |4]\,\langle 1|\right) - \frac{\mu^2}{2s_{14}t}\left(|4\rangle\,[1| + |1]\,\langle 4|\right) \tag{5-215}$$

Solutions for the channel s_{34}

$$l_2^\mu \gamma_\mu = t\left(|2\rangle\,[1| + |1]\,\langle 2|\right) - \frac{\mu^2}{2s_{12}t}\left(|1\rangle\,[2| + |2]\,\langle 1|\right) \tag{5-216}$$

$$\bar{l}_2^\mu \gamma_\mu = t\left(|1\rangle\,[2| + |2]\,\langle 1|\right) - \frac{\mu^2}{2s_{12}t}\left(|2\rangle\,[1| + |1]\,\langle 2|\right) \tag{5-217}$$

Solutions for the channel s_{14}

$$l_3^\mu \gamma_\mu = t\left(|2\rangle\,[3| + |3]\,\langle 2|\right) - \frac{\mu^2}{2s_{23}t}\left(|3\rangle\,[2| + |2]\,\langle 3|\right) \tag{5-218}$$

$$\bar{l}_3^\mu \gamma_\mu = t\left(|3\rangle\,[2| + |2]\,\langle 3|\right) - \frac{\mu^2}{2s_{23}t}\left(|2\rangle\,[3| + |3]\,\langle 2|\right) \tag{5-219}$$

Right Turning solutions for loop momenta

Solutions for the channel s_{12}

$$l_3^\mu \gamma_\mu = l_3^\flat + \bar{l} \tag{5-220}$$

$$l_3^\flat = t\left(|4\rangle\,[3| + |3]\,\langle 4|\right) \tag{5-221}$$

$$\bar{l} = -\frac{\mu^2}{2s_{34}t}\left(|3\rangle\,[4| + |4]\,\langle 3|\right) \tag{5-222}$$

$$\bar{l}_3^\mu \gamma_\mu = \bar{l}_3^\flat + \bar{\bar{l}} \tag{5-223}$$

$$\bar{l}_3^\flat = t\left(|3\rangle\,[4| + |4]\,\langle 3|\right) \tag{5-224}$$

$$\bar{\bar{l}} = -\frac{\mu^2}{2s_{34}t}\left(|4\rangle\,[3| + |3]\,\langle 4|\right) \tag{5-225}$$

Solutions for the channel s_{34}

$$l_1^\mu \gamma_\mu = l_3^\flat + \bar{l} \tag{5-226}$$

$$l_1^\flat = t\left(|2\rangle\,[1| + |1]\,\langle 2|\right) \tag{5-227}$$

$$\bar{l} = -\frac{\mu^2}{2s_{12}t}\left(|1\rangle\,[2| + |2]\,\langle 1|\right) \tag{5-228}$$

$$\bar{l}_1^\mu \gamma_\mu = \bar{l}_3^\flat + \bar{\bar{l}} \tag{5-229}$$

$$\bar{l}_1^\flat = t\left(|1\rangle\,[2| + |2]\,\langle 1|\right) \tag{5-230}$$

$$\bar{\bar{l}} = -\frac{\mu^2}{2s_{12}t}\left(|2\rangle\,[1| + |1]\,\langle 2|\right) \tag{5-231}$$

Solutions for the channel s_{23}

$$l_4^\mu \gamma_\mu = t\left(|4\rangle\,[1| + |1]\,\langle 4|\right) - \frac{\mu^2}{2s_{14}t}\left(|1\rangle\,[4| + |4]\,\langle 1|\right) \tag{5-232}$$

$$l_4^\flat = t\left(|1\rangle\,[4| + |4]\,\langle 1|\right) \tag{5-233}$$

$$\bar{l} = -\frac{\mu^2}{2s_{14}t}\left(|4\rangle\,[1| + |1]\,\langle 4|\right) \tag{5-234}$$

$$\bar{l}_4^\mu \gamma_\mu = t\left(|1\rangle\,[4| + |4]\,\langle 1|\right) - \frac{\mu^2}{2s_{14}t}\left(|4\rangle\,[1| + |1]\,\langle 4|\right) \tag{5-235}$$

$$\bar{l}_4^\flat = t\left(|4\rangle\,[1| + |1]\,\langle 4|\right) \tag{5-236}$$

$$\bar{\bar{l}} = -\frac{\mu^2}{2s_{14}t}\left(|1\rangle\,[4| + |4]\,\langle 1|\right) \tag{5-237}$$

Solutions for the channel s_{14}

$$l_2^\mu \gamma_\mu = t\left(|2\rangle\,[3| + |3]\,\langle 2|\right) - \frac{\mu^2}{2s_{23}t}\left(|3\rangle\,[2| + |2]\,\langle 3|\right) \tag{5-238}$$

$$l_2^\flat = t\left(|2\rangle\,[3| + |3]\,\langle 2|\right) \tag{5-239}$$

$$\bar{l} = -\frac{\mu^2}{2s_{23}t}\left(|3\rangle\,[2| + |2]\,\langle 3|\right) \tag{5-240}$$

$$\bar{l}_2^\mu \gamma_\mu = t\left(|3\rangle\,[2| + |2]\,\langle 3|\right) - \frac{\mu^2}{2s_{23}t}\left(|2\rangle\,[3| + |3]\,\langle 2|\right) \tag{5-241}$$

$$\bar{l}_2^\flat = t\left(|3\rangle\,[2| + |2]\,\langle 3|\right) \tag{5-242}$$

$$\bar{\bar{l}} = -\frac{\mu^2}{2s_{14}t}\left(|2\rangle\,[3| + |3]\,\langle 2|\right) \tag{5-243}$$

$A_4^{Triangle}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$

Left Turning Consider the triangle contribution from the channel s_{12}

$$\begin{aligned} C_{3;12}^{[2]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) &= A_4^{tree}\left(-l_1^0, 1_g^+, 2_g^-, l_3^0\right) A_4^{tree}\left(-l_3^0, 3_{\bar{q}}^-, 4_q^+, l_1^0\right) i l_4^2 \\ &= i\frac{\langle 2\,|l_1|\,1]^2}{s_{12}\,\langle 1\,|l_1|\,1]} i\,\langle 3\,|l_1|\,4] = -\frac{\langle 2\,|l_4 - 4|\,1]^2\,\langle 3\,|l_4|\,4]}{s_{12}\,\langle 1\,|l_4 - 4|\,1]} \end{aligned} \tag{5-244}$$

Writing the explicit solution for l_4,

$$\begin{aligned} &\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{\mu^2[4|3]\langle 3|4\rangle\left(-\frac{\mu^2[3|1]\langle 2|4\rangle}{2s_{34}t} + t[4|1]\langle 2|3\rangle - \langle 2|4|1]\right)^2}{2s_{12}s_{34}t\left(-\frac{\mu^2[3|1]\langle 1|4\rangle}{2s_{34}t} + t[4|1]\langle 1|3\rangle - \langle 1|4|1]\right)}\right]\right](l_4) = \\ &= -\frac{[4|1]\langle 2|4\rangle(2\langle 1|4\rangle\langle 2|3\rangle - \langle 1|3\rangle\langle 2|4\rangle)}{[2|1]\langle 1|2\rangle\langle 1|4\rangle^2} = \frac{\langle 24\rangle\langle 23\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(2 + \frac{s_{13}}{s_{23}}\right) \end{aligned} \tag{5-245}$$

and its conjugate solution,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{2t[4|3]\langle 3|4\rangle\left(-\frac{\mu^2[4|1]\langle 2|3\rangle}{2s_{34}t}+t[3|1]\langle 2|4\rangle-\langle 2|4|1]\right)^2}{s_{12}\left(-\frac{\mu^2[4|1]\langle 1|3\rangle}{2s_{34}t}+t[3|1]\langle 1|4\rangle-\langle 1|4|1]\right)}\right]\right](\bar{l}_4)=-\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 1|2\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{23}}{s_{13}} \tag{5-246}$$

The rational contribution from the triangle in the channel s_{12},

$$\boxed{C^{[2]}_{3;12}\left(1^+_g,2^-_g,3^-_{\bar{q}},4^+_q\right)=-\frac{1}{2}c_{4;0}\left(+,-;+-\right)\left(2+\frac{s_{13}}{s_{23}}-\frac{s_{23}}{s_{13}}\right)} \tag{5-247}$$

The contribution from the channel s_{34},

$$\begin{aligned}C^{[2]}_{3;34}\left(1^+_g,2^-_g,3^-_{\bar{q}},4^+_q\right)&=A_4^{tree}\left(-l_1^0,1^+_g,2,^-l_3^0\right)A_4^{tree}\left(-l_3^0,3^-_{\bar{q}},4^+_q,l_1^0\right)l_2^2\\&=-i\frac{\langle 2\left|l_1\right|1]^2}{s_{12}}i\frac{\langle 3\left|l_1\right|4]}{\langle 4\left|l_1\right|4]}=\frac{\langle 2\left|l_2\right|1]^2}{s_{12}}\frac{\langle 3\left|l_2+1\right|4]}{\langle 4\left|l_2+1\right|4]}\end{aligned} \tag{5-248}$$

Writing the explicit solution for l_2 and its conjugate solution,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{\mu^4[2|1]^2\langle 1|2\rangle^2\left(-\frac{\mu^2[4|2]\langle 1|3\rangle}{2s_{12}t}+t[4|1]\langle 2|3\rangle+\langle 3|1|4]\right)}{2{s_{12}}^3t^2\left(-\frac{\mu^2[4|2]\langle 1|4\rangle}{2s_{12}t}+t[4|1]\langle 2|4\rangle+\langle 4|1|4]\right)}\right]\right](l_2)=-\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{12}^2}{s_{13}s_{23}} \tag{5-249}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{4t^2[2|1]^2\langle 1|2\rangle^2\left(-\frac{\mu^2[4|1]\langle 2|3\rangle}{2s_{12}t}+2t[4|2]\langle 1|3\rangle+\langle 3|1|4]\right)}{s_{12}\left(-\frac{\mu^2[4|1]\langle 2|4\rangle}{2s_{12}t}+2t[4|2]\langle 1|4\rangle+\langle 4|1|4]\right)}\right]\right](\bar{l}_2)=0 \tag{5-250}$$

The rational contribution from the triangle in the channel s_{34},

$$\boxed{C^{[2]}_{3;34}\left(1^+_g,2^-_g,3^-_{\bar{q}},4^+_q\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\left(2+\frac{s_{13}}{s_{23}}+\frac{s_{23}}{s_{13}}\right)} \tag{5-251}$$

The channel s_{23}

$$\begin{aligned}C^{[2]}_{3;23}\left(1^+_g,2^-_g,3^-_{\bar{q}},4^+_q\right)&=A_4^{tree}\left(-l_2^0,2^-,3^-_{\bar{q}}l_4\right)A_4^{tree}\left(-l_4,4^+_q,1^+_g,l_2^0\right)l_1^2\\&=i\frac{\langle 2\left|l_1\right|1]^2}{s_{12}\langle 1\left|l_1\right|1]}i\frac{\langle 3\left|l_1\right|4]}{\langle 4\left|l_1\right|4]}l_1^2=-\frac{\langle 2\left|l_2\right|1]^2\langle 3\left|l_4\right|4]}{s_{12}\langle 2\left|l_2\right|2]}\end{aligned} \tag{5-252}$$

Writing the explicit solution for l_1 and its conjugate solution,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{\mu^4[4|1]^3\langle 1|2\rangle^2\langle 3|4\rangle}{2s_{12}{s_{14}}^2t\left(\frac{\mu^2[4|2]\langle 1|2\rangle}{2s_{14}t}-t[2|1]\langle 2|4\rangle-\langle 2|1|2]\right)}\right]\right](l_1)=-\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{12}}{s_{13}} \tag{5-253}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{\mu^2t[4|1]^3\langle 1|2\rangle^2\langle 3|4\rangle}{2s_{12}s_{14}\left(\frac{\mu^2[2|1]\langle 2|4\rangle}{2s_{14}t}-t[4|2]\langle 1|2\rangle-\langle 2|1|2]\right)}\right]\right](\bar{l}_1)=0 \tag{5-254}$$

The rational contribution from the triangle in the channel s_{23},

$$\boxed{C^{[2]}_{3;23}\left(1^+_g,2^-_g,3^-_{\bar q},4^+_q\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}}{s_{13}}} \tag{5-255}$$

The contribution from the channel s_{14},

$$\begin{aligned}C^{[2]}_{3;14}\left(1^+_g,2^-_g,3^-_{\bar q},4^+_q\right)&=A^{tree}_4\left(-l^0_2,2^-,3^-_{\bar q}l_4\right)A^{tree}_4\left(-l_4,4^+_q,1^+_g,l^0_2\right)l^2_3\\&=\frac{\langle 2\left|l_2\right|1]^2}{s_{12}\langle 2\left|l_2\right|2]}\frac{\langle 3\left|l_4\right|4]}{\langle 4\left|l_4\right|4]}l^2_3=-\frac{\langle 2\left|l_2\right|1]^2}{s_{12}}\frac{\langle 3\left|l_4\right|4]}{\langle 4\left|l_4\right|4]}\end{aligned} \tag{5-256}$$

Writing the explicit solution for $l_1$and its conjugate solution,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{\mu^4[2|1]^2[4|3]\langle 2|3\rangle^3}{4s_{12}s_{23}{}^2t\left(-\frac{\mu^2[4|2]\langle 3|4\rangle}{2s_{23}t}+t[4|3]\langle 2|4\rangle-\langle 4|3|4]\right)}\right]\right](l_3)=-\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{12}}{s_{13}} \tag{5-257}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{\mu^2t[2|1]^2[4|3]\langle 2|3\rangle^3}{s_{12}s_{23}\left(-\frac{\mu^2[4|3]\langle 2|4\rangle}{2s_{23}t}+t[4|2]\langle 3|4\rangle-\langle 4|3|4]\right)}\right]\right](\bar l_3)=0 \tag{5-258}$$

The rational contribution from the triangle in the channel s_{41},

$$\boxed{C^{[2]}_{3;41}\left(1^+_g,2^-_g,3^-_{\bar q},4^+_q\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}}{s_{13}}} \tag{5-259}$$

Right Turning Channel s_{12}

$$\begin{aligned}C^{[2]}_{3;12}\left(1^+_g,2^-_g,3^-_{\bar q},4^+_q\right)&=-iA^{tree}_4\left(-L_2,2^-,1^+,L_4\right)A^{tree}_4\left(-L_4,4^+_q,3^-_{\bar q},L_2\right)l^2_3\\&=-\frac{[1\left|L_2\right|2\rangle}{s_{12}\langle 2\left|L_2\right|2]\mu^2}\left(2\left\langle 3\left|l^\flat_3\bar l\right|2\right\rangle-\mu^2\langle 32\rangle\right)\left(2\left[1\left|\bar l l^\flat_3\right|4\right]-\mu^2[14]\right)\end{aligned} \tag{5-260}$$

Writing the explicit solution for l_3 and its conjugate,

$$\begin{aligned}&\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{\left(\mu^2\langle 2|3\rangle-\frac{\mu^2[4|3]\langle 2|3\rangle\langle 3|4\rangle}{s_{34}}\right)\left(\mu^2[4|1]-\frac{\mu^2[4|1][4|3]\langle 3|4\rangle}{s_{34}}\right)\left(-\frac{\mu^2[4|1]\langle 2|3\rangle}{2s_{34}t}+t[3|1]\langle 2|4\rangle-\langle 2|3|1]\right)}{\mu^2s_{12}\left(-\frac{\mu^2[4|2]\langle 2|3\rangle}{2s_{34}t}+t[3|2]\langle 2|4\rangle-\langle 2|3|2]\right)}\right]\right](l_3)=\\&=i\frac{[4|1]^2\langle 2|3\rangle}{[2|1][4|2]\langle 1|2\rangle}=-i\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 1|2\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{23}}{s_{13}}\end{aligned} \tag{5-261}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{\mu^2[4|1]\langle 2|3\rangle\left(-\frac{\mu^2[3|1]\langle 2|4\rangle}{2s_{34}t}+t[4|1]\langle 2|3\rangle-\langle 2|3|1]\right)}{s_{12}\left(-\frac{\mu^2[3|2]\langle 2|4\rangle}{2s_{34}t}+t[4|2]\langle 2|3\rangle-\langle 2|3|2]\right)}\right]\right](\bar l_3)=-i\frac{\langle 24\rangle\langle 23\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{14}}{s_{23}} \tag{5-262}$$

By averaging the results, we obtain

$$\boxed{C_{3;12}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}}{s_{13}}} \tag{5-263}$$

Channel s_{34}

$$\begin{aligned}C_{3;34}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)&=-iA_4^{tree}\left(-L_2,2^-,1^+,L_4\right)A_4^{tree}\left(-L_4,4_q^+,3_{\bar{q}}^-,L_2\right)l_1^2\\&=-i\frac{\left[1\left|L_2\right|2\right\rangle}{s_{12}\left\langle 3\left|L_2\right|3\right]\mu^2}\left(2\left\langle 3\left|\overline{L}L_1^\flat\right|2\right\rangle-\mu^2\left\langle 32\right\rangle\right)\left(2\left[1\left|L_1^\flat\overline{L}\right|4\right]-\mu^2\left[14\right]\right)\end{aligned} \tag{5-264}$$

Writing the explicit solution for l_1 and its conjugate,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[-i\frac{t[2|1]\langle 1|2\rangle\left(\mu^2\langle 2|3\rangle-\frac{\mu^2[2|1]\langle 1|2\rangle\langle 2|3\rangle}{s_{12}}\right)\left(\mu^2[4|1]-\frac{\mu^2[2|1][4|1]\langle 1|2\rangle}{s_{12}}\right)}{\mu^2 s_{12}\left(-\frac{\mu^2[3|1]\langle 2|3\rangle}{2s_{12}t}+t[3|2]\langle 1|3\rangle+\langle 3|2|3]\right)}\right]\right](l_1)=0 \tag{5-265}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[i\frac{\mu^4[2|1][4|1]\langle 1|2\rangle\langle 2|3\rangle}{2s_{12}{}^2t\left(-\frac{\mu^2[3|2]\langle 1|3\rangle}{2s_{12}t}+t[3|1]\langle 2|3\rangle+\langle 3|2|3]\right)}\right]\right](\bar{l}_1)=-i\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{12}}{s_{13}} \tag{5-266}$$

By averaging the results, we obtain

$$\boxed{C_{3;34}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{12}}{s_{13}}} \tag{5-267}$$

Channel s_{23}

$$\begin{aligned}C_{3;23}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)&=A_4^{tree}\left(-l_3^0,3_{\bar{q}}^-,2^-,L_1\right)A_4^{tree}\left(-L_1,1^+,4_q^+,l_3^0\right)l_4^2\\&=\frac{\left[1\left|L_1\right|2\right\rangle}{s_{12}\left\langle 3\left|l_3\right|3\right]\mu^2}\left(2\left\langle 3\left|\overline{L}L_1^\flat\right|2\right\rangle-\mu^2\left\langle 32\right\rangle\right)\left(2\left[1\left|L_1^\flat\overline{L}\right|4\right]-\mu^2\left[14\right]\right)\end{aligned} \tag{5-268}$$

Writing the explicit solution for l_4 and its conjugate,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{t[4|1]\langle 1|2\rangle\left(\frac{\mu^2[4|1]\langle 1|2\rangle\langle 3|4\rangle}{s_{14}}+\mu^2\langle 2|3\rangle\right)\left(\mu^2[4|1]-\frac{\mu^2[4|1]^2\langle 1|4\rangle}{s_{14}}\right)}{\mu^2 s_{12}\left(\frac{\mu^2[3|1]\langle 3|4\rangle}{2s_{14}t}-t[4|3]\langle 1|3\rangle-\langle 3|4|3]\right)}\right]\right](l_4)=0 \tag{5-269}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{\mu^2[4|1]^2\langle 1|2\rangle\left(\mu^2\langle 2|3\rangle-2\left(\frac{\mu^2[4|1]\langle 1|3\rangle\langle 2|4\rangle}{2s_{14}}-\frac{\mu^2\langle 2|1|4]\langle 1|3\rangle}{2s_{14}t}\right)\right)}{2s_{12}s_{14}t\left(\frac{\mu^2[4|3]\langle 1|3\rangle}{2s_{14}t}-t[3|1]\langle 3|4\rangle-\langle 3|4|3]\right)}\right]\right](\bar{l}_4)=i\frac{\langle 23\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 3|4\rangle\langle 41\rangle}\frac{s_{14}}{s_{13}} \tag{5-270}$$

By averaging the results, we obtain

$$\boxed{C_{3;23}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{14}}{s_{13}}} \tag{5-271}$$

Channel s_{14}

$$\begin{aligned} C_{3;14}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right) &= A_4^{tree}\left(-l_3^0,3_{\bar{q}}^-,2^-,L_1\right)A_4^{tree}\left(-L_1,1^+,4_q^+,l_3^0\right)l_2^2 \\ &= \frac{[1\,|L_1|\,2\rangle}{s_{12}\,\langle 3\,|l_3|\,3]\,\mu^2}\left(2\left\langle 3\left|\overline{L}L_1^\flat\right|2\right\rangle-\mu^2\langle 32\rangle\right)\left(2\left[1\left|L_1^\flat\overline{L}\right|4\right]-\mu^2[14]\right) \end{aligned} \tag{5-272}$$

Writing the explicit solution for l_2 and its conjugate,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\frac{t[4|1]\langle 1|2\rangle\left(\frac{\mu^2[4|1]\langle 1|2\rangle\langle 3|4\rangle}{s_{14}}+\mu^2\langle 2|3\rangle\right)\left(\mu^2[4|1]-\frac{\mu^2[4|1]^2\langle 1|4\rangle}{s_{14}}\right)}{\mu^2 s_{12}\left(\frac{\mu^2[3|1]\langle 3|4\rangle}{2s_{14}t}-t[4|3]\langle 1|3\rangle-\langle 3|4|3]\right)}\right]\right](\bar{l}_2)=0 \tag{5-273}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[-\frac{t[2|1]\langle 2|3\rangle^2\left(\mu^2[4|1]-\frac{\mu^2[4|2]\langle 3|2|1]}{s_{23}t}\right)}{s_{12}\left(-\frac{\mu^2[3|1]\langle 1|2\rangle}{2s_{23}t}+t[2|1]\langle 1|3\rangle-\langle 1|2|1]\right)}\right]\right](l_2)=i\frac{\langle 2|3\rangle^3\langle 24\rangle}{\langle 1|2\rangle\langle 2|3\rangle\langle 34\rangle\langle 41\rangle}\frac{s_{23}}{s_{13}} \tag{5-274}$$

By averaging the results, we obtain

$$\boxed{C_{3;14}^{[2]}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right)=\frac{1}{2}c_{4;0}\left(+,-;+-\right)\frac{s_{14}}{s_{13}}} \tag{5-275}$$

$A_4^{Triangle}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)$

Left Turning We define C_{1234} as the product of tree level amplitudes,

$$C_{1234}=A_3^{tree}\left(-l_1^0,1_g^-,l_2^0\right)A_3^{tree}\left(-l_2^0,2_g^+,l_3^0\right)A_3^{tree}\left(-l_3^0,3_{\bar{q}}^-,L_4\right)A_3^{tree}\left(-L_4,4_q^+,l_1^0\right) \tag{5-276}$$

$$=\frac{\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}} \tag{5-277}$$

Channel s_{12}

$$C_{134}=\frac{i}{\langle 1|l_1|1]}C_{1234}=\frac{i\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}\langle 1|l_1|1]} \tag{5-278}$$

Writing the explicit solution for l_4 and its conjugate,

$$\inf_{\mu^2}\left[\inf_{t^0}[C_{134}]\right](l_4)=-\frac{i\left(s_{12}{}^2-s_{13}{}^2\right)\langle 1|3\rangle^3}{s_{13}{}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-279}$$

$$\inf_{\mu^2}\left[\inf_{t^0}[C_{134}]\right](\bar{l}_4)=\frac{i\langle 1|3\rangle^3}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-280}$$

By averaging the results, we obtain

$$\boxed{C_{3;12}^{[2]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(\frac{s_{12}{}^2}{s_{13}{}^2} - 2\right)} \tag{5-281}$$

Channel s_{34}

$$C_{123} = -\frac{i}{\langle 4|l_1|4]}C_{1234} = -\frac{i\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}\langle 4|l_1|4]} \tag{5-282}$$

Writing the explicit solution for l_2 and its conjugate,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[C_{123}\right]\right](l_2) = \frac{is_{12}{}^2\langle 1|3\rangle^3}{s_{13}{}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} + 0 \tag{5-283}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[C_{123}\right]\right](\bar{l}_2) = 0 \tag{5-284}$$

By averaging the results, we obtain

$$\boxed{C_{3;34}^{[2]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}^2}{s_{13}^2}} \tag{5-285}$$

Channel s_{23}

$$C_{124} = -\frac{i}{\langle 2|l_2|2]}C_{1234} = -\frac{i\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}\langle 2|l_2|2]} \tag{5-286}$$

Writing the explicit solution for l_2 and its conjugate,

$$\inf_{\mu^2}\left[\inf_{t^0}\left[C_{124}\right]\right](l_1) = \frac{is_{12}s_{14}\langle 1|3\rangle^3}{s_{13}{}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-287}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[C_{124}\right]\right](\bar{l}_1) = 0 \tag{5-288}$$

By averaging the results, we obtain

$$\boxed{C_{3;23}^{[2]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}s_{14}}{s_{13}{}^2}} \tag{5-289}$$

Channel s_{14}

$$C_{234} = \frac{i}{\langle 4|l_4|4]}C_{1234} = -\frac{i\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}\langle 4|l_4|4]} \tag{5-290}$$

Writing the explicit solution for l_3 and its conjugate,

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{234}\right]\right](l_3) = \frac{is_{12}s_{14}\langle 1|3\rangle^3}{s_{13}{}^2\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-291}$$

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{234}\right]\right](\bar{l}_3) = 0 \tag{5-292}$$

By averaging the results, we obtain

$$\boxed{C_{3;41}^{[2]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{12}s_{14}}{s_{13}{}^2}}$$

Right Turning Consider the product of tree amplitudes

$$C_{4321} = A_3^{tree}\left(-L_4, 4_q^+, l_3^0\right) A_3^{tree}\left(-l_3^0, 3_{\bar{q}}^-, L_2\right) A_4^{tree}\left(-L_2, 2^+, L_1\right) A_4^{tree}\left(-L_1, 1^-, L_4\right) =$$
$$= \frac{1}{s_{12}\mu^2}\langle 1|\, L_1\, |2]\left(2\left[4\left|l_3^\flat \bar{l}\right|2\right] - \mu^2\,[42]\right)\left(2\left\langle 1\left|\bar{l} l_3^\flat\right|3\right\rangle - \mu^2\,\langle 13\rangle\right) \quad (5\text{-}293)$$

Channel s_{12}

$$C_{432} = \frac{i}{\langle 2|L_2|2]} C_{4321} = -\frac{i\langle 1|L_2|2]\left(\mu^2[4|2] + 2[2|l l_3^\flat|4]\right)\left(\mu^2\langle 1|3\rangle + 2\langle 3|l_3^\flat l|1\rangle\right)}{\mu^2 s_{12}\langle 2|L_2|2]} \quad (5\text{-}294)$$

Writing the explicit solution for l_3 and its conjugate,

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right](l_3) = -\frac{i s_{14}\langle 1|3\rangle^3}{s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \quad (5\text{-}295)$$

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right](\bar{l}_3) = -\frac{i\langle 1|3\rangle^3}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \quad (5\text{-}296)$$

By averaging the results, we obtain

$$\boxed{C_{3;12}^{[2]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = -\frac{1}{2} c_{4;0}\left(-,+;+-\right)\frac{s_{12}}{s_{13}}} \quad (5\text{-}297)$$

Channel s_{34}

$$C_{421} = \frac{i}{\langle 3|L_2|3]} C_{4321} = -\frac{i\langle 1|L_2|2]\left(\mu^2[4|2] + 2[2|l l_2^\flat|4]\right)\left(\mu^2\langle 1|3\rangle + 2\langle 3|l_2^\flat l|1\rangle\right)}{\mu^2 s_{12}\langle 3|L_2|3]} \quad (5\text{-}298)$$

Writing the explicit solution for l_1 and its conjugate,

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right](l_1) = -\frac{i s_{12}\langle 1|3\rangle^3}{s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \quad (5\text{-}299)$$

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right](\bar{l}_1) = 0 \quad (5\text{-}300)$$

By averaging the results, we obtain

$$\boxed{C_{3;34}^{[2]}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = \frac{1}{2} c_{4;0}\left(-,+;+-\right)\frac{s_{12}}{s_{13}}} \quad (5\text{-}301)$$

Channel s_{23}

$$C_{431} = \frac{i}{\langle 3|L_2|3]} C_{4321}$$
$$= -\frac{i\langle 1|L_2|2]\left(\mu^2[4|2] + 2[2|l l_2^\flat|4]\right)\left(\mu^2\langle 1|3\rangle + 2\langle 3|l_2^\flat l|1\rangle\right)}{\mu^2 s_{12}\langle 3|L_2|3]} \quad (5\text{-}302)$$

Writing the explicit solution for l_4 and its conjugate,

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right](l_4) = -\frac{is_{14}\langle 1|3\rangle^3}{s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-303}$$

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right]\left(\bar{l}_4\right) = 0 \tag{5-304}$$

By averaging the results, we obtain

$$\boxed{C^{[2]}_{3;23}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = \frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{14}}{s_{13}}} \tag{5-305}$$

Channel s_{14}

$$\begin{aligned} C_{431} &= \frac{i}{\langle 1|L_1|1]}C_{4321} \\ &= -\frac{i\langle 1|L_1|2]\left(\mu^2[4|2] + 2[2|l_1^\flat l|4]\right)\left(\mu^2\langle 1|3\rangle + 2\langle 3|ll_2^\flat|1\rangle\right)}{\mu^2 s_{12}\langle 1|L_1|1]} \end{aligned} \tag{5-306}$$

Writing the explicit solution for l_2 and its conjugate,

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right](l_2) = -\frac{is_{14}\langle 1|3\rangle^3}{s_{13}\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} \tag{5-307}$$

$$\inf_{t^0}\left[\inf_{\mu^2}\left[C_{432}\right]\right]\left(\bar{l}_2\right) = 0 \tag{5-308}$$

By averaging the results, we obtain

$$\boxed{C^{[2]}_{3;14}\left(1^-_g, 2^+_g, 3^-_{\bar{q}}, 4^+_q\right) = \frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{14}}{s_{13}}} \tag{5-309}$$

5.2.3. Bubble Contributions

Left Turning solutions for loop momenta

Solutions for the channel s_{12} The loop momentum can be written as

$$l_1 = yK_1^\flat + \frac{S_1\left(1-y\right)}{\gamma}\chi + t\left|K_1^\flat\right\rangle[\chi| + \frac{\left(y\left(1-y\right)S_1 - \mu^2\right)}{\gamma t}|\chi\rangle\left[K_1^\flat\right|, \tag{5-310}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat = 2, \qquad \chi = 1 \tag{5-311}$$

$$l_1 = yK_2 + \left(1-y\right)K_1 + t\,|2\rangle\,[1| + \frac{\left(y\left(1-y\right)s_{12} - \mu^2\right)}{s_{12}t}|1\rangle\,[2|, \tag{5-312}$$

For the triangle contribution to the bubble we need the third on-shell condition to find y. Writting y in terms of t, we find,

$$y_{\pm} = \alpha_{1,\pm} t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} \tag{5-313}$$

$$\alpha_{1,\pm} = \frac{s_{24} - s_{14} \pm (s_{24} + s_{14})}{2\langle 1|4|2]} = \begin{cases} \alpha_{1,+} = \frac{s_{24}}{\langle 1|4|2]} \\ \alpha_{1,-} = -\frac{s_{14}}{\langle 1|4|2]} \end{cases} \tag{5-314}$$

$$\alpha_{2,\pm} = \frac{1}{2}(1 \pm 1) = \begin{cases} \alpha_{2,+} = 1 \\ \alpha_{2,-} = 0 \end{cases} \tag{5-315}$$

$$\alpha_{3,\pm} = \pm\mu^2 \frac{\langle 1|4|2]}{s_{12}^2} = \begin{cases} \alpha_{3,+} = \mu^2 \frac{\langle 1|4|2]}{s_{12}^2} \\ \alpha_{3,-} = -\mu^2 \frac{\langle 1|4|2]}{s_{12}^2} \end{cases} \tag{5-316}$$

Solutions for the channel s_{23} The loop momentum can be written as

$$l_2 = yK_1^\flat + \frac{S_1(1-y)}{\gamma}\chi + t\left|K_1^\flat\right\rangle[\chi| + \frac{(y(1-y)S_1 - \mu^2)}{\gamma t}|\chi\rangle\left[K_1^\flat\right|, \tag{5-317}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat = 3, \qquad \chi = 2 \tag{5-318}$$

$$l_2 = yK_3 + (1-y)K_2 + t|3\rangle[2| + \frac{(y(1-y)s_{23} - \mu^2)}{s_{23}t}|2\rangle[3|, \tag{5-319}$$

For the triangle contribution to the bubble we need the third on-shell condition to find y. Writting y in terms of t, we find,

$$y_{\pm} = \alpha_{1,\pm} t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} \tag{5-320}$$

$$\alpha_{1,\pm} = \begin{cases} \alpha_{1,+} = \frac{s_{24}}{\langle 2|1|3]} \\ \alpha_{1,-} = -\frac{s_{14}}{\langle 2|1|3]} \end{cases} \tag{5-321}$$

$$\alpha_{2,\pm} = \frac{1}{2}(1 \pm 1) = \begin{cases} \alpha_{2,+} = 1 \\ \alpha_{2,-} = 0 \end{cases} \tag{5-322}$$

$$\alpha_{3,\pm} = \pm\mu^2 \frac{\langle 1|4|2]}{s_{12}^2} = \begin{cases} \alpha_{3,+} = \mu^2 \frac{\langle 2|1|3]}{s_{23}^2} \\ \alpha_{3,-} = -\mu^2 \frac{\langle 1|4|2]}{s_{23}^2} \end{cases} \tag{5-323}$$

Right Turning solutions for loop momenta

Solutions for the channel s_{12} The loop momentum can be written as

$$l_4 = yK_1^\flat + \frac{S_1(1-y)}{\gamma}\chi + t\left|K_1^\flat\right\rangle[\chi| + \frac{(y(1-y)S_1 - \mu^2)}{\gamma t}|\chi\rangle\left[K_1^\flat\right|, \tag{5-324}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat = 2, \qquad \chi = 1 \tag{5-325}$$

$$l_4 = yK_2 + (1-y)\,K_1 + t\,|2\rangle\,[1| + \frac{\left(y\,(1-y)\,s_{12} - \mu^2\right)}{s_{12}t}\,|1\rangle\,[2|\,, \tag{5-326}$$

For the triangle contribution to the bubble we need the third on-shell condition to find y. Writting y in terms of t, we find,

$$y_\pm = \alpha_{1,\pm}t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} \tag{5-327}$$

$$\alpha_{1,\pm} = \frac{s_{24} - s_{14} \pm (s_{24} + s_{14})}{2\,\langle 1\,|4|\,2]} = \begin{cases} \alpha_{1,+} = \frac{s_{24}}{\langle 1|4|2]} \\ \alpha_{1,-} = -\frac{s_{14}}{\langle 1|4|2]} \end{cases} \tag{5-328}$$

$$\alpha_{2,\pm} = \frac{1}{2}\,(1 \pm 1) = \begin{cases} \alpha_{2,+} = 1 \\ \alpha_{2,-} = 0 \end{cases} \tag{5-329}$$

$$\alpha_{3,\pm} = \pm\mu^2 \frac{\langle 1\,|4|\,2]}{s_{12}^2} = \begin{cases} \alpha_{3,+} = \mu^2 \frac{\langle 1|4|2]}{s_{12}^2} \\ \alpha_{3,-} = -\mu^2 \frac{\langle 1|4|2]}{s_{12}^2} \end{cases} \tag{5-330}$$

Solutions for the channel s_{23} The loop momentum can be written as

$$l_1 = yK_1^\flat + \frac{S_1\,(1-y)}{\gamma}\chi + t\left|K_1^\flat\right\rangle[\chi| + \frac{\left(y\,(1-y)\,S_1 - \mu^2\right)}{\gamma t}\,|\chi\rangle\left[K_1^\flat\right|, \tag{5-331}$$

where $K_1^\flat$ and χ are given by

$$K_1^\flat = 3, \qquad \chi = 2 \tag{5-332}$$

$$l_1 = yK_3 + (1-y)\,K_2 + t\,|3\rangle\,[2| + \frac{\left(y\,(1-y)\,s_{23} - \mu^2\right)}{s_{23}t}\,|2\rangle\,[3|\,, \tag{5-333}$$

For the triangle contribution to the bubble we need the third on-shell condition to find y. Writting y in terms of t, we find,

$$y_\pm = \alpha_{1,\pm}t + \alpha_{2,\pm} + \frac{1}{t}\alpha_{3,\pm} \tag{5-334}$$

$$\alpha_{1,\pm} = \begin{cases} \alpha_{1,+} = \frac{s_{24}}{\langle 2|1|3]} \\ \alpha_{1,-} = -\frac{s_{14}}{\langle 2|1|3]} \end{cases} \tag{5-335}$$

$$\alpha_{2,\pm} = \frac{1}{2}\,(1 \pm 1) = \begin{cases} \alpha_{2,+} = 1 \\ \alpha_{2,-} = 0 \end{cases} \tag{5-336}$$

$$\alpha_{3,\pm} = \pm\mu^2 \frac{\langle 1\,|4|\,2]}{s_{12}^2} = \begin{cases} \alpha_{3,+} = \mu^2 \frac{\langle 2|1|3]}{s_{23}^2} \\ \alpha_{3,-} = -\mu^2 \frac{\langle 1|4|2]}{s_{23}^2} \end{cases} \tag{5-337}$$

$A_4^{bubble}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$

Left Turning

Channel s_{12}

$$C_{13} = \frac{i}{l_4^2} C_{134} = \frac{i}{\langle 4|l_1|4]} \left(\frac{i\langle 3|l_1|4]\langle 2|l_1|1]^2}{s_{12}\langle 1|l_1|1]} \right) = -\frac{\langle 3|l_1|4]\langle 2|l_1|1]^2}{s_{12}\langle 4|l_1|4]\langle 1|l_1|1]} \tag{5-338}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\inf_{y}[C_{13}]\right]\right] = 0 \tag{5-339}$$

and the triangle contributions,

$$C_{234} = i\frac{\langle 3|l_1|4]\langle 2|l_1|1]^2}{s_{12}\langle 1|l_1|1]} \tag{5-340}$$

$$\inf_{\mu^2}\left[\inf_{t\to T_m}[C_{234}]\right] = 0 \tag{5-341}$$

$$\boxed{C_{2;12}^{[2]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-342}$$

Channel s_{23}

$$C_{24} = \frac{i}{\langle 1\,|l_2|\,1]} \left(-\frac{i\,\langle 2\,|l_2|\,1]^2\,\langle 3\,|l_4|\,4]}{s_{12}\,\langle 2\,|l_2|\,2]} \right) \tag{5-343}$$

$$\inf_{\mu^2}\left[\inf_{t^0}\left[\inf_{y}[C_{24}]\right]\right] = 0 \tag{5-344}$$

and the triangle contributions,

$$C_{124} = -\frac{i\,\langle 2\,|l_2|\,1]^2\,\langle 3\,|l_4|\,4]}{s_{12}\,\langle 2\,|l_2|\,2]} \tag{5-345}$$

$$\inf_{\mu^2}\left[\inf_{t\to T_m}[C_{124}]\right] = 0 \tag{5-346}$$

finally,

$$\boxed{C_{2;23}^{[2]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-347}$$

Right Turning

Channel s_{12}

$$C_{42} = \frac{i}{\langle 1\,|L_4|\,1]}\frac{i}{\langle 3\,|L_2|\,3]}C_{4321}$$
$$= \frac{[1\,|L_2|\,2\rangle}{s_{12}\,\langle 1\,|L_4|\,1]\,\langle 3\,|L_2|\,3]\,\mu^2}\left(2\left\langle 3\left|l_2^\flat \bar{l}\right|2\right\rangle - \mu^2\,\langle 32\rangle\right)\left(2\left[1\left|\bar{l}l_4^\flat\right|4\right] - \mu^2\,[14]\right) \quad (5\text{-}348)$$
$$\inf_{\mu^2}\left[\inf_{t^0}\left[\inf_{y}\left[C_{42}\right]\right]\right] = 0 \quad (5\text{-}349)$$

and the triangle contributions,

$$C_{432} = \frac{i}{\langle 1\,|L_4|\,1]}C_{4321}$$
$$= -i\frac{[1\,|L_2|\,2\rangle}{s_{12}\,\langle 1\,|L_4|\,1]\,\mu^2}\left(2\left\langle 3\left|l_2^\flat \bar{l}\right|2\right\rangle - \mu^2\,\langle 32\rangle\right)\left(2\left[1\left|\bar{l}l_4^\flat\right|4\right] - \mu^2\,[14]\right) \quad (5\text{-}350)$$
$$\inf_{\mu^2}\left[\inf_{t\to T_m}\left[C_{432}\right]\right] = 0 \quad (5\text{-}351)$$

finally,

$$\boxed{C_{2;12}^{[2]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = 0} \quad (5\text{-}352)$$

Channel s_{23}

$$C_{31} = \frac{i}{\langle 2\,|L_1|\,2]}\frac{i}{\langle 4\,|l_3|\,4]}C_{4321}$$
$$= \frac{[1\,|L_2|\,2\rangle}{s_{12}\,\langle 2\,|L_1|\,2]\,\langle 4\,|l_3|\,4]\,\mu^2}\left(2\left\langle 3\left|l_3^\flat \bar{l}\right|2\right\rangle - \mu^2\,\langle 32\rangle\right)\left(2\left[1\left|\bar{l}l_3^\flat\right|4\right] - \mu^2\,[14]\right) \quad (5\text{-}353)$$
$$\inf_{\mu^2}\left[\inf_{t^0}\left[\inf_{y}\left[C_{31}\right]\right]\right] = 0 \quad (5\text{-}354)$$

and the triangle contributions,

$$C_{431} = \frac{i}{\langle 2\,|L_1|\,2]}C_{4321}$$
$$= -i\frac{[1\,|L_2|\,2\rangle}{s_{12}\,\langle 2\,|L_1|\,2]\,\mu^2}\left(2\left\langle 3\left|l_3^\flat \bar{l}\right|2\right\rangle - \mu^2\,\langle 32\rangle\right)\left(2\left[1\left|\bar{l}l_3^\flat\right|4\right] - \mu^2\,[14]\right) \quad (5\text{-}355)$$
$$\inf_{\mu^2}\left[\inf_{t\to T_m}\left[C_{431}\right]\right] = 0 \quad (5\text{-}356)$$

finally,

$$\boxed{C_{2;23}^{[2]}\left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) = 0} \quad (5\text{-}357)$$

$A_4^{Bubbble}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right)$

Left Turning

Channel s_{12}

$$C_{13} = \frac{i}{l_4^2} C_{134} = \frac{i}{\langle 4|l_1|4]} \left(\frac{i\langle 3|l_1|4] \left[2\left|l_1\right|1\right\rangle^2}{s_{12}\langle 1|l_1|1]} \right) = -\frac{\langle 3|l_1|4] \left[2\left|l_1\right|1\right\rangle^2}{s_{12}\langle 4|l_1|4]\langle 1|l_1|1]} \tag{5-358}$$

$$\inf_{\mu^2} \left[\inf_{t^0} \left[\inf_{y} \left[C_{13} \right] \right] \right] = 0 \tag{5-359}$$

and the triangle contributions,

$$C_{234} = i\frac{\langle 3|l_1|4] \left[2\left|l_1\right|1\right\rangle^2}{s_{12}\langle 1|l_1|1]} \tag{5-360}$$

$$\inf_{\mu^2} \left[\inf_{t \to T_m} \left[C_{234} \right] \right] = 0 \tag{5-361}$$

$$\boxed{C^{[2]}_{2;12}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-362}$$

Channel s_{23}

$$C_{24} = \frac{i}{\langle 1\left|l_2\right|1]} \left(-\frac{i\left[2\left|l_2\right|1\right\rangle^2 \langle 3\left|l_4\right|4]}{s_{12}\langle 2\left|l_2\right|2]} \right) \tag{5-363}$$

$$\inf_{\mu^2} \left[\inf_{t^0} \left[\inf_{y} \left[C_{24} \right] \right] \right] = 0 \tag{5-364}$$

and the triangle contributions,

$$C_{124} = -\frac{i\left\langle 2\left|l_2\right|1\right]^2 \langle 3\left|l_4\right|4]}{s_{12}\langle 2\left|l_2\right|2]} \tag{5-365}$$

$$\inf_{\mu^2} \left[\inf_{t \to T_m} \left[C_{124} \right] \right] = 0 \tag{5-366}$$

finally,

$$\boxed{C^{[2]}_{2;23}\left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-367}$$

Right Turning

Channel s_{12}

$$\begin{aligned} C_{42} &= \frac{i}{\langle 1\left|L_4\right|1]} \frac{i}{\langle 3\left|L_2\right|3]} C_{4321} \\ &= -\frac{\langle 1|\, L_1\, |2]}{s_{12}\langle 1\left|L_4\right|1] \langle 3\left|L_2\right|3]\, \mu^2} \left(2\left[4\left|l_4^\flat \bar{l}\right|2\right] - \mu^2 [42] \right) \left(2\left\langle 1\left|\bar{l} l_2^\flat\right|3\right\rangle - \mu^2 \langle 13\rangle \right) \end{aligned} \tag{5-368}$$

$$\inf_{\mu^2} \left[\inf_{t^0} \left[\inf_{y} \left[C_{42} \right] \right] \right] = 0 \tag{5-369}$$

and the triangle contributions,

$$C_{432} = \frac{i}{\langle 1 \left| L_4 \right| 1]} C_{4321}$$
$$= i \frac{\langle 1 | \, L_1 \, |2]}{s_{12} \langle 1 \left| L_4 \right| 1] \, \mu^2} \left(2 \left[4 \left| l_4^\flat \bar{l} \right| 2 \right] - \mu^2 \, [42] \right) \left(2 \left\langle 1 \left| \bar{l} l_2^\flat \right| 3 \right\rangle - \mu^2 \, \langle 13 \rangle \right) \tag{5-370}$$
$$\inf_{\mu^2} \left[\inf_{t \to T_m} [C_{432}] \right] = 0 \tag{5-371}$$

finally,

$$\boxed{C_{2;12}^{[2]} \left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-372}$$

Channel s_{23}

$$C_{31} = \frac{i}{\langle 2 \left| L_1 \right| 2]} \frac{i}{\langle 4 \left| l_3 \right| 4]} C_{4321}$$
$$= - \frac{\langle 1 | \, L_1 \, |2]}{s_{12} \langle 2 \left| L_1 \right| 2] \, \langle 4 \left| l_3 \right| 4] \, \mu^2} \left(2 \left[4 \left| l_3^\flat \bar{l} \right| 2 \right] - \mu^2 \, [42] \right) \left(2 \left\langle 1 \left| \bar{l} l_3^\flat \right| 3 \right\rangle - \mu^2 \, \langle 13 \rangle \right) \tag{5-373}$$
$$\inf_{\mu^2} \left[\inf_{t^0} \left[\inf_{y} [C_{31}] \right] \right] = 0 \tag{5-374}$$

and the triangle contributions,

$$C_{431} = \frac{i}{\langle 4 \left| l_3 \right| 4]} C_{4321}$$
$$= i \frac{\langle 1 | \, L_1 \, |2]}{s_{12} \langle 4 \left| l_3 \right| 4] \, \mu^2} \left(2 \left[4 \left| l_3^\flat \bar{l} \right| 2 \right] - \mu^2 \, [42] \right) \left(2 \left\langle 1 \left| \bar{l} l_3^\flat \right| 3 \right\rangle - \mu^2 \, \langle 13 \rangle \right) \tag{5-375}$$
$$\inf_{\mu^2} \left[\inf_{t \to T_m} [C_{431}] \right] = 0 \tag{5-376}$$

finally,

$$\boxed{C_{2;23}^{[2]} \left(1_g^-, 2_g^+, 3_{\bar{q}}^-, 4_q^+\right) = 0} \tag{5-377}$$

5.3. Full amplitudes

5.3.1. $A_4^{1-loop} \left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right)$

Left Turning

Our full left turning amplitude is given by

$$A_4^{1-loop} \left(1_g^+, 2_g^-, 3_{\bar{q}}^-, 4_q^+\right) =$$
$$= r_\Gamma c_{4;0} \left(+, -; +-\right) \left\{ \frac{1}{s_{12} s_{14}} \left(\frac{2}{\epsilon^2} \left[(-s_{12})^{-\epsilon} + (-s_{14})^{-\epsilon} \right] - \log^2 \left(\frac{s_{12}}{s_{14}} \right) - \pi^2 \right) \left(s_{12} s_{14} + \frac{1}{2} \frac{s_{12}^2 s_{14}}{s_{13}} \right) + \right.$$
$$\left. - \frac{1}{\epsilon^2} \left[- \frac{s_{14}}{s_{13}} (-s_{12})^{-\epsilon} + \frac{s_{12}}{s_{13}} (-s_{14})^{-\epsilon} \right] + \frac{3}{2} \left[\frac{1}{\epsilon} (-s_{12})^{-\epsilon} + 2 \right] + \mathcal{R} \right\} \tag{5-378}$$

Where $\mathcal{R}$ are the rational terms,

$$\mathcal{R} = \frac{1}{4}c_{4;0}\left(+,-;+-\right)\left[-\left(2+\frac{s_{13}}{s_{23}}-\frac{s_{23}}{s_{13}}\right)+\left(2+\frac{s_{13}}{s_{23}}+\frac{s_{23}}{s_{13}}\right)+\frac{s_{12}}{s_{13}}+\frac{s_{12}}{s_{13}}\right] = -\frac{1}{2}c_{4;0}\left(+,-;+-\right) \tag{5-379}$$

the amplitude takes the form,

$$\boxed{\begin{aligned} A_4^{1-loop}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right) &= r_\Gamma c_{4;0}\left(+,-;+-\right)\left(-s_{12}\right)^{-\epsilon}\times \\ &\times\left[\frac{3}{\epsilon^2}+\frac{3}{2\epsilon}-\frac{2}{\epsilon}\log\left(\frac{s_{14}}{s_{12}}\right)+3-\frac{1}{2}-\pi^2-\frac{1}{2}\frac{s_{12}}{s_{13}}\left(\log^2\left(\frac{s_{12}}{s_{14}}\right)+\pi^2\right)\right] \end{aligned}} \tag{5-380}$$

Right Turning

Now, let's study the right turning amplitude,

$$\begin{aligned} &A_4^{1-loop}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right) = \\ &= r_\Gamma c_{4;0}\left(+,-;+-\right)\left\{\frac{1}{s_{12}s_{14}}\left(\frac{2}{\epsilon^2}\left[\left(-s_{12}\right)^{-\epsilon}+\left(-s_{14}\right)^{-\epsilon}\right]-\log^2\left(\frac{s_{12}}{s_{14}}\right)-\pi^2\right)\left(-\frac{1}{2}\frac{s_{12}^2 s_{14}}{s_{13}}\right)+\right. \\ &\quad\left.-\frac{1}{\epsilon^2}\left[\frac{s_{14}}{s_{13}}\left(-s_{12}\right)^{-\epsilon}-\frac{s_{12}}{s_{13}}\left(-s_{14}\right)^{-\epsilon}\right]+\frac{3}{2}\left[\frac{\left(-s_{12}\right)^{-\epsilon}}{\epsilon}+2\right]+\mathcal{R}\right\} \end{aligned} \tag{5-381}$$

Here $\mathcal{R}$ is given by,

$$\mathcal{R} = \frac{1}{2}\frac{s_{14}}{s_{13}}c_{4;0}\left(+,-;+-\right)+\frac{1}{2}\frac{s_{12}}{s_{13}}c_{4;0}\left(+,-;+-\right) = -\frac{1}{2}c_{4;0}\left(+,-;+-\right) \tag{5-382}$$

with this,

$$\boxed{\begin{aligned} A_4^{1-loop}\left(1_g^+,2_g^-,3_{\bar{q}}^-,4_q^+\right) &= r_\Gamma c_{4;0}\left(+,-;+-\right)\left(-s_{12}\right)^{-\epsilon}\times \\ &\times\left[\frac{1}{\epsilon^2}-\frac{3}{2\epsilon}-3-\frac{1}{2}+\frac{1}{2}\frac{s_{12}}{s_{13}}\left(\log^2\left(\frac{s_{12}}{s_{14}}\right)+\pi^2\right)\right] \end{aligned}} \tag{5-383}$$

5.3.2. $A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)$

Left turning

Box and triangle contributions,

$$\begin{aligned} &A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right)\Big|_{Box+Triangle} = \\ &= r_\Gamma c_{4;0}\left(-,+;+-\right)\left(-s_{12}\right)^{-\epsilon}\left\{\frac{3}{\epsilon^2}-\frac{2}{\epsilon}\log\left(\frac{-s_{14}}{-s_{12}}\right)+\frac{1}{2}\left(\frac{s_{14}^3}{s_{13}^3}+1\right)\log^2\left(\frac{s_{12}}{s_{14}}\right)+\frac{1}{2}\pi^2\left(\frac{s_{14}^3}{s_{13}^3}-1\right)\right\} \end{aligned} \tag{5-384}$$

Bubble contributions

$$A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right)\Big|_{Bubble}$$
$$= r_\Gamma c_{4;0}\left(-,+;+-\right)\left(-s_{12}\right)^{-\epsilon}\left\{\frac{3}{2\epsilon}+3-\frac{3}{2}\log\left(\frac{s_{14}}{s_{12}}\right)+\left(\frac{s_{14}}{s_{13}}\right)^2\log\left(\frac{s_{14}}{s_{12}}\right)-\frac{1}{2}\frac{s_{14}}{s_{13}}\log\left(\frac{s_{14}}{s_{12}}\right)\right\} \tag{5-385}$$

And the rational contribution, $\mathcal{R}$ is given, by,

$$\mathcal{R}=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{14}}{s_{13}} \tag{5-386}$$

The full left turning contribution to the amplitude amounts to

$$A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right)=r_\Gamma c_{4;0}\left(-,+;+-\right)\left(-s_{12}\right)^{-\epsilon}\times$$
$$\times\left\{\frac{3}{\epsilon^2}+\frac{3}{2\epsilon}-\frac{2}{\epsilon}\log\left(\frac{-s_{14}}{-s_{12}}\right)+\frac{7}{2}-\frac{1}{2}-\frac{1}{2}\pi^2+\frac{1}{2}\log^2\left(\frac{s_{12}}{s_{14}}\right)-\frac{3}{2}\log\left(\frac{s_{14}}{s_{12}}\right)\right.$$
$$\left.+\frac{1}{2}\frac{s_{14}}{s_{13}}\left[\left(1+\left(\frac{s_{14}}{s_{13}}\right)\log\left(\frac{s_{14}}{s_{12}}\right)\right)^2-\log\left(\frac{s_{14}}{s_{12}}\right)+\left(\frac{s_{14}}{s_{13}}\right)^2\pi^2\right]\right\} \tag{5-387}$$

This agrees with the application of the transition rules from the HV to DR scheme as in KST (eq. (5.35))

Right turning

Box and triangle contributions

$$A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right)\Big|_{Box+Triangle}$$
$$= r_\Gamma c_{4;0}\left(-,+;+-\right)\left(-s_{12}\right)^{-\epsilon}\left[-\frac{1}{\epsilon^2}-\frac{1}{2}\frac{s_{12}^3}{s_{13}^3}\log^2\left(\frac{s_{12}}{s_{14}}\right)-\frac{1}{2}\frac{s_{12}^3}{s_{13}^3}\pi^2\right] \tag{5-388}$$

Bubble contributions,

$$A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar q}^-,4_q^+\right)\Big|_{Bubble}=$$
$$= r_\Gamma c_{4;0}\left(-,+;+-\right)\left(-s_{12}\right)^{-\epsilon}\left[-\frac{3}{2\epsilon}-3+\left(\frac{s_{12}}{s_{13}}\right)^2\log\left(\frac{s_{14}}{s_{12}}\right)-\frac{1}{2}\frac{s_{12}}{s_{13}}\log\left(\frac{s_{14}}{s_{12}}\right)\right] \tag{5-389}$$

And the rational contribution, $\mathcal{R}$ is given, by,

$$\mathcal{R}=\frac{1}{2}c_{4;0}\left(-,+;+-\right)\frac{s_{14}}{s_{13}}=-\frac{1}{2}c_{4;0}\left(-,+;+-\right)\left(1+\frac{s_{12}}{s_{13}}\right) \tag{5-390}$$

The final result for the right turning contribution for the amplitude is,

$$A_4^{1-loop}\left(1_g^-,2_g^+,3_{\bar{q}}^-,4_q^+\right) = r_\Gamma c_{4;0}\left(-,+;+-\right)\left(-s_{12}\right)^{-\epsilon}\times$$
$$\times\left\{-\frac{1}{\epsilon^2}-\frac{3}{2\epsilon}-3-\frac{1}{2}-\frac{1}{2}\frac{s_{12}}{s_{13}}\left[\left(1-\frac{s_{12}}{s_{13}}\log\left(\frac{s_{14}}{s_{12}}\right)\right)^2+\log\left(\frac{s_{14}}{s_{12}}\right)+\left(\frac{s_{12}}{s_{13}}\right)^2\pi^2\right]\right\} \quad (5\text{-}391)$$

These results are in agreement with [18]. For this check we used the transition rules HV to DR,

$$c_{4;1}^{DR}\left(-,\pm;\mp,+\right)-c_{4;1}^{HV}\left(-,\pm;\mp,+\right)=c_\Gamma c_{4;1}^{HV}\left(-,\pm;\mp,+\right)\frac{1}{2}\left(N_c-\frac{1}{N_c}\right) \quad (5\text{-}392)$$

because the amplitudes computed by Kunszt *et al* appear in the HV renormalization scheme and we have worked in FDH regularization scheme.

6. One-loop amplitude of Higgs with partons

Higgs production in the gluon–gluon fusion mechanism is mediated by triangular loops of heavy quarks. In the SM, only the top quark and, to a lesser extent, the bottom quark will contribute to the amplitude. The decreasing Hgg form factor with rising loop mass is counterbalanced by the linear growth of the Higgs coupling with the quark mass. In this section we discuss the analytical features of the process.

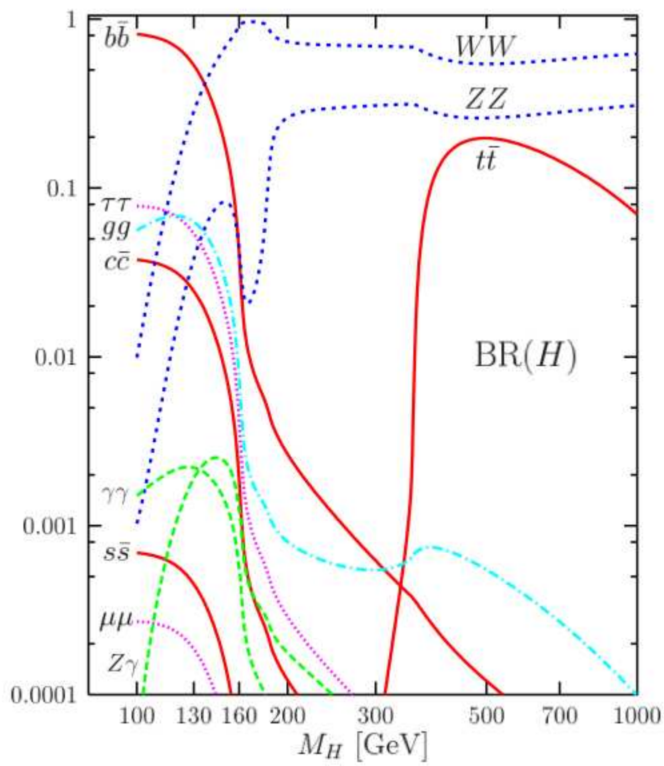


Figure **6-1**.: The decay branching ratios of the Standard Model Higgs boson and its production cross sections in the main channels at the LHC [60].

The process $gg \to H$ by far dominant process, where

$$\begin{aligned} 1\text{fb}^{-1} &\Rightarrow \mathcal{O}\left(10^4\right) \text{ events @LHC} \\ &\Rightarrow \mathcal{O}\left(10^3\right) \text{ events @Tevatron} \end{aligned}$$

we obtain huge cross sections for QCD processes.

In this chapter we show another application of our formalism, where we are looking for the amplitude for the process $ggg \to H$, however, we do not have an interaction vertex of two gluons with one Higgs, then we need to go to higher orders to get relevant information. Indeed, to study this process $gg \to H$ we should contruct an effective vertex of gluons and Higgs, which gives us a process to one-loop. The most interesting part is to study the process $ggg \to H$ because with an effective vertex we get to compute a process to two-loops. It is worth to mention that this is a first step of analysis of the study of the Higgs production at NLO accurancy in QCD. We mention the papers [68], [69] in which by using the code NINJA based on an integrand reduction constructed on the lines studied in this thesis the cross sections and the differential distributions are calculated for the process $gg \to H + q + \bar{q} + g$. The agreement with traditional techniques implemented in SHERPA and GOSAM is of very precised.

6.1. Effective vertex

From the standard model H does not couple to massless particles at tree-level. This suggests us that the process $\gamma\gamma \to H$ to lower order must be treated to one-loop, this loop has to be fermionic due to the Higgs/photon couples to fermions[60].
Moreover, to developing one-loop diagram calculation it becomes complicated, for this reason we need to take certain approaches such as: H momentum is small (i.e. $M_H \ll M_{loop}$), it implies top mass going to infinite ($m_t \to \infty$). In addition, this approach is correct because the processes at the LHC are given by 95% when there is a top quark loop and 5% with a bottom quark[61, 62, 63, 64].

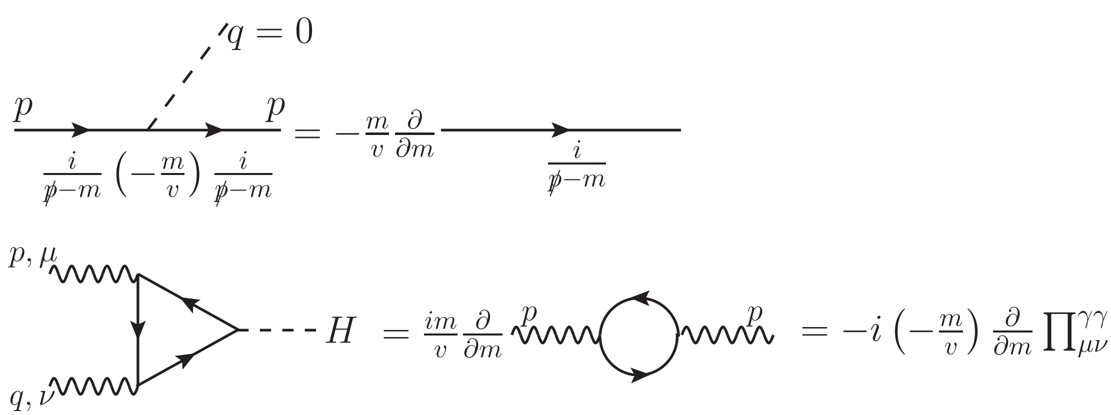


Figure **6-2**.: Effective vertex for the process $\gamma\gamma \to H$, Here v is the vacuum expectation value ($v = 1/\sqrt{\sqrt{2}G_F} = 246\text{GeV}$)

To compute this amplitude we need the photonic selft energy due to the bubble configuration [12]. Here we consider the approximation when the transfered momentum in the top loop is much larger than the Higgs momentum (see fig. **6-2**), the first triangle configuration can be studied as the derivarive of a bubble.

Computing the amplitude for the bubble configuration and calculating the derivative of the fermionic photon self-energy,

$$-i\prod_{\mu\nu}^{\gamma\gamma}\left(p^2\right) = N_c\int\frac{d^4k}{(2\pi)^4}(-1)\,\mathrm{Tr}\left\{(-iee_f\gamma_\mu)\frac{i}{\not{k}-m}(-iee_f\gamma_\nu)\frac{i}{(\not{p}+\not{k})-m}\right\} \tag{6-1}$$

$$\prod_{\mu\nu}^{\gamma\gamma}\left(p^2\right) = -iN_c e^2 e_f^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left\{\gamma_\mu(\not{k}+m)\gamma_\nu\left(\not{p}+\not{k}\right)+m\right\}}{\left[(p+k)^2-m^2\right](k^2-m^2)} \tag{6-2}$$

with $N_c = 3\,(1)$ for quarks (leptons) and e_f the electric charge for the fermions in the loop. Applying rules for traces of gamma matrices and writing the denominator with Feynman parameters,

$$\begin{aligned}\mathrm{Tr}\left\{\gamma_\mu(\not{k}+m)\gamma_\nu\left(\not{p}+\not{k}\right)+m\right\} &= \mathrm{Tr}\left\{\gamma_\mu\not{k}\gamma_\nu\left(\not{p}+\not{k}\right)+m^2\gamma_\mu\gamma_\nu\right\}\\ &= 4\left[2k_\mu k_\nu+\left(m^2-k^2-p\cdot k\right)g_{\mu\nu}\right]\end{aligned} \tag{6-3}$$

$$\frac{1}{\left[(p+k)^2-m^2\right](k^2-m^2)} = \int_0^1\frac{dx}{\left[k^2+2p\cdot kx+p^2x-m^2\right]^2} = \int_0^1\frac{dx}{\left[(k+px)^2+p^2x(1-x)-m^2\right]^2} \tag{6-4}$$

the fermionic photon self-energy takes the form,

$$\prod_{\mu\nu}^{\gamma\gamma}\left(p^2\right) = -iN_c e^2 e_f^2\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{4\left[2k_\mu k_\nu+\left(m^2-k^2-p\cdot k\right)g_{\mu\nu}\right]}{\left[(k+px)^2+p^2x(1-x)-m^2\right]^2} \tag{6-5}$$

This integral is solved with the following procedure

- shifting $k \to k + px$,
- doing the Wick rotation $k_0 \to ik_0$ for Euclidean space (i.e. $k^2 \to -k^2$)
- And, using the symmetry relation,

$$k_\mu k_\nu = \frac{1}{4}g_{\mu\nu}k^2 \tag{6-6}$$

- We write the integral in spherical coordinates, where we have the identity [1]

$$\int d^4k\,F\left(k^2\right) = 2\pi^2\int_0^\infty dy\,yF(y) \tag{6-7}$$

With these prescriptions,

$$\begin{aligned}\prod_{\mu\nu}^{\gamma\gamma}\left(p^2\right) =& -iN_c e_f^2 e^2\times 4\times\pi^2\times\frac{i}{16\pi^4}\times\int_0^1 dx\int_0^\infty dy\,y\\ &\frac{\left[\frac{1}{2}y+m^2-x(1-x)p^2\right]g_{\mu\nu}+2x(1-x)\left[g_{\mu\nu}p^2-p_\mu p_\nu\right]}{\left[y+m^2-p^2x(1-x)\right]^2}\end{aligned} \tag{6-8}$$

[1] Here the solid angle has been expressed as $\int d\Omega_n = \frac{2^n\pi^{n/2}\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}$

Due to gauge invariance, photon is transverse ($\propto g_{\mu\nu}p^2 - p_\mu p_\nu$): the first term ($\propto g_{\mu\nu}$) vanishes and we study the remaining,

$$\prod_{\mu\nu}^{\gamma\gamma}\left(p^2\right) = \frac{N_c e_f^2 e^2}{4\pi^2}\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\int_0^1 dx\int_0^\infty dy\, y\frac{2x\left(1-x\right)}{\left[y+m^2-p^2x\left(1-x\right)\right]^2} \tag{6-9}$$

We now compute the $H\gamma\gamma$ vertex. It is important to see that external photons are on-shell and $p_{1,2}\neq p$ (and must be symmetrize, i.e. $A_{\mu\nu}^{H\gamma\gamma}\to 2A_{\mu\nu}^{H\gamma\gamma}$) but $p^2 = p_1\cdot p_2 = \frac{1}{2}M_H^2$.
We write down our amplitude

$$\begin{aligned} A_{\mu\nu}^{H\gamma\gamma} &= -2\frac{m}{v}\frac{\partial}{\partial m}\prod_{\mu\nu}^{\gamma\gamma}\left(p_1,p_2\right) = -\frac{4m^2}{v}\frac{\partial}{\partial m^2}\prod_{\mu\nu}^{\gamma\gamma}\left(p_1,p_2\right) \\ &= -\frac{2m^2}{v}\frac{N_c e_f^2 e^2}{\pi^2}\left(g_{\mu\nu}p_1\cdot p_2 - p_{1\mu}p_{2\nu}\right)\int_0^1 dx\int_0^\infty \frac{-2x\left(1-x\right)ydy}{\left[y+m^2-p^2x\left(1-x\right)\right]^3} \end{aligned} \tag{6-10}$$

As we mentioned before, we are studying $m_t\to\infty$, this implies $m^2 \gg p^2\left(M_H^2\right)$. Taking into account this prescription inside the integral and integrate out over x and y,

$$\int x\left(1-x\right)dx = \frac{1}{6}, \qquad\qquad \int\frac{ydy}{\left(y+m^2\right)^3} = \frac{1}{2}m^2 \tag{6-11}$$

Finally,

$$A_{\mu\nu}^{H\gamma\gamma} = \frac{2}{3v}N_c e_f^2\frac{\alpha}{\pi}\left(p_1\cdot p_2 g_{\mu\nu} - p_{1\mu}p_{2\nu}\right) \tag{6-12}$$

The amplitude obtained is finite and there is not tree level contribution, then the approximation $m_f \gg M_H$ is in practice good up to $M_H\sim 2m_f$. By the way, only top quarks contribute, other f have negligible Yukawa coupling.
The calculation made for photon can be used also for gluons if we make the changes:

- $Q_e \to g_s T^a$,
- which $\alpha\to\alpha_s$
- and, $N_c \to \mathrm{Tr}\left\{T^aT^b\right\}$

Taking into accout the result obtained in eq. (6-12) we can construct an effective Lagrangian for infinitely heavy quarks.
Writing the effective $H\gamma\gamma$ Lagrangian[61],

$$\mathcal{L}\left(H\gamma\gamma\right) = \frac{1}{4}\left(\sqrt{2}G_F\right)^{1/2} e_q^2\beta'\left(1+\delta\right)HF_{\mu\nu}F^{\mu\nu} \tag{6-13}$$

with $\beta' = 2\left(\alpha/\pi\right)\left(1+\alpha_s/\pi\right)$ and the Higgs-quark vertex correction $\delta = 2\alpha_s/\pi$ the $H\gamma\gamma$ coupling can be readily derived:

$$\mathcal{L}\left(H\gamma\gamma\right) = \frac{\left(\sqrt{2}G_F\right)^{1/2}\alpha e_q^2}{2\pi}\left(1-\frac{\alpha_s}{\pi}\right)HF_{\mu\nu}F^{\mu\nu}$$

The generalization to the Hgg coupling follow from $\beta' = \frac{1}{3}(\alpha_s/\pi)\left(1+\frac{19}{4}\alpha_s/\pi\right)$ and $\delta = 2\alpha_s/\pi$ that,

$$\mathcal{L}(Hgg) = \frac{\left(\sqrt{2}G_F\right)^{1/2}\alpha_s}{12\pi}\left(1+\frac{11}{4}\frac{\alpha_s}{\pi}\right)HF^a_{\mu\nu}F^{\mu\nu}_a \tag{6-14}$$

This effective vertex is in agreement with Adler et al[65]

6.2. Higgs production in association with one jet

The SM Higgs boson can be produced in association with a large transverse momentum jet (j), $pp \to jH + X$, via the following subprocesses: $gg \to gH, gq \to qH, g\bar{q} \to \bar{q}H$, and $q\bar{q} \to gH$, the mass of the Higgs boson might be reconstructed from its $\tau^+\tau^-$ decay channel [66].

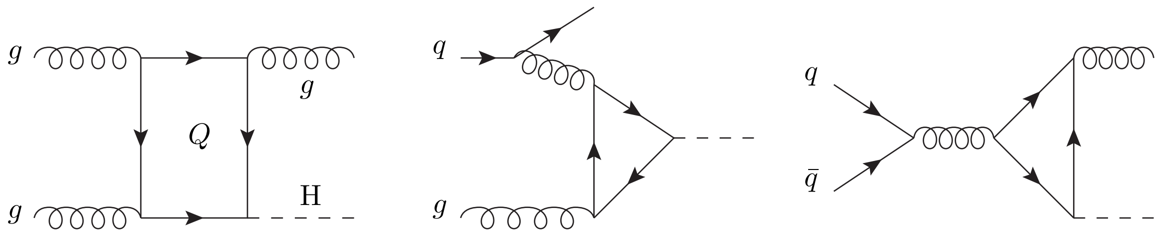


Figure **6-3**.: Diagrams for real QCD corrections to $gg \to H$.

If the quark mass in the loop diagrams is much larger than that of the Higgs boson, $m_q \gg m_H$, the ggH and $gggH$ couplings can be obtained from the low energy theorem of the axial anomaly [65] or from the exact calculation of $gg \to H$ at the limit of $m_q/m_H \gg 1$ (as we did in the previos section).

In this section we are going to compute the amplitude

$$gg \to gH \tag{6-15}$$

that represents a two-loop amplitude, due to we have taken into account the effective vertex of gluon-gluon fusion in a Higgs.

6.2.1. $A_4^{1-loop}(1^+, 2^+, 3^+, H)$

In our calculations we will use the effective vertex,

$$\mathcal{L}(Hgg) = \frac{C}{2} H F^a_{\mu\nu} F^{\mu\nu}_a \tag{6-16}$$

with $C = \frac{\left(\sqrt{2}G_F\right)^{1/2}\alpha_s}{6\pi}\left(1 + \frac{11}{4}\frac{\alpha_s}{\pi}\right)$. Where the Higgs two-gluon color-ordered vertex is,

$$-2i\left(g^{\mu_1\mu_2} k_1 \cdot k_2 - k_1^{\mu_2} k_2^{\mu_1}\right),$$

6.2.2. Tree Level Amplitudes

To compute tree-level amplitudes, first we need the three-point amplitudse and then by using BCFW recursive relation we obtain the remaining four-point amplitudes,

$$A_4^{tree}\left(-l_4^0, H, l_2^0\right) = -im_H^2 \tag{6-17}$$

$$A_3^{tree}\left(H, 1^+, 2^+\right) = i\left[12\right]^2 = i\frac{m_H^4}{\langle 12\rangle^2} \tag{6-18}$$

$$A_3^{tree}\left(H, 1^-, 2^-\right) = i\langle 12\rangle^2 = i\frac{m_H^4}{\left[12\right]^2} \tag{6-19}$$

$$A_3^{tree}\left(H, 1^\pm, 2^\mp\right) = 0 \tag{6-20}$$

$$A_4^{tree}\left(-l_4^0, H, 1^+, l_2^0\right) = i\frac{s_{l_1 l_4}\langle q\left|l_2\right|1\right]}{\langle 1\left|l_1\right|1\right]\langle q1\rangle} = i\frac{m_H^2\langle q\left|l_2\right|1\right]}{\langle 1\left|l_1\right|1\right]\langle q1\rangle} \tag{6-21}$$

$$A_4^{tree}\left(1^+, 2^+, 3^+, H\right) = i\frac{\left[3\hat{2}\right]^3}{\left[l_1\hat{2}\right]\left[3l_1\right]}\frac{i}{s_{23}}i\left[\hat{1}l_1\right]^2 = \frac{im_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{6-22}$$

6.2.3. Cut constructible amplitude

Quadrupole cut coefficient

We consider the configuration showed in fig. **6-4**

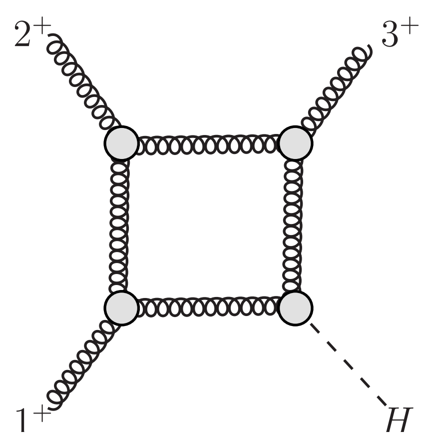


Figure **6-4**.: Box configuration for the process $A_4^{1-loop}\left(1^+, 2^+, 3^+, H\right)$

If we take the solution for the loop momentum en terms of l_2, we obtain,

$$l_2^\mu = c\langle 1\left|\gamma^\mu\right|2\right] \tag{6-23}$$

where c is find by taking the on-shell conditions.
In the following calculations we consider the MHV$-\overline{\text{MHV}}$ sequence because this is the only sequence that does not vanish.

$C_{4;123H}^{[0]}\left(1^+, 2^+, 3^+, H\right)$

The product of the tree-level amplitudes,

$$C_{123H} = A_3^{tree}\left(-l_1^-, 1_g^+, l_2^+\right) A_3^{tree}\left(-l_2^-, 2_g^+, l_3^-\right) A_3^{tree}\left(-l_3^+, 3^+, l_4^-\right) A_3^{tree}\left(-l_4^+, H, l_1^+\right)$$
$$= \frac{m_H^4[3|l_3|l_2|1]^3}{\langle 2|l_2|l_1|l_4|3]\langle 2|l_3|l_4|l_1|1]} = -\frac{m_H^4[2|1][3|2]}{\langle 1|3\rangle} \quad \text{(6-24)}$$

$$\boxed{C_{4;123H}^{[0]}\left(1^+, 2^+, 3^+, H\right) = \frac{1}{2}A_4^{tree} s_{12}s_{23}} \quad \text{(6-25)}$$

We always have been studying Color-ordered amplitudes then is necessary to study all possible configurations where the Higgs may appear, this is because this particle does not have color information.

Taking into account eq. (6-25), we find other posible Higgs configurations by shifting the label of each particle, for example consider $C_{4;12H3}^{[0]}\left(1^+, 2^+, 3^+, H\right)$ and do the shift over eq. (6-25) in the following way,

$$1 \to 3$$
$$2 \to 1$$
$$3 \to 2$$

$$\boxed{C_{4;12H3}^{[0]}\left(1^+, 2^+, 3^+, H\right) = \frac{1}{2}A_4^{tree} s_{13}s_{12}} \quad \text{(6-26)}$$

with the same procedure

$$\boxed{C_{4;1H23}^{[0]}\left(1^+, 2^+, 3^+, H\right) = \frac{1}{2}A_4^{tree} s_{23}s_{13}} \quad \text{(6-27)}$$

Triple cut coefficient

For these triangle contributions we need to distinguish two posible configurations, the first one in which the Higgs is in the four-point block with an external gluon (and internal gluons in $4 - 2\epsilon$ dimensions). The another one is when there is only gluons in the four-point block.

- We study the situation where the Higgs is in the four-point block, for intance, consider the loop momentum solution for the channel s_{1H}, then with the mass-shell conditions l_3 can be written as,

$$l_3^\mu = t\left\langle 2\left|\gamma^\mu\right|3\right], \qquad \bar{l}_3^\mu = t\left\langle 3\left|\gamma^\mu\right|2\right] \quad \text{(6-28)}$$

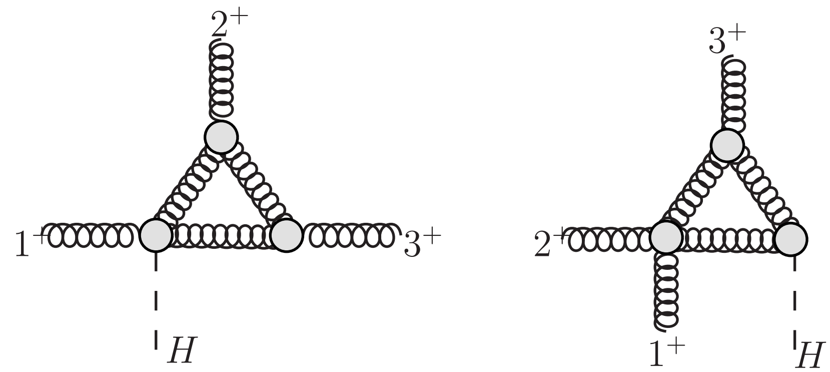


Figure **6-5**.: Triangle configuration for the process $A_4^{1-loop}\left(1^+,2^+,3^+,H\right)$

- Now, consider the block where all particles are gluons. With the mass shell conditions for the channel s_{23}we obtain an explicit solution for l_4,

$$l_4^\mu = \frac{m_H^2}{\gamma}K_3^\mu + t\left\langle K_4^\flat\left|\gamma^\mu\right|3\right], \qquad \bar{l}_4^\mu = \frac{m_H^2}{\gamma}K_3^\mu + t\left\langle 3\left|\gamma^\mu\right|K_4^\flat\right], \tag{6-29}$$

$$\gamma = 2\left(K_4^\flat\cdot K_3\right) = s_{12} - m_H^2. \tag{6-30}$$

Here $K_4^\flat$ has been defined previously as,

$$K_4^\flat = H - \frac{m_H^2}{\gamma}K_3 = -\left(K_1+K_2+K_3\right) - \frac{m_H^2}{\gamma}K_3 = -\left[K_1+K_2+\left(1+\frac{m_H^2}{\gamma}\right)K_3\right] \tag{6-31}$$

With eqs. (6-28) and eq. (6-29) we can compute all triangle contributions,
$C^{[0]}_{3;123H;12}\left(1^+,2^+,3^+,H\right)$
The product of tree-level amplitudes,

$$\begin{aligned} C_{134} &= A_4^{tree}\left(-l_1^-,1^+,2^+,l_3^-\right)A_3^{tree}\left(-l_3^+,3^+,l_4^-\right)A_3^{tree}\left(-l_4^+,H,l_1^+\right) \\ &= -\frac{im_H^4\langle l_1|l_3|3]^3}{\langle 1|2\rangle\langle l_1|1\rangle\langle l_1|l_4|3]\langle l_1|l_4|l_3|2\rangle} = \frac{im_H^4\langle l_1|l_3|3]}{\langle 1|2\rangle\langle l_1|1\rangle\,\langle 32\rangle} = \frac{im_H^4\,[21]}{\langle 1|2\rangle\,\langle 32\rangle}\frac{\langle 1\left|l_3\right|3]}{\langle 1\left|l_3\right|2]} \\ &= -\frac{im_H^4\,[21]}{\langle 1|2\rangle\,\langle 32\rangle}\frac{t\langle 1|3\rangle[3|\mathrm{K}4]}{-t\langle 1|3\rangle[2|\mathrm{K}4] + \frac{m^2[3|2]\langle 1|3\rangle}{\gamma} + [3|2]\langle 1|3\rangle} \end{aligned} \tag{6-32}$$

taking the $\mathrm{Int_t}$,

$$\begin{aligned} &\Longrightarrow \inf_{t^0}\left(-\frac{im_H^4\,[21]}{\langle 1|2\rangle\,\langle 32\rangle}\frac{t\langle 1|3\rangle[3|\mathrm{K}4]}{-t\langle 1|3\rangle[2|\mathrm{K}4] + \frac{m^2[3|2]\langle 1|3\rangle}{\gamma} + [3|2]\langle 1|3\rangle}\right) \\ &= \frac{im_H^4\,[21]}{\langle 1|2\rangle\,\langle 32\rangle}\frac{\langle 1|3\rangle[3|\mathrm{K}4]\,\langle K4|1\rangle}{\langle 1|3\rangle[2|\mathrm{K}4]\,\langle K4|1\rangle} = \frac{im_H^4}{\langle 1|2\rangle\,\langle 2|3\rangle\,\langle 3|1\rangle}\left(s_{13}+s_{23}\right) \end{aligned} \tag{6-33}$$

finally,

$$\boxed{C^{[0]}_{3;123H;12}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\left(s_{13}+s_{23}\right)} \tag{6-34}$$

Using symmetry (as before) we obtain all possible coefficients for the scalar triangle

$$\boxed{C^{[0]}_{3;123H;23}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\left(s_{13}+s_{12}\right)} \tag{6-35}$$

$$\boxed{C^{[0]}_{3;12H3;12}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\left(s_{23}+s_{13}\right)} \tag{6-36}$$

$$\boxed{C^{[0]}_{3;123H;31}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\left(s_{23}+s_{12}\right)} \tag{6-37}$$

$$\boxed{C^{[0]}_{3;1H23;12}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\left(s_{12}+s_{13}\right)} \tag{6-38}$$

$$\boxed{C^{[0]}_{3;1H23;31}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\left(s_{12}+s_{23}\right)} \tag{6-39}$$

Now, we compute the coefficients that come from the four-point block where there is a Higgs. We compute the coefficient $C^{[0]}_{3;123H;H1}\left(1^+,2^+,3^+,H\right)$, the product of tree-level amplitudes,

$$\begin{aligned}&A_4^{tree}\left(-l_4^+,H,1^+,l_2^+\right)A_3^{tree}\left(-l_2^-,2^+,l_3^+\right)A_3^{tree}\left(-l_3^-,3^+,l_4^-\right)=\\&=\frac{\langle 3\left|l_4\right|1]\,[32]^3}{\langle 1\left|l_3\right|2]\,\langle 1\left|l_4\right|2]\,\langle 2\left|l_4\right|1]}=\frac{im_H^4\,[32]^3\,[3|1]\langle 2|3\rangle}{[3|2]\langle 2|3|1]\langle 1|2\rangle(t[3|2]\langle 1|2\rangle-\langle 1|3|2])}\end{aligned} \tag{6-40}$$

taking the Inf_t,

$$\inf_{t^0}\left(\frac{im_H^4\,[32]^3\,[3|1]\langle 2|3\rangle}{[3|2]\langle 2|3|1]\langle 1|2\rangle(t[3|2]\langle 1|2\rangle-\langle 1|3|2])}\right)=0 \tag{6-41}$$

then coefficient,

$$\boxed{C^{[0]}_{3;123H;H1}\left(1^+,2^+,3^+,H\right)=0} \tag{6-42}$$

Using symmetry,

$$\boxed{C^{[0]}_{3;123H;3H}\left(1^+,2^+,3^+,H\right)=0} \tag{6-43}$$

$$\boxed{C^{[0]}_{3;12H3;2H}\left(1^+,2^+,3^+,H\right)=0} \tag{6-44}$$

$$\boxed{C^{[0]}_{3;12H3;H3}\left(1^+,2^+,3^+,H\right)=0} \tag{6-45}$$

$$\boxed{C^{[0]}_{3;1H23;1H}\left(1^+,2^+,3^+,H\right)=0} \tag{6-46}$$

$$\boxed{C^{[0]}_{3;1H23;H2}\left(1^+,2^+,3^+,H\right)=0} \tag{6-47}$$

Double cut coefficient

For the double cut, we need to consider two topologies for this calculation,

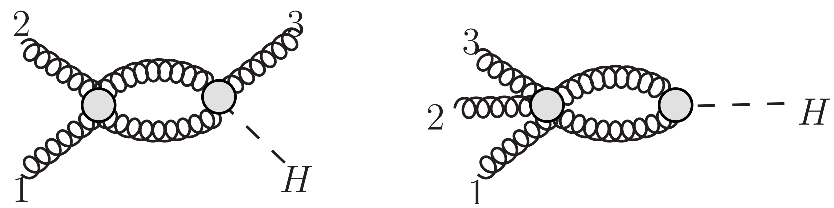


Figure **6-6**.: Triangle configuration for the process $A_4^{1-loop}\left(1^+,2^+,3^+,H\right)$

First, we compute the coefficient $C^{[0]}_{2;123H;12}\left(1^+,2^+,3^+,H\right)$,
Writting down the product of tree-amplitudes,

$$C_{13} = A_4^{tree}\left(-l_1,1,2,l_3\right) A_4^{tree}\left(-l_3,3,H,l_1\right) = \frac{m^4[2|1]^3}{\langle 3|l_3|2]\langle 3|l_1 l_3 l_1|1]} \tag{6-48}$$

the solution for l_1 and l_3 are given by,

$$l_3^\mu = l_1^\mu - K_1^\mu - K_2^\mu \tag{6-49}$$

$$l_1^\mu = yK_2^\mu + (1-y)\,K_1^\mu + \frac{t}{2}\left\langle 2\left|\gamma^\mu\right|1\right] + \frac{y\,(1-y)}{2t}\left\langle 1\left|\gamma^\mu\right|2\right] \tag{6-50}$$

using these solutions on C_{13}, we obtain,

$$\inf_{y^m\rightarrow Y_m}\left[C_{12}\right] = 0 \tag{6-51}$$

Now, let's study the triangle contributions to the bubble coefficients, where we have another one on-shell condition and y can be find as,

$$y_+ = \frac{t[2|3]\langle 3|2\rangle}{\langle 1|3|2]} + 1, \qquad y_- = -\frac{t[1|3]\langle 3|1\rangle}{\langle 1|3|2]} \tag{6-52}$$

taking into account the solutions for y, we obtain,

$$\inf_{t^m\rightarrow T_m}\left[C_{13}\right] = 0 \tag{6-53}$$

Finally, we do not have bubble contribution,

$$\boxed{C^{[0]}_{2;123H;12}\left(1^+,2^+,3^+,H\right) = 0} \tag{6-54}$$

Using symmetry,

$$\boxed{C^{[0]}_{2;123H;23}\left(1^+,2^+,3^+,H\right)=0} \tag{6-55}$$

$$\boxed{C^{[0]}_{2;12H3;12}\left(1^+,2^+,3^+,H\right)=0} \tag{6-56}$$

$$\boxed{C^{[0]}_{2;12H3;H3}\left(1^+,2^+,3^+,H\right)=0} \tag{6-57}$$

$$\boxed{C^{[0]}_{2;1H23;1H}\left(1^+,2^+,3^+,H\right)=0} \tag{6-58}$$

$$\boxed{C^{[0]}_{2;1H23;H2}\left(1^+,2^+,3^+,H\right)=0} \tag{6-59}$$

6.2.4. Rational Terms

Box contributions

The product of tree amplitudes is given by,

$$\begin{aligned} C_{123H} &= A_3^{tree}\left(-l_1^0,1^+,l_2^0\right)A_3^{tree}\left(-l_2^0,2^+,l_3^0\right)A_3^{tree}\left(-l_3^0,3^+,l_4^0\right)A_3^{tree}\left(-l_4^0,H,l_1^0\right) \\ &= -\frac{\mu^2\langle 2|l2|1]\langle 1|l3|2]\langle 1|l3|3]}{\langle 1|2\rangle^2\langle 1|3\rangle} = -\frac{\mu^4\langle 2|1|2]\langle 1|l3|3]}{\langle 1|2\rangle^2\langle 1|3\rangle} \end{aligned} \tag{6-60}$$

writing the solution for l_3 as,

$$l_3^\mu = \frac{c}{2}\left\langle 3\left|\gamma^\mu\right|2\right] - \frac{\mu^2}{2s_{23}c}\left\langle 2\left|\gamma^\mu\right|3\right] \tag{6-61}$$

$$c = \pm\mu\sqrt{\frac{\left\langle 2\left|1\right|3\right]}{\left\langle 3\left|1\right|2\right]}} \tag{6-62}$$

with l_3, C_{123H} becomes,

$$C_{123H} = \frac{\mu^4 c\langle 2|1|2]\,\langle 31\rangle\,[32]}{\langle 1|2\rangle^2\langle 1|3\rangle} \propto \mu^5 \tag{6-63}$$

with this result

$$\mathrm{Inf}_{\mu^4}\left[C_{1234}\right]=0 \tag{6-64}$$

Finally,

$$\boxed{C^{[0]}_{4;123H}\left(1^+,2^+,3^+,H\right)=0} \tag{6-65}$$

$$\boxed{C^{[0]}_{4;12H3}\left(1^+,2^+,3^+,H\right)=0} \tag{6-66}$$

$$\boxed{C^{[0]}_{4;1H23}\left(1^+,2^+,3^+,H\right)=0} \tag{6-67}$$

Triangle contributions

First we study the configuration where the Higgs is in the four-point block.
For this case, the loop momentum can be write as

$$l_3^\mu = t\,\langle 2\,|\gamma^\mu|\,3] - \frac{\mu^2}{4s_{23}t}\,\langle 3\,|\gamma^\mu|\,2]\,, \qquad \bar{l}_3^\mu = t\,\langle 3\,|\gamma^\mu|\,2] - \frac{\mu^2}{4s_{23}t}\,\langle 2\,|\gamma^\mu|\,3] \tag{6-68}$$

we have taking into account the mass shell conditions,

$$l_3^2 = \mu^2, \qquad l_4^2 = (l_3 - 3)^2 = \mu^2, \qquad l_2^2 = (l_3 + 2)^2 = \mu^2$$

The another one contribution for this process is when there are four gluons in the four-point block
The solution for the loop momentun, l_4,

$$l_4^\mu = \frac{m_H^2}{\gamma} K_3^\mu + t\left\langle K_4^\flat\,|\gamma^\mu|\,3\right] - \frac{\mu^2}{4\gamma t}\left\langle 3\,|\gamma^\mu|\,K_4^\flat\right] \tag{6-69}$$

$$\bar{l}_4^\mu = \frac{m_H^2}{\gamma} K_3^\mu + t\left\langle 3\,|\gamma^\mu|\,K_4^\flat\right] - \frac{\mu^2}{4\gamma t}\left\langle K_4^\flat\,|\gamma^\mu|\,3\right] \tag{6-70}$$

where the mass shell conditions are given by,

$$l_4^2 = \mu^2, \qquad l_1^2 = (l_4 - H)^2 = \mu^2, \qquad l_3^2 = (l_4 + 3)^2 = \mu^2$$

With these prescriptions, we compute the rational contributions from the triangles,
First, we study the contribution from the triangle $C^{[2]}_{3;123H;12}\left(1^+,2^+,3^+,H\right)$

$$\begin{aligned}
2C^{[2]}_{3;12}\left(1^+,2^+,3^+,H\right) &= \inf_{t^0}\left[\inf_{\mu^2}\left[A_4^{tree}\left(-l_1,1,2,l_3\right)A_3^{tree}\left(-l_3,3,l_4\right)A_3^{tree}\left(-l_4,H,l_3\right)\right]\right] \\
&= \inf_{t^0}\inf_{\mu^2}\left[\frac{i\mu^4[2|1][3|\mathrm{K4}]}{2ts_{\mathrm{K43}}\langle 1|2\rangle\left(\frac{\mu^2\langle 1|3\rangle[1|\mathrm{K4}]}{2ts_{\mathrm{K43}}} - 2t[3|1]\langle \mathrm{K4}|1\rangle + \frac{m^2\langle 1|3|1]}{s_{\mathrm{K43}}} + \langle 1|2|1] + \langle 1|3|1]\right)}\right]_{l_4}^{\bar{l}_4} \\
&= 0 + \frac{i[2|1][3|K_4]}{\langle 1|2\rangle\langle 1|3\rangle[1|K_4]} = -\left(s_{13}+s_{23}\right)i\frac{m_H^2}{\langle 1|2\rangle\,\langle 23\rangle\,\langle 3|1\rangle}
\end{aligned} \tag{6-71}$$

finally

$$\boxed{C^{[2]}_{3;123H;12}\left(1^+,2^+,3^+,H\right) = \frac{1}{2}A_4^{tree}\frac{s_{13}+s_{23}}{m_H^2}} \tag{6-72}$$

Using symmetry to obtain other contributions,

$$\boxed{C^{[2]}_{3;123H;23}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{13}+s_{12}}{m_H^2}} \tag{6-73}$$

$$\boxed{C^{[2]}_{3;12H3;12}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{13}+s_{23}}{m_H^2}} \tag{6-74}$$

$$\boxed{C^{[2]}_{3;12H3;12}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{12}+s_{23}}{m_H^2}} \tag{6-75}$$

$$\boxed{C^{[2]}_{3;1H23;31}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{12}+s_{23}}{m_H^2}} \tag{6-76}$$

$$\boxed{C^{[2]}_{3;1H23;23}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{12}+s_{13}}{m_H^2}} \tag{6-77}$$

Now, we study the triangles that include the Higgs in the four-point particle, Consider $C^{[2]}_{3;123H;1H}\left(1^+,2^+,3^+,H\right)$, the product of tree-level amplitudes,

$$\begin{aligned}2C^{[2]}_{3;123H;1H}\left(1^+,2^+,3^+,H\right)&=\inf_{t^0}\inf_{\mu^2}\left(-\frac{im^2\langle 3|l_3|2]\langle 2|l_4|1]\langle 2|l_4|3]}{\langle 1|2\rangle\langle 2|3\rangle^2\langle 1|l_2|1]}\right)\\&=\inf_{t^0}\inf_{\mu^2}\left(-\frac{im^2\langle 2|l_3|3](\langle 2|l_3|1]-\langle 2|3|1])\langle 3|l_3|2]}{\langle 1|2\rangle\langle 2|3\rangle^2(\langle 1|l_3|1]+\langle 1|2|1])}\right)=\frac{im_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\frac{s_{23}}{m_H^2}\end{aligned} \tag{6-78}$$

finally,

$$\boxed{C^{[2]}_{3;123H;1H}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\frac{s_{23}}{m_H^2}} \tag{6-79}$$

The remaining contributions,

$$\boxed{C^{[2]}_{3;123H;H3}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\frac{s_{12}}{m_H^2}} \tag{6-80}$$

$$\boxed{C^{[2]}_{3;12H3;3H}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\frac{s_{12}}{m_H^2}} \tag{6-81}$$

$$\boxed{C^{[2]}_{3;12H3;2H}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\frac{s_{13}}{m_H^2}} \tag{6-82}$$

$$\boxed{C^{[2]}_{3;1H23;1H}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\frac{s_{23}}{m_H^2}} \tag{6-83}$$

$$\boxed{C^{[2]}_{3;1H23;2H}\left(1^+,2^+,3^+,H\right)=-\frac{1}{2}A_4^{tree}\frac{s_{13}}{m_H^2}} \tag{6-84}$$

The remaining bubble contribution also vanishes.

Bubble contributions

For the bubble contribution to the rational part, we need to compute only one diagrams, the other posible diagrams can be obtained using symmetries.

First we compute the pure bubble contribution,
Sewing two tree level amplitudes,

$$C_{13} = A_4^{tree}(-l_1, 1, 2, l_3)\, A_4^{tree}(-l_3, 3, H, l_1) = \frac{i\mu^2\langle 2|l3|1|l3|3|l3|2]}{\langle 1|2\rangle\langle 2|l3|2|3\rangle\langle 3|l3|3|1\rangle} \tag{6-85}$$

Writting the explicit solution for l_1,

$$l_1 = yK_2^\mu + (1-y)\,K_1^\mu + \frac{t}{2}\langle 2\,|\gamma^\mu|\,1] + \frac{1}{2t}\left((1-y)\,y - \frac{\mu^2}{s_{12}}\right)\langle 1\,|\gamma^\mu|\,2] \tag{6-86}$$

using momentum conservation and taking Inf_y,

$$\text{Inf}_{\mu^2}\left[\text{Inf}_{t^0}\left(\text{Inf}_y\left[C_{13}\right]\right)\right] = \frac{1}{2}\frac{is_{12}}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|1\rangle} = \frac{1}{2}A_4^{tree}s_{12} \tag{6-87}$$

This result has been obtained from the terms of y^0 and y^1.
Now, we compute the triangle contribution to the bubble.
Studying the only one triangle that gives contribution,

$$C_{123} = A_4^{tree}(-l_1, 1, 2, l_3)\, A_3^{tree}(-l_3, 3, l_4)\, A_3^{tree}(-l_4, H, l_1) = \frac{\mu^2\langle 2|l3|1|l3|3|l3|2]}{\langle 1|2\rangle\langle 1|3\rangle\langle 2|l3|2|3\rangle} \tag{6-88}$$

As we know, for triangles we have another one on-shell condition,

$$l_4^2 = (l_1 + H)^2 = (l_1 - 1 - 2 - 3)^2 = \mu^2 \tag{6-89}$$

then, using this condition y is determined in terms of t,

$$y_+ = -\frac{\mu^2\langle 1|3|2]}{{s_{12}}^2 t} - \frac{ts_{13}}{\langle 1|3|2]} + 1, \qquad y_- = \frac{\mu^2\langle 1|3|2]}{{s_{12}}^2 t} + \frac{ts_{23}}{\langle 1|3|2]} \tag{6-90}$$

putting the explicit solution for l_1 and taking into account y,

$$\text{Inf}_{t^m \to T_m}\left[C_{123}\right] = 0 \tag{6-91}$$

Finally, the bubble coefficient becomes,

$$\boxed{C_{123H;12}^{[2]}\left(1^+, 2^+, 3^+, H\right) = \frac{1}{2}A_4^{tree}s_{12}} \tag{6-92}$$

Using symmetry,

$$\boxed{C^{[2]}_{123H;23}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{23}}{m_H^4}} \tag{6-93}$$

$$\boxed{C^{[2]}_{12H3;12}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{12}}{m_H^4}} \tag{6-94}$$

$$\boxed{C^{[2]}_{12H3;13}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{13}}{m_H^4}} \tag{6-95}$$

$$\boxed{C^{[2]}_{1H23;13}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{13}}{m_H^4}} \tag{6-96}$$

$$\boxed{C^{[2]}_{123H;23}\left(1^+,2^+,3^+,H\right)=\frac{1}{2}A_4^{tree}\frac{s_{23}}{m_H^4}} \tag{6-97}$$

6.2.5. Full Amplitude

Considering only the box coefficient,

$$\begin{aligned}A_4^{1-loop}\left(1^+,2^+,3^+,H\right)_{Box}&=\Bigg\{\frac{2r_\Gamma}{st}\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}-\left(-m_H^2\right)^{-\epsilon}\right]\\&\quad-\frac{2r_\Gamma}{st}\left[\mathrm{Li}_2\left(1-\frac{m_H^2}{s}\right)+\mathrm{Li}_2\left(1-\frac{m_H^2}{t}\right)+\frac{1}{2}\log^2\left(\frac{s}{t}\right)-\frac{\pi^2}{6}\right]\Bigg\}\\&\quad\times\left(\frac{1}{2}A_4^{tree}s_{12}s_{23}+\frac{1}{2}A_4^{tree}s_{12}s_{13}+\frac{1}{2}A_4^{tree}s_{13}s_{23}\right)\\&=r_\Gamma A_4^{tree}\Bigg\{\frac{1}{\epsilon^2}\left[2\left(-s_{12}\right)^{-\epsilon}+2\left(-s_{13}\right)^{-\epsilon}+2\left(-s_{23}\right)^{-\epsilon}-3\left(-m_H^2\right)^{-\epsilon}\right]-\frac{\pi^2}{2}\\&\quad-\left[2\mathrm{Li}_2\left(1-\frac{m_H^2}{s_{12}}\right)+2\mathrm{Li}_2\left(1-\frac{m_H^2}{s_{13}}\right)+2\mathrm{Li}_2\left(1-\frac{m_H^2}{s_{23}}\right)\right]\\&\quad-\frac{1}{2}\left[\log^2\left(\frac{s_{12}}{s_{23}}\right)+\log^2\left(\frac{s_{12}}{s_{13}}\right)+\log^2\left(\frac{s_{13}}{s_{23}}\right)\right]\Bigg\}\end{aligned} \tag{6-98}$$

and the triangle coefficient,

$$\begin{aligned}A_4^{1-loop}\left(1^+,2^+,3^+,H\right)_{Triangle}&=\frac{r_\Gamma}{\epsilon^2}\frac{\left(-K_1^2\right)^{-\epsilon}-\left(-K_2^2\right)^{-\epsilon}}{\left(-K_1^2\right)-\left(-K_2^2\right)}\times\\&\times-\frac{1}{2}A_4^{tree}\left[(s_{13}+s_{23})+(s_{13}+s_{12})+(s_{23}+s_{13})+(s_{23}+s_{12})+(s_{12}+s_{13})+(s_{12}+s_{23})\right]\\&=-\frac{r_\Gamma}{\epsilon^2}\left[\left(-s_{12}\right)^{-\epsilon}+\left(-s_{23}\right)^{-\epsilon}+\left(-s_{13}\right)^{-\epsilon}-3\left(-m_H^2\right)^{-\epsilon}\right]\end{aligned} \tag{6-99}$$

The cut constructible amplitude is given by,

$$\begin{aligned}
A_4^{1-loop}\left(1^+,2^+,3^+,H\right)_{\text{Cut-Const}} &= r_\Gamma A_4^{tree}\left\{\frac{1}{\epsilon^2}\left[(-s_{12})^{-\epsilon}+(-s_{13})^{-\epsilon}+(-s_{23})^{-\epsilon}\right]-\frac{\pi^2}{2}\right.\\
&\quad-\left[2\text{Li}_2\left(1-\frac{m_H^2}{s_{12}}\right)+2\text{Li}_2\left(1-\frac{m_H^2}{s_{13}}\right)+2\text{Li}_2\left(1-\frac{m_H^2}{s_{23}}\right)\right]\\
&\quad\left.-\frac{1}{2}\left[\log^2\left(\frac{s_{12}}{s_{23}}\right)+\log^2\left(\frac{s_{12}}{s_{13}}\right)+\log^2\left(\frac{s_{13}}{s_{23}}\right)\right]\right\}\\
&= r_\Gamma A_4^{tree}\left\{\frac{1}{\epsilon^2}\left[(-s_{12})^{-\epsilon}+(-s_{13})^{-\epsilon}+(-s_{23})^{-\epsilon}\right]-\frac{\pi^2}{2}\right.\\
&\quad+\left[2\text{Li}_2\left(1-\frac{s_{12}}{m_H^2}\right)+2\text{Li}_2\left(1-\frac{s_{13}}{m_H^2}\right)+2\text{Li}_2\left(1-\frac{s_{23}}{m_H^2}\right)\right]\\
&\quad\left.+\left[\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)+\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{13}}{m_H^2}\right)+\log\left(\frac{s_{13}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)\right]\right\}
\end{aligned}\tag{6-100}$$

Where we have used[12, 70],

$$\text{Li}_2\left(1-\frac{m_H^2}{s_{12}}\right) = -\text{Li}_2\left(1-\frac{s_{12}}{m_H^2}\right)-\frac{1}{2}\log^2\frac{s_{12}}{m_H^2}\tag{6-101}$$

$$\log^2\left(\frac{s_{12}}{s_{23}}\right) = \log^2\left(\frac{s_{12}}{m_H^2}\right)+\log^2\left(\frac{s_{23}}{m_H^2}\right)-2\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)\tag{6-102}$$

Finally, the full amplitude is,

$$\begin{aligned}
A_4^{1-loop}\left(1^+,2^+,3^+,H\right) &= r_\Gamma A_4^{tree}\left\{\frac{1}{\epsilon^2}\left[(-s_{12})^{-\epsilon}+(-s_{13})^{-\epsilon}+(-s_{23})^{-\epsilon}\right]-\frac{\pi^2}{2}\right.\\
&\quad+\left[2\text{Li}_2\left(1-\frac{s_{12}}{m_H^2}\right)+2\text{Li}_2\left(1-\frac{s_{13}}{m_H^2}\right)+2\text{Li}_2\left(1-\frac{s_{23}}{m_H^2}\right)\right]\\
&\quad+\left[\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)+\log\left(\frac{s_{12}}{m_H^2}\right)\log\left(\frac{s_{13}}{m_H^2}\right)+\log\left(\frac{s_{13}}{m_H^2}\right)\log\left(\frac{s_{23}}{m_H^2}\right)\right]\\
&\quad\left.-\frac{1}{3}\frac{s_{12}s_{13}+s_{12}s_{23}+s_{13}s_{23}}{m_H^2}+1\right\}
\end{aligned}\tag{6-103}$$

This result is in agreement with [19].

7. Conclusions

We have presented developments in calculating perturbative scattering amplitudes in QCD theory. The proposed methods can be readily extended to any gauge theory. In particular, we have discussed computational techniques for gauge boson and fermion scattering amplitudes at tree and one-loop level. The main focus of the thesis has been the study of the on-shell methods, that exploit the analytic properties of the amplitudes. The great success of these methods has revealed in many computational advances and simplifications in perturbative computation of amplitudes in non-Abelian gauge theory.

Color decomposition has led to great simplification due to the color-ordered primitive amplitudes. The spinor-helicity formalism has revealed a great simplicity in amplitudes. The most striking example is the class of MHV amplitudes. At one-loop level, in addition to the color decomposition and an expansion in terms of scalar integrals has required the computation of their coefficients, which are rational functions of the momenta and polarization. Those coefficients are computed by the generalized unitarity technique, based on the tree level amplitudes obtained by multiple unitarity cuts.

Useful recursion relations like BCFW allows for the calculation of the tree level amplitudes more efficiently than Feynman diagrams, the simplification and the lack of redundacy, which becomes proibitive for a multipartonic process, is actually based on the appropriate continuation of momenta to take complex values. The behaviour at the singularities required by unitarity makes the remaining job. This is based on working explicitly on the amplitudes which are by construction gauge invariant, instead of Feynman diagrams, which provides gauge invariant sums, but in their intermediate steps are not gauge invariant.

At one-loop level, we have explored the unitarity methods and we have provided a new formalism with extended helicity spinors and consequently extended polarization vectors, which allows for fully reconstructing the one loop scattering amplitude in their rational part as well as in their cut-constructible part. The main message of this thesis is that there is an unified formalism in which the cut-constructible part and the rational part of a scattering amplitude can be found at once. It is enough just to give off-shellness to the internal momentum in a natural way related to the dimensional regularization and then to perform multiple unitarity cuts for massive internal legs, where for a massless theory such a mass is exactly the off-shellness. In this thesis there are very non-trivial examples that such a prescription really works since we have been able to reproduce important $2 \to 2$ partonic amplitudes at the next-to-leading order as well as the Higgs with one jet in gluon-gluon fusion. The novelty of our approach is that our formalism works in the gauge theory at hands in its purely four-dimensional formulation, meaning that we do not need neither the supersymmetric decomposition or the extension of the spinors in higher dimensions, which, for such problems, was typical of previous approaches . Our generalized spinors needed to take into account the polarization of the cut internal legs are purely four-dimensional. So by regularizing our gauge theory in dimensional regularization and adopting the FDH (four dimensional helicity) scheme we have provided a formalism that can implement such a procedure.

There are many outlooks for this job; they involve the two-loop implementation of our formalism but also a complete automation at one loop. For the latter we expect that our formalism can be mixed with the semianalitical technique like OPEN-LOOP [71]. The last one being particularly attractive, because our proposed formalism could solve the problem of finding the full dependence of μ^2 of the integrand obtained by the OPEN-LOOP technique for a next-to-leading order scattering amplitude.

A. Color Algebra

We study the group $SU(3)$ that has $3^2 - 1 = 1$ generators and is represented by,

$$SU(3) = \left\{ U \in M_{3\times 3}(\mathbb{C}) \,/ UU^\dagger = U^\dagger U = I_{3\times 3}, \mathrm{det}U = 1 \right\} \tag{A-1}$$

These generators are traceless and hermitian.
The standard choice for the generators of the fundamental representation are

$$t^a = \frac{1}{2}\lambda^a \tag{A-2}$$

where λ^a are the Gell-Mann matrices,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{A-3}$$

Is clear to see that t^1, t^2, t^3 generate an $SU(2)$ subalgebra.
The properties of the group are defined by the commutator,

$$\left[t^a, t^b\right] = if^{abc}t^c \tag{A-4}$$

with f^{abc} the structure constants that are completely antisymmetric in the indices,

$$f^{123} = 1\ , \quad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}\ , \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2} \tag{A-5}$$

In the $SU(3)$ we can consider two different representations:

- The adjoint representation, where the color indices are denoted by $a, b, c, a_i, \ldots \in \{1, 2, \ldots, 8\}$.
  Here the dimension of the adjoint representation is $d(G) = N^2 - 1 = 8$.
  By the way, the antisymmetry of f^{abc} can be showed by studying the operator t^2 ($t^2 = t^a t^a$) that commutes with all group generators[1]

$$\left[t^b, t^a t^a\right] = \left(if^{bac}t^c\right)t^a + t^a\left(if^{bac}t^c\right) = if^{bac}\left\{t^c, t^a\right\} = 0 \tag{A-6}$$

this implies that t^2 takes a constant value on each irreducible representation,

$$f^{acd}f^{bcd} = C_2\left(G\right)\delta^{ab} \tag{A-7}$$

here $C_2\left(G\right)$ is the Casimir operator for the adjoint representation.

- The fundamental representation $N_c = 3$ with its conjugate representation $\bar{N}_c = \bar{3}$ for quarks and antiquarks respectively.
  Fundamental color indices are denoted by $i_1, i_2, \ldots \in \{1,2,3\}$ and anti-fundamental by $\bar{j}_1, \bar{j}_2, \ldots \in \{1,2,3\}$.
  Then the matrix representation of T^2 is proportional to the unit matrix,

$$t^a t^a = C_2\left(r\right)\mathbb{I} \tag{A-8}$$

Furthemore, we have the relation

$$\mathrm{tr}\left[t^a t^b\right] = C\left(r\right)\delta^{ab} \tag{A-9}$$

if we contract eq. (A-9) with δ^{ab} and take the trace to eq. (A-8), we get,

$$d\left(r\right)C_2\left(r\right) = d\left(G\right)C\left(r\right) \tag{A-10}$$

$d\left(r\right)$ and $d\left(G\right)$ are the dimensions of the fundamental and adjoint representations respectively
All t^a's satisty,

$$\mathrm{tr}\left[t^a t^b\right] = \frac{1}{2}\delta^{ab} \tag{A-11}$$

then $C\left(N\right)$ and $C_2\left(N\right)$ are given by,

$$C\left(N\right) = \frac{1}{2}, \qquad C_2\left(N\right) = \frac{N^2-1}{2N} \tag{A-12}$$

In $N = 3$, $C\left(3\right) = \frac{1}{2}, C_2\left(3\right) = \frac{4}{3}$.

Fact 1. *Fierz identity*

$$\left(T^a\right)_{i_1}^{\bar{j}_1}\left(T^a\right)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} - \frac{1}{N_c}\delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} \tag{A-13}$$

Proof. The set of $\{\mathbb{I}, T^a\}$ spans $M_3\left(\mathbb{C}\right)$, which means any 3×3 complex matrix can be expanded in terms of

$$\mathbb{X} = X_0\mathbb{I} + X_a T^a, \tag{A-14}$$

For any representation of $SU\left(N\right)$ algebra $\{T^a\}$ satisfying relation (A-9) we get,

$$X_0 = \frac{1}{N_c}\mathrm{Tr}\left[X\right], \qquad X_a = \frac{1}{C}\mathrm{Tr}\left[XT^a\right] \tag{A-15}$$

putting (A-15) in (A-14),

$$X_{ij} = \frac{1}{N_c} X_{kk} \delta_{ij} + \frac{1}{C} X_{lk} T^a_{kl} T^a_{ij}, \tag{A-16}$$

Factoring the matrix X,

$$\delta_{il} \delta_{kj} = \frac{1}{N_c} \delta_{ij} \delta_{kl} + \frac{1}{C} T^a_{ij} T^a_{kl} \tag{A-17}$$

Finally,

$$\frac{1}{C} T^a_{ij} T^a_{kl} = \delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \tag{A-18}$$

with $C = 1$ we recover (A-13) □

B. Numerical evaluation for Spinors

B.1. Spinor Identities

Fact 2. *Relation between left-handed and right-handed spinors*

$$u_R(p) = \left(i\sigma^2\right) u_L^* \tag{B-1}$$

Proof. From eqs. (2-37),

$$(p_o\sigma_o - \vec{p}\cdot\vec{\sigma})\, u_R = 0 \tag{B-2}$$

$$(p_o\sigma_o + \vec{p}\cdot\vec{\sigma})\, u_L = 0 \tag{B-3}$$

conjugating (B-3),

$$\begin{aligned}[(p_o\sigma_o + \vec{p}\cdot\vec{\sigma})\, u_L]^* = (p_o\sigma_o + \vec{p}\cdot\vec{\sigma}^*)\, u_L^* = \left(p_o\sigma_o - \left(i\sigma^2\right)\vec{p}\cdot\vec{\sigma}\left(i\sigma^2\right)\right) u_L^* &= 0\\ \left(i\sigma^2\right) p\cdot\sigma \left(i\sigma^2\right) u_L^* &= 0\\ \rightarrow p\cdot\sigma\left(i\sigma^2\right) u_L^* &= 0\end{aligned} \tag{B-4}$$

Comparing eq. (B-4) with (B-2), we obtain,

$$u_R(p) = \left(i\sigma^2\right) u_L^* \tag{B-5}$$

□

Fact 3. *Fierz Identity.*
The general Fierz identity can be written as an equation,

$$\left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right) = \sum_{C,D} C^{AB}{}_{CD}\left(\bar{u}_1\Gamma^C u_2\right)\left(\bar{u}_3\Gamma^D u_2\right) \tag{B-6}$$

where, u_i represents the spinor for the particle with momentum p_i and Γ's are any 16 combinations of Dirac Matrices that have the normalization

$$Tr[\Gamma^A\Gamma^B] = 4\delta^{AB} \tag{B-7}$$

$$\Gamma^A = \left\{1, \gamma^0, i\gamma^i, \frac{1}{2}\left[\gamma^0,\gamma^i\right], \frac{i}{2}\left[\gamma^i,\gamma^j\right], \gamma^5, i\gamma^0\gamma^5, \gamma^i\gamma^5\right\} \tag{B-8}$$

with $i, j = 1, 2, 3$ and $i \neq j$
If we take,

$$\left(\Gamma^A\right)_{a\bar{a}}\left(\Gamma^B\right)_{b\bar{b}} = \sum_{C,D} C^{AB}{}_{CD}\left(\Gamma^C\right)_{a\bar{b}}\left(\Gamma^D\right)_{b\bar{a}} \tag{B-9}$$

and multiply by $\left(\Gamma^C\right)_{\bar{b}a}$ *and* $\left(\Gamma^D\right)_{\bar{a}b}$ *in both sides, this becomes*

$$(\Gamma^C)_{\bar{b}a}(\Gamma^A)_{a\bar{a}}(\Gamma^D)_{\bar{a}b}(\Gamma^B)_{b\bar{b}} = 16C^{AB}\ {}_{EF} \tag{B-10}$$

$$C^{AB}\ {}_{CD} = \frac{1}{16}\,Tr[\Gamma^C\Gamma^A\Gamma^D\Gamma^B] \tag{B-11}$$

from this relation we obtain

$$\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\left[\gamma_\mu\right]_{ab} = 2\left[u_L\left(p_2\right)\bar{u}_L\left(p_1\right) + u_R\left(p_1\right)\bar{u}_R\left(p_2\right)\right]_{ab} \tag{B-12}$$

Proof. Consider two arbitrary spinors a and b, using (B-6), $\left(\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\right)\left(a_L\gamma^\mu b_L\right)$ can be expanded in terms of Dirac Matrices as,

$$\begin{aligned}
\left(\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\right)\left(\bar{a}_L\gamma^\mu b_L\right) &= \left(\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\right)\left(\bar{b}_R\gamma^\mu a_R\right)\\
&= \sum_{C=1}^{16} C_C\left(\bar{u}_L\left(p_1\right)\Gamma^C b_L\right)\left(a_L\Gamma_C u_L\left(p_2\right)\right)\\
&= \left(\bar{u}_L\left(p_1\right)a_R\right)\left(\bar{b}_R u_L\left(p_2\right)\right) - 2\left(\bar{u}_L\left(p_1\right)\gamma^\mu a_R\right)\left(\bar{b}_R\gamma_\mu u_L\left(p_2\right)\right)\\
&\quad - 2\left(\bar{u}_L\left(p_1\right)\gamma^\mu\gamma^5 a_R\right)\left(\bar{b}_R\gamma_\mu\gamma_5 u_L\left(p_2\right)\right) + \left(\bar{u}_L\left(p_1\right)\gamma^5 a_R\right)\left(\bar{b}_R\gamma^5 u_L\left(p_2\right)\right)\\
&= \left(\bar{u}_L\left(p_1\right)a_R\right)\left(\bar{b}_R u_L\left(p_2\right)\right) + \left(\bar{u}_L\left(p_1\right)\gamma^5 a_R\right)\left(\bar{b}_R\gamma^5 u_L\left(p_2\right)\right)\\
&= 2\left(\bar{u}_L\left(p_1\right)a_R\right)\left(\bar{b}_R u_L\left(p_2\right)\right) = 2\bar{a}_L\left[u_R\left(p_1\right)\bar{u}_R\left(p_2\right)\right]b_L
\end{aligned} \tag{B-13}$$

$$\left(\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\right)\left(a_R\gamma^\mu b_R\right) = 2a_R\left[u_L\left(p_2\right)\bar{u}_L\left(p_1\right)\right]b_R \tag{B-14}$$

Finally,

$$\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\left[\gamma_\mu\right]_{ab} = 2\left[u_L\left(p_2\right)\bar{u}_L\left(p_1\right) + u_R\left(p_1\right)\bar{u}_R\left(p_2\right)\right]_{ab} \tag{B-15}$$

□

writting this result in our shorthand notation,

$$\langle p\left|\gamma^\mu\right|q]\,\langle k\left|\gamma_\mu\right|l] = 2\langle pk\rangle[lq]\,, \qquad \langle p\left|\gamma^\mu\right|q]\,[k\left|\gamma_\mu\right|l\rangle = 2\langle pl\rangle[kq] \tag{B-16}$$

Fact 4. *Relationship between* $u_L\left(p\right)$ *and* $u_R\left(p\right)$

$$u_L^\dagger\left(p\right)\bar{\sigma}^\mu u_L\left(q\right) = u_R^\dagger\left(q\right)\sigma^\mu u_R\left(p\right) \tag{B-17}$$

Proof. using the identity (B-5) we find,

$$\begin{aligned}
u_L^\dagger\left(p\right)\bar{\sigma}^\mu u_L\left(q\right) &= u_L^\dagger\left(p\right)\bar{\sigma}^\mu\left(-\left(i\sigma^2\right)^2\right)u_L\left(q\right)\\
&= u_L^\dagger\left(p\right)\left(-i\sigma^2\right)\sigma^{\mu T}\left(i\sigma^2\right)u_L\left(q\right)\\
&= u_R^T\left(p\right)\sigma^{\mu T}u_R^*\left(q\right)\\
&= u_R^\dagger\left(q\right)\sigma^\mu u_R\left(p\right)
\end{aligned} \tag{B-18}$$

□

Fact 5. *Schouten Identity.*
Schouten identity is given by,

$$\langle ij\rangle\langle kl\rangle+\langle ik\rangle\langle lj\rangle+\langle il\rangle\langle jk\rangle=0 \tag{B-19}$$
$$[ij]\,[kl]+[ik]\,[lj]+[il]\,[jk]=0 \tag{B-20}$$

Proof. multiply (B-19) by $[jk]\,[il]$,

$$\begin{aligned}
&\langle ij\rangle\langle kl\rangle[jk]\,[il]+\langle ik\rangle\langle lj\rangle[jk]\,[il]+\langle il\rangle\langle jk\rangle[jk]\,[il]=\\
&=-\langle ij\rangle[jk]\langle kl\rangle[li]-\langle ik\rangle[kj]\langle jl\rangle[li]+s_{il}s_{jk}\\
&=-\langle i\,|jkl|\,i]-\langle i\,|kjl|\,i]+s_{il}s_{jk}=-\langle i\,|jkl|\,i]+\langle i\,|jkl|\,i]-s_{kj}s_{il}+s_{il}s_{jk}=0
\end{aligned}$$

with this,

$$\langle ij\rangle\langle kl\rangle+\langle ik\rangle\langle lj\rangle+\langle il\rangle\langle jk\rangle=0$$

and

$$[ij]\,[kl]+[ik]\,[lj]+[il]\,[jk]=0$$

□

Fact 6. *Completeness relation for polarization vectors*

$$\varepsilon_+^\mu\varepsilon_+^{\nu*}+\varepsilon_-^\mu\varepsilon_-^{\nu*}=-g^{\mu\nu}+\frac{k^\mu p^\nu+k^\nu p^\mu}{p\cdot k} \tag{B-21}$$

Proof. Writing the explicit form for the polarization vectors,

$$\begin{aligned}
\varepsilon_+^\mu\varepsilon_+^{\nu*}+\varepsilon_-^\mu\varepsilon_-^{\nu*}&=-\frac{\langle q\,|\gamma^\mu|\,k]}{\sqrt{2}\langle qk\rangle}\frac{[q\,|\gamma^\nu|\,k\rangle}{\sqrt{2}\,[qk]}-\frac{[q\,|\gamma^\mu|\,k\rangle}{\sqrt{2}\,[qk]}\frac{\langle q\,|\gamma^\nu|\,k]}{\sqrt{2}\langle qk\rangle}\\
&=\frac{1}{4q\cdot k}\langle q\,|\gamma^\mu|\,k]\,[q\,|\gamma^\nu|\,k\rangle+[q\,|\gamma^\mu|\,k\rangle\langle q\,|\gamma^\nu|\,k]\\
&=\frac{1}{16p\cdot k}\left\{\mathrm{Tr}\left[\frac{(1+\gamma^5)}{2}\not{k}\gamma^\mu\frac{(1+\gamma^5)}{2}\not{p}\gamma^\nu\right]+\mathrm{Tr}\left[\frac{(1-\gamma^5)}{2}\not{k}\gamma^\mu\frac{(1-\gamma^5)}{2}\not{p}\gamma^\nu\right]\right\}\\
&=\frac{1}{4p\cdot k}\mathrm{Tr}\left[\not{k}\gamma^\mu\not{p}\gamma^\nu\right]\\
&=-g^{\mu\nu}+\frac{k^\mu p^\nu+k^\nu p^\mu}{p\cdot k}
\end{aligned} \tag{B-22}$$

□

C. Mathematica Implementation of S@M

In this appendix we present the most used functions of the `s@m` package. The following information have been taken from the paper of Maitre and Matrolia[42],

```
<< Spinors`

------- SPINORS @ MATHEMATICA (S@M) -------

            Version: S@M 1.0 (3-APR-2007)

Authors:
        Daniel Maitre (SLAC),
        Pierpaolo Mastrolia (University of Zurich)

A list of all functions provided by the package
is stored in the variable
        $SpinorsFunctions
```

C.1. Most used functions on S@M

In `s@m` objects called spinors are considering to be the solution of the massless Dirac equation, That is not a restriction on the usabilily of the package, since solutions of the massive Dirac equation can be constructed from massless spinors,[42].

- `DeclareSpinor`
  The function `DeclareSpinor` can be called with one or a sequence of arguments. It declares its arguments to be spinors. If undeclared variables are used as spinors, some automatic properties will not be applied and most functions cannot be used.

```
DeclareSpinor[a, b, c, d, e, f, g, h, i]
{a, b, c, d, e, f, g, h, i} added to the list of spinors
```

  Integer labels for spinors do not have to be declared.

- `s[i,j]`
  The function `s[i,j]` represents the kinematic invariant given by the square of the sum of two momenta,

$$s_{ij} = (p_i + p_j)^2$$

  Since the scalar product `s` is symmetric in its arguments, they are automatically sorted.
  This function also accepts more than two arguments for multi-particle invariants,

```
s[a, b]
s[4, 1, 3]
```

$s_{a\,b}$

$s_{1\;\;3\;\;4}$

Slashed matrices

Slashed matrices are in general contractions of Lorentz momenta with gamma matrices $\not{P} = P^{\mu}\gamma_{\mu}$:

- Sm

  The object Sm is used for slashed matrices corresponding to previously declared spinors. Sm can be called with one argument, being a spinor label. In particular, slashed matrices associated to spinors (declared through DeclareSpinor) are automatically declared. One can use the symbol of the spinor, say S to represent the corresponding slashed matrix by means of Sm[s].

```
Sm[a]
Sm[a + b]
```

Sm(a)

Sm(a) + Sm(b)

  The object Sm[s] is linear in its argument.

- SmBA

  The object SmBA represents slashed matrices formed by the tensor product of two spinors, like

$$|b]\,\langle a| + |a\rangle\,[b|$$

  The arguments a and b are spinors labels. SmBA is linear in both arguments. If the two arguments are equal, SmBA[a,a] is automatically replaced by Sm[a].

```
SmBA[1, 2]
SmBA[b + c, a]
SmBA[a, a]
```

SmBA(1, 2)

SmBA(b, a) + SmBA(c, a)

Sm(a)

C.2. Spinor Products

Spinor products are represented in S@M by four different objects: Spaa, Spab, Spba and Spbb, according to the following table.

$\langle a \ldots b\rangle$	$\langle a \ldots b]$	$[a \ldots b\rangle$	$[a \ldots b]$
Spaa[a,...,b]	Spaab[a,...,b]	Spba[a,...,b]	Spbb[a,...,b]

The left- and right-most arguments are spinors and the intermediate arguments are slashed matrices or objects that can be interpreted as slashed matrices such as spinors. Spaa and Spbb expect an even number of (declared) arguments whereas Spab and Spba expect an odd number of (declared) arguments.

- Standard order
  The spinor products have a normal ordering for their arguments. If the rightmost and leftmost elements are spinors, the middle elements are slashed matrices (or can be interpreted as such) and in addition if the spinors are not in the standard order, the spinor product is ordered using the identities

$$\begin{aligned} \langle ba\rangle &= -\langle ab\rangle\,, & [ba] &= -[ab] \\ \langle b\,|Q\ldots P|\,a\rangle &= -\langle a\,|P\ldots Q|\,b\rangle\,, & [b\,|Q\ldots P|\,a] &= -[a\,|P\ldots Q|\,b] \\ [b\,|P|\,a\rangle &= \langle a\,|P|\,b]\,, & [b\,|Q\ldots P|\,a\rangle &= \langle a\,|P\ldots Q|\,b] \end{aligned}$$

  A special case of these identities is the on-shell condition

$$\langle aa\rangle = 0, \qquad [aa] = 0$$

```
Spbb[a, a]
Spaa[b, a]
Spbb[a, P, Q, b]
Spaa[a, b] Spbb[b, a]

0

-⟨a|b⟩

[a|P|Q|b]

[b|a]⟨a|b⟩
```

  The normal ordering of the Spaa and Spbb products are opposite so that the products $\langle ab\rangle\,[ba]$ are displayed in this usual way.

C.3. Spinor Manipulations

- ExpandSToSpinors, ConvertSpinorsToS
  The functions ExpandSToSpinors, ConvertSpinorsToS convert invariants s to spinor products and conversely.

```
ExpandSToSpinors[s[1, 2] s[2, 3]]
[2|1][3|2]⟨1|2⟩⟨2|3⟩

ConvertSpinorsToS[%]
s1  2 s2  3
```

- SpOpen, SpClose
  The function SpOpen decomposes spinor chains containing any slashed matrix that corresponds to a massless spinor with products of smaller spinor chains, by applying the definition of such a matrix in terms of its opposite chirality spinors,

$$\not{k} = |k]\,\langle k| + |k\rangle\,[k|$$

  The function SpClose has the reverse effect as that of SpOpen. It attempts to replace products of spinor products with spinor chains containing slashed matrices. Both the functions can take either one or two arguments. The first argument is the expression to be manipulated; the second argument must be a spinor. With one argument, the functions open or close wherever possible.

```
Spaa[1, 2, 3, 4, 5, 6]
⟨1|2|3|4|5|6⟩

SpOpen[%]
[3|2][5|4]⟨1|2⟩⟨3|4⟩⟨5|6⟩

SpClose[%]
⟨1|2|3|4|5|6⟩
```

If there are different possibilities of reconstructing the spinor chain, `SpClose` does not search for the longest possible spinor chain. The result will depend on the ordering of the spinor labels and might not be invariant under relabeling of the spinor labels.
If a spinor is given as a second argument, `SpOpen` and `SpClose` will only open or close spinor chains containing this specified spinor

```
Spaa[1, 2, 3, 4, 5, 6]
⟨1|2|3|4|5|6⟩

SpOpen[%, 2]
⟨6|5|4|3|2]⟨1|2⟩

SpClose[%, 2]
⟨1|2|3|4|5|6⟩
```

- `Schouten`
  The function `Schouten` applies the Schouten identities

$$\langle ij\rangle\,\langle kl\rangle + \langle ik\rangle\,\langle lj\rangle + \langle il\rangle\,\langle jk\rangle = 0$$
$$[ij]\,[kl] + [ik]\,[lj] + [il]\,[jk] = 0$$

`Schouten[x,i,j,k,l]`
The function with four spinor arguments will search `x` for occurrences of the products $\langle ij\rangle\,\langle kl\rangle$ or $[ij]\,[kl]$ and replace it using the above identities.
`Schouten[x,i,j,k]`
The function with three spinor arguments will search for occurrences of the spinor product $\langle ij\rangle$ or $[ij]$ and will try to use the Schouten identity to combine it with the spinor `k`.

```
Spaa[1, 2] Spaa[3, 4]
⟨1|2⟩⟨3|4⟩

Schouten[%, 1, 2, 3, 4]
⟨1|3⟩⟨2|4⟩ - ⟨1|4⟩⟨2|3⟩
```

C.4. Special Functions

In the following calculations we have to take the Inf in different ways. Then we define the most used functions,

```
Inft[Expr_] := Coefficient[Series[Expr, {t, ∞, 5}],
  t, 0]
Infμ2[Expr_] := Coefficient[Series[Expr, {μ, ∞, 5}],
  μ, 2]
Infμ4[Expr_] := Coefficient[Series[Expr, {μ, ∞, 5}],
  μ, 4]
Infy[Expr_, n_] :=
 Coefficient[Series[Expr, {y, ∞, 5}], y, n]
InfT[Expr_, n_] :=
 Coefficient[Series[Expr, {t, ∞, 5}], t, n]
```

D. BCFW with mathematica

In this appendix we show the numerical code implemented in Mathematica to compute amplitudes using BCFW recursive formula (amplitudes (2-127) and (2-130))
We generate momenta for our particles ($p_i, i = 1, 2, 3, 4, 5, 6$) and for the transferred momentum K,

```
GenMomenta[{1, 2, 3, 4, 5, 6}]
DeclareSpinor[K]
Momenta for the spinors 1, 2, 3, 4, 5, 6 generated.
{K} added to the list of spinors
```

The MHV and Anti-MHV amplitudes are defined as,

```
gMHV[{a_, b_, c__}, {l_, m_}] :=
 i Spaa[l, m]^4 / Times @@ Spaa @@@ Partition[{a, b, c, a}, 2, 1]
BgMHV[{a_, b_, c__}, {l_, m_}] :=
 (-1)^Length[{a, b, c}] i Spbb[l, m]^4 / Times @@ Spbb @@@ Partition[{a, b, c, a}, 2, 1]
```

For the amplitude $A_6\left(1^-, 2^-, 3^-, 4^+, 5^+, 6^+\right)$, we only compute the contribution that comes from the first BCFW diagram

Amplitude A_6(---+++)

```
A1 = gMHV[{1, 2, -K}, {1, 2}] i/s12 gMHV[{K, 3, 4, 5, 6}, {K, 3}]
```

$$-\frac{i\langle 1|2\rangle^3\langle K|3\rangle^3}{s12\langle 3|4\rangle\langle 4|5\rangle\langle 5|6\rangle\langle K|1\rangle\langle K|2\rangle\langle K|6\rangle}$$

```
int1 =
 Factor[
   Factor[
     Simplify[
       SpClose[Factor[SpOpen[A1 //. Spaa[K, a_] → (-Spab[a, -Sm[3] - Sm[4] - Sm[5] - Sm[6], 6]) /
               Spbb[K, 6]] //. {Spaa[a_, 6] → Spaa[a, 60] + z Spaa[a, 1]}] //.
            {Spaa[a_, 60] → Spaa[a, 6], z → Spbb[2, 1]/Spbb[2, 6]} // Factor] //.
          {Spab[a_, 3, b_] → Spab[a, -Sm[1] - Sm[2] - Sm[4] - Sm[5] - Sm[6], b],
           s12 → Spaa[1, 2] Spbb[2, 1]} // SpOpen // ConvertSpinorsToS // SpClose] //.
      {Spbb[a_, 3, b__] → Spbb[a, -Sm[1] - Sm[2] - Sm[4] - Sm[5] - Sm[6], b]} // SpOpen //
     ConvertSpinorsToS] //. -s[3, 6] - s[4, 6] - s[5, 6] → s[1, 6] + s[2, 6] // SpClose] //.
  Spab[a_, 1, b_] → Spab[a, -Sm[3] - Sm[2] - Sm[4] - Sm[5] - Sm[6], b] // Factor
```

$$\left(i\left(\langle 3|4|6] + \langle 3|5|6]\right)^3\right)/\left(\left(s_{1\ 2} + s_{1\ 6} + s_{2\ 6}\right)[2|1][6|1]\left(\langle 5|3|2] + \langle 5|4|2]\right)\langle 3|4\rangle\langle 4|5\rangle\right)$$

And the amplitude $A_6\left(1^+, 2^-, 3^+, 4^-, 5^+, 6^-\right)$

Amplitude (+-+-+-)

```
ClearAll[A1]
```

First BCFW diagram

```
A1 = BgMHV[{1, 2, -K}, {1, -K}] i/s12 BgMHV[{K, 3, 4, 5, 6}, {3, 5}]
```

$$-\frac{i\,[5\,|3]^4\,[1\,|K]^3}{\mathrm{s12}\,[2\,|1]\,[4\,|3]\,[5\,|4]\,[6\,|5]\,[2\,|K]\,[3\,|K]\,[6\,|K]}$$

```
int1 =
 SpClose[
   Factor[
    Factor[A1 //. {Spbb[6, a_] → SPBB[6, a] - z Spbb[1, a], z → -Spaa[2, 1]/Spaa[2, 6]}] //.
       {SPBB → Spbb, Spbb[a_, K] → Spba[a, Sm[1] + Sm[2], 6]/Spaa[K, 6]} // Factor // SpOpen //
     ConvertSpinorsToS] //. s12 → Spaa[1, 2] Spbb[2, 1]] //.
  {Spab[a_, 1, b_] → -Spab[a, Sm[2] + Sm[3] + Sm[4] + Sm[5] + Sm[6], b]} // Factor
```

$$\frac{i\,[5\,|3]^4\,\langle 2\,|6\rangle^4}{(s_{1\ 2}+s_{1\ 6}+s_{2\ 6})\,[4\,|3]\,[5\,|4]\,(\langle 2\,|3\,|5]+\langle 2\,|4\,|5])\,(\langle 6\,|4\,|3]+\langle 6\,|5\,|3])\,\langle 1\,|2\rangle\,\langle 1\,|6\rangle}$$

The second BCFW diagram,

```
ClearAll[A1]
```

Second BCFW diagram

```
A1 = BgMHV[{1, 2, 3, -K}, {1, 3}] i/s123 gMHV[{K, 4, 5, 6}, {4, 6}]
```

$$-\frac{i\,[3\,|1]^4\,\langle 4\,|6\rangle^4}{\mathrm{s123}\,[2\,|1]\,[3\,|2]\,\langle 4\,|5\rangle\,\langle 5\,|6\rangle\,[1\,|K]\,[3\,|K]\,\langle K\,|4\rangle\,\langle K\,|6\rangle}$$

```
int1 =
 Factor[Factor[A1 //. {Spbb[a_, K] → Spba[a, -Sm[4] - Sm[5], 6]/Spaa[K, 6]}] //.
    {Spaa[K, a_] → Spba[1, Sm[2] + Sm[3], a]/Spbb[1, K],
     Spab[a_, 2, b_] → Spab[a, -Sm[1] - Sm[3] - Sm[4] - Sm[5] - Sm[6], b]}] //.
  {Spab[a_, 5, b_] → Spab[a, -Sm[1] - Sm[2] - Sm[3] - Sm[4] - Sm[6], b], s123 → s[1, 2] + s[1, 3] + s[2, 3]} //
 Factor
```

$$-\frac{i\,[3\,|1]^4\,\langle 4\,|6\rangle^4}{(s_{1\ 2}+s_{1\ 3}+s_{2\ 3})\,[2\,|1]\,[3\,|2]\,(\langle 4\,|2\,|1]+\langle 4\,|3\,|1])\,(\langle 6\,|1\,|3]+\langle 6\,|2\,|3])\,\langle 4\,|5\rangle\,\langle 5\,|6\rangle}$$

E. Building blocks in Scalar QCD for scalars in $(4-2\epsilon)$-dimensions

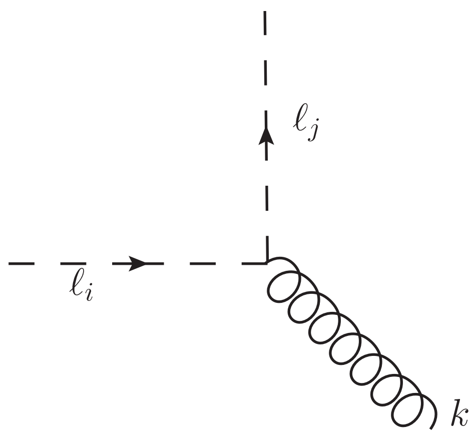
The Lagrangian is written as,

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \left(D_\mu\varphi\right)\left(D^\mu\varphi^*\right) - m^2\varphi\varphi^* \tag{E-1}$$

$$D_\mu\varphi = \partial_\mu\varphi - \frac{ig}{\sqrt{2}}T^a A^a_\mu\varphi \tag{E-2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{g}{2}f^{abc}A^b_\mu A^c_\nu \tag{E-3}$$

E.1. Three-point tree level amplitudes



$$A_3\left(-\ell_i, k^+, \ell_j\right) = -\frac{i}{\sqrt{2}}\left(\ell_i + \ell_j\right)\cdot\varepsilon_+(k) = -\frac{2i}{\sqrt{2}}\ell_j\cdot\varepsilon_+(k) = -i\frac{\langle q_k\,|\ell_j|\,k]}{\langle q_k k\rangle} \tag{E-4}$$

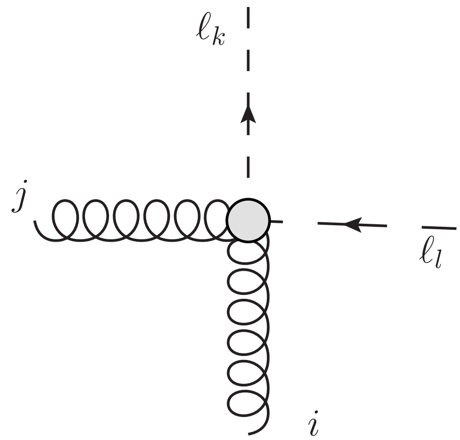
$$A_3\left(-\ell_i, k^+, \ell_j\right) = -i\frac{\langle q_k\,|\ell_j|\,k]}{\langle q_k k\rangle} \tag{E-5}$$

using parity,

$$A_3\left(-\ell_i, k^-, \ell_j\right) = i\frac{[q_k\,|\ell_j|\,k\rangle}{[q_k k]} \tag{E-6}$$

E.2. Four-point tree level amplitudes

To compute these amplitudes we use the BCFW method with the shift $[j, i\rangle$



Putting together the tree amplutes,

$$A_4\left(-\ell_l, i^+, j^+, \ell_k\right) = \sum_{\substack{\text{Internal}\\\text{States}}} \left(-i\frac{\left\langle q_i \left|\hat{P}\right| \hat{i}\right]}{\left\langle q_i \hat{i}\right\rangle}\right) \frac{i}{P^2-\mu^2} \left(-i\frac{\left\langle q_k \left|\ell_k\right| \hat{j}\right]}{\left\langle q_k \hat{j}\right\rangle}\right) \tag{E-7}$$

$$= -\frac{1}{P^2-\mu^2} \frac{\left\langle \hat{j} \left|\hat{P}\right| \hat{i}\right]}{\left\langle \hat{j}\hat{i}\right\rangle} \frac{\left\langle \hat{i} \left|\ell_k\right| \hat{j}\right]}{\left\langle \hat{i}\hat{j}\right\rangle} = \frac{i}{P^2-\mu^2} \frac{\left\langle \hat{j} \left|\hat{P}\hat{i}\ell_k\right| \hat{j}\right]}{\left\langle \hat{i}\hat{j}\right\rangle^2} \tag{E-8}$$

here the reference vectors are $q_k = \hat{i}$ and $q_i = \hat{j}$.
For this process the momentum conservation is given by,

$$-\ell_l + \hat{p}_i + \hat{p}_j + \ell_k = 0$$

where p_i, p_j represent the 4-mometa for external gluons, P the transferred momentum that is defined as,

$$\hat{P} = \ell_l - \hat{p}_i = \hat{p}_j + \ell_k \tag{E-9}$$

and the mass-shell condition, $\hat{P}^2 = \mu^2$, implies,

$$2\left(\ell_l \cdot \hat{i}\right) = 2\left(\ell_k \cdot \hat{j}\right) = 0 \tag{E-10}$$

the numerator in eq. (E-8) takes the form,

$$\left\langle \hat{j} \left|P\hat{i}\ell_k\right| \hat{j}\right] = \left\langle \hat{j} \left|\hat{P}\hat{i}\hat{P}\right| \hat{j}\right] = 2\left(\hat{P}\cdot\hat{i}\right)\left\langle \hat{j} \left|\hat{P}\right| \hat{j}\right] - \mu^2 \left\langle \hat{j} \left|\hat{i}\right| \hat{j}\right] = 4\left(\hat{P}\cdot\hat{i}\right)\left(\hat{P}\cdot\hat{j}\right) - 2P^2\left(\hat{i}\cdot\hat{j}\right) \tag{E-11}$$

$$= 4\left(\hat{\ell}_l\cdot\hat{i}\right)\left(\hat{\ell}_k\cdot\hat{j}\right) - 2\mu^2\left(\hat{i}\cdot\hat{j}\right) = -\mu^2\left\langle \hat{i}\hat{j}\right\rangle\left[\hat{j}\hat{i}\right] = -\mu^2\left\langle ij\right\rangle\left[ji\right] \tag{E-12}$$

finally, we obtain

$$A_4\left(-\ell_l, i^+, j^+, \ell_k\right) = i\mu^2 \frac{[ij]}{\langle ij\rangle \langle i\left|\ell_l\right| i]} \tag{E-13}$$

here $\langle i\,|\ell_l|\,i] = \hat{P}^2 - \mu^2$.

Following the same procedure done before, we compute the amplitude $A_4\left(-\ell_l, i^+, j^-, \ell_k\right)$

$$A_4\left(-\ell_l, i^+, j^-, \ell_k\right) = \sum_{\substack{\text{Internal}\\ \text{States}}} \left(-i\frac{\left\langle q_i\,|P|\,\hat{i}\right]}{\left\langle q_i\hat{i}\right\rangle}\right)\frac{i}{P^2-\mu^2}\left(i\frac{\left[q_j\,|\ell_k|\,\hat{j}\right\rangle}{\left[q_j\hat{j}\right]}\right) = \frac{i}{P^2-\mu^2}\frac{\left\langle \hat{j}\,|P|\,\hat{i}\right]}{\left\langle \hat{j}\hat{i}\right\rangle}\frac{\left[\hat{i}\,|\ell_k|\,\hat{j}\right\rangle}{\left[\hat{i}\hat{j}\right]} \tag{E-14}$$

$$= i\frac{\left\langle \hat{j}\,|\ell_l|\,\hat{i}\right]^2}{s_{ij}\left\langle i\,|\ell_l|\,i\right]} = i\frac{\left\langle j\,|\ell_l|\,i\right]^2}{s_{ij}\left\langle i\,|\ell_l|\,i\right]} \tag{E-15}$$

$$A_4\left(-\ell_l, i^+, j^-, \ell_k\right) = i\frac{\left\langle j\,|\ell_l|\,i\right]^2}{s_{ij}\left\langle i\,|\ell_l|\,i\right]} \tag{E-16}$$

the remaining amplitudes can be easily found by using parity

$$A_4\left(-\ell_l, i^-, j^-, \ell_k\right) = i\mu^2\frac{\langle ij\rangle}{[ij]\left\langle i\,|\ell_l|\,i\right]} \tag{E-17}$$

$$A_4\left(-\ell_l, i^-, j^+, \ell_k\right) = i\frac{\left[j\,|\ell_l|\,i\right\rangle^2}{s_{ij}\left\langle i\,|\ell_l|\,i\right]} \tag{E-18}$$

F. Building blocks in Fermionic QCD for fermions in $(4-2\epsilon)$-dimensions

In fermionic QCD the Lagrangian can be witten as,

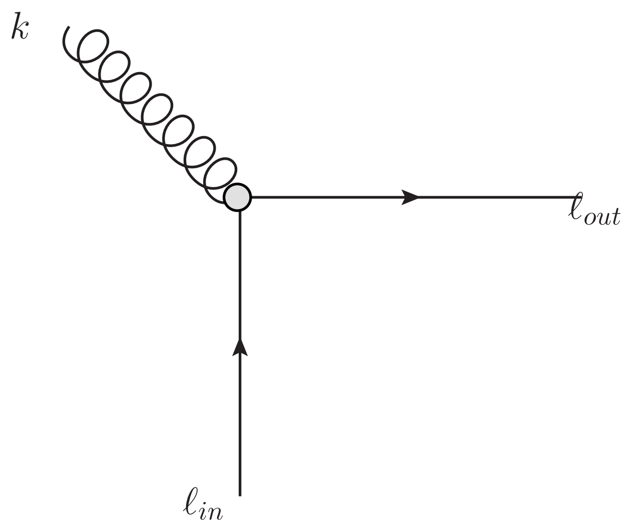
$$\mathcal{L}_{QCD} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \bar{\psi}_i\left(i\not{D} - m\delta_{ij}\right)\psi_j \tag{F-1}$$

$$D_{ij}^{\mu} = \partial^{\mu}\delta_{ij} + \frac{ig_s}{\sqrt{2}}T_{ij}^a A_a \tag{F-2}$$

$$F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a - g_s f^{abc}A_{\mu}^b A_{\nu}^c \tag{F-3}$$

In the following calculations we will use the QR brackets,

F.1. Three-point tree level amplitudes



$$A\left(-\ell_{in}, k^+, \ell_{out}\right) = -\frac{i}{\sqrt{2}}\bar{u}\left(\ell_{out}\right)\varepsilon_+\left(k\right)u\left(\ell_{in}\right) \tag{F-4}$$

$$= -\frac{i}{\langle q_k k\rangle}\left\{\ell_{out}\right|\left(|q_k\rangle\left[k\right| + |k]\left\langle q_k\right|\right)\left|\ell_{in}\right\} \tag{F-5}$$

$$= -\frac{i}{\langle q_k k\rangle}\left(\langle\ell_{out}q_k\rangle\left[k\ell_{in}\right] + \left[\ell_{out}k\right]\langle q_k\ell_{in}\rangle\right) \tag{F-6}$$

here q_k is an arbitrary null 4-vector
Using parity

$$A\left(-\ell_{in}, k^-, \ell_{out}\right) = \frac{i}{[q_k k]}\left([\ell_{out} q_k]\langle k \ell_{in}\rangle + \langle \ell_{out} k\rangle [q_k \ell_{in}]\right) \tag{F-7}$$

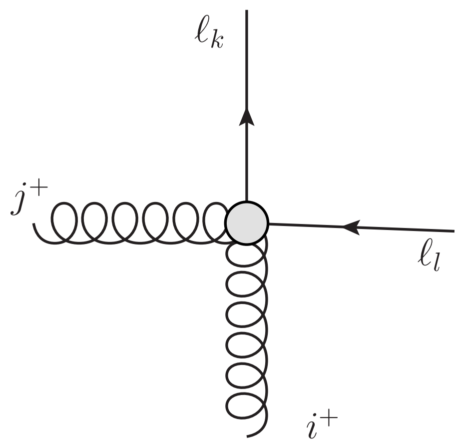
F.2. Four-point tree level amplitudes

To compute these amplitudes we use the BCFW method with the shift $[j, i\rangle$

$$i \to \hat{i} = i - z\eta \tag{F-8}$$

$$j \to \hat{j} = j + z\eta \tag{F-9}$$

$$\eta = |i\rangle [j| + |i] \langle j| \tag{F-10}$$



Following the BCFW calculation:

$$A_4\left(-\ell_{in}, i^+, j^+, \ell_{out}\right) = \sum_{\substack{\text{Internal}\\ \text{States}}} \left(-\frac{i}{\sqrt{2}}\bar{u}\left(\ell_{out}\right)\varepsilon_+\left(\hat{j}\right)u\left(\hat{P}\right)\right)\frac{i}{P^2-\mu^2}\left(-\frac{i}{\sqrt{2}}\bar{u}\left(\hat{P}\right)\varepsilon_+\left(\hat{i}\right)u\left(\ell_{in}\right)\right)$$

$$= \left(-\frac{i}{\left\langle \hat{i}\hat{j}\right\rangle}\left(\left\langle \ell_{out}\hat{i}\right\rangle\left[\hat{j}\hat{P}\right] + \left[\ell_{out}\hat{j}\right]\left\langle \hat{i}\hat{P}\right\rangle\right)\right)\frac{i}{P^2-\mu^2}\left(-\frac{i}{\left\langle \hat{j}\hat{i}\right\rangle}\left(\left\langle \hat{P}\hat{j}\right\rangle\left[\hat{i}\ell_{in}\right] + \left[\hat{P}\hat{i}\right]\left\langle \hat{j}\ell_{in}\right\rangle\right)\right)$$

$$= \frac{1}{\left\langle \hat{i}\hat{j}\right\rangle^2}\frac{i}{P^2-\mu^2}\left(\left\langle \ell_{out}\hat{i}\right\rangle\left[\hat{j}\hat{P}\right] + \left[\ell_{out}\hat{j}\right]\left\langle \hat{i}\hat{P}\right\rangle\right)\left(\left\langle \hat{P}\hat{j}\right\rangle\left[\hat{i}\ell_{in}\right] + \left[\hat{P}\hat{i}\right]\left\langle \hat{j}\ell_{in}\right\rangle\right)$$

$$= \frac{1}{\left\langle \hat{i}\hat{j}\right\rangle^2}\frac{i}{P^2-\mu^2}\left\{\left\langle \ell_{out}\hat{i}\right\rangle\left[\hat{j}\hat{P}\right]\left\langle \hat{P}\hat{j}\right\rangle\left[\hat{i}\ell_{in}\right] + \left\langle \ell_{out}\hat{i}\right\rangle\left[\hat{j}\hat{P}\right]\left[\hat{P}\hat{i}\right]\left\langle \hat{j}\ell_{in}\right\rangle + \right.$$

$$\left. + \left[\ell_{out}\hat{j}\right]\left\langle \hat{i}\hat{P}\right\rangle\left\langle \hat{P}\hat{j}\right\rangle\left[\hat{i}\ell_{in}\right] + \left[\ell_{out}\hat{j}\right]\left\langle \hat{i}\hat{P}\right\rangle\left[\hat{P}\hat{i}\right]\left\langle \hat{j}\ell_{in}\right\rangle\right\}$$

$$= \frac{1}{\left\langle \hat{i}\hat{j}\right\rangle^2}\frac{i}{P^2-\mu^2}\left\{2\left(\hat{P}\cdot\hat{j}\right)\left\langle \ell_{out}\hat{i}\right\rangle\left[\hat{i}\ell_{in}\right] + \left\langle \ell_{out}\left|\hat{i}\mu\hat{j}\right|\ell_{in}\right\rangle + \left[\ell_{out}\left|\hat{j}\mu\hat{i}\right|\ell_{in}\right] + 2\left(\hat{P}\cdot\hat{i}\right)\left[\ell_{out}\hat{j}\right]\left\langle \hat{j}\ell_{in}\right\rangle\right\}$$

$\hat{P}$ is given by, $\hat{P} = \ell_{in} - \hat{p}_i = -\ell_{out} - \hat{p}_j$, then the pole takes the form

$$\hat{P} = \ell_{in} - \hat{p}_i = -\ell_{out} - \hat{p}_j \tag{F-11}$$

$$\hat{P}^2 = \mu^2 - 2\left(\ell_{in}\cdot\hat{p}_i\right) = \mu^2 - 2\ell_{in}\cdot\left(p_i - z\eta\right) = \mu^2 \Longrightarrow z = \frac{\ell_{in}\cdot p_i}{\ell_{in}\cdot\eta} \tag{F-12}$$

$$\text{or } z = -\frac{\ell_{out}\cdot j}{\ell_{out}\cdot\eta} \tag{F-13}$$

The products $2\left(\hat{P}\cdot\hat{j}\right)$ and $2\left(\hat{P}\cdot\hat{i}\right)$ are:

$$2\left(\hat{P}\cdot\hat{p}_j\right) = 2\left(-\ell_{out} - \hat{p}_j\right)\cdot\hat{p}_j = -2\ell_{out}\cdot\hat{p}_j = -2\ell_{out}\cdot\left(p_j + z\eta\right) = 0 \tag{F-14}$$

$$2\left(\hat{P}\cdot\hat{p}_i\right) = 0 \tag{F-15}$$

The amplitude becomes:

$$\begin{aligned}
A_4\left(-\ell_{in}, i^+, j^+, \ell_{out}\right) &= \frac{1}{\left\langle\hat{i}\hat{j}\right\rangle^2}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left(\left\langle\ell_{out}\left|\hat{i}\mu\hat{j}\right|\ell_{in}\right\rangle + \left[\ell_{out}\left|\hat{j}\mu\hat{i}\right|\ell_{in}\right]\right)\\
&= \frac{1}{\left\langle\hat{i}\hat{j}\right\rangle^2}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\ell_{out}\right|\left(\hat{i}\mu\hat{j}\omega_+ + \hat{j}\mu\hat{i}\omega_-\right)\left|\ell_{in}\right\}\\
&= -\frac{1}{\left\langle\hat{i}\hat{j}\right\rangle^2}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\ell_{out}\right|\left(\omega_+\mu\hat{i}\hat{j} + \omega_-\mu\hat{j}\hat{i}\right)\left|\ell_{in}\right\}\\
&= -\frac{1}{\left\langle\hat{i}\hat{j}\right\rangle^2}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\ell_{out}\right|\left(-\omega_+\mu\hat{j}\hat{i} + \omega_-\mu\hat{j}\hat{i} + \omega_+\mu s_{ij}\right)\left|\ell_{in}\right\}\\
&= \frac{1}{\left\langle\hat{i}\hat{j}\right\rangle^2}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\ell_{out}\right|\mu\gamma^5\hat{j}\hat{i}\left|\ell_{in}\right\} + \frac{\left[\hat{i}\hat{j}\right]}{\left\langle\hat{i}\hat{j}\right\rangle}\frac{i}{\left(\ell_{in}-\hat{p}_i\right)^2-\mu^2}\left\{\ell_{out}\right|\omega_+\mu\left|\ell_{in}\right\}
\end{aligned} \tag{F-16}$$

Consider the term $\left\{\ell_{out}\right|\mu\gamma^5\hat{j}\hat{i}\left|\ell_{in}\right\}$,

$$\begin{aligned}
\left\{\ell_{out}\right|\mu\gamma^5\hat{j}\hat{i}\left|\ell_{in}\right\} &= \left\{\ell_{out}\right|\hat{j}\mu\gamma^5\hat{i}\left|\ell_{in}\right\} = \left\{\ell_{out}\right|\hat{j}\mu\gamma^5\hat{i}\left|\ell_{in}\right\} = \left\{\ell_{out}\right|\left(P+\ell_{out}\right)\mu\gamma^5\left(-P+\ell_{in}\right)\left|\ell_{in}\right\}\\
&= \left\{\ell_{out}\right|\left(P+\ell_{out}+\mu-\mu\right)\mu\gamma^5\left(-P+\ell_{in}+\mu-\mu\right)\left|\ell_{in}\right\}\\
&= \left\{\ell_{out}\right|\left(P-\mu\right)\mu\gamma^5\left(-P+\mu\right)\left|\ell_{in}\right\} = \left\{\ell_{out}\right|\left(P^2-\mu^2\right)\mu\gamma^5\left|\ell_{in}\right\} = 0
\end{aligned} \tag{F-17}$$

Finally,

$$A_4\left(-\ell_{in}, i^+, j^+, \ell_{out}\right) = \frac{[ij]}{\langle ij\rangle}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\ell_{out}\right|\omega_+\mu\left|\ell_{in}\right\} = -i\frac{[ij]}{\langle ij\rangle}\frac{\langle\ell_{out}\left|\mu\right|\ell_{in}\rangle}{\langle i\left|\ell_{in}\right|i]} \tag{F-18}$$

In agreement with [6]

Using parity,

$$\mathcal{A}_4\left(-\ell_{in}, i^-, j^-, \ell_{out}\right) = \frac{\langle ij\rangle}{[ij]}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\ell_{out}\right|\omega_-\mu\left|\ell_{in}\right\} = -i\frac{\langle ij\rangle}{[ij]}\frac{\left[\ell_{out}\left|\mu\right|\ell_{in}\right]}{\left\langle i\left|\ell_{in}\right|i\right]} \tag{F-19}$$

Now, we compute the amplitude $A_4\left(-\ell_{in}, i^+, j^-, \ell_{out}\right)$.

$$\begin{aligned}
A_4\left(-\ell_{in}, i^+, j^-, \ell_{out}\right) &= \sum_{\substack{\text{Internal}\\ \text{States}}} A_3\left(-\ell_{in}, i^+, \hat{P}\right)\frac{i}{P^2-\mu^2}A_3\left(-\hat{P}, j^-, \ell_{out}\right)\\
&= \frac{1}{\langle ji\rangle}\left(\left\langle \hat{P}j\right\rangle\left[i\ell_{in}\right]+\left[\hat{P}i\right]\left\langle j\ell_{in}\right\rangle\right)\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\frac{1}{[ij]}\left(\left[\ell_{out}i\right]\left\langle j\hat{P}\right\rangle+\left\langle\ell_{out}j\right\rangle\left[i\hat{P}\right]\right)\\
&= \frac{1}{\langle ij\rangle[ji]}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left(\left\langle \hat{P}j\right\rangle\left[i\ell_{in}\right]+\left[\hat{P}i\right]\left\langle j\ell_{in}\right\rangle\right)\left(\left[\ell_{out}i\right]\left\langle j\hat{P}\right\rangle+\left\langle\ell_{out}j\right\rangle\left[i\hat{P}\right]\right)\\
&= \frac{1}{\langle ij\rangle[ji]}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\{\left\langle \hat{P}j\right\rangle\left[i\ell_{in}\right]\left[\ell_{out}i\right]\left\langle j\hat{P}\right\rangle+\left\langle \hat{P}j\right\rangle\left[i\ell_{in}\right]\left\langle\ell_{out}j\right\rangle\left[i\hat{P}\right]+\right.\\
&\qquad\left.\left[\hat{P}i\right]\left\langle j\ell_{in}\right\rangle\left[\ell_{out}i\right]\left\langle j\hat{P}\right\rangle+\left[\hat{P}i\right]\left\langle j\ell_{in}\right\rangle\left\langle\ell_{out}j\right\rangle\left[i\hat{P}\right]\right\}\\
&= \frac{1}{\langle ij\rangle[ji]}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left(\left\langle \hat{P}j\right\rangle\left[i\ell_{in}\right]\left\langle\ell_{out}j\right\rangle\left[i\hat{P}\right]+\left[\hat{P}i\right]\left\langle j\ell_{in}\right\rangle\left[\ell_{out}i\right]\left\langle j\hat{P}\right\rangle\right)\\
&= \frac{1}{\langle ij\rangle[ji]}\frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\left\langle j\left|\hat{P}\right|i\right]\left(\left[\ell_{out}i\right]\left\langle j\ell_{in}\right\rangle+\left\langle\ell_{out}j\right\rangle\left[i\ell_{in}\right]\right)\\
&= \frac{1}{\langle ij\rangle[ji]}\frac{i}{\left\langle i\left|\ell_{in}\right|i\right]}\left\langle j\left|\ell_{in}\right|i\right]\left(\left[\ell_{out}i\right]\left\langle j\ell_{in}\right\rangle+\left\langle\ell_{out}j\right\rangle\left[i\ell_{in}\right]\right)\\
&= \frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\varepsilon_+(i)\cdot\ell_{in}\left\{\ell_{out}\left|\varepsilon_-(j)\right|\ell_{in}\right\}
\end{aligned} \tag{F-20}$$

here we have taking into account the folloing relations,

$$\left\langle \hat{P}j\right\rangle\left\langle j\hat{P}\right\rangle = \left\langle j\left|\mu\right|j\right\rangle \propto \left\langle j\left|\gamma^5\right|j\right\rangle = 0 \tag{F-21}$$

$$\left[\hat{P}i\right]\left[i\hat{P}\right] = \left[i\left|\mu\right|i\right] \propto \left[i\left|\gamma^5\right|i\right] = 0 \tag{F-22}$$

Using parity,

$$A_4\left(-\ell_{in}, i^-, j^+, \ell_{out}\right) = \frac{1}{\langle ij\rangle[ji]}\frac{i}{\left\langle i\left|\ell_{in}\right|i\right]}\left[j\left|\ell_{in}\right|i\right\rangle\left(\left\langle\ell_{out}i\right\rangle\left[j\ell_{in}\right]+\left[\ell_{out}j\right]\left\langle i\ell_{in}\right\rangle\right) \tag{F-23}$$

$$= \frac{i}{\left(\ell_{in}-p_i\right)^2-\mu^2}\varepsilon_-(i)\cdot\ell_{in}\left\{\ell_{out}\left|\varepsilon_+(j)\right|\ell_{in}\right\} \tag{F-24}$$

G. The scalar integral functions

In this appendix we present the explicit results for scalar integral functions.

- The constant r_Γ is given by

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}, \tag{G-1}$$

  we are working in $D = 4 - 2\epsilon$ dimensions.

- The dilogarithm functios is defined as

$$\mathrm{Li}_2(x) = -\int_0^1 \frac{\log(1-xz)}{z} dz = -\int_0^x \log(1-z)\, dz, \tag{G-2}$$

  satisfyind the following identities

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2\left(1-\frac{1}{x}\right) = -\frac{1}{2}\log^2(x) \tag{G-3}$$

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2(x) = -\log(x)\log(1-x) + \frac{\pi^2}{6} \tag{G-4}$$

- In the following scalar integral functions, the indices in figures, labelling the cyclically ordered external momenta p_i, increase in the clockwise direction.

G.1. The Scalar Bubble integral

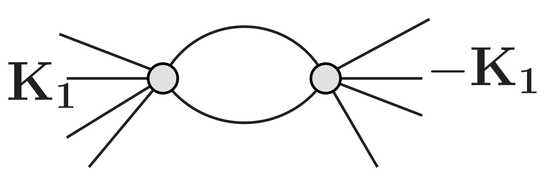


Figure **G-1**.: The scalar bubble integral with a leg of mass K_1^2.

The scalar bubble integral with massive leg K_1 given in figure **G-1** is defined as

$$I_2 = -i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2\,(l-K_1)^2}, \tag{G-5}$$

and is given by

$$I_2 = \frac{r_\Gamma}{\epsilon(1-2\epsilon)}\left(-K_1^2\right)^{-\epsilon} = r_\Gamma\left(\frac{1}{\epsilon} - \log\left(-K_1^2\right) + 2\right) \tag{G-6}$$

G.2. The Scalar Triangle integral

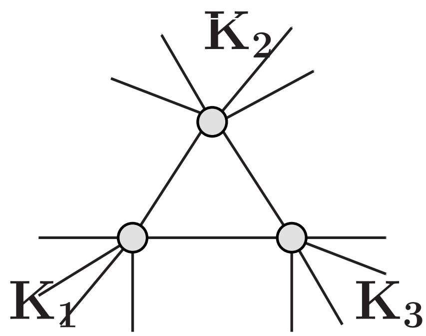


Figure **G-2**.: The scalar triangle with its three legs of mass K_1^2, K_2^2, K_3^2.

The general form of the scalar triangle integral with the masses of its legs labelled K_1^2, K_2^2 and K_3^2 given in figure **G-2** is defined as

$$I_3 = i\,(4\pi)^{2-\epsilon}\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{1}{l^2\,(l-K_1)^2\,(l+K_3)^2}, \tag{G-7}$$

and separates into three cases depending upon the masses of these external legs.

1. If $K_2^2 = K_3^2 = 0$ and $K_1^2 \neq 0$, then the scalar triangle is called "one-mass", and it is

$$I_3^{1m} = \frac{r_\Gamma}{\epsilon^2}\left(-K_1^2\right)^{-1-\epsilon} \tag{G-8}$$

2. If $K_3^2 = 0$ and $K_1^2, K_2^2 \neq 0$, then the scalar triangle is called "two-mass", and it is

$$I_3^{2m} = \frac{r_\Gamma}{\epsilon^2}\frac{\left(-K_1^2\right)^{-\epsilon} - \left(-K_2^2\right)^{-\epsilon}}{\left(-K_1^2\right) + \left(-K_2^2\right)} \tag{G-9}$$

$$= \frac{r_\Gamma}{\left(-K_1^2\right) + \left(-K_2^2\right)}\left(-\frac{\log\left(-K_1^2\right) - \log\left(-K_2^2\right)}{\epsilon} + \frac{\log^2\left(-K_1^2\right) - \log^2\left(-K_2^2\right)}{2}\right) \tag{G-10}$$

3. Finally if all three legs are massive then the integral is as given by

$$I_3^{3m} = \frac{i}{\sqrt{\triangle_3}}\sum_{j=1}^{3}\mathrm{Li}_2\left(-\left(\frac{1+i\delta_j}{1-i\delta_j}\right)\right) - \mathrm{Li}_2\left(-\left(\frac{1-i\delta_j}{1+i\delta_j}\right)\right) \tag{G-11}$$

where

$$\Delta_3 = -\left(K_1^2\right)^2 - \left(K_2^2\right)^2 - \left(K_3^2\right)^2 + 2K_1^2K_2^2 + 2K_2^2K_3^2 + 2K_3^2K_1^2 \tag{G-12}$$

$$\delta_j = \frac{2K_j^2 - \left(K_1^2 + K_2^2 + K_3^2\right)}{\sqrt{\Delta_3}} \tag{G-13}$$

G.3. The Scalar Box integral

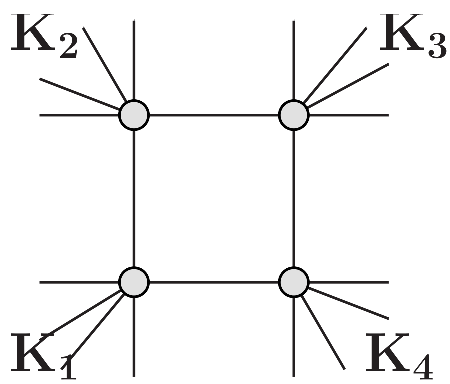


Figure **G-3**.: The scalar box with its four legs of mass K_1^2, K_2^2, K_3^2 and K_4^2.

Let $s = (K_1 + K_2)^2$ and $t = (K_1 + K_4)^2$
The general form of the scalar box integral with the masses of its legs labelled K_1^2, K_2^2, K_3^2 and K_4^2 given in figure **G-3** is defined as

$$I_3 = -i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2\,(l-K_1)^2\,(l-K_1-K_2)^2\,(l+K_4)^2}, \tag{G-14}$$

and separates into six cases depending upon the masses of these external legs.

1. If all four momenta are massless, i.e. $K_1^2 = K_2^2 = K_3^2 = K_4^2 = 0$ (a special case for four-point amplitudes), then the box integral is given by

$$I_4^{0m} = \frac{r_\Gamma}{st}\left(\frac{2}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon}\right] - \log^2\left(\frac{s}{t}\right) - \pi^2\right) \tag{G-15}$$

2. If only one of the four momenta, say K_1, is massive, and the other are massless, i.e. $K_2^2 = K_3^2 = K_4^2 = 0$, then the box is called "one-mass", and it is given by

$$\begin{aligned} I_4^{1m} = {} & \frac{2r_\Gamma}{st}\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon} - \left(K_1^2\right)^{-\epsilon}\right] \\ & - \frac{2r_\Gamma}{st}\left[\mathrm{Li}_2\left(1 - \frac{K_1^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{K_1^2}{t}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{\pi^2}{6}\right] \end{aligned} \tag{G-16}$$

3. In the "two-mass-easy" box, the massless legs are diagonally opposite. If $K_2^2 = K_4^2 = 0$ while the other two legs are massive, the integral is

$$\begin{aligned} I_4^{2m\,e} = & \frac{2r_\Gamma}{st - K_1^2 K_3^2}\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon} - \left(K_1^2\right)^{-\epsilon} - \left(K_3^2\right)^{-\epsilon}\right] \\ & - \frac{2r_\Gamma}{st - K_1^2 K_3^2}\left[\mathrm{Li}_2\left(1 - \frac{K_1^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{K_1^2}{t}\right) + \mathrm{Li}_2\left(1 - \frac{K_3^2}{s}\right)\right. \\ & \left. + \mathrm{Li}_2\left(1 - \frac{K_3^2}{t}\right) - \mathrm{Li}_2\left(1 - \frac{K_1^2 K_3^2}{st}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right)\right] \end{aligned} \tag{G-17}$$

4. In the "two-mass-hard" box, the massless legs are adjacent. If $K_3^2 = K_4^2 = 0$ while the other two legs are massive, the integral is

$$\begin{aligned} I_4^{2m\,h} = & \frac{2r_\Gamma}{st}\frac{1}{\epsilon^2}\left[\frac{1}{2}(-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2}\left(K_1^2\right)^{-\epsilon} - \frac{1}{2}\left(K_2^2\right)^{-\epsilon}\right] \\ & - \frac{2r_\Gamma}{st}\left[-\frac{1}{2}\log\left(\frac{s}{K_1^2}\right)\log\left(\frac{s}{K_2^2}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \mathrm{Li}_2\left(1 - \frac{K_1^2}{t}\right) - \mathrm{Li}_2\left(1 - \frac{K_2^2}{t}\right)\right] \end{aligned} \tag{G-18}$$

5. If exactly one leg is massless, say $K_4^2 = 0$, then we have the "three-mass" box, given by

$$\begin{aligned} I_4^{3m} = & \frac{2r_\Gamma}{st - K_1^2 K_3^2}\frac{1}{\epsilon^2}\left[\frac{1}{2}(-s)^{-\epsilon} + \frac{1}{2}(-t)^{-\epsilon} - \frac{1}{2}\left(K_1^2\right)^{-\epsilon} - \frac{1}{2}\left(K_3^2\right)^{-\epsilon}\right] \\ & - \frac{2r_\Gamma}{st - K_1^2 K_3^2}\left[-\frac{1}{2}\log\left(\frac{s}{K_1^2}\right)\log\left(\frac{s}{K_2^2}\right) - \frac{1}{2}\log\left(\frac{t}{K_2^2}\right)\log\left(\frac{t}{K_3^2}\right)\right. \\ & \left. + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \mathrm{Li}_2\left(1 - \frac{K_1^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{K_3^2}{s}\right) - \mathrm{Li}_2\left(1 - \frac{K_1^2 K_3^2}{st}\right)\right] \end{aligned} \tag{G-19}$$

6. Finally, the "four-mass" box, which is finite, is given by

$$\begin{aligned} I_4^{4m} = & \frac{1}{a\left(x_1 - x_2\right)}\sum_{j=1}^{2}(-1)^j\left[-\frac{1}{2}\log^2\left(-x_j\right)\right. \\ & - \mathrm{Li}_2\left(1 + \frac{-K_3^2 - i\varepsilon}{-s - i\varepsilon}x_j\right) - \eta\left(-x_k, \frac{-K_3^2 - i\varepsilon}{-s - i\varepsilon}\right)\log\left(1 + \frac{-K_3^2 - i\varepsilon}{-s - i\varepsilon}x_j\right) \\ & - \mathrm{Li}_2\left(1 + \frac{-t - i\varepsilon}{-K_1^2 - i\varepsilon}x_j\right) - \eta\left(-x_k, \frac{-t - i\varepsilon}{-K_1^2 - i\varepsilon}\right)\log\left(1 + \frac{-t - i\varepsilon}{-K_1^2 - i\varepsilon}x_j\right) \\ & \left. + \log\left(-x_j\right)\left[\log\left(-K_1^2 - i\varepsilon\right) + \log\left(-s - i\varepsilon\right) - \log\left(-K_4^2 - i\varepsilon\right) - \log\left(-K_2^2 - i\varepsilon\right)\right]\right] \end{aligned} \tag{G-20}$$

Here we have defined

$$\eta\left(x, y\right) = 2\pi i\left[\vartheta\left(-\mathrm{Im}\ x\right)\vartheta\left(-\mathrm{Im}\ y\right)\vartheta\left(\mathrm{Im}\ (xy)\right) - \vartheta\left(\mathrm{Im}\ x\right)\vartheta\left(\mathrm{Im}\ y\right)\vartheta\left(-\mathrm{Im}\ (xy)\right)\right] \tag{G-21}$$

and x_1 and x_2 are the roots of a quadratic polynomial

$$ax^2 + bx + c + i\varepsilon d = a\,(x - x_1)\,(x - x_2) \tag{G-22}$$

with

$$a = tK_3^2, \qquad b = st + K_1^2 K_3^2 - K_2^2 K_4^2, \qquad c = sK_1^2, \qquad d = -K_2^2. \tag{G-23}$$

H. The higher dimensional integrals

Consider the master integrals in $D = 4 - 2\epsilon$-dimensions,

$$\int \frac{d^{4-2\epsilon}\ell}{(4\pi)^{D/2}} \frac{1}{\left(\ell^2 - m_1^2\right)\left((\ell - K_1)^2 - m_2^2\right)\cdots\left((\ell + K_n)^2 - m_n^2\right)} = (4\pi)^{2-\epsilon} \int \frac{d^4p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\left(\mu^2\right)^r}{\mathcal{D}_n} \tag{H-1}$$

Fact 7. *we have the identity*

$$I_n\left[\left(\mu^2\right)^r\right] = (4\pi)^{2-\epsilon} \int \frac{d^4p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\left(\mu^2\right)^r}{\mathcal{D}_n} = -\epsilon\,(1-\epsilon)\,(2-\epsilon)\cdots(r-1-\epsilon)\, I_n^{D=2r+4-2\epsilon} \tag{H-2}$$

Proof. Suppose that we have an integrand as:

$$\begin{aligned} I_n\left[\left(\mu^2\right)^r\right] &= \int \frac{d^4p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \left(\mu^2\right)^r f\left(p^\alpha, u^2\right) \\ &= \int \frac{d^4p}{(2\pi)^4} \int d\Omega_{-1-2\epsilon} \int_0^\infty \frac{d\mu}{(2\pi)^{-2\epsilon}} \mu^{-1-2\epsilon+2r} f\left(p^\alpha, u^2\right) \\ &= \int \frac{d^4p}{(2\pi)^4} \int d\Omega_{-1-2\epsilon} \int_0^\infty \frac{d\mu^2}{2\,(2\pi)^{-2\epsilon}} \left(\mu^2\right)^{-1-\epsilon+r} f\left(p^\alpha, u^2\right) \\ &= \frac{(2\pi)^{2r-2\epsilon} \int d\Omega_{-1-2\epsilon}}{\int d\Omega_{2r-1-2\epsilon}} \int \frac{d^4p}{(2\pi)^4} \int_0^\infty \frac{d\mu^2}{2\,(2\pi)^{2r-2\epsilon}} \left(\mu^2\right)^{-1-\epsilon+r} f\left(p^\alpha, u^2\right) \end{aligned} \tag{H-3}$$

In general, odd powers of μ^α cancel from one-loop integrals. The μ^α integration, of eq. (H-3) is formally in a sub-space that does not overlap with any contribution of the momenta associeted with the loop. For this reason any contribution to the numerator that is odd in the vector μ^α will no contribute to the integral. Accordingly, we shall only need to consider the cases where the numerator depends on μ^2.
Multiplying eq. (H-3) by $\frac{\int d\Omega_{-1+2r-2\epsilon}}{\int d\Omega_{-1+2r-2\epsilon}}$,

$$\begin{aligned} I_n\left[\left(\mu^2\right)^r\right] &= \frac{(2\pi)^{2r} \int d\Omega_{-1-2\epsilon}}{\int d\Omega_{-1+2r-2\epsilon}} \int \frac{d^4p}{(2\pi)^4} \int d\Omega_{-1+2r-2\epsilon} \int_0^\infty \frac{d\mu^2}{2\,(2\pi)^{2r-2\epsilon}} \left(\mu^2\right)^{-1-\epsilon+r} f\left(p^\alpha, u^2\right) \\ &= \frac{(2\pi)^{2r} \int d\Omega_{-1-2\epsilon}}{\int d\Omega_{-1+2r-2\epsilon}} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} \left(\mu^2\right)^{-1-\epsilon+r} f\left(p^\alpha, u^2\right) \end{aligned} \tag{H-4}$$

and,

$$\int d\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \Longrightarrow \int d\Omega_{2r-1-2\epsilon} = \frac{2\pi^{r-\epsilon}}{\Gamma(r-\epsilon)} \tag{H-5}$$

The integral amounts

$$I_n\left[\left(\mu^2\right)^r\right]=\frac{(2\pi)^{2r}\,\frac{2\pi^{-\epsilon}}{\Gamma(-\epsilon)}}{\frac{2\pi^{r-\epsilon}}{\Gamma(r-\epsilon)}}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}}\,f\left(p^\alpha,u^2\right)$$
$$=-\epsilon\left(1-\epsilon\right)\left(2-\epsilon\right)\cdots\left(r-1-\epsilon\right)\left(4\pi\right)^r\int\frac{d^4p}{(2\pi)^4}\int\frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}}\,f\left(p^\alpha,u^2\right)$$
$$=-\epsilon\left(1-\epsilon\right)\left(2-\epsilon\right)\cdots\left(r-1-\epsilon\right)\left(4\pi\right)^r\int\frac{d^{4+2r-2\epsilon}P}{(2\pi)^{4+2r-2\epsilon}}\,f\left(p^\alpha,u^2\right)$$
$$=-\epsilon\left(1-\epsilon\right)\left(2-\epsilon\right)\cdots\left(r-1-\epsilon\right)\left(4\pi\right)^r\int\frac{d^{4+2r-2\epsilon}P}{(2\pi)^{4+2r-2\epsilon}}\,f\left(p^\alpha,u^2\right)\quad\text{(H-6)}$$

In general,

$$I_n^D\left[\left(\mu^2\right)^r\right]=-\epsilon\left(1-\epsilon\right)\left(2-\epsilon\right)\cdots\left(r-1-\epsilon\right)I_n^{D+2r}$$
$$=I_n^{D+2r}\prod_{k=0}^{r-1}\left(\frac{D-4}{2}+k\right)$$
$$=\frac{1}{2^r}I_n^{D+2r}\prod_{k=0}^{r-1}\left(D-4+2k\right)\quad\text{(H-7)}$$

□

In particular using the identity (H-7), we find

$$I_n^{D=4-2\epsilon}\left[\mu^2\right]=-\epsilon I_n^{D=6-2\epsilon},\qquad\text{and}\qquad I_n^{D=4-2\epsilon}\left[\mu^4\right]=-\epsilon\left(1-\epsilon\right)I_n^{D=8-2\epsilon}.\quad\text{(H-8)}$$

although the loop momentum has been shifted to higher dimension, the external momenta remain always in 4-dimensions.

Fact 8. *Recursive relations*
For $n\leq 6$, the master integrals can be written in terms of the $(4-2\epsilon)$-dimensional integrals via the integral recursion relations[57],

$$I_n^{D=6-2\epsilon}=\frac{1}{\left(n-5+2\epsilon\right)c_0}\left[2I_n^{D=4-2\epsilon}-\sum_{i=1}^{n}c_iI_{n-1}^{(i),D=4-2\epsilon}\right],\quad\text{(H-9)}$$
$$I_n^{D=8-2\epsilon}=\frac{1}{\left(n-7+2\epsilon\right)c_0}\left[2I_n^{D=6-2\epsilon}-\sum_{i=1}^{n}c_iI_{n-1}^{(i),D=6-2\epsilon}\right],\quad\text{(H-10)}$$
$$c_i=\sum_{j=1}^{5}S_{ij}^{-1},$$
$$c_0=\sum_{i=1}^{n}c_i$$
$$S_{ij}\equiv\frac{1}{2}\left(m_i^2+m_j^2-p_{ij}^2\right)$$

Consider the triangle integral, $I_{3;k_3}^{D=6-2\epsilon}$,

$$\epsilon I_3^{D=6-2\epsilon} = -\frac{1}{2}\epsilon \left[2I_3^{D=4-2\epsilon} - \sum_{i=1}^{n} c_i I_2^{(i),D=4-2\epsilon} \right] = \frac{1}{2} \tag{H-11}$$

the box integral, $I_{4;k_4}^{D=8-2\epsilon}$,

$$I_4^{D=8-2\epsilon} = -\frac{1}{3c_0}\epsilon \left(1-\epsilon\right) \left[2I_4^{D=6-2\epsilon} - \sum_{i=1}^{n} c_i I_3^{(i),D=6-2\epsilon} \right] = \frac{1}{3} I_3^{D=6-2\epsilon} = \frac{1}{6} \tag{H-12}$$

finally, the bubble integral, $I_{2;k_2}^{D=6-2\epsilon}$,

$$\begin{aligned} I_2^{D=6-2\epsilon}\left(p^2, m_1, m_2\right) &= \int d^D q \, \frac{1}{d_0 d_1} \\ d_0 &= q^2 + m_1^2 - i\varepsilon \\ d_1 &= (q+p)^2 + m_2^2 - i\varepsilon \end{aligned}$$

$$\begin{aligned} \int d^D q \, \frac{1}{d_0 d_1} &= \int_0^1 dx \int d^D q \, \frac{1}{\left[x d_1 + (1-x)\, d_0\right]^2} \\ &= \int_0^1 dx \int d^D q \, \frac{1}{\left[(q+px)^2 + \Delta\right]^2} \\ \Delta &= p^2 x^2 - p^2 x + m_2^2 x + m_1^2 + i\varepsilon \\ \int d^D q \, \frac{1}{d_0 d_1} &= \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \, \Delta^{\frac{D}{2}-2} \\ \lim_{D\to 6-2\varepsilon} \int d^D q \, \frac{1}{d_0 d_1} &= \Gamma\left(-1+\varepsilon\right) \int_0^1 dx \, \Delta^{1-\varepsilon} \\ &= \frac{\Gamma(\varepsilon)}{(-1+\varepsilon)} \int_0^1 dx \, \Delta^{1-\varepsilon} \\ &= -\frac{1}{6} \frac{\Gamma(\varepsilon)}{(-1+\varepsilon)} \left[p^2 - 3\left({m_1}^2 + {m_2}^2\right)\right] \\ &= \lim_{\varepsilon\to 0} -\frac{1}{6} \frac{\Gamma(\varepsilon)}{(-1+\varepsilon)} \left[p^2 - 3\left({m_1}^2 + {m_2}^2\right)\right] \\ &= \frac{1}{6\varepsilon} \left[p^2 - 3\left({m_1}^2 + {m_2}^2\right)\right] \end{aligned} \tag{H-13}$$

With this, the bubble scalar integral in $D = 6 - 2\epsilon$ is given by:

$$I_{2;K_2}^{D}\left[\mu^2\right] = \frac{D-4}{2} I_{2;k_2}^{D+2}\left[1\right] \tag{H-14}$$

$$I_{2;K_2}^{D=6-2\epsilon}\left[\mu^2\right] = -\frac{1}{6}\left[p^2 - 3\left({m_1}^2 + {m_2}^2\right)\right] \tag{H-15}$$

I. Integral Coefficient extraction details

I.1. Box contribution

Remembering that

$$\frac{1}{(l^2-\mu^2)} \to 2\pi i \delta\left(l^2-\mu^2\right) \tag{I-1}$$

$$\begin{aligned} D_0 = (4\pi)^{D/2} \int \frac{d^D l_1}{(2\pi)^D} (-2\pi i)^4 \prod_{i=1}^{4} \delta\left(l_i^2\right) A_1 A_2 A_3 A_4 \\ = (4\pi)^{D/2} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \int d^4 l_1 \prod_{i=1}^{4} \delta\left(\bar{l}_i^2-\mu^2\right) A_1 A_2 A_3 A_4 \end{aligned} \tag{I-2}$$

treating μ as complex variable and partial fractioning off terms with poles at finite μ, this last integral can be rewritten as:

$$D_0 = (4\pi)^{D/2} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \sum_{\sigma} \left[\mathrm{Inf}_{\mu^2} \left[A_1 A_2 A_3 A_4 \left(\bar{l}_1^{\sigma}\right)\right] + \sum_{\mathrm{poles}\{i\}} \frac{\mathrm{Res}_{\mu^2=\mu_i^2} A_1 A_2 A_3 A_4 \left(\bar{l}_1^{\sigma}\right)}{\mu^2-\mu_i^2} \right] \tag{I-3}$$

this equation represents a sum over the residues of all poles $\{i\}$ at finite μ_k and a contribution at infinity.
The $\left[\mathrm{Inf}_{\mu^2}\right]$ is defined so that

$$\lim_{\mu^2\to\infty} \left(\left[\mathrm{Inf}_t\, A_1 A_2 A_3 A_4\right]\left(\mu^2\right) - A_1\left(\mu^2\right) A_2\left(\mu^2\right) A_3\left(\mu^2\right) A_4\left(\mu^2\right)\right) = 0 \tag{I-4}$$

The operator Inf yields a pole-free rational function reproducing the large$-\mu^2$ behavior of an amplitude [52].The first term represents the pure quadrupole cut coefficient and the second the contribution that comes from the pentagon, however, this contribution is not taken into account,

$$C_4^{[4]} = \frac{i}{2} \sum_{\sigma} \mathrm{Inf}_{\mu^2} \left[A_1 A_2 A_3 A_4 \left(\bar{l}_1^{\sigma}\right)\right]_{\mu^4} \tag{I-5}$$

I.2. Triangle contribution

The triangle integral is:

$$C_0 = (4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{A_1^{(j)}(K_1,l)\, A_2^{(j)}(K_2,l)\, A_3^{(j)}(K_3,l)}{(l^2-\mu^2)\left((l-K_1)^2-\mu^2\right)\left((l+K_2)^2-\mu^2\right)} \tag{I-6}$$

with the mass-shell conditions,

$$C_0 = i\int \frac{d^4 l}{(2\pi)^4}(2\pi i)^3 \delta\left(l^2-\mu^2\right)\delta\left((l-K_1)^2-\mu^2\right)\delta\left((l+K_2)^2-\mu^2\right) A_1^{(j)}(K_1,l)\, A_2^{(j)}(K_2,l)\, A_3^{(j)}(K_3,l)$$
$$= (2\pi i)^3\, i\int \frac{dt}{(2\pi)^4} J_t A_1^{(j)}(t)\, A_2^{(j)}(t)\, A_3^{(j)}(t) \qquad \text{(I-7)}$$

treating t as complex variable,

$$C_0 = (2\pi i)^3\, i\int \frac{dt}{(2\pi)^4} J_t \left(\left[\mathrm{Inf}_t\, A_1^{(j)} A_2^{(j)} A_3^{(j)}\right](t) + \sum_{\{k\}}\left[\frac{\mathrm{Res}_{t=t_k} A_1^{(j)} A_2^{(j)} A_3^{(j)}}{t-t_k}\right]\right) \qquad \text{(I-8)}$$

In C_0 is important to see that the term corresponding to the sum of residues does not give any contribution to the triple cut, since this sum only gives contributions to the box coefficient. By the way, triangle contributions of the triple cut come exclusively from the terms at infinity,

$$C_0 = (2\pi i)^3\, i\int \frac{dt}{(2\pi)^4} J_t \left[\mathrm{Inf}_t\, A_1^{(j)} A_2^{(j)} A_3^{(j)}\right](t)$$

For the integral we get the relation,

$$i\,(-2\pi i)^3 \int \frac{dt}{(2\pi)^4} J_t t^n = 0 \quad \text{For } n \neq 0$$

Then, the coefficient takes the form,

$$C_3^{[2]} = -\frac{1}{2}\sum_{\sigma} \mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_1^{(j)} A_2^{(j)} A_3^{(j)}\right]_{t^0}\right]_{\mu^2}$$

We must also sum over the two solutions, σ, for the loop momentum. For the massless case when $S_1 = 0$ and $S_3 = 0$ it is sufficient to sum over the two solutions for γ_{13}.

I.3. Bubble contribution

The bubble integral is:

$$
\begin{aligned}
&(4\pi)^{2-\epsilon}(-2\pi i)^2 \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \prod_{i=1}^{2} \delta(l_i^2) A_1 A_2 \\
&= -(4\pi)^{2-\epsilon} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \int dt dy J_{t,y} \sum_{\sigma} \left([\mathrm{Inf}_y A_1 A_2(\bar{l}_1^\sigma)](y) + \sum_{\mathrm{poles}\{j\}} \frac{\mathrm{Res}_{y=y_j} A_1 A_2(\bar{l}_1^\sigma)}{y-y_j} \right) \\
&= -(4\pi)^{2-\epsilon} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \int dt dy J_{t,y} \\
&\quad \times \sum_{\sigma} \left([\mathrm{Inf}_t [\mathrm{Inf}_y A_1 A_2(\bar{l}_1^\sigma)](y)](t) + \left[\mathrm{Inf}_t \left(\sum_{\mathrm{poles}\{j\}} \frac{\mathrm{Res}_{y=y_j} A_1 A_2(\bar{l}_1^\sigma)}{y-y_j} \right) \right](t) \right. \\
&\quad \left. + \sum_{\mathrm{poles}\{l\}} \frac{\mathrm{Res}_{t=t_l}[\mathrm{Inf}_y A_1 A_2(\bar{l}_1^\sigma)](y)}{t-t_l} + \sum_{\mathrm{poles}\{j\},\{l\}} \frac{\mathrm{Res}_{t=t_l}\left[\frac{\mathrm{Res}_{y=y_j} A_1 A_2(\bar{l}_1^\sigma))}{y-y_j}\right]}{t-t_l} \right), \qquad \text{(I-9)}
\end{aligned}
$$

Here we have integrated over the delta functions and done change of variables from l^μ to t and y, similar to our treatment of the triple cut. The last term of the final expression has two additional propagators on shell, and its numerator has no dependence on t or y. It corresponds to box contributions and it is not taken into account. The second and third terms correspond to triangle contributions to the bubble coefficient. the first term is the pure bubble contribution to the bubble coefficient.

To compute the triangle contribution to the bubble coefficient we fix y to parametrize the loop momentum, however, this parametrization of the loop momentum may differ from that of section I.2. With the new parametrization, integrals over positive powers of t that before vanished now they will not necessarily vanish here,

$$\int dt J_t' t^n \neq 0, \qquad \text{(I-10)}$$

where J_t' is the Jacobian corresponding our new loop momentum parametrization. These tensor integrals are not in our integral basis and must be reduced. Passarino-Veltman reduction shows us that there are both scalar triangle and scalar bubble integrals within these tensor integrals,

$$\int dt J_t' t^n = C_3 I_3^{4;\mathrm{cut}} + C_2 I_2^{4;\mathrm{cut}},$$

then we must extract the bubble integral coefficients.

we get the value of y by imposing a third on-shell condition in our momentum parametrization,

$$y_\pm = \frac{B_1 \pm \sqrt{B_1^2 + 4B_0 B_2}}{2B_2}, \qquad \text{(I-11)}$$

$$B_2 = S_1 \left\langle \chi \left| \not{K}_3 \right| K_1^\flat \right] \qquad \text{(I-12)}$$

$$B_1 = \bar{\gamma} t \left\langle K_1^\flat \left| \not{K}_3 \right| K_1^\flat \right] - S_1 t \left\langle \chi \left| \not{K}_3 \right| \chi \right] + S_1 \left\langle \chi \left| \not{K}_3 \right| K_1^\flat \right], \qquad \text{(I-13)}$$

$$B_0 = \bar{\gamma} t^2 \left\langle K_1^\flat \left| \not{K}_3 \right| \chi \right] - \mu^2 \left\langle \chi \left| \not{K}_3 \right| K_1^\flat \right] + \bar{\gamma} t S_3 + t S_1 \left\langle \chi \left| \not{K}_3 \right| \chi \right]. \qquad \text{(I-14)}$$

as we saw befores, the triple cut is given by

$$i(4\pi)^{2-\epsilon}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\sum_{\sigma_y}\int dtJ'_t[\text{Inf}_tA_1A_2A_3(\bar{l}_1^{\sigma_y})](t), \tag{I-15}$$

where the sum is over the solutions in eq. (I-11).
Studying the behavior at infinite of the product of three tree amplitudes, we obtain the following information,

- The t^0 term gives us the double-cut scalar triangle,
- while positive powers of t return, among other pieces, cut scalar bubble coefficients. We evaluate the integrals over positive powers of t and retain only the contributing bubble integral to our particular double cut,

$$\begin{aligned}(4\pi)^2\int dtJ'_tt^j &= T_jI_2^{4;\text{cut}}\\ &= -\left(\frac{S_1}{\bar{\gamma}}\right)^j\frac{\langle\chi^-|\not{K}_3|K_1^{\flat,-}\rangle^j(K_1\cdot K_3)^{j-1}}{\Delta^j}\left(\sum_{l=1}^{j}\text{C}_{jl}\frac{S_3^{l-1}}{(K_1\cdot K_3)^{l-1}}\right)I_2^{4;\text{cut}},\end{aligned} \tag{I-16}$$

where[35, 53]

$$\text{C}_{11}=\frac{1}{2}, \tag{I-17}$$

$$\text{C}_{21}=\frac{3}{8},\qquad \text{C}_{22}=\frac{3}{8}, \tag{I-18}$$

$$\text{C}_{31}=-\frac{1}{12}\frac{\Delta}{(K_1\cdot K_3)^2}\left(1-4\frac{\mu^2}{S_1}\right)+\frac{5}{16},\qquad \text{C}_{32}=\frac{5}{8},\qquad \text{C}_{33}=\frac{5}{16}, \tag{I-19}$$

$$\Delta=(K_1\cdot K_3)^2-S_1S_3, \tag{I-20}$$

$$I_2^{4;\text{cut}}=(-i)(4\pi)^2(-2\pi i)^2\int\frac{d^4\bar{l}_1}{(2\pi)^4}\prod_{i=1}^{2}\delta(\bar{l}_i). \tag{I-21}$$

Some relevant terms of t^n are[35, 49]

$$T_0=0 \tag{I-22}$$

$$T_1=-\frac{S_1\left\langle\chi\left|\not{K}_3\right|K_1^\flat\right]}{2\bar{\gamma}\Delta}, \tag{I-23}$$

$$T_2=-\frac{3S_1\left\langle\chi\left|\not{K}_3\right|K_1^\flat\right]^2}{8\bar{\gamma}^2\Delta^2}(S_1S_3+K_1\cdot K_3S_1), \tag{I-24}$$

$$T_3=-\frac{\left\langle\chi\left|\not{K}_3\right|K_1^\flat\right]^3}{48\bar{\gamma}^3\Delta^3}\left(15S_1^3S_3^2+30K_1\cdot K_3S_1^3S_3+11(K_1\cdot K_3)^2S_1^3+4S_1^4S_3+16\mu^2S_1^2\Delta\right), \tag{I-25}$$

then have the contribution of the single residue terms to the bubble coefficient is given by

$$-\frac{1}{2}\sum_{\sigma_y}[\text{Inf}_tA_1A_2A_3](t)|_{t^j\to T_j}, \tag{I-26}$$

After averaging over solutions. Single residue terms arise from every possible third leg that can go on-shell.

Lastly we need the bubble coefficients due to the first term in eq. (I-9),

$$-(4\pi)^{2-\epsilon}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\int dtdyJ_{t,y}\sum_{\sigma}[\mathrm{Inf}_t[\mathrm{Inf}_y A_1A_2(\bar{l}_1^{\sigma})](y)](t).$$

Once again we expand around infinity to evaluate these integrals. Because of our choice of momentum parametrization, integrals of positive order in t vanish. This is not the case with y, so we must keep terms of the form t^0y^m. These evaluate to[35]

$$\begin{aligned}Y_m &= (2\pi)^2\int dtdyJ_{t,y}y^m\\ &= \frac{1}{m+1}\sum_{i=0}^{\lfloor m/2\rfloor}\begin{pmatrix} m-i \\ i\end{pmatrix}\left(\frac{-\mu^2}{S_1}\right)^i.\end{aligned} \tag{I-27}$$

In our calculations, the most relevant integrals are

$$Y_0=1,\quad Y_1=m\frac{1}{2},\quad Y_2=\frac{1}{3}\left(1-\frac{\mu^2}{S_1}\right),\quad Y_3=\frac{1}{4}\left(1-2\frac{\mu^2}{S_1}\right),\quad Y_4=\frac{1}{5}\left(1-3\frac{\mu^2}{S_1}+\frac{\mu^4}{S_1^2}\right). \tag{I-28}$$

The term is then

$$\begin{aligned}&-(4\pi)^{2-\epsilon}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\int dtdyJ_{t,y}\sum_{\sigma}[\mathrm{Inf}_t[\mathrm{Inf}_y A_1A_2(\bar{l}_1^{\sigma})](y)](t)\\ &\quad=-(4\pi)^{2-\epsilon}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\sum_{\sigma}\left[[\mathrm{Inf}_t[\mathrm{Inf}_y A_1A_2(\bar{l}_1^{\sigma})](y)](t)|_{t\to 0,y^m\to Y_m}\right]\int dtdyJ_{t,y},\end{aligned} \tag{I-29}$$

in which we identify the double-cut scalar bubble integral,

$$i(4\pi)^{2-\epsilon}\int\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\int dtdyJ_{t,y}=(-i)(4\pi)^{2-\epsilon}(-2\pi i)^2\int\frac{d^{4-2\epsilon}l_1}{(2\pi)^{4-2\epsilon}}\prod_{i=1}^{2}\delta(l_i^2). \tag{I-30}$$

Adjusting a factor of i, we can easily identify the bubble coefficient from the first term of eq. (I-9) as

$$-i[\mathrm{Inf}_t[\mathrm{Inf}_y A_1A_2](y)](t)|_{t\to 0,y^m\to Y_m}. \tag{I-31}$$

We then have for our total bubble coefficient, in the case where our calculation falls into a polynomial in μ^2,

$$C_2^{[0]}+\mu^2C_2^{[2]}=-i[\mathrm{Inf}_t[\mathrm{Inf}_y A_1A_2](y)](t)|_{t\to 0,y^m\to Y_m}-\frac{1}{2}\sum_{C_{\mathrm{tri}}}\sum_{\sigma_y}[\mathrm{Inf}_t A_1A_2A_3](t)|_{t^j\to T_j}, \tag{I-32}$$

where C_{tri} denotes a sum over all possible triangles attainable from cutting one more leg of our two-particle cut. Most generally, we have

$$C_2^{[0]}=-\,i[\mathrm{Inf}_t[\mathrm{Inf}_y A_1A_2](y)](t)|_{t\to 0,y^m\to Y_m}-\frac{1}{2}\sum_{C_{\mathrm{tri}}}\sum_{\sigma_y}[\mathrm{Inf}_t A_1A_2A_3](t)|_{t^j\to T_j} \tag{I-33}$$

J. Quadrupole, triple and double cut with Mathematica

In this appendix we show how compute quadrupole, triple and double cut coefficients using `s@m` in Mathematica.

J.1. Quadrupole cut coefficient for one-loop five gluons amplitude in pure YM

By sewing tree-level amplitudes,

```
C5 = gMHV[{-11, 1, 2, 13}, {1, 2}] BgMHV[{-13, 3, 14}, {3, 14}] gMHV[{-14, 4, 12}, {-14, 12}]
   BgMHV[{-12, 5, 11}, {-12, 5}] // SpClose
```

$$-\frac{\langle 1\,|2\rangle^3\,[5\,|12\,|14\,|3]^3}{\langle 1\,|11\,|5]\,\langle 2\,|13\,|14\,|4\rangle\,\langle 4\,|12\,|11\,|13\,|3]}$$

Using momentun conservation and writting the explicit solution for l_4,

```
l2 = l4 - Sm[4];
l3 = l4 + Sm[3];
l1 = l4 - Sm[5] - Sm[4];
l4 = t SmBA[4, 3];
```

Finally, with the Schouten identity,

$$\text{Schouten}\left[\text{C5 //. t} \to \frac{\text{Spaa[4, 5]}}{\text{Spaa[3, 5]}}\ \text{// SpOpen // Factor, 1, 4, 3, 5}\right]\ \text{//.}$$
$$\left\{\text{Spbb[5, 4]} \to \frac{\text{s[4, 5]}}{\text{Spaa[4, 5]}},\ \text{Spbb[4, 3]} \to \frac{\text{s[3, 4]}}{\text{Spaa[3, 4]}}\right\}$$

$$\frac{s_{3\ 4}\, s_{4\ 5}\,\langle 1\,|2\rangle^3}{\langle 1\,|5\rangle\,\langle 2\,|3\rangle\,\langle 3\,|4\rangle\,\langle 4\,|5\rangle}$$

We recover the result (3-93).

J.2. Triple cut coefficient for gluon production by quark anti-quark annihilation

Sewing tree level amplitudes,

```
A1 = QMHV[14, 1, 11] BqMHV[4, 13, 14] QMHV[12, -13, 3] BqMHV[11, 2, 12] // SpClose
```

$$\frac{\langle 1\,|11\,|2]^2\,\langle 3\,|13\,|4]^2}{\langle 3\,|12\,|11\,|14\,|4]}$$

$$\text{A} = \frac{i}{\langle 3\,|\,12\,|\,3]}\ \text{A1}$$

$$\frac{i\,\langle 1\,|11\,|2]^2\,\langle 3\,|13\,|4]^2}{\langle 3\,|12\,|3]\,\langle 3\,|12\,|11\,|14\,|4]}$$

Using mometum conservation and writting the explicit solutions for l_1,

```
l3 = l2 + Sm[3];
l2 = l1 + Sm[2];
l4 = l1 - Sm[1];
l1 = t SmBA[1, 2];
```

Taking $\mathrm{Inf_t}$ (see appendix C.4)

```
Inft[A] // SpOpen // Factor // ConvertSpinorsToS
```

$$\frac{i\, s_{1\ 2}\,([3\,|1]\,[4\,|2]-[3\,|2]\,[4\,|1])^2}{[3\,|1]^3\,[4\,|1]}$$

Finally, using Schouten identity and spinor identities,

```
SpOpen[SpClose[Schouten[%, 3, 1, 4, 2] //. {Spbb[3, 1] → s[1, 3]/Spaa[1, 3], Spbb[2, 1] → s[1, 2]/Spaa[1, 2]},
    1] //. {Spab[2, 1, 4] → -Spab[2, 3, 4]}] //. {Spbb[4, 3] → s[1, 2]/Spaa[3, 4]}
```

$$\frac{i\, s_{1\ 2}{}^4\,\langle 1\,|3\rangle^3}{s_{1\ 3}{}^3\,\langle 1\,|2\rangle\,\langle 2\,|3\rangle\,\langle 3\,|4\rangle}$$

We recover the result (3-123).

J.3. Double cut coefficient for gluon production by quark anti-quark annihilation

First we compute the pure bubble coefficient.
Defining and sewing tree-level amplitudes,

```
A12 = i (Spaa[l2, 1]^3 * Spaa[l4, 1]) / (Spaa[l2, 2] Spaa[2, 1] Spaa[1, l4] Spaa[l4, l2])
```

$$-\frac{i\,\langle l2\,|1\rangle^3}{\langle 1\,|2\rangle\,\langle l2\,|2\rangle\,\langle l2\,|l4\rangle}$$

```
A34 = SpClose[QMHV[l2, -l3, 3] i/⟨3 | l2 | 3] BqMHV[4, l3, l4]] //.
    Spab[a_, l3, b_] -> Spab[a, Sm[l2] + Sm[3], b] // SpOpen
```

$$-\frac{i\,[4\,|l2]^2}{[3\,|l2]\,[4\,|l4]}$$

```
C24 = SpClose[SpClose[SpClose[A12 * A34] * Spbb[l2, 4]] / Spbb[l2, 4]] //.
    Spbb[a_, l4, b__] → Spbb[a, Sm[l2] + Sm[3] + Sm[4], b] // SpOpen
```

$$-\frac{[4\,|l2]^2\,\langle l2\,|1\rangle^3}{[4\,|3]\,\langle 1\,|2\rangle\,[3\,|l2]\,\langle l2\,|2\rangle\,\langle l2\,|3\rangle}$$

Writing the explicit solution for l_2 and l_4,

```
C24 //. {Spaa[l4, a__] → t Spaa[2, a] + (1 - y) Spaa[1, a],
    Spaa[l2, a_] → Spaa[2, a] - y/t Spaa[1, a], Spbb[a_, l2] → (y - 1) Spbb[a, 2] + t Spbb[a, 1],
    Spbb[a_, l4] → y/t Spbb[a, 2] + Spbb[a, 1]} // Factor
```

$$-\frac{t^2\,\langle 1\,|2\rangle\,(t\,[4\,|1]+y\,[4\,|2]-[4\,|2])^2}{y\,[4\,|3]\,(t\,[3\,|1]+y\,[3\,|2]-[3\,|2])\,(t\,\langle 2\,|3\rangle-y\,\langle 1\,|3\rangle)}$$

Taking Inf_y,

```
Infy[int3, 1]
0
```

Now we compute the contribution that comes from the triangles.
following the same procedure done before for the triple cut

```
A432 = SpClose[A12 * BqMHV[12, -13, 3] * QMHV[4, 13, 14]] //. 13 → 12 + Sm[3] // SpOpen
```

$$\frac{i\,[3\,|12]\,\langle 12\,|1\rangle^3\,\langle 14\,|3\rangle^2}{\langle 1\,|2\rangle\,\langle 12\,|2\rangle\,\langle 14\,|4\rangle\,\langle 12\,|14\rangle}$$

Writing the explicit solution for l_2 and l_4,

```
A432 //. {Spaa[14, a_] → t Spaa[2, a] + (1 - y) Spaa[1, a],
    Spaa[12, a_] → Spaa[2, a] - y/t Spaa[1, a], Spbb[a_, 12] → (y - 1) Spbb[a, 2] + t Spbb[a, 1]} //
  Factor
```

$$-(i\,t\,(t\,[3\,|1]+y\,[3\,|2]-[3\,|2])\,(t\,\langle 2\,|3\rangle-y\,\langle 1\,|3\rangle+\langle 1\,|3\rangle)^2)/(y\,(t\,\langle 2\,|4\rangle-y\,\langle 1\,|4\rangle+\langle 1\,|4\rangle))$$

Using some spinors identies, we get

```
Atotal =
 ConvertSpinorsToS[
    SpClose[ConvertSpinorsToS[Ab[ym] // SpOpen // Factor // ExpandSToSpinors], 4] //.
       Spab[a_, 4, b_] → Spab[a, -Sm[1] - Sm[2] - Sm[3], b] // SpOpen // Factor] //.
   {s[1, 3] → -s[1, 4] - s[1, 2], s[2, 3] → s[1, 4], s[2, 4] → s[1, 3]} // Factor
```

$$\frac{i\,[3\,|2]^2\,\langle 1\,|3\rangle^3}{s_{1\ \ 4}\,\langle 1\,|4\rangle}$$

Another one contribution comes from the following triangle,
By sewing tree-level amplitudes

```
A12 = -i (Spbb[2, 14]^3 * Spbb[2, 12]) / (Spbb[12, 2] Spbb[2, 1] Spbb[1, 14] Spbb[14, 12])
```

$$\frac{i\,[2\,|14]^3}{[2\,|1]\,[1\,|14]\,[14\,|12]}$$

```
A432 = SpClose[A12 * QMHV[12, -13, 3] * BqMHV[4, 13, 14]] //. 13 → 12 + Sm[3] // SpOpen
```

$$-\frac{i\,[4\,|12]^2\,[2\,|14]^3\,\langle 12\,|3\rangle}{[2\,|1]\,[1\,|14]\,[4\,|14]\,[14\,|12]}$$

Writing explicit solutions,

```
A432 //. {Spaa[14, a_] → t Spaa[2, a] + (1 - y) Spaa[1, a],
    Spaa[12, a_] → Spaa[2, a] - y/t Spaa[1, a], Spbb[a_, 12] → (y - 1) Spbb[a, 2] + t Spbb[a, 1],
    Spbb[a_, 14] → y/t Spbb[a, 2] + Spbb[a, 1]} // Factor
```

$$\frac{i\,t\,(t\,[4\,|1]+y\,[4\,|2]-[4\,|2])^2\,(t\,\langle 2\,|3\rangle-y\,\langle 1\,|3\rangle)}{y\,(t\,[4\,|1]+y\,[4\,|2])}$$

Using some identities,

```
Atotal =
 Factor[Schouten[Ab[yp] // SpOpen // ExpandSToSpinors // Factor, 1, 4, 2, 3] //
     ConvertSpinorsToS] //. {s[1, 4] → -s[2, 4] - s[1, 2]} // Factor
```

$$\frac{i\, s_{1\ 2}^2\, t^3\, (t\langle 1|2\rangle\langle 3|4\rangle + \langle 1|3\rangle\langle 1|4\rangle)}{\langle 1|4\rangle (t\langle 2|4\rangle + \langle 1|4\rangle)(s_{1\ 2}\, t - [4|2]\langle 1|4\rangle)}$$

Studying the contribution that comes from $\mathrm{Inf}_{t^m \to T_m}$ for the case of T_1,

```
Ab1 =
 SpClose[
   ConvertSpinorsToS[
     Factor[
       Factor[
         SpClose[ConvertSpinorsToS[Schouten[t1 InfT[Atotal, 1], 1, 3, 2, 4] // Factor] //.
                 {Spaa[1, 4] → Spab[1, 4, 2]/Spbb[4, 2], Spab[1, 4, 2] → -Spab[1, 3, 2]} // SpOpen //
               Factor // ConvertSpinorsToS, 4] //. Spab[3, 4, 2] → -Spab[3, 1, 2] // SpOpen //
           ConvertSpinorsToS // Factor] //. {s[2, 3] + s[2, 4] → -s[1, 2], s[2, 4] → s[1, 3]}] *
       Spaa[2, 3]] / (Spaa[2, 3] * Spbb[4, 3]), 4] * s[1, 2] / Spaa[3, 4] //.
   {Spab[2, 4, 3] → -Spab[2, 1, 3], s[2, 3] → s[1, 4], Spab[2, 1, 3] → Spaa[2, 1] s[1, 3]/Spaa[3, 1]}
```

$$-\frac{2\, i\, s_{1\ 2}\, s_{1\ 4}\, \langle 1|3\rangle^3}{s_{1\ 3}^2\, \langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle}$$

And T_2,

```
Ab2 =
 SpOpen[
   ConvertSpinorsToS[
     SpClose[
       ExpandSToSpinors[
         ConvertSpinorsToS[
           SpClose[SpOpen[t2 InfT[Atotal, 2] //. Spab[a_, 4, b_] →
                  Spab[a, -Sm[1] - Sm[2] - Sm[3], b]], 2] //.
               Spab[a_, 2, b_] → Spab[a, -Sm[1] - Sm[4] - Sm[3], b] // SpOpen] //.
             s[3, 4] → s[1, 2]] / Spbb[4, 3], 4] * Spbb[4, 3] * Spaa[3, 4]] / Spaa[3, 4] //.
     {Spab[a_, 4, b_] → Spab[a, -Sm[1] - Sm[2] - Sm[3], b], s[3, 4] → s[1, 2]}] //.
  {Spbb[3, 1] → s[1, 3]/Spaa[1, 3], Spbb[3, 2] → s[1, 4]/Spaa[2, 3]}
```

$$\frac{3\, i\, s_{1\ 4}\, \langle 1|3\rangle^3}{s_{1\ 3}\, \langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle}$$

Finally, we get

```
(Ab1 + Ab2)/2 // Expand
```

$$\frac{3\, i\, s_{1\ 4}\, \langle 1|3\rangle^3}{2\, s_{1\ 3}\, \langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle} - \frac{i\, s_{1\ 2}\, s_{1\ 4}\, \langle 1|3\rangle^3}{s_{1\ 3}^2\, \langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle}$$

K. Rational Contributions for amplitudes of four gluons in Scalar QCD

K.1. Quadrupole cut coefficients

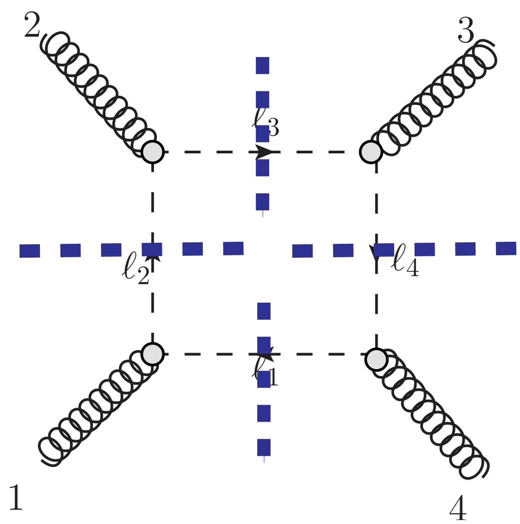


Figure **K-1**.: Quadrupole cut for the process of four gluons in scalar QCD.

K.1.1. $C_4^{[4]}\left(1^+, 2^+, 3^+, 4^+\right)$

The product of the tree amplitudes:

$$
\begin{aligned}
A_3^{tree}\left(-\ell_1, 1^+, \ell_2\right) A_3^{tree}\left(-\ell_2, 2^+, \ell_3\right) A_3^{tree}\left(-\ell_3, 3^+, \ell_4\right) A_3^{tree}\left(-\ell_4, 4^+, \ell_1\right) = \\
= \frac{\langle 2\,|\ell_2|\,1]}{\langle 21\rangle}\frac{\langle 1\,|\ell_3|\,2]}{\langle 12\rangle}\frac{\langle 4\,|\ell_4|\,3]}{\langle 43\rangle}\frac{\langle 3\,|\ell_1|\,4]}{\langle 34\rangle} \\
= \frac{1}{\langle 21\rangle\langle 12\rangle\langle 43\rangle\langle 34\rangle}\langle 2\,|\ell_2 1 \ell_2|\,2]\,\langle 3\,|\ell_1 4 \ell_1|\,3] \\
= \frac{\mu^4}{\langle 21\rangle\langle 12\rangle\langle 43\rangle\langle 34\rangle}\langle 2\,|1|\,2]\,\langle 3\,|4|\,3] = \mu^4\frac{[12]\,[43]}{\langle 12\rangle\langle 43\rangle}
\end{aligned}
\tag{K-1}
$$

We have used momentum conservation in each corner (see figure **K-1**), we obtain

$$\ell_2 \not{p}_1 \ell_2 = -\ell_2 \ell_2 \not{p}_1 = -\mu^2 \not{p}_1 \tag{K-2}$$

$$\ell_1 \not{p}_4 \ell_1 = -\ell_1 \ell_1 \not{p}_4 = -\mu^2 \not{p}_4 \tag{K-3}$$

The coefficient then takes the form

$$C_4^{[4]}\left(1^+,2^+,3^+,4^+\right) = i\frac{[12]\,[43]}{\langle 12\rangle\,\langle 43\rangle} \tag{K-4}$$

K.1.2. $C_4^{[4]}\left(1^-,2^+,3^+,4^+\right)$ and $C_4^{[2]}\left(1^-,2^+,3^+,4^+\right)$

The product of the tree amplitudes:

$$\begin{aligned}
&A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right) = \\
&= -\frac{[2\,|\ell_2|\,1\rangle}{[21]}\frac{\langle 1\,|\ell_3|\,2]}{\langle 12\rangle}\frac{\langle 4\,|\ell_4|\,3]}{\langle 43\rangle}\frac{\langle 3\,|\ell_1|\,4]}{\langle 34\rangle} \\
&= -\frac{\langle 1\,|\ell_2|\,2]^2}{\langle 12\rangle\,[21]}\frac{\langle 4\,|\ell_1|\,3]}{\langle 43\rangle}\frac{\langle 3\,|\ell_1|\,4]}{\langle 34\rangle} \\
&= -\frac{1}{\langle 12\rangle\,[21]\,\langle 43\rangle\,\langle 34\rangle}\langle 1\,|\ell_2|\,2]^2\,\langle 3\,|\ell_1 4\ell_1|\,3] \\
&= \frac{\mu^2}{\langle 12\rangle\,[21]\,\langle 43\rangle\,\langle 34\rangle}\langle 1\,|\ell_2|\,2]^2\,\langle 3\,|4|\,3] = \mu^2\frac{[43]}{\langle 12\rangle\,[21]\,\langle 43\rangle}\langle 1\,|\ell_2|\,2]^2
\end{aligned} \tag{K-5}$$

Consider the solution for ℓ_2 as:

$$\ell_2 = c\,\langle 2\,|\gamma^\mu|\,1] - \frac{\mu^2}{4s_{12}c}\langle 1\,|\gamma^\mu|\,2] \tag{K-6}$$

$$c_\pm = \frac{-2s_{12}s_{14} \pm \sqrt{4s_{12}^2s_{14}^2 + 16\mu^2 s_{12}\,\langle 1\,|4|\,2]\,\langle 2\,|4|\,1]}}{8s_{12}\,\langle 2\,|4|\,1]} \tag{K-7}$$

And the term, $\langle 1\,|\ell_2|\,2]$

$$\langle 1\,|\ell_2|\,2] = 2c\,\langle 12\rangle\,[12] = -2s_{12}c \tag{K-8}$$

$$\langle 1\,|\ell_2|\,2]^2 = (-2s_{12}c)^2 \tag{K-9}$$

uisng the explicit solution for c,

$$\langle 1\,|\ell_2|\,2]^2 = \frac{1}{2}\frac{s_{12}^2s_{14}^2}{\langle 2\,|4|\,1]^2} + \mu^2 s_{12}\frac{\langle 1\,|4|\,2]}{\langle 2\,|4|\,1]} + \ldots = \frac{[42]^2}{[41]^2}\left(\frac{1}{2}\frac{s^2t^2}{u^2} + \mu^2\frac{[42]^2}{[41]^2}\frac{st}{u}\right) \tag{K-10}$$

the product of the tree amplitudes becomes

$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} = \mu^2\frac{[43]}{\langle 12\rangle\,[21]\,\langle 43\rangle}\frac{[42]^2}{[41]^2}\left(\frac{1}{2}\frac{s^2t^2}{u^2} + \mu^2\frac{st}{u}\right) \tag{K-11}$$

$$= \mu^2\frac{[42]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]}\left(\frac{1}{2}\frac{s^2t^2}{u^2} + \mu^2\frac{st}{u}\right) \tag{K-12}$$

the coefficients of μ^4 and μ^2:

$$\mu^4 : \frac{[42]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]}\frac{st}{u} \tag{K-13}$$

$$\mu^2 : \frac{1}{2}\frac{[42]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]}\frac{s^2t^2}{u^2} \tag{K-14}$$

Finally we obtain,

$$C_4^{[4]}\left(1^-,2^+,3^+,4^+\right) = i\frac{[42]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\frac{st}{u} \tag{K-15}$$

$$C_4^{[2]}\left(1^-,2^+,3^+,4^+\right) = \frac{i}{2}\frac{[42]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\frac{s^2t^2}{u^2} \tag{K-16}$$

K.1.3. $C_4^{[4]}\left(1^-,2^-,3^+,4^+\right)$

The product of the tree amplitudes:

$$\begin{aligned}
&A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^-,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right) = \\
&= \frac{[2\,|\ell_2|\,1\rangle}{[21]}\frac{[1\,|\ell_2|\,2\rangle}{[12]}\frac{\langle 4\,|\ell_4|\,3]}{\langle 43\rangle}\frac{\langle 3\,|\ell_1|\,4]}{\langle 34\rangle} \\
&= \frac{1}{[12]\,[21]\,\langle 43\rangle\langle 34\rangle}\langle 1\,|\ell_2 2\ell_2|\,1]\,\langle 3\,|\ell_1 4\ell_1|\,3] \\
&= \frac{\mu^4}{[12]\,[21]\,\langle 43\rangle\langle 34\rangle}\langle 1\,|2|\,1]\,\langle 3\,|4|\,3] = \mu^4\frac{\langle 12\rangle\,[34]}{[12]\,\langle 34\rangle}
\end{aligned}$$

easily the coefficient,

$$C_4^{[4]}\left(1^-,2^-,3^+,4^+\right) = i\frac{\langle 12\rangle\,[34]}{[12]\,\langle 34\rangle} = -A_4^{tree}\left(1^-,2^-,3^+,4^+\right)\frac{t}{s} \tag{K-17}$$

K.1.4. $C_4^{[4]}\left(1^-,2^+,3^-,4^+\right)$

The product of the tree amplitudes:

$$\begin{aligned}
&A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^-,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right) = \\
&= \frac{[2\,|\ell_2|\,1\rangle}{[21]}\frac{\langle 1\,|\ell_2|\,2]}{\langle 12\rangle}\frac{[4\,|\ell_1|\,3\rangle}{[43]}\frac{\langle 3\,|\ell_1|\,4]}{\langle 34\rangle} = \frac{1}{[21]\,\langle 12\rangle\,[43]\,\langle 34\rangle}\langle 1\,|\ell_2|\,2]^2\,\langle 3\,|\ell_1|\,4]^2
\end{aligned} \tag{K-18}$$

Studying the products $\langle 1\,|\ell_2|\,2]^2$ and $\langle 3\,|\ell_1|\,4]^2$,

$$\langle 1\,|\ell_2|\,2]^2 = \langle 1\,|\ell_1|\,2]^2 = \mu^2 s_{12}\frac{\langle 1\,|4|\,2]}{\langle 2\,|4|\,1]} \tag{K-19}$$

$$\langle 3\,|\ell_1|\,4]^2 = \mu^2 s_{34}\frac{\langle 3\,|2|\,4]}{\langle 4\,|2|\,3]} \tag{K-20}$$

the product of tree amplitudes

$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} = \mu^4\frac{\langle 1\,|4|\,2]}{\langle 2\,|4|\,1]}\frac{\langle 3\,|2|\,4]}{\langle 4\,|2|\,3]} \tag{K-21}$$

$$= \mu^4\frac{\langle 14\rangle\,[42]^2\,\langle 32\rangle}{\langle 24\rangle^2\,[41]\,[23]}\times\frac{\langle 12\rangle\,[21]}{\langle 12\rangle\,[21]} \tag{K-22}$$

$$= \mu^4\frac{\langle 12\rangle\,\langle 34\rangle\,[42]^2}{\langle 24\rangle^2\,[21]\,[43]} \tag{K-23}$$

and the coefficient,

$$C_4^{[4]}\left(1^-,2^+,3^-,4^+\right) = i\frac{\langle 12\rangle\langle 34\rangle[42]^2}{\langle 24\rangle^2[21][43]} == -A_4^{tree}\left(1^-,2^+,3^-,4^+\right)\frac{st}{u^2} \tag{K-24}$$

we also have a contribution from μ^2:

$$C_4^{[2]}\left(1^-,2^+,3^-,4^+\right) = \frac{[4|1][4|2]\langle 1|2\rangle^2\langle 1|4\rangle\langle 3|4\rangle}{[4|3]\langle 2|4\rangle^3} = -\frac{s^2t^2}{u^3}A_4^{tree}\left(1^-,2^+,3^-,4^+\right) \tag{K-25}$$

K.2. Triple cut coefficients

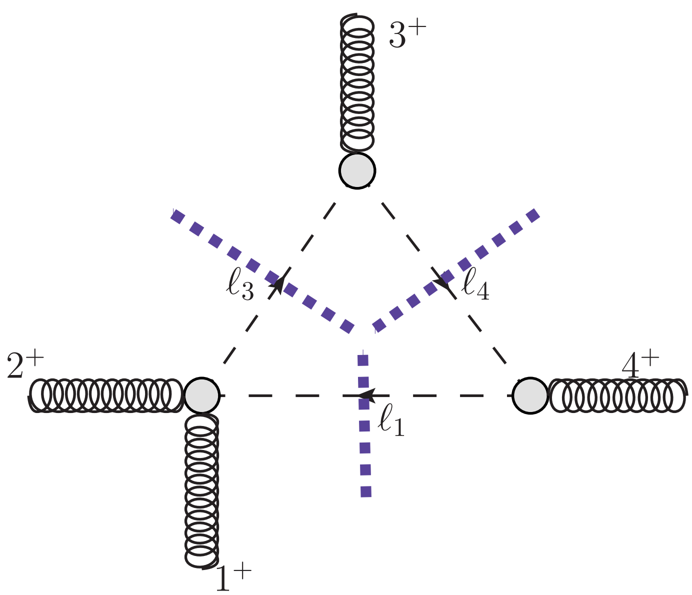


Figure **K-2**.: Triple cut for the process of four gluons in scalar QCD.

K.2.1. $C_{3;12}^{[2]}\left(1^+,2^+,3^+,4^+\right)$ Coefficients

The product of the three tree amplitudes is:

$$A_4^{tree}\left(-\ell_1,1^+,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right) =$$

$$= -i\mu^2\frac{[12]}{\langle 12\rangle\langle 1|\ell_1|1]}\frac{\langle 4|\ell_4|3]}{\langle 43\rangle}\frac{\langle 3|\ell_1|4]}{\langle 34\rangle} = -i\mu^2\frac{[12]}{\langle 12\rangle\langle 1|\ell_1|1]}\frac{\langle 4|\ell_4|3]}{\langle 43\rangle}\frac{\langle 3|\ell_4|4]}{\langle 34\rangle}$$

$$= -i\mu^2\frac{[12]}{\langle 12\rangle\langle 43\rangle\langle 34\rangle}\frac{\langle 4|\ell_4 3\ell_4|4]}{\langle 1|\ell_1|1]} = i\mu^4\frac{[12]}{\langle 12\rangle\langle 43\rangle\langle 34\rangle}\frac{\langle 4|3|4]}{\langle 1|\ell_1|1]} \tag{K-26}$$

and, the triple cut coefficient takes the form

$$C_{3;1,2}^{[2]}\left(1^+,2^+,3^+,4^+\right) = 0 \tag{K-27}$$

this result is because we have $\ell_1 = \ell_4 - 4 = t\langle 1|\gamma^\mu|2] + \frac{\mu^2}{4s_{12}t}\langle 2|\gamma^\mu|1] - K_4^\mu$ and the power of t and t^{-1} arise.

The contributions form other channels also vanish.

K.2.2. $C_3^{[2]}\left(1^-,2^+,3^+,4^+\right)$ Coefficients

$C_{3;12}^{[2]}\left(1^-,2^+,3^+,4^+\right)$

The product of tree amplitudes

$$A_4^{tree}\left(-\ell_1,1^-,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)=\frac{i\langle 1|\ell_1|2]^2\langle 3|\ell_1|4]\langle 4|\ell_4|3]}{s_{12}\langle 3|4\rangle^2\langle 1|\ell_1|1]} \tag{K-28}$$

now, let's consider the solution for $\ell_1=\ell_4-4$,

- By using the solution of l_4,

$$l_4^\mu=t_4\left\langle 3\left|\gamma^\mu\right|4\right]-\frac{\mu^2}{4s_{34}t_4}\left\langle 4\left|\gamma^\mu\right|3\right] \tag{K-29}$$

we obtain:

$$\begin{aligned}\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4^{tree}\left(-\ell_1,1^-,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)\right]_{t^0}\right]_{\mu^2}&=\\ =\frac{i[3|2](2[3|1][4|2]-[3|2][4|1])[4|3]\langle 1|4\rangle}{s_{12}[3|1]^2\langle 3|4\rangle}&\\ =\frac{i[34][4|2]^2s_{14}s_{23}}{\langle 12\rangle\langle 3|4\rangle[12][14]^2s_{24}s_{13}}\left(-s_{24}+s_{12}\right)=\frac{i[34][4|2]^2s_{23}}{\langle 12\rangle\langle 3|4\rangle[12][14]^2}\left(\frac{-s_{14}s_{24}+s_{14}s_{12}}{s_{24}^2}\right)&\end{aligned} \tag{K-30}$$

Here the sum $-s_{14}s_{24}+s_{14}s_{12}$ can be written as,

$$-s_{14}s_{24}+s_{14}s_{12}=\frac{1}{2}\left(-\left(s_{24}^2+s_{14}^2-s_{12}^2\right)+\left(s_{12}^2+s_{14}^2-s_{24}^2\right)\right)=\frac{1}{2}\left(-2s_{24}^2+2s_{12}^2\right) \tag{K-31}$$

with this simplication,

$$\begin{aligned}\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4^{tree}\left(-\ell_1,1^-,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)\right]_{t^0}\right]_{\mu^2}&=\\ =-\frac{i[34][4|2]^2s_{23}}{\langle 12\rangle\langle 3|4\rangle[12][14]^2}\left(1-\frac{s_{12}^2}{s_{24}^2}\right)&\end{aligned}$$

- Now we study the contribution from the conjugate solution

$$\begin{aligned}\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4\left(-l_1,1^-,2^+,l_3\right)A_3\left(-l_3,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)\right]_{t^0}\right]_{\mu^2}&=\\ =\frac{i[4|2]^2[4|3]\langle 1|4\rangle}{s_{12}[4|1]\langle 3|4\rangle}=-\frac{i[24]^2[34]s_{23}}{\langle 12\rangle\langle 3|4\rangle[12][14]^2}&\end{aligned}$$

the total coefficient is

$$\begin{aligned}C_{3;12}^{[2]}\left(1^-,2^+,3^+,4^+\right)&=\frac{1}{2}\frac{i[24]^2[34]s_{23}}{\langle 12\rangle\langle 3|4\rangle[12][14]^2}\left(2-\frac{s_{12}^2}{s_{24}^2}\right)\\ &=-\frac{i}{2}\frac{[32]}{\langle 23\rangle\langle 24\rangle\langle 34\rangle[21][31]}\left(s_{12}^2-2s_{24}^2\right)=\frac{i}{2}\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\frac{st}{u}\left(s^2-2u^2\right)\frac{1}{su}\end{aligned} \tag{K-32}$$

$C^{[2]}_{3;23}\,(1^-,2^+,3^+,4^+)$

The product of tree amplitudes

$$A_4^{tree}\left(-\ell_2,2^+,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)=\frac{i\mu^2[3|2][4|\ell_2|1\rangle\langle1|\ell_1|4]}{[4|1]\langle1|4\rangle\langle2|3\rangle\langle2|\ell_2|2]}\qquad\text{(K-33)}$$

- Consider the solution for ℓ_1^μ

$$\ell_1^\mu=t\,\langle1\,|\gamma^\mu|\,4]-\frac{\mu^2}{4s_{14}t}\,\langle4\,|\gamma^\mu|\,1]\qquad\text{(K-34)}$$

we obtain:

$$\begin{aligned}&\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4^{tree}\left(-\ell_2,2^+,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)\right]_{t^0}\right]_{\mu^2}=\\&=\frac{i[3|2][4|1]\langle2|1|2]\langle1|4\rangle}{[2|1]^2\langle2|3\rangle\langle2|4\rangle^2}=-\frac{[24]^2}{[21]\,\langle23\rangle\,\langle34\rangle\,[41]}\frac{t^2s}{u^2}\end{aligned}\qquad\text{(K-35)}$$

- With the solution ℓ_1^*:

$$\ell_1^{\mu*}=t\,\langle4\,|\gamma^\mu|\,1]-\frac{\mu^2}{4s_{14}t}\,\langle1\,|\gamma^\mu|\,4]\,,\qquad\text{(K-36)}$$

we obtain,

$$\mathrm{Inf}_\mu\left[\mathrm{Inf}_t\left[A_1^{tree}A_2^{tree}A_3^{tree}\right]_{t^0}\right]_{\mu^2}=0\qquad\text{(K-37)}$$

Finally, the coefficient takes the form:

$$C^{[2]}_{3;23}\left(1^-,2^+,3^+,4^+\right)=\frac{[24]^2}{[12]\,\langle23\rangle\,\langle34\rangle\,[41]}\frac{st}{u}\frac{ts}{su}\qquad\text{(K-38)}$$

$C^{[2]}_{3;34}\,(1^-,2^+,3^+,4^+)$

The product of tree amplitudes,

$$\begin{aligned}&A_4^{tree}\left(-\ell_3,3^+,4^+,\ell_1\right)A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right)=\\&=-i\mu^2\frac{[34]}{\langle34\rangle\,\langle3\,|\ell_3|\,3]}\frac{[3\,|\ell_2|\,1\rangle}{[q_1 1]}\frac{\langle3\,|\ell_3|\,2]}{\langle q_2 2\rangle}=-i\mu^2\frac{[34]}{\langle34\rangle\,[31]\,\langle32\rangle\,\langle3\,|\ell_3|\,3]}\,\langle1\,|\ell_2 3\ell_2|\,2]\\&=-i\mu^2\frac{[34]}{\langle34\rangle\,[31]\,\langle32\rangle\,\langle3\,|\ell_3|\,3]}\left(\langle3\,|\ell_2|\,3]\,\langle1\,|\ell_2|\,2]-\mu^2\,\langle1\,|3|\,2]\right)\end{aligned}$$

- The solution for ℓ_2 is:

$$\ell_2^\mu=t\,\langle1\,|\gamma^\mu|\,2]-\frac{\mu^2}{4s_{12}t}\,\langle2\,|\gamma^\mu|\,1]\qquad\text{(K-39)}$$

we obtain,

$$\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_1^{tree}A_2^{tree}A_3^{tree}\right]_{t^0}\right]_{\mu^2}=i\frac{s_{12}\,[43]\,[32]}{\langle23\rangle\,\langle34\rangle\,[31]^2}\qquad\text{(K-40)}$$

- The solution ℓ_2^* does not contribute to this calculation because we obtain powers of $t^n, n > 1$. Taking into account this argue, the coefficient is:

$$C_{3;34}^{[2]}\left(1^-,2^+,3^+,4^+\right) = \frac{i}{2}\frac{s_{12}\,[43]\,[32]}{\langle 23\rangle\,\langle 34\rangle\,[31]^2} = \frac{i}{2}\frac{[24]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]}\frac{st}{u}\frac{st}{ut} \tag{K-41}$$

$C_{3;41}^{[2]}\left(1^-,2^+,3^+,4^+\right)$

The product of the tree amplitudes

$$A_4^{tree}\left(-\ell_4,4^+,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right) = \frac{i\langle 3|l_3|2]\langle 1|l_4|4]^2\langle 2|l_4|3]}{s_{14}\langle 23\rangle^2\langle 4|l_4|4]} \tag{K-42}$$

Taking into account that $l_4 = l_3 - 3$

- By using the solution for l_3^μ,

$$l_3^\mu = t_3\,\langle 2\,|\gamma^\mu|\,3] - \frac{\mu^2}{4s_{23}t_3}\,\langle 3\,|\gamma^\mu|\,2] \tag{K-43}$$

  we obtain:

$$\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_1^{tree}A_2^{tree}A_3^{tree}\right]_{t^0}\right]_{\mu^2} = i\frac{\langle 12\rangle\,s_{24}^2}{\langle 2|3\rangle\,\langle 24\rangle^2\,\langle 3|4\rangle[4|1]} \tag{K-44}$$

- and the conjugate solution,

$$l_3^\mu = t_3\,\langle 3\,|\gamma^\mu|\,2] - \frac{\mu^2}{4s_{23}t_3}\,\langle 2\,|\gamma^\mu|\,3] \tag{K-45}$$

  we obtain:

$$\begin{aligned}&\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_1^{tree}A_2^{tree}A_3^{tree}\right]_{t^0}\right]_{\mu^2} = \\ &= -\frac{i[3|2]^2\left([4|3]\langle 1|2\rangle^2\langle 3|4\rangle - 2[4|3]\langle 1|2\rangle\langle 1|3\rangle\langle 2|4\rangle\right)}{s_{14}s_{23}\langle 2|4\rangle^2} = i\frac{\langle 1|2\rangle}{\langle 23\rangle\,\langle 2|4\rangle^2\,\langle 34\rangle\,[41]}\left(s_{24}^2 - s_{23}^2\right)\end{aligned}$$

The total contribution is:

$$C_{3;41}^{[2]}\left(1^-,2^+,3^+,4^+\right) = \frac{i}{2}\frac{\left(s_{23}^2 - 2s_{24}^2\right)\langle 12\rangle}{\langle 23\rangle\,\langle 24\rangle^2\,\langle 34\rangle\,[41]} = \frac{i}{2}\frac{[24]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]}\frac{st}{u}\left(t^2 - 2u^2\right)\frac{1}{ut} \tag{K-46}$$

K.2.3. $C_3^{[2]}\left(1^-,2^-,3^+,4^+\right)$ Coefficients

$C_{3;12}^{[2]}\left(1^-,2^-,3^+,4^+\right)$

$$\begin{aligned}A_4^{tree}\left(-\ell_1,1^-,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right) &= -i\mu^2\frac{\langle 12\rangle}{[12]\,[1\,|\ell_1|\,1\rangle}\frac{\langle 4\,|\ell_4|\,3]}{\langle 43\rangle}\frac{\langle 3\,|\ell_1|\,4]}{\langle 34\rangle} \\ &= -i\mu^2\frac{\langle 12\rangle}{[12]\,\langle 43\rangle\,\langle 34\rangle\,[1\,|\ell_3|\,1\rangle}\,\langle 4\,|\ell_4 3\ell_4|\,4] == i\mu^4\frac{\langle 12\rangle\,[34]}{[12]\,\langle 34\rangle}\frac{1}{[1\,|\ell_1|\,1\rangle}\end{aligned}$$

- the solution for ℓ_4

$$\ell_4^\mu = t\left\langle 3\left|\gamma^\mu\right|4\right] - \frac{\mu^2}{4s_{12}t}\left\langle 3\left|\gamma^\mu\right|4\right] \tag{K-47}$$

$$\ell_1^\mu = \ell_4^\mu - 4 = t\left\langle 3\left|\gamma^\mu\right|4\right] - \frac{\mu^2}{4s_{12}t}\left\langle 3\left|\gamma^\mu\right|4\right] - K_4^\mu \tag{K-48}$$

$$\begin{aligned}\left[1\left|\ell_1\right|1\right\rangle &= 2t\left\langle 3\left|1\right|4\right] - \frac{\mu^2}{2s_{12}t}\left\langle 3\left|1\right|4\right] - s_{14} \\ &= \frac{1}{2s_{12}t}\left(4s_{12}\left\langle 3\left|1\right|4\right]t^2 - 2s_{12}s_{14}t - \mu^2\left\langle 3\left|\gamma^\mu\right|4\right]\right)\end{aligned} \tag{K-49}$$

$$\begin{aligned}\frac{1}{\left[1\left|\ell_1\right|1\right\rangle} &= -\frac{1}{\frac{1}{2\mu^2\left\langle 3\left|\gamma^\mu\right|4\right]s_{12}t}\left(1 - \frac{4s_{12}\left\langle 3\left|1\right|4\right]t^2 - 2s_{12}s_{14}t}{\mu^2\left\langle 3\left|\gamma^\mu\right|4\right]}\right)} \\ &= -2\mu^2\left\langle 3\left|\gamma^\mu\right|4\right]s_{12}t\left(1 - \frac{4s_{12}\left\langle 3\left|1\right|4\right]t^2 - 2s_{12}s_{14}t}{\mu^2\left\langle 3\left|\gamma^\mu\right|4\right]}\right)\end{aligned} \tag{K-50}$$

- With the conjugate solution we also obtain

$$\mathrm{Inf}_t\left[\frac{1}{\left[1\left|\ell_1\right|1\right\rangle}\right]_{t^0} = 0 \tag{K-51}$$

The coefficient:

$$C_{3;12}^{[2]}\left(1^-, 2^-, 3^+, 4^+\right) = 0 \tag{K-52}$$

$C_{3;23}^{[2]}\left(1^-, 2^-, 3^+, 4^+\right)$

The productof tree amplitudes,

$$\begin{aligned}&A_4^{tree}\left(-\ell_2, 2^+, 3^+, \ell_4\right)A_3^{tree}\left(-\ell_4, 4^+, \ell_1\right)A_3^{tree}\left(-\ell_1, 1^-, \ell_2\right) = \\ &\quad = i\frac{\left[3\left|\ell_2\right|2\right\rangle^2}{s_{32}\left\langle 2\left|\ell_2\right|2\right]}\frac{\left\langle 3\left|\ell_1\right|4\right]}{\langle 34\rangle}\frac{\left[3\left|\ell_2\right|1\right\rangle}{[31]} = i\frac{\left[3\left|\ell_2\right|2\right\rangle^2}{s_{32}\left\langle 2\left|\ell_2\right|2\right][31]\langle 34\rangle}\left[4\left|\ell_1 3\ell_1\right|1\right\rangle \\ &\quad = i\frac{\left[3\left|\ell_2\right|2\right\rangle^2}{s_{32}\left\langle 2\left|\ell_2\right|2\right][31]\langle 34\rangle}\left(\left[3\left|\ell_1\right|3\right\rangle\left[4\left|\ell_1\right|1\right\rangle - \mu^2\left[4\left|3\right|1\right\rangle\right)\end{aligned} \tag{K-53}$$

- The solution for ℓ_1

$$\ell_1^\mu = t\left\langle 1\left|\gamma^\mu\right|4\right] - \frac{\mu^2}{4s_{14}t}\left\langle 4\left|\gamma^\mu\right|1\right] \tag{K-54}$$

  the contribution,

$$\mathrm{Inf}_t\left[\frac{\left[3\left|\ell_1\right|3\right\rangle\left[4\left|\ell_1\right|1\right\rangle\left[3\left|\ell_2\right|2\right\rangle^2}{\left\langle 2\left|\ell_2\right|2\right]}\right]_{t^0} \propto \mu^4 \tag{K-55}$$

- For the conjugate solution, we have the same behavior

With this,

$$C_{3;23}^{[2]}\left(1^-, 2^-, 3^+, 4^+\right) = 0 \tag{K-56}$$

$C^{[2]}_{3;34}\left(1^-,2^-,3^+,4^+\right)$

We have symmentry between the channels: $C^{[2]}_{3;34}\left(1^-,2^-,3^+,4^+\right)=C^{[2]}_{3;12}\left(1^-,2^-,3^+,4^+\right)$,

$$C^{[2]}_{3;34}\left(1^-,2^-,3^+,4^+\right)=0 \tag{K-57}$$

$C^{[2]}_{3;41}\left(1^-,2^-,3^+,4^+\right)$

We have symmentry between the channels: $C^{[2]}_{3;23}\left(1^-,2^-,3^+,4^+\right)=C^{[2]}_{3;41}\left(1^-,2^-,3^+,4^+\right)$,

$$C^{[2]}_{3;41}\left(1^-,2^-,3^+,4^+\right)=0 \tag{K-58}$$

K.2.4. $C^{[2]}_{3}\left(1^-,2^+,3^-,4^+\right)$ Coefficients

$C^{[2]}_{3;12}\left(1^-,2^+,3^-,4^+\right)$

The product of the tree amplitudes

$$A_4\left(-l_1,1^-,2^+,l_3\right)A_3\left(-l_3,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)=-\frac{i[4|l_4|3\rangle\langle 1|l_1|2]^2\langle 3|l_1|4]}{s_{12}[4|3]\langle 3|4\rangle\langle 1|l_1|1]} \tag{K-59}$$

Taking into account that $l_1=l_4-4$

- By using the solution for l_4

$$l_4^\mu=t_4\left\langle 3\left|\gamma^\mu\right|4\right]-\frac{\mu^2}{4s_{34}t_4}\left\langle 4\left|\gamma^\mu\right|3\right] \tag{K-60}$$

  we obtain:

$$\begin{aligned}&\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4\left(-l_1,1^-,2^+,l_3\right)A_3\left(-l_3,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)\right]_{t^0}\right]_{\mu^2}=\\&=-\frac{2i([3|1][4|2]-[3|2][4|1])^2\langle 1|3\rangle}{[2|1][3|1]^3\langle 1|2\rangle}=-A_4^{tree}\frac{s^2t}{u^3}\end{aligned} \tag{K-61}$$

- The conjugate solution

$$\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4\left(-l_1,1^-,2^+,l_3\right)A_3\left(-l_3,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)\right]_{t^0}\right]_{\mu^2}=0 \tag{K-62}$$

The total contribution:

$$C^{[2]}_{3;12}\left(1^-,2^+,3^-,4^+\right)=-A_4^{tree}\frac{s^2t}{u^3} \tag{K-63}$$

$C_{3;23}^{[2]}\left(1^-,2^+,3^-,4^+\right)$

The product of the tree amplitudes

$$A_4\left(-l_2,2^+,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)A_3\left(-l_1,1^-,l_2\right)=-\frac{i[4|l_2|1\rangle\langle 1|l_1|4]\langle 3|l_2|2]^2}{s_{23}[4|1]\langle 1|4\rangle\langle 2|l_2|2]} \tag{K-64}$$

Taking into account that $l_2=l_1-1$

- By using the solution for l_1

$$l_1^\mu=t_1\left\langle 4\left|\gamma^\mu\right|1\right]-\frac{\mu^2}{4s_{14}t_1}\left\langle 1\left|\gamma^\mu\right|4\right] \tag{K-65}$$

  we obtain:

$$\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4\left(-l_1,1^-,2^+,l_3\right)A_3\left(-l_3,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)\right]_{t^0}\right]_{\mu^2}=$$
$$=-\frac{2i[4|2](\langle 1|3\rangle\langle 2|4\rangle-\langle 1|2\rangle\langle 3|4\rangle)^2}{s_{23}\langle 2|4\rangle^3}=\frac{2is_{23}\langle 13\rangle\langle 23\rangle}{\langle 24\rangle^3\,[41]}=A_4^{tree}\frac{st^2}{u^3}$$

- The conjugate solution

$$\mathrm{Inf}_{\mu^2}\left[\mathrm{Inf}_t\left[A_4\left(-l_2,2^+,3^-,l_4\right)A_3\left(-l_4,4^+,l_1\right)A_3\left(-l_1,1^-,l_2\right)\right]_{t^0}\right]_{\mu^2}=0 \tag{K-66}$$

The total contribution:

$$C_{3;23}^{[2]}\left(1^-,2^+,3^-,4^+\right)=A_4^{tree}\frac{st^2}{u^3} \tag{K-67}$$

$C_{3;34}^{[2]}\left(1^-,2^-,3^+,4^+\right)$

We have symmentry between the channels: $C_{3;34}^{[2]}\left(1^-,2^+,3^-,4^+\right)$ and $C_{3;12}^{[2]}\left(1^-,2^+,3^-,4^+\right)$,

$$C_{3;34}^{[2]}\left(1^-,2^+,3^-,4^+\right)=-\frac{is_{12}s_{24}}{[3|1]^2\,\langle 24\rangle^2}=-A_4^{tree}\frac{s^2t}{u^3} \tag{K-68}$$

$C_{3;41}^{[2]}\left(1^-,2^+,3^-,4^+\right)$

We have symmentry between the channels: $C_{3;23}^{[2]}\left(1^-,2^+,3^-,4^+\right)$ and $C_{3;41}^{[2]}\left(1^-,2^+,3^-,4^+\right)$

$$C_{3;41}^{[2]}\left(1^-,2^+,3^-,4^+\right)=\frac{is_{23}\langle 13\rangle\langle 23\rangle}{\langle 24\rangle^3\,[41]}=A_4^{tree}\frac{st^2}{u^3} \tag{K-69}$$

L. Rational Contributions for amplitudes of four gluons in Fermionic QCD

L.1. Quadrupole cut coefficients

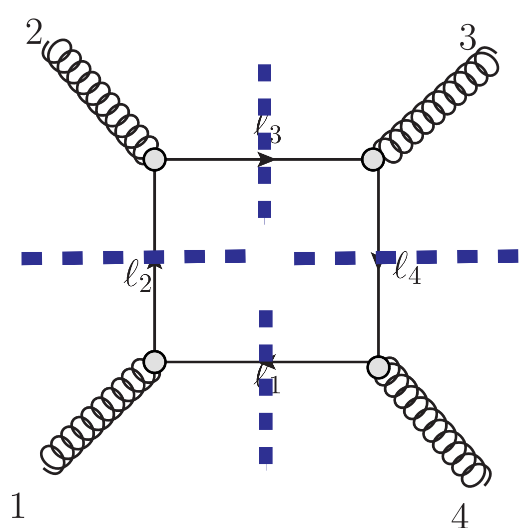


Figure **L-1**.: Quadrupole cut for the process of four gluons in fermionic QCD.

L.1.1. $C_4^{[4]}\left(1^+,2^+,3^+,4^+\right)$

$$A_3^{tree}\left(-\ell_1,1^+,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)=$$
$$=\frac{1}{\langle q_1 1\rangle\langle q_2 2\rangle\langle q_3 3\rangle\langle q_4 4\rangle}\left(\langle\ell_2 q_1\rangle[1\ell_1]+[\ell_2 1]\langle q_1\ell_1\rangle\right)\left(\langle\ell_3 q_2\rangle[2\ell_2]+[\ell_3 2]\langle q_2\ell_2\rangle\right)\times$$
$$\times\left(\langle\ell_4 q_3\rangle[3\ell_3]+[\ell_4 3]\langle q_3\ell_3\rangle\right)\left(\langle\ell_1 q_4\rangle[4\ell]+[\ell_1 4]\langle q_4\ell_4\rangle\right)\quad\text{(L-1)}$$

Let's study the products $\left(\langle\ell_2 q_1\rangle\left[1\ell_1\right]+\left[\ell_2 1\right]\langle q_1\ell_1\rangle\right)\left(\langle\ell_3 q_2\rangle\left[2\ell_2\right]+\left[\ell_3 2\right]\langle q_2\ell_2\rangle\right)$ and $\left(\langle\ell_4 q_3\rangle\left[3\ell_3\right]+\left[\ell_4 3\right]\langle q_3\ell_3\rangle\right)\left(\langle\ell_1 q_4\rangle\left[4\ell\right]+\left[\ell_1 4\right]\langle q_4\ell_4\rangle\right)$:

$$\begin{aligned}
&\left(\langle\ell_2 q_1\rangle\left[1\ell_1\right]+\left[\ell_2 1\right]\langle q_1\ell_1\rangle\right)\left(\langle\ell_3 q_2\rangle\left[2\ell_2\right]+\left[\ell_3 2\right]\langle q_2\ell_2\rangle\right)=\\
&=\left(\langle\ell_2 2\rangle\left[1\ell_1\right]+\left[\ell_2 1\right]\langle 2\ell_1\rangle\right)\left(\langle\ell_3 1\rangle\left[2\ell_2\right]+\left[\ell_3 2\right]\langle 1\ell_2\rangle\right)\\
&=\langle\ell_2 2\rangle\left[1\ell_1\right]\langle\ell_3 1\rangle\left[2\ell_2\right]+\langle\ell_2 2\rangle\left[1\ell_1\right]\left[\ell_3 2\right]\langle 1\ell_2\rangle+\left[\ell_2 1\right]\langle 2\ell_1\rangle\langle\ell_3 1\rangle\left[2\ell_2\right]+\left[\ell_2 1\right]\langle 2\ell_1\rangle\left[\ell_3 2\right]\langle 1\ell_2\rangle\\
&=2\left(\ell_2\cdot 2\right)\left[\ell_1\left|1\right|\ell_3\right\rangle+\left[\ell_1\left|1\right|\ell_2\right\rangle\left[\ell_3\left|2\right|\ell_2\right\rangle+\left[\ell_2\left|1\right|\ell_3\right\rangle\left[\ell_2\left|2\right|\ell_1\right\rangle+2\left(\ell_2\cdot 1\right)\left[\ell_3\left|2\right|\ell_1\right\rangle\\
&=\left[\ell_1\left|1\right|\ell_2\right\rangle\left[\ell_3\left|2\right|\ell_2\right\rangle+\left[\ell_2\left|1\right|\ell_3\right\rangle\left[\ell_2\left|2\right|\ell_1\right\rangle=-\langle 12\rangle\left[21\right]\left\{\ell_1\left|\omega_+\mu\right|\ell_3\right\}\\
&=\left[\ell_1\left|1\omega_+\mu 2\right|\ell_3\right]+\left\langle\ell_1\left|2\omega_-\mu 1\right|\ell_3\right\rangle=-\langle 12\rangle\left[21\right]\left\langle\ell_1\left|\mu\right|\ell_3\right\rangle
\end{aligned}\tag{L-2}$$

$$\left(\langle\ell_4 q_3\rangle\left[3\ell_3\right]+\left[\ell_4 3\right]\langle q_3\ell_3\rangle\right)\left(\langle\ell_1 q_4\rangle\left[4\ell\right]+\left[\ell_1 4\right]\langle q_4\ell_4\rangle\right)=-\langle 34\rangle\left[43\right]\left\langle\ell_3\left|\mu\right|\ell_1\right\rangle\tag{L-3}$$

With this,

$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree}=\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\left\langle\ell_1\left|\mu\right|\ell_3\right\rangle\left\langle\ell_3\left|\mu\right|\ell_1\right\rangle\tag{L-4}$$

$$=\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\left\langle\ell_1\left|\mu\omega_+\mu\mu\right|\ell_1\right\rangle\tag{L-5}$$

$$=-\mu^2\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\left\langle\ell_1\left|\ell_1\right|\ell_1\right]\tag{L-6}$$

$$=-2\mu^4\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\tag{L-7}$$

The coefficient is given by

$$C_4^{[4]}\left(1^+,2^+,3^+,4^+\right)=-2i\frac{[12]\,[34]}{\langle 12\rangle\langle 34\rangle}\tag{L-8}$$

L.1.2. $C_4^{[4]}\left(1^-,2^-,3^+,4^+\right)$ and $C_4^{[2]}\left(1^-,2^-,3^+,4^+\right)$

$$\begin{aligned}
&A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)=\\
&=\frac{1}{[q_1 1]\,[q_2 2]\,\langle q_3 3\rangle\langle q_4 4\rangle}\left(\left[\ell_2 q_1\right]\langle 1\ell_1\rangle+\langle\ell_2 1\rangle\left[q_1\ell_1\right]\right)\left(\left[\ell_3 q_2\right]\langle 2\ell_2\rangle+\langle\ell_3 2\rangle\left[q_2\ell_2\right]\right)\times\\
&\times\left(\langle\ell_4 q_3\rangle\left[3\ell_3\right]+\left[\ell_4 3\right]\langle q_3\ell_3\rangle\right)\left(\langle\ell_1 q_4\rangle\left[4\ell\right]+\left[\ell_1 4\right]\langle q_4\ell_4\rangle\right)
\end{aligned}$$

Using parity in eq. (L-2)

$$\left(\left[\ell_2 q_1\right]\langle 1\ell_1\rangle+\langle\ell_2 1\rangle\left[q_1\ell_1\right]\right)\left(\left[\ell_3 q_2\right]\langle 2\ell_2\rangle+\langle\ell_3 2\rangle\left[q_2\ell_2\right]\right)=-\langle 12\rangle\left[21\right]\left[\ell_1\left|\mu\right|\ell_3\right]\tag{L-9}$$

$$\left(\langle\ell_4 q_3\rangle\left[3\ell_3\right]+\left[\ell_4 3\right]\langle q_3\ell_3\rangle\right)\left(\langle\ell_1 q_4\rangle\left[4\ell\right]+\left[\ell_1 4\right]\langle q_4\ell_4\rangle\right)=-\langle 34\rangle\left[43\right]\left\langle\ell_3\left|\mu\right|\ell_1\right\rangle\tag{L-10}$$

we obtain,

$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} = \frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}[\ell_1|\mu|\ell_3]\langle\ell_3|\mu|\ell_1\rangle \tag{L-11}$$
$$= \frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}\mathrm{Tr}\left\{\left(\frac{1+\gamma^5}{2}\right)\ell_1\mu\ell_3\mu\right\} \tag{L-12}$$
$$= \mu^2\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}\mathrm{Tr}\left\{\left(\frac{1+\gamma^5}{2}\right)\ell_1\ell_3\right\} = \mu^2\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}2(\ell_1\cdot\ell_3) \tag{L-13}$$
$$= \mu^2\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}\left(2\mu^2 - s_{12}\right) \tag{L-14}$$
$$= 2\mu^4\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle} - \mu^2\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}s_{12} \tag{L-15}$$

Let's call $s = s_{12}$ and $t = s_{14}$.
The coefficients of the power μ^4 and μ^2:

$$\mu^4 : 2\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle} = -2\frac{t}{s}A_4^{tree} \tag{L-16}$$
$$\mu^2 : -\frac{\langle 12\rangle [43]}{[12]\langle 43\rangle}s_{12} = tA_4^{tree} \tag{L-17}$$

The coeffcients take the form:

$$C_4^{[4]}\left(1^-,2^-,3^+,4^+\right) = -2\frac{t}{s}A_4^{tree} \tag{L-18}$$
$$C_4^{[2]}\left(1^-,2^-,3^+,4^+\right) = tA_4^{tree} \tag{L-19}$$

L.1.3. $C_4^{[4]}\left(1^-,2^+,3^-,4^+\right)$ and $C_4^{[2]}\left(1^-,2^+,3^-,4^+\right)$

$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} = \frac{1}{[q_11]\langle q_22\rangle [q_33]\langle q_44\rangle}\left([\ell_2q_1]\langle 1\ell_1\rangle + \langle\ell_21\rangle [q_1\ell_1]\right)\left(\langle\ell_3q_2\rangle [2\ell_2] + [\ell_32]\langle q_2\ell_2\rangle\right)\times$$
$$\times\left([\ell_4q_3]\langle 3\ell_3\rangle + \langle\ell_43\rangle [q_3\ell_3]\right)\left(\langle\ell_1q_4\rangle [4\ell] + [\ell_14]\langle q_4\ell_4\rangle\right) \tag{L-20}$$

Let's study the products $\left([\ell_2q_1]\langle 1\ell_1\rangle + \langle\ell_21\rangle [q_1\ell_1]\right)\left(\langle\ell_3q_2\rangle [2\ell_2] + [\ell_32]\langle q_2\ell_2\rangle\right)$ and $\left([\ell_4q_3]\langle 3\ell_3\rangle + \langle\ell_43\rangle [q_3\ell_3]\right)\left(\langle\ell_1q_4\rangle [4\ell] + [\ell_14]\langle q_4\ell_4\rangle\right)$:

$$\left([\ell_2q_1]\langle 1\ell_1\rangle + \langle\ell_21\rangle [q_1\ell_1]\right)\left(\langle\ell_3q_2\rangle [2\ell_2] + [\ell_32]\langle q_2\ell_2\rangle\right) =$$
$$= \left([\ell_22]\langle 1\ell_1\rangle + \langle\ell_21\rangle [2\ell_1]\right)\left(\langle\ell_31\rangle [2\ell_2] + [\ell_32]\langle 1\ell_2\rangle\right)$$
$$= [\ell_22]\langle 1\ell_1\rangle\langle\ell_31\rangle [2\ell_2] + [\ell_22]\langle 1\ell_1\rangle [\ell_32]\langle 1\ell_2\rangle + \langle\ell_21\rangle [2\ell_1]\langle\ell_31\rangle [2\ell_2] + \langle\ell_21\rangle [2\ell_1][\ell_32]\langle 1\ell_2\rangle$$
$$= -[\ell_22]^2\langle 1\ell_1\rangle\langle\ell_31\rangle + \langle 1|\ell_2|2]\langle 1\ell_1\rangle [\ell_32] + \langle 1|\ell_2|2][2\ell_1]\langle\ell_31\rangle - \langle\ell_21\rangle^2[2\ell_1][\ell_32]$$
$$= \langle 1|\ell_1|2]\left(\langle 1\ell_1\rangle [\ell_32] + \langle 1\ell_3\rangle [\ell_12]\right) = \langle 12\rangle [21]\varepsilon_+(2)\cdot\ell_1\bar{u}(\ell_1)\not\varepsilon_-(1)u(\ell_3) \tag{L-21}$$

$$\left([\ell_4q_3]\langle 3\ell_3\rangle + \langle\ell_43\rangle [q_3\ell_3]\right)\left(\langle\ell_1q_4\rangle [4\ell] + [\ell_14]\langle q_4\ell_4\rangle\right) = \langle 34\rangle [43]\varepsilon_+(4)\cdot\ell_1\bar{u}(\ell_3)\not\varepsilon_-(3)u(\ell_1) \tag{L-22}$$

for this calculation,

$$[\ell_2 2]^2 = -[2|\mu|2] = 0 \tag{L-23}$$

$$\langle \ell_2 1\rangle^2 = -\langle 1|\mu|1\rangle = 0 \tag{L-24}$$

With the sum over internal states, we can obtain the traces:

$$\begin{aligned} A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} &= \varepsilon_+(2)\cdot\ell_1\bar{u}(\ell_1)\not\varepsilon_-(1)\,u(\ell_3)\,\varepsilon_+(4)\cdot\ell_1\bar{u}(\ell_3)\not\varepsilon_-(3)\,u(\ell_1) &\text{(L-25)}\\ &= \varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\mathrm{Tr}\left\{(\ell_1+\mu)\not\varepsilon_-(1)(\ell_3+\mu)\not\varepsilon_-(3)\right\} &\text{(L-26)}\\ &= \varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\left[\mathrm{Tr}\{\ell_1\varepsilon_-(1)\ell_3\varepsilon_-(3)\}+\mathrm{Tr}\{\mu\varepsilon_-(1)\mu\varepsilon_-(3)\}\right] &\text{(L-27)}\\ &= \varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\left[\mathrm{Tr}\{\ell_1\varepsilon_-(1)\ell_3\varepsilon_-(3)\}+\mu^2\mathrm{Tr}\{\varepsilon_-(1)\varepsilon_-(3)\}\right] &\text{(L-28)}\\ &= 4\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\left[[\ell_1\cdot\varepsilon_-(1)\,\ell_3\cdot\varepsilon_-(3)-\ell_1\cdot\ell_3\varepsilon_-(1)\cdot\varepsilon_-(3)+\right] \\ &\quad +\ell_1\cdot\varepsilon_-(3)\,\ell_3\cdot\varepsilon_-(1)+\mu^2\varepsilon_-(1)\cdot\varepsilon_-(3)\big] &\text{(L-29)}\\ &= 4\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\left[2\ell_1\cdot\varepsilon_-(1)\,\ell_1\cdot\varepsilon_-(3)+\left(-\ell_1\cdot\ell_3+\mu^2\right)\varepsilon_-(1)\cdot\varepsilon_-(3)\right] &\text{(L-30)} \end{aligned}$$

From momentum conservation, we know that

$$\ell_1 = \ell_3 + p_1 + p_2 \tag{L-31}$$

$$\Rightarrow -\ell_1\cdot\ell_3 + \mu^2 = p_1\cdot p_2 \tag{L-32}$$

Then, the product of four tree-level amplitudes is given by:

$$\begin{aligned} A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} &= 2\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\left[4\ell_1\cdot\varepsilon_-(1)\,\ell_1\cdot\varepsilon_-(3)+s_{12}\varepsilon_-(1)\cdot\varepsilon_-(3)\right] &\text{(L-33)}\\ &= 8\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\ell_1\cdot\varepsilon_-(1)\,\ell_1\cdot\varepsilon_-(3)+2s_{12}\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\varepsilon_-(1)\cdot\varepsilon_-(3) &\text{(L-34)} \end{aligned}$$

The first term was obtained when we did this process in the scalar loop

$$8\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\ell_1\cdot\varepsilon_-(1)\,\ell_1\cdot\varepsilon_-(3) = 2\mu^4\frac{\langle 12\rangle\langle 34\rangle[42]^2}{\langle 24\rangle^2[21][43]} \tag{L-35}$$

$$\begin{aligned} 2s_{12}\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\varepsilon_-(1)\cdot\varepsilon_-(3) &= 2s_{12}\frac{\langle 1|\ell_1|2]}{\sqrt{2}\langle 12\rangle}\frac{\langle 3|\ell_1|4]}{\sqrt{2}\langle 34\rangle}\frac{[2|\gamma^\mu|1\rangle}{\sqrt{2}[21]}\frac{[4|\gamma^\mu|3\rangle}{\sqrt{2}[43]} &\text{(L-36)}\\ &= \frac{1}{2\langle 34\rangle[43]}\langle 1|\ell_1|2]\langle 3|\ell_1|4][2|\gamma^\mu|1\rangle[4|\gamma^\mu|3\rangle &\text{(L-37)}\\ &= \frac{\langle 13\rangle[42]}{\langle 34\rangle[43]}\langle 1|\ell_1|2]\langle 3|\ell_1|4] &\text{(L-38)} \end{aligned}$$

$$\text{The explicit solution: } \ell_1^\mu = c\langle 1|\gamma^\mu|4] + \frac{\mu^2}{4s_{14}c}\langle 4|\gamma^\mu|1] \tag{L-39}$$

$$\begin{aligned} \langle 1|\ell_1|2]\langle 3|\ell_1|4] &= \left(\frac{\mu^2}{2s_{14}c}\right)^2\langle 14\rangle[12]\langle 34\rangle[14] = &\text{(L-40)}\\ &= -\left(\frac{\mu^2}{2s_{14}c}\right)^2[12]\langle 34\rangle s_{14} &\text{(L-41)} \end{aligned}$$

and c is:

$$c = \pm\frac{\mu}{2}\sqrt{\frac{\langle 4\left|2\right|1]}{s_{14}\langle 1\left|2\right|4]}} \tag{L-42}$$

$$(2c)^2 = \mu^2\frac{\langle 4\left|2\right|1]}{s_{14}\langle 1\left|2\right|4]} \tag{L-43}$$

with this:

$$\langle 1\left|\ell_1\right|2]\langle 3\left|\ell_1\right|4] = -\mu^2\frac{\langle 1\left|2\right|4]}{\langle 4\left|2\right|1]}[12]\langle 34\rangle \tag{L-44}$$

$$2s_{12}\varepsilon_+(2)\cdot\ell_1\varepsilon_+(4)\cdot\ell_1\varepsilon_-(1)\cdot\varepsilon_-(3) = \mu^2\frac{\langle 13\rangle[42]\langle 12\rangle[24]}{\langle 42\rangle[43]} = -\mu^2\frac{\langle 13\rangle^2[42]^2}{s_{13}} \tag{L-45}$$

the product of the tree-level ampitudes is given by:

$$A_1^{tree}A_2^{tree}A_3^{tree}A_4^{tree} = 2\mu^4\frac{\langle 12\rangle\langle 34\rangle[42]^2}{\langle 24\rangle^2[21][43]} - \mu^2\frac{\langle 13\rangle^2[42]^2}{[13]^2} \tag{L-46}$$

$$C_4^{[4]}\left(1^-,2^+,3^-,4^+\right) = 2i\frac{\langle 12\rangle\langle 34\rangle[42]^2}{\langle 24\rangle^2[21][43]} = -2\frac{st}{u^2}A_4^{tree} \tag{L-47}$$

$$C_4^{[2]}\left(1^-,2^+,3^-,4^+\right) = -i\frac{\langle 13\rangle^2[42]^2}{u} = \frac{st}{u}A_4^{tree} \tag{L-48}$$

Another way

Previously we obtained the coefficient for the amplitude of $C_{4;12}^{[4]}(1^-,2^-,3^+,4^+)$ putting the tree level scattering amplitudes in this form:

$$C_{4;12}^{[4]}\left(1^-,2^-,3^+,4^+\right) = \sum A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^-,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right) \tag{L-49}$$

$$= \sum A_4^{tree}\left(-\ell_1,1^-,2^-,\ell_3\right)i\ell_2^2A_4^{tree}\left(-\ell_3,3^+,4^+\ell_1\right)i\ell_4^2 \tag{L-50}$$

so, if we put the tree amplitudes in this way:

$$C_{4;12}^{[4]}\left(1^-,2^-,3^+,4^+\right) = \sum A_3^{tree}A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^-,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right) \tag{L-51}$$

$$= \sum A_4^{tree}\left(-\ell_4,4^+,1^-,\ell_2\right)i\ell_1^2A_4^{tree}\left(-\ell_2,2^-,3^+\ell_4\right)i\ell_3^2, \tag{L-52}$$

we will obtain:

$$C_{4;12}^{[4]}\left(1^-,2^-,3^+,4^+\right) = \sum A_4^{tree}\left(-\ell_4,4^+,1^-,\ell_2\right)i\ell_1^2A_4^{tree}\left(-\ell_2,2^-,3^+\ell_4\right)i\ell_3^2, = -2\frac{t}{s}A_4^{tree}\left(1^-,2^-,3^+,4^+\right) \tag{L-53}$$

Now we study the term $C_4^{[4]}\left(1^-,2^-,3^+,4^+\right)$:

$$C_4^{[4]}\left(1^-,2^-,3^+,4^+\right) = -2\frac{t}{s}A_4^{tree}\left(1^-,2^-,3^+,4^+\right) = -2i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{t}{s} \tag{L-54}$$

$$-2\frac{t}{s}A_4^{tree}\left(i^-,j^-,k^+,l^+\right) = -2i\frac{\langle ij\rangle^4}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}\frac{\langle jk\rangle[kj]}{\langle ij\rangle[ji]} \tag{L-55}$$

$$\sum A_4^{tree}\left(-\ell_4,l^+,i^-,\ell_2\right)A_4^{tree}\left(-\ell_2,j^-,k^+\ell_4\right) = -2i\frac{\langle ij\rangle^4}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}\frac{\langle jk\rangle[kj]}{\langle ij\rangle[ji]} \tag{L-56}$$

$$\sum A_4^{tree}\left(-\ell_4,4^+,1^-,\ell_2\right)A_4^{tree}\left(-\ell_2,3^-,2^+\ell_4\right) = -2i\frac{\langle 13\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\frac{\langle 32\rangle[23]}{\langle 13\rangle[31]} \tag{L-57}$$

$$= 2i\frac{\langle 13\rangle^4}{\langle 23\rangle\langle 41\rangle}\frac{\langle 32\rangle[23]}{\langle 13\rangle^2\langle 24\rangle[31]}\times\frac{\langle 12\rangle\langle 34\rangle}{\langle 12\rangle\langle 34\rangle} \tag{L-58}$$

$$= 2i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 32\rangle[23]\langle 12\rangle\langle 34\rangle}{\langle 13\rangle^2\langle 24\rangle[31]} \tag{L-59}$$

$$= 2iA_4^{tree}\left(1^-,2^+,3^-,4^+\right)\frac{\langle 32\rangle[23]\langle 12\rangle\langle 34\rangle}{\langle 13\rangle^2\langle 24\rangle[31]}\times\frac{[31]}{[31]} \tag{L-60}$$

$$\Rightarrow C_4^{[4]}\left(1^-,2^+,3^-,4^+\right) = -2iA_4^{tree}\left(1^-,2^+,3^-,4^+\right)\frac{st}{u^2} \tag{L-61}$$

and the term $C_4^{[2]}\left(1^-,2^-,3^+,4^+\right)$:

$$C_4^{[2]}\left(1^-,2^-,3^+,4^+\right) = tA_4^{tree}\left(1^-,2^-,3^+,4^+\right) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}t \tag{L-62}$$

$$tA_4^{tree}\left(i^-,j^-,k^+,l^+\right) = i\frac{\langle ij\rangle^4}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}\langle jk\rangle[kj] \tag{L-63}$$

$$\sum A_4^{tree}\left(-\ell_4,l^+,i^-,\ell_2\right)A_4^{tree}\left(-\ell_2,j^-,k^+\ell_4\right) = i\frac{\langle ij\rangle^4}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}\langle jk\rangle[kj] \tag{L-64}$$

$$\sum A_4^{tree}\left(-\ell_4,4^+,1^-,\ell_2\right)A_4^{tree}\left(-\ell_2,3^-,2^+\ell_4\right) = i\frac{\langle 13\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\langle 32\rangle[23]\times\frac{\langle 12\rangle\langle 34\rangle}{\langle 12\rangle\langle 34\rangle} \tag{L-65}$$

$$= -i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 32\rangle[23]\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} \tag{L-66}$$

$$= -A_4^{tree}\left(1^-,2^+,3^-,4^+\right)\frac{t\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}\times\frac{[31][42]}{[31][42]} \tag{L-67}$$

$$= A_4^{tree}\left(1^-,2^+,3^-,4^+\right)[31]\langle 13\rangle\langle 12\rangle[21]\frac{t}{u^2} \tag{L-68}$$

$$\Rightarrow C_4^{[2]}\left(1^-,2^+,3^-,4^+\right) = A_4^{tree}\left(1^-,2^+,3^-,4^+\right)\frac{st}{u} \tag{L-69}$$

L.1.4. $C_4^{[4]}\left(1^-,2^+,3^+,4^+\right)$

$$\begin{aligned}
C_4^{[4]}\left(1^-,2^+,3^+,4^+\right) &= \sum A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)\\
&= \left[\frac{i}{[21]}\left([\ell_2 2]\langle 1\ell_1\rangle+\langle\ell_2 1\rangle[2\ell_1]\right)\right]\left[-\frac{i}{\langle 12\rangle}\left(\langle\ell_4 1\rangle[2\ell_2]+[\ell_4 2]\langle 1\ell_2\rangle\right)\right]\times\\
&\times\left[-\frac{i}{\langle 43\rangle}\left(\langle\ell_4 4\rangle[3\ell_3]+[\ell_4 3]\langle 4\ell_3\rangle\right)\right]\left[-\frac{i}{\langle 34\rangle}\left(\langle\ell_1 3\rangle[4\ell_4]+[\ell_1 4]\langle 3\ell_4\rangle\right)\right]\\
&= -\frac{1}{[21]\langle 12\rangle\langle 43\rangle\langle 34\rangle}\left([\ell_2 2]\langle 1\ell_1\rangle+\langle\ell_2 1\rangle[2\ell_1]\right)\left(\langle\ell_4 1\rangle[2\ell_2]+[\ell_4 2]\langle 1\ell_2\rangle\right)\times\\
&\times\left(\langle\ell_4 4\rangle[3\ell_3]+[\ell_4 3]\langle 4\ell_3\rangle\right)\left(\langle\ell_1 3\rangle[4\ell_4]+[\ell_1 4]\langle 3\ell_4\rangle\right)\\
&= -\frac{[43]}{[21]\langle 12\rangle\langle 43\rangle}\langle 1|\ell_1|2]\left(\langle 1\ell_1\rangle[\ell_3 2]+\langle 1\ell_3\rangle[\ell_1 2]\right)\langle\ell_3|\mu|\ell_1\rangle\\
&= -\frac{[43]}{[21]\langle 12\rangle\langle 43\rangle}\langle 1|\ell_1|2]\langle\ell_3|\mu|\ell_1\rangle\left(\langle 1\ell_1\rangle[\ell_3 2]+\langle 1\ell_3\rangle[\ell_1 2]\right)\\
&= 2\mu^2\frac{[43]}{[21]\langle 12\rangle\langle 43\rangle}\langle 1|\ell_1|2]\left([2|\ell_3|1\rangle+[2|\ell_1|1\rangle\right)\\
&= 2\mu^2\frac{[43]}{[21]\langle 12\rangle\langle 43\rangle}\langle 1|\ell_1|2]^2\\
&= 2\mu^4 i\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\frac{st}{u}
\end{aligned}\tag{L-70}$$

Previously we obtained the same result of the scalar loop.

$$C_4^{[4]}\left(1^-,2^+,3^+,4^+\right) = 2i\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\frac{st}{u}\tag{L-71}$$

L.2. Triple Cut Coefficients

L.2.1. $C_3^{[2]}\left(1^+,2^+,3^+,4^+\right)$

The product of tree amplitudes

$$\begin{aligned}
&A_4^{tree}\left(-\ell_1,1^+,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)=\\
&= i\frac{[12]}{\langle 12\rangle\langle 34\rangle^2}\frac{\langle\ell_3|\mu|\ell_1\rangle}{\langle 1|\ell_1|1]}\left(\langle\ell_4 4\rangle[3\ell_3]+[\ell_4 3]\langle 4\ell_3\rangle\right)\left(\langle\ell_1 3\rangle[4\ell_4]+[\ell_1 4]\langle 3\ell_4\rangle\right) = i\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\frac{\langle\ell_3|\mu|\ell_1\rangle^2}{\langle 1|\ell_1|1]}
\end{aligned}\tag{L-72}$$

the term in the numerator is given by:

$$\langle\ell_3|\mu|\ell_1\rangle^2 = \langle\ell_3|\mu|\ell_1\rangle\langle\ell_1|\ell_3|\ell_3] = \mathrm{Tr}\left\{\left(\frac{1-\gamma^5}{2}\right)\ell_3\mu\mu\ell_3\right\} = 2\mu^4\tag{L-73}$$

the solution for ℓ_1,

$$\ell_1^\mu = t\langle 3|\gamma^\mu|4] - \frac{\mu^2}{4s_{34}t}\langle 4|\gamma^\mu|3] - K_4^\mu\tag{L-74}$$

$$\ell_1^{\mu *} = t\langle 4|\gamma^\mu|3] - \frac{\mu^2}{4s_{34}t}\langle 3|\gamma^\mu|4] - K_4^\mu\tag{L-75}$$

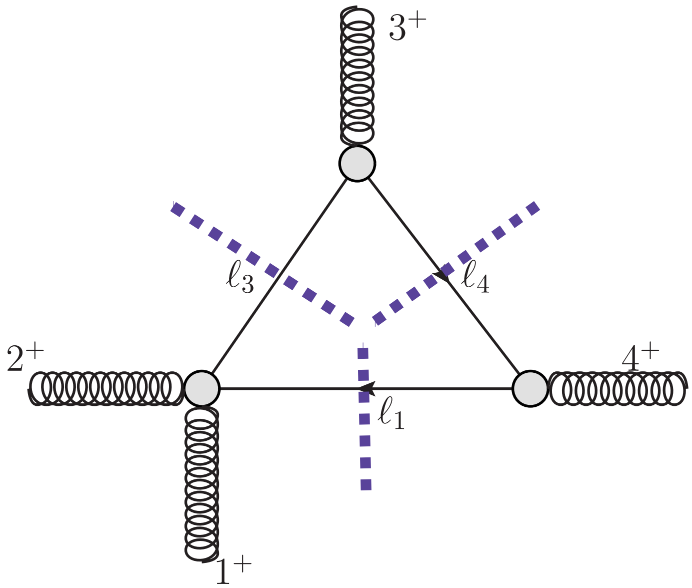


Figure **L-2.**: Triple cut for the process of four gluons in fermionic QCD.

and the term in the denominator

$$\langle 1\left|\ell_1\right|1] = 2t\langle 13\rangle[41] - \frac{\mu^2}{2s_{34}t}\langle 14\rangle[31] - s_{14} = -\frac{\mu^2\langle 14\rangle[31]}{2s_{34}t}\left(1 - \frac{4s_{34}\langle 13\rangle[41]t^2 - 2s_{34}s_{14}t}{\mu^2\langle 14\rangle[31]}\right) \tag{L-76}$$

$$\langle 1\left|\ell_1^*\right|1] = 2t\langle 14\rangle[31] - \frac{\mu^2}{2s_{34}t}\langle 13\rangle[41] - s_{14} = -\frac{\mu^2\langle 13\rangle[41]}{2s_{34}t}\left(1 - \frac{4s_{34}\langle 14\rangle[31]t^2 - 2s_{34}s_{14}t}{\mu^2\langle 14\rangle[41]}\right) \tag{L-77}$$

We find the coefficient,

$$\frac{\langle \ell_3\left|\mu\right|\ell_1\rangle^2}{\langle 1\left|\ell_1\right|1]} = -\frac{4s_{34}t\mu^2}{\langle 13\rangle[41]}\frac{1}{\left(1 - \frac{4s_{34}\langle 14\rangle[31]t^2 - 2s_{34}s_{14}t}{\mu^2\langle 14\rangle[41]}\right)} = -\frac{4s_{34}t\mu^2}{\langle 13\rangle[41]}\left(1 + \frac{4s_{34}\langle 14\rangle[31]t^2 - 2s_{34}s_{14}t}{\mu^2\langle 14\rangle[41]}\right) \tag{L-78}$$

$$\inf_{t^0}\left[\frac{\langle \ell_3\left|\mu\right|\ell_1\rangle^2}{\langle 1\left|\ell_1\right|1]}\right] = \inf_{t^0}\left[\frac{\langle \ell_3\left|\mu\right|\ell_1\rangle^2}{\langle 1\left|\ell_1^*\right|1]}\right] = 0 \tag{L-79}$$

finally,

$$C^{[2]}_{3;12}\left(1^+,2^+,3^+,4^+\right) = 0 \tag{L-80}$$

in the same way

$$C^{[2]}_{3;23}\left(1^+,2^+,3^+,4^+\right) = C^{[2]}_{3;34}\left(1^+,2^+,3^+,4^+\right) = C^{[2]}_{3;41}\left(1^+,2^+,3^+,4^+\right) = 0 \tag{L-81}$$

L.2.2. $C_3^{[2]}(1^-,2^+,3^+,4^+)$

$C_{3;12}^{[2]}(1^-,2^+,3^+,4^+)$

The product of the three tree amplitudes is:

$$\begin{aligned} &A_4^{tree}\left(-\ell_1,1^-,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)= \\ &\qquad = -i\frac{1}{\langle 12\rangle\,[21]\,\langle 34\rangle\,\langle 43\rangle}\frac{[2\,|\ell_1|\,1\rangle}{\langle 1\,|\ell_1|\,1]}\left(\langle\ell_3 1\rangle\,[2\ell_1]+[\ell_3 2]\,\langle 1\ell_1\rangle\right)\langle\ell_3\,|\mu|\,\ell_1\rangle \end{aligned} \quad \text{(L-82)}$$

where the second term in the prodcut is given by,

$$\begin{aligned} \left(\langle\ell_3 1\rangle\,[2\ell_1]+[\ell_3 2]\,\langle 1\ell_1\rangle\right)\langle\ell_3\,|\mu|\,\ell_1\rangle &= \langle 1\ell_3\rangle\,\langle\ell_3\,|\mu|\,\ell_1\rangle\,[\ell_1 2]+[2\ell_3]\,\langle\ell_3\,|\mu|\,\ell_1\rangle\,\langle\ell_1 1\rangle \\ &= \langle 1\,|\mu\mu\ell_1|\,2]+\langle 1\,|\mu\mu\ell_3|\,2]=-2\mu^2\,\langle 1\,|\ell_1|\,2] \end{aligned} \quad \text{(L-83)}$$

the product of tree amplitudes than becomes

$$A_4^{tree}\left(-\ell_1,1^-,2^+,\ell_3\right)A_3^{tree}\left(-\ell_3,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)=-2i\mu^2\frac{[43]}{\langle 12\rangle\,[21]\,\langle 34\rangle}\frac{[2\,|\ell_1|\,1\rangle^2}{\langle 1\,|\ell_1|\,1]} \quad \text{(L-84)}$$

This result was obtained in the scalar loop.

$$C_{3;12}^{[2]}\left(1^-,2^+,3^+,4^+\right)=i\frac{[24]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]}\frac{st}{u}\left(s^2-2u^2\right)\frac{1}{su}$$

$C_{3;23}^{[2]}(1^-,2^+,3^+,4^+)$

$$\begin{aligned} A_4^{tree}\left(-\ell_2,2^+,3^+,\ell_4\right)A_3^{tree}\left(-\ell_4,4^+,\ell_1\right)A_3^{tree}\left(-\ell_1,1^-,\ell_2\right) &= -i\frac{[23]}{\langle 23\rangle\,\langle 14\rangle\,[41]}\frac{\langle\ell_4\,|\mu|\,\ell_2\rangle}{\langle 2\,|\ell_2|\,2]} \\ &\times\left(\langle\ell_1 1\rangle\,[4\ell_4]+[\ell_1 4]\,\langle 1\ell_4\rangle\right)\left([\ell_2 4]\,\langle 1\ell_1\rangle+\langle\ell_2 1\rangle\,[4\ell_1]\right) \end{aligned} \quad \text{(L-85)}$$

$$\begin{aligned} \left(\langle\ell_1 1\rangle\,[4\ell_4]+[\ell_1 4]\,\langle 1\ell_4\rangle\right)\left([\ell_2 4]\,\langle 1\ell_1\rangle+\langle\ell_2 1\rangle\,[4\ell_1]\right) &= \langle\ell_1 1\rangle\,[4\ell_4]\,[\ell_2 4]\,\langle 1\ell_1\rangle+\langle\ell_1 1\rangle\,[4\ell_4]\,\langle\ell_2 1\rangle\,[4\ell_1]+ \\ &+[\ell_1 4]\,\langle 1\ell_4\rangle\,[\ell_2 4]\,\langle 1\ell_1\rangle+[\ell_1 4]\,\langle 1\ell_4\rangle\,\langle\ell_2 1\rangle\,[4\ell_1] \end{aligned} \quad \text{(L-86)}$$

$$= \langle 1\,|\ell_1|\,4]\left(\langle\ell_2 1\rangle\,[4\ell_4]+[\ell_2 4]\,\langle 1\ell_4\rangle\right) \quad \text{(L-87)}$$

$$A_1^{tree}A_2^{tree}A_3^{tree}=-i\frac{[23]}{\langle 23\rangle\,\langle 14\rangle\,[41]}\frac{\langle 1\,|\ell_1|\,4]}{\langle 2\,|\ell_2|\,2]}\langle\ell_4\,|\mu|\,\ell_2\rangle\left(\langle\ell_2 1\rangle\,[4\ell_4]+[\ell_2 4]\,\langle 1\ell_4\rangle\right) \quad \text{(L-88)}$$

$$\begin{aligned} \langle\ell_4\,|\mu|\,\ell_2\rangle\left(\langle\ell_2 1\rangle\,[4\ell_4]+[\ell_2 4]\,\langle 1\ell_4\rangle\right) &= [4\ell_4]\,\langle\ell_4\,|\mu|\,\ell_2\rangle\,\langle\ell_2 1\rangle+\langle 1\ell_4\rangle\,\langle\ell_4\,|\mu|\,\ell_2\rangle\,[\ell_2 4] && \text{(L-89)} \\ &= [4\,|\ell_4\mu\mu|\,1\rangle+[4\,|\ell_2\mu\mu|\,1\rangle=-\mu^2\,[4\,|\ell_4|\,1\rangle-\mu^2\,[4\,|\ell_2|\,1\rangle && \text{(L-90)} \\ &= -2\mu^2\,[4\,|\ell_1|\,1\rangle && \text{(L-91)} \end{aligned}$$

$$A_1^{tree}A_2^{tree}A_3^{tree}=2i\mu^2\frac{[23]}{\langle 23\rangle\,\langle 14\rangle\,[41]}\frac{\langle 1\,|\ell_1|\,4]^2}{\langle 2\,|\ell_2|\,2]} \quad \text{(L-92)}$$

The explicit solution for ℓ_1^μ

$$\ell_1^\mu = t\left<1\left|\gamma^\mu\right|4\right] - \frac{\mu^2}{4s_{14}t}\left<4\left|\gamma^\mu\right|1\right] \quad \text{(L-93)}$$

$$\ell_2^\mu = t\left<1\left|\gamma^\mu\right|4\right] - \frac{\mu^2}{4s_{14}t}\left<4\left|\gamma^\mu\right|1\right] - K_1^\mu \quad \text{(L-94)}$$

$$\left<1\left|\ell_1\right|4\right]^2 = \frac{\mu^4}{4t^2} \quad \text{(L-95)}$$

$$\left<2\left|\ell_2\right|2\right] = 2t\left<1\left|2\right|4\right] - \frac{\mu^2}{2s_{14}t}\left<4\left|2\right|1\right] - s_{12} = -\frac{\mu^2\left<4\left|2\right|1\right]}{2s_{14}t}\left(1 - \frac{4s_{14}\left<1\left|2\right|4\right]t^2 - 2s_{12}s_{14}t}{\mu^2\left<4\left|2\right|1\right]}\right) \quad \text{(L-96)}$$

$$\frac{\left<1\left|\ell_1\right|4\right]^2}{\left<2\left|\ell_2\right|2\right]} = -\frac{\mu^2 s_{14}}{2\left<4\left|2\right|1\right]t}\frac{1}{\left(1 - \frac{4s_{14}\left<1|2|4\right]t^2 - 2s_{12}s_{14}t}{\mu^2\left<4|2|1\right]}\right)} = -\frac{\mu^2 s_{14}}{2\left<4\left|2\right|1\right]t}\left(1 + \frac{4s_{14}\left<1\left|2\right|4\right]t^2 - 2s_{12}s_{14}t}{\mu^2\left<4\left|2\right|1\right]}\right) \quad \text{(L-97)}$$

$$\inf\left[\frac{\left<1\left|\ell_1\right|4\right]^2}{\left<2\left|\ell_2\right|2\right]}\right]_{t^0} = \frac{s_{12}s_{14}^2}{\left<4\left|2\right|1\right]^2} \quad \text{(L-98)}$$

$$\inf\left[\inf\left[A_1^{tree}A_2^{tree}A_3^{tree}\right]_{t^0}\right]_{\mu^2} = 2i\frac{[23]}{\left<23\right>}\frac{s_{12}s_{23}}{\left<4\left|2\right|1\right]^2} \quad \text{(L-99)}$$

$$C_{3;23}^{[2]}\left(1^-,2^+,3^+,4^+\right) = -2\frac{[24]^2}{[12]\left<23\right>\left<34\right>[41]}\frac{st}{u}\frac{ts}{su} \quad \text{(L-100)}$$

The another solution of ℓ_1^* does not give contribution because $\inf\left[A\left(-\ell_2,2^+,3^+,\ell_4\right)A\left(-\ell_4,4^+,\ell_1\right)A\left(-\ell_1,1^-,\ell_2\right)\right]_{t^0} =$ 0

$C_{3;34}^{[2]}\left(1^-,2^+,3^+,4^+\right)$

The product of tree amplitudes

$$A_4^{tree}\left(-\ell_3,3^+,4^+,\ell_1\right)A_3^{tree}\left(-\ell_1,1^-,\ell_2\right)A_3^{tree}\left(-\ell_2,2^+,\ell_3\right) =$$
$$= -i\frac{[34]}{\left<34\right>s_{12}}\frac{\left<\ell_1\left|\mu\right|\ell_3\right>}{\left<3\left|\ell_3\right|3\right]}\left([\ell_2 2]\left<1\ell_1\right> + \left<\ell_2 1\right>[2\ell_1]\right)\left(\left<\ell_3 1\right>[2\ell_2] + [\ell_3 2]\left<1\ell_2\right>\right) \quad \text{(L-101)}$$

studying each term

$$\left([\ell_2 2]\left<1\ell_1\right> + \left<\ell_2 1\right>[2\ell_1]\right)\left(\left<\ell_3 1\right>[2\ell_2] + [\ell_3 2]\left<1\ell_2\right>\right) = \left<1\left|\ell_2\right|2\right]\left(\left<1\ell_1\right>[\ell_3 2] + [2\ell_1]\left<\ell_3 1\right>\right) \quad \text{(L-102)}$$

with this, the amplitude,

$$A_1^{tree}A_2^{tree}A_3^{tree} == -i\frac{[34]}{\left<34\right>\left<12\right>[21]}\frac{\left<1\left|\ell_2\right|2\right]}{\left<3\left|\ell_3\right|3\right]}\left<\ell_1\left|\mu\right|\ell_3\right>\left(\left<1\ell_1\right>[\ell_3 2] + [2\ell_1]\left<\ell_3 1\right>\right), \quad \text{(L-103)}$$

where,

$$\begin{aligned}\langle \ell_1 |\mu| \ell_3\rangle \left(\langle 1\ell_1\rangle [\ell_3 2] + [2\ell_1]\langle \ell_3 1\rangle\right) &= \langle 1\ell_1\rangle \langle \ell_1 |\mu| \ell_3\rangle [\ell_3 2] + [2\ell_1]\langle \ell_1 |\mu| \ell_3\rangle \langle \ell_3 1\rangle \\ &= \langle 1 |\mu\mu\ell_3| 2] + \langle 1 |\mu\mu\ell_1| 2] \\ &= -2\mu^2 \langle 1 |\ell_2| 2]\end{aligned} \tag{L-104}$$

Consider the solution for ℓ_2^μ

$$\begin{aligned}\ell_2^\mu &= t\langle 1 |\gamma^\mu| 2] - \frac{\mu^2}{4s_{12}t}\langle 2 |\gamma^\mu| 1] \\ \ell_3^\mu &= t\langle 1 |\gamma^\mu| 2] - \frac{\mu^2}{4s_{12}t}\langle 2 |\gamma^\mu| 1] - K_2^\mu \\ \langle 1 |\ell_2| 2]^2 &= \frac{\mu^4}{4t^2} \\ \langle 3 |\ell_3| 3] &= 2t\langle 1 |3| 2] - \frac{\mu^2}{2s_{12}t}\langle 2 |3| 1] - s_{23} = -\frac{\mu^2 \langle 2 |3| 1]}{2s_{12}t}\left(1 - \frac{4s_{12}\langle 1 |3| 2] t^2 - 2s_{12}s_{23}t}{\mu^2 \langle 2 |3| 1]}\right) \\ \frac{[1 |\ell_2| 2\rangle^2}{\langle 3 |\ell_3| 3]} &= -\frac{\mu^2 s_{12}}{2\langle 1 |3| 1] t}\left(1 + \frac{4s_{12}\langle 2 |3| 1] t^2 - 2s_{12}s_{23}t}{\mu^2 \langle 2 |3| 1]}\right) \\ \inf\left[\frac{[1 |\ell_2| 2\rangle^2}{\langle 3 |\ell_3| 3]}\right]_{t^0} &= \frac{s_{12}^2 s_{23}}{\langle 2 |3| 1]^2}\end{aligned}$$

With these prescriptions,

$$\begin{aligned}\inf\left[\inf\left[A\left(-\ell_3, 3^+, 4^+, \ell_1\right) A\left(-\ell_1, 1^-, \ell_2\right) A\left(-\ell_2, 2^+, \ell_3\right)\right]_{t^0}\right]_{\mu^2} &= 2i\mu^2 \frac{[34]}{\langle 34\rangle \langle 12\rangle [21]}\frac{s_{12}^2 s_{23}}{\langle 2 |3| 1]^2} \\ C_{3;34}^{[2]}\left(1^-, 2^+, 3^+, 4^+\right) &= 2i\frac{[42]^2}{[12]\langle 23\rangle \langle 34\rangle [41]}\frac{st}{u}\frac{st}{ut}\end{aligned}$$

$C_{3;41}^{[2]}\left(1^-, 2^+, 3^+, 4^+\right)$

The product of the tree amplitudes

$$\begin{aligned}&A_4^{tree}\left(-\ell_4, 4^+, 1^-, \ell_2\right) A_3^{tree}\left(-\ell_2, 2^+, \ell_3\right) A_3^{tree}\left(-\ell_3, 3^+, \ell_4\right) = \\ &\qquad = -\frac{1}{s_{14}}\frac{i}{\langle 4 |\ell_4| 4]}\langle 1 |\ell_4| 4]\left([\ell_2 4]\langle 1\ell_4\rangle + \langle \ell_2 1\rangle [4\ell_4]\right)\frac{[23]}{\langle 23\rangle}\langle \ell_4 |\mu| \ell_2\rangle \\ &\qquad = -\frac{i}{s_{14}}\frac{[23]}{\langle 23\rangle}\frac{\langle \ell_4 |\mu| \ell_2\rangle}{\langle 4 |\ell_4| 4]}\langle 1 |\ell_4| 4]\left([\ell_2 4]\langle 1\ell_4\rangle + \langle \ell_2 1\rangle [4\ell_4]\right) = i\frac{\mu^2}{s_{14}}\frac{[23]}{\langle 23\rangle}\frac{\langle 1 |\ell_4| 4]^2}{\langle 4 |\ell_4| 4]}\end{aligned} \tag{L-105}$$

We have already obtained this result,

$$C_{3;41}^{[2]}\left(1^-, 2^+, 3^+, 4^+\right) = -i\frac{[24]^2}{[12]\langle 23\rangle \langle 34\rangle [41]}\left(t^2 - 2u^2\right)\frac{st}{u}\frac{1}{ut} \tag{L-106}$$

L.2.3. $C_3^{[2]}\left(1^-,2^-,3^+,4^+\right)$

$C_{3;12}^{[2]}\left(1^-,2^-,3^+,4^+\right)$

$$
\begin{aligned}
A\left(-\ell_1,1^-,2^-,\ell_3\right)A\left(-\ell_3,3^+,\ell_4\right)A\left(-\ell_4,4^+,\ell_1\right) &= i\frac{\langle 12\rangle\,[34]}{[12]\,\langle 34\rangle}\frac{[\ell_3\,|\mu|\,\ell_1]\,\langle\ell_1\,|\mu|\,\ell_3\rangle}{\langle 1\,|\ell_1|\,1]} \\
[\ell_3\,|\mu|\,\ell_1]\,\langle\ell_1\,|\mu|\,\ell_3\rangle &= \mathrm{Tr}\left\{\left(\frac{1-\gamma^5}{2}\right)\ell_3\mu\ell_1\mu\right\} = \mu^2\mathrm{Tr}\left\{\left(\frac{1-\gamma^5}{2}\right)\ell_3\ell_1\right\} \\
&= 2\mu^2\ell_3\cdot\ell_1 = \mu^2\left(2\mu^2 - s_{12}\right) \\
A\left(-\ell_1,1^-,2^-,\ell_3\right)A\left(-\ell_3,3^+,\ell_4\right)A\left(-\ell_4,4^+,\ell_1\right) &= i\mu^2\left(2\mu^2 - s_{12}\right)\frac{\langle 12\rangle\,[34]}{[12]\,\langle 34\rangle}\frac{1}{[1\,|\ell_1|\,1\rangle} \\
\mathrm{inf}\left[\frac{1}{[1\,|\ell_1|\,1\rangle}\right]_{t^0} &= 0 \\
C_{3;12}^{[2]}\left(1^-,2^-,3^+,4^+\right) &= 0
\end{aligned}
$$

The other coefficients vanish

M. Completeness relation of generalized spinors

Writting explicitely $u_\lambda(k,q)$ and $\bar{u}_\lambda(k,q)$ for $\lambda=\pm$, we find

$$\begin{aligned}
&\sum_{\lambda=\pm} u_\lambda(k,q)\,\bar{u}_\lambda(k,q) = \\
&= \left(\left|k^\flat\right\rangle + \frac{(m+i\mu)}{\left[k^\flat q\right]}\left|q\right]\right)\left(\left[k^\flat\right| + \frac{(m-i\mu)}{\left\langle q k^\flat\right\rangle}\left\langle q\right|\right) + \left(\left|k^\flat\right] + \frac{(m-i\mu)}{\left\langle k^\flat q\right\rangle}\left|q\right\rangle\right)\left(\left\langle k^\flat\right| + \frac{(m+i\mu)}{\left[q k^\flat\right]}\left[q\right|\right) \\
&\qquad = \not{k}^\flat + \frac{m^2+\mu^2}{2k^\flat\cdot q}\not{q} + \frac{(m+i\mu)}{\left[k^\flat q\right]}\left(\left|q\right]\left[k^\flat\right| - \left|k^\flat\right]\left[q\right|\right) + \frac{(m-i\mu)}{\left\langle q k^\flat\right\rangle}\left(\left|k^\flat\right\rangle\left\langle q\right| - \left|q\right\rangle\left\langle k^\flat\right|\right) \\
&\qquad = \not{k}^\flat + \frac{m^2+\mu^2}{2k^\flat\cdot q}\not{q} + \frac{(m+i\mu)}{2k^\flat\cdot q}\frac{1-\gamma_5}{2}\left(\not{q}\not{k}^\flat + \not{k}^\flat\not{q}\right) + \frac{(m-i\mu)}{2k^\flat\cdot q}\frac{1+\gamma_5}{2}\left(\not{k}^\flat\not{q} + \not{q}\not{k}^\flat\right) \\
&\qquad = \not{k}^\flat + \frac{m^2+\mu^2}{2k^\flat\cdot q}\not{q} + m - i\mu\gamma^5
\end{aligned} \tag{M-1}$$

finally,

$$\sum_{\lambda=\pm} u_\lambda(k,q)\,\bar{u}_\lambda(k,q) = \not{k} + m - i\mu\gamma^5 \tag{M-2}$$

in the previous calculation, we used

$$\begin{aligned}
\frac{1}{\left[k^\flat q\right]}\left(\left|q\right]\left[k^\flat\right| - \left|k^\flat\right]\left[q\right|\right) &= \frac{1}{2k^\flat\cdot q}\left(\left|q\right]\left\langle q k^\flat\right\rangle\left[k^\flat\right| + \left|k^\flat\right]\left\langle k^\flat q\right\rangle\left[q\right|\right) \\
&= \frac{1}{2k^\flat\cdot q}\left\{\left(\left|q\right]\left\langle q\right|\right)\left(\left|k^\flat\right\rangle\left[k^\flat\right|\right) + \left(\left|k^\flat\right]\left\langle k^\flat\right|\right)\left(\left|q\right\rangle\left[q\right|\right)\right\} \\
&= \frac{1}{2k^\flat\cdot q}\left\{\frac{1-\gamma_5}{2}\not{q}\frac{1+\gamma_5}{2}\not{k}^\flat + \frac{1-\gamma_5}{2}\not{q}\frac{1+\gamma_5}{2}\not{k}^\flat\right\} \\
&= \frac{1}{2k^\flat\cdot q}\frac{1-\gamma_5}{2}\left\{\not{q},\not{k}^\flat\right\} = \frac{1-\gamma_5}{2}
\end{aligned} \tag{M-3}$$

N. Three point amplitudes with gluons in $4-2\epsilon$ dimensions

N.1. $A_3\left(1^+, l_2^-, -l_1^+\right)$

$$A_3\left(1^+, l_2^-, -l_1^+\right) = \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}\left(K_1 - l_2\right)^\lambda + g^{\nu\lambda}\left(l_2 + l_1\right)^\mu + g^{\lambda\mu}\left(-l_1 - K_1\right)^\nu\right]\varepsilon_\mu^+\left(1\right)\varepsilon_\nu^-\left(l_2\right)\varepsilon_\lambda^+\left(-l_1\right) \tag{N-1}$$

$$= \frac{ig}{\sqrt{2}}\left[\varepsilon^+\left(1\right)\cdot\varepsilon^-\left(l_2\right)\left(K_1 - l_2\right)\cdot\varepsilon^+\left(-l_1\right) + \varepsilon^-\left(l_2\right)\cdot\varepsilon^+\left(-l_1\right)\left(l_2 + l_1\right)\cdot\varepsilon^+\left(1\right) + \right.$$
$$\left. + \varepsilon^+\left(1\right)\cdot\varepsilon^+\left(-l_1\right)\left(-l_1 - K_1\right)\cdot\varepsilon^-\left(l_2\right)\right] \tag{N-2}$$

studying each term in (N-2),

$$\varepsilon^+\left(1\right)\cdot\varepsilon^-\left(l_2\right) = -\frac{\left\langle q l_2^\flat\right\rangle\left[\bar{l}_2 1\right]}{\left\langle q1\right\rangle\mu} \tag{N-3}$$

$$\left(K_1 - l_2\right)\cdot\varepsilon^+\left(-l_1\right) = 2\frac{\left[l_1^\flat\left|1\right|\bar{l}_1\right\rangle}{\sqrt{2}\mu} \tag{N-4}$$

$$\varepsilon^-\left(l_2\right)\cdot\varepsilon^+\left(-l_1\right) = -\frac{\left\langle l_2^\flat\bar{l}_1\right\rangle\left[l_1^\flat\bar{l}_2\right]}{\mu^2} \tag{N-5}$$

$$\left(l_2 + l_1\right)\cdot\varepsilon^+\left(1\right) = 2\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\left\langle q1\right\rangle} \tag{N-6}$$

$$\varepsilon^+\left(1\right)\cdot\varepsilon^+\left(-l_1\right) = -\frac{\left\langle q\bar{l}_1\right\rangle\left[l_1^\flat 1\right]}{\left\langle q1\right\rangle\mu} = 0 \tag{N-7}$$

$$\left(-l_1 - K_1\right)\cdot\varepsilon^-\left(l_2\right) = 2\frac{\left\langle l_2^\flat\left|1\right|\bar{l}_2\right]}{\sqrt{2}\mu} \tag{N-8}$$

(N-2) amounts,

$$A_3\left(1^+, l_2^-, -l_1^+\right) = -\frac{ig}{\mu^2\left\langle q1\right\rangle}\left[\left[l_1^\flat\left|1\right|\bar{l}_1\right\rangle\left\langle q l_2^\flat\right\rangle\left[\bar{l}_2 1\right] + \left\langle l_2^\flat\bar{l}_1\right\rangle\left[l_1^\flat\bar{l}_2\right]\left\langle q\left|l_1^\flat\right|1\right]\right] \tag{N-9}$$

we choose the reference vector of $p_1$as,

$$q = \bar{l}_1 = \bar{l}_2 = \bar{l} \tag{N-10}$$

finally the amplitude,

$$\begin{aligned} A_3\left(1^+, l_2^-, -l_1^+\right) &= -\frac{ig}{\mu^2\left\langle \bar{l}1\right\rangle}\left[\left[l_1^\flat\left|1\right|\bar{l}\right\rangle\left\langle \bar{l}l_2^\flat\right\rangle\left[\bar{l}1\right] + \left\langle l_2^\flat\bar{l}\right\rangle\left[l_1^\flat\bar{l}\right]\left\langle \bar{l}\left|l_1^\flat\right|1\right]\right] \\ &= \frac{ig}{\mu^2\left\langle \bar{l}1\right\rangle}\left[\left[l_1^\flat 1\right]\left\langle \bar{l}l_2^\flat\right\rangle\left\langle 1\bar{l}\right\rangle\left[\bar{l}1\right] - \left\langle l_2^\flat\bar{l}\right\rangle\left[l_1^\flat\bar{l}\right]\left\langle \bar{l}l_1^\flat\right\rangle\left[l_1^\flat 1\right]\right] \\ &= -\frac{ig}{\mu^2}\frac{\left[l_1^\flat 1\right]\left\langle \bar{l}l_2^\flat\right\rangle}{\left\langle \bar{l}1\right\rangle}\left\langle \bar{l}\left|1-l_1^\flat\right|\bar{l}\right] = \frac{ig}{\mu^2}\frac{\left[l_1^\flat 1\right]\left\langle \bar{l}l_2^\flat\right\rangle}{\left\langle \bar{l}1\right\rangle}\left\langle \bar{l}\left|l_2^\flat\right|\bar{l}\right] \\ &= ig\frac{\left[l_1^\flat 1\right]\left[l_2^\flat\bar{l}\right]}{\left\langle \bar{l}1\right\rangle}\times\frac{\left[l_1^\flat 1\right]^3}{\left[l_1^\flat 1\right]^3} = -ig\frac{\left[1l_1^\flat\right]^4}{\left[1l_2^\flat\right]\left[l_2^\flat l_1^\flat\right]\left[l_1^\flat 1\right]} \end{aligned} \tag{N-11}$$

N.2. $A_3\left(1^+, l_2^-, l_1^-\right)$

$$A_3\left(1^+, l_2^-, -l_1^-\right) = \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}\left(K_1-l_2\right)^\lambda + g^{\nu\lambda}\left(l_2+l_1\right)^\mu + g^{\lambda\mu}\left(-l_1-K_1\right)^\nu\right]\varepsilon_\mu^+(1)\,\varepsilon_\nu^-(l_2)\,\varepsilon_\lambda^-(-l_1) \tag{N-12}$$

$$\begin{aligned} = \frac{ig}{\sqrt{2}}&\left[\varepsilon^+(1)\cdot\varepsilon^-(l_2)\left(K_1-l_2\right)\cdot\varepsilon^-(-l_1) + \varepsilon^-(l_2)\cdot\varepsilon^-(-l_1)\left(l_2+l_1\right)\cdot\varepsilon^+(1) + \right. \\ &\left. + \varepsilon^+(1)\cdot\varepsilon^-(-l_1)\left(-l_1-K_1\right)\cdot\varepsilon^-(l_2)\right] \end{aligned} \tag{N-13}$$

each term,

$$\varepsilon^+(1)\cdot\varepsilon^-(l_2) = -\frac{\left\langle ql_2^\flat\right\rangle\left[\bar{l}_2 1\right]}{\left\langle q1\right\rangle\mu} \tag{N-14}$$

$$\left(K_1-l_2\right)\cdot\varepsilon^-(-l_1) = 2\frac{\left\langle l_1^\flat\left|1\right|\bar{l}_1\right]}{\sqrt{2}\mu} \tag{N-15}$$

$$\varepsilon^-(l_2)\cdot\varepsilon^-(-l_1) = -\frac{\left\langle l_2^\flat l_1^\flat\right\rangle\left[\bar{l}_1\bar{l}_2\right]}{\mu^2} = 0 \tag{N-16}$$

$$\left(l_2+l_1\right)\cdot\varepsilon^+(1) = 2\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\left\langle q1\right\rangle} \tag{N-17}$$

$$\varepsilon^+(1)\cdot\varepsilon^-(-l_1) = \frac{\left\langle ql_1^\flat\right\rangle\left[\bar{l}_1 1\right]}{\left\langle q1\right\rangle\mu} \tag{N-18}$$

$$\left(-l_1-K_1\right)\cdot\varepsilon^-(l_2) = 2\frac{\left\langle l_2^\flat\left|1\right|\bar{l}_2\right]}{\sqrt{2}\mu} \tag{N-19}$$

The amplitude takes the form,

$$A_3\left(1^+, l_2^-, -l_1^-\right) = -\frac{ig}{\left\langle q1\right\rangle\mu^2}\left[\left\langle ql_2^\flat\right\rangle\left[\bar{l}_2 1\right]\left\langle l_1^\flat\left|1\right|\bar{l}_1\right] - \left\langle ql_1^\flat\right\rangle\left[\bar{l}_1 1\right]\left\langle l_2^\flat\left|1\right|\bar{l}_2\right]\right] \tag{N-20}$$

with the reference vector,

$$q = \bar{l}_2 = \bar{l}_1 = \bar{l} \tag{N-21}$$

we obtain,

$$\begin{aligned}
A_3\left(1^+, l_2^-, -l_1^-\right) &= -\frac{ig}{\left\langle \bar{l}1\right\rangle \mu^2}\left[-\left\langle \bar{l}l_2^\flat\right\rangle \left[\bar{l}1\right] \left\langle l_1^\flat \left| l_2^\flat \right| \bar{l}\right] - \left\langle \bar{l}l_1^\flat\right\rangle \left[\bar{l}1\right] \left\langle l_2^\flat \left| l_1^\flat \right| \bar{l}\right]\right] \\
&= \frac{ig}{\mu^2}\frac{\left\langle l_1^\flat l_2^\flat\right\rangle \left[\bar{l}1\right]}{\left\langle \bar{l}1\right\rangle}\left[\left\langle \bar{l}l_2^\flat\right\rangle \left[l_2^\flat \bar{l}\right] - \left\langle \bar{l}l_1^\flat\right\rangle \left[l_1^\flat \bar{l}\right]\right] = -\frac{ig}{\mu^2}\left\langle l_2^\flat l_1^\flat\right\rangle \left[\bar{l}1\right]^2 \\
&= -\frac{ig}{\mu^2}\left\langle l_2^\flat l_1^\flat\right\rangle \left[\bar{l}1\right]^2 \times \frac{\left\langle l_2^\flat l_1^\flat\right\rangle^3}{\left\langle l_2^\flat l_1^\flat\right\rangle^3} = ig\frac{\left\langle l_2^\flat l_1^\flat\right\rangle^4}{\left\langle 1l_2^\flat\right\rangle \left\langle l_2^\flat l_1^\flat\right\rangle \left\langle l_1^\flat 1\right\rangle}
\end{aligned} \tag{N-22}$$

N.3. $A_3\left(1^-, l_2^+, l_1^-\right)$

By parity in eq. (N-11)

$$A_3\left(1^-, l_2^+, -l_1^-\right) = ig\frac{\left\langle 1l_1^\flat\right\rangle^4}{\left\langle 1l_2^\flat\right\rangle \left\langle l_2^\flat l_1^\flat\right\rangle \left\langle l_1^\flat 1\right\rangle} \tag{N-23}$$

N.4. $A_3\left(1^-, l_2^+, l_1^+\right)$

By parity in eq. (N-22)

$$A_3\left(1^-, l_2^+, -l_1^-\right) = ig\frac{\left[l_2^\flat l_1^\flat\right]^4}{\left[1l_2^\flat\right] \left[l_2^\flat l_1^\flat\right] \left[l_1^\flat 1\right]} \tag{N-24}$$

N.5. $A_3\left(1^+, l_2^+, -l_1^0\right)$

$$A_3\left(1^+, l_2^+, -l_1^0\right) = \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}\left(K_1 - l_2\right)^\lambda + g^{\nu\lambda}\left(l_2 + l_1\right)^\mu + g^{\lambda\mu}\left(-l_1 - K_1\right)^\nu\right] \varepsilon_\mu^+(1)\, \varepsilon_\nu^+(l_2)\, \varepsilon_\lambda^0(-l_1) \tag{N-25}$$

$$\begin{aligned}
&= \frac{ig}{\sqrt{2}}\left[\varepsilon^+(1)\cdot\varepsilon^+(l_2)\left(K_1 - l_2\right)\cdot\varepsilon^0(-l_1) + \varepsilon^+(l_2)\cdot\varepsilon^0(-l_1)\left(l_2 + l_1\right)\cdot\varepsilon^+(1) + \right. \\
&\quad \left. + \varepsilon^+(1)\cdot\varepsilon^0(-l_1)\left(-l_1 - K_1\right)\cdot\varepsilon^+(l_2)\right]
\end{aligned} \tag{N-26}$$

studying each term,

$$\varepsilon^+(1)\cdot\varepsilon^+(l_2) = \frac{\langle q\bar{l}_2\rangle\left[l_2^\flat 1\right]}{\langle 1q\rangle\,\mu} = 0 \tag{N-27}$$

$$(K_1 - l_2)\cdot\varepsilon^0(-l_1) = 0 \tag{N-28}$$

$$\varepsilon^+(l_2)\cdot\varepsilon^0(-l_1) = \frac{\left[l_2^\flat\left|1\right|\bar{l}_2\right\rangle}{\sqrt{2}\mu^2} \tag{N-29}$$

$$(l_2 + l_1)\cdot\varepsilon^+(1) = 2\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\,\langle q1\rangle} \tag{N-30}$$

$$\varepsilon^+(1)\cdot\varepsilon^0(-l_1) = -\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\,\langle q1\rangle\,\mu} \tag{N-31}$$

$$(-l_1 - K_1)\cdot\varepsilon^+(l_2) = 2\frac{\left[l_2^\flat\left|1\right|\bar{l}_2\right\rangle}{\sqrt{2}\mu} \tag{N-32}$$

the amplitude,

$$\begin{aligned} A_3\left(1^+, l_2^+, -l_1^0\right) &= \frac{ig}{\sqrt{2}}\left[\frac{\left[l_2^\flat\left|l_1^\flat\right|\bar{l}_2\right\rangle}{2\mu^2}2\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\,\langle q1\rangle} - \frac{\left\langle q\left|l_1^\flat\right|1\right]}{2\,\langle q1\rangle\,\mu}2\frac{\left[l_2^\flat\left|1\right|\bar{l}_2\right\rangle}{\sqrt{2}\mu}\right] \\ &= \frac{ig}{\mu^2}\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\langle q1\rangle}\left[\left[l_2^\flat\left|l_1^\flat\right|\bar{l}_2\right\rangle - \left[l_2^\flat\left|1\right|\bar{l}_2\right\rangle\right] = \frac{ig}{\mu^2}\frac{\left\langle q\left|l_1^\flat\right|1\right]\left[l_2^\flat\left|-l_2^\flat\right|\bar{l}_2\right\rangle}{\langle q1\rangle} = 0 \end{aligned} \tag{N-33}$$

N.5.1. $A_3\left(1^+, l_2^-, -l_1^0\right)$

$$A_3\left(1^+, l_2^-, -l_1^0\right) = \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}(K_1 - l_2)^\lambda + g^{\nu\lambda}(l_2 + l_1)^\mu + g^{\lambda\mu}(-l_1 - K_1)^\nu\right]\varepsilon_\mu^+(1)\,\varepsilon_\nu^-(l_2)\,\varepsilon_\lambda^+(-l_1) \tag{N-34}$$

$$\begin{aligned} &= \frac{ig}{\sqrt{2}}\left[\varepsilon^+(1)\cdot\varepsilon^-(l_2)\,(K_1 - l_2)\cdot\varepsilon^0(-l_1) + \varepsilon^-(l_2)\cdot\varepsilon^0(-l_1)\,(l_2 + l_1)\cdot\varepsilon^+(1) + \right. \\ &\quad \left. + \,\varepsilon^+(1)\cdot\varepsilon^0(-l_1)\,(-l_1 - K_1)\cdot\varepsilon^-(l_2)\right] \end{aligned} \tag{N-35}$$

each term,

$$\varepsilon^+(1)\cdot\varepsilon^-(l_2) = -\frac{\langle ql_2^\flat\rangle\left[\bar{l}_2 1\right]}{\langle q1\rangle\,\mu} \tag{N-36}$$

$$(K_1 - l_2)\cdot\varepsilon^0(-l_1) = 0 \tag{N-37}$$

$$\varepsilon^-(l_2)\cdot\varepsilon^0(-l_1) = \frac{\left\langle l_2^\flat\left|l_1^\flat\right|\bar{l}_2\right]}{\sqrt{2}\mu^2} \tag{N-38}$$

$$(l_2 + l_1)\cdot\varepsilon^+(1) = 2\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\,\langle q1\rangle} \tag{N-39}$$

$$\varepsilon^+(1)\cdot\varepsilon^0(-l_1) = -\frac{\left\langle q\left|l_1^\flat\right|1\right]}{\sqrt{2}\,\langle q1\rangle\,\mu} \tag{N-40}$$

$$(-l_1 - K_1)\cdot\varepsilon^-(l_2) = 2\frac{\left\langle l_2^\flat\left|1\right|\bar{l}_2\right]}{\sqrt{2}\mu} \tag{N-41}$$

finally,

$$A_3\left(1^+,l_2^-,-l_1^0\right)=\frac{ig}{\sqrt{2}\mu^2}\frac{\langle l_2^\flat\left|l_1^\flat\right|\bar{l}_2]}{\langle q1\rangle}\left[\frac{\langle q\left|l_1^\flat\right|1]}{\langle q1\rangle}-\frac{\langle q\left|l_1^\flat\right|1]}{\langle q1\rangle}\right]=0 \tag{N-42}$$

N.5.2. $A_3\left(1^-,l_2^-,-l_1^0\right)$

By parity in eq. (N-33)

$$A_3\left(1^-,l_2^-,-l_1^0\right)=0 \tag{N-43}$$

N.5.3. $A_3\left(1^-,l_2^+,-l_1^0\right)$

By parity in eq. (N-35)

$$A_3\left(1^-,l_2^+,-l_1^0\right)=0 \tag{N-44}$$

N.5.4. $A_3\left(1^+,l_2^0,-l_1^0\right)$

Finally, consider a gluon in 4 dimensions and the other two in $4-2\epsilon$ dimension with zero polarization,

$$A_3\left(1^+,l_2^0,-l_1^0\right)=\frac{ig}{\sqrt{2}}\left[g^{\mu\nu}\left(K_1-l_2\right)^\lambda+g^{\nu\lambda}\left(l_2+l_1\right)^\mu+g^{\lambda\mu}\left(-l_1-K_1\right)^\nu\right]\varepsilon_\mu^+\left(1\right)\varepsilon_\nu^0\left(l_2\right)\varepsilon_\lambda^0\left(-l_1\right) \tag{N-45}$$

$$=\frac{ig}{\sqrt{2}}\left[\varepsilon^+\left(1\right)\cdot\varepsilon^0\left(l_2\right)\left(K_1-l_2\right)\cdot\varepsilon^0\left(-l_1\right)+\varepsilon^0\left(l_2\right)\cdot\varepsilon^0\left(-l_1\right)\left(l_2+l_1\right)\cdot\varepsilon^+\left(1\right)+\right.$$
$$\left.+\,\varepsilon^+\left(1\right)\cdot\varepsilon^0\left(-l_1\right)\left(-l_1-K_1\right)\cdot\varepsilon^0\left(l_2\right)\right] \tag{N-46}$$

each term is given by,

$$\varepsilon^+\left(1\right)\cdot\varepsilon^0\left(l_2\right)=\frac{\langle q\left|l_2^\flat-\bar{l}\right|1]}{\sqrt{2}\mu\left\langle q1\right\rangle} \tag{N-47}$$

$$\left(K_1-l_2\right)\cdot\varepsilon^0\left(-l_1\right)=0 \tag{N-48}$$

$$\varepsilon^0\left(l_2\right)\cdot\varepsilon^0\left(-l_1\right)=1 \tag{N-49}$$

$$\left(l_2+l_1\right)\cdot\varepsilon^+\left(1\right)=2\frac{\langle q\left|l_1^\flat+\bar{l}\right|1]}{\sqrt{2}\left\langle q1\right\rangle}=\frac{\sqrt{2}\left\langle q\left|l_1\right|1\right]}{\langle q1\rangle} \tag{N-50}$$

$$\varepsilon^+\left(1\right)\cdot\varepsilon^0\left(-l_1\right)=-\frac{\langle q\left|l_1^\flat-\bar{l}\right|1]}{2\left\langle q1\right\rangle} \tag{N-51}$$

$$\left(-l_1-K_1\right)\cdot\varepsilon^0\left(l_2\right)=0 \tag{N-52}$$

the amplitude takes the form,

$$A_3\left(1^+,l_2^0,-l_1^0\right)=ig\frac{\langle q\left|l_1\right|1]}{\langle q1\rangle} \tag{N-53}$$

N.5.5. $A_3\left(1^-, l_2^0, -l_1^0\right)$

By parity in eq. (N-53)

$$A_3\left(1^-, l_2^0, -l_1^0\right) = ig\frac{[q\,|l_1|\,1\rangle}{[q1]} \tag{N-54}$$

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